FSRM/MSDS 581 Homework 5
Due on Wednesday 11:59pm EST, 10/14

1. Suppose (X_1, X_2) follow

$$f_{X_1,X_2}(x_1, x_2) = e^{-(x_1+x_2)}, \text{ for } x_1 > 0, x_2 > 0$$

Find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/(X_1 + X_2)$.

2. Let $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$. Assume X_1 and X_2 are independent. Use MGF to show that $Y = X_1 + X_2 \sim Poisson(\lambda_1 + \lambda_2)$

3. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean μ and variance σ^2. Let N be a random variable following a poisson distribution with mean λ and variance λ. Let $Y = X_1 + X_2 + \ldots + X_N$.

   - (a) Show that $E(Y \mid N = n) = n\mu$ and $var(Y \mid N = n) = n\sigma^2$
   - (b) Let $Z = E(Y \mid N)$, the conditional expectation of Y given N. Find the mean and variance of Z.
   - (c) Show that $E(Y) = \lambda\mu$
   - (d) Find $var(Y)$

4. For a randomly selected family with two parents and one male child and one female child, let X_1 be the average height of the parents in inches. Let X_2 be the height of the (adult) male child and X_3 be the height of the (adult) female child. Assume that (X_1, X_2, X_3) follow a multivariate normal distribution with mean and covariance matrix

$$\mu = \begin{pmatrix} 65 \\ 70 \\ 60 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 10^2 & \frac{1}{2}10^2 & \frac{1}{2}10^2 \\ \frac{1}{2}10^2 & 10^2 & \frac{1}{4}10^2 \\ \frac{1}{2}10^2 & \frac{1}{4}10^2 & 10^2 \end{bmatrix}$$

- (a) What are the marginal distributions of X_2 and X_3? What are their corresponding means?

- (b) What is the correlation between the average parent height and the height of the male child? what is the correlation between the average parent hight and the height of the female child?

- (c) What is the correlation between the heights of the two children?

- (d) Conditioned on the mid-parent height $X_1 = 75$ (tall parents), what is the joint distribution of (X_2, X_3)?

- (e) From your answer to (d), what is the mean height of the male child given $X_1 = 75$? What is the mean height of the female child given $X_1 = 75$? Compare them with your answer in (d) and provide possible reasons.

- (f) From your answer to (d), what is the correlation between the heights of the children once we know $X_1 = 75$? [**This is called conditional correlation.**] Comparing it with your answer in (c), are you surprised? Provide some possible explanations.

5. Let W, X, Y, Z be four random variables. Define:

$$E(Y|X = x, Z = z) = \int y f_{Y|X,Z}(y|X = x, Z = z) dy$$

$$Var(Y|X) = E\Big[\Big(Y - E(Y|X)\Big)^2 \Big| X\Big]$$

$$Var(Y|X, Z) = E\Big[\Big(Y - E(Y|X, Z)\Big)^2 \Big| X, Z\Big]$$

$$Cov(X, Y|Z) = E\Big[\Big(X - E(X|Z)\Big)\Big(Y - E(Y|Z)\Big) \Big| Z\Big]$$

$$Cov(X, Y|W, Z) = E\Big[\Big(X - E(X|W, Z)\Big)\Big(Y - E(Y|W, Z)\Big) \Big| W, Z\Big]$$

Prove the following equations.

(a) $E(f(X)|X) = f(X)$, $Var(f(X)|X) = 0$,

$E[f(X)Y|X] = f(X)E[Y|X]$, $Var[f(X)Y|X] = f^2(X)Var(Y|X)$,

where $f(\cdot)$ is an arbitrary function.

(b) $E(Y) = E[E(Y|X)]$

$Var(Y) = Var[E(Y|X)] + E[Var(Y|X)]$,

$Cov(X,Y) = Cov[E(X|Z), E(Y|Z)] + E[Cov(X,Y|Z)]$

(c) $E(Y|X) = E[E[Y|X,Z]|X]$,

$Var(Y|X) = Var[E(Y|X,Z)|X] + E[Var(Y|X,Z)|X]$,

$Cov(X,Y|Z) = Cov[E(X|W,Z), E(Y|W,Z)|Z] + E[Cov(X,Y|W,Z)|Z]$