

# TOPIC 4 NONLINEAR PROGRAMMING

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# Quadratic Programming

- A special type of NLP is called **quadratic programming**
  - Allows quadratic terms in the objective
- The general form of **QP** is
  - $\min_x x^T Q x + c^T x$
  - s.t.  $A x \leq b, x \geq 0$
- If  $x$  has  $n$  decision variables, then  $Q$  is an  $n \times n$  matrix, and  $c$  is  $n \times 1$  (just like for LP)
- In gurobi, the only extra step is to tell  $Q$  to the objective

# Quadratic Form

- What is  $(x_1 \quad x_2) \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + (c_1 \quad c_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ?

# Portfolio Optimization

- Some of you may have seen Markowitz portfolio optimization before
  - Minimize risk (volatility) subject to mean return constraint
  - It is the basis for the finance project
- We will pose it as a QP
- Ordinary Least Squares Regression is also a QP
  - This will be the basis for the non-finance project

# Portfolio Optimization

- Our objective is to pick portfolio weights to get the portfolio's variance/standard deviation to be as small as possible
- Our constraints are
  - Portfolio weights must sum to 1
  - No short selling stock (no negative weights)
  - The mean return of the portfolio must be bigger than some threshold,  $R$

# Portfolio Variance

- We can write the portfolio variance using the quadratic form we saw earlier
  - $Var(\sum_{i=1}^n w_i r_i) = w^T \Sigma w$
  - $\Sigma$  is the covariance matrix:  $\Sigma_{ij} = Cov(r_i, r_j)$
- Why:  $cov(aX+bY) = a^2 var(X) + b^2 var(Y) + 2 ab cov(X,Y)$

# Portfolio Mean

- The portfolio's mean return is
  - $E(\sum_{i=1}^n w_i r_i) = \sum_{i=1}^n w_i E[r_i] = \bar{r}^T w$
  - $\bar{r}$  is the vector of each stock's mean return

# Portfolio Optimization

- We can write the optimization problem as
- $\min_w w^T \Sigma w$
- s.t.
  - $\bar{r}^T w \geq R$
  - $1^T w = 1$
  - $w \geq 0$



# Example

- Assume there are 3 stocks with mean returns:
  - (10.73%, 7.37%, 6.27%)
- The covariance matrix between the 3 stocks is
  - $\Sigma = \begin{pmatrix} 0.0278 & 0.00387 & 0.000207 \\ 0.00387 & 0.0111 & -0.000195 \\ 0.000207 & -0.000195 & 0.00116 \end{pmatrix}$
- Find the portfolio with the smallest possible variance that achieves a mean return of at least 7%

# Quadratic Programming

- If you choose to do the finance project 3, you should look at how I pose this as a general NLP and solve it using scipy
- Topic4NLP-03.ipynb is posted on canvas
- For the project you just slightly modify the objective and mean return constraint in that code