VELOCITY UPDATE

Old velocity definition, only toroidal component of the magnetic field:

$$\vec{V}_{\perp} = \frac{\vec{B}_{\zeta} e^{\zeta} \wedge \vec{\nabla}\Phi}{B_0^2}$$

New velocity definition, all magnetic field components:

$$\vec{V}_E = \frac{\vec{B}_0 \wedge \vec{\nabla}\Phi}{B_0^2}$$

Using the definition of the magnetic field:

$$\vec{B}_0 = \frac{2\rho\beta_*}{a^2}\vec{\nabla}\rho + I\vec{\nabla}\theta - J\vec{\nabla}\zeta$$

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial\rho}\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\zeta$$

So:

$$\begin{aligned}
\vec{B}_0 \wedge \vec{\nabla}\Phi &= \left(\frac{2\rho\beta_*}{a^2}\vec{\nabla}\rho + I\vec{\nabla}\theta - J\vec{\nabla}\zeta\right) \wedge \left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\zeta\right) \\
&= \frac{2\rho\beta_*}{a^2}\left(\frac{\partial\Phi}{\partial\theta}\vec{\nabla}\rho \wedge \vec{\nabla}\theta + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\rho \wedge \vec{\nabla}\zeta\right) \\
&\quad + I\left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\theta \wedge \vec{\nabla}\rho + \frac{\partial\Phi}{\partial\zeta}\vec{\nabla}\theta \wedge \vec{\nabla}\zeta\right) \\
&\quad - J\left(\frac{\partial\Phi}{\partial\rho}\vec{\nabla}\zeta \wedge \vec{\nabla}\rho + \frac{\partial\Phi}{\partial\theta}\vec{\nabla}\zeta \wedge \vec{\nabla}\theta\right) \\
&= \left(I\frac{\partial\Phi}{\partial\zeta} + J\frac{\partial\Phi}{\partial\theta}\right)\vec{\nabla}\theta \wedge \vec{\nabla}\zeta - \left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\zeta} + J\frac{\partial\Phi}{\partial\rho}\right)\vec{\nabla}\rho \wedge \vec{\nabla}\zeta + \left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\theta} - I\frac{\partial\Phi}{\partial\rho}\right)\vec{\nabla}\rho \wedge \vec{\nabla}\theta
\end{aligned}$$

Using the next vectorial identities:

$$i = j \wedge k \qquad j = k \wedge i \qquad k = i \wedge j$$

$$\frac{1}{\sqrt{g}}\frac{1}{\rho R_0}\hat{e}_{\rho} = \vec{\nabla}\theta \wedge \vec{\nabla}\zeta \qquad \frac{1}{\sqrt{g}}\frac{1}{R_0}\hat{e}_{\theta} = \vec{\nabla}\rho \wedge \vec{\nabla}\zeta \qquad \frac{1}{\sqrt{g}}\frac{1}{\rho}\hat{e}_{\zeta} = \vec{\nabla}\rho \wedge \vec{\nabla}\theta$$

So:

$$\vec{B}_0 \wedge \vec{\nabla}\Phi = \frac{1}{\sqrt{g}}\left[\frac{1}{\rho R_0}\left(I\frac{\partial\Phi}{\partial\zeta} + J\frac{\partial\Phi}{\partial\theta}\right)\hat{e}_{\rho} - \left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\zeta} + J\frac{\partial\Phi}{\partial\rho}\right)\hat{e}_{\theta} + \frac{1}{\rho}\left(\frac{2\rho\beta_*}{a^2}\frac{\partial\Phi}{\partial\theta} - I\frac{\partial\Phi}{\partial\rho}\right)\hat{e}_{\zeta}\right]$$

Also:

$$B_0^2 = B^\rho B_\rho + B^\theta B_\theta + B^\zeta B_\zeta \quad ==> \quad B_0^2 = \frac{2\psi'}{a^2 R_0 \sqrt{g}}\left(J - \iota I\right)$$

So:

$$\Rightarrow \vec{V}_E = \frac{a^2 R_0 \sqrt{g}}{2\psi'(J-\iota I)}\frac{1}{\sqrt{g}}\left\{\left[\frac{I}{\rho}\frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta} + \frac{J}{R_0}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\right]\hat{e}^\rho \right.$$
$$\left. - \left[\frac{J}{R_0}\frac{\partial\Phi}{\partial\rho} + \frac{2\rho\beta_*}{a^2}\frac{1}{R_0}\frac{\partial\Phi}{\partial\zeta}\right]\hat{e}^\theta + \left[\frac{2\rho\beta_*}{a^2}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} - \frac{I}{\rho}\frac{\partial\Phi}{\partial\rho}\right]\hat{e}^\zeta\right\}$$

After normalization:

$$\Rightarrow \vec{V}_E = \frac{1}{(J-\iota I)}\frac{R_0}{\tau_A}\left\{\left[\frac{I}{\rho}\frac{\partial\Phi}{\partial\zeta} + \frac{J}{\rho}\frac{\partial\Phi}{\partial\theta}\right]\hat{e}^\rho - \left[J\frac{\partial\Phi}{\partial\rho} + \rho\beta\frac{\partial\Phi}{\partial\zeta}\right]\hat{e}^\theta + \left[\frac{\rho\beta_*}{\varepsilon}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} - \frac{I}{\varepsilon\rho}\frac{\partial\Phi}{\partial\rho}\right]\hat{e}^\zeta\right\}$$

Updated definition of the toroidal component of the vorticity including the toroidal component of the velocity:

$$U = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}\left(g_{\rho\theta}v^\rho + g_{\theta\theta}v^\theta + g_{\theta\zeta}v^\zeta\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}\left(g_{\rho\rho}v^\rho + g_{\rho\theta}v^\theta + g_{\rho\zeta}v^\zeta\right)\right]$$
$$= \frac{1}{\tau_A}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}\frac{g_{\rho\theta}}{(J-\iota I)}\left(\frac{I}{\rho}\frac{\partial\Phi}{\partial\zeta} + \frac{J}{\rho}\frac{\partial\Phi}{\partial\theta}\right)\right.$$
$$\left. - \rho_m\rho\sqrt{g}\frac{g_{\theta\theta}}{(J-\iota I)}\left(J\frac{\partial\Phi}{\partial\rho} + \rho\beta\frac{\partial\Phi}{\partial\zeta}\right) + \rho_m\rho\sqrt{g}\frac{g_{\theta\zeta}}{\varepsilon(J-\iota I)}\left(\frac{\rho\beta_*}{\varepsilon}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} - \frac{I}{\varepsilon\rho}\frac{\partial\Phi}{\partial\rho}\right)\right]$$
$$- \frac{1}{\tau_A}\frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}\frac{g_{\rho\rho}}{(J-\iota I)}\left(\frac{I}{\rho}\frac{\partial\Phi}{\partial\zeta} + \frac{J}{\rho}\frac{\partial\Phi}{\partial\theta}\right)\right.$$
$$\left. - \rho_m\sqrt{g}\frac{g_{\rho\theta}}{(J-\iota I)}\left(J\frac{\partial\Phi}{\partial\rho} + \rho\beta\frac{\partial\Phi}{\partial\zeta}\right) + \rho_m\sqrt{g}\frac{g_{\rho\zeta}}{\varepsilon(J-\iota I)}\left(\frac{\rho\beta_*}{\varepsilon}\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta} - \frac{I}{\varepsilon\rho}\frac{\partial\Phi}{\partial\rho}\right)\right]$$

Regrouping terms and normalizing:

$$U = \frac{\partial\Phi}{\partial\rho}\left\{-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}g_{\theta\theta}\frac{J}{(J-\iota I)}\right] - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}g_{\theta\zeta}\frac{I}{\varepsilon(J-\iota I)}\right]\right.$$
$$\left. + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\rho\sqrt{g}g_{\rho\theta}\frac{J}{(J-\iota I)}\right] + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\rho\sqrt{g}g_{\rho\zeta}\frac{I}{\varepsilon(J-\iota I)}\right]\right\}$$
$$+ \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\left\{\frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}g_{\theta\zeta}\frac{\beta_*}{\varepsilon(J-\iota I)}\right] + \frac{\partial}{\partial\rho}\left[\rho_m\rho\sqrt{g}g_{\rho\theta}\frac{J}{(J-\iota I)}\right]\right.$$
$$\left. - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}g_{\rho\rho}\frac{J}{(J-\iota I)}\right] - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}g_{\rho\zeta}\frac{\rho\beta_*}{\varepsilon(J-\iota I)}\right]\right\}$$
$$+ \frac{\partial\Phi}{\partial\zeta}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\sqrt{g}g_{\rho\theta}\frac{I}{(J-\iota I)}\right] - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho_m\rho^2\sqrt{g}g_{\theta\theta}\frac{\beta_*}{(J-\iota I)}\right]\right.$$
$$\left. - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}g_{\rho\rho}\frac{I}{\rho(J-\iota I)}\right] + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\rho_m\sqrt{g}g_{\rho\theta}\frac{\rho\beta_*}{(J-\iota I)}\right]\right\}$$

$$+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho\partial\theta}\left\{\rho_m\rho\sqrt{g}g_{\theta\zeta}\frac{\beta_*}{\varepsilon(J-\iota I)}+\rho_m\sqrt{g}g_{\rho\theta}\frac{J}{(J-\iota I)}+\rho_m\sqrt{g}g_{\rho\theta}\frac{J}{(J-\iota I)}+\rho_m\sqrt{g}g_{\rho\zeta}\frac{I}{\varepsilon(J-\iota I)}\right\}$$
$$+\frac{\partial\Phi}{\partial\rho\partial\zeta}\left\{\frac{\rho_m}{\rho}\sqrt{g}g_{\rho\theta}\frac{I}{(J-\iota I)}-\rho_m\rho\sqrt{g}g_{\theta\theta}\frac{\beta_*}{(J-\iota I)}\right\}$$
$$+\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta\partial\zeta}\left\{\rho\rho_m\sqrt{g}g_{\rho\theta}\frac{\beta_*}{(J-\iota I)}-\frac{\rho_m}{\rho}\sqrt{g}g_{\rho\rho}\frac{I}{(J-\iota I)}\right\}$$
$$+\frac{\partial^2\Phi}{\partial\rho^2}\left\{-\rho_m\sqrt{g}g_{\theta\theta}\frac{J}{(J-\iota I)}-\rho_m\sqrt{g}g_{\theta\zeta}\frac{I}{\varepsilon(J-\iota I)}\right\}$$
$$+\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2}\left\{-\rho_m\sqrt{g}g_{\rho\rho}\frac{J}{(J-\iota I)}-\rho_m\sqrt{g}g_{\rho\zeta}\frac{\rho\beta_*}{\varepsilon(J-\iota I)}\right\}$$

Implementation (subroutine dlstar):

```
!               Move equilibrium variables to dynamic

                call eqtodyn(wk1,jbgrr,0.0_IDP,1.0_IDP)
                call eqtodyn(wk2,jbgrt,0.0_IDP,1.0_IDP)
                call eqtodyn(wk3,jbgtt,0.0_IDP,1.0_IDP)
                call eqtodyn(wk4,jbgrz,0.0_IDP,1.0_IDP)
                call eqtodyn(wk5,jbgtz,0.0_IDP,1.0_IDP)
                call eqtodyn(wk6,bst,0.0_IDP,1.0_IDP)

!               d2Phi / drho2 components

        do l=1,lmax
                do j=1,mjm1
                        wk7(j,l)=del2cp(j)*(ff(j+1,l)-ff(j,l))+del2cm(j)*(ff(j-1,l)-ff(j,l))
                end do
        end do
        do l=1,lmax
                wk8(:,l)=-denseq*(feq*wk3(:,l)+(rinv*cureq*wk5(:,l)/eps))/(feq-  qqinv*cureq)
        end do
        call mult(ss,wk7,itypf,wk8,1,c1,c2)

!               d2Phi / dtheta2 components

        do l=1,lmax
                xm=mm(l)
                wk7(1:mj,l)=-denseq(1:mj)*feq(1:mj)*(rinv(1:mj)*xm)**2*ff(1:mj,l)/(feq(1:mj)-
qqinv(1:mj)*cureq(1:mj))
        end do
        do l=1,lmax
                wk7(0,l)=0.
        end do
        call mult(ss,wk1,1,wk7,itypf,1.0_IDP,c2)
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        do l=1,lmax
                xm=mm(l)
                wk8(1:mj,l)=-denseq(1:mj)*r(1:mj)*(rinv(1:mj)*xm)**2*ff(1:mj,l)/(eps*(feq(1:mj)-
qqinv(1:mj)*cureq(1:mj)))
        end do
        do l=1,lmax
                wk8(0,l)=0.
        end do
        call mult(wk9,wk6,-1,wk8,itypf,0.0_IDP,c2)
        call mult(ss,wk4,-1,wk9,-itypf,1.0_IDP,c2)

! !             d2Phi / drhodzeta components

        call dbydr(wk7,ff,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk7(0,l)=0.
        end do
        call dbydth(wk8,wk7,itypf,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk8(0,l)=0.
        end do
        call mult(wk9,wk3,1,wk6,-1,0.0_IDP,c2)
        do l=1,lmax
                wk10(:,l)=denseq*(cureq*rinv*wk2(:,l)-r*wk9(:,l))/(feq-qqinv*cureq)
        end do
        call mult(ss,wk8,-itypf,wk10,-1,1.0_IDP,c2)

! !             d2Phi / dthetadzeta components

        call dbydzt(wk7,ff,itypf,0.0_IDP,1.0_IDP)
        call dbydth(wk8,wk7,-itypf,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk8(0,l)=0.
        end do
        call mult(wk9,wk2,-1,wk6,-1,0.0_IDP,c2)
        do l=1,lmax
                wk10(:,l)=denseq*(r*wk9(:,l)-rinv*cureq*wk1(:,l))/(feq-qqinv*cureq)
        end do
        call mult(ss,wk8,itypf,wk10,1,1.0_IDP,c2)

! !             d2Phi / drhodtheta components

        call dbydth(wk7,ff,itypf,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk7(0,l)=0.
        end do
        call dbydr(wk8,wk7,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk8(0,l)=0.
        end do
        call mult(wk7,wk5,1,wk6,-1,0.0_IDP,c2)
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	do l=1,lmax
			wk10(:,l)=denseq*((cureq*rinv*wk4(:,l)/eps)+2*feq*wk2(:,l)+
(r*wk7(:,l)/eps))/(feq-qqinv*cureq)
	end do
	call mult(ss,wk8,-itypf,wk10,-1,1.0_IDP,c2)

! !		dPhi / drho components

	do l=1,lmax
		wk7(:,l)=denseq*r*feq*wk3(:,l)/(feq-qqinv*cureq)
	end do
	call dbydr(wk8,wk7,0.0_IDP,1.0_IDP,0)
	do l=1,lmax
		wk7(:,l)=denseq*cureq*wk5(:,l)/(eps*(feq-qqinv*cureq))
	end do
	call dbydr(wk9,wk7,0.0_IDP,1.0_IDP,0)
	do l=1,lmax
		wk7(:,l)=denseq*feq*wk2(:,l)/(feq-qqinv*cureq)
	end do
	call dbydth(wk10,wk7,-1,0.0_IDP,1.0_IDP,2)
	do l=1,lmax
		wk7(:,l)=denseq*rinv*cureq*wk4(:,l)/(eps*(feq-qqinv*cureq))
	end do
	call dbydth(wk11,wk7,-1,0.0_IDP,1.0_IDP,2)
	do l=1,lmax
		wk7(:,l)=-rinv*(wk8(:,l)+wk9(:,l)) + wk10(:,l) + wk11(:,l)
	end do
	call dbydr(wk8,ff,0.0_IDP,1.0_IDP,0)
	do l=1,lmax
		wk8(0,l)=0.
	end do
	call mult(ss,wk8,itypf,wk7,1,1.0_IDP,c2)

! !		dPhi / dtheta components

	call mult(wk9,wk6,-1,wk5,-1,0.0_IDP,c2)
	do l=1,lmax
		wk7(:,l)=denseq*r*wk9(:,l)/(eps*(feq-qqinv*cureq))
	end do
	call dbydr(wk8,wk7,0.0_IDP,1.0_IDP,0)
	do l=1,lmax
		wk7(:,l)=denseq*feq*wk2(:,l)/(feq-qqinv*cureq)
	end do
	call dbydr(wk9,wk7,0.0_IDP,1.0_IDP,0)
	do l=1,lmax
		wk7(:,l)=denseq*feq*wk1(:,l)/(feq-qqinv*cureq)
	end do
	call dbydth(wk10,wk7,1,0.0_IDP,1.0_IDP,2)
	call mult(wk11,wk6,-1,wk4,-1,0.0_IDP,c2)
	do l=1,lmax
		wk7(:,l)=denseq*r*wk11(:,l)/(eps*(feq-qqinv*cureq))
	end do
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        call dbydth(wk11,wk7,1,0.0_IDP,1.0_IDP,2)
        do l=1,lmax
                wk7(:,l)=wk8(:,l) + wk9(:,l) - wk10(:,l) - wk11(:,l)
        end do
        call dbydth(wk8,ff,itypf,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk8(0,l)=0.
        end do
        call mult(ss,wk8,-itypf,wk7,-1,1.0_IDP,c2)

! !             dPhi / dzeta components

        do l=1,lmax
                wk7(:,l)=denseq*cureq*wk2(:,l)/(feq-qqinv*cureq)
        end do
        call dbydr(wk8,wk7,0.0_IDP,1.0_IDP,0)
        call mult(wk9,wk6,-1,wk3,1,0.0_IDP,c2)
        do l=1,lmax
                wk7(:,l)=denseq*r*r*wk9(:,l)/(feq-qqinv*cureq)
        end do
        call dbydr(wk9,wk7,0.0_IDP,1.0_IDP,0)
        do l=1,lmax
                wk7(:,l)=rinv*denseq*cureq*wk1(:,l)/(feq-qqinv*cureq)
        end do
        call dbydth(wk10,wk7,1,0.0_IDP,1.0_IDP,2)
        call mult(wk11,wk6,-1,wk2,-1,0.0_IDP,c2)
        do l=1,lmax
                wk7(:,l)=r*denseq*wk11(:,l)/(feq-qqinv*cureq)
        end do
        call dbydth(wk11,wk7,1,0.0_IDP,1.0_IDP,2)
        do l=1,lmax
                wk7(:,l)=rinv*(wk8(:,l) - wk9(:,l)) - wk10(:,l) + wk11(:,l)
        end do
        call dbydzt(wk8,ff,itypf,0.0_IDP,1.0_IDP)
        call mult(ss,wk8,-itypf,wk7,-1,1.0_IDP,c2)

        do l=1,lmax
                ss(mj,l)=(ss(mjm1,l)*(r(mj)-r(mj-2))-ss(mj-2,l)*(r(mj)-r(mjm1)))/(r(mjm1)-r(mj-2))
        end do
        do l=1,lmax
                if (mm(l) == 0) ss(0,l)=(r(2)**2*ss(1,l)-r(1)**2*ss(2,l))/(r(2)**2-r(1)**2)
        end do

        wk1(:,0)=0.
        wk2(:,0)=0.
        wk3(:,0)=0.
        ss(:,0)=0.
        ff(:,0)=0.
```