1.

a. $\mathcal{H} = \mathcal{H}(f(q_1, p_1), q_2, q_3, \dots, q_n, p_2, p_3, \dots, p_n)$

$$\{\mathcal{H}, f(q_1, p_1)\} = \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial f}{\partial q_i}$$

$$\frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial q_i}\delta_{i1} \quad , \quad \frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial p_i}\delta_{i1}$$

$$\{\mathcal{H}, f\} = \frac{\partial \mathcal{H}}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial \mathcal{H}}{\partial p_1}\frac{\partial f}{\partial q_1} = \frac{\partial \mathcal{H}}{\partial f}\frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial \mathcal{H}}{\partial f}\frac{\partial f}{\partial p_1}\frac{\partial f}{\partial q_1}$$

$$= \frac{\partial \mathcal{H}}{\partial f}\left[\frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1}\right] = 0$$

b. $V = \dfrac{\vec{a}\cdot\vec{r}}{r^3}$ , $\vec{a} = a_z\hat{z}$

$$V = \frac{\vec{a}\cdot\hat{r}}{r^2} = \frac{a_z\cos\theta}{r^2}$$

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - \frac{a_z\cos\theta}{r^2} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta) - \frac{a_z\cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mr^2\dot{\varphi}\sin^2\theta = L_\varphi \rightarrow \dot{\varphi} = \frac{L_\varphi}{mr^2\sin^2\theta} \rightarrow \dot{\varphi}^2 = \frac{L_\varphi^2}{mr^2\sin^2\theta}$$

$$\mathcal{L} = \frac{L_\varphi^2}{2} + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{a_z\cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} = p_r$$

$$\mathcal{L}' = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{a_z\cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta$$

$$\mathcal{H} = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + \frac{r^2p_\theta^2}{m^2r^4}\right) + \frac{a_z\cos\theta}{r^2}$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{a_z\cos\theta}{r^2} = \frac{p_r^2}{2m} + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + a_z\cos\theta\right]$$

$$\frac{p_\theta^2}{2m} + a_z\cos\theta = f(\theta, p_\theta) = cte$$

$E = cte$

2. $H = \frac{1}{2}p^2 - \frac{1}{2q^2}$    $f_1 = pq - 2Ht$

a.

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + \{f_1, H\} = \frac{\partial f_1}{\partial t} + \{pq, H\} - 2t\{H, H\}$$

$$= \frac{\partial f_1}{\partial t} + \{pq, H\}$$

$$\frac{\partial f_1}{\partial t} = -2H$$

$$\{pq, H\} = p\frac{\partial H}{\partial p} - q\frac{\partial H}{\partial q} = p^2 - \frac{1}{q^2} = 2H$$

$$\frac{df_1}{dt} = -2H + 2H = 0$$

b. $Q = \lambda q$, y $P = \lambda^{-1}p$

$\{Q, P\} = \lambda\lambda^{-1} - 0 = 1$ , $\{Q, Q\} = 0$ , $\{P, P\} = 0$

$Q = (1+\epsilon)q$    $P = (1+\epsilon)^{-1}p$    $(1+\epsilon)^{-1} \approx 1-\epsilon$

$Q = q + \epsilon q$    $P = p - \epsilon p$

$q + \epsilon\frac{\partial G}{\partial P}$    $P = p - \epsilon\frac{\partial G}{\partial q}$

$q = \frac{\partial G}{\partial P}$    $p = \frac{\partial G}{\partial q} \to pq = G_2$

$q = \frac{\partial G}{\partial P} \to qP = G_1 \to$    $G = pq$

3. $V(\vec{r}) = \frac{\vec{a} \cdot \vec{r}}{r^3}$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$$

a. $\{\vec{r}\,\vec{p}, \mathcal{H}\} = \{r p_r + \theta p_\theta + \varphi p_\varphi, \mathcal{H}\}$

Como $\mathcal{H}$ no depende de $\varphi$

$$\frac{p_r^2}{2m} = \frac{-1}{r^2} C_1 + E$$

$$\{\vec{r}\,\vec{p}, \mathcal{H}\} = \{r p_r + \theta p_\theta, \mathcal{H}\}$$

$$\pm p_r = \sqrt{\frac{-2m}{r^2} C_1 + 2mE}$$

$$\{r p_r, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p_r} p_r - r\frac{\partial \mathcal{H}}{\partial r}$$

$$\frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m} \qquad \frac{\partial \mathcal{H}}{\partial r} = -\frac{2}{r^3}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$$

$$\{r p_r, \mathcal{H}\} = \frac{p_r^2}{m} + \frac{2}{r^2}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right] = 2\mathcal{H}$$

$$\{r p_r - 2\mathcal{H}t, \mathcal{H}\} = \{r p_r, \mathcal{H}\} - 2\{\mathcal{H}t, \mathcal{H}\} = 2\mathcal{H} - 2\{\mathcal{H}, \mathcal{H}\} - 2\frac{\partial \mathcal{H}t}{\partial t}$$

$$= 2\mathcal{H} - 2\mathcal{H} = 0$$

Como el corchete de poisson es igual a cero con $r p_r - 2\mathcal{H}t$, eso implica que es una cantidad conservada en el tiempo

$$p_i = P_i + \epsilon \frac{\partial (r p_r - 2\mathcal{H}t)}{\partial q_i} \qquad q_i = Q_i - \epsilon \frac{\partial (r p_r - 2\mathcal{H}t)}{\partial p_i}$$

$$p_r = P_r + \epsilon\left[p_r + \frac{2}{r^3}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]\right]$$

$$q_r + \epsilon\left[r - \frac{2 p_r t}{m}\right] = Q_r$$

$$S = \pm r\sqrt{\frac{-2m}{r^2} C_1 + 2mE} - 2Et$$