# Lecture 7: Generalised Method of Moments

Econometric Methods – Warsaw School of Economics

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# Outline

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1. Introduction
2. GMM estimator
3. Application: Gali&Gertler's hybrid Phillips curve (1999)

# Instrumental variables: general idea

- OLS estimaton based on the general underlying assumption that $E\left(\mathbf{X}_{T\times k}^{T}\varepsilon_{T\times 1}\right)=\mathbf{0}_{k\times 1}$ (by Gauss-Markov).
- It may be broken i.a. for the following reasons:
  - just by construction of the economic model;
  - two-way causality between $\boldsymbol{y}_t$ and a subset of $\boldsymbol{x}_t$;
  - non-random sample selection.
- Solution: find variables that are truly orthogonal to $\varepsilon_{T\times 1}$ ("instrumental variables").

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# OLS and IV: alternative approach

- OLS: residuals uncorrelated with regressors, $E\left(\mathbf{X}_{T\times k}^{T}\varepsilon_{T\times 1}\right)=\mathbf{0}_{k\times 1}$
  - these are $k$ "moment conditions" from which we infer the estimates
  - $\mathbf{X}^T\mathbf{y}=\mathbf{X}^T\mathbf{X}\beta+\underbrace{\mathbf{X}^T\varepsilon}_{\mathbf{0}}\Rightarrow\hat{\beta}^{OLS}=\left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$
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- what if we get more than $l\,(>k)$ "moment conditions", i.e. more than we actually need? $\mathbf{Z}_{T\times l}^{T}\varepsilon_{T\times 1}=\mathbf{0}_{l\times 1}$
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# Limitations of IV estimation

- $\beta^{IV} = \left(\mathbf{Z}^T\mathbf{X}\right)^{-1}\mathbf{Z}^T\mathbf{y}$
  - $\mathbf{Z}$ must have the same number of columns as $\mathbf{X}$ for this operation to be feasible
- in IV, there must be as many instrumental variables as regressors (some of which can instrumentalise themselves)
- what are $\mathbf{Z}$?
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  - cannot solve $l$ equations (moment conditions) for $k < l$ unknowns
- ...so take quadratic form $\left(\mathbf{Z}^T\varepsilon\right)^T\mathbf{W}\left(\mathbf{Z}^T\varepsilon\right)$ instead!: $\mathbf{W}_{l\times l}$
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# How to obtain $\mathbf{W}$?

- in general: any symmetric, positive-definite matrix sized $l \times l$
- according to ***Hansen (1982)***, you should assign the lowest weights to the moment conditions with lowest precision of estimation, which implies

$$\mathbf{V} = \frac{1}{T}\mathbf{Z}^T\boldsymbol{\Omega}\mathbf{Z}$$

- with $\boldsymbol{\Omega}$ – variance-covariance matrix of the estimator
  - under white noise: $\hat{\boldsymbol{\Omega}} = \hat{\sigma}^2\boldsymbol{I}$
  - under non-spherical disturbances – White or Newey-West versions (see: serial correlation / heteroskedasticity lectures)

# GMM as iterative procedure

- $\hat{\varepsilon}$ unknown in advance!
    1. estimate under the assumption of white noise
    2. calculate the residuals
    3. update $\hat{\mathbf{W}}$
- various implementations of this algorithm in the software (i.a. as a two-step or iterative estimator – cf. GLS)

# Derivation of GMM estimator (1)

Let us minimise the quadratic form with respect to $\beta$:

$$\begin{aligned}\varepsilon^T \mathbf{Z}\mathbf{W}\mathbf{Z}^T\varepsilon &= (\mathbf{y}-\beta\mathbf{X})^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T(\mathbf{y}-\beta\mathbf{X}) = \\ &= \mathbf{y}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{y}-\mathbf{y}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X}\beta \\ &\quad -\beta^T\mathbf{X}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{y}+\beta^T\mathbf{X}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X}\beta \\ &\to \mathit{min}\end{aligned}$$

**From matrix calculus:**

$$\frac{\partial \mathbf{x}^T\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{x}^T\left(\mathbf{A}^T+\mathbf{A}\right)$$
$$\frac{\partial \mathbf{x}^T\mathbf{a}}{\partial \mathbf{x}} = \mathbf{a}^T$$

Compute the derivative and set it equal to 0.

# Derivation of GMM estimator (2)

- $-\mathbf{y}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X} - \mathbf{y}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X} + \beta^T\left(\mathbf{X}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X} + \mathbf{X}^T\mathbf{Z}\mathbf{W}\mathbf{Z}^T\mathbf{X}\right) = 0$
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**Nonlinear GMM:**

– you can also (numerically) minimise the quadratic form for a nonlinear model;
– it is advisable to write the moment conditions “as close to linear as possible”.

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# Variance-covariance of GMM estimator

- GMM estimator: consistent, asymptotically normally distributed
- asymptotic variance-covariance estimator in a linear model:

$$Var\left(\beta^{GMM}\right) = \frac{1}{T}\left[\frac{1}{T}\left(\mathbf{X}^T\mathbf{Z}\right)\mathbf{W}^{-1}\frac{1}{T}\left(\mathbf{Z}^T\mathbf{X}\right)\right]^{-1}$$

# $J$ statistic

- the non-zero value of the minimised quadratic form is interpretable
  - how far are we from fulfilling the (excessive) orthogonality conditions?
  - the lower the value, the lower the "distance" to perfect fulfilment of excessive conditions
- divided by $T$, it is $\chi^2$-distributed with degrees of freedom equal to $l - k$ (instruments in excess of parameters)

**$J$-test of orthogonality**

$$J\left(\hat{\beta}, \hat{\mathbf{\Omega}}^{-\mathbf{1}}\right) = \frac{1}{T}\varepsilon\left(\hat{\beta}\right)^{T}\mathbf{Z}\hat{\mathbf{\Omega}}^{-\mathbf{1}}\mathbf{Z}^{T}\varepsilon\left(\hat{\beta}\right)$$

$H_0$ : all orthogonality conditions fulfilled

$H_1$ : some orthogonality conditions not fulfilled

# Outline

# Hybrid Phillips curve

- *Gali, Gertler (1999)*:
  $$\pi_t = \gamma_b \pi_{t-1} + \gamma_f E_t \pi_{t+1} + \lambda mc_t + \varepsilon_t$$
  where $\pi_t$ – inflation rate, $mc_t$ – real marginal cost, $\varepsilon_t$ – error term.
- Why GMM?
  - there is an unobservable variable on the right-hand side, $E_t \pi_{t+1}$
  - we can just replace it with observable $\pi_{t+1} = E_t \pi_{t+1} + v_{t+1}$, where $v_t$ – expectations error
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# Orthogonality conditions

- $E_t[\varepsilon_t \mathbf{z}_t] = E_t[(\pi_t - \gamma_b \pi_{t-1} - \gamma_f E_t \pi_{t+1} - \lambda mc_t)\mathbf{z}_t] =$
  $E_t[(\pi_t - \gamma_b \pi_{t-1} - \gamma_f \pi_{t+1} - \lambda mc_t)\mathbf{z}_t] = \mathbf{0}$
  - the expected value on the orthogonality condition allows to drop the expected value on $\pi_{t+1}$
- in this context, we interpret the instrument set **Z** as **variables that allow to forecast inflation one period ahead without systematic errors**
- there should be more than 3 instruments to use GMM here

# Example

**(Imperfect) replication of Gali-Gertler results:**

- Read the paper by Gali and Gertler.
- Consider the following set of variables as linear one-period-ahead predictors:
  - *inflation (4 lags), log real ULC (1 lag), output gap (1 lag), short- vs long-term interest rate spread (1 lag), log-differences of wage index (1 lag), log-differences of commodity price index (1 lag)*
- Define the instruments and the initial weight matrix $\mathbf{W}$.
- Estimate the model. Discuss the results. Do they confirm Your intuition? If not, look for the (economic) solution in the article.