1.

a. $\mathcal{H} = \mathcal{H}(f(q_1, p_1), q_2, q_3, \dots, q_n, p_2, p_3, \dots, p_n)$

$$\{\mathcal{H}, f(q_1, p_1)\} = \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial f}{\partial q_i}$$

$$\frac{\partial f}{\partial q_i} = \frac{\partial f}{\partial q_i}\delta_{i1} \quad , \quad \frac{\partial f}{\partial p_i} = \frac{\partial f}{\partial p_i}\delta_{i1}$$

$$\{\mathcal{H}, f\} = \frac{\partial \mathcal{H}}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial \mathcal{H}}{\partial p_1}\frac{\partial f}{\partial q_1} = \frac{\partial \mathcal{H}}{\partial f}\frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial \mathcal{H}}{\partial f}\frac{\partial f}{\partial p_1}\frac{\partial f}{\partial q_1}$$

$$\frac{\partial \mathcal{H}}{\partial f}\left[\frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1} - \frac{\partial f}{\partial q_1}\frac{\partial f}{\partial p_1}\right] = 0$$

b. $V = \frac{\vec{a} \cdot \vec{r}}{r^3}$, $\vec{a} = a_z \hat{z}$

$$V = \frac{\vec{a} \cdot \hat{r}}{r^2} = \frac{a_z \cos\theta}{r^2}$$

$$\mathcal{L} = \frac{1}{2} m \dot{r}^2 - \frac{a_z \cos\theta}{r^2} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta) - \frac{a_z \cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \sin^2\theta = L_\varphi \to \dot{\varphi} = \frac{L_\varphi}{m r^2 \sin^2\theta} \to \dot{\varphi}^2 = \frac{L_\varphi^2}{m r^2 \sin^2\theta}$$

$$\mathcal{L} = \frac{L_\varphi^2}{2} + \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{a_z \cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} = p_r$$

$$\mathcal{L}' = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{a_z \cos\theta}{r^2}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} = p_\theta$$

$$\mathcal{H} = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{1}{2} m\left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4}\right) + \frac{a_z \cos\theta}{r^2}$$

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} + \frac{a_z \cos\theta}{r^2} = \frac{p_r^2}{2m} + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$$

$$\frac{p_\theta^2}{2m} + a_z \cos\theta = f(\theta, p_\theta) = \text{cte}$$

$E = \text{cte}$

2. $\mathcal{H} = \frac{1}{2}p^2 - \frac{1}{2q^2}$ $\qquad f_1 = pq - 2\mathcal{H}t$

a.

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + \{f_1, \mathcal{H}\} = \frac{\partial f_1}{\partial t} + \{pq, \mathcal{H}\} - 2t\{\mathcal{H}, \mathcal{H}\}$$

$$= \frac{\partial f_1}{\partial t} + \{pq, \mathcal{H}\}$$

$$\frac{\partial f_1}{\partial t} = -2\mathcal{H}$$

$$\{pq, \mathcal{H}\} = p\frac{\partial \mathcal{H}}{\partial p} - q\frac{\partial \mathcal{H}}{\partial q} = p^2 - \frac{1}{q^2} = 2\mathcal{H}$$

$$\frac{df_1}{dt} = -2\mathcal{H} + 2\mathcal{H} = 0$$

b. $Q = \lambda q$, y $P = \lambda^{-1}p$

$$\{Q, P\} = \lambda\lambda^{-1} - 0 = 1, \quad \{Q, Q\} = 0, \quad \{P, P\} = 0$$

Supongamos $F_2 = qP + \epsilon G(q, P)$

$$p = \frac{\partial F_2}{\partial q} = P + \epsilon\frac{\partial G}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} = q + \epsilon\frac{\partial G}{\partial P}$$

$$\lambda P = P + \epsilon\frac{\partial G}{\partial q} \qquad \lambda q = q + \epsilon\frac{\partial G}{\partial P}$$

$$(\lambda - 1)P = \epsilon\frac{\partial G}{\partial q} \qquad (\lambda - 1)q = \epsilon\frac{\partial G}{\partial P}$$

$$P = \frac{\partial G}{\partial q} \qquad q = \frac{\partial G}{\partial P} \qquad \epsilon = \lambda - 1$$

$$Pq = G \qquad qP = G \rightarrow G = Pq \rightarrow G = pq \rightarrow f_1 = G - 2\mathcal{H}t$$

3. $V(\vec{r}) = \frac{\vec{a} \cdot \vec{r}}{r^3}$ $H = \frac{p_r^2}{2m} + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$

a. $\{\vec{r}\cdot\vec{p}, H\} = \{r p_r + \theta p_\theta + \varphi p_\varphi, H\}$

Como H no depende de φ

$\{\vec{r}\cdot\vec{p}, H\} = \{r p_r + \theta p_\theta, H\}$

$\{r p_r, H\} = \frac{\partial H}{\partial p_r} p_r - r\frac{\partial H}{\partial r}$

$\{\theta p_\theta, H\} = \frac{\partial H}{\partial p_\theta} p_\theta - \theta\frac{\partial H}{\partial \theta}$

$\frac{\partial H}{\partial p_r} = \frac{p_r}{m}$ $\frac{\partial H}{\partial r} = -\frac{1}{r^3}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$

$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2}$ $\frac{\partial H}{\partial \theta} = -\frac{a_z \sin\theta}{r^2}$

$\{r p_r, H\} = \frac{p_r^2}{m} + \frac{1}{r^2}\left[\frac{p_\theta^2}{2m} + a_z \cos\theta\right]$

$\{\theta p_\theta, H\} = \frac{p_\theta^2}{m r^2} + \frac{\theta\, a_z \sin\theta}{r^2}$