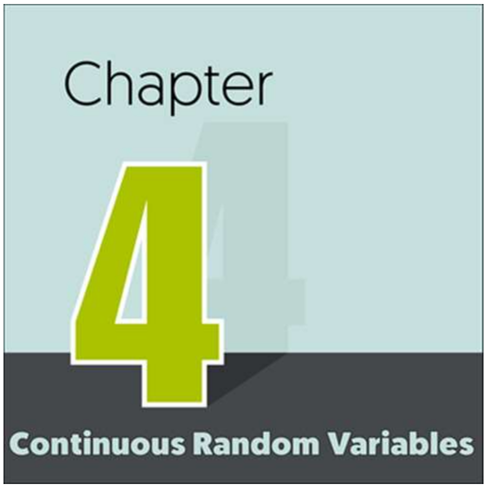


In the last chapter we studied discrete random variables. Discrete random variable can only take on, or produce, a finite or countable number of possible values. That is, the support is finite or countable.

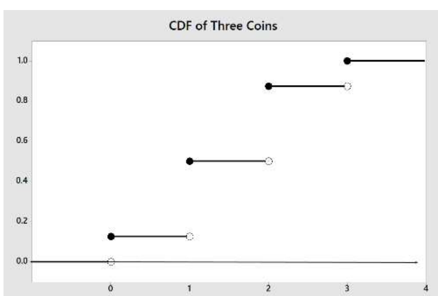
In this chapter, we study a totally different class of random variables called **Continuous Random Variables**. Like discrete random variables, continuous random variables turn a sample space into the real numbers. Like discrete random variables, associated with a continuous random variable is the distribution. It is the CDF and the support that distinguishes discrete random variables from continuous random variables.

Suppose that the support of a random variable was an interval or a union of intervals. This random variable would not be discrete since the support is uncountable. An example of such a random variable would be a random variable that measures time, height, weight or any measurement that can take on any value over an interval. So not every random variable is discrete.

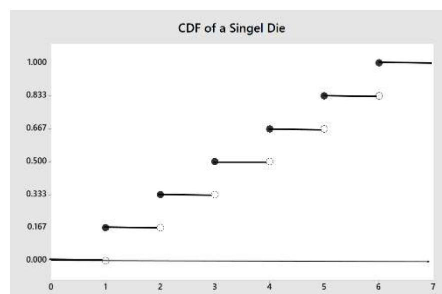
4.1 Continuous Distributions – the CDF

Recall that for a random variable X, we defined the pmf as $f(x) = P(X = x)$ and then we defined the **Cumulative Distribution Function** (CDF) as $F(x) = P(X \le x)$. In the discrete case, our CDF was a non-decreasing function that suddenly stepped up at each value in our support.

x	F(x)
0	1/8
1	4/8
2	7/8
3	1



x	F(x)
1	1/6
2	2/6
3	3/6
4	4/6
5	5/6
6	1



Neither CDF listed above is function that is **not** continuous, but rather a right-continuous step function. In the case of a discrete random variable, the CDF can never be a continuous function. We now introduce a new type of random variable with a continuous CDF.

Definition: *Let X be a random variable with CDF* $F(x) = P(X \le x)$. *If F(x) is a continuous function, then X is said to be a continuous random variable.*

Note that the definition of the CDF has not changed. It is still $F(x) = P(X \le x)$. The CDF still has (and must have) the same properties as before:

$\lim_{x \to -\infty} F(x) = 0$ As we go to negative infinity, the remaining probability to the left goes to 0.

$\lim_{x \to \infty} F(x) = 1$ The probability goes to 1 in limit, but may never actually equal 1.

$A \le B \Rightarrow F(A) \le F(B)$ This means that the CDF is non-decreasing.

The only difference is that $F(x)$ is a continuous function. We now consider several examples and exercises to get a feel for this new type of random variable.

Example: *Suppose that we were interested in the random variable X which measures how long individuals could hold their breath. The set of possible outcomes would clearly be uncountable. Therefore, X would not be a discrete random variable. It seems clear that the CDF would have no discontinuities. Thus, X is a continuous random variable.*

Since time, weight, height and other types of measurements, often take on an uncountably infinite of possibilities, random variables related to these measurements would be continuous random variables.

Example: Here are some examples of functions that can serve as a CDF for some random variable. Each of these represents a CDF for some random variable since they are all non-decreasing continuous functions that also do not violate either of the two limit statements. (Plot them.)

$$F(x) = \begin{cases} 0 & x < 1 \\ x - 1 & 1 \le x \le 2 \\ 1 & 2 < x \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \le x \end{cases}$$

$$F(x) = \frac{e^x}{1 + e^x} \quad -\infty < x < \infty$$

Exercise: *If the random variable X has CDF* $F(x)=\begin{cases} 0 & x \le a \\ \dfrac{x^2-3}{x^2+7} & a < x < \infty \end{cases}$ *, determine the value of a.*

Example: *Consider the random variable X with CDF* $F(x)=\begin{cases} 0 & x<1 \\ x-1 & 1\le x \le 2 \\ 1 & 2<x \end{cases}$.

Determine $P(1.2 < X \le 1.3)$.

Example: *(Continuing with the same CDF as above) Determine* $P(1.2 \le X \le 1.3)$.

$P(1.2 \le X \le 1.3) = F(1.3) - F(1.2^-) = F(1.3) - F(1.2) = .3 - .2 = .1$

This is certainly a friendly feature of continuous distributions:

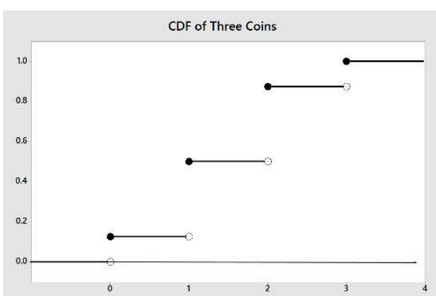
$P(A \le X \le B) = P(A \le X < B) = P(A < X \le B) = P(A < X < B)$

Exercise: *(Continuing with the same CDF as above) Determine* $P(0 \le X \le 1.4)$

Exercise: *(Continuing with the same CDF as above) Determine* $P(1.5 \le X \le 3)$

A quick note:

x	F(x)
0	1/8
1	4/8
2	7/8
3	1



With discrete random variables, we could use the CDF to determine answers to questions like $P(X = x)$. $P(X = x) = F(x) - F(x^-)$. Here, $P(X = 1.5) = F(1.5) - F(1.5^-) = .5 - .5 = 0$. Also, $P(X = 1) = F(1) - F(1^-) = .5 - .125 = .375$

For a continuous random variable *X* with CDF $F(x)$, the same is true. Since $F(x)$ is continuous, $F(x)$ has no steps (no jumps), so $P(X = x) = F(x) - F(x^-) = F(x) - F(x) = 0$. We will need to investigate this.

Continuing our investigation into the statement that $P(X = x) = 0$ for all x-values yields more interesting ideas.

Example: Consider the random variable *X* with CDF $F(x)=\begin{cases}0 & x<1\\ x-1 & 1\le x\le 2\\ 1 & 2<x\end{cases}$.

As already noted, $P(1.2\le X\le 1.3)=.1$. So, the probability of getting any exact value between 1.2 and 1.3 is zero, yet we amass some probability along the way. Looking more generically at the same idea:

$P(x\le X\le x+\Delta x)=F(x+\Delta x)-F(x^{-})=F(x+\Delta x)-F(x)=(x+\Delta x-1)-(x-1)=\Delta x$. We see that the probability is "evenly spread" amongst the x-values between 1 and 2. If we look at the ratio of the probability of being in an interval to the length of the interval we get:

$\dfrac{P(x\le X\le x+\Delta x)}{(x+\Delta x)-x}=\dfrac{\Delta x}{\Delta x}=1$. Note that this is $F'(x)$.

Exercise: *Let's play the same way with a random variable X that has CDF* $F(x)=\begin{cases}0 & x<0\\ x^2 & 0\le x\le 1\\ 1 & 1<x\end{cases}$

4.2 Continuous Distributions – the pdf

Definition: *Let X be a continuous random variable X with CDF* $F(x)$. *The probability density function (pdf) of X is defined as* $f(x)=F'(x)$ *for all x where* $F'(x)$ *exists.*

Definition: *The support of a continuous random variable X is defined to be all x such that* $f(x)>0$.

Example: Given the CDF of a random variable *X*, determine the pdf of *X*, the support of *X* and then plot the pdf.

$$F(x)=\begin{cases}0 & x<0\\ x^2 & 0\le x\le 1\\ 1 & 1<x\end{cases} \qquad f(x)=F'(x)=\begin{cases}2x & 0\le x\le 1\\ 0 & \text{Otherwise}\end{cases}$$

Exercise: Given the CDF of a random variable *X*, determine the pdf of *X*, the support of *X* and then plot the pdf.

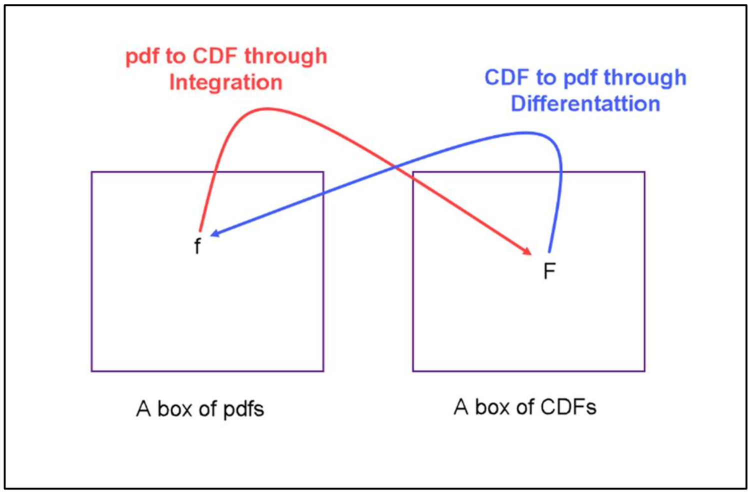
$$F(x)=\begin{cases}0 & x<0\\ 1-e^{-x} & 0\le x\end{cases}$$

Exercise: Given the CDF of a random variable *X*, determine the pdf of *X*, the support of *X* and then plot the pdf.

$$F(x)=\begin{cases}0 & x<0\\ .5x & 0\le x\le 1\\ .5x^2 & 1<x\le\sqrt{2}\\ 1 & \sqrt{2}<x\end{cases}$$

Theorem: Given a random variable X with CDF $F(x)$ and pmf $f(x)$ (determined from $F(x)$, $P(X\le x)=F(x)=\int_{-\infty}^{x} f(t)dt$. Note then: $P(A\le X\le B)=F(x)=\int_{A}^{B} f(x)dx=F(B)-F(A)$

Proof: The proof is a direct result of the Fundamental Theorem of Calculus.



Note: The functions $F(x)$ are all continuous. The functions $f(x)$ might not be. This is why we reference the CDF when trying to define what is meant by a continuous random variable.

Example: *Determine the value of k, so that* $f(x)=\begin{cases} kx & 1\le x\le 5 \\ 0 & \text{otherwise} \end{cases}$ *is a pdf for some random variable X.*

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} 0\,dx + \int_{1}^{5} x\,dx + \int_{5}^{\infty} 0\,dx = \left.\frac{x^2}{2}\right|_{x=1}^{x=5} = \frac{25}{2} - \frac{1}{2} = 12$ This must integrate to 1, so $k = \frac{1}{12}$.

Example: *For the above random variable, determine* $F(x)$.

For $1 < x < 5$, $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{1}^{x} \frac{t}{12}\,dt = \left.\frac{1}{12}\frac{t^2}{2}\right|_{t=1}^{x} = \frac{x^2-1}{24}$ Thus, $F(x)=\begin{cases} 0 & x<1 \\ \frac{x^2-1}{24} & 1\le x\le 5 \\ 1 & 5<x \end{cases}$

Exercise: *Given the pdf for X is* $f(x)=\begin{cases} \frac{2x}{15} & 1\le x\le 4 \\ 0 & \text{otherwise} \end{cases}$ *, determine the following:*

a) Determine $F(x)$

b) Determine $P(X \le 2)$

c) Graph $F(x)$

Obviously, weakness in differentiation and integration will end any hope of securing a good grade on the next exam. Also, it gets much worse!!

4.3 The Expected Value of a Continuous Random Variable

As in the case with discrete random variables, we wish to determine the theoretical mean (average) of continuous random variables. That is, if our random variable produces endless amounts of data, what should we expect the mean of the data to be near.

Definition: *The mean (expected value) of a continuous random variable* X *, is defined by the formula* $\mu = \int_{-\infty}^{\infty} xf(x)dx$ *(provided the integral converges)*

Example: *Determine the mean of X if the pdf is given by* $f(x) = \begin{cases} \dfrac{x^2}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{-1} xf(x)dx + \int_{-1}^{2} xf(x)dx + \int_{2}^{\infty} xf(x)dx = \int_{-\infty}^{-1} x(0)dx + \int_{-1}^{2} x\frac{x^2}{3}dx + \int_{2}^{\infty} x(0)dx$$

$$= \int_{-1}^{2} x\frac{x^2}{3}dx = \left.\frac{x^4}{12}\right|_{x=-1}^{x=2} = \frac{16}{12} - \frac{1}{12} = 1.25$$

Exercise: *Determine the mean of X if the pdf is given by* $f(x) = \begin{cases} \dfrac{x}{8} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

Example: *Determine the mean of X if the pdf is given by* $f(x) = \dfrac{1}{\pi(1+x^2)}$ with support $-\infty < x < \infty$

$E[X] = \int_{-\infty}^{\infty} \dfrac{x}{\pi(1+x^2)}dx = \dfrac{1}{\pi}\int_{-\infty}^{0} \dfrac{x}{1+x^2}dx + \dfrac{1}{\pi}\int_{0}^{\infty} \dfrac{x}{1+x^2}dx$ The original exists only of each of the other two integrals converge (see definition of $\int_{-\infty}^{\infty} f(x)dx$ in your calculus book). We now evaluate the second improper integral. We will include $(1/\pi)$ at the end if necessary.

$$\int_{0}^{\infty} \frac{x}{1+x^2}dx = \frac{1}{2}\lim_{t\to\infty}\int_{0}^{t} \frac{2x}{1+x^2}dx = \frac{1}{2}\lim_{t\to\infty}\left[\ln(1+x^2)\right]_{x=0}^{x=t} = \frac{1}{2}\lim_{t\to\infty}\left[\ln(1+t^2)\right] - \ln(1) = \frac{1}{2}\lim_{t\to\infty}\left[\ln(1+t^2)\right] \to \infty$$

Since this integral diverges, the original integral diverges. This random variable does not have a mean.

4.4 The Expected Value of a Function of a Random Variable

Definition: *The variance of a random variable X is defined by* $Var[X] = E[(X-\mu)^2]$

Theorem: *Suppose that X is a random variable with pdf* $f(x)$ *and let* $Y = g(X)$ *be some function of X.*

Then, $E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$

Using the above theorem, $Var[X] = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$.

This leads to the shortcut formula for the variance: $Var[X] = E[X^2] - E^2[X]$ as seen below.

$$Var[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - 2\mu \int_{-\infty}^{\infty} xf(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$$

Example: *Determine the variance of X if the pdf is given by* $f(x) = \begin{cases} \dfrac{x^2}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

We have already determined $E[X] = 1.25$. $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-1}^{2} x^2 \dfrac{x^2}{3} dx = \dfrac{x^5}{15}\Big|_{x=-1}^{x=2} = \dfrac{32}{15} + \dfrac{1}{15} = \dfrac{33}{15}$

So, $Var[X] = \dfrac{33}{15} - \left(\dfrac{5}{4}\right)^2 = .6375$

Example: *Determine the variance of X if the pdf is given by* $f(x) = \begin{cases} \dfrac{x}{8} & 0 < x < 4 \\ 0 & \text{otherwise} \end{cases}$

4.5 The Uniform Distribution

Definition: *A continuous random variable with pdf:* $f(x)=\begin{cases}\dfrac{1}{B-A} & A<x<B\\ 0 & \text{otherwise}\end{cases}$ *is said to be a Uniformly Distributed Random Variable and the pdf is the* ***Uniform Distribution***.

Theorem: If $X \sim \text{Unif}[A,B]$, then $E[X]=\dfrac{A+B}{2}$

Proof: Clearly the mean is the midpoint of A and B. $E[X]=\int_A^B \dfrac{x}{B-A}dx=\dfrac{1}{B-A}\left(\dfrac{x^2}{2}\right)\Bigg|_A^B=\dfrac{B^2-A^2}{2(B-A)}=\dfrac{A+B}{2}$

Theorem: If $X \sim \text{Unif}[A,B]$, then $\text{Var}[X]=\dfrac{(B-A)^2}{12}$

Proof: $E[X^2]=\int_A^B \dfrac{x^2}{B-A}dx=\dfrac{1}{B-A}\left(\dfrac{x^3}{3}\right)\Bigg|_A^B=\dfrac{B^3-A^3}{3(B-A)}=\dfrac{A^2+AB+B^2}{3}$

$$\text{Var}[X]=\frac{A^2+AB+B^2}{3}-\left(\frac{A+B}{2}\right)^2=\frac{4A^2+4AB+4B^2}{12}-\frac{3A^2+6AB+3B^2}{12}=\frac{(B-A)^2}{12}$$

Theorem: *If* $X \sim \text{Unif}[A,B]$, *then* $F(x)=\begin{cases}0 & x\le A\\ \dfrac{x-A}{B-A} & A<x<B\\ 1 & B\le x\end{cases}$

Proof: $F[x]=\int_A^x \dfrac{dt}{B-A}=\dfrac{t}{B-A}\Bigg|_A^x=\dfrac{x-A}{B-A}$

Exercise: *If* $X \sim \text{Unif}[-1,5]$, *determine the mean, variance, and CDF of X.*

Exercise: *If* $X \sim \text{Unif}[-1,5]$, *determine* $P(-.5<X<2.5)$

4.6 The Exponential Distribution

Definition: *A continuous random variable with pdf:* $f(x)=\begin{cases} 0 & x\le 0 \\ be^{-bx} & 0<x \end{cases}$ *is said to be an Exponentially Distributed Random Variable and the pdf is the* ***Exponential Distribution****.*

Theorem: *If* $X \sim \text{Exp}[b]$*, then* $E[X]=\dfrac{1}{b}$

Proof: $E[X]=\int_0^\infty bxe^{-bx}dx$ (Integration by parts)

Theorem: *If* $X \sim \text{Exp}[b]$*, then* $\text{Var}[X]=\dfrac{1}{b^2}$

Proof: $E[X^2]=\int_0^\infty bx^2e^{-bx}dx=\dfrac{2}{b^2}$ (Integration by parts twice – do it)

Theorem*: If* $X \sim \text{Exp}[b]$*, then* $F(x)=\begin{cases} 0 & x\le 0 \\ 1-e^{-bx} & 0<x \end{cases}$

Proof: $F[x]=\int_0^x be^{-bt}dt=-e^{-bt}\Big|_{t=0}^{t=x}=-e^{-bx}-(-e^0)=1-e^{-bx}$

Example: *X is known to follow an exponential distribution with parameter* $b=3$*. Determine the mean, variance, standard deviation, and CDF of X.*

$\mu=\dfrac{1}{3}$, $\sigma^2=\dfrac{1}{9}$, $\sigma=\dfrac{1}{3}$, $F(x)=1-e^{-3x}$

Exercise: *X is known to follow an exponential distribution with* $\mu=2$*. Determine the variance, standard deviation, and CDF of X.*

Exercise: *X is known to follow an exponential distribution with* $\mu=2$*. Determine* $P(3<X)$ *and* $P(-2<X<2)$*.*