Signal Processing Systems

Machine learning for signal processing *[5LSL0]*

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Technische Universiteit
Eindhoven
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Where innovation starts

Optimum linear filters and adaptive filters

Content

Focus on single channel adaptive algorithms using FIR structures

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Focus on single channel adaptive algorithms using FIR structures

- Minimum Mean Squared Error
- Gradient Descent Algorithm
- Adaptive (N)LMS
- Newton algorithm
- Recursive Least Squares (RLS)

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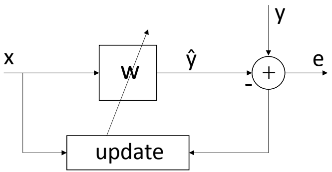
General Adaptive Filter model



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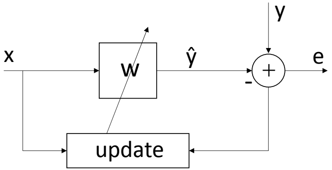
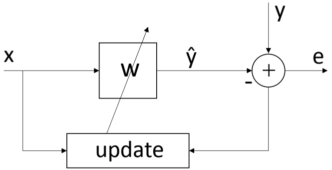


Notes:

- Input signal x and desired response y correlated

General Adaptive Filter model



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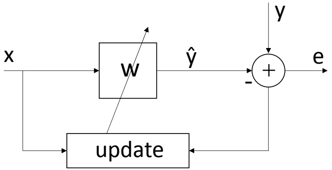
  - All signals have zero average
  - Filter w: FIR

General Adaptive Filter model

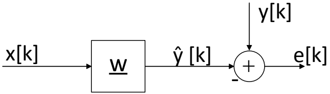


Notes:

- Input signal x and desired response y correlated
- Pragmatic choices:
  - All signals have zero average
  - Filter w: FIR
- Calculation of weight of filter w:
  - Use quadratic cost function: $J = f(e^2)$
  - First fixed weights (MMSE), then adaptive

Fixed weights: MMSE

General Minimum Mean Squared Error (MMSE) model:



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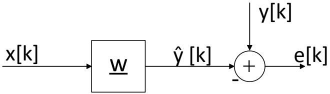

Goal:

Given N samples $\underline{\mathbf{x}}[k] = (x[k], x[k-1], \cdots, x[k-N+1])^t$ calculate coefficients fixed filter $\underline{\mathbf{w}} = (w_0, w_1, \cdots, w_{N-1})^t$ such that Mean Squared Error (MSE) $J = E\left\{e^2[k]\right\} = E\{(y[k] - \hat{y}[k])^2\}$ is minimized.

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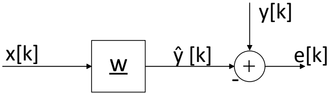

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MMSE Optimization problem:

$$\boxed{\begin{array}{c}\text{Given FIR samples } x[k-i] \text{ for } i = 0, 1, \cdots N-1 \\ \underline{\mathbf{w}}_o = \arg\min_{\underline{\mathbf{w}}} \left(E\left\{e^2[k]\right\}\right)\end{array}}$$

Fixed weights: MMSE

$$
\begin{aligned}
J &= E\{(y[k] - \underline{\mathbf{w}}^t \cdot \underline{\mathbf{x}}[k]) \cdot (y[k] - \underline{\mathbf{x}}^t[k] \cdot \underline{\mathbf{w}})\} \\
&= E\{y^2[k]\} - \underline{\mathbf{w}}^t E\{\underline{\mathbf{x}}[k]y[k]\} - E\{y[k]\underline{\mathbf{x}}^t[k]\}\underline{\mathbf{w}} + \underline{\mathbf{w}}^t E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\}\underline{\mathbf{w}}
\end{aligned}
$$

Fixed weights: MMSE

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$$\Rightarrow \boxed{J = E\{y^2[k]\} - \underline{\mathbf{w}}^t \underline{\mathbf{r}}_{yx} - \underline{\mathbf{r}}_{yx}^t \underline{\mathbf{w}} + \underline{\mathbf{w}}^t \mathbf{R}_x \underline{\mathbf{w}}}$$

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$$

with cross correlation $\rho_{yx}[\tau] = E\{y[k]x[k-\tau]\}$:

$$
\underline{\mathbf{r}}_{yx} = E\{y[k]\underline{\mathbf{x}}[k]\} = (\rho_{yx}[0], \rho_{yx}[1], \cdots, \rho_{yx}[N-1])^t
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$$\underline{\mathbf{r}}_{yx} = E\{y[k]\underline{\mathbf{x}}[k]\} = (\rho_{yx}[0], \rho_{yx}[1], \cdots, \rho_{yx}[N-1])^t$$

and autocorrelation: $\rho_x[\tau] = E\{x[k]x[k-\tau]\} = \rho_x[-\tau]$

$$\mathbf{R}_x = E\{\underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\} = \begin{pmatrix} \rho_x[0] & \rho_x[1] & \cdots & \rho_x[N-1] \\ \rho_x[1] & \rho_x[0] & \cdots & \rho_x[N-2] \\ \vdots & \vdots & \vdots & \vdots \\ \rho_x[N-1] & \rho_x[N-2] & \cdots & \rho_x[0] \end{pmatrix}$$

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$\Rightarrow$ **Normal Equations** $\boxed{\mathbf{R}_x \cdot \underline{\mathbf{w}} = \underline{\mathbf{r}}_{yx}}$

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General expression: $\boxed{J = J_{min} + (\underline{\mathbf{w}} - \underline{\mathbf{w}}_o)^t \cdot \mathbf{R}_x \cdot (\underline{\mathbf{w}} - \underline{\mathbf{w}}_o)}$

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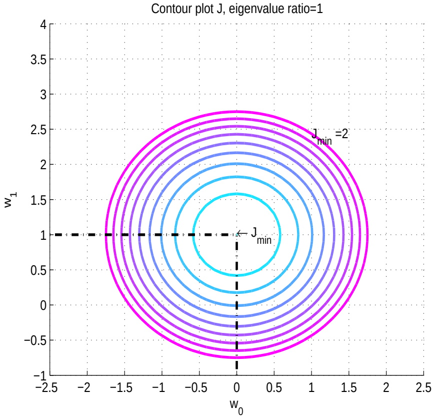
From general expression $\Rightarrow$ J quadratic in $\underline{\mathbf{w}}$ thus $\underline{\mathbf{w}}_o$ really minimum

Fixed weights: MMSE

Contour plots $J = J_{min} + (\underline{\mathbf{w}} - \underline{\mathbf{w}}_o)^t \cdot \mathbf{R}_x \cdot (\underline{\mathbf{w}} - \underline{\mathbf{w}}_o)$



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Fixed weights: MMSE

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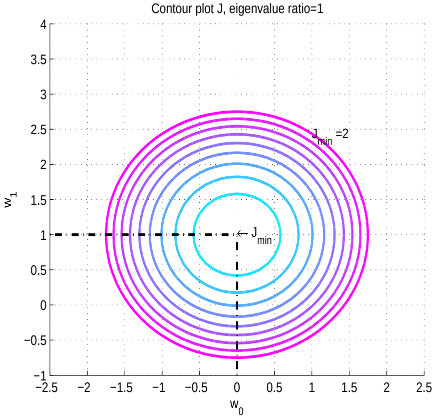
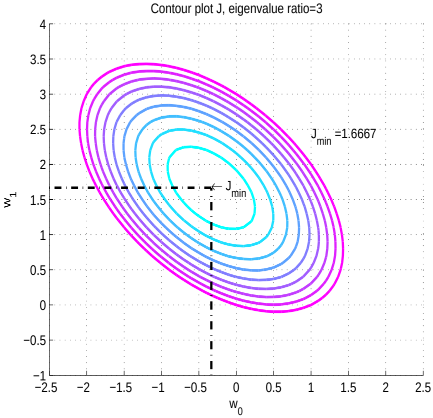
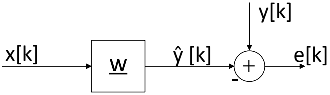




Eigenvalues: see Appendix

Least Squares (LS)



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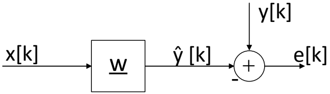

Different quadratic cost functions:

- Mean Square Error (MSE):

$$J_{mse} = E\{e^2[k]\} = E\{(y[k] - \underline{\mathbf{w}}^t \underline{\mathbf{x}}[k])^2\}$$

$\Rightarrow$ Minimum MSE (MMSE) = Wiener

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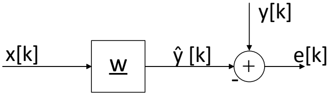

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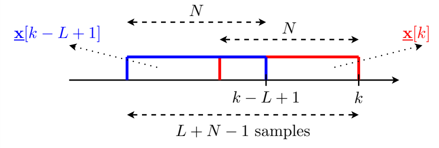
- **Least Square (LS):** If statistical information is not available $\Rightarrow$

  Use criterion based on data (thus without $E\{\cdot\}$)

Collect L (≥ 1) data vectors $\underline{\mathbf{x}}[k-i]$ (each of length N)



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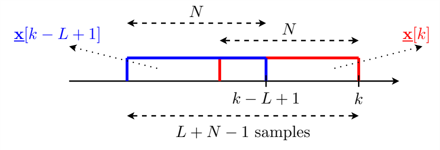

Available data (for $i = 0, 1, \cdots, L-1$):

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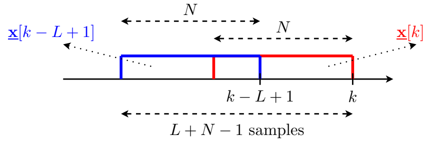

Available data (for $i = 0, 1, \cdots, L-1$):

- *Input signal samples/ vectors* $\underline{\mathbf{x}}[k-i]$

$$\underline{\mathbf{x}}[k-i] = (x[k-i], x[k-i-1], \cdots, x[k-i-N+1])^t$$

- *Reference signal samples:* $y[k-i]$

- *Residual signal samples:* $e[k-i] = y[k-i] - \underline{\mathbf{x}}^t[k-i] \cdot \underline{\mathbf{w}}$

Notation:

$$
\mathbf{X}[k] = \begin{pmatrix} \underline{\mathbf{x}}^t[k] \\ \underline{\mathbf{x}}^t[k-1] \\ \vdots \\ \underline{\mathbf{x}}^t[k-L+1] \end{pmatrix} \qquad \underline{\mathbf{w}} = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{pmatrix}
$$

$$
\underline{\mathbf{y}}[k] = \begin{pmatrix} y[k] \\ y[k-1] \\ \vdots \\ y[k-L+1] \end{pmatrix} \qquad \underline{\mathbf{e}}[k] = \begin{pmatrix} e[k] \\ e[k-1] \\ \vdots \\ e[k-L+1] \end{pmatrix}
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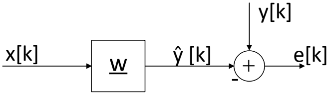
Simplified notation (skip time indices):

$$\underline{\mathbf{e}} = \underline{\mathbf{y}} - \mathbf{X} \cdot \underline{\mathbf{w}}$$



LS problem formulation:

$$\boxed{\underline{\mathbf{w}}_{ls,o} = \arg\min_{\underline{\mathbf{w}}} \left|\underline{\mathbf{y}} - \mathbf{X} \cdot \underline{\mathbf{w}}\right|^2}$$

$$J_{ls} \;=\; \sum_{i=0}^{L-1} e^2[k-i] = \underline{\mathbf{e}}^t \cdot \underline{\mathbf{e}} = (\underline{\mathbf{y}}^t - \underline{\mathbf{w}}^t \mathbf{X}^t) \cdot (\underline{\mathbf{y}} - \mathbf{X}\underline{\mathbf{w}})$$

$$
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&= \underline{\mathbf{y}}^t \underline{\mathbf{y}} + \underline{\mathbf{w}}^t \mathbf{X}^t \mathbf{X} \underline{\mathbf{w}} - \underline{\mathbf{w}}^t \mathbf{X}^t \underline{\mathbf{y}} - \underline{\mathbf{y}}^t \mathbf{X} \underline{\mathbf{w}}
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\end{aligned}
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Minimum by setting gradient equal to zero:

$$
\frac{\mathrm{d}J_{ls}}{\mathrm{d}\underline{\mathbf{w}}} = \underline{\bigtriangledown}_{ls} = -2(\mathbf{X}^t \underline{\mathbf{y}} - \mathbf{X}^t \mathbf{X} \cdot \underline{\mathbf{w}}) = \underline{\mathbf{0}}
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With $\overline{\mathbf{R}} = \mathbf{X}^t \mathbf{X}$ and $\underline{\mathbf{r}} = \mathbf{X}^t \underline{\mathbf{y}}$

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With $\overline{\mathbf{R}} = \mathbf{X}^t \mathbf{X}$ and $\underline{\overline{\mathbf{r}}} = \mathbf{X}^t \underline{\mathbf{y}}$

$\Rightarrow$ Normal Equations $\boxed{\overline{\mathbf{R}}_x \cdot \underline{\mathbf{w}} = \underline{\overline{\mathbf{r}}}_{yx}}$

$\Rightarrow$ Wiener filter $\boxed{\underline{\mathbf{w}}_{ls,o} = \overline{\mathbf{R}}_x^{-1} \cdot \underline{\overline{\mathbf{r}}}_{yx}}$

LS: Correspondence with MMSE

Use time-averaging (ergodicity):

$$\hat{\mathbf{R}}_x = \frac{1}{L}\sum_{i=0}^{L-1} \underline{\mathbf{x}}[k-i] \cdot \underline{\mathbf{x}}^t[k-i] = \frac{1}{L}\mathbf{X}^t \cdot \mathbf{X} = \frac{1}{L}\overline{\mathbf{R}}_x$$

$$\hat{\underline{\mathbf{r}}}_{yx} = \frac{1}{L}\sum_{i=0}^{L-1} \underline{\mathbf{x}}[k-i] \cdot y[k-i] = \frac{1}{L}\mathbf{X}^t \cdot \underline{\mathbf{y}} = \frac{1}{L}\overline{\underline{\mathbf{r}}}_{yx}$$

LS: Correspondence with MMSE

Use time-averaging (ergodicity):

$$\hat{\mathbf{R}}_x = \frac{1}{L}\sum_{i=0}^{L-1} \underline{\mathbf{x}}[k-i] \cdot \underline{\mathbf{x}}^t[k-i] = \frac{1}{L}\mathbf{X}^t \cdot \mathbf{X} = \frac{1}{L}\overline{\mathbf{R}}_x$$

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with $\hat{\mathbf{R}}_x$ estimate of $\mathbf{R}_x$ and $\hat{\underline{\mathbf{r}}}_{yx}$ estimate of $\underline{\mathbf{r}}_{yx}$

$$\Rightarrow \quad \hat{\underline{\mathbf{w}}}_{mmse} = \left(\frac{1}{L}\overline{\mathbf{R}}_x\right)^{-1} \cdot \left(\frac{1}{L}\overline{\underline{\mathbf{r}}}_{yx}\right) = \overline{\mathbf{R}}_x^{-1} \cdot \overline{\underline{\mathbf{r}}}_{yx} = \underline{\mathbf{w}}_{ls}$$

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Finally note that for ergodic processes:

$$\lim_{L\to\infty}\frac{1}{L}\overline{\mathbf{R}}_x = \mathbf{R}_x\ ;\ \lim_{L\to\infty}\frac{1}{L}\overline{\underline{\mathbf{r}}}_{yx} = \underline{\mathbf{r}}_{yx}\ ;\ \lim_{L\to\infty}\underline{\mathbf{w}}_{ls} = \underline{\mathbf{w}}_{mmse}$$

Steepest gradient descent (GD)

Problem: Optimal Wiener involves $\mathbf{R}_x^{-1}$

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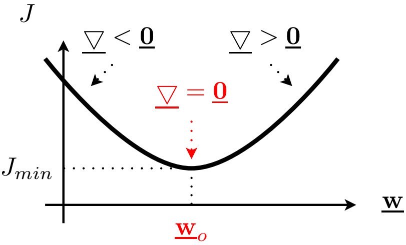
Goal: Decrease J each new iteration


J
$\underline{\nabla} < \underline{\mathbf{0}}$
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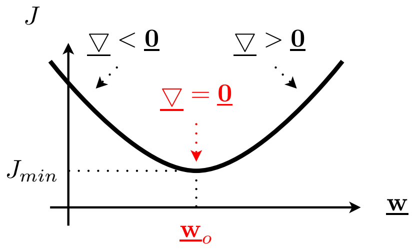
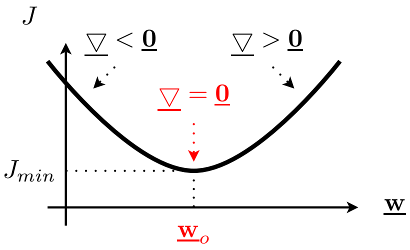
$\underline{\nabla} = \underline{\mathbf{0}}$
J_{min}
$\underline{\mathbf{w}}$
$\underline{\mathbf{w}}_o$



GD principle: Update in negative gradient direction

$\Leftrightarrow \underline{\mathbf{w}} \doteq \underline{\mathbf{w}} - \alpha \underline{\nabla}$ with adaptation constant $\alpha \geq 0$



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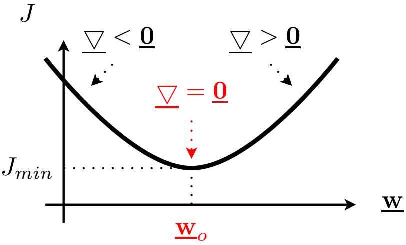

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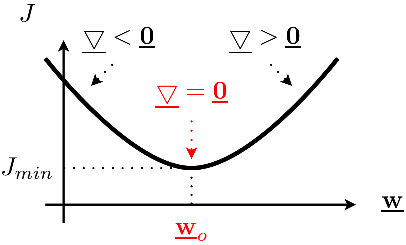

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Notes: 1) No matrix inversion needed! 2) Usually $\underline{\mathbf{w}}[0] = \underline{\mathbf{0}}$

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For exact proof we need **stability analysis**

Stability GD

Define difference weight vector: $\underline{\mathbf{d}}[k] = \underline{\mathbf{w}}[k] - \underline{\mathbf{w}}_o$

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Note:

When stable $\Rightarrow \underline{\mathbf{d}}[\infty] = \underline{\mathbf{0}} \Rightarrow \underline{\mathbf{w}}[\infty] \simeq$ Wiener

<u>How</u> do weights converge:

Stability GD

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Use eigenvalue decomposition (see Appendix):

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$$\begin{aligned} \Rightarrow \; (\mathbf{I} - 2\alpha\mathbf{R}_x)^k &= (\mathbf{Q}\mathbf{Q}^h - 2\alpha\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^h)^k \\ &= \mathbf{Q}(\mathbf{I} - 2\alpha\mathbf{\Lambda})^k\mathbf{Q}^h \end{aligned}$$

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Stability GD

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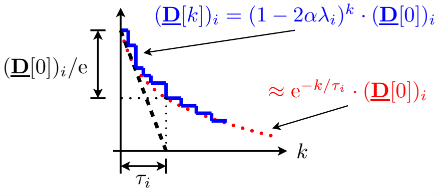
Convergence GD



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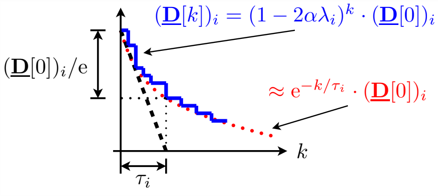

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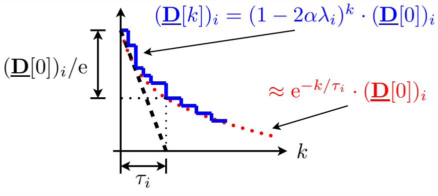

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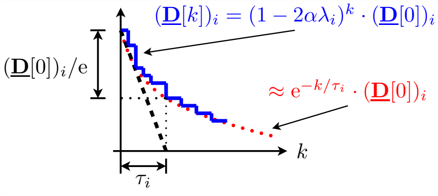

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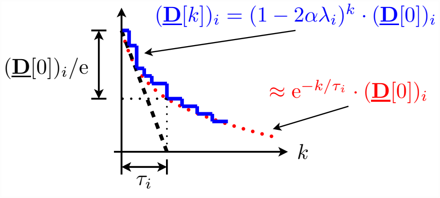

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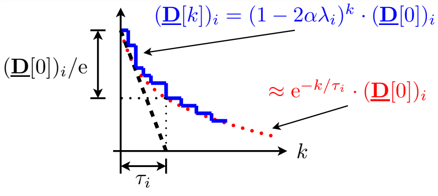
Note:

Overall time constant depends on eigenvalue spread $\Gamma_x = \lambda_{max}/\lambda_{min}$.
Thus, the larger Γ_x the longer it takes for adaptation.

Convergence GD



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Q: What happens for white noise?

Convergence GD

Example with $\Gamma_x = \lambda_{max}/\lambda_{min} = 3$

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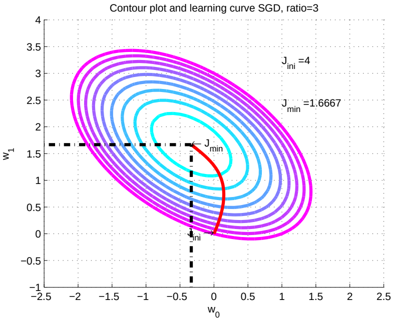
Learning curve in contour plot J



Convergence GD

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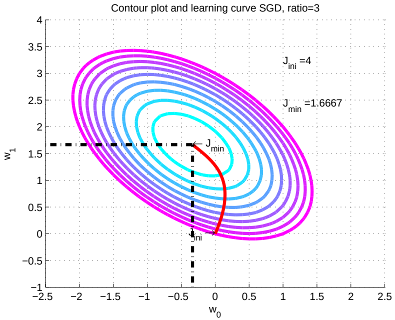
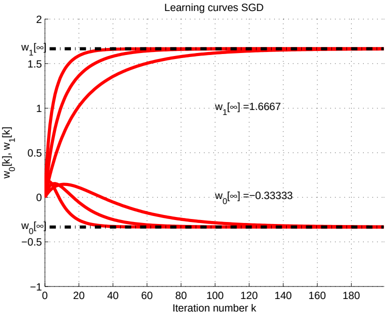


Learning curves for different α



Least Mean Square (LMS)

Motivation: GD not practical. Gradient assumes known $\mathbf{R}_x$ and $\underline{\mathbf{r}}_{yx}$

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LMS principle: Use instantaneous estimate of gradient:

$$\begin{aligned}\underline{\hat{\bigtriangledown}}[k] &= -2\left(y[k]\underline{\mathbf{x}}[k] - \underline{\mathbf{x}}[k]\underline{\mathbf{x}}^t[k]\underline{\mathbf{w}}[k]\right) \\ &= -2\underline{\mathbf{x}}[k]\left(y[k] - \underline{\mathbf{x}}^t[k]\underline{\mathbf{w}}[k]\right) = -2\underline{\mathbf{x}}[k]e[k]\end{aligned}$$

Least Mean Square (LMS)

Motivation: GD not practical. Gradient assumes known $\mathbf{R}_x$ and $\underline{\mathbf{r}}_{yx}$

LMS principle: Use instantaneous estimate of gradient:

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With $\underline{\mathbf{w}} \doteq \underline{\mathbf{w}} - \alpha\underline{\hat{\nabla}} \Rightarrow$ LMS algorithm (Widrow, 1975):

$$\boxed{\begin{array}{lcl} k = 0 & : & \underline{\mathbf{w}}[0] = \underline{\mathbf{0}} \quad \text{(usually)} \\ k > 0 & : & \hat{y}[k] = \underline{\mathbf{w}}^t[k] \cdot \underline{\mathbf{x}}[k] \\ & & e[k] = y[k] - \hat{y}[k] \\ & & \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha\underline{\mathbf{x}}[k]e[k] \end{array}}$$

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Note: $\underline{\mathbf{w}}^t[k]\cdot\underline{\mathbf{x}}[k]$ is "convolution" and $\underline{\mathbf{x}}[k]e[k]$ "correlation"

Normalized LMS

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In practice $\hat{\sigma}_x^2[k] \Rightarrow$ time-varying step size. E.g.:

- $\hat{\sigma}_x^2[k] = \beta\hat{\sigma}_x^2[k-1] + (1-\beta)\frac{\underline{\mathbf{x}}^t[k]\underline{\mathbf{x}}[k]}{N}$ with $0 < \beta < 1$
- $\hat{\sigma}_x^2[k] = \frac{\underline{\mathbf{x}}^t[k]\underline{\mathbf{x}}[k]}{N} + \epsilon$ with ϵ some small constant

Newton

Convergence gradient based algorithms depends on coloration input:

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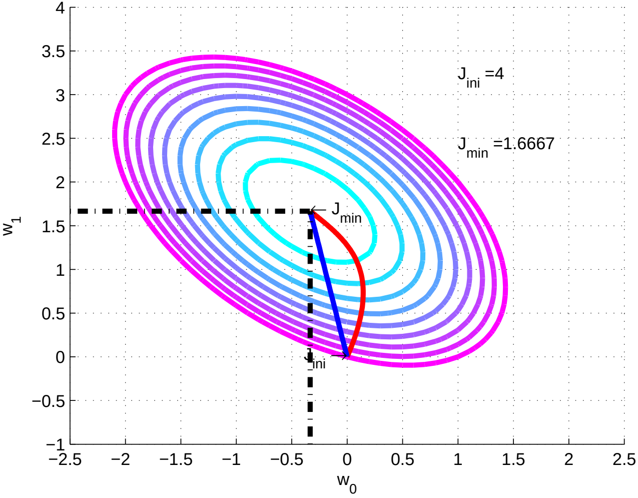
- Newton $\equiv$ GD with white noise input!

Learning curves in contour plot: Newton vs. GD



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$\Rightarrow$ Complexity Newton algorithm huge

$\Rightarrow$ Need for efficient solution with estimate of $\mathbf{R}_x$

$\Rightarrow$ Different algorithms, e.g. RLS.

Recursive Least Squares (RLS)

For data block length L fixed, Least Squares problem becomes:

$$\min_{\underline{\mathbf{w}}[k]} |\underline{\mathbf{y}}[k] - \mathbf{X}[k] \cdot \underline{\mathbf{w}}[k]|^2 \quad \Rightarrow \quad \underline{\mathbf{w}}_{LS}[k] = \left(\mathbf{X}^t[k]\mathbf{X}[k]\right)^{-1} \left(\mathbf{X}^t[k]\underline{\mathbf{y}}[k]\right)$$

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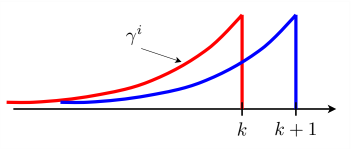

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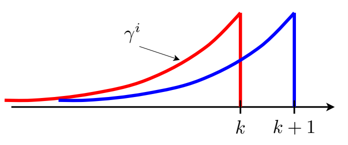

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$$\mathbf{X}[k] = \begin{pmatrix} \gamma^0 \underline{\mathbf{x}}^t[k] \\ \cdots \\ \gamma^i \underline{\mathbf{x}}^t[k-i] \\ \cdots \\ \gamma^k \underline{\mathbf{x}}^t[0] \end{pmatrix} \text{ and } \underline{\mathbf{y}}[k] = \begin{pmatrix} \gamma^0 y[k] \\ \cdots \\ \gamma^i y[k-i] \\ \cdots \\ \gamma^k y[0] \end{pmatrix}$$

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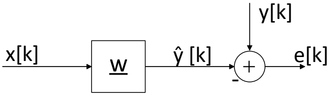
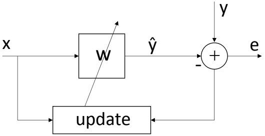
- Decorrelation takes place in algorithm

Summary



	MMSE	LS
Auto correlation	$\mathbf{R}_x = E\{\underline{\mathbf{x}}[k] \cdot \underline{\mathbf{x}}^t[k]\}$	$\overline{\mathbf{R}}_x = \mathbf{X}^t \cdot \mathbf{X}$
Cross correlation	$\underline{\mathbf{r}}_{yx} = E\{y[k] \cdot \underline{\mathbf{x}}[k]\}$	$\overline{\underline{\mathbf{r}}}_{yx} = \mathbf{X}^t \cdot \underline{\mathbf{y}}$
Error J	$E\{e^2[k]\}$	$\sum_{i=0}^{L-1} e^2[k-i]$
Criterion	$\min_{\underline{\mathbf{w}}}\{E\{e^2[k]\}\}$	$\min_{\underline{\mathbf{w}}} \lvert\underline{\mathbf{y}} - \mathbf{X} \cdot \underline{\mathbf{w}}\rvert^2$
Opt. solution $\underline{\mathbf{w}}_o$	$\mathbf{R}_x^{-1} \cdot \underline{\mathbf{r}}_{yx}$	$\overline{\mathbf{R}}_x^{-1} \cdot \overline{\underline{\mathbf{r}}}_{yx}$
Min. error J_{min}	$E\{y^2\} - \underline{\mathbf{r}}_{yx}^t \mathbf{R}_x^{-1} \underline{\mathbf{r}}_{yx}$	$\underline{\mathbf{y}}^t \underline{\mathbf{y}} - \overline{\underline{\mathbf{r}}}_{yx}^t \overline{\mathbf{R}}_x^{-1} \overline{\underline{\mathbf{r}}}_{yx}$

Summary



Simple adaptive algorithms (no decorrelation):

$$\text{GD} \quad : \quad \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha(\underline{\mathbf{r}}_{yx} - \mathbf{R}_x \underline{\mathbf{w}}[k])$$

$$\text{(N)LMS} \quad : \quad \underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + \frac{2\alpha}{\hat{\sigma}_x^2} \underline{\mathbf{x}}[k] e^*[k]$$

Summary

Algorithms with improved convergence:

LMS/Newton : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1}\underline{\mathbf{x}}[k]r[k]$

Newton : $\underline{\mathbf{w}}[k+1] = \underline{\mathbf{w}}[k] + 2\alpha \mathbf{R}_x^{-1} \cdot \left(\underline{\mathbf{r}}_{yx} - \mathbf{R}_x \underline{\mathbf{w}}[k]\right)$

RLS :

$$\underline{\mathbf{g}}[k+1] = \frac{\overline{\mathbf{R}}_x^{-1}[k]\underline{\mathbf{x}}[k+1]}{\gamma^2 + \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}_x^{-1}[k]\underline{\mathbf{x}}[k+1]}$$

$$\overline{\mathbf{R}}_x^{-1}[k+1] = \gamma^{-2}\left(\overline{\mathbf{R}}_x^{-1}[k] - \underline{\mathbf{g}}[k+1] \cdot \underline{\mathbf{x}}^t[k+1]\overline{\mathbf{R}}_x^{-1}[k]\right)$$

$$\overline{\underline{\mathbf{r}}}_{yx}[k+1] = \gamma^2 \overline{\underline{\mathbf{r}}}_{yx}[k] + \underline{\mathbf{x}}[k+1] \cdot y[k+1]$$

$$\underline{\mathbf{w}}[k+1] = \overline{\mathbf{R}}_x^{-1}[k+1] \cdot \underline{\mathbf{r}}_{yx}[k+1]$$

Appendix
Optimum Linear Filters & Adaptive Signal Processing

Content appendix

- Eigenvalue problem

Eigenvalue problem

Procedure: With eigenvalues λ_i and eigenvectors $\underline{\mathbf{q}}_i$:

$$\mathbf{R} \cdot \underline{\mathbf{q}}_i = \lambda_i \cdot \underline{\mathbf{q}}_i \;\; \Rightarrow \;\; (\mathbf{R} - \lambda_i \mathbf{I}) \cdot \underline{\mathbf{q}}_i = \underline{\mathbf{0}} \text{ for } i = 0, 1, \cdots, N-1$$

Eigenvalue problem

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With $\mathbf{Q} = (\underline{\mathbf{q}}_0, \cdots, \underline{\mathbf{q}}_{N-1})$ and $\mathbf{\Lambda} = diag\{\lambda_0, \cdots, \lambda_{N-1}\}$

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Property: *Eigenvectors* $\underline{\mathbf{q}}_i$ *orthogonal* $\Rightarrow$

$$\mathbf{Q}^h \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^h = c \cdot \mathbf{I} \quad \text{with} \quad c \quad \text{some constant}$$

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Main result:

Diagonalization: $\boxed{\mathbf{Q}^h\mathbf{R}\mathbf{Q} = \mathbf{\Lambda} \quad \Leftrightarrow \quad \mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^h}$

Eigenvalue problem

Example MA(1):

$x[k] = i[k] + ai[k-1]$ with $E\{i[k]\} = 0$ and $E\{i^2[k]\} = \sigma_i^2 \Rightarrow$

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Eigenvalues problem det $(\mathbf{R} - \lambda\mathbf{I}) = 0$ for $N = 2$ (with $\gamma = \rho[1]/\rho[0]$):

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix} \;;\; \mathbf{Q} = \left(\underline{\mathbf{q}}_0, \underline{\mathbf{q}}_1\right) = c \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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Notes:

- Vector $\underline{\mathbf{q}}_0$ orthogonal to $\underline{\mathbf{q}}_1$ since $\underline{\mathbf{q}}_0^t \cdot \underline{\mathbf{q}}_1 = 0$
- For white noise ($a = 0$): $\mathbf{\Lambda} = \mathbf{I}$
- For MA(1) with $N > 2$: $\mathbf{R}$ is tri-diagonal

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Eigenvalue problem

Example: Eigenvalues and psd for white noise, MA(1) and AR(1)

Eigenvalue problem

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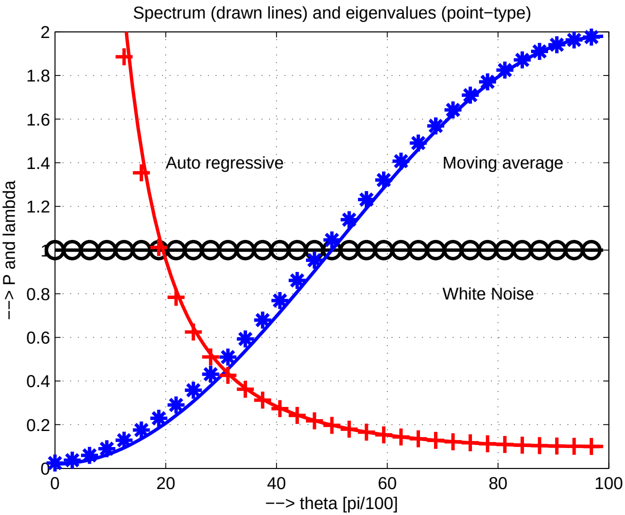

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