Green's functions further examples
(For the video)

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July 19, 2021

This write-up is for anyone who wants a deeper understanding of how to find and use the Green's function. *Unless I can find a relatively novel way of presenting all these in a video format, I might not make a separate video about this on the channel. Hope that you understand.* I used the oscillator example first because it uses single-variable calculus.

1 The oscillator example

The equation for a driven oscillator (undamped to make our lives easier) is as in the video:

$$m\frac{d^2x}{dt^2} + \omega^2 x = F(t) \tag{1}$$

Finding the Green's function in this case is not too difficult if you know how to solve basic differential equations. The Green's function $G(t;\tau)$ satisfies the following:

$$m\frac{d^2G(t;\tau)}{dt^2} + \omega^2 G(t;\tau) = \delta(t-\tau), \tag{2}$$

when viewed as a function of t. The usual convention is to further assume that $\tau > 0$ in addition to the homogenous conditions as in the video (i.e. $x(0) = x'(0) = 0$), simply because we ultimately would want to see how the oscillator behaves after applying the force, and we originally would want the oscillator to be at rest, only applying some force after $t > 0$.

Now we know that RHS of (2) is **0 almost everywhere**, and the solution in that case has the following form: $A\sin(kt) + B\sin(kt)$, where $k = \sqrt{\omega^2/m}$. However, because it is 0 except at $t = \tau$, it is possible that the solution on the left of τ and that on the right of τ are different, and in fact they are different. And since we are being clever, avoiding further unnecessarily complicated algebra, we reconsider the constants so that we have the following solution:

$$G(t;\tau) = \begin{cases} A(\tau)\sin(k(t-\tau)) + B(\tau)\cos(k(t-\tau)) & \text{for } t < \tau \\ C(\tau)\sin(k(t-\tau)) + D(\tau)\cos(k(t-\tau)) & \text{for } t > \tau \end{cases} \tag{3}$$

Note in particular that all the "constants" could be functions of τ because in this equation, τ is seen as a constant when taking derivatives with respect to t. The initial conditions and that $\tau > 0$ imply that $A(\tau) = B(\tau) = 0$, and now we only need to find $C(\tau)$ and $D(\tau)$.

We want **continuity**, i.e. at $t = \tau$, the Green's function should be 0, and so this implies $D(\tau) = 0$. (Note that if we haven't changed from kt to $k(t-\tau)$, we wouldn't be able to eliminate a "constant" that way.)

The next thing to note is that integrating around τ gives 1, i.e.

$$m\left(\frac{dG}{dt}(\varepsilon;\tau) - \frac{dG}{dt}(-\varepsilon;\tau)\right) + \omega^2\int_{-\varepsilon}^{\varepsilon} G(t;\tau)dt = 1,$$

but since G is already continuous in t around τ, the integral obviously is continuous as well, and so we get

$$\lim_{\varepsilon\to 0}\left(G'(\varepsilon;\tau) - G'(-\varepsilon;\tau)\right) = \frac{1}{m} \tag{4}$$

We know that $G'(-\varepsilon;\tau) = 0$ (assuming $\varepsilon > 0$), so that gives $C(\tau) = 1/mk$. So we now rewrite (3) as follows:

$$G(t;\tau) = \begin{cases} 0 & \text{for } t < \tau \\ \dfrac{1}{mk}\sin(k(t-\tau)) & \text{for } t > \tau \end{cases} \tag{5}$$

Now we want to think the general applied force in (1). Remember that in the video, the way to solve it would be

$$x(t) = \int_0^\infty F(\tau)G(t;\tau)d\tau. \tag{6}$$

The lower limit being 0 is because we assumed $\tau > 0$ to begin with. However, the Green's function is 0 if $t < \tau$, or equivalently, $\tau > t$, so there is no need to integrate up to ∞, just t, and so we have

$$x(t) = \frac{1}{mk}\int_0^t F(\tau)\sin(k(t-\tau))d\tau \tag{7}$$

And this concludes our oscillator example.

2 Electrostatics example

This requires knowledge of multivariable calculus. The equation we need to deal with is this:

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0} \tag{8}$$

Again, we want to find the Green's function first, i.e.

$$-\nabla^2 G(\mathbf{x};\xi) = \delta(\mathbf{x}-\xi) \tag{9}$$

As said in the video, we use the symmetry in the situation, and we should know that the Green's function should only depend on $r = |\mathbf{x}-\xi|$, so let's say $G(\mathbf{x};\xi) = G(r)$, and in this case, $\nabla G(r) = G'(r)\mathbf{e}_r$. We now integrate (9) in a ball B surrounding $\mathbf{x}$ with radius R. Then

$$\begin{aligned} \int_B \nabla^2 G(r)d\xi &= -\int_B \delta(\mathbf{x}-\xi)d\xi \\ \int_{\partial B} \nabla G(r)dS &= -1 && \text{(Divergence theorem)} \\ 4\pi R^2 G'(R) &= -1 \\ G'(R) &= \frac{1}{4\pi R}, \end{aligned}$$

but this is true for all R, so that is the Green's function, and so the solution to (8) is simply (by applying the principle of Green's functions in the video):

$$\phi(\mathbf{x}) = \int_{\mathbb{R}^3} \frac{\rho(\xi)}{4\pi\varepsilon_0|\mathbf{x}-\xi|}d\xi \tag{10}$$