46770 Integrated energy grids

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Lecture 2 – Optimization (a short review)

Balancing renewable generation

We ended last lecture, stating that we need optimization to decide how to combine different flexibility options.

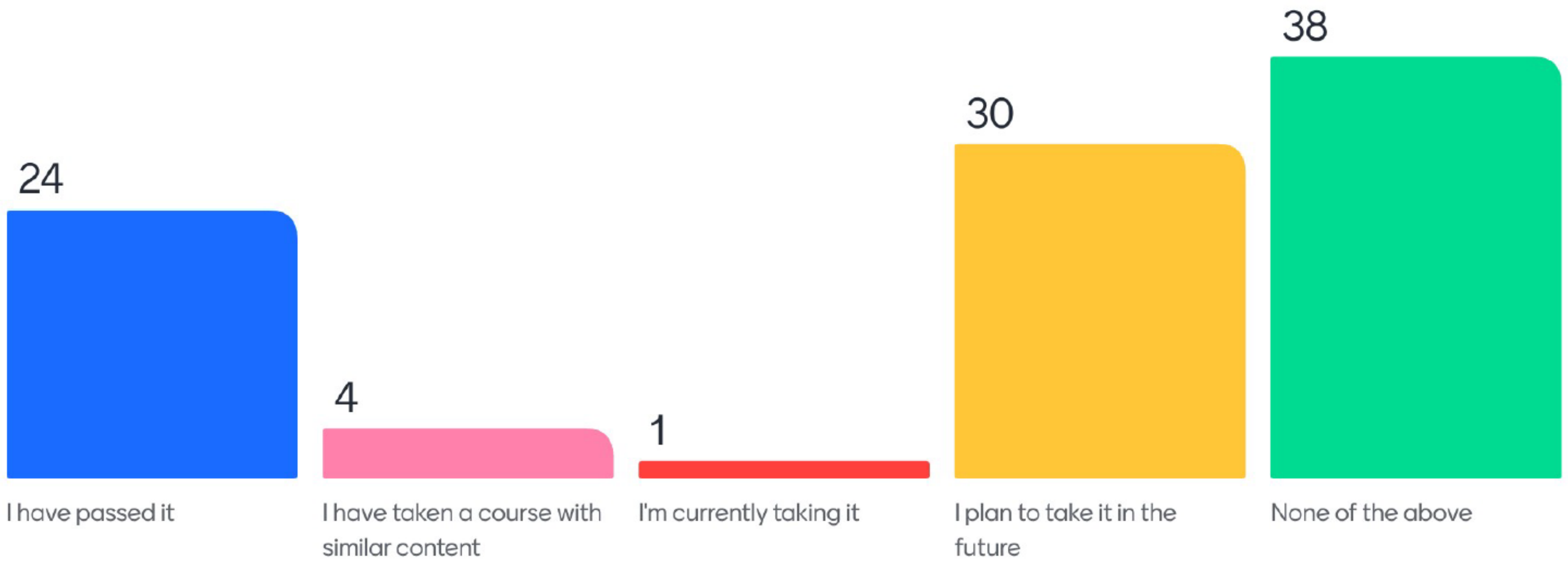
1. Back-up generation and curtailment
2. Storage
3. Regional integration of renewables
4. Demand-side management
5. Sector-coupling

Temporal and spatial balancing (together with demand-side management and sector-coupling) must be simultaneously considered -> **This requires optimization!**

This lecture provides a review of the optimization theory that we will use in this course.

For a comprehensive discussion, check the nice DTU course 46750 Optimization in Modern Power Systems.

Regarding the course "Optimization in Modern Power Systems (46750)"



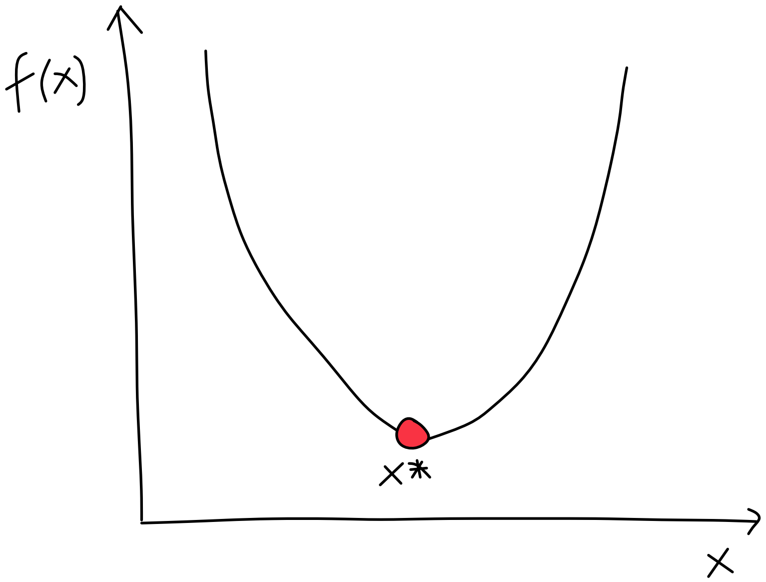
Learning goals for this lecture

- Formulate and solve optimization problems subject to constraints
- Interpret the meaning of Lagrange/KKT multipliers associated with the constraints of an optimization problem and analyze their values
- Formulate and solve simple economic dispatch problems
- Describe convex optimization problems

Optimization theory

One-dimension optimization problem

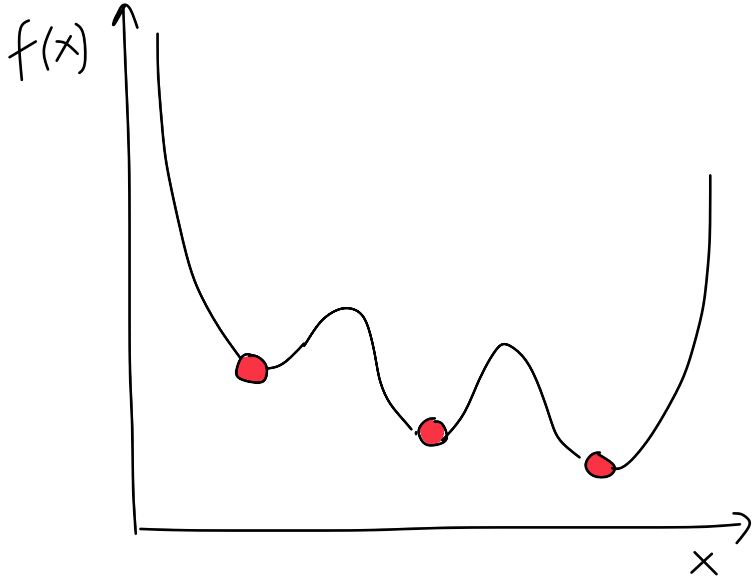
To calculate the optimum in one-dimension problems, we calculate the derivative and make it equal to zero



$$\min_x f(x)$$

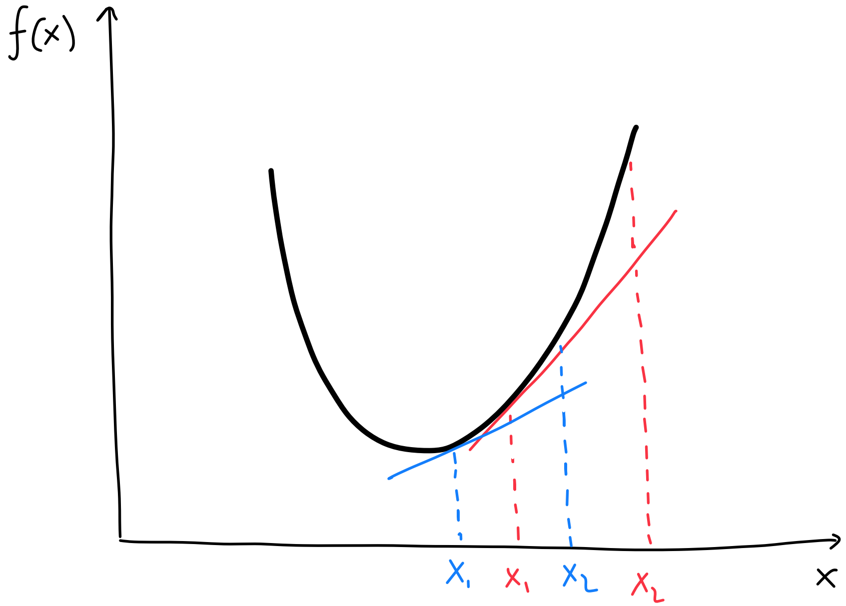
$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = 0$$

This method fails if the function is not convex

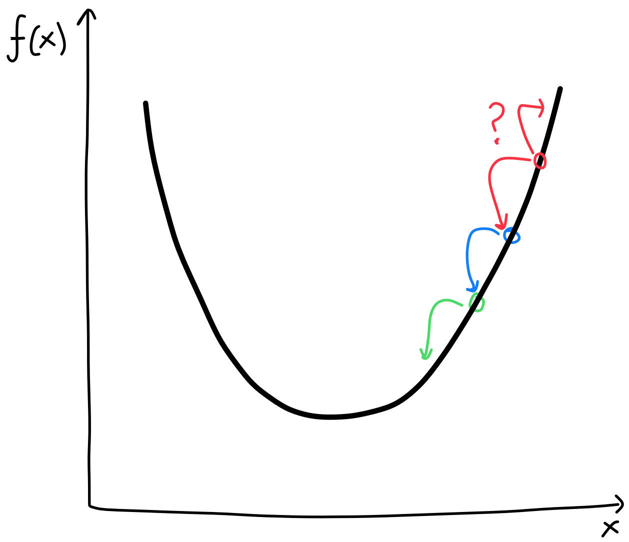


One-dimension optimization problem

Convex function: $f(x)$ is above all its tangents,
i.e. for all $x_1, x_2 \in \boldsymbol{R}^N$: $f(x_2) \geq (x_2 - x_1)\frac{\partial f(x_1)}{\partial x}$



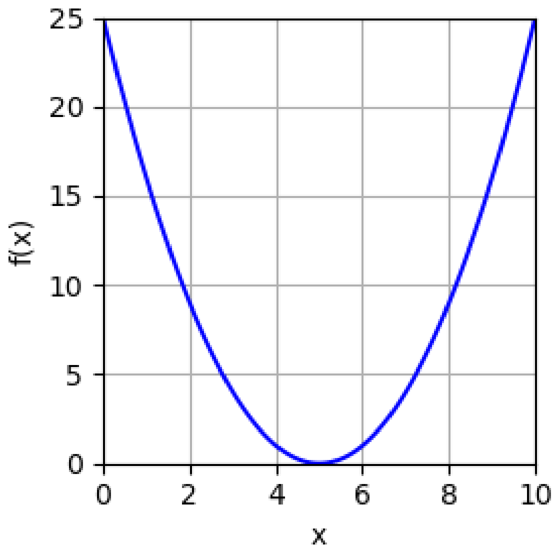
Convexity guarantees the convergence of gradient descent algorithms to a global optimum.



Examples of convex and non-convex functions

Convex function:

$$f(x) = (x-5)^2$$

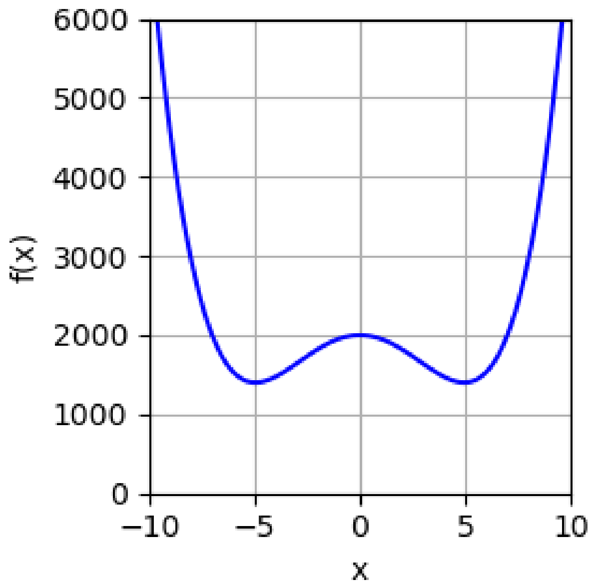


$$\frac{\partial f}{\partial x} = 2(x-5) = 0$$

$$x^* = 5 \qquad f(x^*) = 0$$

Non-convex function:

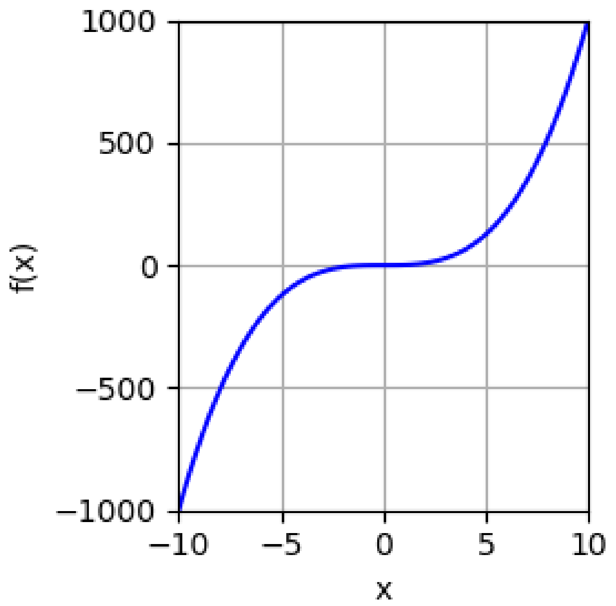
$$f(x) = x^4 - 7x^2 + 2000$$



Two local minima.
Algorithms that look for global minima do not work

Non-convex function:

$$f(x) = x^3$$

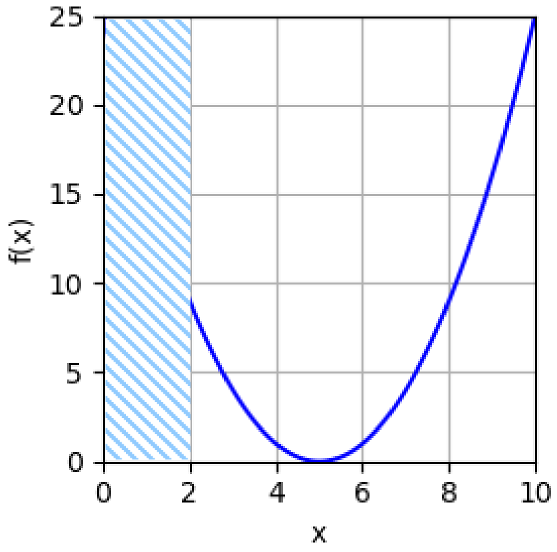


Derivative equals to zero indicates a saddle point.

And the function is not bounded

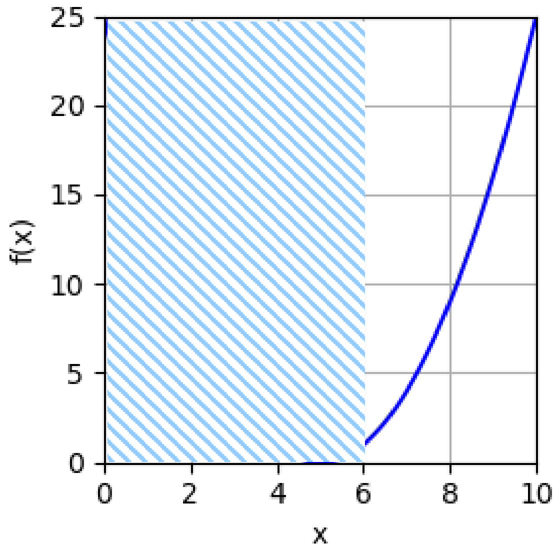
One-dimension optimization problem with constraint

$$\begin{cases} f(x) = (x-5)^2 \\ \text{subject to:} \\ \quad x \geq 2 \end{cases}$$



The constraint is **not binding**.
It has **no effect on the solution**.

$$\begin{cases} f(x) = (x-5)^2 \\ \text{subject to:} \\ \quad x \geq 6 \end{cases}$$



The constraint is **binding**.
It modifies the solution

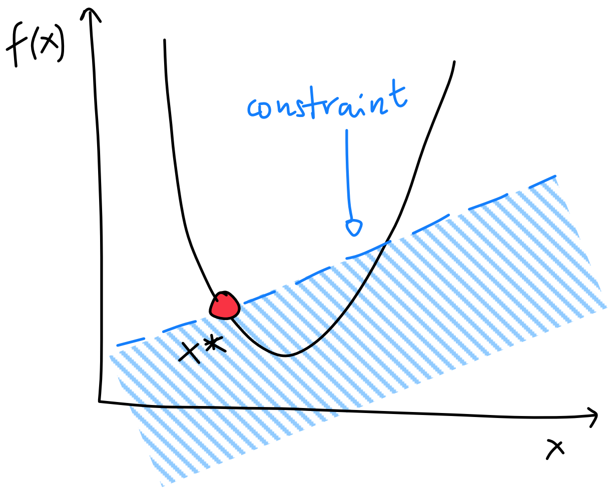
General formulation of optimization problems (I)

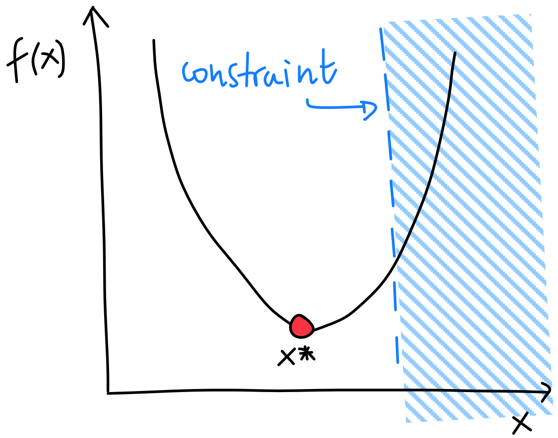
$$\begin{cases} \min\limits_{x} f(x) & \text{Objective function} \\ \text{subject to} & \\ h_i(x) = c_i \;\leftrightarrow\; \lambda_i & \text{Equality and} \\ g_j(x) \geq d_j \;\leftrightarrow\; \mu_j & \text{inequality constraints} \end{cases}$$

λ_i and μ_j are the Lagrange or Karush-Kuhn-Tucker (KKT) multipliers

x, y, z are called primary variables and λ_i , μ_j are called dual variables

A constraint can be binding (affecting the optimal solution x^*) or not-binding





General formulation of optimization problems (II)

In energy optimization problems, the objective function typically attempts to minimize total costs, the constraints impose physical requirements (supply demand, maximum transmission capacity…) and the optimization variables are the energy produced by every generator.

In this course, we will always build minimization problems (for consistency, because we will follow the approach introduced in course 46750 Optimization in Modern Power Systems)

Non-standard problem formulation

$$\max_x f(x)$$

subject to

$$h_i(x) = c_i \;\leftrightarrow\; \lambda_i$$
$$g_j(x) \leq d_j \;\leftrightarrow\; \mu_j$$

Objective function

Equality and inequality constraints

Standard problem formulation

$$\min_x -f(x)$$

subject to

$$h_i(x) = c_i \;\leftrightarrow\; \lambda_i$$
$$-g_j(x)+d_j \geq 0 \leftrightarrow\; \mu_j$$

Using the Lagrangian to solve optimization problems (I)

$$\begin{cases} \min_x f(x) \\ \\ \text{subject to} \\ h_i(x) - c_i = 0 \ \leftrightarrow \ \lambda_i \\ g_j(x) - d_j \geq 0 \ \leftrightarrow \ \mu_j \end{cases}$$

Objective function

Equality and inequality constraints

We can solve optimization problems by building the Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j]$$

and making its partial derivative with respect to x, λ, μ equal to zero.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i \frac{\partial h_i}{\partial x} - \sum_j \mu_j \frac{\partial g_j}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \mu_j} = 0$$

Using the Lagrangian to solve optimization problems (II)

The Lagrange or Karush-Kuhn-Tucker (KKT) multipliers λ_i, μ_j represents the sensitivity of the optimal objective function value with respect to a small change in the right-hand side of the constraint c_i they are associated with $\lambda_i \sim \frac{\partial f}{\partial c_i} \quad \mu_j \sim \frac{\partial f}{\partial d_j}$

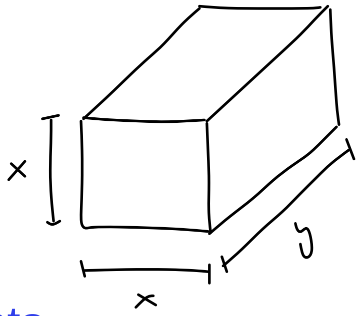
They have relevant meaning in energy optimization problems. They are also called shadow prices.

Here, we build the Lagrangian function with negative signs so that later we can obtain $\lambda_i \sim \frac{\partial f}{\partial c_i}$ but different $\pm$ criteria exist in the literature.

Example 1

Minimize the volume of a box subject to a constraint that limits the base perimeter plus length

$$\begin{cases} \min_x f(x,y) = x^2 y & \text{Objective function} \\ \text{subject to} \\ 4\mathrm{x} + y - 24 = 0 \leftrightarrow \lambda & \text{Equality and inequality constraints} \end{cases}$$



To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x,\lambda,\mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j \left[g_j(x) - d_j\right] = x^2 y - \lambda(4\mathrm{x} + y - 24)$$

We derive the Lagrangian and make the derivative equal to zero

$$\frac{\partial \mathcal{L}}{\partial x} = 2xy - \lambda \cdot 4 = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x^2 - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 4x + y - 24 = 0$$

$$4\lambda = 2\mathrm{xy}$$

$$\lambda = x^2$$

$$4x^2 = 2\mathrm{xy}$$

$$2x = y$$

$$4x + 2x - 24 = 0$$

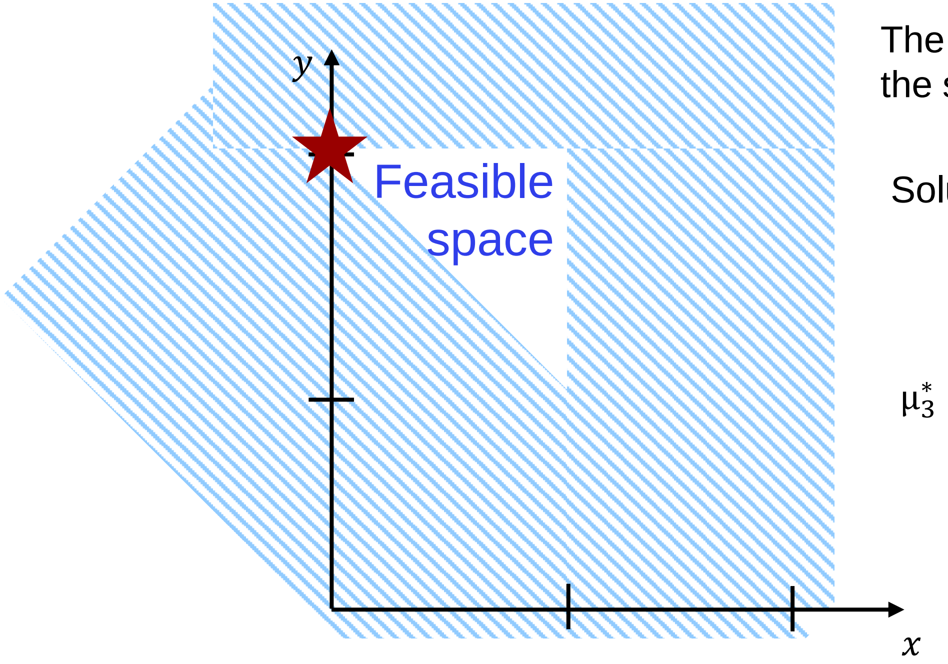
$$x^* = \frac{24}{6} = 4$$

$$y^* = 8$$

$$\lambda^* = 16$$

Example 2

$$\begin{cases} \min_{x,y} f(x) = 8x \\ \text{subject to} \\ x + y \geq 2 \quad \leftrightarrow \quad \mu_1 \\ -y + 2 \geq 0 \quad \leftrightarrow \quad \mu_2 \\ -x + 1 \geq 0 \quad \leftrightarrow \quad \mu_3 \end{cases}$$



The constraints define the feasible space. If the space is empty, the problem is unfeasible

Solution: $x^* = 0$

$y^* = 2$

$f^* = 0$

$\mu_3^* = 0$ because this constraint is not binding

To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j] = 8x - \mu_1(x + y - 2) - \mu_2(-y + 2) - \mu_3(-x + 1)$$

We derive the Lagrangian and make the derivative equal to zero

$$\frac{\partial \mathcal{L}}{\partial x} = 8 - \mu_1^* + \mu_3^* = 0 \qquad \mu_3^* = 0 \quad \mu_1^* = 8$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 - \mu_1^* + \mu_2^* = 0 \qquad \mu_1^* = \mu_2^*$$

$\mu_1^* = 8$ represents the change in the objective value of the optimal solution with respect to a small change in the constraint

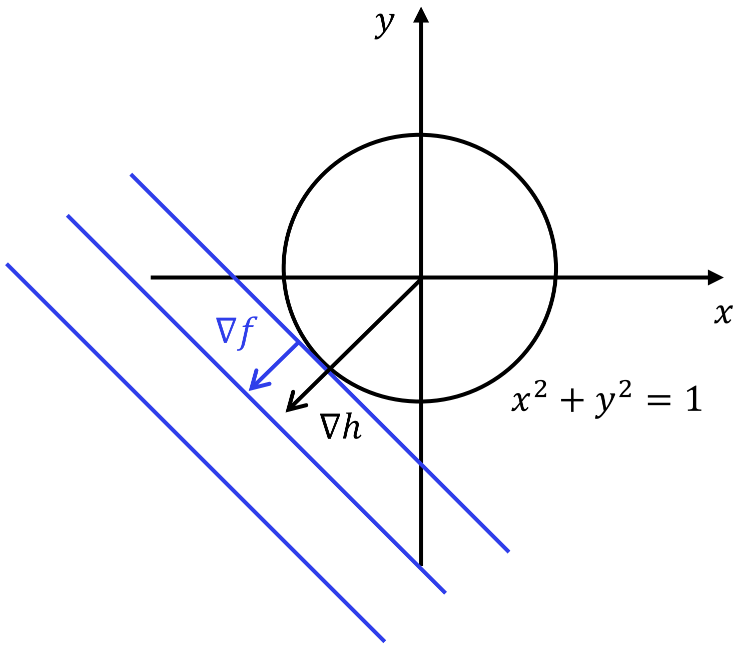
If now, we assume $x + y \geq 2 - \varepsilon$

The solution will be $x^* = -\varepsilon$

$f^* = 8(-\varepsilon) = -8\varepsilon \qquad \frac{\partial f}{\partial d_j} = 8 \sim \mu_1^*$

Example 3: Graphical interpretation of Lagrangian function

$$\begin{cases} \min_{x,y} f(x,y) = x + y \\ \text{subject to:} \\ h(x,y) = x^2 + y^2 - 1 = 0 \end{cases}$$



At the point where the objective function is maximized and the constraint is fulfilled, the gradients of the objective function and the constrains are parallel $\nabla f = \lambda \nabla h$

$$\nabla f - \lambda \nabla h = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \qquad x = \frac{1}{2\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - 2\lambda y = 0 \qquad y = \frac{1}{2\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 - 1 = 0$$

$$\frac{1}{2\lambda^2} = 1 \qquad \lambda^* = \pm\frac{1}{\sqrt{2}} \qquad x^* = \pm\frac{\sqrt{2}}{2} \qquad y^* = \pm\frac{\sqrt{2}}{2}$$

Here, we can see again that the Lagrange multiplier corresponds to the sensitivity of the optimal objective function value with respect to a small change in the equality constraint they are associated with.

Sufficient and necessary Karush-Kuhn-Tucker (KKT) conditions

KKT are necessary conditions for an optimal solution.

In some cases, KKT are also sufficient conditions (e.g. linear optimization problems)

KKT conditions are typically defined for maximization problems, for a minimization problem in the standard form that we use in this course, KKT conditions are the following

$$\frac{\partial \mathcal{L}(x^*, \lambda^*, \mu^*)}{\partial x} = \frac{\partial f(x^*)}{\partial x} - \sum_i \lambda_i^* \frac{\partial h_i(x^*)}{\partial x} - \sum_j \mu_j^* \frac{\partial g_j(x^*)}{\partial x} = 0$$

1st order condition or stationarity

$$h_i(x^*) - c_i = 0$$

$$g_j(x^*) - d_j \geq 0$$

Primal feasibility conditions

$$\mu_j^* \geq 0$$

Dual feasibility conditions

$$\mu_j^*(g_j(x^*) - d_j) = 0$$

Complementary slackness conditions

On the meaning of KKT conditions

For each inequality constraint

$$\mu_j^* \geq 0$$ Dual feasibility conditions

$$\mu_j^*(g_j(x^*) - d_j) = 0$$ Complementary slackness conditions

Only two options are possible:

(a) The constraint is **not binding**. It has no effect on the solution. We can remove the constraint

$$\mu_j^* = 0 \qquad \frac{\partial f(x^*)}{\partial x} = 0$$

(b) The constraint is **binding**. It modifies the solution. We cannot improve the objective along the constraint

$$\mu_j^* \neq 0 \qquad g_j(x^*) - d_j = 0 \qquad \mu_j^* > 0 \qquad \frac{\partial f(x^*)}{\partial x} = \mu_j^* \frac{\partial g(x^*)}{\partial x}$$

Economic dispatch (or one-node dispatch optimization)

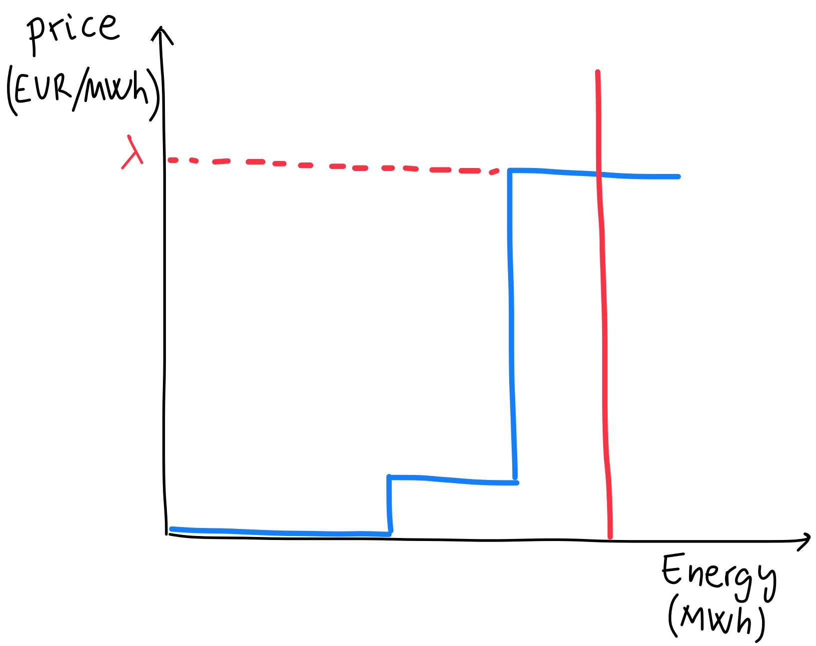
Types of optimization problems and course structure

<table>
<tr><th></th><th>One node</th><th colspan="4">Network</th></tr>
<tr><td rowspan="2">One time step</td><td rowspan="2">Economic dispatch or
One-node dispatch optimization
(Lecture 2)</td><td colspan="2">Power</td><td rowspan="2">Gas flow
(Lecture 6)</td><td rowspan="2">Heat flow
(Lecture 7)</td></tr>
<tr><td>Linearized
AC power flow
(Lecture 4)</td><td>AC power flow
(Lecture 5)</td></tr>
<tr><td>Multiple time steps</td><td>Multi-period optimization

Join capacity and dispatch optimization in one node
(Lecture 8)</td><td colspan="4">Join capacity and dispatch optimization in a network
(Lecture 10)</td></tr>
</table>

Economic dispatch example

	Wind	Solar	Gas
Variable cost o_s (EUR/MWh)	0	5	50
Installed Capacity G_s (MW)	2	1	1
CF_s	0.5	0.5	1
Calculate generation g_s (assuming demand d = 1.5 MWh)	1	0.5	0
Calculate generation g_s (assuming demand d = 2 MWh)	1	0.5	0.5



Economic dispatch is also called merit-order dispatch (because we rank the generator based on their merit)

Economic dispatch is used to decide which generators produce energy in a market. It assumes ideal lossless ("copperplate") network within the area belonging to the market. After the economic dispatch, the system operators run AC power flow (including N-1 security constraint). If they identify anything that is physically impossible, redispatch takes place (e.g. via ancillary markets)

Economic dispatch or one-node dispatch optimization (I)

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s. The economic dispatch consists in calculating the optimal dispatch (how much energy is being produced by each generator g_s) to supply the demand d in a certain hour while minimizing the total system cost.

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda$$

$$g_s \geq 0 \ \leftrightarrow \ \underline{\mu_s}$$

$$-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}$$

For renewable generators, the installed capacity is multiplied by the capacity factor: $-g_s + CF_s G_s \leq 0 \ \leftrightarrow \overline{\mu_s}$

Economic dispatch or one-node dispatch optimization (II)

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s. The economic dispatch consists in calculating the optimal dispatch (how much energy is being produced by each generator g_s) to supply the demand d in a certain hour while minimizing the total system cost.

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda$$

$$g_s \geq 0 \;\leftrightarrow\; \underline{\mu_s}$$

$$-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}$$

To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = \sum_s o_s g_s - \lambda \left(\sum_s g_s - d\right) - \sum_s \overline{\mu_s}(-g_s + G_s)$$

We derive the Lagrangian and make the derivative equal to zero

$$\frac{\partial \mathcal{L}}{\partial g_s} = o_s - \lambda^* + \overline{\mu_s^*} = 0$$

The inequality constraint can be binding ($\overline{\mu_s} > 0$) when the installed capacity is limiting the generation or not-binding ($\overline{\mu_s} = 0$).

The most expensive generator s whose capacity is not binding sets the price because for that generator $s1$ $\quad \overline{\mu_{s1}^*} = 0 \quad \lambda^* = o_{s1}$

Economic dispatch or one-node dispatch optimization (III)

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda$$

$$g_s \geq 0 \ \leftrightarrow \ \underline{\mu_s}$$

$$-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}$$

λ represents the change in the objective function at the optimal solution with respect to a small change in the constraint.

Small change in constraint : $d^* = d^* + 1\ \mathrm{MWh}$

Change in objective function :

$$System\ cost^* = System\ cost + \Delta System\ cost$$

λ represents the cost of 1 MWh, i.e. the electricity price

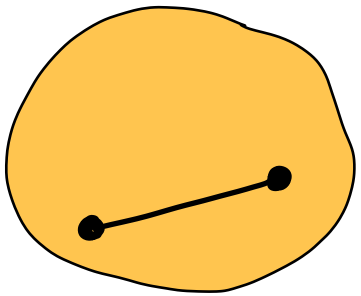
So far, this set of equations excludes the consideration of any network constraints (e.g. line limits), and any additional security constraints, and additional generation constraints (CO2 emissions, ramp limits …)

Convex optimization problems

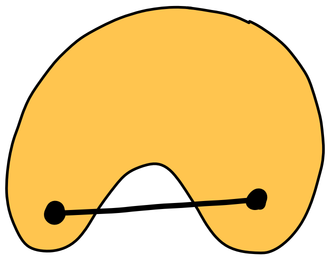
Convex optimization problem

Convex set : A convex set X contains every line segment between two points in the set,
i.e. for all $x_1, x_2 \in X \in \boldsymbol{R}^N$ and $\alpha \in [0,1]$: $\alpha x_1 + (1-\alpha)x_2 \in X$

convex



non-convex



$$\left\{\begin{array}{l} \min\limits_{x} f(x) \\ \text{subject to} \\ \; h_i(x) - c_i = 0 \;\leftrightarrow\; \lambda_i \\ g_j(x) - d_j \geq 0 \;\leftrightarrow\; \mu_j \end{array}\right.$$

An optimization problem is convex if:

- The objective function $f(x)$ is convex
- The equality constraints $h_i(x)$ are affine (i.e. composed of a linear function and a constant)
- The inequality constrains $g_j(x)$ are convex

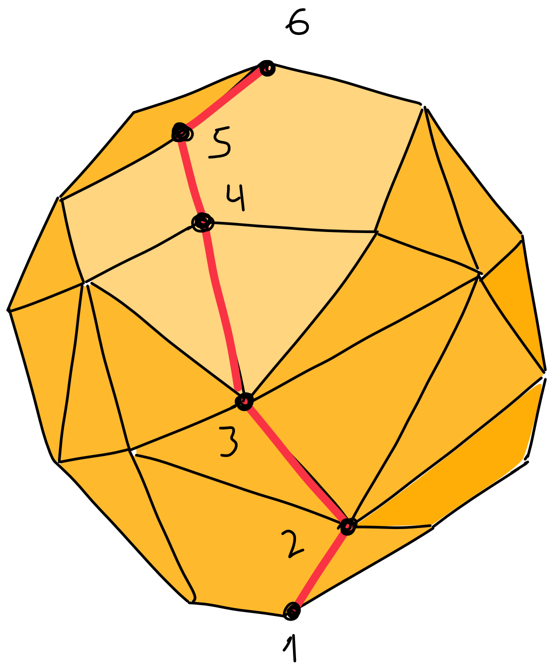
Convex optimization problems have **one single optimum which is the global optimum**

Optimization algorithms and computational challenges (I)

If all the constraints are affine, the feasible space is a multi-dimensional polyhedron with only flat sides.

The optimum always occurs at one of the corners or vertices.

The simplex algorithm operates by descending the vertices of the polyhedron.

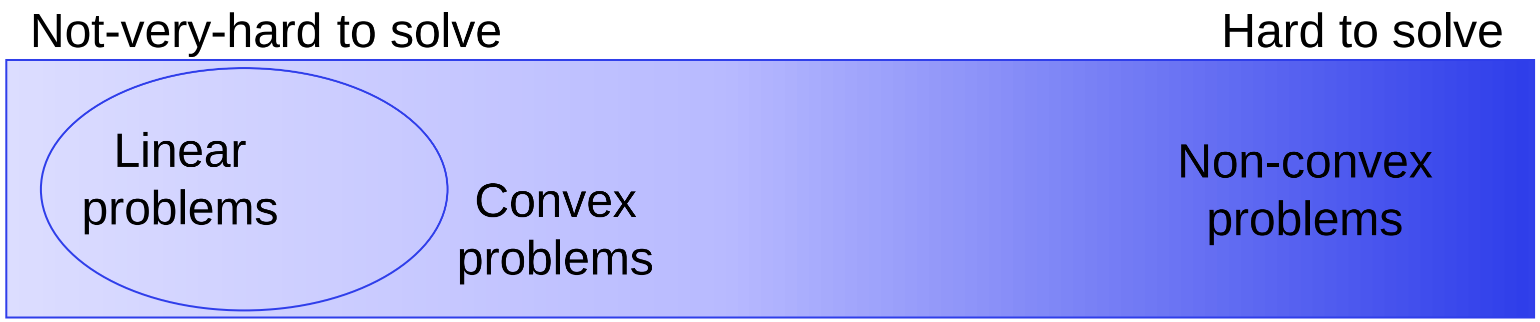


Optimization algorithms and computational challenges (II)

Convex optimization problems are efficiently solvable because it is not necessary to check numerous local minima to find the global minimum.

Usually convex-problems admit polynomial-time algorithms.

Polynomial-time algorithms require a number of operation that is polynomial in both the number of variables and the constraints.

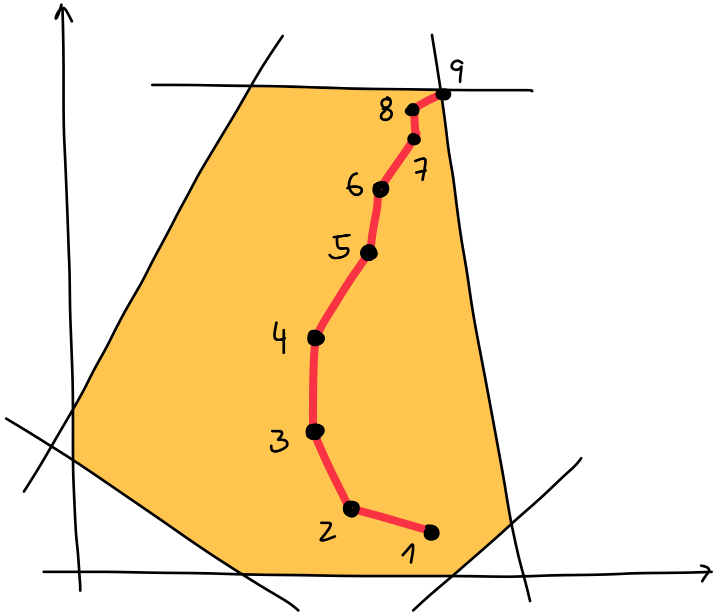


Barrier interior-point methods

We can use interior-point methods which converge faster.

The objective is to iteratively approach the optimal solution from the interior of the feasible space.

A barrier term is added to the objective function that penalizes solutions that come close to the boundary.

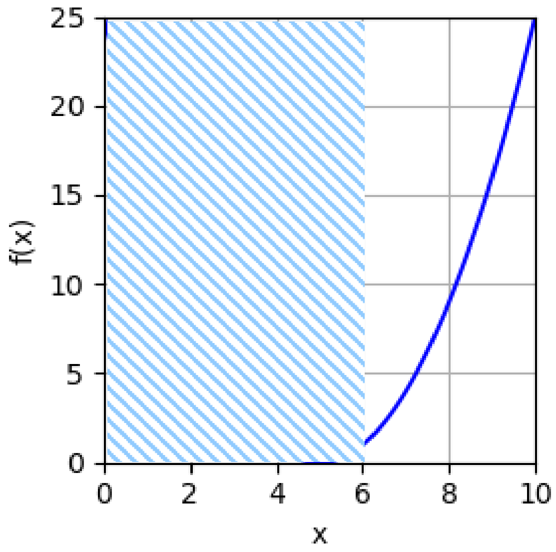


Karmarkar's interior-point method can run in polynomial time for linear problems.

Barrier interior-point methods. Example

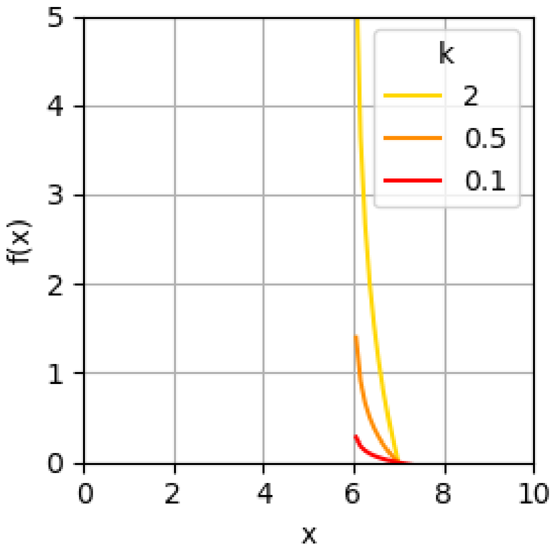
Let's recover our initial optimization problem

$$\begin{cases} f(x) = (x-5)^2 \\ \text{subject to:} \\ \quad x \geq 6 \end{cases}$$



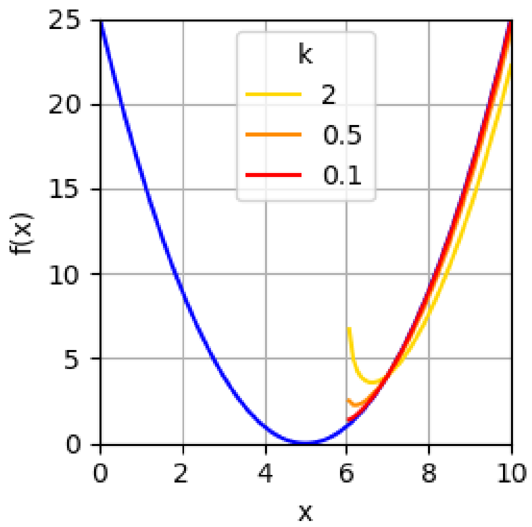
We can add a barrier term that increases when the solution approaches the boundary

$$\text{penalty} = -kln(x-6)$$



Now, we can substitute the constraint by the barrier term

$$f(x) = (x-5)^2 - \underbrace{kln(x-6)}_{\text{barrier term}}$$

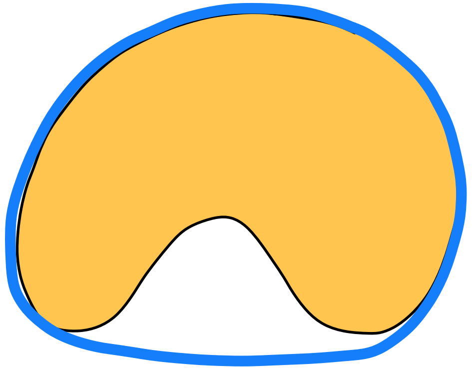


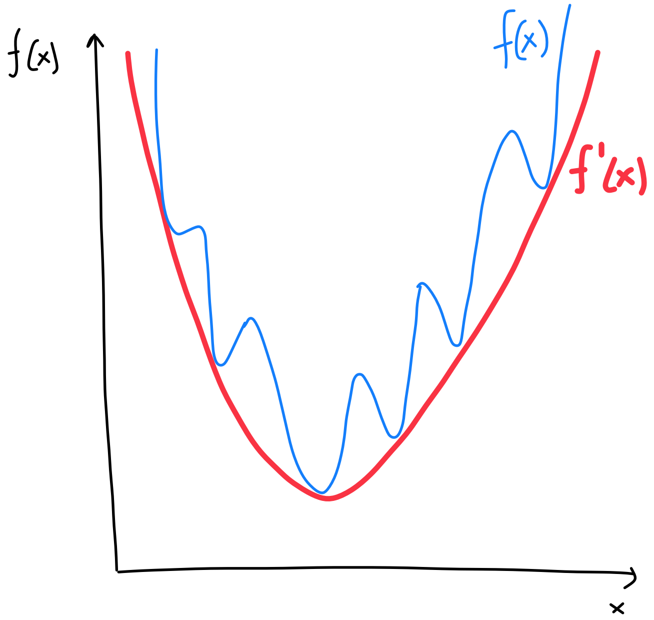
By reducing the penalty, the algorithm converges to the optimal point at the boundary

Convex relaxation

To solve a non-convex optimization problem, we typically create a convex relaxation problem, and we say that we "convexify" the problem.

Convex relaxation: we create an alternative problem by relaxing some of the constraints.





Short videos

Visually Explained: Convexity and The Principle of Duality
https://www.youtube.com/watch?v=d0CF3d5aEGc&list=PLP3dxscCx69JuM3Ppv9CD49IN1Z_a2Avq&index=9

Visually Explained: The Karush–Kuhn–Tucker (KKT) Conditions and the Interior Point Method for Convex Optimization https://www.youtube.com/watch?v=uh1Dk68cfWs&t=1169s

Understanding Lagrange Multipliers Visually

https://www.youtube.com/watch?v=5A39Ht9Wcu0

Problems for this lecture

Complete Multiple-choice test in Lecture 2 in DTU Learn

Review intro to linopy
https://martavp.github.io/integrated-energy-grids/intro-linopy.html

To be presented next day:

Problems 2.1 and 2.2 (**Group 4**)
Problems 2.3 (**Group 5**)

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