FSRM/MSDS 581 Homework 8
Due on Wednesday 11:59pm EST, 11/18

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d following the F-distribution F_{d_1,d_2}, where d_1, d_2 are unknown parameters (real numbers). Use the first two moments to obtain the MOM estimator of $\theta = (d_1, d_2)$.

2. Let (x_i, Y_i) $(i = 1, \ldots, n)$ be independent following the regression model (without intercept)

$$Y_i = \beta_1 X_i + \varepsilon_i$$

where ε_i i.i.d with mean 0 and variance 1. The unknown parameter is $\theta = \beta_1$.

Let

$$\begin{aligned} g_1^*(x_i, Y_i, \theta) &= Y_i - \beta_1 x_i \\ g_2^*(x_i, Y_i, \theta) &= x_i(Y_i - \beta_1 x_i) \\ g_3^*(x_i, Y_i, \theta) &= x_i^2(Y_i - \beta_1 x_i) \end{aligned}$$

(a) Show $E(g_j^*(x_i, Y_i, \theta)) = 0$ for $j = 1, 2, 3$.

(b) Obtain the sample average $m_j^*(\theta)$ corresponding to $g_j^*(x_i, Y_i, \theta)$ for $j = 1, 2, 3$.

(c) Let the weight matrix W be a diagonal matrix of 1's. write down the objective function for GMM

$$\phi(\theta) = -[\boldsymbol{m}^*(\theta)]^T W[\boldsymbol{m}^*(\theta)].$$

(d) Find the GMM estimator for θ by maximizing $\phi(\theta)$. (Hint: $\phi(\theta)$ is a quadratic function of β_1. You should be able to find the solution analytically).

(e) Suggest another function that can be used in the GMM.

3. Let X_1, $X_2, \ldots, X_n$ be i.i.d following Uniform(θ_1, θ_2).

   (a) Use factorization theorem to show that $T = (X_{(1)}, X_{(n)})$ is sufficient for $\theta = (\theta_1, \theta_2)$, where $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$.

   (b) Show $T = (X_{(1)}, X_{(n)})$ is minimal sufficient.

   (c) Find MLE of θ_1 and θ_2.

4. Let (x_i, Y_i) $(i = 1, \ldots, n)$ be independent following the **Poisson regression** model (without intercept)

$$Y_i \sim Poission(\lambda x_i)$$

   (This is often used to model number of events occurred given a covariate. For example, the number of traffic accidents in a month as Y_i at the i-th intersection vs the number of cars going through the same intersection in the same period, as x_i)

   (a) Since $E(Y_i) = \lambda x_i$, we can write $Y_i = \lambda x_i + e_i$ where $E(e_i) = 0$. Find the least square estimator for λ.

   (b) Write the likelihood function $L(\lambda)$ (treat $x_1, \ldots, x_n$ as non-random)

   (c) Find MLE for λ.

   (d) Suppose we use Gamma(α, β) with pdf

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \quad \text{for} \quad \lambda > 0$$

   as the prior distribution. Find the posterior distribution given observations $(y_1, x_1), \ldots, (y_n, x_n)$. And find the corresponding Bayes estimator for λ. [Hint: The prior is a conjugate prior so the posterior distribution is also a Gamma distribution. The mean of the Gamma(a, b) distribution is a/b]