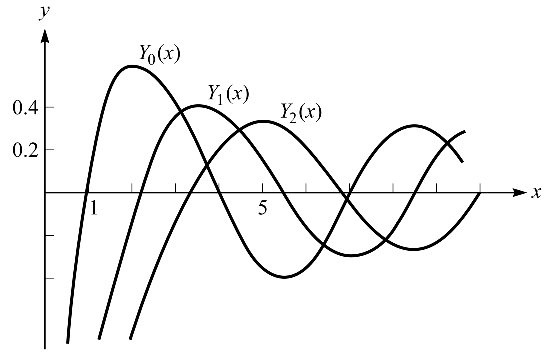

Figure 7.4. Bessel functions of the second kind.

These linearly independent functions are known as Bessel functions of the third kind of order α or first and second Hankel functions of order α (Hermann Hankel, 1839–1873, German mathematician).

To illustrate how Bessel functions enter into the analysis of physical problems, we consider one example in classical physics: small oscillations of a hanging chain, which was first considered as early as 1732 by Daniel Bernoulli.

Hanging flexible chain

Fig. 7.5 shows a uniform heavy flexible chain of length l hanging vertically under its own weight. The x-axis is the position of stable equilibrium of the chain and its lowest end is at $x = 0$. We consider the problem of small oscillations in the vertical xy plane caused by small displacements from the stable equilibrium position. This is essentially the problem of the vibrating string which we discussed in Chapter 4, with two important differences: here, instead of being constant, the tension T at a given point of the chain is equal to the weight of the chain below that point, and now one end of the chain is free, whereas before both ends were fixed. The analysis of Chapter 4 generally holds. To derive an equation for y, consider an element dx, then Newton's second law gives

$$\left(T\frac{\partial y}{\partial x}\right)_2 - \left(T\frac{\partial y}{\partial x}\right)_1 = \rho dx \frac{\partial^2 y}{\partial t^2}$$

or

$$\rho dx \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)dx,$$

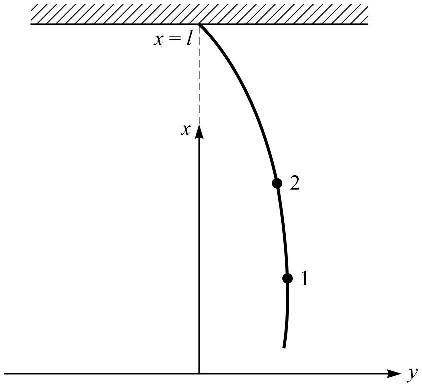

Figure 7.5. A flexible chain.

from which we obtain

$$\rho \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T \frac{\partial y}{\partial x}\right).$$

Now $T = \rho g x$. Substituting this into the above equation for y, we obtain

$$\frac{\partial^2 y}{\partial t^2} = g \frac{\partial y}{\partial x} + gx \frac{\partial^2 y}{\partial x^2},$$

where y is a function of two variables x and t. The first step in the solution is to separate the variables. Let us attempt a solution of the form $y(x, t) = u(x)f(t)$. Substitution of this into the partial differential equation yields two equations:

$$f''(t) + \omega^2 f(t) = 0, \quad xu''(x) + u'(x) + (\omega^2/g)u(x) = 0,$$

where ω^2 is the separation constant. The differential equation for $f(t)$ is ready for integration and the result is $f(t) = \cos(\omega t - \delta)$, with δ a phase constant. The differential equation for $u(x)$ is not in a recognizable form yet. To solve it, first change variables by putting

$$x = gz^2/4, \quad w(z) = u(x),$$

then the differential equation for $u(x)$ becomes Bessel's equation of order zero:

$$zw''(z) + w'(z) + \omega^2 z w(z) = 0.$$

Its general solution is

$$w(z) = AJ_0(\omega z) + BY_0(\omega z)$$

or

$$u(x) = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right) + BY_0\left(2\omega\sqrt{\frac{x}{g}}\right).$$

Since $Y_0(2\omega\sqrt{x/g}) \to -\infty$ as $x \to 0$, we are forced by physics to choose $B = 0$ and then

$$y(x,t) = AJ_0\left(2\omega\sqrt{\frac{x}{g}}\right)\cos(\omega t - \delta).$$

The upper end of the chain at $x = l$ is fixed, requiring that

$$J_0\left(2\omega\sqrt{\frac{\ell}{g}}\right) = 0.$$

The frequencies of the normal vibrations of the chain are given by

$$2\omega_n\sqrt{\frac{\ell}{g}} = \alpha_n,$$

where α_n are the roots of J_0. Some values of $J_0(x)$ and $J_1(x)$ are tabulated at the end of this chapter.

Generating function for $J_n(x)$

The function

$$\Phi(x,t) = e^{(x/2)(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{7.87}$$

is called the generating function for Bessel functions of the first kind of integral order. It is very useful in obtaining properties of $J_n(x)$ for integral values of n which can then often be proved for all values of n.

To prove Eq. (7.87), let us consider the exponential functions $e^{xt/2}$ and $e^{-xt/2}$. The Laurent expansions for these two exponential functions about $t = 0$ are

$$e^{xt/2} = \sum_{k=0}^{\infty}\frac{(xt/2)^k}{k!},\; e^{-xt/2} = \sum_{m=0}^{\infty}\frac{(-xt/2)^k}{m!}.$$

Multiplying them together, we get

$$e^{x(t-t^{-1})/2} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{k!m!}\left(\frac{x}{2}\right)^{k+m} t^{k-m}. \tag{7.88}$$

It is easy to recognize that the coefficient of the t^0 term which is made up of those terms with $k = m$ is just $J_0(x)$:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}x^{2k} = J_0(x).$$

Similarly, the coefficient of the term t^n which is made up of those terms for which $k - m = n$ is just $J_n(x)$:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+n)!k!2^{2k+n}} x^{2k+n} = J_n(x).$$

This shows clearly that the coefficients in the Laurent expansion (7.88) of the generating function are just the Bessel functions of integral order. Thus we have proved Eq. (7.87).

Bessel's integral representation

With the help of the generating function, we can express $J_n(x)$ in terms of a definite integral with a parameter. To do this, let $t = e^{i\theta}$ in the generating function, then

$$\begin{aligned} e^{x(t-t^{-1})/2} &= e^{x(e^{i\theta}-e^{-i\theta})/2} = e^{ix\sin\theta} \\ &= \cos(x\sin\theta) + i\sin(x\cos\theta). \end{aligned}$$

Substituting this into Eq. (7.87) we obtain

$$\begin{aligned} \cos(x\sin\theta) + i\sin(x\cos\theta) &= \sum_{n=-\infty}^{\infty} J_n(x)(\cos\theta + i\sin\theta)^n \\ &= \sum_{-\infty}^{\infty} J_n(x)\cos n\theta + i\sum_{-\infty}^{\infty} J_n(x)\sin n\theta. \end{aligned}$$

Since $J_{-n}(x) = (-1)^n J_n(x), \cos n\theta = \cos(-n\theta)$, and $\sin n\theta = -\sin(-n\theta)$, we have, upon equating the real and imaginary parts of the above equation,

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos 2n\theta,$$

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin(2n-1)\theta.$$

It is interesting to note that these are the Fourier cosine and sine series of $\cos(x\sin\theta)$ and $\sin(x\sin\theta)$. Multiplying the first equation by $\cos k\theta$ and integrating from 0 to π, we obtain

$$\frac{1}{\pi}\int_0^{\pi} \cos k\theta\cos(x\sin\theta)d\theta = \begin{cases} J_k(x), & \text{if } k = 0, 2, 4, \ldots \\ 0, & \text{if } k = 1, 3, 5, \ldots \end{cases}.$$