Math Review – Exam for Linear Algebra

Fall 2020

1. The exam lasts **2 hours: 10am-12pm 10/10/2020 EST**.
2. The exam is closed-book and closed-notes. You can use your calculator to solve the cubic equation in 7(a).
3. Please place your camera facing toward you and your desk and unmute your mic.
4. Write your answers on white blank papers (no grid or lines).
5. Scan your answers into **ONE pdf file** and submit it by **12:15pm**.
6. Show all your work; answers without procedure will NOT be credited.
7. Write CLEARY!!!
8. 55 points in total.

1. Please indicate if you want to include your homework submissions in the assessment

○ Yes. Your final grade = 15% Quiz + 30% Homework + 55% Exam.

○ No. Your final grade = 15% Quiz + 85% Exam.

How I calculate your final grade:

If	Q1	Q2	Q3	HW1	HW2	HW3	Exam	final
Yes	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$x_1 + \cdots + x_7$
No	x_1	x_2	x_3				x_7	$x_1 + x_2 + x_3 + x_7 \times (85/55)$

2. (6 points) Find $\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^k$ for any positive integer k. (4 points for $k = 2, 3$)

3. (10 points) Let A, P be $n \times n$ matrices and P is invertible. Let $B = PAP^{-1} - P^{-1}AP$.

   (a) (3 points) Verify $tr(C + D) = tr(C) + tr(D)$ for any square matrices C, D.

   (b) (5 points) Using (a) to find $tr(B)$.

   (c) (2 points) Find the sum of eigenvalues of B.

4. (6 points)

   (a) (2 points) Let V be the set of all 3×3 real matrices with rank 1. Provide an example to show V is not a vector space.

(b) (4 points) Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 real matrices. For any $A \in \mathbb{R}^{2\times 2}$, define a mapping f from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ as

$$f(A) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show f is a linear transformation.

5. (13 points) Let

$$A = \begin{pmatrix} 1 & 1 & -1 & -2 \\ 5 & 8 & -2 & -3 \\ 3 & 9 & 3 & 8 \end{pmatrix}$$

(a) (6 points) Using Gram-Schmidt process to find an **orthogonal** basis for the **row** space of A.

(b) (2 points) Find the nullity of A.

(c) (3 points) Find the orthogonal projection of vector $\boldsymbol{w} = (-9, 13, 7, -5)^\top$ on the row space of A.

(d) (2 points) Using (c) to find the orthogonal projection of vector $\boldsymbol{w} = (-9, 13, 7, -5)^\top$ on the null space of A.

6. (5 points) Let $\boldsymbol{u} \in \mathbb{R}^n$ be an eigenvector of A corresponding to eigenvalue λ. For any vector $\boldsymbol{v} \in \mathbb{R}^n$, show $\boldsymbol{u}$ is also an eigenvector of $A + \boldsymbol{u}\boldsymbol{v}^\top$ and find its corresponding eigenvalue.

7. (15 points) Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{pmatrix}.$$

(a) (5 points) Find all eigenvalues of A and their corresponding eigenvectors.

(b) (4 points) Diagonalize A.

(c) (3 points) Find A^{-k} for any positive integer k. You can write it as a product of three matrices, that is, $A^{-k} = BCD$.

(d) (3 points) Find the determinant of matrix $B = 2AA^\top$.