VMEC COORDINATES

The contravariant representation of the magnetic field in VMEC coordinates is given by the expression

$$\mathbf{B} = \nabla\psi \times \nabla\theta + \iota\nabla\phi \times \nabla\psi + \nabla\psi \times \nabla\lambda \tag{1}$$

The toroidal flux is then

$$\iint \mathbf{B} \cdot d\boldsymbol{\sigma}_\phi = \iint \mathbf{B} \cdot \nabla\phi |\mathcal{J}| ds d\theta = \iint \frac{d\psi}{ds} \mathcal{J}^{-1} |\mathcal{J}| ds d\theta = -2\pi\psi \tag{2}$$

since the coordinate system is left-handed ($\mathcal{J} < 0$).

The poloidal flux will be

$$\iint \mathbf{B} \cdot d\boldsymbol{\sigma}_\theta = \iint \mathbf{B} \cdot \nabla\theta |\mathcal{J}| ds d\phi = \iint \iota \frac{d\psi}{ds} \mathcal{J}^{-1} |\mathcal{J}| ds d\phi = -2\pi\chi, \tag{3}$$

where $\iota = d\chi/d\psi$.

The covariant representation is

$$\mathbf{B} = B_s \nabla s + B_\theta \nabla\theta + B_\phi \nabla\phi \tag{4}$$

Then,

$$\mu_0 J^s = \frac{1}{\mathcal{J}} \left(\frac{\partial B_\phi}{\partial\theta} - \frac{\partial B_\theta}{\partial\phi} \right) \tag{5}$$

$$\mu_0 J^\theta = \frac{1}{\mathcal{J}} \left(\frac{\partial B_s}{\partial\phi} - \frac{\partial B_\phi}{\partial s} \right) \tag{6}$$

$$\mu_0 J^\phi = \frac{1}{\mathcal{J}} \left(\frac{\partial B_\theta}{\partial s} - \frac{\partial B_s}{\partial\theta} \right) \tag{7}$$

Since $J^s = \mathbf{B} \cdot \nabla s = 0$, we can write

$$B_\theta = I(s) + \frac{\partial\nu}{\partial\theta} \qquad B_\phi = g(s) + \frac{\partial\nu}{\partial\phi}, \tag{8}$$

where ν is a periodic function in θ and ϕ.

The toroidal current is then

$$\iint \mathbf{J} \cdot d\boldsymbol{\sigma}_\phi = \iint \mathbf{J} \cdot \nabla\phi |\mathcal{J}| ds d\theta = -\frac{1}{\mu_0} \iint \frac{dI}{ds} ds d\theta = -\frac{2\pi I}{\mu_0}, \tag{9}$$

and the poloidal current,

$$\iint \mathbf{J} \cdot d\boldsymbol{\sigma}_\theta = \iint \mathbf{J} \cdot \nabla\theta |\mathcal{J}| ds d\phi = \frac{1}{\mu_0} \iint \frac{dg}{ds} ds d\phi = \frac{2\pi}{\mu_0} \left[g(s) - g(0) \right] \tag{10}$$

We can write the magnetic field as

$$\mathbf{B} = \psi' \nabla s \times \nabla \lambda_V, \quad \text{where} \quad \lambda_V = \theta - \iota\phi + \lambda(s, \theta, \phi) \tag{11}$$

or

$$\mathbf{B} = \beta_V \nabla s + \nabla \nu_V, \quad \text{where} \quad \nu_V = I\theta + g\phi + \nu \;\text{ and }\; \beta_V = B_s - I'\theta - g'\phi - \frac{\partial \nu}{\partial s}. \tag{12}$$

Here, the symbol $'$ denotes d/ds. From the VMEC output, we get ι (iota), $\psi'/2\pi$ (phip), I (buco), g (bvco), λ (lmnc/s), B_θ (bsubtmnc/s) and B_ϕ (bsubzmnc/s).

BOOZER COORDINATES

Angular coordinates θ_B, ζ_B, such that

$$\mathbf{B} = \psi' \nabla s \times \nabla \lambda_B \quad \text{where} \quad \lambda_B = \theta_B - \iota\phi_B,$$

$$\mathbf{B} = \beta_B \nabla s + \nabla \nu_B \quad \text{where} \quad \nu_B = I\theta_B + g\phi_B.$$

A solution (which always exists) is obtained from

$$\lambda_B = \lambda_V, \nu_B = \nu_V, \beta_B = \beta_V,$$

so, from (11) and (12), we have

$$\left.\begin{array}{l} \theta_B - \iota\phi_B = \theta - \iota\phi + \lambda \\ I\theta_B + g\phi_B = I\theta + g\phi + \nu \end{array}\right\}$$

Defining

$$\theta_B = \theta + \tilde{\theta}_B, \quad \phi_B = \phi + \tilde{\phi}_B$$

we get

$$\left.\begin{array}{l} \tilde{\theta}_B - \iota\tilde{\phi}_B = \lambda \\ I\tilde{\theta}_B + g\tilde{\phi}_B = \nu \end{array}\right\} \quad \Rightarrow \quad \tilde{\theta}_B = \frac{g\lambda + \iota\nu}{g + \iota I}, \quad \tilde{\phi}_B = \frac{\nu - I\lambda}{g + \iota I}$$

For stellarator symmetry [λ and ν odd under the transformation $(\theta, \phi) \to (-\theta, -\phi)$], and by requiring that θ_B and ϕ_B keep the symmetry, the above solution is unique.

Then, we have, for Boozer coordinates,

$$\mathbf{B} = \nabla\psi \times \nabla\theta_B - \iota\nabla\psi \times \nabla\phi_B = \nabla\psi \times \nabla\theta_B + \iota\nabla\phi_B \times \nabla\psi,$$

$$\mathbf{B} = \beta_* \nabla s + I\nabla\theta_B + g\nabla\phi_B, \;\text{ where }\; \beta_* = \beta_B + I'\theta_B + g'\phi_B.$$

$$B^i = (0, \iota\psi'/\mathcal{J}_B, \psi'/\mathcal{J}_B)\,, B_i = (\beta_*, I, g)\,. \tag{13}$$

Then,

$$\mathcal{J}_B = \frac{\psi'(g+\iota I)}{B^2} < 0 \quad (\mathrm{left-handed}). \tag{14}$$

Note that if $B^\phi > 0$, $\psi' < 0$ ($\nabla\psi \times \nabla\theta_B \cdot \nabla\phi_B > 0$) and $g > 0$. If $B^\phi < 0$, $\psi' > 0$ ($\nabla\psi \times \nabla\theta_B \cdot \nabla\phi_B < 0$) and $g < 0$. In any case, $\psi' g < 0$.

From BOOZ_XFORM output, we get gmncb and gmnsb, the Fourier components of $\mathcal{J}_B/|\psi'|$. In the original boozer.f subroutine, gmncb and gmnsb where the Fourier components of $\mathcal{J}_B/\psi'$. It was modified so $\mathcal{J}_B$ has always the right sign.

FIELD LINES IN BOOZER COORDINATES

Field lines of $\mathbf{B}$:

$$\begin{cases} s = \text{constant.} \\ \psi'(\theta_B - \iota\phi_B) = \text{constant.} \end{cases} \Rightarrow \quad \theta_B = \iota\phi_B + \text{constant.}$$

Field lines of $\nabla s \times \mathbf{B}$:

$$\begin{cases} s = \text{constant.} \\ I\theta_B + g\phi_B = \text{constant.} \end{cases} \Rightarrow \quad \phi_B = -\frac{I}{g}\theta_B + \text{constant.}$$

So, in Boozer coordinates, the field lines of $\mathbf{B}$ and the lines orthogonal to $\mathbf{B}$ on the magnetic surfaces are straight lines.

EQUILIBRIUM

$$\nabla p = \mathbf{J} \times \mathbf{B} \;\Rightarrow\; \frac{\partial p}{\partial s} = \mathcal{J}_B \left(J^\theta B^\phi - J^\phi B^\theta\right).$$

Since

$$\mu_0 J^\theta = \frac{1}{\mathcal{J}_B}\left(\frac{\partial B_s}{\partial \phi_B} - \frac{\partial B_\phi}{\partial s}\right),\; \mu_0 J^\phi = \frac{1}{\mathcal{J}_B}\left(\frac{\partial B_\theta}{\partial s} - \frac{\partial B_s}{\partial \theta_B}\right),$$

$$\mu_0 p' = \left(B^\theta \frac{\partial}{\partial\theta_B} + B^\phi \frac{\partial}{\partial \phi_B}\right) B_s - \left(B^\theta \frac{\partial B_\theta}{\partial s} + B^\phi \frac{\partial B_\phi}{\partial s}\right)$$

From (13),

$$\mu_0 \mathcal{J}_B \frac{p'}{\psi'} = \left(\iota\frac{\partial}{\partial\theta_B} + \frac{\partial}{\partial\phi_B}\right)\beta_* - (g' + \iota I') \tag{15}$$

By averaging in the poloidal and toroidal angles, we get

$$\mu_0 \left(\mathcal{J}_B\right)_{00} p' = -\psi' \left(g' + \iota I'\right). \tag{16}$$

On the other hand, the volume between two nearby flux surfaces is given by

$$dV = ds \iint |\mathcal{J}_B|\, d\theta_B d\phi_B \;\Rightarrow\; V' = -4\pi^2 \left(\mathcal{J}_B\right)_{00},$$

so the (averaged) equilibrium equation is

$$\mu_0 p' V' = 4\pi^2 \psi' \left(g' + \iota I' \right). \tag{17}$$

From the VMEC output, we get $V'/4\pi^2$ (vp).

For modes $(m, n) \neq (0, 0)$, equation (15) implies

$$(\beta_*)^s_{mn} = \frac{\mu_0 p' \left(\mathcal{J}_B\right)^c_{mn}}{\psi'(m\iota - n)}, \quad (\beta_*)^c_{mn} = -\frac{\mu_0 p' \left(\mathcal{J}_B\right)^s_{mn}}{\psi'(m\iota - n)}, \tag{18}$$

where we have used the Fourier decomposition

$$\beta_* = \sum_{m,n} (\beta_*)^s_{mn} \sin\left(m\theta_B - n\phi_B\right) + \sum_{m,n} (\beta_*)^c_{mn} \cos\left(m\theta_B - n\phi_B\right) \tag{19}$$

RIGHT-HANDED COORDINATE SYSTEM

By changing the toroidal coordinate to $\zeta_B = -\phi_B$, the system of Boozer coordinates becomes right-handed so $\sqrt{g_B} = -\mathcal{J}_B > 0$. Then, the contravariant representation of the magnetic field is given by the expression

$$\mathbf{B} = \nabla\psi \times \nabla\theta_B - \iota \nabla\zeta_B \times \nabla\psi = \nabla\psi \times \nabla\theta_B + \nabla\zeta_B \times \nabla\chi \tag{20}$$

Here, $2\pi\psi$ is the toroidal flux, $2\pi\chi$ is the poloidal flux, and $d\chi/d\psi = -\iota$. Note that the definition of χ has the opposite sign as the one defined for the left-handed system. The rotational transform is the same, so field lines are straight lines with slope $d\theta_B/d\zeta_B = -\iota$ (FAR convention).

The toroidal current is now $2\pi I/\mu_0$, and

$$B^i = \left(0, -\iota\psi'/\sqrt{g_B}, \psi'/\sqrt{g_B}\right), B_i = \left(\beta_*, I, J\right), \tag{21}$$

where $J = -g$ and

$$\sqrt{g_B} = \frac{\psi'(J - \iota I)}{B^2} > 0 \quad (\mathrm{right-handed}). \tag{22}$$

From (15),

$$\mu_0 \sqrt{g_B} \frac{p'}{\psi'} = \left(\frac{\partial}{\partial\zeta_B} - \iota \frac{\partial}{\partial\theta_B} \right) \beta_* - \left(J' - \iota I'\right) \tag{23}$$

For modes $(m, n) \neq (0, 0)$, equation (23) implies

$$(\beta_*)^s_{mn} = -\frac{\mu_0 p' \left(\sqrt{g_B}\right)^c_{mn}}{\psi'(m\iota - n)}, \quad (\beta_*)^c_{mn} = \frac{\mu_0 p' \left(\sqrt{g_B}\right)^s_{mn}}{\psi'(m\iota - n)}, \tag{24}$$

where we have used the Fourier decomposition

$$\beta_* = \sum_{m,n} (\beta_*)^s_{mn} \sin(m\theta_B + n\zeta_B) + \sum_{m,n} (\beta_*)^c_{mn} \cos(m\theta_B + n\zeta_B) \tag{25}$$

(FAR convention).

CYLINDRICAL-LIKE COORDINATES

Starting from a VMEC equilibrium in Boozer coordinates, we define a radial coordinate ρ, which value at the edge is a. Since s in VMEC is dimensionless, proportional to the toroidal flux, and normalized to 1 at the edge, we take $s = (\rho/a)^2$. The reciprocal base vectors in the cylindrical-like system are $(\nabla\rho, \rho\nabla\theta, R_0\nabla\zeta)$, where R_0 is the major radius. From now on, θ is θ_B, and ζ is ζ_B.

In this system,

$$\begin{aligned} g_{\rho\rho} =& \left(\frac{\partial R}{\partial\rho}\right)^2 + \left(\frac{\partial Z}{\partial\rho}\right)^2 + R^2\left(\frac{\partial\zeta}{\partial\rho}\right)^2, \\ g_{\rho\theta} =& \frac{\partial R}{\partial\rho}\frac{1}{\rho}\frac{\partial R}{\partial\theta} + \frac{\partial Z}{\partial\rho}\frac{1}{\rho}\frac{\partial Z}{\partial\theta} + R^2\frac{\partial\zeta}{\partial\rho}\frac{1}{\rho}\frac{\partial\zeta}{\partial\theta}, \\ g_{\theta\theta} =& \left(\frac{1}{\rho}\frac{\partial R}{\partial\theta}\right)^2 + \left(\frac{1}{\rho}\frac{\partial Z}{\partial\theta}\right)^2 + R^2\left(\frac{1}{\rho}\frac{\partial\zeta}{\partial\theta}\right)^2, \end{aligned} \tag{26}$$

and

$$\frac{1}{\sqrt{g}} = \nabla\rho\cdot\rho\nabla\theta\times R_0\nabla\zeta = \frac{a^2R_0}{2}\nabla s\cdot\nabla\theta\times\nabla\zeta, \tag{27}$$

where, from (22),

$$\nabla s\cdot\nabla\theta\times\nabla\zeta = \frac{B^2}{\psi'(J-\iota I)}$$

Since $2\pi|\psi|$ is the absolute value of the toroidal magnetic flux, a possible choice for the radial variable ρ is

$$2\pi|\psi| = \pi B_0\rho^2 \quad \Rightarrow \quad |\psi| = \frac{B_0\rho^2}{2}, \tag{28}$$

which represents the toroidal flux in a cylinder with toroidal field B_0. Since s is proportional to ψ and is normalized to one at the edge,

$$|\psi'| = \frac{d|\psi|}{ds} = \frac{d}{ds}\left(\frac{B_0a^2}{2}s\right) = \frac{B_0a^2}{2}, \tag{29}$$

so the minor radius, a, is defined by Eq. (29). The major radius, R_0, and B_0 are defined as the averaged R and B at the magnetic axis, respectively.

Then,

$$\sqrt{g} = \frac{2}{a^2 R_0}\sqrt{g_B} = \frac{2\psi'}{a^2 R_0}\frac{(J - \text{ɨ} I)}{B^2} = \frac{B_0}{R_0}\mathrm{sign}(\psi')\frac{(J - \text{ɨ} I)}{B^2}. \tag{30}$$

Differential operators

$$(\nabla\Phi)_\rho = \frac{\partial\Phi}{\partial\rho}, \quad (\nabla\Phi)_\theta = \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}, \quad (\nabla\Phi)_\zeta = \frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta}, \tag{31}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\sqrt{g}A^\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\sqrt{g}A^\theta\right) + \frac{1}{R_0}\frac{\partial}{\partial\zeta}\left(\sqrt{g}A^\zeta\right)\right], \tag{32}$$

$$\begin{aligned}
(\nabla\times\mathbf{A})^\rho &= \frac{1}{\sqrt{g}}\left(\frac{1}{\rho}\frac{\partial A_\zeta}{\partial\theta} - \frac{1}{R_0}\frac{\partial A_\theta}{\partial\zeta}\right),\\
(\nabla\times\mathbf{A})^\theta &= \frac{1}{\sqrt{g}}\left(\frac{1}{R_0}\frac{\partial A_\rho}{\partial\zeta} - \frac{\partial A_\zeta}{\partial\rho}\right),\\
(\nabla\times\mathbf{A})^\zeta &= \frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial\left(\rho A_\theta\right)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\theta}\right],
\end{aligned} \tag{33}$$

$$\begin{aligned}
\nabla^2\Phi =& \frac{1}{\sqrt{g}}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\sqrt{g}\left(g^{\rho\rho}\frac{\partial\Phi}{\partial\rho} + g^{\rho\theta}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + g^{\rho\zeta}\frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta}\right)\right] + \right.\\
&\frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\sqrt{g}\left(g^{\rho\theta}\frac{\partial\Phi}{\partial\rho} + g^{\theta\theta}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + g^{\theta\zeta}\frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta}\right)\right] + \\
&\left.\frac{1}{R_0}\frac{\partial}{\partial\zeta}\left[\sqrt{g}\left(g^{\rho\zeta}\frac{\partial\Phi}{\partial\rho} + g^{\theta\zeta}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + g^{\zeta\zeta}\frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta}\right)\right]\right\}.
\end{aligned} \tag{34}$$

From (21), we get the equilibrium magnetic field components

$$\begin{aligned}
B^\rho_{eq} &= \mathbf{B}_{eq}\cdot\nabla\rho = \frac{a^2}{2\rho}\mathbf{B}_{eq}\cdot\nabla s = 0,\\
B^\theta_{eq} &= \mathbf{B}_{eq}\cdot\rho\nabla\theta = -\frac{2\rho\,\text{ɨ}}{a^2 R_0}\frac{\psi'}{\sqrt{g}},\\
B^\zeta_{eq} &= \mathbf{B}_{eq}\cdot R_0\nabla\theta = \frac{2}{a^2}\frac{\psi'}{\sqrt{g}},
\end{aligned} \tag{35}$$

$$\begin{aligned}
B_\rho^{eq} &= \mathbf{B}_{eq}\cdot\frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{B}_{eq}\cdot\frac{2\rho}{a^2}\frac{\partial\mathbf{r}}{\partial s} = \frac{2\rho\beta_*}{a^2},\\
B_\theta^{eq} &= \mathbf{B}_{eq}\cdot\frac{1}{\rho}\frac{\partial\mathbf{r}}{\partial\theta} = \frac{I}{\rho},\\
B_\zeta^{eq} &= \mathbf{B}_{eq}\cdot\frac{1}{R_0}\frac{\partial\mathbf{r}}{\partial\zeta} = \frac{J}{R_0},
\end{aligned} \tag{36}$$

where we have used relation (27).

<u>Relations between metric elements</u>.

From (35),

$$\mathbf{B}_{eq} = \frac{2\psi'}{a^2\sqrt{g}}\left(\mathbf{e}_\zeta - \frac{\rho\,\iota\!\!\!-}{R_0}\mathbf{e}_\theta\right), \tag{37}$$

so

$$\mathbf{e}_\zeta = \frac{\rho\,\iota\!\!\!-}{R_0}\mathbf{e}_\theta + \frac{a^2\sqrt{g}}{2\psi'}\mathbf{B}_{eq} = \frac{\rho\,\iota\!\!\!-}{R_0}\mathbf{e}_\theta + \frac{a^2\sqrt{g}}{2\psi'}\left(\frac{2\rho\beta_*}{a^2}\mathbf{e}^\rho + \frac{I}{\rho}\mathbf{e}^\theta + \frac{J}{R_0}\mathbf{e}^\zeta\right), \tag{38}$$

where we have used the expression (36) of $\mathbf{B}_{eq}$ covariant components.

Then,

$$\begin{aligned}
g_{\rho\zeta} &= \frac{\rho\,\iota\!\!\!-}{R_0}g_{\rho\theta} + \frac{\rho\beta_*}{\psi'}\sqrt{g}, \\
g_{\theta\zeta} &= \frac{\rho\,\iota\!\!\!-}{R_0}g_{\theta\theta} + \frac{a^2}{2\psi'}\frac{I}{\rho}\sqrt{g}, \\
g_{\zeta\zeta} &= \frac{\rho\,\iota\!\!\!-}{R_0}g_{\theta\zeta} + \frac{a^2}{2\psi'}\frac{J}{R_0}\sqrt{g} = \left(\frac{\rho\,\iota\!\!\!-}{R_0}\right)^2 g_{\theta\theta} + \frac{a^2}{2\psi'}\frac{J+\iota\!\!\!-I}{R_0}\sqrt{g}.
\end{aligned} \tag{39}$$

On the other hand,

$$\left(g^{ij}\right) = \frac{1}{g}\begin{pmatrix} g_{\theta\theta}g_{\zeta\zeta} - g_{\theta\zeta}^2 & g_{\rho\zeta}g_{\theta\zeta} - g_{\rho\theta}g_{\zeta\zeta} & g_{\rho\theta}g_{\theta\zeta} - g_{\rho\zeta}g_{\theta\theta} \\ g_{\rho\zeta}g_{\theta\zeta} - g_{\rho\theta}g_{\zeta\zeta} & g_{\rho\rho}g_{\zeta\zeta} - g_{\rho\zeta}^2 & g_{\rho\theta}g_{\rho\zeta} - g_{\rho\rho}g_{\theta\zeta} \\ g_{\rho\theta}g_{\theta\zeta} - g_{\rho\zeta}g_{\theta\theta} & g_{\rho\theta}g_{\rho\zeta} - g_{\rho\rho}g_{\theta\zeta} & g_{\rho\rho}g_{\theta\theta} - g_{\rho\theta}^2 \end{pmatrix}.$$

Substituting expressions (39) for the metric elements $g^{\rho j}$, we get

$$\begin{aligned}
g^{\rho\rho} &= \frac{g_{\theta\theta}}{\sqrt{g}}\left[\left(\frac{\rho\,\iota\!\!\!-}{R_0}\right)^2\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{J+\iota\!\!\!-I}{R_0}\right] - \left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\right)^2 \\
&= \frac{a^2}{2\psi'}\frac{J-\iota\!\!\!-I}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} - \left(\frac{a^2}{2\psi'}\frac{I}{\rho}\right)^2,
\end{aligned}$$

$$\begin{aligned}
g^{\rho\theta} &= \left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\right)\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\right) - \frac{g_{\rho\theta}}{\sqrt{g}}\left[\left(\frac{\rho\,\iota\!\!\!-}{R_0}\right)^2\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{J+\iota\!\!\!-I}{R_0}\right] \\
&= -\frac{a^2}{2\psi'}\frac{J}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\,\iota\!\!\!-}{R_0}\frac{\rho\beta_*}{\psi'}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\frac{\rho\beta_*}{\psi'}, \\
g^{\rho\zeta} &= \frac{g_{\rho\theta}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\right) - \frac{g_{\theta\theta}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\right) = \frac{a^2}{2\psi'}\frac{I}{\rho}\frac{g_{\rho\theta}}{\sqrt{g}} - \frac{\rho\beta_*}{\psi'}\frac{g_{\theta\theta}}{\sqrt{g}},
\end{aligned}$$

so

$$
\begin{aligned}
g^{\rho\rho} &= \frac{a^2}{2\psi'}\frac{J-\iota\!\!\!-I}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} - \left(\frac{a^2}{2\psi'}\frac{I}{\rho}\right)^2, \\
g^{\rho\theta} &= -\frac{a^2}{2\psi'}\frac{J}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\,\iota\!\!\!-}{R_0}\frac{\rho\beta_*}{\psi'}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\frac{\rho\beta_*}{\psi'}, \\
g^{\rho\zeta} &= \frac{a^2}{2\psi'}\frac{I}{\rho}\frac{g_{\rho\theta}}{\sqrt{g}} - \frac{\rho\beta_*}{\psi'}\frac{g_{\theta\theta}}{\sqrt{g}}.
\end{aligned} \tag{40}
$$

The remaining metric elements can relate to each other through

$$
\begin{aligned}
g^{\theta\zeta} &= \frac{g_{\rho\theta}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\right) - \frac{g_{\rho\rho}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{I}{\rho}\right) = -\frac{\rho\,\iota\!\!\!-}{R_0}g^{\zeta\zeta} - \frac{a^2}{2\psi'}\frac{I}{\rho}\frac{g_{\rho\rho}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\frac{g_{\rho\theta}}{\sqrt{g}}, \\
g^{\theta\theta} &= \frac{g_{\rho\rho}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\theta\zeta}}{\sqrt{g}} + \frac{a^2}{2\psi'}\frac{J}{R_0}\right) - \frac{g_{\rho\zeta}}{\sqrt{g}}\left(\frac{\rho\,\iota\!\!\!-}{R_0}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\right) = -\frac{\rho\,\iota\!\!\!-}{R_0}g^{\theta\zeta} + \frac{a^2}{2\psi'}\frac{J}{R_0}\frac{g_{\rho\rho}}{\sqrt{g}} - \frac{\rho\beta_*}{\psi'}\frac{g_{\rho\zeta}}{\sqrt{g}} \\
&= \left(\frac{\rho\,\iota\!\!\!-}{R_0}\right)^2 g^{\zeta\zeta} + \frac{a^2}{2\psi'}\frac{J+\iota\!\!\!-I}{R_0}\frac{g_{\rho\rho}}{\sqrt{g}} - 2\frac{\rho\,\iota\!\!\!-}{R_0}\frac{\rho\beta_*}{\psi'}\frac{g_{\rho\theta}}{\sqrt{g}} - \left(\frac{\rho\beta_*}{\psi'}\right)^2.
\end{aligned} \tag{41}
$$

Since

$$
\begin{aligned}
B^\zeta_{eq} = \frac{2}{a^2}\frac{\psi'}{\sqrt{g}} &= \frac{2\rho\beta_*}{a^2}g^{\rho\zeta} + \frac{I}{\rho}g^{\theta\zeta} + \frac{J}{R_0}g^{\zeta\zeta} \\
&= \frac{2\rho\beta_*}{a^2}\left(\frac{a^2}{2\psi'}\frac{I}{\rho}\frac{g_{\rho\theta}}{\sqrt{g}} - \frac{\rho\beta_*}{\psi'}\frac{g_{\theta\theta}}{\sqrt{g}}\right) + \frac{I}{\rho}\left(-\frac{a^2}{2\psi'}\frac{I}{\rho}\frac{g_{\rho\rho}}{\sqrt{g}} + \frac{\rho\beta_*}{\psi'}\frac{g_{\rho\theta}}{\sqrt{g}}\right) + \frac{J-\iota\!\!\!-I}{R_0}g^{\zeta\zeta} \;\Rightarrow
\end{aligned}
$$

$$
\frac{J-\iota\!\!\!-I}{R_0}g^{\zeta\zeta} = \frac{2}{a^2}\frac{\psi'}{\sqrt{g}} + \frac{a^2}{2\psi'}\left(\frac{I}{\rho}\right)^2\frac{g_{\rho\rho}}{\sqrt{g}} - 2\frac{I\beta_*}{\psi'}\frac{g_{\rho\theta}}{\sqrt{g}} + 2\frac{(\rho\beta_*)^2}{a^2\psi'}\frac{g_{\theta\theta}}{\sqrt{g}} \tag{42}
$$

Expressions (40), (42), and (41) give g^{ij} as functions of g_{ij}.

Equilibrium current density.

From (36),

$$
\mu_0 J^\rho_{eq} = 0,\ \mu_0 J^\theta_{eq} = -\frac{1}{\sqrt{g}}\left(\frac{1}{R_0}\frac{dJ}{d\rho} - \frac{2\rho}{a^2}\frac{1}{R_0}\frac{\partial\beta_*}{\partial\zeta}\right),\ \mu_0 J^\zeta_{eq} = \frac{1}{\sqrt{g}}\left(\frac{1}{\rho}\frac{dI}{d\rho} - \frac{2\rho}{a^2}\frac{1}{\rho}\frac{\partial\beta_*}{\partial\theta}\right). \tag{43}
$$

DIMENSIONLESS MAGNITUDES

We start with the covariant components of the equilibrium magnetic field. Since J goes like RB_T, we normalize them like

$$J = \frac{2\psi' R_0}{a^2}\bar{J} = \mathrm{sign}(\psi')B_0R_0\bar{J}, \quad I = \frac{2\psi' R_0}{a^2}\bar{I}, \quad \beta_* = \frac{\psi' R_0}{a^2}\bar{\beta}_* \tag{44}$$

The signs are chosen to ensure that $\bar{J}$ is positive defined, and that $\bar{I}$ represents the toroidal current in the direction of the toroidal field.

The major radius R and the magnetic field B are normalized to B_0 and R_0, respectively. From (30) and (44),

$$\sqrt{g} = \frac{B_0}{R_0}\mathrm{sign}(\psi')\frac{(J - \iota I)}{B^2} = \frac{(\bar{J} - \iota\bar{I})}{\bar{B}^2} \tag{45}$$

The pressure is normalized to its value at the magnetic axis, p_0. This value is related to B_0 by the definition of β_0,

$$\beta_0 = \frac{2\mu_0 p_0}{B_0^2}. \tag{46}$$

From (24), (30) and (44)

$$\begin{aligned}\frac{\psi' R_0}{a^2}\left(\bar{\beta}_*\right)^s_{mn} = -\frac{\beta_0 B_0^2}{2}\frac{a^2 R_0}{2}\frac{\bar{p}'\left(\sqrt{g}\right)^c_{mn}}{\psi'(m\iota - n)} \Rightarrow \left(\bar{\beta}_*\right)^s_{mn} &= -\frac{\beta_0}{2\bar{\rho}}\frac{d\bar{p}}{d\bar{\rho}}\frac{\left(\sqrt{g}\right)^c_{mn}}{(m\iota - n)} \\ \left(\bar{\beta}_*\right)^c_{mn} &= \frac{\beta_0}{2\bar{\rho}}\frac{d\bar{p}}{d\bar{\rho}}\frac{\left(\sqrt{g}\right)^s_{mn}}{(m\iota - n)},\end{aligned} \tag{47}$$

where $\bar{\rho} = \rho/a$ ($s = \bar{\rho}^2$).

From now on, we will omit the bar over the dimensionless magnitudes. Then, the equilibrium magnetic field components are

$$B^i_{eq} = \frac{2\psi'}{a^2}\left(0, -\varepsilon\frac{\rho\iota}{\sqrt{g}}, \frac{1}{\sqrt{g}}\right), \qquad B_i^{eq} = \frac{2\psi'}{a^2}\left(\frac{\rho\beta_*}{\varepsilon}, \frac{I}{\varepsilon\rho}, J\right), \tag{48}$$

and the equilibrium current density,

$$J^i_{eq} = \frac{2\psi'}{\mu_0 a^3}\left[0, -\frac{1}{\sqrt{g}}\left(\frac{dJ}{d\rho} - \rho\frac{\partial\beta_*}{\partial\zeta}\right), \frac{1}{\varepsilon\sqrt{g}}\left(\frac{1}{\rho}\frac{dI}{d\rho} - \frac{\partial\beta_*}{\partial\theta}\right)\right], \tag{49}$$

where $\varepsilon = a/R_0$, and $2|\psi'|/a^2 = B_0$.

The Jacobian is

$$\sqrt{g} = \frac{(J - \iota I)}{B^2} \tag{50}$$

and the relations between metric elements are now

$$
\begin{aligned}
g_{\rho\zeta} &= \varepsilon\rho\,\iota\!\!\!-\, g_{\rho\theta} + \frac{\rho\beta_*}{\varepsilon}\sqrt{g}, \\
g_{\theta\zeta} &= \varepsilon\rho\,\iota\!\!\!-\, g_{\theta\theta} + \frac{I}{\varepsilon\rho}\sqrt{g}, \\
g_{\zeta\zeta} &= (J + \iota\!\!\!-\, I)\sqrt{g} + (\varepsilon\rho\,\iota\!\!\!-\,)^2 g_{\theta\theta}.
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
g^{\rho\rho} &= (J - \iota\!\!\!-\, I)\frac{g_{\theta\theta}}{\sqrt{g}} - \left(\frac{I}{\varepsilon\rho}\right)^2, \\
g^{\rho\theta} &= -J\frac{g_{\rho\theta}}{\sqrt{g}} + \rho^2\iota\!\!\!-\,\beta_*\frac{g_{\theta\theta}}{\sqrt{g}} + \frac{I\beta_*}{\varepsilon^2}, \\
g^{\rho\zeta} &= \frac{1}{\varepsilon}\left(\frac{I}{\rho}\frac{g_{\rho\theta}}{\sqrt{g}} - \rho\beta_*\frac{g_{\theta\theta}}{\sqrt{g}}\right) \\
(J - \iota\!\!\!-\, I)g^{\zeta\zeta} &= \frac{1}{\sqrt{g}} + \left(\frac{I}{\varepsilon\rho}\right)^2\frac{g_{\rho\rho}}{\sqrt{g}} - 2\frac{I\beta_*}{\varepsilon^2}\frac{g_{\rho\theta}}{\sqrt{g}} + \frac{(\rho\beta_*)^2}{\varepsilon^2}\frac{g_{\theta\theta}}{\sqrt{g}} \\
g^{\theta\zeta} &= -\frac{1}{\varepsilon}\left(\frac{I}{\rho}\frac{g_{\rho\rho}}{\sqrt{g}} - \rho\beta_*\frac{g_{\rho\theta}}{\sqrt{g}}\right) - \varepsilon\rho\,\iota\!\!\!-\, g^{\zeta\zeta} \\
g^{\theta\theta} &= (J + \iota\!\!\!-\, I)\frac{g_{\rho\rho}}{\sqrt{g}} - 2\rho^2\iota\!\!\!-\,\beta_*\frac{g_{\rho\theta}}{\sqrt{g}} - \left(\frac{\rho\beta_*}{\varepsilon}\right)^2 + (\varepsilon\rho\,\iota\!\!\!-\,)^2 g^{\zeta\zeta}
\end{aligned}
\tag{52}
$$