Unit 2
Probability

ESIGELEC

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PROBABILTY

Introduction

Events. Operations with events.

Probability. Basic concept and properties

Conditional probability

Law of total probability

Independence of events

Bayes’ theorem

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Summary

- In this unit you will learn the mathematical basis that will allow you to give a formal structure to the statistical inference, to be later studied:
- More specifically, our goals in this unit are:
  - Knowing the concept of event and its basic operations.
  - Knowing the concept of probability and its properties.
  - Computing probabilities of events.
  - Knowing the concept of conditional probabilities, and computing them.
  - Computing probabilities of a partition of the sample space.
  - Understanding the concept of independence of events.
  - Applying Bayes' theorem.
  - Using calculus of probabilities in simple problems.

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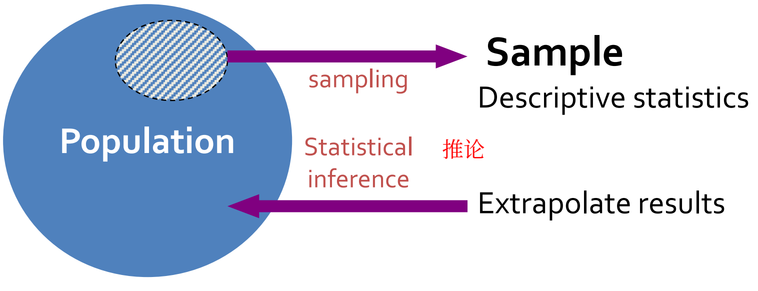
Introduction

- In the previous unit you learned how to represent a sample of data coming from the observation of a random variable over a population.
- Before going back to the population, some basic mathematical concepts are needed.
- Let's enter the calculus of probabilities!!



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Events

- A random variable is the result of a random experiment, and can be coded numerically.

Example: You want to know how many times you get six when rolling three times a 6-side dice:
- Which is the random experiment?
- Which is the random variable of interest?
- Which is the population?

roll the dice 3 times

the time s we get 6

the number we get

each time

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- Sample space: The set of all possible outcomes that can happen in the random experiment under consideration. In other words, the set constituted by all possible values of the random variable considered. It is often represented by *E or Ω*.

- **Exercise :** What is the sample space of the previous example (number of sixes when rolling 3 dices)? SOL: E={0,1,2,3} degree

- **Exercise :** What is the sample space of the random variable "height of a student"? SOL: E = [0.5,2.5] continuous

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- Event: Anything that can happen after the random experience.

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- Formally, an event is any subset of the sample space *E.* They are usually denoted by capital letters (*A*, *B*, *C*, etc.).
- The elements of an event *A* are the possible outcomes that this event contains. If any of the outcomes in the event has happened after the random experience, you say that the event has occurred. Otherwise, you will say that the event has not occurred.

- **Examples :** «You get 1 six» | «You get 2 sixes» | «You get at least 1 six» | «You get an even number of sixes»

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- **Exercise .** Mention at least 2 events related with the following random experiments:
  - Number of points when rolling a dice.
  - Getting one card from a French deck of cards (52 cards, organized in 4 groups: Clubs, Diamonds, Hearts, Spades).
  - Measuring the number of accesses to a given web server in one hour.
  - Measuring the energy consumption in an industrial plant.
  - Counting the number of defective screws in a 1000-item box, which has been randomly chosen from a batch.
  - Counting the number of calls received by a switchboard on a given day.

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- Types of events
  - Elementary (simple) event : constituted by only one outcome (element of the sample space).
  - Compound event : Consists of more than one elementary event. at least
- Some special events :
  - Sure event: It coincides with the sample space *E: always occurs!*
  - Impossible event : It coincides with the empty space $\emptyset$: never occurs!

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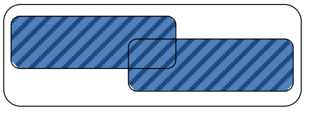
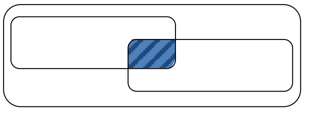
- Operations on events:
  - Union of two events : $A \cup B$

    «*A* or *B*» , «ESIGELEC student or Univ-Rouen student»



  - Intersection : $A \cap B$

    «*A* and *B*» , «ESIGELEC student and Univ-Rouen student»



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- Operations on events :
  - Complementary event of A: $\overline{A}$

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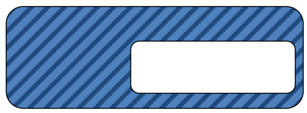
«No A» , «No ESIGELEC student»



A and $\bar{A}$ satisfy:

$$A \cup \bar{A} = E \quad \text{and} \quad A \cap \bar{A} = \varnothing$$

(A given person can either be a ESIGELEC student or not be a ESIGELEC student, and it cannot «be and not be» at the same time.)

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- **Exercise :** What event is the complementary of the complementary of A?

SOL: A

- **Exercise :** Given the experiment "randomly select a package in a delivery company", following events

  A = «Urgent delivery» , B = «Delivery with origin Valencia»

  express in words the following events:

$$A \cup B, A \cap B, \bar{A} \cap \bar{B}\ , \bar{A} \cup \bar{B}, \bar{A} \cap B, A \cap \bar{B}$$

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- **Exercise :** Given the following events

  A = «Defective piece» , B = «Piece from provider 2»

  express in words the **complementary** events of the following events:

  $A \cup B$ $\quad\quad$ $A \cap B$

- De Morgan's laws:

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B} \ , \ \overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

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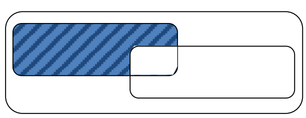
- Operations on events:
  - Difference of events : $B - A$, $B \setminus A$

    «*B* and not *A*» , Complementary of A with respect to B



There is a relation with the intersection operation:

$$B - A = B \cap \bar{A}$$

«Being an ESIGELEC student but NOT a Univ-Rouen student»

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- Relations between events:
  - A and B are incompatible (or disjoint or mutually exclusive) if A and B cannot occur at the same time, that is:

$$A \cap B = \varnothing$$

  - A implies B, $A \Rightarrow B$: Event A is contained in event B, that is:

$$A \subseteq B \text{ (}A\text{ \textbf{is a subevent of} }B\text{)}$$ A happens , so B hanppens

- **Exercise :** Give three examples of events that imply other events.

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- **Example . Incompatible events:**
  - Random experiment : Rolling a dice
  - Population : All possible (infinite) throws of the dice.
  - Sample space : $E = \{ 1 , 2 , 3 , 4 , 5 , 6 \}$
  - Event A : A = «Getting an even value»
  - Event B : B = «Getting a value lower than 4»

Are events A and B incompatible? That is, if you get an even value, does this imply that the value cannot be lower than 4? Give an event that is incompatible with B

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Probability. Basic concepts and properties

- You all have an intuitive idea of what probability is: a measure of how easy/difficult it is for an event to occur. Whenever an event has a low probability of occurring, you understand that this event is not likely to happen.
- Probability is applied to events.
- Given an event A, a number between 0 and 1 is assigned to it, called its probability, $P(A)$.
- Probability is a generalization of the relative frequency concept. Unlike the frequency, which is applied to samples, probability is applied to populations.

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- The mathematical concept of probability coincides with the intuitive idea you already have.
- Therefore, you will see that all its properties are consistent with the mathematical theory.

From now on :

LOTS OF LOGIC AND COMMON SENSE

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- Basic properties:
  - $P(A) \geq 0$, for any event A
  - $P(E) = 1$ (sure event has probability 1)
  - If A and B are incompatible, then

$$P(A \cup B) = P(A) + P(B)$$

The other properties of probability follow after the three above.

1) $P(\bar{A}) = 1 - P(A)$;
2) $P(A) \leq 1$;
3) $P(\emptyset) = 0$;
4) If $A \subseteq B$, then $P(A) \leq P(B)$

A implies B

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- Probability of the union of events:
  - You have already seen that if A and B are incompatible events ($P(A \cap B)=0$),

$$P(A \cup B) = P(A) + P(B)$$

  - For any two events A and B,

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

  - This formula can be generalized to more than two events.

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- Probability in finite symmetric sample spaces :
  - Laplace's rule : In an experiment with a finite number of outcomes, each equally likely to occur, (rolling a dice, tossing a coin, getting a card from a deck, etc.), probabilities of events can be computed using the following formula:

$$\boxed{P(A) = \frac{\text{Number of successful outcomes}}{\text{Number of possible outcomes}}}$$

each even has the same possibility

- **Example :** What is the probability of getting heads when tossing a coin?
  SOL: 0.5

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Range Experiment: Flip a coin

E={Head, Tail}

- **Exercise :** If you flip a true coin, what is the probability of getting heads? Suppose now that you flip a manipulated coin so that the probability of getting heads is twice the probability of getting tails. What is now the probability of getting heads?
  SOL: 0.5; 0.667 1/2=0.5 , 2/3=0.667

- **Exercise :** What is the probability of getting 5 when rolling a dice, knowing that the dice has been manipulated so that number 1 is twice more likely to occur than the other numbers, and the dice is white?
  SOL: 1/7

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Conditional probability

- **Examples :**
  - What is the probability of getting to class late? What is this probability if you know that there was a traffic accident on your route?
  - What is the probability of getting two when you roll a dice? What would be this probability if you know that the result was an even number? What if you know that the result was an odd number?
  - What is the probability of passing this course? Is this probability different if you study hard?
  - What is the probability of getting a defective piece? What is this probability if you know that the piece was produced by the “good” provider?

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- At times, the probability of a certain event can be modified if you know that certain another event has occurred.
- Conditional probability of A given the event B : The probability of A if B is known to occur. It is expressed as:

$$P(A \mid B) \quad \text{or} \quad P(A / B)$$

- REMARK: You are computing the probability of A, not that of B!!! Event B “has already occurred”.

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A={2}, B={2, 4, 6} P (A|VB) =1/3

- **Exercise :** What is the probability of getting a two, when throwing a dice, if you know that the result was an even number? Identify the events in this exercise and formally express the asked probability.
  SOL: 1/3

Queen, not club

- **Exercise :** What is the probability of getting a club, when extracting one card from a French deck of cards? 4/52
  4/12
  - What is this probability if you know that the card was a face card?
  - On the other hand, what is this probability if you know that the card was a "spades"? 1/13

  NOTE that sometimes knowing that a given event B has occurred does not influence the probability of event A . In such cases you will say that both events are independent.
  SOL: 4/52; 4/12;1/13 p(Q)=P(Q\S): Q and S are independant

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- Calculating the conditional probability of A given B is like only considering the part of A that falls within B , that is $A \cap B$, because the part of A that falls out of B can no longer occur. In other words, you are calculating how large $A \cap B$ is with respect to B.
- If you know the probabilities of B and $A \cap B$, then you can calculate $P(A / B)$ as follows:

$$\boxed{P(A / B) = \frac{P(A \cap B)}{P(B)}}$$

- As a result of this formula, you can calculate the probability of the intersection of two events A and B :

$$\boxed{P(A \cap B) = P(B) \cdot P(A | B)}$$

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- Independence of events : If prior knowledge on the likelihood (probability) of *B* does not affect the likelihood of A, you say that these two events are independent events.

- **Example :** The probability that today doesn't rain does NOT depend on what my mother had for lunch yesterday.

- Formally, this is expressed as:

$$\boxed{P(A/B) = P(A)}$$

- Therefore, in the case of independent events (and only in this case) you have:

$$\boxed{P(A \cap B) = P(A)\, P(B)}$$

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B1: 1st is blue, B2: 2nd is blue P(B1 intersects B2) = P(B1).P(B2|B1)=3/12 * 2/11

P(A)=3/12, P(B)=2/11,, 0.25*2/11=0.0454

- **Exercise :** There are 12 balls in a bag: 5 red, 4 green, 3 blue. What is the probability that, when getting two balls without replacement, both are blue? (you get one ball, keep it outside the bag, and then get another one) SOL: 0.045

- **Exercise :** What is the probability that the next person entering the class has the same birthday as you? (For simplicity, assume that there is no February 29). SOL: 1/365

  REMARK : A similar problem is the «birthday problem» : «*In a group of n randomly chosen people, what is the probability that two of them have the same birthday?*». It can be solved with the probability tools you have already studied.

A: at least 2 of us have the same birthday, P(A)=1- complementary of A
1/(n-1) * 1/365

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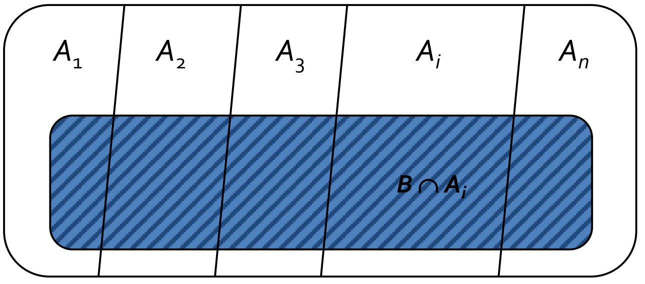
Law of total probability

- You sometimes cannot compute the probability of event B, but you do know its conditional probability given several events A_i, in such a way that all A_i form a partition of the sample space.



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- Low (or formula) of total probability: The probability of event B can be decomposed into the sum of each "piece" in which the partition divides event B :

$$\boxed{P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \dots + P(B/A_n)P(A_n)}$$

A partition of the sample space is a set of pairwise disjoint events whose union is the entire sample space

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- **Example :** There are 3 providers of a certain product. Some 10% of units produced by provider A are defective; provider B defective rate is 5%; and for provider C this rate is 7%.

  Our company buys 50% of the units you need to provider A, 30% to provider B, and the remaining 20% to provider C.

  What is the probability that a randomly chosen unit is defective?

  SOL: 0.079

- **Example :** In a bag I have both a true coin, and a coin that has been manipulated so that the probability of getting heads is twice as much as the probability of getting tails. If I randomly pick one of these two coins, and I toss it, what is the probability of getting heads? SOL: 7/12

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- **Exercise :** A human resources department uses a test to hire staff. You know that 60% of applicants pass the test. If an applicant passes the test, they have to go through a training period at the company before being hired. Out of these applicants who pass the test, some 80% of them successfully complete the training period at the company, and are later hired by the company. In order to check the validity of the test, they once also put through the training period the applicants who did not pass the text. They saw that only 50% of these applicants could successfully complete the training period.

If no test was used, and all applicants were hired, what percentage of them would successfully complete the training period?

SOL: 0.68

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Independence of events

- The probability of occurring for a certain event may change if another event has occurred or not. Or it might not.
- If the probability of an event A does not vary when you have prior information about the occurrence of another event B , you say that both events are independent. This happens if and only if :

$$\boxed{P(A/B) = P(A)}$$

- REMARK : Do not confuse incompatible events with independent events.

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- Probability of unions of independent events :
- $P(A \cup B)=P(A) + P(B) - P(A)P(B)$ (prove this equality!)

$$P(A \cup B \cup C) = [\text{comple. event}] = 1 - P(\overline{A \cup B \cup C}) = [\text{Morgan}] =$$

$$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C}) = [\text{since they are independent}] = 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C}) =$$

$$= [\text{complementary event}] = 1 - (1 - P(A))(1 - P(B))(1 - P(C))$$

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Bayes' theorem

- An interesting, simple, and useful result.

THOMAS BAYES

(1702?-1761)

- **Example :** You do know the proportion of defective pieces produced by each machine. If you found a defective piece, what would be the probability that this piece was made by machine 2?
- **Example :** You know the probability of passing this course if you study, and this probability if you don't study (really close to 0). But, if you randomly choose a student who has passed the course, what is the probability that he/she passed without studying?

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- You will use Bayes' theorem when you know $P(B|A)$ and you want to calculate $P(A|B)$.
- Bayes' theorem : Given a partition of the sample space A_1, ..., A_n and a given event B, you have that:

$$\boxed{P(A_j|B) = \frac{P(A_j)P(B|A_j)}{P(B)}}$$

or equivalently, applying the Law of total probability:

$$\boxed{P(A_j|B) = \frac{P(A_j)P(B|A_j)}{P(A_1)P(B|A_1)+\cdots+P(A_n)P(B|A_n)}}$$

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- In general, you use Bayes' theorem when you know $P(A)$, the a priori probability of an event A, and you are asked $P(A|B)$, the a posteriori probability of the same event A, but knowing that B has occurred.

- **Example :** Let's go back to the example of the three providers seen in section 2.5.
  a) A randomly chosen piece is defective. What is the probability that this piece comes from provider A? SOL: 0.633
  b) You know that the defective piece does NOT come from provider B. Recalculate the probability in a) given the new information. SOL: 0.781

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- **Exercise :** In a bag there are 5 balls: 3 red and 2 green. You extract one ball from the bag and, without looking at it, you keep it out of the bag.

  Afterwards, you extract another ball, which turns out to be green.
  a) What is the probability that the first ball was green? SOL: 0.25
  b) Recalculate the probability in a) assuming that the second ball was red. SOL: 0.5

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