

『해석학』 5강. 리만적분』 보충자료

Thm. $f, g \in \mathfrak{R}[a, b] \Rightarrow fg \in \mathfrak{R}[a, b]$

증명>

- $f, g \in \mathfrak{R}[a, b] \Rightarrow f+g, f-g \in \mathfrak{R}[a, b]$
- **Lemma 1.** $f \in \mathfrak{R}[a, b] \Rightarrow |f| \in \mathfrak{R}[a, b]$

증명>

- $f \in \mathfrak{R}[a, b] \Rightarrow \forall \epsilon > 0, \exists \wp \text{ s.t. } U(\wp, f) - L(\wp, f) < \epsilon$
- $U(\wp, |f|) - L(\wp, |f|)$

$$\begin{aligned}
 &= \sum_{i=1}^n (\sup |f| [x_{i-1}, x_i] - \inf |f| [x_{i-1}, x_i]) \Delta x_i \\
 &\leq \sum_{i=1}^n (\sup f [x_{i-1}, x_i] - \inf f [x_{i-1}, x_i]) \Delta x_i \\
 &\quad (\because \forall x, y \in [a, b], |f(x)| - |f(y)| \leq |f(x) - f(y)| : \text{삼각부등식}) \\
 &= U(\wp, f) - L(\wp, f) \\
 &< \epsilon
 \end{aligned}$$

- **Lemma 2.** $f \in \mathfrak{R}[a, b] \Rightarrow f^2 \in \mathfrak{R}[a, b]$

증명>

- f 가 $[a, b]$ 에서 유계이므로 $\forall x \in [a, b], \exists M > 0 \text{ s.t. } |f(x)| < M$
- Lemma 1에 의해 $f \in \mathfrak{R}[a, b] \Rightarrow |f| \in \mathfrak{R}[a, b]$

$$\therefore \exists \wp \text{ s.t. } U(\wp, |f|) - L(\wp, |f|) = \sum_{i=1}^n (\overline{M}_i - \overline{m}_i) \Delta x_i < \frac{\epsilon}{2M}$$

$$(\text{단, } \overline{M}_i = \sup |f|([x_{i-1}, x_i]), \overline{m}_i = \inf |f|([x_{i-1}, x_i]))$$

- $\sup f^2([x_{i-1}, x_i]) = \overline{M}_i^2$ 이고 $\inf f^2([x_{i-1}, x_i]) = \overline{m}_i^2$ 이므로

$$\begin{aligned}
 U(\wp, f^2) - L(\wp, f^2) &\leq \sum_{i=1}^n (\overline{M}_i^2 - \overline{m}_i^2) \Delta x_i \\
 &= \sum_{i=1}^n (\overline{M}_i + \overline{m}_i)(\overline{M}_i - \overline{m}_i) \Delta x_i \\
 &\leq 2M \sum_{i=1}^n (\overline{M}_i - \overline{m}_i) \Delta x_i \\
 &= 2M \{U(\wp, |f|) - L(\wp, |f|)\} \\
 &< \epsilon
 \end{aligned}$$

- $f+g, f-g \in \mathfrak{R}[a, b] \Rightarrow (f+g)^2, (f-g)^2 \in \mathfrak{R}[a, b] \quad (\because \text{Lemma 2})$
- $fg = \frac{1}{4} \{ (f+g)^2 - (f-g)^2 \}$ 이므로 $fg \in \mathfrak{R}[a, b] \quad \blacksquare$

Thm. $f \in \mathfrak{R}[a, b]$ 이고 α 가 $[a, b]$ 에서 증가하고 미분가능한 함수이며 $\alpha' \in \mathfrak{R}[a, b]$ 이면

$$f \in \mathfrak{R}_\alpha[a, b] \text{ 이고 } \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx \text{ 이다.}$$

증명>

- $f \in \mathfrak{R}[a, b] \wedge \alpha' \in \mathfrak{R}[a, b] \Rightarrow f\alpha' \in \mathfrak{R}[a, b]$
- $\alpha' \in \mathfrak{R}[a, b] \Rightarrow \forall \epsilon > 0, \exists \wp \text{ s.t. } U(\wp, \alpha') - L(\wp, \alpha') < \epsilon$
- $\wp^* = \{x_0, x_1, \dots, x_n\}$ 가 \wp 의 세분이라 하자. 그러면 $\exists y_i \in [x_{i-1}, x_i] \text{ s.t.}$

$$U(\wp^*, f, \alpha) - \sum_{i=1}^n f(y_i)\Delta\alpha_i < \epsilon \quad (i = 1, 2, \dots, n)$$

- 평균값정리에 의해 $\exists t_i \in [x_{i-1}, x_i] \text{ s.t. } \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)\Delta x_i$
($i = 1, 2, \dots, n$)

$$\therefore \sum_{i=1}^n f(y_i)\Delta\alpha_i = \sum_{i=1}^n f(y_i)\alpha'(t_i)\Delta x_i$$

- $M_i^* = \sup\{\alpha'(x) \mid x_{i-1} \leq x \leq x_i\}, m_i^* = \inf\{\alpha'(x) \mid x_{i-1} \leq x \leq x_i\}$ 라 하면
 $|\alpha'(y_i) - \alpha'(t_i)| \leq M_i^* - m_i^*$ 이다.

- $M = \sup\{|f(x)| \mid a \leq x \leq b\}$ 라 하면

$$\begin{aligned} \left| \sum_{i=1}^n f(y_i)\alpha'(y_i)\Delta x_i - \sum_{i=1}^n f(y_i)\alpha'(t_i)\Delta x_i \right| &\leq \sum_{i=1}^n |f(y_i)| |\alpha'(y_i) - \alpha'(t_i)| \Delta x_i \\ &\leq M \sum_{i=1}^n (M_i^* - m_i^*) \Delta x_i \\ &< M \times \epsilon \end{aligned}$$

$$\therefore \sum_{i=1}^n f(y_i)\alpha'(t_i)\Delta x_i \leq \sum_{i=1}^n f(y_i)\alpha'(y_i)\Delta x_i + M\epsilon \leq U(\wp^*, f\alpha') + M\epsilon$$

$$\text{그러므로 } U(\wp^*, f, \alpha) < U(\wp^*, f\alpha') + (M+1)\epsilon$$

- $f\alpha' \in \mathfrak{R}[a, b]$ 이므로 $U(\wp^*, f\alpha') < \int_a^b f(x)\alpha'(x)dx + \epsilon$ 인 \wp 의 세분 \wp^* 가 존재하고

$$\text{따라서 } \overline{\int_a^b f d\alpha} < \int_a^b f(x)\alpha'(x)dx + (M+2)\epsilon$$

$$\therefore \overline{\int_a^b f d\alpha} \leq \int_a^b f(x)\alpha'(x)dx \quad (\because \epsilon \text{은 임의의 양수})$$

- 마찬가지로 $\underline{\int_a^b f d\alpha} \geq \int_a^b f(x)\alpha'(x)dx$

$$\therefore \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx \quad \blacksquare$$