PRESSURE EQUATION EXTENSION IMPLEMENTATION

Extra term in the thermal plasma pressure equation:

$$\frac{\partial p}{\partial t} = -\vec{v}_E \cdot \vec{\nabla} p - \Gamma p \vec{\nabla}\left(\vec{v}_E + \vec{v}_{th,\|}\hat{b}\right)$$

where $\vec{v}_E = \dfrac{\vec{B} \wedge \vec{\nabla}\Phi}{B^2}$

A) First term:

Thus:

$$-\vec{v}_E \cdot \vec{\nabla} p = -\vec{v}_E{}^{\rho} \cdot \left(\vec{\nabla} p\right)_{\rho} - \vec{v}_E{}^{\theta} \cdot \left(\vec{\nabla} p\right)_{\theta} - \vec{v}_E{}^{\zeta} \cdot \left(\vec{\nabla} p\right)_{\zeta}$$

Linear terms only:

$$\vec{B} = \frac{2\rho\beta_*}{a^2}\vec{\nabla}\rho + I\vec{\nabla}\theta - J\vec{\nabla}\zeta$$

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial\rho}\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\zeta$$

$$\Rightarrow -\vec{v}_E \cdot \vec{\nabla} p = -\frac{1}{B^2}\frac{dp_{eq}}{d\rho}\vec{\nabla}\rho \cdot \left[\left(\frac{2\rho\beta_*}{a^2}\vec{\nabla}\rho + I\vec{\nabla}\theta - J\vec{\nabla}\zeta\right) \wedge \left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\zeta\right)\right]$$

$$= -\frac{1}{B^2}\frac{dp_{eq}}{d\rho}\left[I\frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\rho \cdot \vec{\nabla}\theta \wedge \vec{\nabla}\zeta - J\frac{\partial\Phi}{\partial\theta}\vec{\nabla}\rho \cdot \vec{\nabla}\zeta \wedge \vec{\nabla}\theta\right]$$

Using the definition:

$$\frac{1}{\sqrt{g}} = \rho\vec{\nabla}\rho \cdot \vec{\nabla}\theta \wedge R_0\vec{\nabla}\zeta$$

And the vectorial relations:

$$A \cdot (B \wedge C) = B \cdot (C \wedge A) = (A \wedge B) \cdot C$$

$$\Rightarrow -\vec{v}_E \cdot \vec{\nabla} p = -\frac{1}{B^2}\frac{dp_{eq}}{d\rho}\left[I\frac{\partial\Phi}{\partial\zeta}\frac{1}{\rho R_0\sqrt{g}} + J\frac{\partial\Phi}{\partial\theta}\frac{1}{\rho R_0\sqrt{g}}\right]$$

With $\dfrac{1}{\sqrt{g}} = \dfrac{B^2}{J - \tau I}$

After normalization:

$$-\vec{v}_E \cdot \vec{\nabla} p = -\frac{p_0}{\tau_A}\frac{1}{\rho}\frac{dp_{eq}}{d\rho}\frac{1}{J-\iota I}\left[I\,\frac{\partial \Phi}{\partial \zeta} - J\,\frac{\partial \Phi}{\partial \theta}\right]$$

B) Second term:

$$\vec{\nabla}\cdot\vec{v}_E = \vec{\nabla}\left(\frac{\vec{B}\wedge\vec{\nabla}\Phi}{B^2}\right)$$

$$\vec{B}\wedge\vec{\nabla}\Phi = \left(\frac{2\rho\beta_*}{a^2}\vec{\nabla}\rho + I\vec{\nabla}\theta - J\vec{\nabla}\zeta\right)\wedge\left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\zeta\right)$$

$$= \frac{2\rho\beta_*}{a^2}\left(\frac{\partial\Phi}{\partial\theta}\vec{\nabla}\rho\wedge\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\rho\wedge\vec{\nabla}\zeta\right)$$

$$+ I\left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\theta\wedge\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\theta\wedge\vec{\nabla}\zeta\right)$$

$$- J\left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\zeta\wedge\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\zeta\wedge\vec{\nabla}\theta\right)$$

$$= \left(I\,\frac{\partial\Phi}{\partial\zeta} + J\,\frac{\partial\Phi}{\partial\theta}\right)\vec{\nabla}\theta\wedge\vec{\nabla}\zeta - \left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\zeta} + J\,\frac{\partial\Phi}{\partial\rho}\right)\vec{\nabla}\rho\wedge\vec{\nabla}\zeta + \left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\theta} - I\,\frac{\partial\Phi}{\partial\rho}\right)\vec{\nabla}\rho\wedge\vec{\nabla}\theta$$

Using the next definitions:

$$\frac{1}{\sqrt{g}}\frac{1}{\rho R_0}\hat{e}_\rho = \vec{\nabla}\theta\wedge\vec{\nabla}\zeta \quad ; \quad \frac{1}{\sqrt{g}}\frac{1}{R_0}\hat{e}_\theta = \vec{\nabla}\rho\wedge\vec{\nabla}\zeta \quad ; \quad \frac{1}{\sqrt{g}}\frac{1}{\rho}\hat{e}_\zeta = \vec{\nabla}\rho\wedge\vec{\nabla}\theta$$

$$\Rightarrow \vec{B}\wedge\vec{\nabla}\Phi = \frac{1}{\sqrt{g}}\left[\underbrace{\frac{1}{\rho R_0}\left(I\,\frac{\partial\Phi}{\partial\zeta} + J\,\frac{\partial\Phi}{\partial\theta}\right)}_{A^\rho}\hat{e}_\rho - \underbrace{\frac{1}{R_0}\left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\zeta} + J\,\frac{\partial\Phi}{\partial\rho}\right)}_{A^\theta}\hat{e}_\theta + \underbrace{\frac{1}{\rho}\left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\theta} - I\,\frac{\partial\Phi}{\partial\rho}\right)}_{A^\zeta}\hat{e}_\zeta\right]$$

$$\Rightarrow \nabla\cdot\left(\frac{\vec{B}\wedge\vec{\nabla}\Phi}{B^2}\right) = \frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho\sqrt{g}A^\rho}{B^2}\right) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{\rho\sqrt{g}A^\theta}{B^2}\right) + \frac{1}{R_0}\frac{\partial}{\partial\zeta}\left(\frac{\rho\sqrt{g}A^\zeta}{B^2}\right)\right]$$

Normalized final expression:

$$-\Gamma p_{eq}\vec{\nabla}\cdot\left(\frac{\vec{B}\wedge\vec{\nabla}\Phi}{B^2}\right)=-\frac{\Gamma p_{eq}}{J-\iota I}\left[\frac{1}{\rho}\frac{dI}{d\rho}\frac{\partial\Phi}{\partial\zeta}-\frac{1}{\rho}\frac{dJ}{d\rho}\frac{\partial\Phi}{\partial\theta}-\frac{d\beta_*}{d\theta}\frac{\partial\Phi}{\partial\zeta}+\frac{d\beta_*}{d\zeta}\frac{\partial\Phi}{\partial\theta}\right]$$

$$-\Gamma p_{eq}\left[\frac{1}{\rho}\left(I\frac{\partial\Phi}{\partial\zeta}-J\frac{\partial\Phi}{\partial\theta}\right)\frac{1}{\sqrt{g}}\frac{d}{d\rho}\left(\frac{\sqrt{g}}{J-\iota I}\right)\right.$$

$$\left.-\left(\rho\beta_*\frac{\partial\Phi}{\partial\zeta}-J\frac{\partial\Phi}{\partial\rho}\right)\frac{1}{\sqrt{g}}\frac{d}{d\theta}\left(\frac{\sqrt{g}}{J-\iota I}\right)+\left(\beta_*\frac{\partial\Phi}{\partial\theta}-\frac{I}{\rho}\frac{\partial\Phi}{\partial\rho}\right)\frac{1}{\sqrt{g}}\frac{d}{d\zeta}\left(\frac{\sqrt{g}}{J-\iota I}\right)\right]$$

New terms in the pressure equation:

call dbydr0(sd1,preq,0.0_IDP,1.0_IDP,0)

$$sd1=\frac{dp_{eq}}{d\rho}$$

sd2=-rinv*sd1*cureq(:)/(feq(:)-qqinv(:)*cureq(:))

$$sd2=-\frac{1}{\rho}\frac{dp_{eq}}{d\rho}\frac{I}{J-\iota I}$$

call block0(sd2,3,2,0,0,1,1.0_IDP)

$$Sp_1=-\frac{1}{\rho}\frac{dp_{eq}}{d\rho}\frac{I}{J-\iota I}\frac{\partial\Phi}{\partial\zeta}$$

sd2=sd1*feq(:)/(feq(:)-qqinv(:)*cureq(:))

$$sd2=\frac{dp_{eq}}{d\rho}\frac{J}{J-\iota I}$$

call block0(sd2,3,2,1,0,0,1.0_IDP)

$$Sp_2=\frac{dp_{eq}}{d\rho}\frac{J}{J-\iota I}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}$$

call dbydr0(sd1,cureq,0.0_IDP,1.0_IDP,0)

$$sd1=\frac{dI}{d\rho}$$

sd2=-rinv*sd1*preq/(feq-qqinv*cureq)

$$sd2 = -\frac{1}{\rho}\frac{dI}{d\rho}\frac{p_{eq}}{J - \iota I}$$

call block0(sd2,3,2,0,0,1,gamma)

$$Sp_3 = -\frac{1}{\rho}\frac{dI}{d\rho}\frac{\Gamma p_{eq}}{J - \iota I}\frac{\partial \Phi}{\partial \zeta}$$

call dbydr0(sd1,feq,0.0_IDP,1.0_IDP,0)

$$sd1 = \frac{dJ}{d\rho}$$

sd2=sd1*preq/(feq-qqinv*cureq)

$$sd2 = \frac{dJ}{d\rho}\frac{p_{eq}}{J - \iota I}$$

call block0(sd2,3,2,1,0,0,gamma)

$$Sp_4 = \frac{dJ}{d\rho}\frac{\Gamma p_{eq}}{J - \iota I}\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}$$

call dbydtheq(sceq1,bst,-1,0.0_IDP,1.0_IDP,0)

$$sceq1 = \frac{1}{\rho}\frac{\partial \beta_*}{\partial \theta}$$

do l=1,leqmax

sceq1(:,l)=sceq1(:,l)*r*preq/(feq(:)-qqinv(:)*cureq(:))

end do

$$sceq1 = \frac{p_{eq}}{J - \iota I}\frac{\partial \beta_*}{\partial \theta}$$

call blockj(sceq1,1,3,2,0,0,1,gamma)

$$Sp_5 = \frac{\Gamma p_{eq}}{J - \iota I}\frac{\partial \beta_*}{\partial \theta}\frac{\partial \Phi}{\partial \zeta}$$

call dbydzteq(sceq1,bst,-1,0.0_IDP,1.0_IDP)

$$sceq1 = \frac{\partial \beta_*}{\partial \zeta}$$

```
do l=1,leqmax
        sceq1(:,l)=-r*sceq1(:,l)*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = -\frac{\rho p_{eq}}{J-\imath I}\frac{\partial \beta_*}{\partial \zeta}$$

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call blockj(sceq1,1,3,2,1,0,0,gamma)
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$$Sp_6 = -\frac{\Gamma p_{eq}}{J-\imath I}\frac{\partial \beta_*}{\partial \zeta}\frac{\partial \Phi}{\partial \theta}$$

```
sd1=1/(feq(:)-qqinv(:)*cureq(:))
```

$$sd1 = \frac{1}{J-\imath I}$$

```
call dbydr0(sd2,sd1,0.0_IDP,1.0_IDP,0)
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$$sd2 = \frac{d}{d\rho}\left(\frac{1}{J-\imath I}\right)$$

```
sd3=-cureq*rinv*sd2*preq
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$$sd3 = -\frac{Ip_{eq}}{\rho}\frac{d}{d\rho}\left(\frac{1}{J-\imath I}\right)$$

```
call block0(sd3,3,2,0,0,1,gamma)
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$$Sp_7 = -\frac{\Gamma I p_{eq}}{\rho}\frac{d}{d\rho}\left(\frac{1}{J-\imath I}\right)\frac{\partial \Phi}{\partial \zeta}$$

```
sd3=feq*sd2*preq
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$$sd3 = Jp_{eq}\frac{d}{d\rho}\left(\frac{1}{J-\imath I}\right)$$

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call block0(sd3,3,2,1,0,0,gamma)
```

$$Sp_8 = \Gamma J p_{eq}\frac{d}{d\rho}\left(\frac{1}{J-\imath I}\right)\frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}$$

```
do l=1,leqmax
        sceq1(:,l)=-sqgdroj(:,l)*cureq*rinv*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = -\frac{Ip_{eq}}{\rho\left(J - \iota I\right)}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\rho}$$

call blockj(sceq1,1,3,2,0,0,1,gamma)

$$Sp_9 = -\frac{\Gamma Ip_{eq}}{\rho\left(J - \iota I\right)}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\rho}\frac{\partial\Phi}{\partial\zeta}$$

```
do l=1,leqmax
        sceq1(:,l)=sqgdroj(:,l)*feq*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = \frac{Jp_{eq}}{J - \iota I}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\rho}$$

call blockj(sceq1,1,3,2,1,0,0,gamma)

$$Sp_{10} = \frac{\Gamma Jp_{eq}}{J - \iota I}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\rho}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}$$

```
do l=1,leqmax
        sceq1(:,l)=sqgdthojbst(:,l)*r*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = \frac{p_{eq}}{J - \iota I}\frac{\beta_*}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\theta}$$

call blockj(sceq1,1,3,2,0,0,1,gamma)

$$Sp_{11} = \frac{\Gamma p_{eq}}{J - \iota I}\frac{\beta_*}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\theta}\frac{\partial\Phi}{\partial\zeta}$$

```
do l=1,leqmax
        sceq1(:,l)=-sqgdthoj(:,l)*feq*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = -\frac{Jp_{eq}}{J - \iota I}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial\theta}$$

call blockj(sceq1,-1,3,2,0,1,0,gamma)

$$Sp_{12} = -\frac{\Gamma J p_{eq}}{J - \iota I} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \theta} \frac{\partial \Phi}{\partial \rho}$$

```
do l=1,leqmax
    sceq1(:,l)=-sqgdztojbst(:,l)*r*preq/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = \frac{\rho p_{eq}}{J - \iota I} \frac{\beta_*}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \zeta}$$

call blockj(sceq1,1,3,2,1,0,0,gamma)

$$Sp_{13} = \frac{\Gamma p_{eq}}{J - \iota I} \frac{\beta_*}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \zeta} \frac{\partial \Phi}{\partial \theta}$$

```
do l=1,leqmax
    sceq1(:,l)=sqgdztoj(:,l)*cureq*preq*rinv/(feq(:)-qqinv(:)*cureq(:))
end do
```

$$sceq1 = \frac{I p_{eq}}{\rho (J - \iota I)} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \zeta}$$

call blockj(sceq1,-1,3,2,0,1,0,gamma)

$$Sp_{14} = \frac{\Gamma I p_{eq}}{\rho (J - \iota I)} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \zeta} \frac{\partial \Phi}{\partial \rho}$$