Universidad de San Carlos de Guatemala
Escuela de Ciencias Físicas y Matemáticas
Mecánica Estadística
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Tarea 4

1. Problema 1

Dada la varianza $\langle (\Delta E)^2 \rangle = \langle (E_i - \varepsilon)^2 \rangle = \langle E_i^2 \rangle - \varepsilon^2$, entonces, encontramos el segundo momento

$$\frac{\mathrm{d}^2 \mathfrak{z}}{\mathrm{d}\beta^2} = \sum E_i^2 e^{-\beta E_i} = \underbrace{\left(\sum p_i E_i^2\right)}_{\langle E_i^2 \rangle} \mathfrak{z}. \tag{1}$$

Ahora, tomando la segunda derivada de $\ln \mathfrak{z}$, se tiene

$$\frac{\mathrm{d}^2 \ln \mathfrak{z}}{\mathrm{d}\beta^2} = \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{1}{\mathfrak{z}}\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right) = -\frac{1}{\mathfrak{z}^2}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^2 + \frac{1}{\mathfrak{z}}\frac{\mathrm{d}^2\mathfrak{z}}{\mathrm{d}\beta^2},$$

despejando y reemplazando en la fórmula de la varianza

$$\langle (\Delta E)^2 \rangle = \frac{\mathrm{d}^2 \ln \mathfrak{z}}{\mathrm{d}\beta^2} + \frac{1}{\mathfrak{z}^2}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^2 - \left(-\frac{1}{\mathfrak{z}}\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^2$$

$$\boxed{\langle (\Delta E)^2 \rangle = \frac{\mathrm{d}^2 \ln \mathfrak{z}}{\mathrm{d}\beta^2}.}$$

2. Problema 2

Dada la definición de tercer momento, se realiza la expación

$$\langle (\Delta E)^3 \rangle = \sum \underbrace{(E_i - \varepsilon)^3}_{E_i^3 - 3E_i^2\varepsilon + 3E_i\varepsilon^2 - \varepsilon^3} p_i = \langle E_i^3 \rangle - 3\varepsilon \langle E_i^2 \rangle + 3\varepsilon^3 - \varepsilon^3 = \langle E_i^3 \rangle - 3\varepsilon \langle E_i^2 \rangle + 2\varepsilon^3. \tag{2}$$

Ahora, siguiendo la idea del problema anterior

$$-\frac{1}{\mathfrak{z}}\frac{\mathrm{d}^3\mathfrak{z}}{\mathrm{d}\beta^3} = \langle E_i^3 \rangle,$$

entontrando la tercera derivada de $\ln \mathfrak{z}$,

$$\frac{\mathrm{d}^3 \ln \mathfrak{z}}{\mathrm{d}\beta^3} = \frac{2}{\mathfrak{z}^3}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^3 - \frac{3}{\mathfrak{z}^2}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)\frac{\mathrm{d}^2\mathfrak{z}}{\mathrm{d}\beta^2} + \frac{1}{\mathfrak{z}}\frac{\mathrm{d}^3\mathfrak{z}}{\mathrm{d}\beta^3}. \tag{3}$$

Sustituyendo (1), (3) y ε en (2)

$$\langle (\Delta E)^3 \rangle = -\frac{\mathrm{d}^3 \ln \mathfrak{z}}{\mathrm{d}\beta^3} + \frac{2}{\mathfrak{z}^3}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^3 - \frac{3}{\mathfrak{z}^2}\left(\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)\frac{\mathrm{d}^2\mathfrak{z}}{\mathrm{d}\beta^2} - 3\left(-\frac{1}{\mathfrak{z}}\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)\left(\frac{1}{\mathfrak{z}}\frac{\mathrm{d}^2\mathfrak{z}}{\mathrm{d}\beta^2}\right) + 2\left(-\frac{1}{\mathfrak{z}}\frac{\mathrm{d}\mathfrak{z}}{\mathrm{d}\beta}\right)^3,$$

$$\boxed{\langle (\Delta E)^3 \rangle = -\frac{\mathrm{d}^3 \ln \mathfrak{z}}{\mathrm{d}\beta^3}.}$$

3. Problema 3

Dada la definición de calor específico

$$c_V = \frac{\partial \varepsilon}{\partial T},$$

entonces, tomando la energía promedio

$$\varepsilon = -\frac{\partial \ln \mathfrak{z}}{\partial \beta} = k_B T^2 \frac{\partial \ln \mathfrak{z}}{\partial T}.$$

Reemplazando en la definición de calor específico

$$c_V = \frac{\partial}{\partial T}\left(k_B T^2 \frac{\partial \ln \mathfrak{z}}{\partial T}\right),$$

$$\boxed{c_V = 2k_B T \frac{\partial \ln \mathfrak{z}}{\partial T} + k_B T \frac{\partial^2 \ln \mathfrak{z}}{\partial T^2}.}$$

4. Problema 4

Sabiendo que la presión la podemos escribir en términos de la energía libre de Helmholtz

$$P = -\left(\frac{\partial A}{\partial V}\right)_T,$$

tomando a A como

$$A = -\frac{\ln Z}{\beta}, \qquad Z = \mathfrak{z}^n.$$

Reemplazando A en la presión, se tiene

$$P = \left(\ln \mathfrak{z} \cancelto{0}{\frac{\partial T}{\partial V}} \frac{\partial}{\partial T}\frac{1}{\beta} + \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\right)_T,$$

entonces

$$\boxed{P = k_B T\left(\frac{\partial \ln Z}{\partial V}\right)_T = n k_B T\left(\frac{\partial \ln \mathfrak{z}}{\partial V}\right)_T.}$$