Stat 958:587 Homework No.2

Due: 10/28/2021

Note: from now on, feel free to use random number generators for common distributions to do the homework. (Unless the question itself is about it)

In class discussion schedule

Problem 1 and 2. 10/7
Problem 3 and 4. 10/14
Problem 5. 10/21

Problem 1:

Consider the following joint density function $f(\beta, Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75})$, whose fully conditioned distribution functions are:

- $\beta | Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75} \sim N\left(\frac{\sum_{i=1}^{75} \lambda_i Z_i}{\sum_{i=1}^{75} \lambda_i}, \frac{1}{\sum_{i=1}^{75} \lambda_i}\right)$.
- For $i = 1, \dots, 50$,
  $Z_i | \beta, \lambda_1, \dots, \lambda_{75} \sim$ left truncated normal at 0:
  $$f_L(t; \beta, 0, \frac{1}{\lambda_i}) = \begin{cases} 0, & \text{if } t \le 0; \\ \dfrac{e^{-\frac{\lambda_i}{2}(t-\beta)^2}}{\sqrt{\frac{2\pi}{\lambda_i}}[\Phi(\beta\sqrt{\lambda_i})]}, & \text{if } t > 0. \end{cases}$$
  For $i = 51, \dots, 75$,
  $Z_i | \beta, \lambda_1, \dots, \lambda_{75} \sim$ right truncated normal at 0:
  $$f_R(t, \beta, 0, \frac{1}{\lambda_i}) = \begin{cases} \dfrac{e^{-\frac{\lambda_i}{2}(t-\beta)^2}}{\sqrt{\frac{2\pi}{\lambda_i}}[\Phi(-\beta\sqrt{\lambda_i})]}, & \text{if } t \le 0; \\ 0, & \text{if } t > 0. \end{cases}$$
- For $i = 1, \dots, 75$,
  $$\lambda_i | \beta, Z_1, \dots, Z_{75} \sim \Gamma\left(\frac{5}{2}, \frac{2}{4 + (Z_i + \beta)^2}\right).$$

(a) Consider the following proposals on sampling from the first 50 left truncated normal. Are they valid or invalid? Try to briefly explain your answer without doing numerical experiments.

① Sampling $Z_i \sim N(\beta, \frac{1}{\lambda_i})$, if $Z_i > 0$, just use Z_i, otherwise, set $Z_i = 0$.

② Sampling $Z_i \sim N(\beta, \frac{1}{\lambda_i})$, and use $|Z_i|$.

③ Repeat sampling $Z_i \sim N(\beta, \frac{1}{\lambda_i})$ until $Z_i > 0$. Use that Z_i.

(b) Write a Gibbs algorithm to simulate from $f(\beta, Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75})$. Plot the trace and density of: $\beta, Z_1, Z_{60}, \lambda_1$. (Be careful: How will you generate IID samples?)

(c) Based on the Monte Carlo method, the marginal density of β, say $f(\beta)$, can be estimated by

$$\hat{f}(\beta) = \frac{1}{T} \sum_{t=1}^{T} f(\beta | Z_1^{(t)}, \dots, Z_{75}^{(t)}, \lambda_1^{(t)}, \dots, \lambda_{75}^{(t)}),$$

where $(Z_1^{(t)}, \dots, Z_{75}^{(t)}, \lambda_1^{(t)}, \dots, \lambda_{75}^{(t)})$, $t = 1, 2, \dots, T$, are T independent samples. This approximation is based on approximating the distribution of $(Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75})$ $F(Z, \lambda)$ by their empirical distribution $\hat{F}_T(Z, \lambda)$, and then marginalizing them out:

$$\begin{aligned} f(\beta) &= \int f(\beta | Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75})\, dF(Z, \lambda) \\ &\approx \int f(\beta | Z_1, \dots, Z_{75}, \lambda_1, \dots, \lambda_{75})\, d\hat{F}_T(Z, \lambda) \\ &= \sum_{t=1}^{T} f\left(\beta \middle| Z_1^{(t)}, \dots, Z_{75}^{(t)}, \lambda_1^{(t)}, \dots, \lambda_{75}^{(t)}\right) \times \frac{1}{T} := \hat{f}(\beta). \end{aligned}$$

Use your result in (b) and the above approximation to estimate the probability that β lies in [0,1], that is:

$$F(1.0) - F(0) \approx \int_0^{1.0} \hat{f}(\beta) d\beta \approx ?$$

Problem 2:

We want to estimate the integral

$$I = \int_{-\infty}^{\infty} x^3 p(x) dx,$$

where $p(x)$ is the density of $N(2,1)$.

(a) Calculate the exact value of I.

(b) Now we are going to estimate this quantity by a Monte-Carlo approximation based on different samples of size $n = 5000$.

- Using Metropolis-Hasting with proposed distribution $g(\cdot\, | x_k) = N(2,1)$.
- Using Metropolis-Hasting with proposed distribution $g(\cdot\, | x_k) = N(0,1)$.
- Using Metropolis-Hasting with proposed distribution $g(\cdot\, | x_k) = N(x_k, 1)$.
- Using Metropolis-Hasting with proposed distribution $g(\cdot\, | x_k) = N(x_k, 5)$.

Implement the algorithms and compare the estimated (average) acceptance rates and estimates of I that you obtain.

Problem 3:

(a) Write a program to simulate bivariate Gaussian copulas with correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

(b) Write a program to simulate bivariate t-copulas with correlation matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ and ν degrees of freedom.

(c) Use your program in (a) and (b) to simulate 1000 samples of two Gaussian copulas with $\rho = 0.25$ and $\rho = 0.8$, respectively, and two t-copulas with $(\rho = 0.25, \nu = 3)$ and $(\rho = 0.8, \nu = 5)$, respectively. Draw the scatter plots and compare the results. To simplify the problem, use marginal distribution as N(0, 1).

Problem 4:

Suppose $X_1, \cdots, X_n \sim^{iid} Uniform[0, \theta]$. We would like to estimate and make inference for θ.

(a) Find the MLE $\hat{\theta}^{MLE}$ for θ and compute the density of $\hat{\theta}^{MLE}$.

(b) Let the underlying truth be θ=1. Suppose we are only able to get n=60 samples. Generate your own observations $x_1 \cdots x_{60}$. Based on them we can do bootstrap and get the bootstrap MLE $\widehat{\theta^*}$. Bootstrap for $N = 1000$ times and plot the density of $\widehat{\theta^*}$. Compare the distribution of $\hat{\theta}^{MLE}$ and $\widehat{\theta^*}$ according to the density plot you obtain. Are they symmetric of asymmetric?

(c) Based on $\hat{\theta}^{MLE}$ and 1000 $\widehat{\theta^*}$ you get in (b), find $P\left(\hat{\theta}^{MLE} = 1\right)$ and $P\left(\widehat{\theta^*} = \hat{\theta}^{MLE}\right)$.

(d) Based on the 1000 $\widehat{\theta^*}$ obtained in (b), build the bootstrap 90% confidence interval for θ. Simulate the data for $N = 1000$ times and get the 90% confidence interval for θ based on the $\hat{\theta}^{MLE}$. Compare these two intervals.

(e) This example shows that bootstrap does poorly. In fact, try to prove $P\left(\hat{\theta}^{MLE} = 1\right) = 0$ and $P\left(\widehat{\theta^*} = \hat{\theta}^{MLE}\right) = 1 - (1 - 1/n)^n$. You may use these two facts to check whether you have done correctly in (c).

Problem 5:

The following table gives for ten power plant pumps the recorded number of failures Y_i and the length t_i of operation time (in 1000s of hours):

Pump i	1	2	3	4	5	6	7	8	9	10
t_i	94.3	15.7	62.9	126	5.24	31.4	1.05	1.05	2.1	10.5
Y_i	5	1	5	14	3	19	1	1	4	22

The number of failures is assumed to follow a Poisson distribution

$$Y_i \sim \text{Poisson}(\theta_i t_i), \qquad i = 1, \dots, 10,$$

where θ_i is the failure rate for pump i. For Bayesian inference we adopt conjugate gamma prior distribution for the failure rates

$$\theta_i \sim \Gamma(\alpha, \beta), \qquad i = 1, \dots, 10.$$

Instead of specifying values for both hyperparameters α and β, we assume the following hierarchical prior specification

$$\alpha = 1, \qquad \beta \sim \Gamma\left(\frac{1}{5}, 1\right).$$

(a) Compute the full conditional posterior distributions for θ_i and β. (Hint: use the trick of "∝" in calculating Bayesian posterior.)

(b) Based on the results from (a), formulate the Gibbs sampler for this problem.

(c) Implement the Gibbs sampler and apply it to the pump data. Plot the marginal posterior densities of β, θ_1 and θ_2.