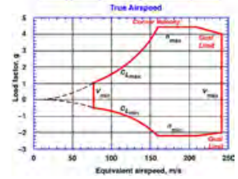
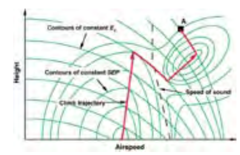
Gliding, Climbing, and Turning Flight Performance

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2016

Learning Objectives

- Conditions for gliding flight
- Parameters for maximizing climb angle and rate
- Review the *V-n* diagram
- Energy height and specific excess power
- Alternative expressions for steady turning flight
- The *Herbst maneuver*





Reading:
Flight Dynamics
Aerodynamic Coefficients, 130-141

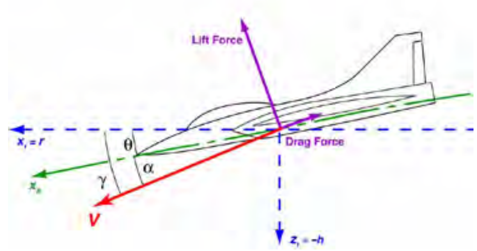
http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Review Questions

- *How does air density decrease with altitude?*
- *What are the different definitions of airspeed?*
- *What is a "lift-drag polar"?*
- *Power and thrust: How do they vary with altitude?*
- *What factors define the "flight envelope"?*
- *What were some features of the first commercial transport aircraft?*
- *What are the important parameters of the "Breguet Range Equation"?*
- *What is a "step climb", and why is it important?*

Gliding Flight

Equilibrium Gliding Flight



$$D = C_D \frac{1}{2}\rho V^2 S = -W \sin\gamma$$

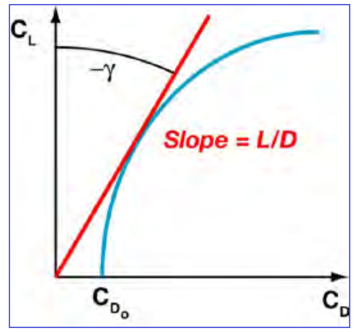
$$C_L \frac{1}{2}\rho V^2 S = W \cos\gamma$$

$$\dot{h} = V \sin\gamma$$

$$\dot{r} = V \cos\gamma$$

Gliding Flight

- Thrust = 0
- Flight path angle < 0 in gliding flight
- Altitude is decreasing
- Airspeed ~ constant
- Air density ~ constant



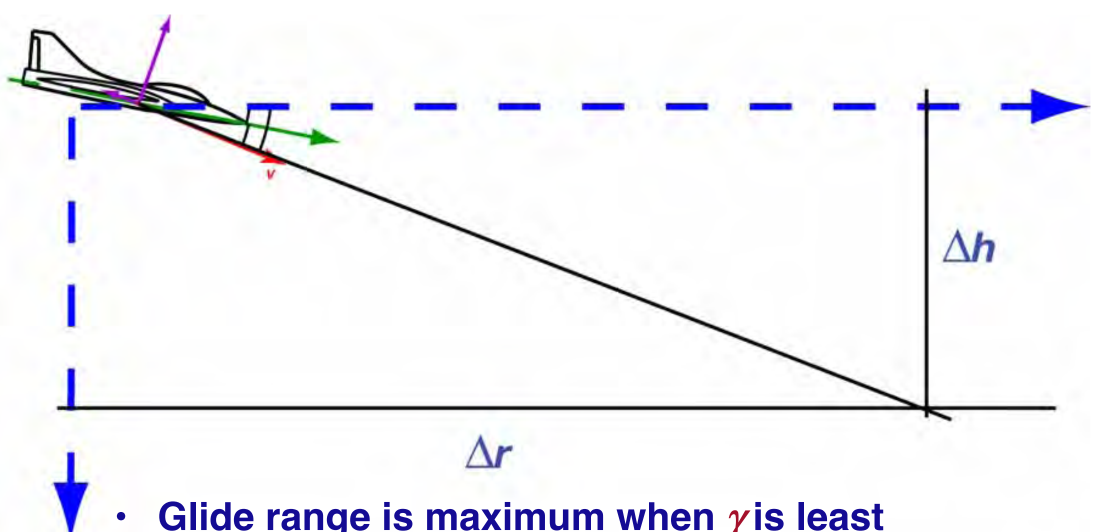
Gliding flight path angle

$$\tan\gamma = -\frac{D}{L} = -\frac{C_D}{C_L} = \frac{\dot{h}}{\dot{r}} = \frac{dh}{dr}; \quad \gamma = -\tan^{-1}\left(\frac{D}{L}\right) = -\cot^{-1}\left(\frac{L}{D}\right)$$

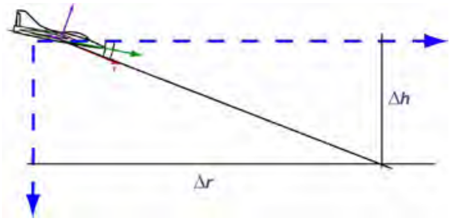
Corresponding airspeed

$$V_{glide} = \sqrt{\frac{2W}{\rho S\sqrt{C_D^2 + C_L^2}}}$$

Maximum Steady Gliding Range



- Glide range is maximum when γ is least negative, i.e., most positive
- This occurs at $(L/D)_{max}$



Maximum Steady Gliding Range

- Glide range is maximum when γ is least negative, i.e., most positive
- This occurs at $(L/D)_{max}$

$$\gamma_{max} = -\tan^{-1}\left(\frac{D}{L}\right)_{min} = -\cot^{-1}\left(\frac{L}{D}\right)_{max}$$

$$\tan\gamma = \frac{\dot{h}}{\dot{r}} = negative\ constant = \frac{(h-h_o)}{(r-r_o)}$$

$$\Delta r = \frac{\Delta h}{\tan\gamma} = \frac{-\Delta h}{-\tan\gamma} = maximum\ when\ \frac{L}{D} = maximum$$

Sink Rate, m/s

- Lift and drag define γ and V in gliding equilibrium

$$D = C_D \frac{1}{2}\rho V^2 S = -W\sin\gamma$$

$$\sin\gamma = -\frac{D}{W}$$

$$L = C_L \frac{1}{2}\rho V^2 S = W\cos\gamma$$

$$V = \sqrt{\frac{2W\cos\gamma}{C_L \rho S}}$$

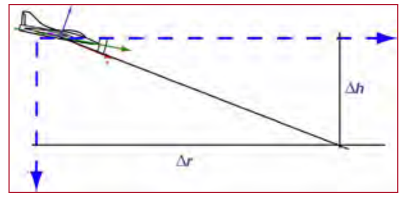
- Sink rate = altitude rate, *dh/dt* (negative)

$$\dot{h} = V\sin\gamma$$

$$= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{D}{W}\right) = -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{L}{W}\right)\left(\frac{D}{L}\right)$$

$$= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\cos\gamma\left(\frac{1}{L/D}\right)$$

Conditions for Minimum Steady Sink Rate



- **Minimum sink rate provides maximum endurance**
- **Minimize sink rate by setting $\partial(dh/dt)/\partial C_L = 0$ ($\cos\gamma \sim 1$)**

$$\dot{h} = -\sqrt{\frac{2W\cos\gamma}{C_L \rho S}}\cos\gamma\left(\frac{C_D}{C_L}\right)$$
$$= -\sqrt{\frac{2W\cos^3\gamma}{\rho S}}\left(\frac{C_D}{C_L^{3/2}}\right) \approx -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_D}{C_L^{3/2}}\right)$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} \quad and \quad C_{D_{ME}} = 4C_{D_o}$$

L/D and V_{ME} for Minimum Sink Rate

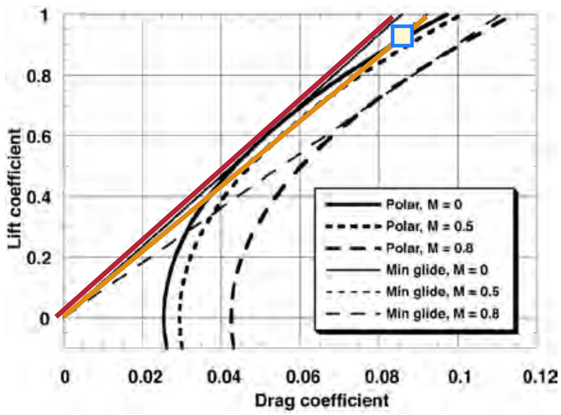
$$\left(L/D\right)_{ME} = \frac{1}{4}\sqrt{\frac{3}{\varepsilon C_{D_o}}} = \frac{\sqrt{3}}{2}\left(L/D\right)_{\max} \approx 0.86\left(L/D\right)_{\max}$$

$$V_{ME} = \sqrt{\frac{2W}{\rho S\sqrt{C_{D_{ME}}^2 + C_{L_{ME}}^2}}} \approx \sqrt{\frac{2\left(W/S\right)}{\rho}\sqrt{\frac{\varepsilon}{3C_{D_o}}}} \approx 0.76V_{L/D_{\max}}$$

L/D for Minimum Sink Rate

- For $L/D < L/D_{max}$, there are two solutions
- Which one produces smaller sink rate?

$$\left(L/D\right)_{ME} \approx 0.86\left(L/D\right)_{max}$$
$$V_{ME} \approx 0.76 V_{L/D_{max}}$$

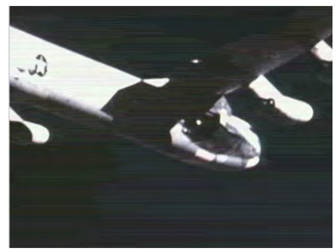


Checklist

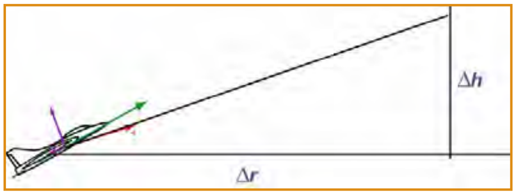
- ❑ *Steady flight path angle?*
- ❑ *Corresponding airspeed?*
- ❑ *Sink rate?*
- ❑ *Maximum-range glide?*
- ❑ *Maximum-endurance glide?*

Historical Factoids
Lifting-Body Reentry Vehicles

X-24A



Climbing Flight



Climbing Flight

- **Flight path angle**

$$\dot{V} = 0 = \frac{(T - D - W\sin\gamma)}{m}$$

$$\sin\gamma = \frac{(T - D)}{W}; \quad \gamma = \sin^{-1}\frac{(T - D)}{W}$$

- **Required lift**

$$\dot{\gamma} = 0 = \frac{(L - W\cos\gamma)}{mV}$$

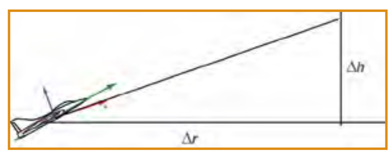
$$L = W\cos\gamma$$

Rate of climb, *dh/dt* = Specific Excess Power

$$\dot{h} = V\sin\gamma = V\frac{(T - D)}{W} = \frac{(P_{thrust} - P_{drag})}{W}$$

$$\textit{Specific Excess Power (SEP)} = \frac{\textit{Excess Power}}{\textit{Unit Weight}} \equiv \frac{(P_{thrust} - P_{drag})}{W}$$

Steady Rate of Climb



Climb rate

$$\dot{h} = V\sin\gamma = V\left[\left(\frac{T}{W}\right) - \frac{(C_{D_o} + \varepsilon C_L^2)\bar{q}}{(W/S)}\right]$$

$$L = C_L\bar{q}S = W\cos\gamma$$

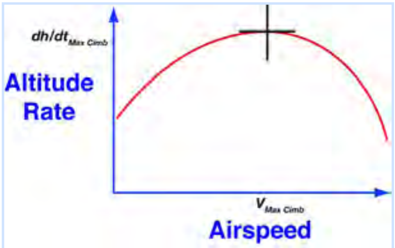
$$C_L = \left(\frac{W}{S}\right)\frac{\cos\gamma}{\bar{q}}$$

$$V = \sqrt{2\left(\frac{W}{S}\right)\frac{\cos\gamma}{C_L\rho}}$$

Note significance of thrust-to-weight ratio and wing loading

$$\dot{h} = V\left[\left(\frac{T}{W}\right) - \frac{C_{D_o}\bar{q}}{(W/S)} - \frac{\varepsilon(W/S)\cos^2\gamma}{\bar{q}}\right]$$

$$= V\left(\frac{T(h)}{W}\right) - \frac{C_{D_o}\rho(h)V^3}{2(W/S)} - \frac{2\varepsilon(W/S)\cos^2\gamma}{\rho(h)V}$$

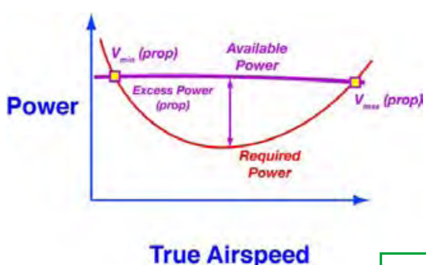


Condition for Maximum Steady Rate of Climb

$$\dot{h} = V\left(\frac{T}{W}\right) - \frac{C_{D_o}\rho V^3}{2(W/S)} - \frac{2\varepsilon (W/S)\cos^2\gamma}{\rho V}$$

Necessary condition for a maximum with respect to airspeed

$$\frac{\partial \dot{h}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T / \partial V}{W}\right)\right] - \frac{3C_{D_o}\rho V^2}{2(W/S)} + \frac{2\varepsilon (W/S)\cos^2\gamma}{\rho V^2}$$



Maximum Steady Rate of Climb:
Propeller-Driven Aircraft

- **At constant power**

$$\frac{\partial P_{thrust}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T / \partial V}{W}\right)\right]$$

- **With $\cos^2\gamma \sim 1$, optimality condition reduces to**

$$\frac{\partial \dot{h}}{\partial V} = 0 = -\frac{3C_{D_o}\rho V^2}{2(W/S)} + \frac{2\varepsilon (W/S)}{\rho V^2}$$

- **Airspeed for maximum rate of climb at maximum power, P_{max}**

$$V^4 = \left(\frac{4}{3}\right)\frac{\varepsilon (W/S)^2}{C_{D_o}\rho^2}; \quad V = \sqrt{2\frac{(W/S)}{\rho}\sqrt{\frac{\varepsilon}{3C_{D_o}}}} = V_{ME}$$

Maximum Steady Rate of Climb:
Jet-Driven Aircraft

Condition for a maximum at constant thrust and $\cos^2\gamma \sim 1$

$$\boxed{\frac{\partial \dot{h}}{\partial V} = 0}$$

$$\boxed{-\frac{3C_{D_o}\rho}{2(W/S)}V^4 + \left(\frac{T}{W}\right)V^2 + \frac{2\varepsilon(W/S)}{\rho} = 0}$$

$$\boxed{-\frac{3C_{D_o}\rho}{2(W/S)}\left(V^2\right)^2 + \left(\frac{T}{W}\right)\left(V^2\right) + \frac{2\varepsilon(W/S)}{\rho} = 0}$$

Quadratic in V^2

Airspeed for maximum rate of climb at maximum thrust, T_{max}

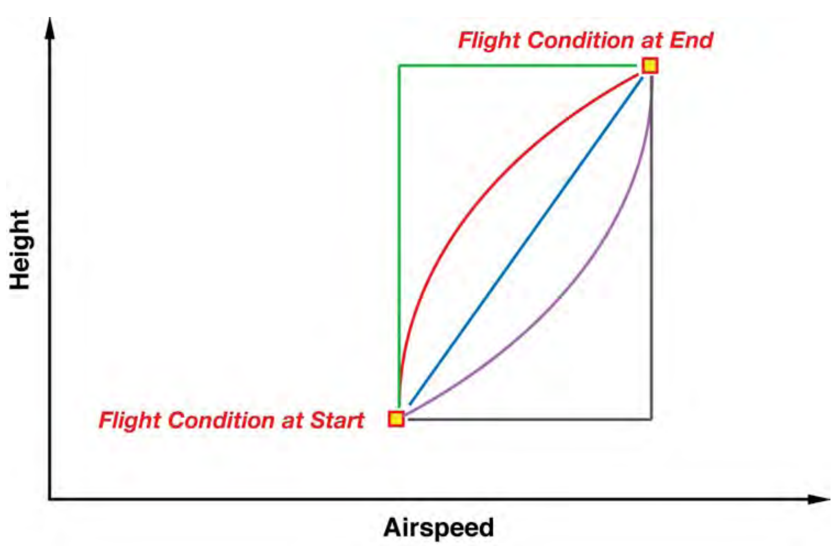
$$\boxed{0 = ax^2 + bx + c \text{ and } \quad V = +\sqrt{x}}$$

Checklist

- ❑ *Specific excess power?*
- ❑ *Maximum steady rate of climb?*
- ❑ *Velocity for maximum climb rate?*

Optimal Climbing Flight

What is the Fastest Way to Climb from One Flight Condition to Another?



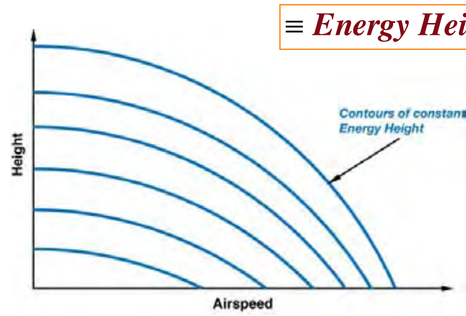
Energy Height

- **Specific Energy**
  - **= (Potential + Kinetic Energy) per Unit Weight**
  - **= Energy Height**

$$\frac{Total\ Energy}{Unit\ Weight} \equiv \boldsymbol{Specific\ Energy}$$

$$= \frac{mgh + mV^2/2}{mg} = h + \frac{V^2}{2g}$$

$$\equiv \boldsymbol{Energy\ Height}, E_h, \quad ft\ or\ m$$



Can trade altitude for airspeed with no change in energy height if thrust and drag are zero

Specific Excess Power

Rate of change of Specific Energy

$$\frac{dE_h}{dt} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}$$

$$= V\sin\gamma + \left(\frac{V}{g}\right)\left(\frac{T - D - mg\sin\gamma}{m}\right) = V\frac{(T-D)}{W}$$

$$= \boldsymbol{Specific\ Excess\ Power}\ (SEP)$$

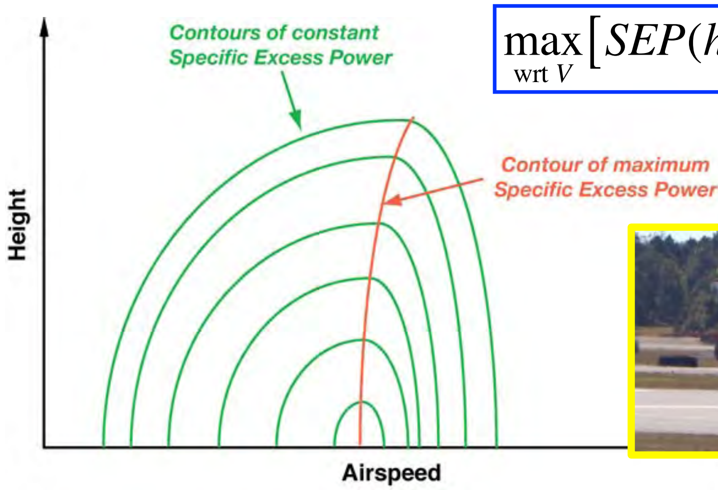
$$= \frac{\text{Excess Power}}{\text{Unit Weight}} \equiv \frac{\left(P_{thrust} - P_{drag}\right)}{W}$$

$$= V\frac{(C_T - C_D)\frac{1}{2}\rho(h)V^2 S}{W}$$

Contours of Constant Specific Excess Power

- Specific Excess Power is a function of altitude and airspeed
- *SEP* is maximized at each altitude, *h*, when

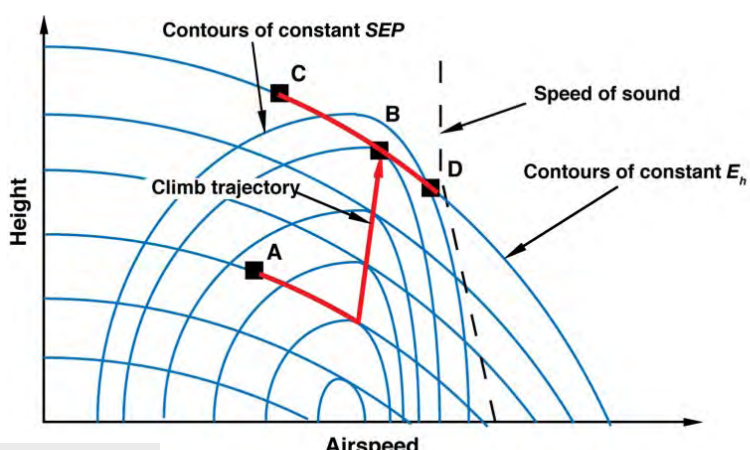
$$\max_{\text{wrt } V}\left[SEP(h)\right] \qquad \frac{d\left[SEP(h)\right]}{dV} = 0$$



Subsonic Minimum-Time Energy Climb

Objective: Minimize time to climb to desired altitude and airspeed

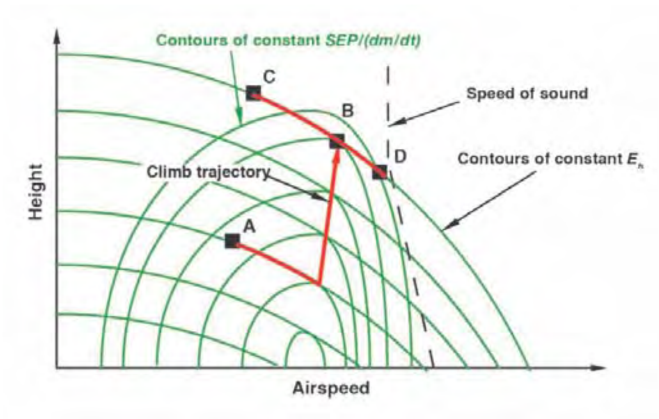
- **Minimum-Time Strategy:**
  - Zoom climb/dive to intercept $SEP_{max}(h)$ contour
  - Climb at $SEP_{max}(h)$
  - Zoom climb/dive to intercept target $SEP_{max}(h)$ contour



Bryson, Desai, Hoffman, 1969

Subsonic Minimum-Fuel Energy Climb

Objective: Minimize fuel to climb to desired altitude and airspeed



- **Minimum-Fuel Strategy:**
  - Zoom climb/dive to intercept $[SEP(h)/(dm/dt)]_{max}$ contour
  - Climb at $[SEP(h)/(dm/dt)]_{max}$
  - Zoom climb/dive to intercept target $[SEP(h)/(dm/dt)]_{max}$ contour

Bryson, Desai, Hoffman, 1969

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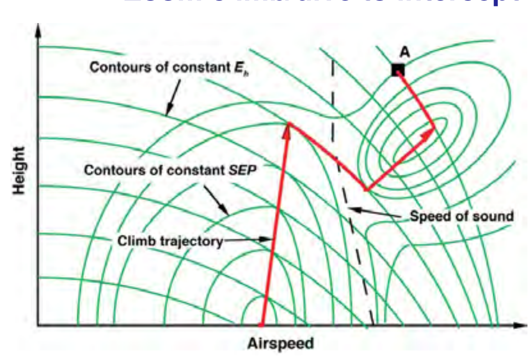
Supersonic Minimum-Time Energy Climb

Objective: Minimize time to climb to desired altitude and airspeed

- **Minimum-Time Strategy:**
  - Intercept subsonic $SEP_{max}(h)$ contour
  - Climb at $SEP_{max}(h)$ to intercept matching zoom climb/dive contour
  - Zoom climb/dive to intercept supersonic $SEP_{max}(h)$ contour
  - Climb at $SEP_{max}(h)$ to intercept target $SEP_{max}(h)$ contour
  - Zoom climb/dive to intercept target $SEP_{max}(h)$ contour



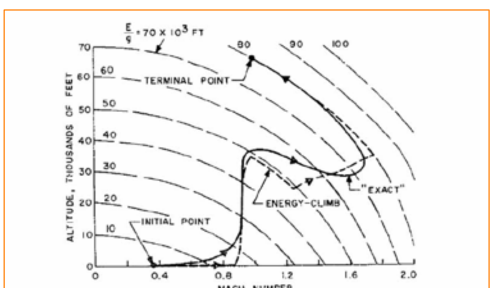


Fig. 5 Comparison of exact and energy-state minimum time-to-climb paths (airplane 1).

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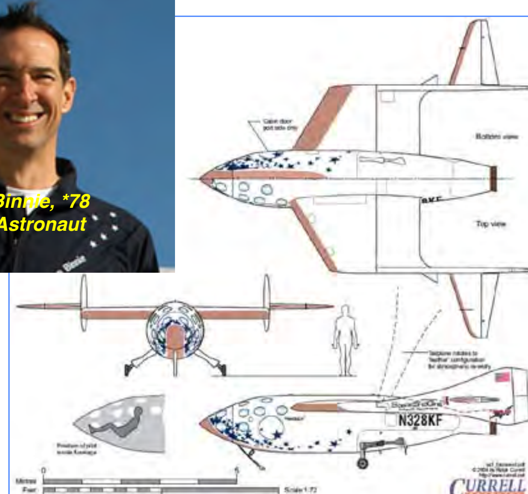
Bryson, Desai, Hoffman, 1969

Checklist

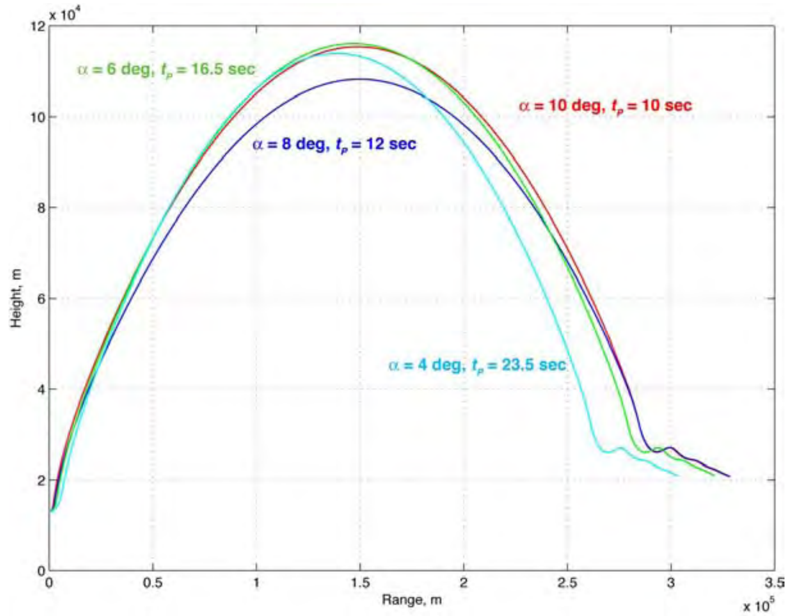
- ❑ ***Energy height?***
- ❑ ***Contours?***
- ❑ ***Subsonic minimum-time climb?***
- ❑ ***Supersonic minimum-time climb?***
- ❑ ***Minimum-fuel climb?***

$$\boxed{\frac{dE_h}{dm_{fuel}} = \frac{dE_h}{dt}\frac{dt}{dm_{fuel}} = \frac{1}{\dot{m}_{fuel}}\left[\frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}\right]}$$

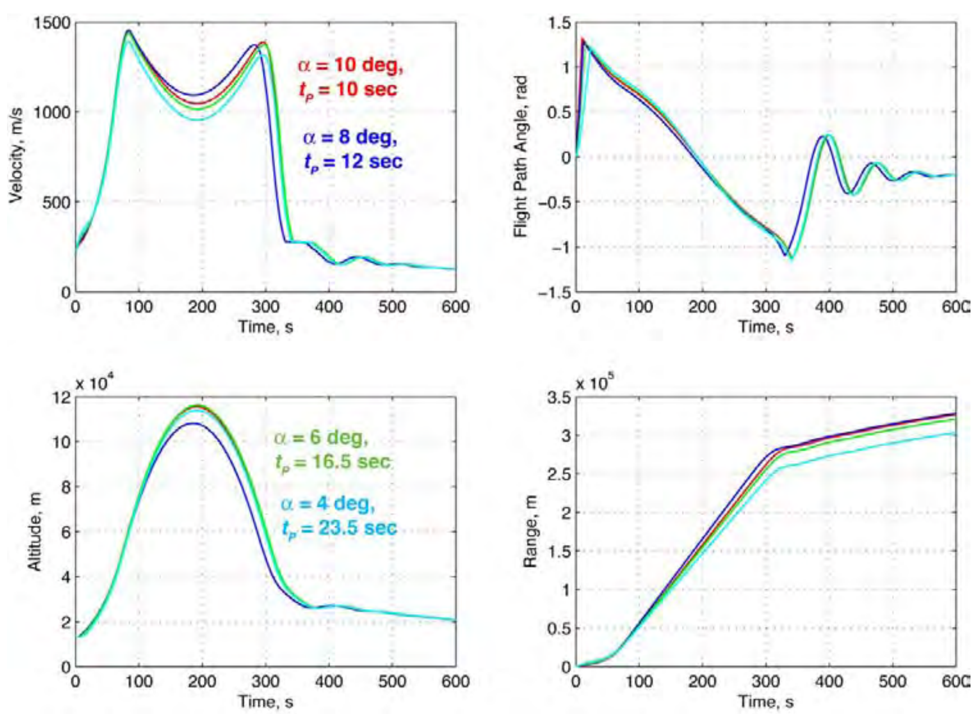
SpaceShipOne
Ansari X Prize, December 17, 2003



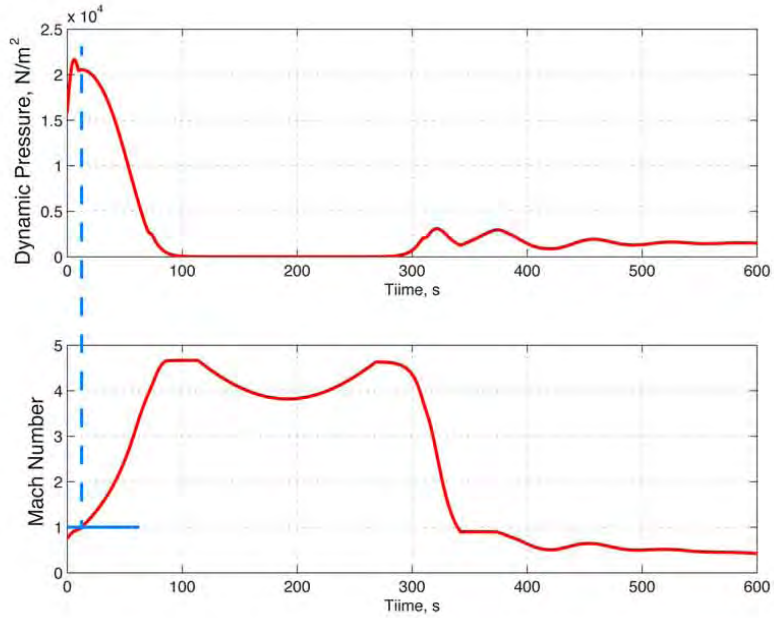
SpaceShipOne Altitude vs. Range
MAE 331 Assignment #4, 2010



SpaceShipOne State Histories



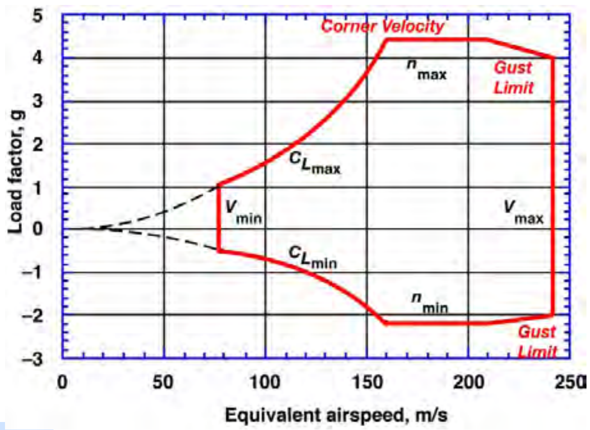
SpaceShipOne Dynamic Pressure and Mach Number Histories



The Maneuvering Envelope

Typical Maneuvering Envelope: *V-n* Diagram

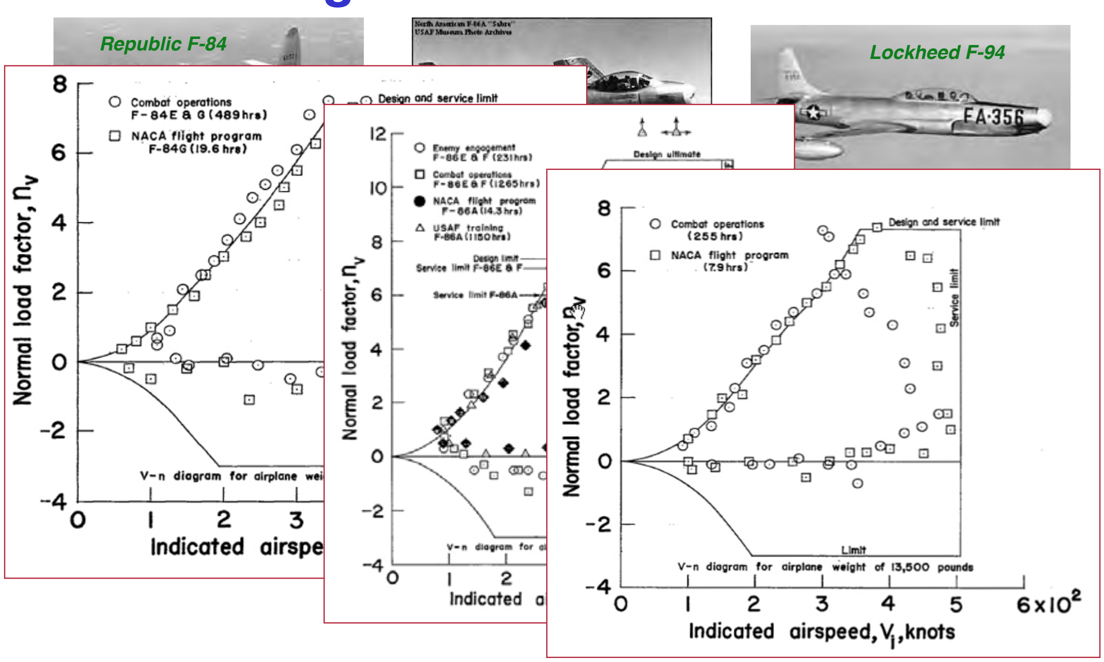
- **Maneuvering envelope**: limits on normal load factor and allowable equivalent airspeed
  - Structural factors
  - Maximum and minimum achievable lift coefficients
  - Maximum and minimum airspeeds
  - Protection against overstressing due to gusts
  - Corner Velocity: Intersection of maximum lift coefficient and maximum load factor



- Typical positive load factor limits
  - Transport: > 2.5
  - Utility: > 4.4
  - Aerobatic: > 6.3
  - Fighter: > 9

- Typical negative load factor limits
  - Transport: < –1
  - Others: < –1 to –3

Maneuvering Envelopes (*V-n Diagrams*) for Three Fighters of the Korean War Era



Turning Flight

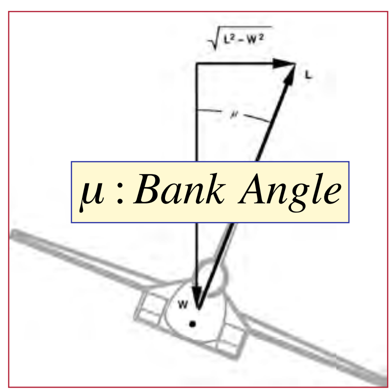
Level Turning Flight

- **Level flight = constant altitude**
- **Sideslip angle = 0**
- **Vertical force equilibrium**

$$\boxed{L\cos\mu = W}$$

- **Load factor**

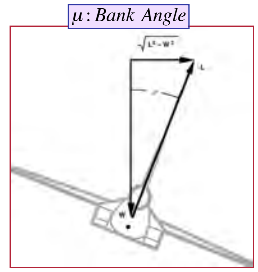
$$\boxed{n = L/W = L/mg = \sec\mu, \text{"}g\text{"}s}$$



- **Thrust required to maintain level flight**

$$T_{req} = \left(C_{D_o} + \varepsilon C_L^{\,2}\right)\frac{1}{2}\rho V^2 S = D_o + \frac{2\varepsilon}{\rho V^2 S}\left(\frac{W}{\cos\mu}\right)^2$$
$$= D_o + \frac{2\varepsilon}{\rho V^2 S}(nW)^2$$

Maximum Bank Angle in Steady Level Flight



Bank angle

$$\cos\mu = \frac{W}{C_L\bar{q}S} = \frac{1}{n} = W\sqrt{\frac{2\varepsilon}{\left(T_{req}-D_o\right)\rho V^2 S}}$$

$$\mu = \cos^{-1}\left(\frac{W}{C_L\bar{q}S}\right) = \cos^{-1}\left(\frac{1}{n}\right) = \cos^{-1}\left[W\sqrt{\frac{2\varepsilon}{\left(T_{req}-D_o\right)\rho V^2 S}}\right]$$

Bank angle is limited by

$$C_{L_{\max}} \ or \ T_{\max} \ or \ n_{\max}$$

Turning Rate and Radius in Level Flight

Turning rate

$$\dot{\xi} = \frac{C_L\bar{q}S\sin\mu}{mV} = \frac{W\tan\mu}{mV} = \frac{g\tan\mu}{V} = \frac{\sqrt{L^2-W^2}}{mV} = \frac{W\sqrt{n^2-1}}{mV} = \frac{\sqrt{\left(T_{req}-D_o\right)\rho V^2 S/2\varepsilon - W^2}}{mV}$$

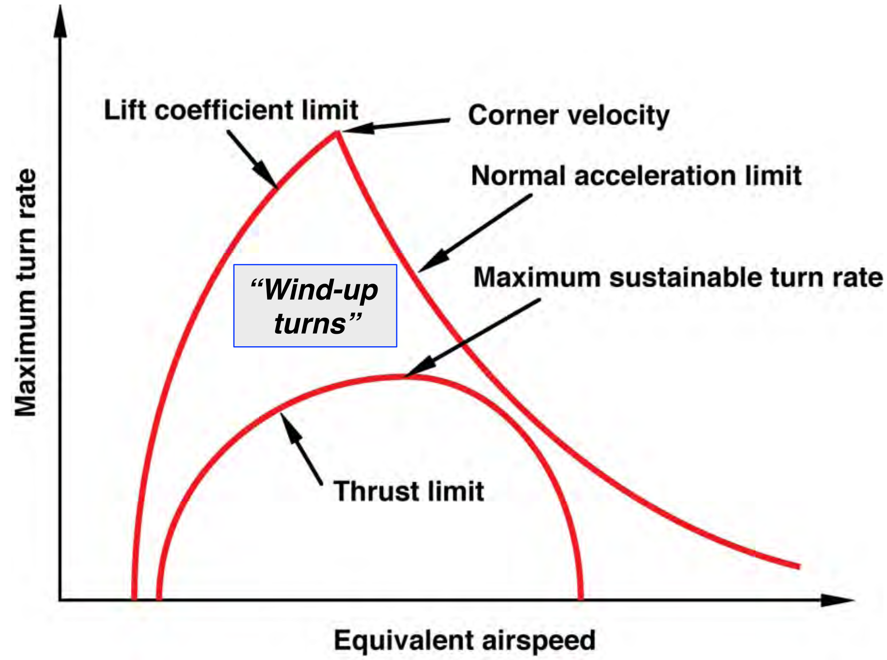
Turning rate is limited by

$$C_{L_{\max}} \ or \ T_{\max} \ or \ n_{\max}$$

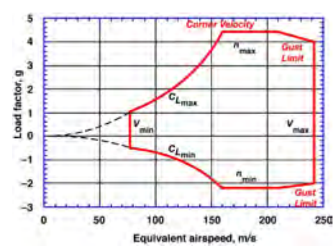
Turning radius

$$R_{turn} = \frac{V}{\dot{\xi}} = \frac{V^2}{g\sqrt{n^2-1}}$$

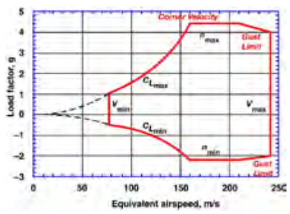
Maximum Turn Rates



Corner Velocity Turn



- **Corner velocity** $$V_{corner} = \sqrt{\frac{2n_{\max}W}{C_{L_{mas}}\rho S}}$$
- **For steady climbing or diving flight** $$\sin\gamma = \frac{T_{\max} - D}{W}$$
- **Turning radius** $$R_{turn} = \frac{V^2\cos^2\gamma}{g\sqrt{n_{\max}^2 - \cos^2\gamma}}$$



Corner Velocity Turn

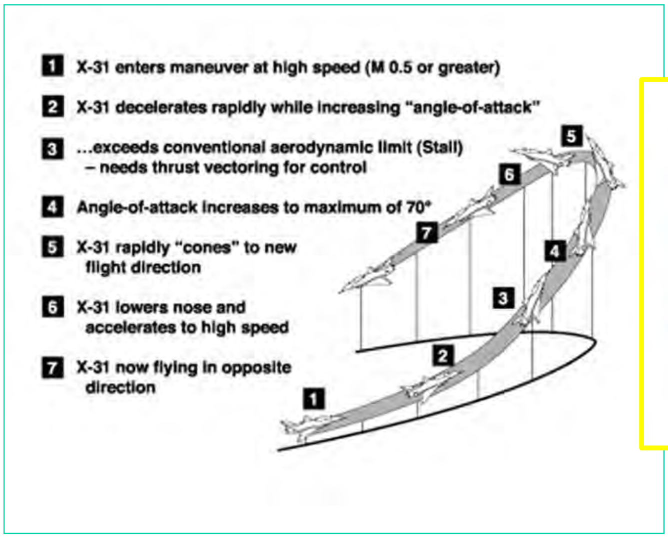
- **Turning rate** $$\dot{\xi} = \sqrt{\frac{g\left(n_{\max}^2 - \cos^2\gamma\right)}{V\cos\gamma}}$$
- **Time to complete a full circle** $$t_{2\pi} = \frac{V\cos\gamma}{g\sqrt{n_{\max}^2 - \cos^2\gamma}}$$
- **Altitude gain/loss** $$\Delta h_{2\pi} = t_{2\pi} V \sin\gamma$$

Checklist

- ❑ *V-n diagram?*
- ❑ *Maneuvering envelope?*
- ❑ *Level turning flight?*
- ❑ *Limiting factors?*
- ❑ *Wind-up turn?*
- ❑ *Corner velocity?*

Herbst Maneuver

- Minimum-time reversal of direction
- Kinetic-/potential-energy exchange
- Yaw maneuver at low airspeed
- *X-31* performing the maneuver



Next Time: Aircraft Equations of Motion

Reading:
Flight Dynamics,
Section 3.1, 3.2, pp. 155-161

Learning Objectives

What use are the equations of motion?
How is the angular orientation of the airplane described?
What is a cross-product-equivalent matrix?
What is angular momentum?
How are the inertial properties of the airplane described?
How is the rate of change of angular momentum calculated?

Supplemental Material

Gliding Flight of the P-51 Mustang

Maximum Range Glide

Loaded Weight = 9,200 lb (3,465 kg)

$$(L/D)_{max} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}} = 16.31$$

$$\gamma_{MR} = -\cot^{-1}\left(\frac{L}{D}\right)_{max} = -\cot^{-1}(16.3) = -3.5°$$

$$(C_D)_{L/D_{max}} = 2C_{D_o} = 0.0326$$

$$(C_L)_{L/D_{max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = 0.531$$

$$V_{L/D_{max}} = \frac{76.49}{\sqrt{\rho}} \text{ m/s}$$

$$\dot{h}_{L/D_{max}} = V\sin\gamma = -\frac{4.68}{\sqrt{\rho}} \text{ m/s}$$

$$R_{h_o=10km} = (16.31)(10) = 163.1\ km$$

Maximum Endurance Glide

Loaded Weight = 9,200 lb (3,465 kg)

$$S = 21.83 \text{ m}^2$$

$$C_{D_{ME}} = 4C_{D_o} = 4(0.0163) = 0.0652$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} = \sqrt{\frac{3(0.0163)}{0.0576}} = 0.921$$

$$(L/D)_{ME} = 14.13$$

$$\dot{h}_{ME} = -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_{D_{ME}}}{C_{L_{ME}}^{3/2}}\right) = -\frac{4.11}{\sqrt{\rho}} \text{ m/s}$$

$$\gamma_{ME} = -4.05°$$

$$V_{ME} = \frac{58.12}{\sqrt{\rho}} \text{ m/s}$$