Coherent State Representation of Photons

Special Topics in Elementary Particle Physics

Utkarsh

Roll no.: 190260044

May 09, 2022

PH 572

Introduction

Consider a macroscopic simple harmonic oscillator, and to keep things simple assume there are no interactions with the rest of the universe. We know how to describe the motion using classical mechanics: for a given initial position and momentum, classical mechanics correctly predicts the future path, as confirmed by experiments with real (admittedly not perfect) systems. But from the Hamiltonian we could also write down Schrödinger's equation, and from that predict the future behavior of the system. Since we already know the answer from classical mechanics and experiment, quantum mechanics must give us the same result in the limiting case of a large system.

It is a worthwhile exercise to see just how this happens. Evidently, we cannot simply follow the classical method of specifying the initial position and momentum—the uncertainty principle won't allow it. What we can do, though, is to take an initial state in which the position and momentum are specified as precisely as possible. Such a state is called a minimum uncertainty state (the details can be found in my earlier lecture on the Generalized Uncertainty Principle).

In fact, the ground state of a simple harmonic oscillator is a minimum uncertainty state. This is not too surprising—it's just a localized wave packet centered at the origin. The system is as close to rest as possible, having only zero-point motion. What is surprising is that there are excited states of the pendulum in which this ground state wave packet swings backwards and forwards indefinitely, a quantum realization of the classical system, and the wave packet is always one of minimum uncertainty. Recall that this doesn't happen for a free particle on a line — in that case, an initial minimal uncertainty wave packet spreads out because the different momentum components move at different speeds. But for the oscillator, the potential somehow keeps the wave packet together, a minimum uncertainty wave packet at all times. These remarkable quasi-classical states are called coherent states, and were discovered by Schrodinger himself. They are important in many quasi-classical contexts, including laser radiation.

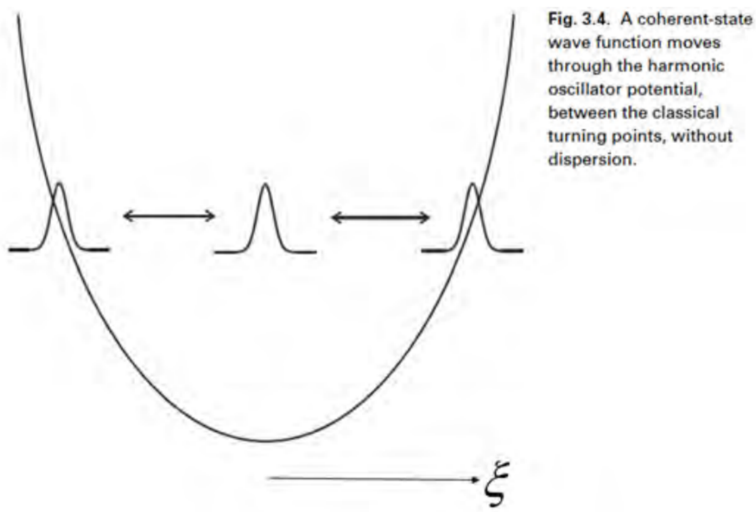

Figure 1: The time evolution of the coherent state. It follows the motion of a classical pendulum or harmonic oscillator

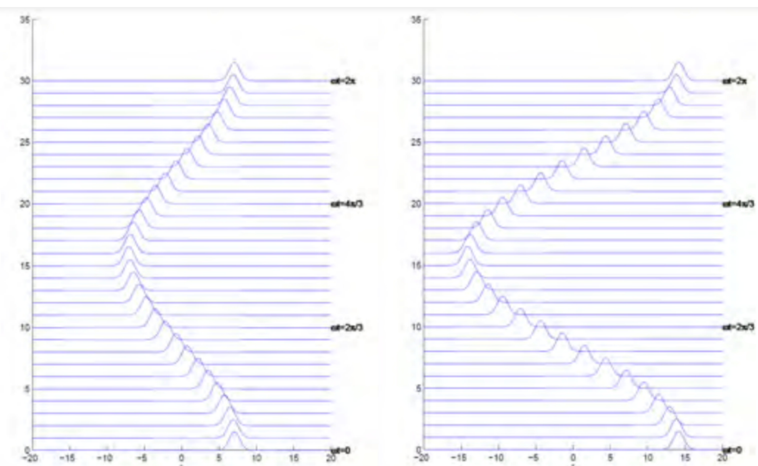

Figure 2: The time evolution of the coherent state for different α's. The left figure is for $\alpha = 5$ while the right figure is for $\alpha = 10$. Recall that $N = |\alpha|^2$.

Deriving Coherent States

Definition

We wish to achieve states which diagonalize the positive frequency part of the quantum field, i.e. a state $|\eta\rangle$, such that $a(k)|\eta\rangle = \eta(k)|\eta\rangle$.

Consider the state $|\eta\rangle = \exp\left[\int d\tilde{k}\eta(k)a^\dagger(k)\right]|0\rangle$

which is equivalent to a coherent superposition of the Fock space number states with 0, 1, 2, ... particles.

Further Results

To understand the form of the coherent states, we take the inner product of two states $|\eta_1\rangle$ and $|\eta_2\rangle$. To calculate this, we apply the Baker-Hausdorff formula, $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{[\hat{A},\hat{B}}{2}} = e^{[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}}$, which is true when $[\hat{A},\hat{B}]$ is a c-number. Since, $[a(k), a^\dagger(k')] \propto \delta^3(k-k')$, we get

$$\langle\eta_1|\eta_2\rangle = \langle 0|[\exp[\int d\tilde{k}\eta_1 a^\dagger]]^\dagger[\exp[\int d\tilde{k}\eta_2 a^\dagger]]|0\rangle$$

$$= \langle 0|\exp[\int d\tilde{k}\eta_1^*(k)\eta_2(k)]\exp[\int d\tilde{k}\eta_2 a(k)^\dagger]\exp[\int d\tilde{k}\eta_1^* a(k)]|0\rangle$$

Since, $\exp[\int d\tilde{k}\eta_1^*(k)\eta_2(k)] = constant$, $\langle 0|\exp[\int d\tilde{k}\eta_2 a(k)^\dagger] = \langle 0|$ and $[\exp[\int d\tilde{k}\eta_1^* a^(k)]]|0\rangle = |0\rangle$. Therefore,

$$\langle\eta_1|\eta_2\rangle = \exp[\int d\tilde{k}\eta_1^*(k)\eta_2(k)] \tag{1}$$

So, the coherent states are not in general orthogonal, and depend on the form of $\eta(k)$'s.

Take the normalized η state as:

$$|\eta) = \exp[-\frac{1}{2}\int d\tilde{k}|\eta(k)|^2]|\eta\rangle \tag{2}$$

which is clear from last relation.

For the normalised states, the inner product of two states is given by:

$$(\eta_1|\eta_2) = \exp[\int d\tilde{k}(\iota Im(\eta_1^*\eta_2) - \frac{1}{2}|\eta_1 - \eta_2|^2] \tag{3}$$

This is clear if we apply the relation $e^A e^B = e^{A+B+[A,B]/2}$.

Since, $a(k)|\eta\rangle = \eta(k)|\eta\rangle = 0$, iff $\eta(k) = 0$. This corresponds to the vacuum state.

The definition of the coherent states is used to show the following:

$$\Phi^{(+)}(x) = \int d\tilde{k} a(k) e^{-\iota k.x}$$

$$\Phi^{(+)}(x)|\eta\rangle = \int d\tilde{k} \eta(k) e^{-\iota k.x}|\eta\rangle$$

Result - Time Evolution of Coherent States

Coherent states remain coherent under time evolution.

$$e^{-\iota Ht}|\eta\rangle = \exp \int d\tilde{k} \eta(k)[e^{-\iota Ht} a^\dagger(k) e^{+\iota Ht}]|0\rangle$$

$$= \exp[\int d\tilde{k} \eta(k) e^{-\iota k_0 t} a^\dagger(k)]|0\rangle = |\eta_t\rangle$$

where $\eta_t(k) = e^{-\iota k_0 t}\eta(k)$

Result - Displacement operator

Consider the unitary operator $D(\eta)$ such that

$$D(\eta_2)|\eta_1) = |\eta_1 + \eta_2)$$

We see that the $D(\eta)$ is given as

$$D(\eta) = \exp\{-\int d\tilde{k}[\eta^*(k)a(k) - \eta(k)a^\dagger(k)]\}$$

To understand this lets first assume the form of the displacement operator and witness its properties.

Now, if $[A,[A,B]] = 0$ and $[B,[A,B]] = 0 \implies e^{A+B} = e^A e^B e^{-[A,B]/2}$. $[a,[a,a^\dagger]] = 0$ and $[a^\dagger,[a,a^\dagger]] = 0$.

Therefore, $D(\eta) = e^{\int d\tilde{k}\eta a^\dagger} e^{-\int d\tilde{k}\eta^* a} e^{-|\eta|^2/2}$

Consider the commutator $[a, D(\eta)] = e^{-|\eta^2|/2}[a, e^{\int d\tilde{k}\eta a^\dagger} e^{-\int d\tilde{k}\eta^* a}]$

$[A, BC] = B[A, C] + [A, B]C$

$$= e^{-|\eta|^2/2}(e^{\int d\tilde{k}\eta a^\dagger}[a, e^{-\int d\tilde{k}\eta^* a}] + [a, e^{\int d\tilde{k}\eta a^\dagger}]e^{-\int d\tilde{k}\eta^* a})$$

Using $[A, [A, B]] = 0$ and $[B, [A, B]] = 0 \implies [A, F(B)] = [A, B]F'(B)$, we get

$[a, D(\eta)] = e^{-|\eta|^2/2}[a, a^\dagger]\eta e^{\int d\tilde{k}\eta a^\dagger} e^{-\int d\tilde{k}\eta^* a} = \eta D(\eta)$

With these tools, we will now show that $D(\eta)$ is indeed the displacement operator, i.e. $|\eta\rangle = D(\eta)|0\rangle$.

Using the last relation, we get

$$a(D(\eta)|0\rangle) = (D(\eta)a + \eta D(\eta))|0\rangle \tag{4}$$

Since, $a|0\rangle = 0$

$$a(D(\eta)|0\rangle) = \eta D(\eta)|0\rangle \tag{5}$$

Thus $D(\eta)$ is the displacement operator.

Result - Field Distribution

Now, to study the field distribution, since $\phi(x)$ is an operator-valued distribution, we smear it with a test function such that:

$$\phi_f = \int d^4x f(x)\phi(x) \tag{6}$$

Now we study the probability distribution of ϕ_f in a state with a fixed number of r particles

$$\rho_r(a) = \langle r|\delta(\phi_f - b)|r\rangle = \int \frac{d\alpha}{2\pi} e^{-\iota\alpha b}\langle r|e^{-\iota\alpha\phi_f}|r\rangle \tag{7}$$

Using the following relations:

$$\phi(x) = \int d\tilde{k}[a(k)e^{-ikx} + a^\dagger(k)e^{ikx}]$$

The fourier transform for $f(x)$ as $\tilde{f}(k) = \int d^4x e^{ikx} f(x)$, and the form for $|\eta\rangle$ in terms of the ground state and anihillation operator, we get

$$\langle\eta_1|e^{i\alpha\phi_f}|\eta_2\rangle = \exp\{\int d\tilde{k}[\eta_1^*\eta_2 + \iota\alpha(\eta_1^*\tilde{f} + \tilde{f}^*\eta_2) - \alpha^2|\tilde{f}|^2/2]\} \tag{8}$$

From this relation, we can derieve the vacuum distribution of the field as

$$\rho_0(b) = \int \frac{d\alpha}{2\pi} \exp[-i\alpha b - \frac{\alpha^2}{2} \int d\tilde{k} |\tilde{f}(k)|^2]$$

$$= [2\pi \int d\tilde{k} |\tilde{f}(k)|^2]^{-\frac{1}{2}} \exp[-\frac{b^2}{2 \int d\tilde{k} |\tilde{f}(k)|^2}] \tag{9}$$

This turns out to be a gaussian distribution with mean square fluctuation

$$\sigma = [\int d\tilde{k} |\tilde{f}(k)|^2]^{\frac{1}{2}}$$

and zero mean.

Note that vacuum is not a zero field state, and it can be shown that vacuum fluctuations are observable.

Another Approach

Instead of considering $\phi^{(+)}(x)$ we may construct a complete set of tested values of $\phi(0, \mathbf{x})$:

$$\phi_n = \int d^3x \phi(0, \mathbf{x}) F_n(\mathbf{x})$$

where $f_n(x) = \delta(x^0) F_n(\mathbf{x})$

Using Fourier transform,

$$\tilde{f}_n(k) = \int d^3x e^{ikx} F_n(\mathbf{x}) = \tilde{F}_n(k)$$

and the normalization relations:

$$\int d\tilde{k}\, \tilde{F}_n^*(\mathbf{k}) \tilde{F}_{n'}(\mathbf{k}) = \delta_{n,n'}$$

$$\tilde{F}_n(-\mathbf{k}) = \tilde{F}_n^*(\mathbf{k})$$

$$\Sigma_n \tilde{F}_n(\mathbf{k}) \tilde{F}_n^*(\mathbf{k}') \equiv (2\pi)^3 2\omega_k \delta^3(\mathbf{k}\text{-}\mathbf{k'})$$

An eigenstate of the operator ϕ_n may be called an eigenstate of the field $\phi(0, \mathbf{x})$ with eigenvalue

$$\phi_c(\mathbf{x}) = \Sigma_n \phi_{n,c} \int d\tilde{k} e^{-ikx} F_n^*(\mathbf{k})$$

such that $\int d^3x \phi_c(\mathbf{x}) F_n(\mathbf{x}) = \phi_{n,c}$

If we denote by $|\phi_c\rangle$ the corresponding state, to obtain its components in the Fock space basis, it is sufficient to know $\langle\eta|\phi_c\rangle$ by solving:

$$\langle\eta|\phi_n|\phi_c\rangle = \phi_{n,c}\langle\eta|\phi_c\rangle$$

for every n.

If we write the identity as

$$I = \int |\phi_c\rangle \Pi_n \frac{d\phi_{n,c}}{\sqrt{2\pi}} \langle\phi_c| \tag{10}$$

we get

$$\langle\eta|\phi_c\rangle = \exp\{-\frac{1}{4}\Sigma_n \phi_{n,c}^2 + \int d\tilde{k}[-\frac{1}{2}\eta^*(\mathbf{k})\eta^*(\textbf{-k}) + \eta^*(\mathbf{k})\Sigma_n \phi_{n,c}\tilde{F}_n(\mathbf{k})]\} \tag{11}$$

Photons

If we focus on one normal mode (with a well defined momentum vector and polarization, and therefore a fixed frequency), the classical plane wave radiation is best approximated in quantum mechanics with a coherent state in that particular mode. This coherent state has maximally well defined electric and magnetic fields (which are equivalent to x and p), and these fields evolve harmonically, as expected. The number of photons in not well defined, but its mean is proportional to the total intensity of the pulse. With a large number of photons, the fractional fluctuation in photon number is small.

This kind of coherent state radiation is what you get in a continuous beam (as opposed to pulsed), high quality, quiet laser. The (unavoidable) fluctuations in photon number limit the precision with which measurements can be made with such a beam. This is known as the quantum limit. Where really high precision is required (in gravity wave detectors, for example), squeezed light can be used to go beyond this limit.

Since any vacuum electromagnetic field configuration can be decomposed into plane wave solutions, any classical electromagnetic field can be thought of as build from coherent states. In Quantum Field Theory, coherent states are used often to describe classical field profiles