REDUCED EQUATIONS

MHD equations:

$$\rho_m\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\mathbf{J}\times\mathbf{B} \tag{1}$$

$$\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p=-\Gamma p\nabla\cdot\mathbf{v} \tag{2}$$

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{J} \tag{3}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{4}$$

$$\nabla\cdot\mathbf{B}=0 \tag{5}$$

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{6}$$

From Ohm's law, and for small resistivity,

$$\mathbf{E}\times\mathbf{B}+(\mathbf{v}\times\mathbf{B})\times\mathbf{B}\simeq 0\ \Rightarrow\ \mathbf{v}_\perp=\frac{\mathbf{E}\times\mathbf{B}}{B^2}\simeq-\frac{\nabla\Phi\times\mathbf{B}}{B^2}, \tag{7}$$

where we are assuming slow variation of the fields (slower than τ_A).

If we assume high aspect ratio and low beta, $\beta\sim\varepsilon\ll 1$, at lowest order,

$$\mathbf{v}_\perp\simeq-\frac{\nabla\Phi\times B_\zeta\mathbf{e}^\zeta}{B^2} \tag{8}$$

If we neglect perturbations of the toroidal magnetic field to eliminate propagation of the fast magnetosonic waves across the magnetic field, the perturbed field will be

$$\mathbf{B}_\perp\simeq\nabla\zeta\times\nabla\Psi\quad\Rightarrow\quad\mathbf{B}_\perp^\rho=-\frac{1}{\sqrt{g}}\frac{1}{R_0}\frac{1}{\rho}\frac{\partial\Psi}{\partial\theta},\quad\mathbf{B}_\perp^\theta=\frac{1}{\sqrt{g}}\frac{1}{R_0}\frac{\partial\Psi}{\partial\rho}. \tag{9}$$

This corresponds to a perturbed vector potential

$$\mathbf{A}_\parallel\equiv-\frac{\Psi}{R_0}\mathbf{e}^\zeta. \tag{10}$$

From (3) and (4), in the direction parallel to the magnetic field,

$$E_\parallel=-\mathbf{b}\cdot\frac{\partial\mathbf{A}_\parallel}{\partial t}-\mathbf{b}\cdot\nabla\Phi=\eta J_\parallel. \tag{11}$$

At lowest order,

$$\mathbf{B}\cdot\mathbf{J}=B_\zeta J^\zeta. \tag{12}$$

By applying the operator $\nabla \times \sqrt{g}$ to the momentum balance equation, we get

$$\frac{\partial}{\partial t}\nabla \times (\rho_m \sqrt{g}\mathbf{v}) + \nabla \times (\rho_m \sqrt{g}\mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla\sqrt{g}\times\nabla p + \nabla\times(\sqrt{g}\mathbf{J}\times\mathbf{B}) \tag{13}$$

Taking the ζ-component, we get

$$\frac{\partial U}{\partial t} = -\sqrt{g}\left[\nabla\times(\rho_m\sqrt{g}\mathbf{v}\cdot\nabla\mathbf{v})\right]^\zeta - \sqrt{g}\left(\nabla\sqrt{g}\times\nabla p\right)^\zeta + \sqrt{g}\left[\nabla\times(\sqrt{g}\mathbf{J}\times\mathbf{B})\right]^\zeta, \tag{14}$$

where

$$U = \sqrt{g}\left[\nabla\times(\rho_m\sqrt{g}\mathbf{v})\right]^\zeta \tag{15}$$

From (1), in the direction parallel to the magnetic field,

$$\rho_m\frac{\partial v_\parallel}{\partial t} = -\rho_m\mathbf{b}\cdot\left[(\mathbf{v}\cdot\nabla)\mathbf{v}\right] - \mathbf{b}\cdot\nabla p \tag{16}$$

Dimensionless equations

To get the dimensionless linear equations, t is normalized to the Alfvén time, τ_A, ρ is normalized to the minor radius, a, which is defined by the relation $2|\psi'| = B_0 a^2$, Ψ is normalized to $2\psi'$, Φ to $2\psi'/\tau_A$, J and I to $2\psi' R_0/a^2$, and β_* to $\psi' R_0/a^2$. The pressure is normalized to its value at the magnetic axis, p_0. This value is related to B_0 by the definition of β_0, $\beta_0 = 2\mu_0 p_0/B_0^2$. The density and resistivity are also normalized to their value at the magnetic axis, ε is the inverse aspect ratio, $\varepsilon = a/R_0$, $S = \tau_R/\tau_A$ is the magnetic Reynolds number, $\tau_R = \mu_0 a^2/\eta(0)$ is the resistive time, and $\tau_A = R_0\sqrt{\mu_0 m_i n_i(0)}/B_0$ is the Alfvén time. Then, equations (11), (14), (2) and (16) are

$$\frac{\partial\tilde{\psi}}{\partial t} = \sqrt{g}B\nabla_\parallel\Phi + \frac{\eta}{S}\tilde{J}^\zeta \tag{17}$$

$$\frac{\partial\tilde{U}}{\partial t} = \sqrt{g}B\nabla_\parallel J^\zeta - \frac{\beta_0}{2\varepsilon^2}\sqrt{g}\left(\nabla\sqrt{g}\times\nabla\tilde{p}\right)^\zeta \tag{18}$$

$$\frac{\partial\tilde{p}}{\partial t} = \frac{dp_{eq}}{d\rho}\frac{1}{\rho}\frac{\partial\tilde{\Phi}}{\partial\theta} - \Gamma p_{eq}\nabla\cdot\mathbf{v} \tag{19}$$

$$\frac{\partial\tilde{v}_{\parallel th}}{\partial t} = -\frac{\beta_0}{2n_{eq}}\nabla_\parallel p \tag{20}$$

where

$$U = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-n_{eq}\sqrt{g}g_{\rho\theta}\frac{\partial\Phi}{\partial\theta} + n_{eq}\rho\sqrt{g}g_{\theta\theta}\frac{\partial\Phi}{\partial\rho}\right) - n_{eq}\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(-\sqrt{g}g_{\rho\rho}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + \sqrt{g}g_{\rho\theta}\frac{\partial\Phi}{\partial\rho}\right), \tag{21}$$

$$J^{\zeta} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-\frac{g_{\rho\theta}}{\sqrt{g}}\frac{\partial\Psi}{\partial\theta} + \rho\frac{g_{\theta\theta}}{\sqrt{g}}\frac{\partial\Psi}{\partial\rho}\right) - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(-\frac{g_{\rho\rho}}{\sqrt{g}}\frac{1}{\rho}\frac{\partial\Psi}{\partial\theta} + \frac{g_{\rho\theta}}{\sqrt{g}}\frac{\partial\Psi}{\partial\rho}\right), \tag{22}$$

$$\sqrt{g}B\nabla_{\parallel}f = \frac{\partial\tilde{f}}{\partial\zeta} - \iota\frac{\partial\tilde{f}}{\partial\theta} - \frac{\partial f_{eq}}{\partial\rho}\frac{1}{\rho}\frac{\partial\tilde{\psi}}{\partial\theta} + \frac{1}{\rho}\frac{\partial f_{eq}}{\partial\theta}\frac{\partial\tilde{\psi}}{\partial\rho}, \tag{23}$$

$$\sqrt{g}\left(\nabla\sqrt{g}\times\nabla\tilde{f}\right)^{\zeta} = \frac{\partial\sqrt{g}}{\partial\rho}\frac{1}{\rho}\frac{\partial\tilde{f}}{\partial\theta} - \frac{1}{\rho}\frac{\partial\sqrt{g}}{\partial\theta}\frac{\partial\tilde{f}}{\partial\rho}, \tag{24}$$

and

$$J^{\zeta}_{eq} = \frac{1}{\varepsilon^2}\left(\frac{1}{\rho}\frac{dI}{d\rho} - \frac{\partial\beta_*}{\partial\theta}\right) \tag{25}$$

Note that to simplify notation we denote $\sqrt{g}J^{\zeta}$ by just J^{ζ}, and ηJ by just η.

GYROFLUID EQUATIONS

A basic model for incorporating Landau closure into reduced MHD was given in Phys. Fluids B **4**, 3316 (1992). Here it was assumed that $k_{\perp}\rho_f \ll 1$, $ik_{\parallel} \to \nabla_{\parallel}$, $i\omega_{Df} \to \Omega_d$, $i\omega_* \to \Omega_*$, $E_{\parallel} = 0$, and $\Gamma_f \to 1$. The subscript "f" refers here to the fast ion component, and Ω_d and Ω_* are operators that will be given bellow. The following two equations are added to the reduced set of equations:

$$\frac{\partial n_f}{\partial t} = -\Omega_d(n_f) - n_{f0}\nabla_{\parallel}v_{\parallel f} - n_{f0}\Omega_d\left(\frac{q_f\Phi}{T_f}\right) + n_{f0}\Omega_*\left(\frac{q_f\Phi}{T_f}\right) \tag{26}$$

$$\frac{\partial}{\partial t}(M_f n_{f0} v_{\parallel f}) = -\left[\Omega_d + \left(\frac{\pi}{2}\right)^{1/2} v_{th,f}\left|\nabla_{\parallel}\right|\right] M_f n_{f0} v_{\parallel f} - T_f\nabla_{\parallel}n_f + q_f n_{f0}\Omega_*\left(\frac{\Psi}{R_0}\right) \tag{27}$$

Here,

$$\begin{aligned} \Omega_d &= \frac{v_{th,f}^2}{\Omega_c}\frac{\mathbf{B}\times\nabla B}{B^2}\cdot\nabla & \qquad v_{th,f}^2 &= \frac{T_f}{M_f} \\ \Omega_* &= \frac{T_f}{q_f B n_{f0}}\nabla n_{f0}\cdot\frac{\mathbf{B}}{B}\times\nabla & \qquad \Omega_c &= \frac{q_f B}{M_f} \end{aligned} \tag{28}$$

where $\mathbf{B}$ is the equilibrium magnetic field.

Using the covariant components of **B**, we get

$$\begin{aligned}
\Omega_d =& \frac{T_f}{2q_f B^4\sqrt{g}}\left[\left(\frac{I}{\rho}\frac{1}{R_0}\frac{\partial B^2}{\partial \zeta} - \frac{J}{R_0}\frac{1}{\rho}\frac{\partial B^2}{\partial \theta}\right)\frac{\partial}{\partial \rho} + \left(\frac{J}{R_0}\frac{\partial B^2}{\partial \rho} - \frac{2\rho\beta_*}{a^2}\frac{1}{R_0}\frac{\partial B^2}{\partial \zeta}\right)\frac{1}{\rho}\frac{\partial}{\partial\theta}\right.\\
&\left.+\left(\frac{2\rho\beta_*}{a^2}\frac{1}{\rho}\frac{\partial B^2}{\partial\theta} - \frac{I}{\rho}\frac{\partial B^2}{\partial \rho}\right)\frac{1}{R_0}\frac{\partial}{\partial\zeta}\right] \qquad (29)\\
\Omega_* =& \frac{T_f}{q_f B^2 n_{f0}}\frac{1}{\sqrt{g}}\frac{dn_{f0}}{d\rho}\left(\frac{I}{\rho}\frac{1}{R_0}\frac{\partial}{\partial\zeta} - \frac{J}{R_0}\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)
\end{aligned}$$

Dimensionless equations

To get the dimensionless equations, n_f is normalized to $n_{f0}(0)$, and $v_{th,f}$ and $v_{\|f}$ are normalized to the Alfvén speed $v_{A0} = R_0/\tau_A$. Then we get

$$\frac{\partial n_f}{\partial t} = -\frac{v_{th,f}^2}{\varepsilon^2\Omega_{cf}}\,\Omega_d(n_f) - n_{f0}\nabla_\| v_{\|f} - n_{f0}\,\Omega_d(\Phi) + n_{f0}\,\Omega_*(\Phi) \qquad (30)$$

$$\frac{\partial v_{\|f}}{\partial t} = -\frac{v_{th,f}^2}{\varepsilon^2\Omega_{cf}}\,\Omega_d(v_{\|f}) - \left(\frac{\pi}{2}\right)^{1/2} v_{th,f}\left|\nabla_\| v_{\|f}\right| - \frac{v_{th,f}^2}{n_{f0}}\nabla_\| n_f + v_{th,f}^2\,\Omega_*(\Psi) \qquad (31)$$

where

$$\begin{aligned}
\Omega_d = \frac{1}{2B^4\sqrt{g}}&\left[\left(\frac{I}{\rho}\frac{\partial B^2}{\partial\zeta} - J\frac{1}{\rho}\frac{\partial B^2}{\partial\theta}\right)\frac{\partial}{\partial\rho} + \left(J\frac{\partial B^2}{\partial\rho} - \rho\beta_*\frac{\partial B^2}{\partial\zeta}\right)\frac{1}{\rho}\frac{\partial}{\partial\theta}\right.\\
&\left.+\left(\rho\beta_*\frac{1}{\rho}\frac{\partial B^2}{\partial\theta} - \frac{I}{\rho}\frac{\partial B^2}{\partial\rho}\right)\frac{\partial}{\partial\zeta}\right] \qquad (32)
\end{aligned}$$

$$\Omega_* = \frac{1}{B^2\sqrt{g}}\frac{1}{n_{f0}}\frac{dn_{f0}}{d\rho}\left(\frac{I}{\rho}\frac{\partial}{\partial\zeta} - J\frac{1}{\rho}\frac{\partial}{\partial\theta}\right) \qquad (33)$$

$$B^2 = \frac{J - \text{ɨ} I}{\sqrt{g}} \qquad (34)$$

Here, $\Omega_{cf} = \Omega_{c0}\tau_A$, where $\Omega_{c0} = q_f B_0/M_f$ is the cyclotron frequency of the fast ions.

We also include the term corresponding to the pressure of the fast ions in the vorticity equation (18), as well as the evolution equation for the parallel velocity of the thermal particles (20) and the effect of an equilibrium toroidal velocity v_{eq}^ζ. In these equations, $v_{\|th}$ and v_{eq}^ζ are normalized to v_{A0}, and $\beta_f = 2\mu_0 n_{f0}(0) T_f/B_0^2$.

$$\frac{\partial\tilde{\psi}}{\partial t} = \sqrt{g}B\nabla_\|\Phi + \frac{\eta}{S}\tilde{J}^\zeta \qquad (35)$$

$$\frac{\partial\tilde{U}}{\partial t} = -v_{eq}^\zeta\frac{\partial\tilde{U}}{\partial\zeta} + \sqrt{g}B\nabla_\| J^\zeta - \frac{\beta_0}{2\varepsilon^2}\sqrt{g}\left(\nabla\sqrt{g}\times\nabla\tilde{p}\right)^\zeta - \frac{\beta_f}{2\varepsilon^2}\sqrt{g}\left(\nabla\sqrt{g}\times\nabla\tilde{n}_f\right)^\zeta \qquad (36)$$

$$\frac{\partial \tilde{p}}{\partial t} = -v_{eq}^{\zeta}\frac{\partial \tilde{p}}{\partial \zeta} + \frac{dp_{eq}}{d\rho}\frac{1}{\rho}\frac{\partial \tilde{\Phi}}{\partial \theta} - \Gamma p_{eq}\nabla\cdot\mathbf{v} \tag{37}$$

$$\frac{\partial \tilde{n}_f}{\partial t} = -v_{eq}^{\zeta}\frac{\partial \tilde{n}_f}{\partial \zeta} - \frac{v_{th,f}^2}{\varepsilon^2\Omega_{cf}}\,\Omega_d(\tilde{n}_f) - n_{f0}\nabla_{\parallel}v_{\parallel f} - n_{f0}\,\Omega_d(\tilde{\Phi}) + n_{f0}\,\Omega_*(\tilde{\Phi}) \tag{38}$$

$$\frac{\partial \tilde{v}_{\parallel f}}{\partial t} = -v_{eq}^{\zeta}\frac{\partial \tilde{v}_{\parallel f}}{\partial \zeta} - \frac{v_{th,f}^2}{\varepsilon^2\Omega_{cf}}\,\Omega_d(\tilde{v}_{\parallel f}) - \sqrt{2}a_1 v_{th,f}\left|\nabla_{\parallel}\tilde{v}_{\parallel f}\right| - 2a_0\frac{v_{th,f}^2}{n_{f0}}\nabla_{\parallel}n_f + v_{th,f}^2\,\Omega_*(\tilde{\psi}) \tag{39}$$

$$\frac{\partial \tilde{v}_{\parallel th}}{\partial t} = -v_{eq}^{\zeta}\frac{\partial \tilde{v}_{\parallel th}}{\partial \zeta} - \frac{\beta_0}{2n_{0,th}}\nabla_{\parallel}p \tag{40}$$

<u>Divergence of $\mathbf{v}$</u>

$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{v}_{\perp} + B\nabla_{\parallel}\left(\frac{v_{\parallel}}{B}\right) \tag{41}$$

At lowest order (8),

$$\nabla\cdot\mathbf{v}_{\perp} = \frac{1}{\sqrt{g}}\left(-\frac{\partial\sqrt{g}}{\partial\rho}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} + \frac{1}{\rho}\frac{\partial\sqrt{g}}{\partial\theta}\frac{\partial\Phi}{\partial\rho}\right) \tag{42}$$

From equation (7),

$$\begin{aligned}
\nabla\cdot\mathbf{v}_{\perp} &= \frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\sqrt{g}v_{\perp}^{\rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\sqrt{g}v_{\perp}^{\theta}\right) + \frac{\partial}{\partial\zeta}\left(\sqrt{g}v_{\perp}^{\zeta}\right)\right] \\
&= -\frac{1}{\sqrt{g}}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{\sqrt{g}}{J-\iota I}\left(J\frac{\partial\Phi}{\partial\theta} - I\frac{\partial\Phi}{\partial\zeta}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\frac{\sqrt{g}}{J-\iota I}\left(\rho\beta_*\frac{\partial\Phi}{\partial\zeta} - J\frac{\partial\Phi}{\partial\rho}\right)\right]\right. \\
&\qquad\qquad \left. + \frac{\partial}{\partial\zeta}\left[\frac{\sqrt{g}}{J-\iota I}\left(\frac{I}{\rho}\frac{\partial\Phi}{\partial\rho} - \beta_*\frac{\partial\Phi}{\partial\theta}\right)\right]\right\} \\
&= -\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\left(\frac{\sqrt{g}J}{J-\iota I}\right)\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\sqrt{g}I}{J-\iota I}\right)\frac{\partial\Phi}{\partial\zeta}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{1}{J-\iota I}\left[\frac{\partial\left(\sqrt{g}\beta_*\right)}{\partial\theta}\frac{\partial\Phi}{\partial\zeta} - J\frac{1}{\rho}\frac{\partial\sqrt{g}}{\partial\theta}\frac{\partial\Phi}{\partial\rho}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{1}{J-\iota I}\left[\frac{I}{\rho}\frac{\partial\sqrt{g}}{\partial\zeta}\frac{\partial\Phi}{\partial\rho} - \frac{\partial\left(\sqrt{g}\beta_*\right)}{\partial\zeta}\frac{\partial\Phi}{\partial\theta}\right]
\end{aligned} \tag{43}$$

$$B\nabla_{\parallel}\left(\frac{v_{\parallel}}{B}\right)=\frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)\left(\frac{v_{\parallel}}{B}\right) \tag{44}$$

Then

$$\begin{aligned}\nabla\cdot\mathbf{v}=&-\frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\rho}\left(\frac{\sqrt{g}J}{J-\iota I}\right)-\frac{1}{J-\iota I}\frac{\partial\left(\rho\sqrt{g}\beta_*\right)}{\partial\zeta}\right]\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\\&+\frac{1}{\sqrt{g}}\frac{1}{J-\iota I}\left(J\frac{1}{\rho}\frac{\partial\sqrt{g}}{\partial\theta}-\frac{I}{\rho}\frac{\partial\sqrt{g}}{\partial\zeta}\right)\frac{\partial\Phi}{\partial\rho}\\&-\frac{1}{\sqrt{g}}\left[\frac{1}{J-\iota I}\frac{\partial\left(\sqrt{g}\beta_*\right)}{\partial\theta}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\sqrt{g}I}{J-\iota I}\right)\right]\frac{\partial\Phi}{\partial\zeta}\\&+\frac{1}{J-\iota I}\left[B\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)v_{\parallel}-v_{\parallel}\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)B\right]\end{aligned} \tag{45}$$

DIAMAGNETIC TERMS

From Ohm's law and Faraday's law, in the direction parallel to the magnetic field,

$$E_{\parallel}=-\mathbf{b}\cdot\frac{\partial\mathbf{A}_{\parallel}}{\partial t}-\mathbf{b}\cdot\nabla\Phi=\eta J_{\parallel}-\frac{\mathbf{b}\cdot\nabla p_e}{en}, \tag{46}$$

where

$$\mathbf{A}_{\parallel}\equiv-\frac{\Psi}{R_0}\mathbf{e}^{\zeta}. \tag{47}$$

The last term in (46) is the diamagnetic contribution to the evolution equation for the poloidal flux Ψ. In dimensionless units, the equation is

$$\frac{\partial\tilde{\Psi}}{\partial t}=\sqrt{g}B\nabla_{\parallel}\Phi-\frac{\beta_{0e}}{2\varepsilon^2\omega_{cy}n}\sqrt{g}B\nabla_{\parallel}p+\frac{\eta}{S}\tilde{J}^{\zeta}, \tag{48}$$

Here, $\omega_{cy}=\omega_{c0}\tau_A$, where $\omega_{c0}=eB_0/m_i$ is the cyclotron frequency of the thermal ions.

Contribution to the momentum balance:

$$\frac{\partial U}{\partial t}=\sqrt{g}\frac{\partial}{\partial t}\left[\nabla\times\left(\rho_m\sqrt{g}\mathbf{v}\right)\right]^{\zeta}=\cdots-\sqrt{g}\left[\nabla\times\left(\rho_m\sqrt{g}(\mathbf{v}_{*i}\cdot\nabla)\mathbf{v}_{\perp}\right)\right]^{\zeta}, \tag{49}$$

where

$$\mathbf{v}_{*i}=\frac{\mathbf{B}\times\nabla p_i}{enB^2} \tag{50}$$

After a lengthy derivation, we get that, in dimensionless units,

$$\frac{\partial U}{\partial t} = \cdots - \frac{\beta_{0i}}{2\varepsilon^2 \omega_{cy}} \sqrt{g} \left[\nabla \times \left(\frac{\sqrt{g}}{B^2} \left(\mathbf{B} \times \nabla p \cdot \nabla \right) \mathbf{v}_\perp \right) \right]^\zeta \tag{51}$$

where

$$\begin{aligned}
&\sqrt{g} \left[\nabla \times \left(\frac{\sqrt{g}}{B^2} \left(\mathbf{B} \times \nabla p \cdot \nabla \right) \mathbf{v}_\perp \right) \right]^\zeta = \\
&\quad \frac{p'J}{B^2} \left(g_{\rho\rho} \frac{1}{\rho^3} \frac{\partial^3 \Phi}{\partial \theta^3} - 2 g_{\rho\theta} \frac{1}{\rho^2} \frac{\partial^3 \Phi}{\partial \rho \partial \theta^2} + g_{\theta\theta} \frac{1}{\rho} \frac{\partial^3 \Phi}{\partial \rho^2 \partial \theta} \right) \\
&- \frac{p'I}{\rho B^2} \left(g_{\rho\rho} \frac{1}{\rho^2} \frac{\partial^3 \Phi}{\partial \theta^2 \partial \zeta} - 2 g_{\rho\theta} \frac{1}{\rho} \frac{\partial^3 \Phi}{\partial \rho \partial \theta \partial \zeta} + g_{\theta\theta} \frac{\partial^3 \Phi}{\partial \rho^2 \partial \zeta} \right) \\
&+ \left[\frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'J}{B^2} g_{\rho\rho} \right) - \rho \frac{\partial}{\partial \rho} \left(\frac{p'J}{\rho B^2} g_{\rho\theta} \right) \right] \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \theta^2} \\
&+ \left[\frac{\partial}{\partial \rho} \left(\frac{p'J}{B^2} g_{\theta\theta} \right) - \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'J}{B^2} g_{\rho\theta} \right) \right] \frac{1}{\rho} \frac{\partial^2 \Phi}{\partial \rho \partial \theta} \\
&- \frac{1}{2} \frac{p'I}{\rho B^2} \left(\frac{\partial g_{\rho\rho}}{\partial \zeta} \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \theta^2} - 2 \frac{\partial g_{\rho\theta}}{\partial \zeta} \frac{1}{\rho} \frac{\partial^2 \Phi}{\partial \rho \partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \zeta} \frac{\partial^2 \Phi}{\partial \rho^2} \right) \\
&+ \left[\frac{\partial}{\partial \rho} \left(\frac{p'I}{\rho B^2} g_{\rho\theta} \right) - \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'I}{\rho B^2} g_{\rho\rho} \right) \right] \frac{1}{\rho} \frac{\partial^2 \Phi}{\partial \theta \partial \zeta} \\
&- \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{p'I}{B^2} g_{\theta\theta} \right) - \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'I}{\rho B^2} g_{\rho\theta} \right) \right] \frac{\partial^2 \Phi}{\partial \rho \partial \zeta} \\
&+ \frac{1}{2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'J}{B^2} \frac{1}{\rho} \frac{\partial g_{\rho\rho}}{\partial \theta} \right) - \frac{\partial}{\partial \rho} \left[\frac{p'J}{\rho^2 B^2} \frac{\partial}{\partial \rho} \left(\rho^2 g_{\theta\theta} \right) \right] + \frac{\partial}{\partial \rho} \left(\frac{p'I}{\rho B^2} \frac{\partial g_{\rho\theta}}{\partial \zeta} \right) \right. \\
&\left. - \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'I}{\rho B^2} \frac{\partial g_{\rho\rho}}{\partial \zeta} \right) - \frac{\partial}{\partial \rho} \left(\frac{p'I}{\varepsilon \rho B^2} \frac{1}{\rho} \frac{\partial g_{\rho\zeta}}{\partial \theta} \right) + \frac{\partial}{\partial \rho} \left[\frac{p'I}{\varepsilon \rho B^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho g_{\theta\zeta} \right) \right] \right\} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \\
&+ \frac{1}{2} \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{p'J}{B^2} \frac{\partial g_{\theta\theta}}{\partial \theta} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left[\frac{p'J}{B^2} \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 g_{\theta\theta} \right) \right] \right. \\
&- \frac{2}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'J}{B^2} \frac{1}{\rho} \frac{\partial g_{\rho\theta}}{\partial \theta} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'I}{\rho B^2} \frac{\partial g_{\rho\theta}}{\partial \zeta} \right) - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{p'I}{B^2} \frac{\partial g_{\theta\theta}}{\partial \zeta} \right) \\
&\left. - \frac{1}{\rho} \frac{\partial}{\partial \theta} \left[\frac{p'I}{\varepsilon \rho B^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho g_{\theta\zeta} \right) \right] + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{p'I}{\varepsilon \rho B^2} \frac{1}{\rho} \frac{\partial g_{\rho\zeta}}{\partial \theta} \right) \right\} \frac{\partial \Phi}{\partial \rho}
\end{aligned} \tag{52}$$

Contribution to the evolution equation for the pressure:

$$\frac{\partial \tilde{p}}{\partial t} = \cdots - \Gamma \frac{p_i}{en} \nabla \tilde{p} \cdot \nabla \times \frac{\mathbf{B}}{B^2} \tag{53}$$

$$\nabla \tilde{p} \cdot \nabla \times \frac{\mathbf{B}}{B^2} = \frac{\mu_0 \mathbf{J} \cdot \nabla \tilde{p}}{B^2} - \mathbf{B} \times \nabla \left(\frac{1}{B^2} \right) \cdot \nabla \tilde{p} = \frac{\mu_0 \mathbf{J} \cdot \nabla \tilde{p}}{B^2} + \frac{2}{B^3} \mathbf{B} \times \nabla B \cdot \nabla \tilde{p}$$

In dimensionless units,

$$\frac{\partial \tilde{p}}{\partial t} = \cdots - \frac{\Gamma \beta_{0i}}{2\varepsilon^2 \omega_{cy}} \frac{p_{i\,eq}}{n} \nabla \tilde{p} \cdot \nabla \times \frac{\mathbf{B}}{B^2} \tag{54}$$

where

$$\begin{aligned}\nabla p \cdot \nabla \times \frac{\mathbf{B}}{B^2} = \frac{1}{J - \iota I} &\left[- \left(\frac{dJ}{d\rho} - \rho \frac{\partial \beta_*}{\partial \zeta} \right) \frac{1}{\rho} \frac{\partial \tilde{p}}{\partial \theta} + \left(\frac{1}{\rho} \frac{dI}{d\rho} - \frac{\partial \beta_*}{\partial \theta} \right) \frac{\partial \tilde{p}}{\partial \zeta} \right] + 2\Omega_d(\tilde{p}) \\ &+ \frac{1}{(J - \iota I)\sqrt{g}} \left\{ \varepsilon^2 \left(\frac{\partial}{\partial \zeta} - \iota \frac{\partial}{\partial \theta} \right) \left(g_{\rho\theta} \frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} - g_{\theta\theta} \frac{\partial \tilde{\psi}}{\partial \rho} \right) \right. \\ &\left. + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left[\sqrt{g} \left(\frac{I}{\rho} \frac{\partial \tilde{\psi}}{\partial \rho} - \beta_* \frac{\partial \tilde{\psi}}{\partial \theta} \right) \right] \right\} \frac{dp_{eq}}{d\rho}\end{aligned} \tag{55}$$

The operator Ω_d is defined in (32).

ION FLR TERMS

Contribution to the equation for the poloidal flux :

$$\frac{\partial \tilde{\psi}}{\partial t} = \cdots + \rho_i^2 \left(\frac{\pi}{2} \right)^{1/2} \frac{v_A^2}{v_{T_e}} |\nabla_\parallel| \nabla_\perp^2 \tilde{\psi} \tag{56}$$

We have the same expression in dimensionless units. The ion Larmor radius ρ_i is normalized to a and the velocities v_A and $v_{T_e} = \sqrt{T_e/m_e}$ are normalized to v_{A0}.

Contribution to the momentum balance (dimensionless units):

$$\frac{\partial U}{\partial t} = \cdots + \omega_r \rho_i^2 \nabla_\perp^2 U, \tag{57}$$

where ω_r is normalized to τ_A^{-1}.

EP FLR TERMS

Contribution to the equation for the EP density:

$$\frac{\partial n_f}{\partial t} = \cdots + n_{f0}(\omega_r - \Omega_*) (1 - \Gamma_f) \frac{q_f \Phi}{T_f} \tag{58}$$

In dimensionless units,

$$\frac{\partial n_f}{\partial t} = \cdots + \varepsilon^2 \omega_r \Omega_{cf} \frac{n_{f0}}{v_{th,f}^2} W - n_{f0} \Omega_*(W). \tag{59}$$

The operator Ω_* is defined in (33) and W is given by the equation

$$(1 - \rho_f^2 \nabla_\perp^2) W + \rho_f^2 \nabla_\perp^2 \Phi = 0, \tag{60}$$

where ρ_f is the EP Larmor radius normalized to a.

Contribution to the equation for the EP parallel velocity:

$$\frac{\partial v_{\parallel f}}{\partial t} = \cdots + \frac{v_{th,f}^2}{n_{f0} B} (\Gamma_f - 1) \frac{\mathbf{B}}{B} \times \nabla \left(\frac{\Psi}{R} \right) \cdot \nabla n_{f0} \tag{61}$$

The combination of this term with the last term of equation (39) yields, in dimensionless units,

$$\frac{\partial v_{\parallel f}}{\partial t} = \cdots + v_{th,f}^2 \Gamma_f \Omega_*(\tilde{\psi}). \tag{62}$$

Here,

$$\Gamma_f \Omega_*(\tilde{\psi}) = \frac{1}{B^2 \sqrt{g}} \frac{1}{n_{f0}} \frac{d n_{f0}}{d\rho} \Gamma_f \left(\frac{I}{\rho} \frac{\partial \tilde{\psi}}{\partial \zeta} - J \frac{1}{\rho} \frac{\partial \tilde{\psi}}{\partial \theta} \right) = \frac{1}{J - \iota I} \frac{1}{n_{f0}} \frac{1}{\rho} \frac{d n_{f0}}{d\rho} \left(I X_1 - J X_2 \right), \tag{63}$$

where X_1 and X_2 are given by the equations

$$(1 - \rho_f^2 \nabla_\perp^2) X_1 - \frac{\partial \tilde{\psi}}{\partial \zeta} = 0, \qquad (1 - \rho_f^2 \nabla_\perp^2) X_2 - \frac{\partial \tilde{\psi}}{\partial \theta} = 0. \tag{64}$$

In dimensionless units, at lowest order,

$$\nabla_\perp = \mathbf{e}^\rho \frac{\partial}{\partial \rho} + \mathbf{e}^\theta \frac{1}{\rho} \frac{\partial}{\partial \theta}$$

and the operator $\nabla_\perp^2$ is given by the expression

$$\nabla_\perp^2 = \frac{1}{\sqrt{g}} \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho g_{\theta\theta} \frac{\partial}{\partial \rho} - g_{\rho\theta} \frac{\partial}{\partial \theta} \right) + \frac{1}{\rho} \frac{\partial}{\partial \theta} \left(- g_{\rho\theta} \frac{\partial}{\partial \rho} + g_{\rho\rho} \frac{1}{\rho} \frac{\partial}{\partial \theta} \right) \right] \tag{65}$$

Using the exact expression of the equilibrium $\mathbf{B}$, we get, in dimensionless units,

$$
\begin{aligned}
\sqrt{g}\nabla_\perp^2 =& \frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\left[(J-\text{ɫ}I)\,g_{\theta\theta}-\left(\frac{I}{\varepsilon\rho}\right)^2\sqrt{g}\right]\frac{\partial}{\partial\rho}\right.\\
&\left.\quad-\left(Jg_{\rho\theta}-\rho^2\text{ɫ}\beta_* g_{\theta\theta}-\frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\frac{\partial}{\partial\theta}+\left(Ig_{\rho\theta}-\rho^2\beta_* g_{\theta\theta}\right)\frac{\partial}{\partial\zeta}\right\}\\
&+\frac{1}{\rho}\frac{\partial}{\partial\theta}\left\{-\left(Jg_{\rho\theta}-\rho^2\text{ɫ}\beta_* g_{\theta\theta}-\frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\frac{\partial}{\partial\rho}+\left[\frac{J}{J-\text{ɫ}I}\left(Jg_{\rho\rho}-\rho^2\text{ɫ}\beta_* g_{\rho\theta}\right)\right.\right.\\
&\left.\quad-\frac{\rho^2\text{ɫ}\beta_*}{J-\text{ɫ}I}\left(Jg_{\rho\theta}-\rho^2\text{ɫ}\beta_* g_{\theta\theta}\right)-\left(\frac{\rho\beta_*}{\varepsilon}\right)^2\sqrt{g}\right]\frac{1}{\rho}\frac{\partial}{\partial\theta}\\
&\left.\quad-\left[\frac{J}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho}-\rho\beta_* g_{\rho\theta}\right)-\frac{\rho^2\text{ɫ}\beta_*}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\right]\frac{\partial}{\partial\zeta}\right\}\\
&+\frac{\partial}{\partial\zeta}\left\{\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\frac{\partial}{\partial\rho}\right.\\
&\quad-\left[\frac{J}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho}-\rho\beta_* g_{\rho\theta}\right)-\frac{\rho^2\text{ɫ}\beta_*}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\right]\frac{1}{\rho}\frac{\partial}{\partial\theta}\\
&\left.\quad+\left[\frac{I}{\rho}\frac{1}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho}-\rho\beta_* g_{\rho\theta}\right)-\frac{\rho\beta_*}{J-\text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\right]\frac{\partial}{\partial\zeta}\right\}
\end{aligned}
\tag{66}
$$

LANDAU ELECTRON-ION DAMPING TERMS

Contribution to the vorticity equation:

$$
\frac{\partial U}{\partial t} = \cdots - \rho_m \frac{B_\zeta}{B^2}\frac{m_i\omega_{ci}^2}{T_e\,\omega_r}\left(S_{ei}\right)_{\text{imag}}\Omega_d^2(\Phi) = \cdots - \frac{q_i^2 n_i B_\zeta}{T_e\omega_r}\left(S_{ei}\right)_{\text{imag}}\Omega_d^2(\Phi) \tag{67}
$$

The coefficients in this term are obtained from the definition of U and the expressions (11) and (41) of Phys. Fluids B **4**, 3869 (1992).

In dimensionless units,

$$
\frac{\partial U}{\partial t} = \cdots - \frac{\beta_{0i}}{2\varepsilon^2\omega_r}\tau_i p_{i\,eq}\left(S_{ei}\right)_{\text{imag}}\Omega_d^2(\Phi), \tag{68}
$$

where $\tau_i = T_i/T_e$, ω_r is normalized to τ_A^{-1} and Ω_d is defined in (32). Here

$$
S_{ei} = Y_{2e} + \tau_i Y_{2i} - i\frac{\nu_e}{\omega_r}\frac{Y_{1e}^2}{D_e} - i\frac{\nu_i}{\omega_r}\frac{Y_{1i}^2}{D_i} + R_{ei}\left(\frac{Y_{1e}}{D_e}-\frac{Y_{1i}}{D_i}\right)^2, \tag{69}
$$

and

$$
\begin{aligned}
&R_{ei}^{-1} = 1 + \frac{Y_{0e}}{D_e} + \tau_i^{-1}\left(1 + \frac{Y_{0i}}{D_i}\right), \quad D_j = 1 + i\frac{\nu_j}{\omega_r}Y_{0j}, \\
&Y_{0j} = \xi_j Z(\zeta_j), \quad \xi_j = \frac{\omega_r}{\sqrt{2}\,|k_\parallel| v_{th,j}} = \frac{\omega_r}{\sqrt{2}\,|n - m\not{t}| v_{th,j}}, \quad \zeta_j = \left(1 + i\frac{\nu_j}{\omega_r}\right)\xi_j, \\
&Y_{1j} = \xi_j\left[\zeta_j + \left(\frac{1}{2} + \zeta_j^2\right)Z(\zeta_j)\right], \quad Y_{2j} = \xi_j\left[\frac{3}{2}\zeta_j + \zeta_j^3 + \left(\frac{1}{2} + \zeta_j^2 + \zeta_j^4\right)Z(\zeta_j)\right]
\end{aligned}
\tag{70}
$$