46770 Integrated energy grids

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Lecture 2 – Optimization (a short review)

Balancing renewable generation

We finished last lecture, stating that we need optimization to decide how to combine different flexibility options.

1. Back-up generation and curtailment

2. Storage

3. Regional integration of renewables

4. Demand-side management

5. Sector-coupling

This lecture provides a review of the optimization theory that we will use in this course.

For a comprehensive discussion, check the nice DTU course 46750 Optimization in Modern Power Systems.

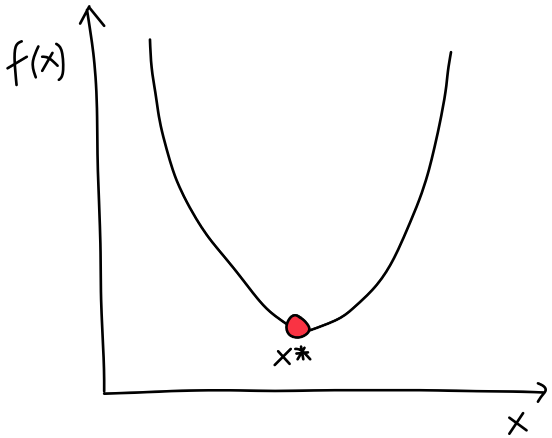
Learning goals for this lecture

- Formulate and solve optimization problems subject to constraints
- Interpret the meaning of the dual variables associated with the constraints of an optimization problem and analyze their values
- Formulate and solve simple economic dispatch problems
- Describe convex optimization problems and convexification approaches

Unconstrained optimization problem

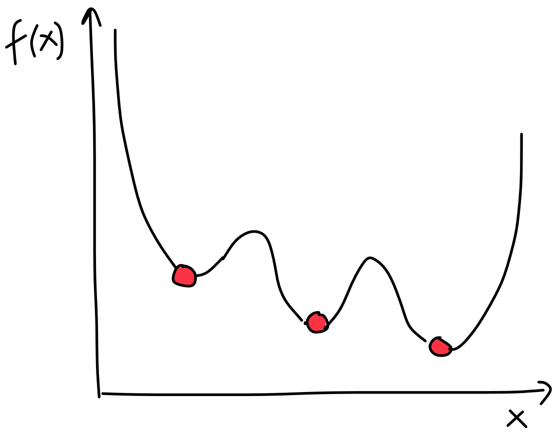
To calculate the optimum in one-dimension problems, we calculate the derivative and make it equal to zero

This method fails if the function is not convex



$$\min_{x} f(x)$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = 0$$



Convex function: $f(x)$ is above all its tangents, i.e. for all $x_1, x_2 \in \boldsymbol{R}^N$: $\mathrm{f}(x_2) \geq (x_2 - x_1)\frac{\partial f(x_1)}{\partial x}$

Add drawing

Convexity guarantees the convergence of gradient descent algorithms to a global optimum.

Unconstrained optimization problem

Add examples 1-d convex funcion and no-convex function, local and global minima, saddle points

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x^*} > 0$$

Optimization problem

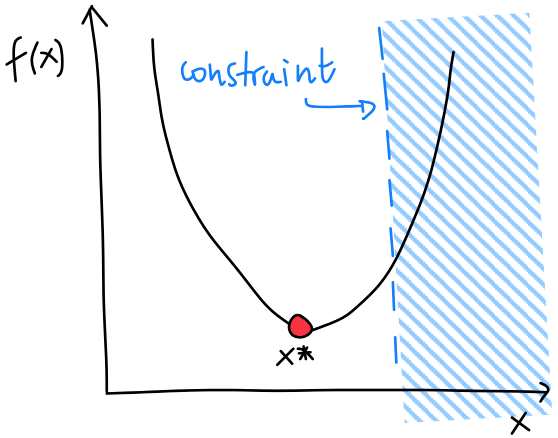
Energy systems models are optimization problems (typically with millions of variables)

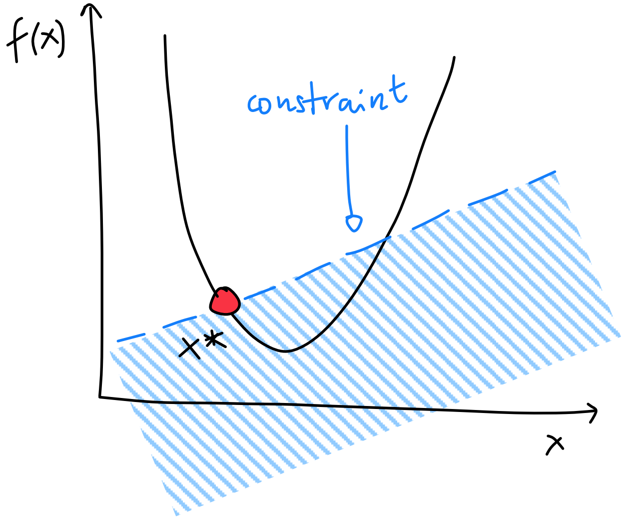
$$\begin{cases} \min_x f(x) \\ \\ \text{subject to} \\ h_i(x) = c_i \;\leftrightarrow\; \lambda_i \\ g_j(x) \geq d_j \;\leftrightarrow\; \mu_j \end{cases}$$

Objective function

Equality and inequality constraints

A constraint can be binding (affecting the optimal solution x^*) or not





Optimization problem

In energy optimization problems the objective function typically attempts to minimize total costs, the constraints impose physical requirements (supply demand, maximum transmission capacity) and the optimization variables are the energy produced by every generator.

We will always build minimization problems (for consistency, we follow the approach introduced in course 46750 Optimization in Modern Power Systems)

Non-standard problem formulation

$$\begin{cases} \max_{x} f(x) \\ \text{subject to} \\ h_i(x) = c_i \ \leftrightarrow \ \lambda_i \\ g_j(x) \leq d_j \ \leftrightarrow \ \mu_j \end{cases}$$

Objective function

Equality and inequality constraints

Standard problem formulation

$$\begin{cases} \min_{x} -f(x) \\ \text{subject to} \\ h_i(x) = c_i \ \leftrightarrow \ \lambda_i \\ -g_j(x)+d_j \geq 0 \leftrightarrow \ \mu_j \end{cases}$$

Optimization problem

$$\begin{cases} \min\limits_{x} f(x) \\ \\ \text{subject to} \\ \quad h_i(x) - c_i = 0 \ \leftrightarrow \ \lambda_i \\ \quad g_j(x) - d_j \geq 0 \ \leftrightarrow \ \mu_j \end{cases}$$

Objective function

Equality and inequality constraints

We can solve optimization problems by building the Lagrangian function $\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j]$

and making its partial derivative with respect to x, λ, μ equal to zero.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i \frac{\partial h_i}{\partial x} - \sum_j \mu_j \frac{\partial g_j}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \mu_j} = 0$$

If $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$, the constraint $h_i(x) - c_i = 0$ will be satisfied

The solution $x^*, y^*, \lambda^*, \mu^*$ fulfills $\mathcal{L}(x^*, y^*, \lambda^*, \mu^*) = f(x^*)$

Example 1

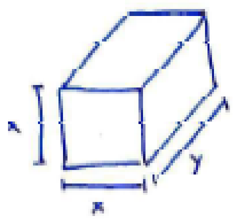
Minimize the volume of a box subject to a constraint that limits the base perimeter plus length

$$\begin{cases} \min_{x} f(x,y) = x^2 y \\ \text{subject to} \\ 4\mathrm{x} + y - 24 = 0 \ \leftrightarrow \lambda \end{cases}$$

Objective function

Equality and inequality constraints

Drawing



To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x,\lambda,\mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j] = x^2 y \ - \lambda(4\mathrm{x} + y - 24\,)$$

We derive the Lagrangian and make the derivative equal to zero

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i \frac{\partial h_i}{\partial x} - \sum_j \mu_j \frac{\partial g_j}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\mathrm{x}y - \lambda \cdot 4 = 0 \qquad 4\lambda = 2\mathrm{xy} \qquad 4x^2 = 2\mathrm{xy}$$

$$\frac{\partial \mathcal{L}}{\partial y} = x^2 - \lambda = 0 \qquad \lambda = x^2 \qquad 2x = y$$

$$4x + 2x - 24 = 0 \qquad x^* = \frac{24}{6} = 4 \quad y^* = 8 \quad \lambda^* = 16$$

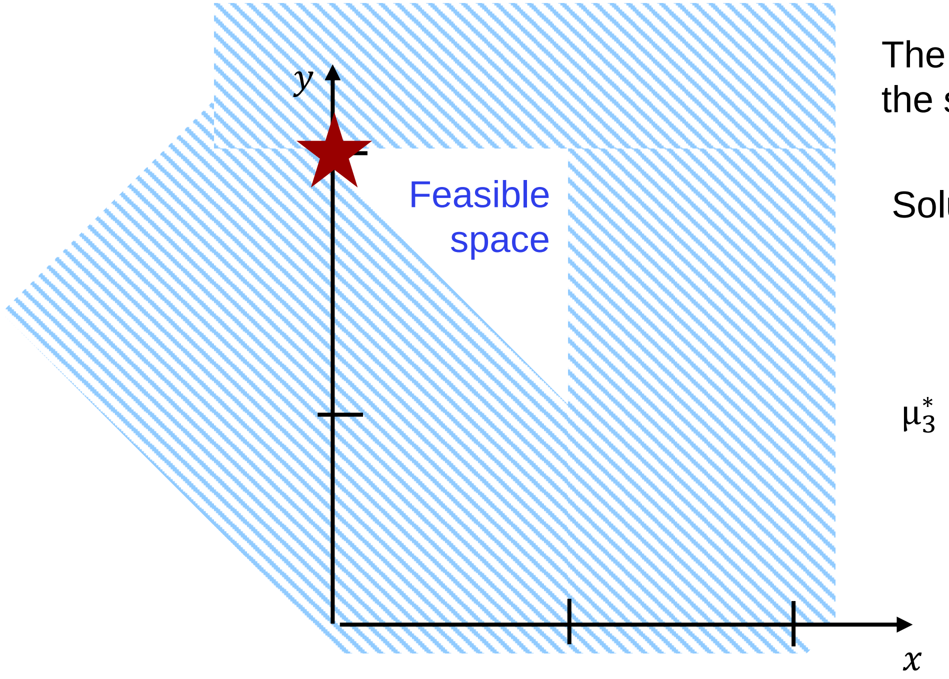
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 4x + y - 24 = 0$$

Example 2

$$\begin{cases} \min\limits_{x,y} f(x) = 8x \\ \text{subject to} \\ \quad x + y \geq 2 \quad \leftrightarrow \quad \mu_1 \\ \quad -y + 2 \geq 0 \quad \leftrightarrow \quad \mu_2 \\ \quad -x + 1 \geq 0 \quad \leftrightarrow \quad \mu_3 \end{cases}$$

Objective function

Equality and inequality constraints



The constraints define the feasible space. If the space is empty, the problem is unfeasible

Solution: $x^* = 0$

$y^* = 2$

$f^* = 0$

$\mu_3^* = 0$ because this constraint is not binding

To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x,\lambda,\mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j] = 8x - \mu_1(x + y - 2) - \mu_2(-y + 2) - \mu_3(-x + 1)$$

We derive the Lagrangian and make the derivative equal to zero $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \sum_i \lambda_i \frac{\partial h_i}{\partial x} - \sum_j \mu_j \frac{\partial g_j}{\partial x} = 0$

$$\frac{\partial \mathcal{L}}{\partial x} = 8 - \mu_1^* + \mu_3^* = 0 \qquad \mu_3^* = 0 \quad \mu_1^* = 8$$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 - \mu_1^* + \mu_2^* = 0 \qquad \mu_1^* = \mu_2^*$$

$\mu_1^* = 8$ represents the change in the objective value of the optimal solution with respect to a small change in the constraint

If now, we assume $x + y \geq 2 - \varepsilon$

The solution will be $x^* = -\varepsilon$

And $f^* = 8(-\varepsilon) = -8\varepsilon$

$$\frac{\partial f}{\partial d_j} = 8 \sim \mu_1^*$$

Optimization problem

$$\begin{cases} \min_x f(x) & \text{Objective function} \\ \text{subject to} & \\ \quad h(x) = c_i \leftrightarrow \lambda_i & \text{Equality and inequality constraints} \\ \quad g_j(x) \geq d_j \leftrightarrow \mu_j & \end{cases}$$

x, y, z are called primary variables and
λ_i , μ_j are called dual variables

.

λ_i and μ_j are Lagrange or Karush-Kuhn-Tucker (KKT) multipliers.

λ_i represents the sensitivity of the optimal objective function value with respect to a small change in the right-hand side of the constraint c_i they are associated with $\lambda_i \sim -\frac{\partial f}{\partial c_i}$ $\mu_j \sim -\frac{\partial f}{\partial d_j}$

They have relevant meaning in energy optimization problems. They are also called shadow prices.

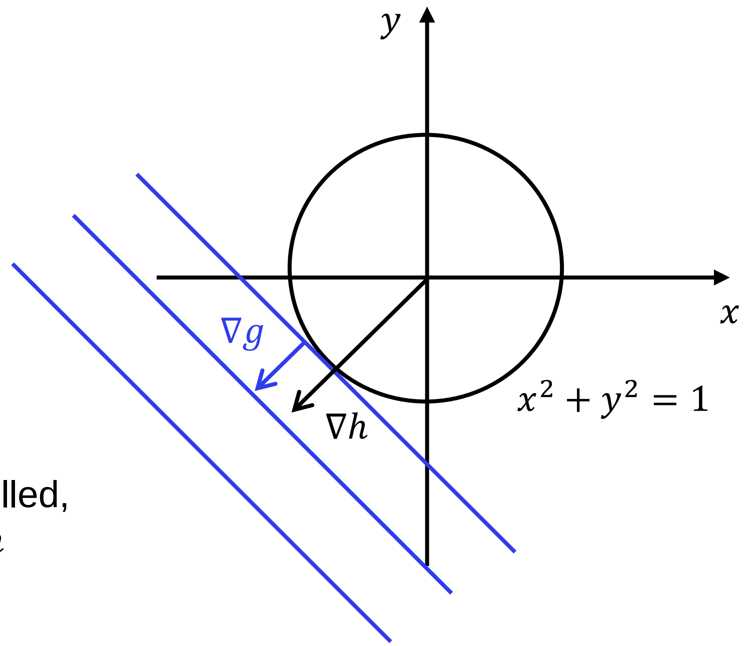
Here, we build the Lagrangian function with negative signs so that later we can obtain $\lambda_i \sim \frac{\partial f}{\partial c_i}$ but different $\pm$ criteria exist in the literature.

Example 3: Graphical interpretation of Lagrangian function

$$\begin{cases} \min\limits_{x,y} f(x,y) = x + y \\ \text{s.t.} \\ h(x,y) = x^2 + y^2 - 1 = 0 \end{cases}$$

Objective function

Equality and inequality constraints



At the point where the objective function is maximized and the constraint is fulfilled, the gradients of the objective function and the constrains are parallel $\nabla f = \lambda \nabla h$

$$\nabla f - \lambda \nabla h = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda x = 0 \qquad x = \frac{1}{2\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - 2\lambda y = 0 \qquad y = \frac{1}{2\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x^2 + y^2 - 1 = 0 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 - 1 = 0$$

$$\frac{1}{2\lambda^2} = 1 \qquad \lambda^* = \pm\frac{1}{\sqrt{2}} \qquad x^* = \pm\frac{\sqrt{2}}{2} \qquad y^* = \pm\frac{\sqrt{2}}{2}$$

Here, we can see again that the Lagrange multiplier corresponds to the sensitivity of the optimal objective function value with respect to a small change in the equality constraint they are associated with.

Sufficient and necessary Karush-Kuhn-Tucker (KKT) conditions

KKT are necessary conditions for an optimal solution.

In some cases, KKT are also sufficient conditions (e.g. linear optimization problems)

$$\frac{\partial \mathcal{L}(x^*,\lambda^*,\mu^*)}{\partial x} = \frac{\partial f(x^*)}{\partial x} + \sum_i \lambda_i^* \frac{\partial h_i(x^*)}{\partial x} + \sum_j \mu_j^* \frac{\partial g_j(x^*)}{\partial x} = 0$$

1st order condition or stationarity

$$\left.\begin{aligned} h_i(x^*) - c_i &= 0 \\ g_j(x^*) - d_j &\leq 0 \end{aligned}\right\}$$

Primal feasibility conditions

$$\mu_j^* \geq 0$$

Dual feasibility conditions

$$\mu_j^*(g_j(x^*) - d_j) = 0$$

Complementary slackness conditions

Sufficient and necessary Karush-Kuhn-Tucker (KKT) conditions

KKT conditions are typically defined for maximization problems, for a minimization problem in the standard form that we use in this course

$$\frac{\partial \mathcal{L}(x^*,\lambda^*,\mu^*)}{\partial x} = \frac{\partial f(x^*)}{\partial x} + \sum_i \lambda_i^* \frac{\partial h_i(x^*)}{\partial x} + \sum_j \mu_j^* \frac{\partial g_j(x^*)}{\partial x} = 0$$

1st order condition or stationarity

$$\left.\begin{aligned} h_i(x^*) - c_i &= 0 \\ g_j(x^*) - d_j &\geq 0 \end{aligned}\right\}$$

Primal feasibility conditions

$$\mu_j^* \geq 0$$

Dual feasibility conditions

$$\mu_j^*(g_j(x^*) - d_j) = 0$$

Complementary slackness conditions

On the meaning of KKT conditions

For each inequality constraint

$\mu_j^* \geq 0$ Dual feasibility conditions

$\mu_j^*(g_j(x^*) - d_j) = 0$ Complementary slackness conditions

Only two options are possible:

(a) Non-binding constraint. We have the same solution as with the unconstraint problem, we can remove the constraint

$\mu_j^* = 0 \qquad \frac{\partial f(x^*)}{\partial x} = 0$

b) Binding constraint: we cannot improve the objective along the constraint

$\mu_j^* \neq 0 \qquad g_j(x^*) - d_j = 0 \qquad \mu_j^* > 0 \qquad \frac{\partial f(x^*)}{\partial x} = \mu_j^* \frac{\partial g(x^*)}{\partial x}$

Costs for different types of generators

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_S . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

This is economic dispatch or merit-order dispatch (because we rank the generator based on their merit)

Economic dispatch assumes ideal lossless (“copperplate”) network. After the economic distpatch, the system operators run AC power flow (including N-1 security constraint). If they identify anything that is physically impossible, redispatch takes place (e.g. via ancillary markets)

Economic dispatch

Non-standard problem formulation

$$\begin{cases} \min_{g_s} \sum_s o_s g_s \\ \text{subject to:} \\ \sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ 0 \leq g_s \leq G_s \end{cases}$$

Standard problem formulation

$$\begin{cases} \min_{g_s} \sum_s o_s g_s \\ \text{subject to:} \\ \sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ g_s \geq 0 \leftrightarrow \underline{\mu_s} \\ -g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s} \end{cases}$$

Economic dispatch

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

$$\begin{cases} \min\limits_{g_s} \sum\limits_s o_s g_s \\ \text{subject to:} \\ \sum\limits_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ g_s \geq 0 \leftrightarrow \underline{\mu_s} \\ -g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s} \end{cases}$$

For renewable generators, the installed capacity is multiplied by the capacity factor: $-g_s + CF_s G_s \leq 0 \leftrightarrow \overline{\mu_s}$

	Wind	Solar	Gas
Variable cost o_s (EUR/MWh)	0	0	50
Installed Capacity G_s (MW)	2	1	1
CF_s	0.5	0.5	1
generation g_s (assuming demand d = 1.5 MWh)	1	0.5	0
generation g_s (assuming demand d = 2 MWh)	1	0.5	0.5

We will explore further this topic in Lecture 8.

Economic dispatch

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

$$
\begin{cases}
\min\limits_{g_s} \sum\limits_{s} o_s g_s \\
\text{subject to:} \\
\sum\limits_{s} g_s - d = 0 \quad \leftrightarrow \quad \lambda \\
g_s \geq 0 \leftrightarrow \underline{\mu_s} \\
-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}
\end{cases}
$$

To find a solution, we start by building the Lagrangian function

$$\mathcal{L}(x, \lambda, \mu) = f(x) - \sum_i \lambda_i [h_i(x) - c_i] - \sum_j \mu_j [g_j(x) - d_j] = \sum_s o_s g_s - \lambda\ (d - \sum_s g_s) - \sum_s \mu_s(-g_s - G_s)$$

We derive the Lagrangian and make the derivative equal to zero

$$\frac{\partial \mathcal{L}}{\partial g_s} = \frac{\partial f}{\partial g_s} - \sum_i \lambda_i \frac{\partial h_i}{\partial g_s} - \sum_j \mu_j \frac{\partial g_j}{\partial g_s} = o_s - \lambda^* + \overline{\mu_s^*} = 0$$

The inequality constraint can be binding ($\mu_s > 0$) when the installed capacity is limiting the generation or not-binding ($\mu_s = 0$).

The most expensive generator s whose capacity is not binding sets the price

$$\overline{\mu_s^*} = 0$$
$$\lambda^* = o_s$$

We will explore further this topic in Lecture 8.

Economic dispatch

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

$$
\begin{cases}
\min\limits_{g_s} \sum\limits_s o_s g_s \\
\text{subject to:} \\
\sum\limits_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\
g_s \geq 0 \ \leftrightarrow \ \underline{\mu_s} \\
-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}
\end{cases}
$$

λ represents the change in the objective function at the optimal solution, with respect to a small change in the constraint.

Small change in constraint : $d^* = d^* + 1\ \text{MWh}$

Change in objective function : $System\ cost^* = System\ cost + \Delta System\ cost$

λ represents the cost of 1 MWh, i.e. spot price in every node and time step

We will explore further this topic in Lecture 8.

Economic dispatch

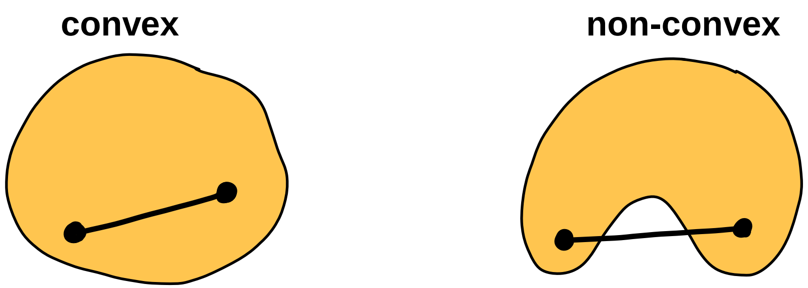
Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

$$\left\{ \begin{array}{l} \min_{g_s} \sum_s o_s g_s \\ \\ \text{subject to:} \\ \sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ g_s \geq 0 \quad \leftrightarrow \quad \underline{\mu_s} \\ -g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s} \end{array} \right.$$

This equation excludes the consideration of any network constraints (e.g. line limits), and any additional security constraints, and additional generation constraints (CO2 emissions, ramp limits)

Convex set

Convex set : A convex set X contains every line segment between two points in the set, i.e. for all $x_1, x_2 \in X \in \boldsymbol{R}^N$ and $\alpha \in [0,1]$: $\alpha x_1 + (1-\alpha)x_2 \in X$



A convex set X defined by constraints

$X = \{x \in \boldsymbol{R}^N\!: s.t.\ h_i(x) = 0;\ g_j(x) \leq 0\}$

is convex if and only if functions $g_j(x)$ are convex over $\boldsymbol{R}^N$ and $h_i(x)$ are affine (i.e. composed of a linear function and a constant)

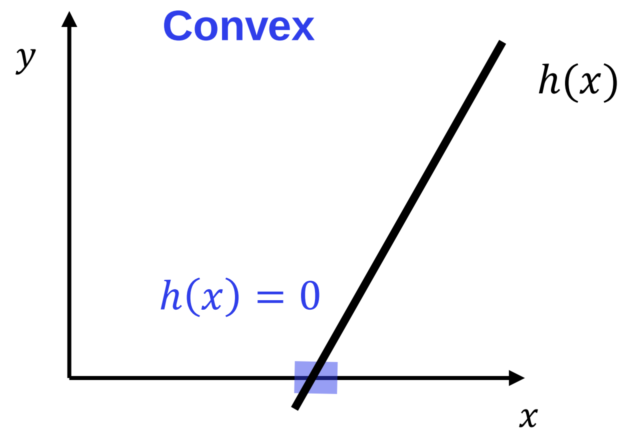
Convex optimization problem

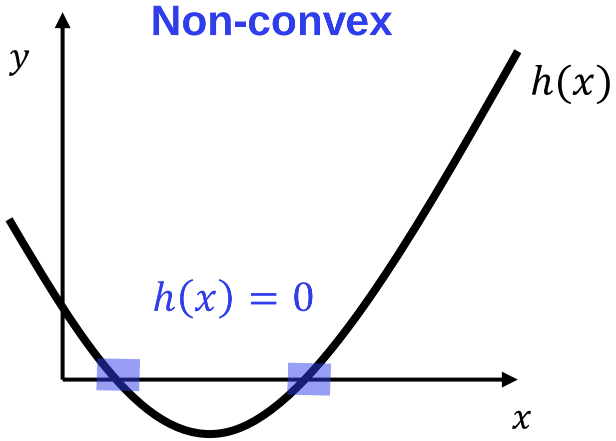
$$\begin{cases} \min\limits_{x} f(x) \\ \text{subject to} \\ \quad h_i(x) - c_i = 0 \ \leftrightarrow \ \lambda_i \\ \quad g_j(x) - d_j \geq 0 \ \leftrightarrow \ \mu_j \end{cases}$$

The optimization problem is convex if:

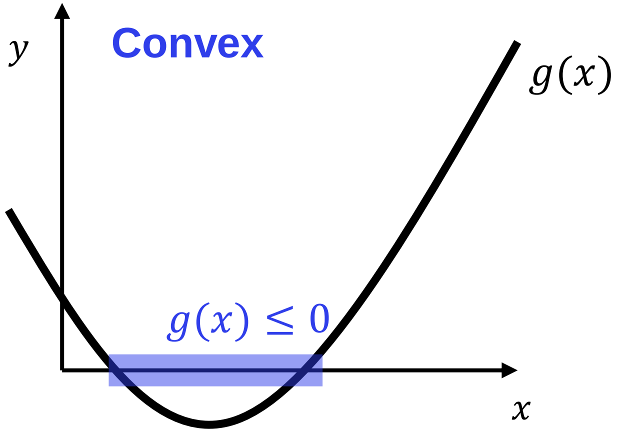
- The objective function f(x) is convex
- The equality constraints $h_i(x)$ are affine (i.e. composed of a linear function and a constant)
- The inequality constrains $g_j(x)$ are convex

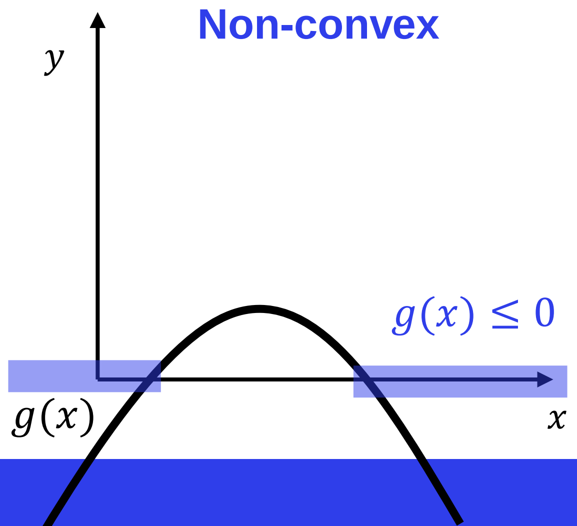
Equality constraint





Inequality constraint



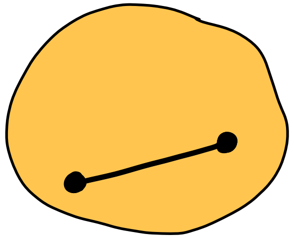


Convex optimization problems has online a single optimum which is the global optimum

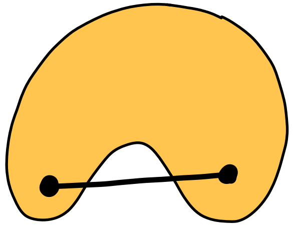
Optimization algorithms and computational challenges

In energy system modelling, we typically work with linear optimization problems. The feasible space is convex, and the solution can be found (without getting trapped in local minima).

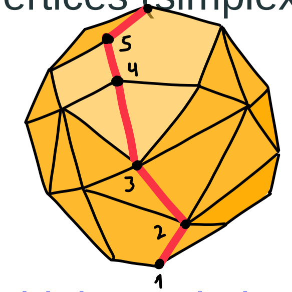
convex



non-convex



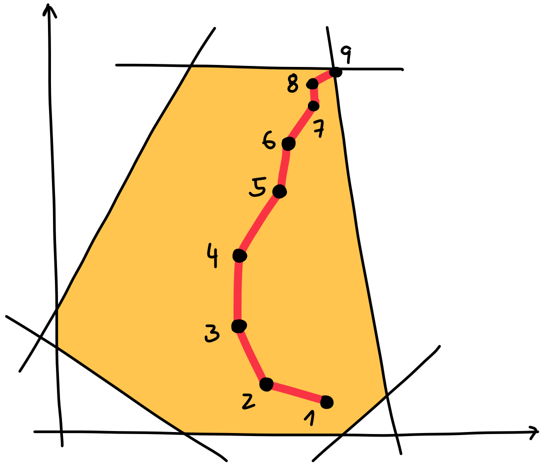
If all the constraints are affine, the feasible space is a multi-dimensional polyhedron. The optimum always occurs at one of the vertices. We can check all the vertices (simplex algorithm)



We use interior-point methods which converge faster.

The objective is to iteratively approach the optimal solution from the interior of the feasible space.

A barrier term is added to the objective function that penalizes solutions that come close to the boundary.



Ideally, we want high resolution in time, space and technologies: Where is the trade-off for space and time resolution?

How do we represent relations that are not linear or not convex? Convex relaxation!

Convexification

Convexification approaches

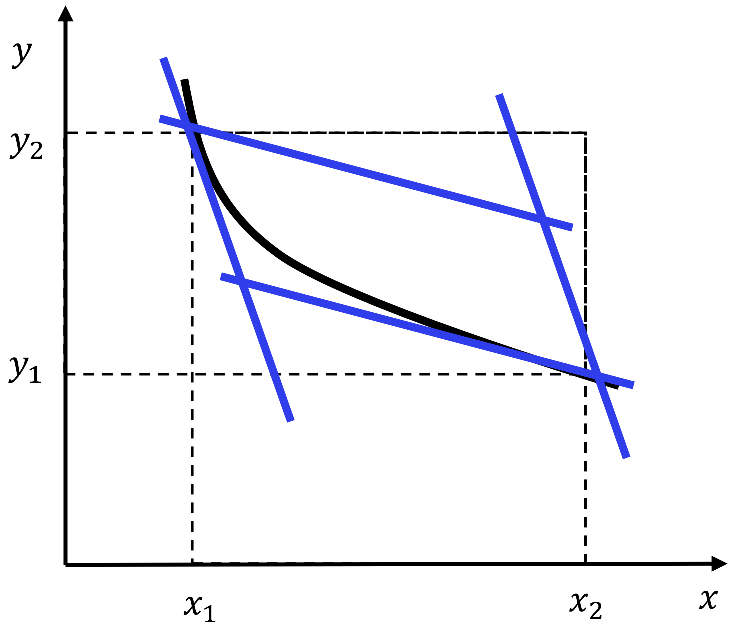
To solve a non-convex optimization problem, we typically create a convex relaxation problem (we say that we convexify the problem)

Convex relaxation : we create an alternative problem by relaxing some of the constraints. The global optimum provides a lower and upper bound to the global solution of the original problem

Example 1: Bilinear equalities :

Assuming the non-convex bilinear equality $xy = w$

and the bounds $x_1 \leq x \leq x_2$ $y_1 \leq y \leq y_2$



Convexification: we assume linear upper and lower bounding inequalities (McCornik envelope)

$w \geq x_1 y + x y_1 - x_1 y_1$

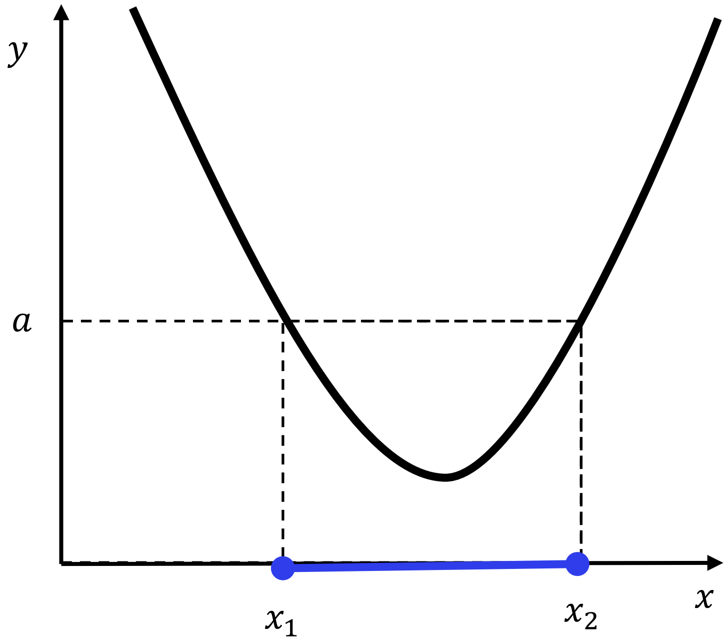
$w \geq x_2 y + x y_1 - x_2 y_2$

$w \leq x_1 y + x y_1 - x_1 y_2$

$w \leq x_2 y + x y_1 - x_2 y_1$

Convexification approaches

Example 2: Quadratic equalities :



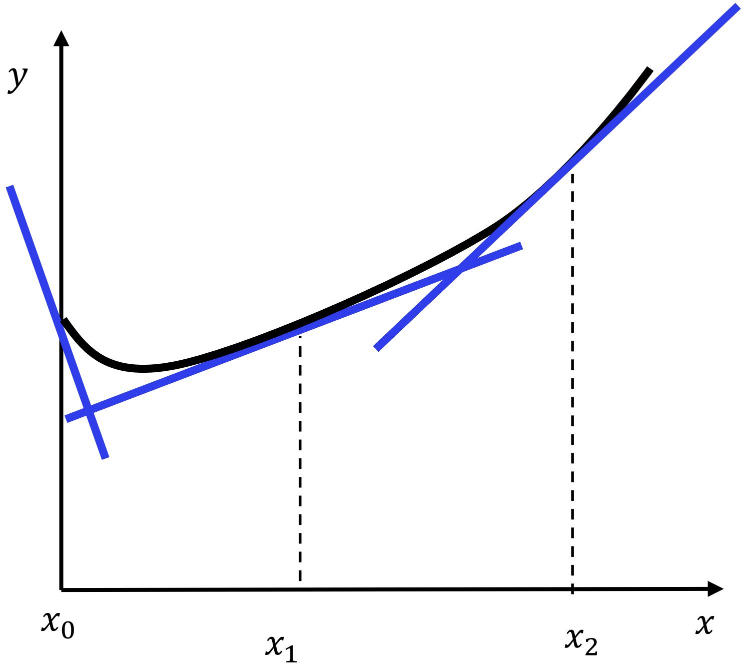
Assuming the non-convex quadratic equality $x^2 = a$ with $a > 0$

Convexification: We can use Second Order Cone Program (SOCP) relaxation

The equality can be relaxed into a convex inequality $x^2 \leq a$

Convexification approaches

Example 3. External approximation of optimization problems:



Assuming a non-linear convex inequality $g(x) \leq 0$

Convexification: Because the inequality is convex, it satisfies that $g(x)$ is above all its tangents, i.e. for all $x_1, x_2 \in \boldsymbol{R}^N$: $\mathrm{g}(x_2) \geq (x_2 - x_1)\frac{\partial f g(x_1)}{\partial x}$.

We can replace the non-linear inequality $g(x) \leq 0$ by a set of linear inequalities

$$\mathrm{g}(x_1) + (x - x_1)\frac{\partial f g(x_1)}{\partial x} \leq 0 \quad \forall x_1$$

Economic dispatch

Assume we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity G_s and a linear variable cost o_s . Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d in a certain hour while minimizing the total system cost.

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0 \quad \leftrightarrow \quad \lambda$$

$$g_s \geq 0 \leftrightarrow \underline{\mu_s}$$

$$-g_s + G_s \geq 0 \quad \leftrightarrow \quad \overline{\mu_s}$$

This equation excludes the consideration of any network constraints (e.g. line limits), and any additional security constraints, and additional generation constraints (CO2 emissions, ramp limits)