The Rational Filter for Image Smoothing

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***Abstract*— A nonlinear operator is presented that is able to effectively attenuate the noise that corrupts an image while introducing small distortions on the image details. It is described by a rational function, i.e., by the ratio of two polynomials in the input variables. Notwithstanding its simplicity, this operator proves to be more powerful than conventional methods for many noise distributions.**

I. Introduction

A LARGE variety of filtering methods have been devised in order to comply with the somewhat contradictory need of smoothing noise without corrupting fine image structures. Many of them are based on the order statistics approach [1]–[3], while others use adaptive averaging masks that take into account only pixels having similar luminance [4]. In some cases, two approaches such as these have also been combined [5].

The method we are proposing belongs to the second cited class; its basic idea is to modulate the coefficients of a linear lowpass filter in order to limit its action in presence of image details. It will be demonstrated that the results achieved by this operator, notwithstanding its simplicity, are better than those yielded by conventional methods for many noise distributions. The proposed operator is formulated so that a rational function, i.e., the ratio of two polynomials in the input variables, describes its input/output relation. In this sense, it belongs to a wide family of powerful and mathematically well tractable functions that are beginning to find use in the digital signal processing area [6]. Other *Rational filters* for image processing are presently under study.

II. The 1-D Rational Filter

Let us start from a 1-D symmetric linear filter of length three. Each output sample y_n is obtained from the vector of the input samples $\boldsymbol{x} = [x_{n-1}, x_n, x_{n+1}]$ as

$$y_n = w(x_{n-1} + x_{n+1}) + (1 - 2w)x_n. \tag{1}$$

Since the sum of its coefficients is one, this operator has unitary gain at zero frequency. If we want it to possess a lowpass behavior, the weight w should be set within the range $0 < w \leq 1/3$. More precisely, for any value of w in the range $0 < w \leq 1/4$, the transfer function of the filter monotonously decreases with increasing frequency, and the cut-off frequency is an inverse function of w. For $w = 1/4$ the gain is zero at π; higher values of w move the zero-gain frequency below π, whereas a lobe of increasing amplitude appears at high frequency.

Linear lowpass operators such as (1), when applied to image data, create unacceptable blurring phenomena. We want to limit this drawback while maintaining as much as possible the noise smoothing capability of the filter. To this end, at each step, we condition the lowpass action of the operator to the absence of relevant signal changes in the present vector $\boldsymbol{x}$. This is achieved in a very simple but effective way: the squared difference of the two samples at the extremes of the filter mask is evaluated. If it is large, it is assumed that the mask is positioned across a signal transition, and the frequency response of the operator is made less selective. It should be observed that the proposed approach is closely related, in spirit, to the one used in [7]. Much improved performance is obtained in our case; according to the experiments we made, this is due to the use of a square norm for detecting signal changes, to a different choice for the set of weights and, not least, to the fact that the luminance gradient is estimated among pixels at the extremes of the mask.

The described approach yields an operator that has the aspect of a rational function in the variables $\{x_n\}$. Different formulations can be devised for this operator [8]; if we consider that it is derived from (1), it can be expressed as the ratio between a third-order polynomial and a second-order one:

$$\begin{aligned} y_n &= \frac{w(x_{n-1} + x_{n+1})}{wk(x_{n-1} - x_{n+1})^2 + 1} \\ &\quad + \left(1 - \frac{2w}{wk(x_{n-1} - x_{n+1})^2 + 1}\right)x_n \end{aligned} \tag{2}$$

$$y_n = \frac{x_{n-1} + x_{n+1} + x_n[k(x_{n-1} - x_{n+1})^2 + 1/w - 2]}{k(x_{n-1} - x_{n+1})^2 + 1/w}. \tag{3}$$

The parameter k should take positive values and is used to control the filter. It is easy to observe that this Rational filter differs from the linear filter of (1) mainly for the scaling, which is introduced on the x_{n-1} and x_{n+1} terms. Indeed, such terms are divided by a factor proportional to the output of the edge-sensing term $(x_{n-1} - x_{n+1})^2$; the weight of the x_n term is accordingly modified, in order to keep the gain constant. More precisely, we can observe that for $k = 0$, we again have the linear filter of (1); on the contrary, in the case $k \to \infty$, the filter has no effect, and $y_n \simeq x_n$. For intermediate values of k, the $(x_{n-1} - x_{n+1})^2$ term perceives the presence of a detail and accordingly reduces the smoothing effect of the operator. In this sense, the rational filter can be interpreted as a linear

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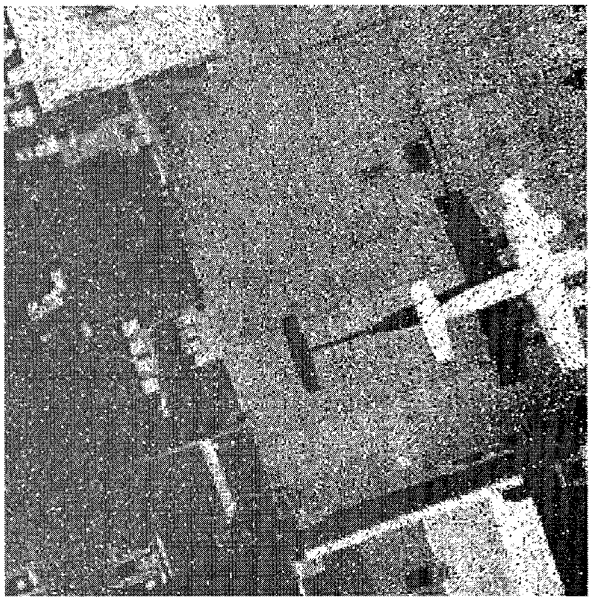
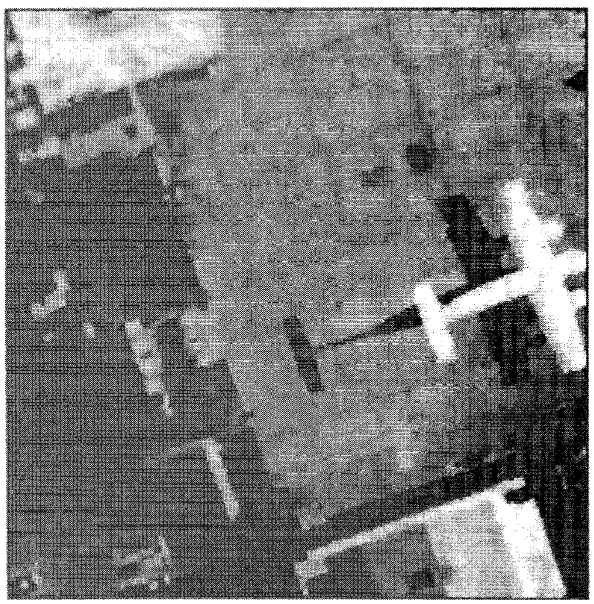


(a)



(b)

Fig. 1. Original (corrupted) and processed image.

lowpass filter the coefficients of which are modulated by the edge-sensitive component.

The edge-preserving capability of this filter makes it well suited to be applied more than once on the input data. If p passes of the filter are performed, the smoothing performance achievable in uniform areas is the same as the one of a linear lowpass filter having size $(2p+1)$. Indeed, in a uniform area, our operator is equivalent to a linear filter having coefficients $\boldsymbol{w}_1 = [w, 1-2w, w]$ for a single filter pass, $\boldsymbol{w}_2 = [w^2, 2w(1-2w), 2w^2+(1-2w)^2, 2w(1-2w), w^2]$ for two passes, and so on. On the contrary, in the vicinity of a detail, the smoothing mask of the equivalent filter takes an asymmetric shape and covers only those pixels that have values similar to the one of the reference pixel.

TABLE I
MEAN-SQUARE ERRORS FOR THE DIFFERENT TECHNIQUES (NUMBERS IN PARENTHESES INDICATE DIFFERENT PARAMETER VALUES USED AT THE SECOND APPLICATION OF THE FILTER)

Noise	Operator	m.s.e.
additive Gaussian	Rational, $w=0.16$, $k=0.01$, $p=3$	183
	ACWM, $\sigma_n^2=600$, T=4, $p=2$	207
	Sigma, 5×5, thr.=95 (15), $p=2$	191
	α-tr., 3×3, $\alpha=2/9$ (3/9), $p=2$	210
	IINGR, 5×5, $p=2$	230
additive contamin.	Rational, $w=0.16$, $k=0.01$, $p=3$	151
	ACWM, $\sigma_n^2=600$, T=7, $p=1$	160
	Sigma, 5×5, thr.=130 (15), $p=2$	257
	α-tr., 3×3, $\alpha=3/9$, $p=1$	157
	IINGR, 5×5, $p=2$	201
additive Laplacian	Rational, $w=0.16$, $k=0.01$, $p=3$	167
	ACWM, $\sigma_n^2=350$, T=5, $p=2$	179
	Sigma, 5×5, thr.=105 (15), $p=2$	214
	α-tr., 3×3, $\alpha=3/9$ (4/9), $p=2$	180
	IINGR, 5×5, $p=2$	209
multiplic. uniform	Rational, $w=0.16$, $k=0.01$, $p=3$	212
	ACWM, $\sigma_n^2=900$, T=1, $p=2$	230
	Sigma, 5×5, thr.=95 (15), $p=2$	213
	α-tr., 3×3, $\alpha=1/9$ (4/9), $p=2$	240
	IINGR, 5×5, $p=2$	271

Finally, we would like to point out that the practical realization of (3) is very simple if we use a small look-up table for evaluating the denominator. We further need only one multiplication, three additions, and one division per output sample.

III. THE 2-D OPERATOR

A practical and effective way to extend the proposed method to 2-D data is to apply it in a 3×3 mask in the 0, 45, 90, 135° directions. We obtain the operator

$$y_{m,n} = \alpha_{m,n}x_{m,n} + z_{m,n} \tag{4}$$

with

$$z_{m,n} = \frac{w(x_{m-1,n}+x_{m+1,n})}{wk(x_{m-1,n}-x_{m+1,n})^2+1} + \frac{w(x_{m,n-1}+x_{m,n+1})}{wk(x_{m,n-1}-x_{m,n+1})^2+1}$$

$$+\frac{w(x_{m-1,n-1}+x_{m+1,n+1})}{wk(x_{m-1,n-1}-x_{m+1,n+1})^2+\sqrt{2}}+\frac{w(x_{m-1,n+1}+x_{m+1,n-1})}{wk(x_{m-1,n+1}-x_{m+1,n-1})^2+\sqrt{2}}, \quad (5)$$

$$\alpha_{m,n}=1-\frac{2w}{wk(x_{m-1,n}-x_{m+1,n})^2+1}-\frac{2w}{wk(x_{m,n-1}-x_{m,n+1})^2+1}-\frac{2w}{wk(x_{m-1,n-1}-x_{m+1,n+1})^2+\sqrt{2}}-\frac{2w}{wk(x_{m-1,n+1}-x_{m+1,n-1})^2+\sqrt{2}}. \quad (6)$$

IV. Computer Simulations

The image we use for this inspection is a 256×256 slice (the fourth quadrant) of the picture "Airfield." Four corrupted test images have been derived from this one. The first two are yielded by adding i.i.d. noise having the contaminated Gaussian probability distribution

$$\nu \sim (1-\lambda)\mathcal{N}(0,\sigma_n)+\lambda\mathcal{N}\left(0,\frac{\sigma_n}{\lambda}\right). \quad (7)$$

Two values of λ have been chosen: $\lambda = 1$, yielding a purely Gaussian noise, and $\lambda = 0.2$, which produces a mixed Gaussian-impulsive distribution. The third corrupted test image is obtained adding zero-mean Laplacian noise on the original data. Finally, uniformly distributed multiplicative noise with mean one yields the fourth image. All the images have a 6 dB SNR. Just as an example, Fig. 1(a) and (b), respectively, show the image corrupted with contaminated Gaussian noise before and after processing with our operator. The parameters w, k and the number of passes p of our method have been set by experimentally minimizing the mean-square error (mse) of the processed image with respect to the original (uncorrupted) one. The parameters we used are optimal for purely Gaussian noise.

Other conventional operators have also been tested in order to compare the obtainable performances. We selected four techniques that are well known for their ability of smoothing noise while preserving image details, i.e., the adaptive center weighted median (ACWM) filter [1], the sigma filter [4], the alpha-trimmed filter [9], and the improved inverse gradient (IINGR) filter [10] (the last one, as its name implies, is an improved version of the operator introduced in [7] and mentioned in Section II). In order to perform the fairest comparison possible, we tested the effects of both single and repeated application of the different operators. We set the mse-optimal parameters of all the above cited methods independently for each of the types of noise. *On the contrary, the same filtering parameters were used in all cases for our Rational filter.* The best mse values obtained from each technique are shown in Table I, together with the corresponding parameters (p indicates the number of passes). As it can be observed, none of the methods was able to yield a mse smaller than ours for any one of the different types of noise. Moreover, the effectiveness of the competitor filters is much more sensitive to the noise distribution.

V. Concluding Remarks

A novel technique for edge-preserving noise smoothing has been presented, which conjugates simplicity and effectiveness. Work still to be performed can be directed to deriving other Rational filters (e.g., filters capable of dealing with purely impulsive noise), possibly using a multiresolution approach.

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