FLR TERMS

Let us start by calculating the $\nabla_\perp^2$-operator in Boozer coordinates.

Perpendicular gradient:

$$\mathbf{B}\times(\nabla f\times\mathbf{B}) = B^2\nabla f - (\mathbf{B}\cdot\nabla f)\mathbf{B}$$

In dimensionless units,

$$(\nabla f\times\mathbf{B})^i = \frac{B_0}{a}\frac{1}{\sqrt{g}}\left(J\frac{1}{\rho}\frac{\partial f}{\partial\theta} - \frac{I}{\rho}\frac{\partial f}{\partial\zeta}, \rho\beta_*\frac{\partial f}{\partial\zeta} - J\frac{\partial f}{\partial\rho}, \frac{I}{\varepsilon\rho}\frac{\partial f}{\partial\rho} - \frac{\beta_*}{\varepsilon}\frac{\partial f}{\partial\theta}\right)$$

$$\begin{aligned}[\mathbf{B}\times(\nabla f\times\mathbf{B})]_i = \frac{B_0^2}{a}\frac{1}{\sqrt{g}}\Bigg[&-\rho\iota\left(\frac{I}{\rho}\frac{\partial f}{\partial\rho} - \beta_*\frac{\partial f}{\partial\theta}\right) - \left(\rho\beta_*\frac{\partial f}{\partial\zeta} - J\frac{\partial f}{\partial\rho}\right),\\ &J\frac{1}{\rho}\frac{\partial f}{\partial\theta} - \frac{I}{\rho}\frac{\partial f}{\partial\zeta}, \varepsilon\rho\iota\left(J\frac{1}{\rho}\frac{\partial f}{\partial\theta} - \frac{I}{\rho}\frac{\partial f}{\partial\zeta}\right)\Bigg]\end{aligned}$$

$$\begin{aligned}(\nabla_\perp f)_i = \frac{1}{a}\frac{1}{\sqrt{g}B^2}\Bigg[&(J-\iota I)\frac{\partial f}{\partial\rho} - \rho\beta_*\left(\frac{\partial f}{\partial\zeta} - \iota\frac{\partial f}{\partial\theta}\right),\\ &J\frac{1}{\rho}\frac{\partial f}{\partial\theta} - \frac{I}{\rho}\frac{\partial f}{\partial\zeta}, \varepsilon\rho\iota\left(J\frac{1}{\rho}\frac{\partial f}{\partial\theta} - \frac{I}{\rho}\frac{\partial f}{\partial\zeta}\right)\Bigg]\end{aligned}$$

Then the differential operator $\nabla_\perp^2 f = \nabla\cdot\nabla_\perp$ is written as

$$\begin{aligned}\nabla_\perp^2 = &\frac{1}{a^2}\frac{1}{\sqrt{g}}\Bigg\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\Bigg[\rho(J-\iota I)\frac{g^{\rho\rho}}{B^2}\frac{\partial}{\partial\rho} - \rho^2\beta_*\frac{g^{\rho\rho}}{B^2}\left(\frac{\partial}{\partial\zeta} - \iota\frac{\partial}{\partial\theta}\right)\\ &+\rho\frac{g^{\rho\theta}+\varepsilon\rho\iota g^{\rho\zeta}}{B^2}\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta} - \frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\Bigg]\\ &+\frac{1}{\rho}\frac{\partial}{\partial\theta}\Bigg[(J-\iota I)\frac{g^{\rho\theta}}{B^2}\frac{\partial}{\partial\rho} - \rho\beta_*\frac{g^{\rho\theta}}{B^2}\left(\frac{\partial}{\partial\zeta} - \iota\frac{\partial}{\partial\theta}\right)\\ &+\frac{g^{\theta\theta}+\varepsilon\rho\iota g^{\theta\zeta}}{B^2}\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta} - \frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\Bigg]\\ &+\varepsilon\frac{\partial}{\partial\zeta}\Bigg[(J-\iota I)\frac{g^{\rho\zeta}}{B^2}\frac{\partial}{\partial\rho} - \rho\beta_*\frac{g^{\rho\zeta}}{B^2}\left(\frac{\partial}{\partial\zeta} - \iota\frac{\partial}{\partial\theta}\right)\\ &+\frac{g^{\theta\zeta}+\varepsilon\rho\iota g^{\zeta\zeta}}{B^2}\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta} - \frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\Bigg]\Bigg\}\end{aligned}$$

Using the relations between upper and lower metric tensor components, we get

$$
\begin{aligned}
\nabla_\perp^2 = & \frac{1}{a^2}\frac{1}{\sqrt{g}}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\left(J-\iota I\right)g_{\theta\theta}\frac{\partial}{\partial\rho}-\rho\sqrt{g}\left(\frac{I}{\varepsilon\rho}\right)^2\frac{\partial}{\partial\rho}\right.\right.\\
& \qquad -\rho^2\beta_*\left(g_{\theta\theta}-\frac{I^2}{\varepsilon^2\rho^2}\frac{\sqrt{g}}{J-\iota I}\right)\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)\\
& \qquad \left.-\rho\left(g_{\rho\theta}-\frac{I\beta_*}{\varepsilon^2}\frac{\sqrt{g}}{J-\iota I}\right)\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta}-\frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\right]\\
& +\frac{1}{\rho}\frac{\partial}{\partial\theta}\left[-\left(Jg_{\rho\theta}-\rho^2\iota\beta_* g_{\theta\theta}-\frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\frac{\partial}{\partial\rho}\right.\\
& \qquad +\frac{\rho\beta_*}{J-\iota I}\left(Jg_{\rho\theta}-\rho^2\iota\beta_* g_{\theta\theta}-\frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)\\
& \qquad \left.+\frac{1}{J-\iota I}\left(Jg_{\rho\rho}-\rho^2\iota\beta_* g_{\rho\theta}-\frac{\rho^2\beta_*^2}{\varepsilon^2}\sqrt{g}\right)\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta}-\frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\right]\\
& +\frac{\partial}{\partial\zeta}\left[\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\frac{\partial}{\partial\rho}\right.\\
& \qquad -\frac{\rho\beta_*}{J-\iota I}\left(\frac{I}{\rho}g_{\rho\theta}-\rho\beta_* g_{\theta\theta}\right)\left(\frac{\partial}{\partial\zeta}-\iota\frac{\partial}{\partial\theta}\right)\\
& \qquad \left.\left.-\frac{1}{J-\iota I}\left(\frac{I}{\rho}g_{\rho\rho}-\rho\beta_* g_{\rho\theta}\right)\left(J\frac{1}{\rho}\frac{\partial}{\partial\theta}-\frac{I}{\rho}\frac{\partial}{\partial\zeta}\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
a^2\sqrt{g}\nabla_\perp^2 = & \frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\left[\left(J - \text{ɫ}I\right)g_{\theta\theta} - \left(\frac{I}{\varepsilon\rho}\right)^2\sqrt{g}\right]\frac{\partial}{\partial\rho}\right. \\
& \qquad \left. - \left(Jg_{\rho\theta} - \rho^2\text{ɫ}\beta_* g_{\theta\theta} - \frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\frac{\partial}{\partial\theta} + \left(Ig_{\rho\theta} - \rho^2\beta_* g_{\theta\theta}\right)\frac{\partial}{\partial\zeta}\right\} \\
& + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left\{-\left(Jg_{\rho\theta} - \rho^2\text{ɫ}\beta_* g_{\theta\theta} - \frac{I\beta_*}{\varepsilon^2}\sqrt{g}\right)\frac{\partial}{\partial\rho} + \left[\frac{J}{J - \text{ɫ}I}\left(Jg_{\rho\rho} - \rho^2\text{ɫ}\beta_* g_{\rho\theta}\right)\right.\right. \\
& \qquad \left. - \frac{\rho^2\text{ɫ}\beta_*}{J - \text{ɫ}I}\left(Jg_{\rho\theta} - \rho^2\text{ɫ}\beta_* g_{\theta\theta}\right) - \left(\frac{\rho\beta_*}{\varepsilon}\right)^2\sqrt{g}\right]\frac{1}{\rho}\frac{\partial}{\partial\theta} \\
& \qquad \left. - \left[\frac{J}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho} - \rho\beta_* g_{\rho\theta}\right) - \frac{\rho^2\text{ɫ}\beta_*}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta} - \rho\beta_* g_{\theta\theta}\right)\right]\frac{\partial}{\partial\zeta}\right\} \\
& + \frac{\partial}{\partial\zeta}\left\{\left(\frac{I}{\rho}g_{\rho\theta} - \rho\beta_* g_{\theta\theta}\right)\frac{\partial}{\partial\rho}\right. \\
& \qquad - \left[\frac{J}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho} - \rho\beta_* g_{\rho\theta}\right) - \frac{\rho^2\text{ɫ}\beta_*}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta} - \rho\beta_* g_{\theta\theta}\right)\right]\frac{1}{\rho}\frac{\partial}{\partial\theta} \\
& \qquad \left. + \left[\frac{I}{\rho}\frac{1}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\rho} - \rho\beta_* g_{\rho\theta}\right) - \frac{\rho\beta_*}{J - \text{ɫ}I}\left(\frac{I}{\rho}g_{\rho\theta} - \rho\beta_* g_{\theta\theta}\right)\right]\frac{\partial}{\partial\zeta}\right\}
\end{aligned} \tag{1}
$$

At lowest order,

$$
\nabla_\perp^2 = \frac{1}{a^2}\frac{1}{\sqrt{g}}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho g_{\theta\theta}\frac{\partial}{\partial\rho} - g_{\rho\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(-g_{\rho\theta}\frac{\partial}{\partial\rho} + g_{\rho\rho}\frac{1}{\rho}\frac{\partial}{\partial\theta}\right)\right] \tag{2}
$$