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Homework 2

COSC 3100
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Problem 1

Determine with proof the asymptotic relationship using Landau notation of the following pairs of functions:

1. $log(n)^2$ and $\sqrt{n}$
   Proof:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{log(n)^2} \tag{1}$$

$$\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{2ln(n)}{ln(2)^2 n}} \tag{2}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}ln(2)^2}{4ln(n)} = \infty \tag{3}$$

$$\therefore \sqrt{n} = \Omega(log(n)^2) \text{ for } n \to \infty \tag{4}$$

2. $\frac{n^2+5}{n+4}$ and n
   Proof:

$$\lim_{n\to\infty} \frac{\frac{n^2+5}{n+4}}{n} \tag{5}$$

$$\lim_{n\to\infty} \frac{\frac{2n}{n+4} - \frac{n^2+5}{(n+4)^2}}{1} \tag{6}$$

$$\lim_{n\to\infty} \frac{1-0}{1} = 1 \tag{7}$$

$$\therefore \frac{n^2+5}{n+4} = \Theta(n) \text{ for } n \to \infty \tag{8}$$

3. e^n and 3^n
   Proof:

$$\lim_{n\to\infty} \frac{e^n}{3^n} \tag{9}$$

$$\lim_{n\to\infty} \frac{e^n}{3^n ln(3)} \tag{10}$$

$$\lim_{n\to\infty} \frac{e^n 3^{-n}}{ln(3)} = \infty \tag{11}$$

$$\therefore e^n = o(3^n) \text{ for } n \to \infty \tag{12}$$

4. n^n and 2^n
   Proof:

$$\lim_{n\to\infty} \frac{n^n}{2^n} \tag{13}$$

$$\lim_{n\to\infty} \frac{n^n(ln(n)+1)}{2^n ln(2)} \tag{14}$$

$$\lim_{n\to\infty} \frac{n^n(ln(n)+1)2^{-n}}{ln(2)} = \infty \tag{15}$$

$$\therefore n^n = \Omega(2^n) \text{ for } n \to \infty \tag{16}$$

5. n^2 and 2^n
   Proof:

$$\lim_{n\to\infty} \frac{n^2}{2^n} \tag{17}$$

$$\lim_{n\to\infty} \frac{2n}{2^n ln(2)} = 0 \tag{18}$$

$$\therefore n^2 = o(2^n) \text{ for } n \to \infty \tag{19}$$

6. $nlog(n)^2$ and $nlog(n^2)$
   Proof:

$$\lim_{n\to\infty} \frac{nlog(n)^2}{nlog(n^2)} \tag{20}$$

$$\lim_{n\to\infty} \frac{log(n)^2}{log(n^2)} \tag{21}$$

$$\lim_{n\to\infty} \frac{\frac{2ln(n)}{ln(2)^2 n}}{\frac{2}{n}} \tag{22}$$

$$\lim_{n\to\infty} \frac{2nln(n)}{2ln(2)^2 n} \tag{23}$$

$$\lim_{n\to\infty} \frac{ln(n)}{ln(2)^2} = \infty \tag{24}$$

$$\therefore nlog(n)^2 = \Omega(nlog(n^2)) \text{ for } n \to \infty \tag{25}$$

Problem 2

Use induction to show that the recurrence $a_i = 2_{a_{i-1}} + 1$ with $a_0 = 0, a_1 = 1$ is solved by $a_i = 2^i - 1$

Proof:

Base Case:

$$a_2 = 2^2 - 1 = 3 \text{ (this works)} \tag{26}$$
$$a_3 = 2^3 - 1 = 7 \text{ (this works)} \tag{27}$$

Inductive Step:

Let $k \geq 2$, we want to show that $a_k = 2^k - 1$ solves the recurrence $a_{k+1} = 2_{a_k} + 1$ for all k.

$$a_{k+1} = 2_{a_k} + 1 \tag{28}$$
$$a_{k+1} = 2(2^k - 1) + 1 \tag{29}$$
$$a_{k+1} = 2^{k+1} - 2 + 1 \tag{30}$$
$$a_{k+1} = 2^{k+1} - 1 \tag{31}$$

This completes the proof by induction.

Problem 3

Use induction to show that the recurrence $a_i = \frac{2}{5}a_{i-1} + \frac{3}{5}a_{i-2}$, $a_0 = 0, a_1 = 1$ is solved by $a_i = -\frac{(-3)^i - 5^i}{8\,5^{i-1}}$

Proof:

Base Case:

$$a_2 = -\frac{(-3)^2 - 5^2}{8 \times 5^1} = \frac{2}{5} \text{ (this works)} \tag{32}$$
$$a_3 = -\frac{(-3)^3 - 5^3}{8 \times 5^2} = \frac{19}{25} \text{ (this works)} \tag{33}$$

Inductive Step:

Let $k \geq 2$, we want to show that $a_k = -\frac{(-3)^k - 5^k}{8\,5^{k-1}}$ solves the recurrence $a_{k+1} = \frac{2}{5}a_k + \frac{3}{5}a_{k-1}$ for all k.

$$\frac{2}{5}\frac{(-3)^{i-1} - 5^{i-1}}{8 \cdot 5^{i-2}} + \frac{3}{5}\frac{(-3)^{i-2} - 5^{i-2}}{8 \cdot 5^{i-3}} \tag{34}$$
$$= \frac{2 \cdot (-3)^{i-1} - 2 \cdot 5^{i-1}}{8 \cdot 5^{i-1}} + \frac{3 \cdot 5 \cdot (-3)^{i-2} - 3 \cdot 5 \cdot 5^{i-2}}{8 \cdot 5^{i-1}} \tag{35}$$

$$= \frac{2 \cdot (-3)^{i-1} - 2 \cdot 5^{i-1}}{8 \cdot 5^{i-1}} + \frac{-5 \cdot (-3)^{i-1} - 3 \cdot 5^{i-1}}{8 \cdot 5^{i-1}} \tag{36}$$

$$= \frac{-3 \cdot (-3)^{i-1} - 5 \cdot (5)^{i-1}}{8 \cdot 5^{i-1}} \tag{37}$$

$$= \frac{(-3)^{i} - 5^{i}}{8 \cdot 5^{i-1}} \tag{38}$$

This completes the proof by induction.

Problem 4

Use the substitution method to show that T(n) = T(n-1) + n + 1 implies that T(n) $\leq C$n^2 as long as $C \geq 1$ and $C \geq T(1)$.

$$T(n) = T(n-1) + n + 1 \tag{39}$$

$$T(n) \leq C(n-1)^2 + n + 1 \tag{40}$$

$$T(n) \leq Cn^2 - 2Cn + C + n + 1 \tag{41}$$

We must also account for $T(1) \leq C(1^2)$, which works.
The induction step works, if right part is less than zero.

$$-2Cn + C + n + 1 \leq 0 \tag{42}$$

$$n + 1 \leq 2Cn + C \tag{43}$$

Base Case:
for n = 0

$$1 \leq C \text{ (this works)} \tag{44}$$

for n = 1

$$2 \leq 3C \text{ (this works)} \tag{45}$$

Let's show that (n+1) works.
We want to show:

$$n + 2 \leq 2Cn + 3C \tag{46}$$

Proof:

$$n + 1 + 1 \leq 2Cn + C + 1 \tag{47}$$

Since $C + 1 \leq 2C$:

$$n + 1 + 1 \leq 2Cn + 2C \tag{48}$$

Since $C + 1 \leq 3C$:

$$n + 1 + 1 \leq 2Cn + 3C \tag{49}$$

$$\therefore n + 2 \leq 2Cn + 3C \tag{50}$$

This completes the proof by induction.