FSRM/MSDS 581 Homework 6
NOT DUE

1. Suppose $X \sim Poisson(\lambda)$. Use Chebyshev's inequality to show that $P(X \geq 2\lambda) \leq 1/\lambda$.

2. Let X has mean μ. We say that X is sub-Gaussian if there exists σ^2 such that

$$log(E[e^{t(X-\mu)}]) \leq \frac{t^2\sigma^2}{2}.$$

   Suppose X has mean μ and is sub-Gaussian, and Y has mean ν and is sub-Gaussian. And suppose X and Y are independent. Show that $X + Y$ is subGaussian.

3. Use Jensen's inequality to show that, for $a_i > 0$, $i = 1, \ldots, n$

$$(a_1 a_2 \ldots a_n)^{1/n} \geq \frac{1}{\frac{1}{n}\left(\frac{1}{a_1} + \ldots + \frac{1}{a_n}\right)}$$

4. Let $Z \sim N(0, 1)$.

   - (a) Use Markov's inequality to show that

$$P(|Z| > t) \leq \frac{E|Z|^k}{t^k}$$

     for all $k > 0$.

   - (b) Compare the answer in (a) with Gaussian Tail Inequality

$$P(|Z| > t) \leq \frac{2e^{-t^2/2}}{t}$$

     when t is very large. [Hint: Which inequality is 'shaper'?]

   - (c) Use computer, plot the exact values of the function $P(|Z| > t)$ on a grid points in $[2, 5]$, in log scale. (e.g. `x=2+(0:30)/10; plot(x,log(1-2*pnorm(x)),ylim=c(-14,0))`). Then plot the bound from the Gaussian Tail Inequality, and the bounds from (a) using $k = 1, 3, 5$, on the same plot in log scale. (e.g. `lines(x,log(2)-x*x/2-log(x))`). Note that $E(|Z|^k) = 2^{k/2}\Gamma((k+1)/2)/\sqrt{\pi}$.

- (d) Comment on the figure in (c).

5. Let X_1, $X_2, \dots,$ be a sequence of random variables such that

$$P\left[X_n = \frac{1}{n}\right] = 1 - \frac{1}{n^2} \quad \text{and} \quad P[X_n = n] = \frac{1}{n^2}$$

Does X_n converges to zero in probability? Does X_n converges to zero in quadratic mean?

6. Use computer, do the following for $n = 1, 3, 9, 27, 81$.

   (a) generate 1000 sets of iid random numbers of size n from $exp(1)$ distribution

   (b) obtain the average of each set (so you have 1000 such averages)

   (c) draw histogram of the 1000 averages

   Compare the histograms. What do they show?