FSRM/MSDS 581 Homework 3
Due on Wednesday 11:59pm EST, 9/30

1. Suppose X follows a Poisson distribution with parameter λ.

   - (a) Find $E\left(\frac{1}{X+1}\right)$.

     [**Hint**]: Use the facts that $\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = 1$ and $(x+1)x! = (x+1)!$

   - (b) Let $Y = 1/(X+1)$. Find $P(Y > 1/4)$

   - (c) Let $Z = \min(2, X)$. Find the pmf of Z.

2. Let $X \sim N(\mu, \sigma^2)$. Let $Y = e^X$. (Y is said to follow a lognormal distribution since $log(Y) = X$ which follows a normal distribution).

   - (a) Find the pdf of Y.

   - (b) Find $E(Y)$

     [**Hint:**] (i) It is easier to use the rule of the lazy statistician; (ii) Re-arrange the integrand inside the integral into a Normal pdf (with different mean and variance) and constant, and note that the integral of a pdf is always one.

   - (c) If σ^2 is small, find a normal distribution approximation for Y, using the delta method.

3. Suppose X follows an exponential distribution $X \sim \text{Exp}(1)$, with pdf $f_X(x) = e^{-x}$ for $x > 0$.

   - (a) Let $Y = \beta X$. Find the pdf of Y. (Note: Y should follow $Y \sim \text{Exp}(\beta)$.

   - (b) Suppose U follows a uniform distribution on $[0, 1]$. Find a function $g(\cdot)$ so that $X^* = g(U) \sim \text{Exp}(1)$.

     [**Hint:**] We showed a general approach to do this in class.

- (c) Find a function $g(\cdot)$ so that $X^* = g(U) \sim \text{Exp}(\beta)$.

  [Hint:] Combine the results in (a) and (b).

- (d) [Use computer] Generate 10,000 random samples from Uniform $[0,1]$. (In R, you can do `x=runif(10000)`). Use the result in (b) to obtain 10,000 random samples from Exp(1). Draw a histogram and compare with the pdf $f_X(x) = e^{-x}$ for $x > 0$. (In R, you can do `hist(y)`).

4. Rejection sampling

The rejection sampling is a basic technique used to generate observations from a distribution. It is also commonly called the acceptance-rejection method or "accept-reject algorithm". It is a basic unit for the modern (and very popular) sampling method Markov Chain Monte Carlo (MCMC).

Given a density function $f(x)$, the rejection sampling generates data points from this distribution using the following procedure.

(i) Choose a proposal density $g(x)$ which we know how to draw sample from (for example, $g(x)$ can be the density of a standard normal distribution) as well as a number $M \geq \sup_x \frac{f(x)}{g(x)}$.

(ii) Generate a random number Y from g and another random number U from $Unif[0,1]$.

(iii) If $U < \frac{f(Y)}{Mg(Y)}$, we set $X = Y$. Otherwise go back to the previous step to draw another new pair of Y and U.

If we want to generate $X_1, \ldots, X_n$ from f, we can apply the above procedure multiple times until we accept n points.

(1) Calculate the probability that a generated Y is accepted.

(2) Show that the points generated using the rejection sampling method follow the distribution $f(x)$.