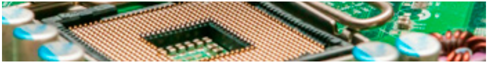
Máster Universitario en Ingeniería Informática

Sistemas Inteligentes

Unit 6. Maximum Entropy Models - Theory

2022/2023



Outline

1. Introduction to Maximum Entropy

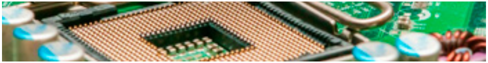
2. Probability estimation with Maximum Entropy

3. Basic concepts

4. Maximum Entropy principle

5. IIS Algorithm

6. References



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Entropy definitions

Entropy of a discrete variable $X \sim p(x)$

$$H(X) = -\sum_{x} p(x) \; \log p(x)$$

Also written as $H(p)$

If the log is to the base 2 then the entropy is measured in bits

Interpretation 1: average number of bits needed to describe X
Interpretation 2: *uncertainty* about the outcome of a stochatic variable X

Properties:

- Entropy is a concave function
- Entropy is maximum when p is uniform $\rightarrow$ maximun *uncertainty*

Introduction to Maximum Entropy

Example [Cover 1991]: Consider a dice with 8 faces (with numbers from 0 to 7), all of them with the same probability

A 3-bit string is necessary as label for each face

The entropy of this random variable is:

$$H(X) = -\sum_{i=1}^{8} p(i) \log p(i) = -\sum_{i=1}^{8} \frac{1}{8} \log \frac{1}{8} = \log 8 = 3 \text{ bits}$$

Intuition: let us supose that we send the winner face to a friend; the average number of bits we need to send to the friend to identify the face is 3 bits

Let us supose that:

x	0	1	2	3	4	5	6	7
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{64}$

Then, the entropy is: $H(X) = 2$ bits

Intuition: the most probable outcomes need less bits to be coded

Consider the following coding for the faces:

$\{0, 10, 110, 1110, 111100, 111101, 111110, 111111\}$

$$\frac{1}{2} + 2\,\frac{1}{4} + 3\,\frac{1}{8} + 4\,\frac{1}{16} + 4 \cdot 6\,\frac{1}{64} = 2$$

Joint entropy of a pair of discrete random variables (X, Y) with a joint distribution $p(x, y)$ [Cover 1991]:

$$H(X, Y) = -\sum_x \sum_y p(x, y) \; \log p(x, y)$$

Expectation of $g(X)$ when $X \sim p(x)$

$$E[g(X)] = \sum_x g(x)p(x)$$

Note that $H(X) = E\left[\frac{1}{\log p(X)}\right]$

Conditional entropy of a pair of discrete random variables (X, Y):

$$
\begin{aligned}
H(Y|X) &= \sum_{x} p(x)\; H(Y|X=x) \\
&= -\sum_{x} p(x) \sum_{y} p(y|x)\;\log p(y|x) \\
&= -\sum_{x} \sum_{y} p(x,y)\;\log p(y|x)
\end{aligned}
$$

Theorem (Chain rule) [Cover 1991]

$$H(X,Y) = H(X) + H(Y|X)$$

Example [Cover 1991]: Let (X, Y) have the following joint distribution:

$Y \diagdown X$	1	2	3	4	$\sum$
1	1/8	1/16	1/32	1/32	1/4
2	1/16	1/8	1/32	1/32	1/4
3	1/16	1/16	1/16	1/16	1/4
4	1/4	0	0	0	1/4
$\sum$	1/2	1/4	1/8	1/8	1

Marginal of X: $(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ Marginal of Y: $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

$$
\begin{aligned}
H(X|Y) &= \sum_{i=1}^{4} p(Y=i)\; H(X|Y=i) \\
&= \frac{1}{4} H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right) + \frac{1}{4} H\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) + \frac{1}{4} H(1, 0, 0, 0) \\
&= \frac{11}{8} = 1.375 \quad \text{bits}
\end{aligned}
$$

Outline

Probability estimation with Maximum Entropy

- **Problem**: estimate $p(y|x)$ where $x = (x_1, \ldots, x_D)$ is D-tuple of discrete (sometimes categoric) observations

$$p(y|x_1, \ldots, x_D)$$

- **Goal**: estimate p given data that follows an empirical distribucion $\tilde{p}$

Probability estimation with Maximum Entropy

Maximum Entropy solution:

- Choose p with maximum entropy (or “uncertainty”) subject to some constraints (given by $\tilde{p}$)
- **Entropy** is a mathematical measure of the uniformity (uncertainty) of a distribution $p(x, y)$

$$H(p) = -\sum_{x,y} p(x, y) \log p(x, y)$$

- For a conditional distribution $p(y|x)$ its conditional entropy is:

$$H(p) = -\sum_{x,y} \widetilde{p}(x) p(y|x) \log p(y|x)$$

- Final form is a log-linear combination

Pending problems:

- Selection of features
- Smoothing: no closed-form solutions for optimal parameters

Probability estimation with Maximum Entropy

Overview of the process:

- Collect (x, y) pairs from training data:
  - y: thing to be predicted
  - x: the context
- Learn the probability p of each (x, y)
- Maximum Entropy strategy:

  Model all that is known and assume nothing about what is unknown
  - to satisfy a set of constraints
  - to assume the most “uniform” distribution

Example 1: MT

"We wish to model an expert translator's decisions concerning the proper French rendering of the English word **in**"

Let p our model of the translator's decisions and let (x, y) a set of samples: $\{(\textit{in}, \textit{dans}), (\textit{in}, \textit{en}), \ldots\}$

"Suppose that the expert translator always chooses among the following five French phrases: $\{\textit{dans}, \textit{en}, \textit{à}, \textit{au cours de}, \textit{pendant}\}$"

$$p(\textit{dans}) + p(\textit{en}) + p(\textit{à}\ ) + p(\textit{au cours de}) + p(\textit{pendant}) \quad = \quad 1$$

Probability estimation with Maximum Entropy

"Suppose we notice that the expert chose either *dans* or *en* 30% of the time"

$$\begin{aligned} p(\textit{dans}) + p(\textit{en}) &= 3/10 \\ p(\textit{dans}) + p(\textit{en}) + p(\textit{à}) + p(\textit{au cours de}) + p(\textit{pendant}) &= 1 \end{aligned}$$

"...in half the cases, the expert chose either *dans* or *à*"

$$\begin{aligned} p(\textit{dans}) + p(\textit{en}) &= 3/10 \\ p(\textit{dans}) + p(\textit{en}) + p(\textit{à}) + p(\textit{au cours de}) + p(\textit{pendant}) &= 1 \\ p(\textit{dans}) + p(\textit{à}) &= 1/2 \end{aligned}$$

Example 2: POS tagging

- Let say we have the following event space

NN	NNS	NNP	NNPS	VBZ	VBD

- Empirical data

3	5	11	13	3	1

- Maximize the entropy

$1/e$	$1/e$	$1/e$	$1/e$	$1/e$	$1/e$

- $E[\mathrm{NN}, \mathrm{NNS}, \mathrm{NNP}, \mathrm{NNPS}, V\mathrm{BZ}, \mathrm{VBD}] = 1$

$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

- N* are more common than V*. Add feature $f_N = \{\mathrm{NN}, \mathrm{NNS}, \mathrm{NNP}, \mathrm{NNPS}\}$ with $E[f_N] = 32/36$

NN	NNS	NNP	NNPS	VBZ	VBD
$8/36$	$8/36$	$8/36$	$8/36$	$2/36$	$2/36$

- Proper nouns are more frequent than common nouns. Add feature $f_P = \{\mathrm{NNP}, \mathrm{NNPS}\}$ with $E[f_P] = 24/36$

NN	NNS	NNP	NNPS	VBZ	VBD
$4/36$	$4/36$	$12/36$	$12/36$	$2/36$	$2/36$

Outline

Basic concepts

Problem: Construct a stochastic model that accurately represents the behaviour of a random process: $p(y|x)$

Training data

Given a training sample (x, y), its empirical probability distribution $\widetilde{p}$ is defined by:

$$\widetilde{p}(x, y) \equiv \frac{1}{N} \times \text{number of times that } (x, y) \text{ occurs in the sample}$$

Features

If *April* is the word following *in*, then the translation of *in* is *en* with frequency $9/10$.

$$f(x, y) = \begin{cases} 1 & \text{if } y = \textit{en} \text{ and } \textit{April} \text{ follows } \textit{in} \\ 0 & \text{otherwise} \end{cases}$$

Constraints

Expected value of f with respect to $\widetilde{p}(x, y)$:

$$\widetilde{p}(f) \equiv \sum_{x,y} \widetilde{p}(x, y) f(x, y)$$

Expected value of f with respect to the model $p(y|x)$:

$$p(f) \equiv \sum_{x,y} \widetilde{p}(x) p(y|x) f(x, y) = \sum_{x,y} \widetilde{p}(x, y) f(x, y)$$

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Maximum Entropy principle

Given n features f_i, we would like p to lie in the subset $\mathcal{C}$ of $\mathcal{P}$ defined by

$$\mathcal{C} \equiv \{p \in \mathcal{P} \mid p(f_i) = \widetilde{p}(f_i) \text{ for } i \in \{1, 2, \ldots, k\}\}$$

Conditional entropy

$$H(p) = -\sum_{x,y} \widetilde{p}(x) p(y|x) \log p(y|x)$$

To select a model from a set $\mathcal{C}$ of allowed probability distributions, choose the model $p_* \in \mathcal{C}$ with maximum entropy $H(p)$:

$$p_* = \arg\max_{p \in \mathcal{C}} H(p)$$

Solution to the primal problem:

http://www.cs.cmu.edu/afs/cs/user/aberger/www/html/tutorial/node7.html#SECTION00024000000000000000

The Maximum Entropy solution p_* has the form:

$$\begin{aligned} p(y|x) &= \frac{1}{Z(x)} \exp\left(\sum_{i=1}^{k} \lambda_i f_i(x,y)\right) \\ Z(x) &= \sum_{y} \exp\left(\sum_{i=1}^{k} \lambda_i f_i(x,y)\right) \end{aligned}$$

where k is the number of features

Let Λ^* be

$$\Lambda^* = \arg\max_{\Lambda} \frac{1}{Z(x)} \exp\left(\sum_{i=1}^{k} \lambda_i f_i(x,y)\right)$$

Then $p_{\Lambda^*} = p_*$

Outline

Solution to the dual problem

http://www.cs.cmu.edu/afs/cs/user/aberger/www/html/tutorial/node10.html#SECTION00030000000000000000

http://luthuli.cs.uiuc.edu/~daf/courses/optimization/papers/berger-iis.pdf

1. Start with some (arbitrary) value for each λ_i
2. Repeat until convergence:
   (a) Solve $\quad \dfrac{\partial \mathcal{B}(\delta)}{\partial \delta_i} = 0 \quad \text{for} \quad \delta_i$
   (b) Set $\lambda_i = \lambda_i + \delta_i$

where:

$$\frac{\partial \mathcal{B}(\delta)}{\partial \delta_i} = \overbrace{\sum_{x,y} \widetilde{p}(x,y) f_i(x,y)}^{\widetilde{p}(f_i)} - \overbrace{\sum_{x,y} \widetilde{p}(x) p_\lambda(y|x) f_i(x,y)}^{p_\lambda(f_i)} e^{\delta_i f^{\#}(x,y)}$$

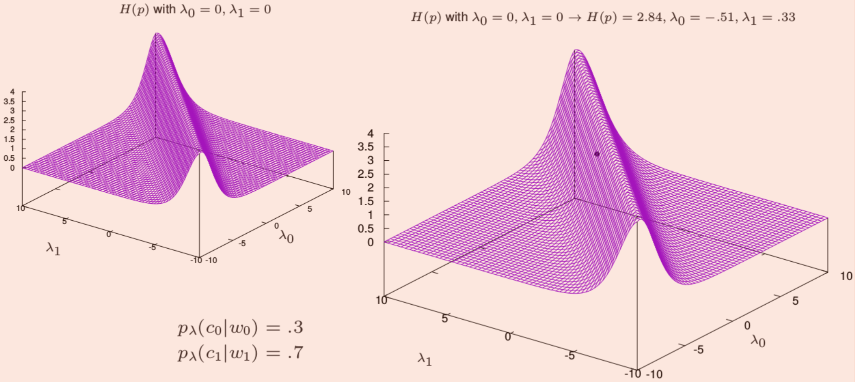
If $f^{\#}(x,y) = M$ then

$$\delta_i = \frac{1}{M} \log \frac{\widetilde{p}(f_i)}{p_\lambda(f_i)}$$

Example to illustrate the optimization problem (just two features)

$$C = \{(w_0,c_0),(w_0,c_0),(w_0,c_0),(w_1,c_1),(w_1,c_1),(w_1,c_1),(w_1,c_1),(w_1,c_1),(w_1,c_1),(w_1,c_1)\}$$

$$\lambda_0 = f(w_0,c_0) \quad \lambda_1 = f(w_1,c_1)$$



$p_\lambda(c_0|w_0) = .3$

$p_\lambda(c_1|w_1) = .7$

Outline

References

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- A. Ratnaparkhi. *Learning to Parse Natural Language with Maximum Entropy Models.* Machine Learning, 34, 151-175, 1999.