Exploring Quantum Geometric Tensor: Understanding Rashba Hamiltonian under Time-Reversal Symmetry and its Violation

In quantum physics, eigenvalues and eigenstates are pivotal in understanding systems. While eigenvalues offer key observables like energy and momentum, eigenstates traditionally aided in probability and transitions. Yet, focusing on eigenstates has become crucial due to advances in quantum concepts such as entanglement and superconductivity. Moreover, quantum geometry, particularly concerning eigenstates, influences quantum transport phenomena like supercurrent flow. This study aims to utilize formalism to derive Quantum Geometric Tensors (QGT) for non-degenerate eigenstates, exploring their behavior in parameter space through adiabatic evolution, and study the Rashba Hamiltonian under Time-reversal symmetry and the broken time-reversal symmetry. QGT components of Rashba Hamiltonian without external magnetic field gives zero Berry curvature as time-reversal symmetry and the combined inversion and spin-flip symmetry leads it to be zero. Whereas, with an external magnetic field results claimed non-trivial quantum geometry.

Introduction

In quantum domain, eigenvalues and eigenstates play a vital role in describing the system that are being studied. However, most of the time we are interested in eigenvalues as they provide observables such as energy, position, momentum etc. which are important for studying the system. On the other hand, eigenstates have been less concerning as they were used to find the probability, expectation values and transition rates[1] Nonetheless, over time the importance of studying about the eigenstates has been recognized with the development of the concepts such as quantum entanglement, Quantum phase transitions, Superconductivity, Topological Physics etc. Therefore, phenomena which directly deal with properties of eigenstates are few aspects of quantum geometry physics. Quantum geometry has shown direct correlation with various quantum transport phenomena. For example, quantum geometry leads to flow of supercurrent in flat band.

Quantum Geometry defines the geometry of the Hilbert space. Classically we can interpret the geometry as the distance between two points on the space. Similarly, we can generally say Quantum geometry is the quantum distance between two quantum states on the geometry of the Hilbert space. Quantum Geometric Tensor (QGT) contains real and imaginary components[1]. Real part of the QGT defines the amplitude distance, while the imaginary part defines the phase distance. Phase distance is related with the well-known Berry Curvature and topological properties of materials such as topological phase transitions and topological defects whereas the real component of the QGT which relates to the amplitude distance, is yet be studied. The idea of the amplitude distance and the phase distance is how the probability amplitudes and phase change between two eigenstates which is defined at two distinct positions on the Hilbert space. Over past few years, several studies have been done to study about QGT. Most of them are related to condensed matter physics. For instance, extracting the Quantum Metric Tensor (GMT), which is the real part of the QGT, has been carried out for degenerate eigenstates, along with extracting the related Berry curvature and its Chern numbers [2], Deriving the Quantum Geometric Tensor (QGT) for two-band systems with and without the Zeeman effects explains the adiabatic limit and the anomalous Hall effect credited to the Berry curvature.[3], Trajectories of Majorana's Stars have been used to obtain QGT, which provides an intuitive picture of the eigenstates in a high-dimensional Hilbert space.[4]. Originally people thought the metric tensor is just mathematical object which cannot be measured but later some studies have been proved that he QGT or its real part GMT can be experimentally measured and consistent with theoretical studies [5]

The main purpose of this study is to use the formalism to obtain QGT, explore QGT for a few toy models, and understand the underlying phenomena. QGTs were obtained for non-degenerate eigenstates by considering the adiabatic evolution of the system in parameter space. The formalism only demonstrates the case for adiabatic evaluation.

Formalism

To obtain the QGT, consider a complete set $\{u(\boldsymbol{k})\}$ of wave functions of some Hilbert space in which the Hamiltonian smoothly depends on m-dimensional real parameter $k = (k_1, k_2, \dots, k_m) \in \mathbb{R}^m$ and let $\| \ \|$ denotes the norm on the Hilbert space so that the distance between two states in the parameter space.

$$ds^2 = d\big(u(k+dk), u(k)\big) = \|u(k+dk) - u(k)\|^2 \tag{1}$$

By the definition of norm we know,

$$\begin{aligned}\|u(k+dk) - u(k)\|^2 &= \langle u(k+dk) - u(k) | u(k+dk) - u(k)\rangle \\ &= \langle \delta u | \delta u\rangle\end{aligned}$$

Let $u(\boldsymbol{k}) \equiv u_{\boldsymbol{k}}$ where the wave function parameterized by quantity k. Up to second order we can write,[6]

$$\|u(k+dk) - u(k)\|^2 = \langle \partial_\mu u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}}\rangle dk^\mu dk^\nu \tag{2}$$

Where $\partial_i = \frac{\partial}{\partial k_i}$ and then $\langle \partial_\mu u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}}\rangle$ can be separated as its real and complex components and then quantum distance can be defined,

$$ds^2 = (\gamma_{\mu\nu} + i\sigma_{\mu\nu}) dk^\mu dk^\nu \tag{3}$$

Since the inner product is Hermitian,

$$\gamma_{\mu\nu} + i\sigma_{\mu\nu} = \gamma_{\nu\mu} - i\sigma_{\nu\mu}$$

Therefore,

$$\gamma_{\mu\nu} = \gamma_{\nu\mu}$$
$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

Antisymmetric of $\sigma_{\mu\nu}$ vanishes $\sigma_{\mu\nu} dk^{\mu} dk^{\nu}$ part of the equation (3) and thus ds^2 reduces to,

$$ds^2 = \gamma_{\mu\nu} dk^{\mu} dk^{\nu} \tag{4}$$

Nevertheless, it can be easily showed that the $\gamma_{\mu\nu}$ is NOT gauge invariant considering gauge transformation $|u_{\boldsymbol{k}}'\rangle = e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle$ where $\alpha(k)$ is smooth function of k and taking $\langle \partial_\mu u_{\boldsymbol{k}}' | \partial_\nu u_{\boldsymbol{k}}' \rangle = \gamma_{\mu\nu}' + i\sigma_{\mu\nu}'$,

$$|\partial_\mu u_{\boldsymbol{k}}'\rangle = e^{i\alpha(k)}|\partial_\mu u_{\boldsymbol{k}}\rangle + i\partial_\mu\alpha\, e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle$$

$$\langle \partial_\mu u_{\boldsymbol{k}}'| = e^{-i\alpha(\lambda)}\langle \partial_\mu u_{\boldsymbol{k}}| - i\partial_\mu\alpha e^{i\alpha(k)}\langle u_{\boldsymbol{k}}|$$

$$|\partial_\nu u_{\boldsymbol{k}}'\rangle = e^{i\alpha(k)}|\partial_\nu u_{\boldsymbol{k}}\rangle + i\partial_\nu\alpha\, e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle$$

$$\langle \partial_\mu u_{\boldsymbol{k}}'|\partial_\nu u_{\boldsymbol{k}}'\rangle = \left[e^{-i\alpha(\lambda)}\langle \partial_\mu u_{\boldsymbol{k}}| - i\partial_\mu\alpha e^{i\alpha(k)}\langle u_{\boldsymbol{k}}|\right]$$
$$\times \left[e^{i\alpha(k)}|\partial_\nu u_{\boldsymbol{k}}\rangle + i\partial_\nu\alpha\, e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle\right]$$

$$\langle \partial_\mu u_{\boldsymbol{k}}'|\partial_\nu u_{\boldsymbol{k}}'\rangle = \langle \partial_\mu u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle + i\langle \partial_\mu u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle\partial_\nu\alpha - i\langle u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle\partial_\mu\alpha + \partial_\mu\alpha\partial_\nu\alpha$$

$$\langle \partial_\mu u_{\boldsymbol{k}}'|\partial_\nu u_{\boldsymbol{k}}'\rangle = \langle \partial_\mu u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle - i\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle\partial_\nu\alpha - i\langle u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle\partial_\mu\alpha + \partial_\mu\alpha\partial_\nu\alpha$$

$$\langle \partial_\mu u_{\boldsymbol{k}}'|\partial_\nu u_{\boldsymbol{k}}'\rangle = \langle \partial_\mu u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle - \beta_\mu\partial_\nu\alpha - \beta_\nu\partial_\mu\alpha + \partial_\mu\alpha\partial_\nu \tag{5}$$

Here β_i has been considered as the Berry connection[7] which is defined by $\beta_i = i\langle u_{\boldsymbol{k}}|\partial_i u_{\boldsymbol{k}}\rangle$ where $i = \mu, \nu$. It's important to notice that Berry connection is purely real, as we have considered a normalized basis $\{u(\boldsymbol{k})\}$ (i.e. $\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle = 1$). It can be proved that the Berry connection is purely real number using the normalization conditions and differentiating with respect parameter quantity.

$$\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle = 1$$

$$\partial_\mu\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle = 0$$

$$\langle \partial_\mu u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle + \langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle = 0$$

$$\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle = -\langle \partial_\mu u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle$$

$$\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle = -\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle^*$$

Which implies $\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle \in Im$ (Pure Imaginary) and hence proved the Berry connection is pure real quantity. Now consider equation (5) which can be written in terms of $\gamma_{\mu\nu}$ and $\gamma_{\mu\nu}'$,

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} - \beta_\mu\,\partial_\nu\,\alpha - \beta_\nu\,\partial_\mu\,\alpha + \partial_\mu\,\alpha\,\partial_\nu\,\alpha \tag{6}$$
$$\sigma_{\mu\nu}' = \sigma_{\mu\nu}$$

Therefore $\gamma_{\mu\nu}$ is NOT gauge invariant as mentioned previously. This disqualifies the tensor as a metric measuring the distance between two quantum states under smooth changes. It can be noticed that the Berry connection the quantity which changes under the gauge transformation.

$$|u_{\boldsymbol{k}}'\rangle = e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle$$

$$|\partial_\mu u_{\boldsymbol{k}}'\rangle = e^{i\alpha(k)}|\partial_\mu u_{\boldsymbol{k}}\rangle + i\partial_\mu\alpha\, e^{i\alpha(k)}|u_{\boldsymbol{k}}\rangle$$

$$\langle u_{\boldsymbol{k}}'|\partial_\mu u_{\boldsymbol{k}}'\rangle = \langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle + i\partial_\mu\alpha\,\langle u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle$$

$$\langle u_{\boldsymbol{k}}'|\partial_\mu u_{\boldsymbol{k}}'\rangle = \langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle + i\partial_\mu\alpha$$

$$i\langle u_{\boldsymbol{k}}'|\partial_\mu u_{\boldsymbol{k}}'\rangle = i\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle - \partial_\mu\alpha$$

$$\beta_\mu' = \beta_\mu - \partial_\mu\alpha \tag{7}$$

Therefore, Berry connection changes as above equation (7) under the gauge transformation. Thus, new gauge invariant metric can be defined as,

$$g_{\mu\nu} = \gamma_{\mu\nu} - \beta_\mu\beta_\nu \tag{8}$$

Considering the same gauge transformation, it can be shown that $g_{\mu\nu}$ is gauge invariant. By equations (6) and (7),

$$g_{\mu\nu}' = \gamma_{\mu\nu} - \beta_\mu\,\partial_\nu\,\alpha - \beta_\nu\,\partial_\mu\,\alpha + \partial_\mu\,\alpha\,\partial_\nu\,\alpha - (\beta_\mu - \partial_\mu\alpha)(\beta_\nu - \partial_\nu\alpha)$$

$$g_{\mu\nu}' = \gamma_{\mu\nu} - \beta_\mu\,\partial_\nu\,\alpha - \beta_\nu\,\partial_\mu\,\alpha + \partial_\mu\,\alpha\,\partial_\nu\,\alpha$$
$$- \left[\beta_\mu\beta_\nu - \beta_\mu\,\partial_\nu\,\alpha - \beta_\nu\,\partial_\mu\,\alpha + \partial_\mu\,\alpha\,\partial_\nu\,\alpha\right]$$

$$g_{\mu\nu}' = \gamma_{\mu\nu} - \beta_\mu\beta_\nu$$

$$g_{\mu\nu}' = g_{\mu\nu}$$

This can be explained as, $\gamma_{\mu\nu}$ measures the distance of the original states in the Hilbert space, while $g_{\mu\nu}$ measures the distance between eigenstates in projected Hilbert space [8]. Now there are all the ingredients to define the Quantum Geometric tensor (QGT).

$$Q_{\mu\nu}(k) = \langle \partial_\mu u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle - \langle \partial_\mu u_{\boldsymbol{k}}|u_{\boldsymbol{k}}\rangle\langle u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle \tag{9}$$

Berry Curvature

We defined Berry connection, which is also known as Berry vector potential, as $\beta_i = i\langle u_{\boldsymbol{k}}|\partial_i u_{\boldsymbol{k}}\rangle$ Where, $i = \mu, \nu$ and $\partial_i = \frac{\partial}{\partial_i}$, which is gauge dependent. A general form for the Berry curvature can be defined in terms of the Berry connection or Berry vector potential as follows,[7]

$$\Omega_n(\boldsymbol{k}) = \nabla_{\boldsymbol{R}} \times \beta_n(\boldsymbol{k}) \tag{10}$$

As $\beta(\boldsymbol{k})$ is referring to a vector potential and the Berry curvature is vector cross product in the parameter space. Intuitively, Berry curvature can be seen as the effective (artificial) magnetic field in the parameter space. Since all the derivations was done considering two-dimensional parameter space, it can be deduced to following form in terms or parameters μ and ν.

$$\Omega_{\mu\nu}(\boldsymbol{k}) = \frac{\partial}{\partial k_\mu}\beta_\nu(\boldsymbol{k}) - \frac{\partial}{\partial k_\nu}\beta_\mu(\boldsymbol{k}) \tag{11}$$

Substituting β_ν and β_μ into the equation (11),

$$\Omega_{\mu\nu}(\boldsymbol{k}) = \partial_\mu[i\langle u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle] - \partial_\nu[i\langle u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle]$$

$$\Omega_{\mu\nu}(\boldsymbol{k}) = i\,[\langle \partial_\mu u_{\boldsymbol{k}}|\partial_\nu u_{\boldsymbol{k}}\rangle + \langle u_{\boldsymbol{k}}|\partial_\mu\partial_\nu u_{\boldsymbol{k}}\rangle - \langle \partial_\nu u_{\boldsymbol{k}}|\partial_\mu u_{\boldsymbol{k}}\rangle - \langle u_{\boldsymbol{k}}|\partial_\mu\partial_\nu u_{\boldsymbol{k}}\rangle]$$

$$\Omega_{\mu\nu}(\boldsymbol{k}) = i\left[\langle \partial_\mu u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}} \rangle - \langle \partial_\nu u_{\boldsymbol{k}} | \partial_\mu u_{\boldsymbol{k}} \rangle\right] \tag{12}$$

Using the equation (12) for the Berry curvature and the Quantum Geometric Tensor which is given by the equation (9), It's straight forward to show that the imaginary part of the QGT is the well-known Berry curvature. Let's consider the equation (9) and notice that $\langle u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}} \rangle$ is pure imaginary, last term of the equation (9) is real. Therefore,

$$Q^*_{\mu\nu}(k) = \langle \partial_\nu u_{\boldsymbol{k}} | \partial_\mu u_{\boldsymbol{k}} \rangle - \langle \partial_\mu u_{\boldsymbol{k}} | u_{\boldsymbol{k}} \rangle \langle u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}} \rangle \tag{13}$$

and subtracting equations (9) and (13), we get

$$Q_{\mu\nu}(k) - Q^*_{\mu\nu}(k) = \langle \partial_\mu u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}} \rangle - \langle \partial_\nu u_{\boldsymbol{k}} | \partial_\mu u_{\boldsymbol{k}} \rangle$$

$$Im\ Q_{\mu\nu}(k) = \frac{\langle \partial_\mu u_{\boldsymbol{k}} | \partial_\nu u_{\boldsymbol{k}} \rangle - \langle \partial_\nu u_{\boldsymbol{k}} | \partial_\mu u_{\boldsymbol{k}} \rangle}{2i} \tag{14}$$

From equation (14),

$$Im\ Q_{\mu\nu}(k) = -\frac{1}{2}\Omega_{\mu\nu}(\boldsymbol{k}) \tag{15}$$

Where $\Omega_{\mu\nu}(\boldsymbol{k})$ is the Berry curvature which gives information about the change in phase in the eigenstates[7]. Since the integral of the Berry curvature over the parameter space gives the Chern number[1], the QGT provides information about topological properties too. As well as the real part of the QGT,

$$Re\ Q_{\mu\nu}(k) = g_{\mu\nu}(\boldsymbol{k}) \tag{16}$$

Here $g_{\mu\nu}$ is known as the Riemannian Metric Tensor [6], [8] or Quantum Metric Tensor (QMT) [9]. Combining Real and Imaginary parts, following expression could be obtained,

$$Q_{\mu\nu}(k) = g_{\mu\nu}(\boldsymbol{k}) - \frac{i}{2}\Omega_{\mu\nu}(\boldsymbol{k}) \tag{17}$$

Adiabatic Evolution

We defined the set of bases $\{u_n(\boldsymbol{k})\}$ for a Hamiltonian depends on time through parameter $k = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m$. If $\boldsymbol{k}(t)$ changes slowly and smoothly along a path in the parameter space, the system is said being evolved adiabatically. According to the quantum adiabatic theorem, the initial state of the system will stay as the instantaneous eigenstate throughout the process.[7] In other words, the system will be on same state as adiabatic evolution confirms the transitions between that instantaneous state and other eigenstates are negligible due to the energy gap. In that case, QGT can be with that instantaneous eigenstate. Even though it doesn't have to be the ground state, for formalism, the ground state is considered the instantaneous eigenstate. Hence, the instantaneous ground state is defined as $\hat{H}(k)|u_0(\boldsymbol{k})\rangle = E_0|u_0(\boldsymbol{k})\rangle$ which is certainly non-degenerate. Considering orthonormal conditions, $\langle u_n(\boldsymbol{k})|u_m(\boldsymbol{k})\rangle = \delta_{nm}$, and assuming an infinitesimal change in parameters δk causes to $\delta\hat{H}$ change in the parameter dependent Hamiltonian. Hence new Hamiltonian can be written as,

$$\hat{H}' = \hat{H} + \delta\hat{H}$$

Which can be re-written in terms of partial derivatives (∂_μ for an instance),

$$\hat{H}' = \hat{H} + \partial_\mu \hat{H} \delta k_\mu$$

Applying non-degenerate perturbation theory up 1 order correction to the eigenstate, and then taking the inner product with $\langle u_m|$,

$$|u_n(k + dk_\mu)\rangle = |u_n(k)\rangle + \sum_{j\neq n} |u_j\rangle \frac{\langle u_j|\partial_\mu \hat{H} \delta k_\mu|u_n\rangle}{E_n - E_j}$$

$$|u_n(k + dk_\mu)\rangle - |u_n\rangle = \delta k_\mu \sum_{j\neq n} |u_j\rangle \frac{\langle u_j|\partial_\mu \hat{H}|u_n\rangle}{E_n - E_j}$$

$$\frac{|u_n(k + dk_\mu)\rangle - |u_n\rangle}{\delta k_\mu} = \sum_{j\neq n} |u_j\rangle \frac{\langle u_j|\partial_\mu \hat{H}|u_n\rangle}{E_n - E_j}$$

$$|\partial_\mu u_n\rangle = \sum_{j\neq n} |u_j\rangle \frac{\langle u_j|\partial_\mu \hat{H}|u_n\rangle}{E_n - E_j}$$

$$\langle u_m|\partial_\mu u_n\rangle = \sum_{j\neq n} \langle u_m|u_j\rangle \frac{\langle u_j|\partial_\mu \hat{H}|u_n\rangle}{E_n - E_j}$$

$$\langle u_m|\partial_\mu u_n\rangle = \frac{\langle u_m|\partial_\mu \hat{H}|u_n\rangle}{E_n - E_m} \tag{18}$$

Now, the QGT can be defined on the ground state and then using the equation (18),

$$Q_{\mu\nu} = \langle \partial_\mu u_0|\partial_\nu u_0\rangle - \langle \partial_\mu u_0|u_0\rangle\langle u_0|\partial_\nu u_0\rangle$$

$$Q_{\mu\nu} = \langle \partial_\mu u_0|((1 - |u_0\rangle\langle u_0|)|\partial_\nu u_0\rangle$$

$$Q_{\mu\nu} = \sum_n \langle \partial_\mu u_0|(|u_n\rangle\langle u_n| - |u_0\rangle\langle u_0|)|\partial_\nu u_0\rangle$$

$$Q_{\mu\nu} = \sum_{n\neq 0} \langle \partial_\mu u_0|u_n\rangle\langle u_n|\partial_\nu u_0\rangle$$

$$Q_{\mu\nu} = \sum_{n\neq 0} \frac{\langle u_0|\partial_\mu \hat{H}|u_n\rangle\langle u_n|\partial_\nu \hat{H}|u_o\rangle}{(E_0 - E_n)^2} \tag{19}$$

Even though both equation (9) and (19) define the QGT, in equation (9), QGT is completely obtained using eigenstates. In that case, partial derivatives of eigenstates must be taken and indeed not a good approach for a system in which eigenstates haven't been taken analytically but numerically. On the other hand, equation (19) defines the QGT in terms of the partial derivatives of the Hamiltonian, which is well-defined, though the system can't be solved analytically [8]. Therefore, the expression for the QGT which is given by the equation (19), was used to further calculations. Most importantly, at least for this moment, only the adiabatic and non-degenerate cases were considered.

Results and Discussion

To start with let's consider two level system described by the Hamiltonian,

$$\hat{H}(\theta, \varphi) = h\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \tag{20}$$

It's straightforward to get eigenstates and eigenvalues.

$$|u_+\rangle = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$|u_-\rangle = \begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix} \tag{21}$$

Eigenvalues are $E_\pm = \pm h$, Where θ and φ are spherical angles and these are parameters that we must consider constructing QGT. Since we are interested in the ground state, let's relabel eigenstates and eigenvalues as ground state and the excited state as follow.

$$|u_1\rangle = |u_+\rangle\,; E_1 = E_+ = +h$$

$$|u_0\rangle = |u_-\rangle\,; E_0 = E_- = -h$$

Using the Hamiltonian defined in equation (20), $\partial_\theta \hat{H}(\theta,\varphi)$ and $\partial_\varphi \hat{H}(\theta,\varphi)$,

$$\partial_\theta \hat{H}(\theta,\varphi) = h\begin{pmatrix} -\sin\theta & \cos\theta\, e^{-i\varphi} \\ \cos\theta\, e^{i\varphi} & \sin\theta \end{pmatrix}$$

$$\partial_\varphi \hat{H}(\theta,\varphi) = h\begin{pmatrix} 0 & -i\sin\theta\, e^{-i\varphi} \\ i\sin\theta\, e^{i\varphi} & 0 \end{pmatrix}$$

Hence, Necessary components that are required to compute the QGT can be derived as follows,

$\langle u_0|\partial_\theta \hat{H}|u_1\rangle$

$$= h\begin{pmatrix} \sin\frac{\theta}{2}\, e^{i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -\sin\theta & \cos\theta\, e^{-i\varphi} \\ \cos\theta\, e^{i\varphi} & \sin\theta \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\varphi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

$$= h\begin{pmatrix} \sin\frac{\theta}{2}\, e^{i\varphi} & -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\varphi} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

$$= -h$$

And,

$\langle u_1|\partial_\varphi \hat{H}|u_0\rangle$

$$= h\begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\varphi} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 0 & -i\sin\theta\, e^{-i\varphi} \\ i\sin\theta\, e^{i\varphi} & 0 \end{pmatrix}\begin{pmatrix} \sin\frac{\theta}{2}\, e^{-i\varphi} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

$$= h\begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\varphi} & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} i\sin\theta\cos\frac{\theta}{2}\, e^{-i\varphi} \\ i\sin\theta\sin\frac{\theta}{2} \end{pmatrix}$$

$$= ih\sin\theta$$

Using the equation (19), all the components of the Quantum Geometric Tensor can be obtained considering μ and ν are running between θ and φ (i.e. $\mu,\nu \leftrightarrow \theta,\varphi$).

$$Q_{\theta\varphi} = \frac{-h \times ih\sin\theta}{(-h-h)^2} = \frac{-i\sin\theta}{4} \tag{a}$$

$$Q_{\varphi\theta} = \frac{-h \times -ih\sin\theta}{(-h-h)^2} = \frac{i\sin\theta}{4} \tag{b}$$

$$Q_{\theta\theta} = \frac{-h \times -h}{(-h-h)^2} = \frac{1}{4} \tag{c}$$

$$Q_{\varphi\varphi} = \frac{-ih\sin\theta \times ih\sin\theta}{(-h-h)^2} = \frac{\sin^2\theta}{4} \tag{d}$$

Now Tensor terms given by the equations (a), (b), (c) & (d) can be used to write the QGT

When $\mu,\nu \leftrightarrow \theta,\varphi$,

$$Q_{\mu\nu} = \begin{pmatrix} Q_{\theta\theta} & Q_{\theta\varphi} \\ Q_{\varphi\theta} & Q_{\varphi\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{-i\sin\theta}{4} \\ \frac{i\sin\theta}{4} & \frac{\sin^2\theta}{4} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & \frac{1}{2}\sin\theta \\ -\frac{1}{2}\sin\theta & 0 \end{pmatrix} \equiv g_{\mu\nu} - \frac{i}{2}\Omega_{\mu\nu}$$

$$g_{\mu\nu} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \qquad \Omega_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix} \tag{22}$$

Considering the ground state of the Hamiltonian (20), the components of the QGT are $g_{\theta\varphi} = 0$, $g_{\theta\theta} = \frac{1}{4}$, $g_{\varphi\varphi} = \frac{1}{4}\sin^2\theta$, and $\Omega_{\theta\varphi} = \frac{1}{2}\sin\theta$. These are consistent with the literatures [5], [8]. Specially, an experimental measurement of QGT using qubit in diamond has been compared with these theoretically predicted results and similar results have been obtained [5].

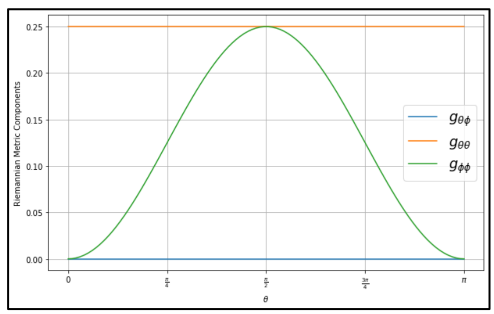

Figure 1 The Graph of Riemannian Metric Components vs angle (θ). There are three components which can be extracted from the real part of the QGT. These are well-known Riemannian metric components [6]. $g_{\varphi\varphi} = \frac{1}{4}sin^2\theta$ are *the only components which depends on the angle θ while other two components are constants.*

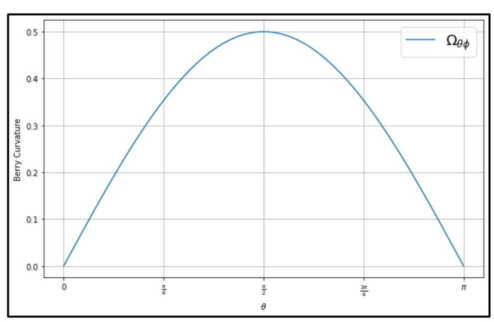

Figure 2 The Graph of Berry Curvature vs angle (θ) represents how Berry curvature changes as function of angle with z-axis. Berry Curvature is zero at 0 and π which is minimum value, whereas $\theta = \pi/2$ maximize the Berry curvature.

One can easily see that for θ angles around 0 or π the Riemannian metric component $g_{\varphi\varphi}$ and the Berry curvature $\Omega_{\theta\varphi}$ are zero or small and this can be interpreted as the trivial quantum geometry which means for two level system, the effect from on state to the other state is minimal around small angles or around π. Thus, amplitude distance and the phase distance are zero or very small. But when $\theta = \frac{\pi}{2}$, we have non-trivial geometry; hence, there is effect on one state from the other state giving non-zero amplitude distance and phase distance on a path in the φ direction.

Rashba Hamiltonian

The Rashba Hamiltonian is frequently observed to exhibit additional momentum-dependent splitting in the band structures of materials near surfaces or interfaces. These splitting are typically attributed to the Rashba effect, arising from an asymmetric electrical potential that introduces a term analogous to spin-orbit coupling. The Rashba Hamiltonian can be represented as follows,

$$\hat{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{2\alpha_R}{\hbar}\left(S_x p_y - S_y p_x\right) \tag{23}$$

For the simplicity let $m = 1$ and $\hbar = 1$, and then the Hamiltonian can be written in matrix form,

$$\hat{H} = \begin{pmatrix} \frac{p_x^2 + p_y^2}{2} & \alpha_R\left(p_y + ip_x\right) \\ \alpha_R\left(p_y - ip_x\right) & \frac{p_x^2 + p_y^2}{2} \end{pmatrix}$$

$$\hat{H}\left(k_x, k_y\right) = \begin{pmatrix} \frac{k_x^2 + k_y^2}{2} & \alpha_R\left(k_y + ik_x\right) \\ \alpha_R\left(k_y - ik_x\right) & \frac{k_x^2 + k_y^2}{2} \end{pmatrix}$$

$$\hat{H}\left(k_x, k_y\right) = \begin{pmatrix} \frac{k^2}{2} & \alpha_R\left(k_y + ik_x\right) \\ \alpha_R\left(k_y - ik_x\right) & \frac{k^2}{2} \end{pmatrix} \tag{24}$$

Where $k = \sqrt{k_x^2 + k_y^2}$. It's straightforward to get exact eigenvalues and exact eigenstates,

$$|u_1\rangle = |u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{k_y - ik_x}{k} \end{pmatrix}$$

$$|u_0\rangle = |u_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{k_y + ik_x}{k} \\ -1 \end{pmatrix} \tag{25}$$

Eigenvalues are $E_\pm = \frac{k^2}{2} \pm \alpha_R k$

$$\partial_{k_x}\hat{H} = \begin{pmatrix} k_x & i\alpha_R \\ -i\alpha_R & k_x \end{pmatrix}$$

$$\partial_{k_y}\hat{H} = \begin{pmatrix} k_y & \alpha_R \\ \alpha_R & k_y \end{pmatrix}$$

Hence the components of the QGT can be obtained using the equation (19). Detailed derivation can be founded in ***Appendix A***
.

$$Q_{k_x k_y} = \frac{-k_x k_y}{4k^4} \tag{e}$$

$$Q_{k_y k_x} = \frac{-k_x k_y}{4k^4} \tag{f}$$

$$Q_{k_x k_x} = \frac{k_y^2}{4k^4} \tag{g}$$

$$Q_{k_y k_y} = \frac{k_x^2}{4k^4} \tag{h}$$

Now Tensor terms given by the equations (e), (f), (g) & (h) can be used to write the QGT , when $\mu, \nu \leftrightarrow k_x, k_y$,

$$\begin{aligned} Q_{\mu\nu} &= \begin{pmatrix} Q_{k_x k_x} & Q_{k_x k_y} \\ Q_{k_y k_x} & Q_{k_y k_y} \end{pmatrix} \\ &= \frac{1}{4k^4}\begin{pmatrix} k_y^2 & -k_x k_y \\ -k_x k_y & k_x^2 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\equiv g_{\mu\nu} - \frac{i}{2}\Omega_{\mu\nu} \end{aligned}$$

$$g_{\mu\nu} = \frac{1}{4k^4}\begin{pmatrix} k_y^2 & -k_x k_y \\ -k_x k_y & k_x^2 \end{pmatrix}, \qquad \Omega_{\mu\nu} = 0 \tag{26}$$

Considering the ground state of the Hamiltonian (24), the components of the QGT are,

$$g_{k_x k_y} = g_{k_y k_x} = \frac{-k_x k_y}{4k^4}$$

$$g_{k_x k_x} = \frac{k_y^2}{4k^4}$$

$$g_{k_y k_y} = \frac{k_x^2}{4k^4}$$

$$\Omega_{k_x k_y} = 0$$

If we have time-reversal and inversion symmetry, Berry curvature vanishes identically all the k-points. However, Rashba term comes with broken inversion symmetry but combined inversion and spin flip symmetry.

$$\Omega_\uparrow(k) = -\Omega_\downarrow(-k) \tag{27}$$

$$\Omega_\uparrow(k) = \Omega_\downarrow(-k) \tag{28}$$

Equation (27) and (28) represent the time-reversal symmetry and combined inversion and spin flip symmetry which forces to the Berry curvature to be zero all over the parameter space. This confirms momentum is not good quantum number for the Rashba Hamiltonian as it is not independent but coupled with the spin.

Notice that the Rashba Hamiltonian defined by the equation (23) has time-reversal symmetry and this time-reversal symmetry can be broken by applying a magnetic field along $\hat{z}$ direction. Hence, new Hamiltonian can be defined as follows. Considering, the applied magnetic field along $\hat{z}$ direction weak enough to apply perturbation theory, corrections to the eigenstates and eigen energies up to leading order term can be written as follows,

$$\widehat{H'} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{2\alpha_R}{\hbar}\left(S_x p_y - S_y p_x\right) - \frac{2}{\hbar}B_0 S_z \tag{29}$$

Assuming $m = 1$ and $\hbar = 1$, then the Hamiltonian can be written as,

$$\hat{H}'(k_x,k_y) = \begin{pmatrix} \frac{k^2}{2} - B_0 & \alpha_R(k_y + ik_x) \\ \alpha_R(k_y - ik_x) & \frac{k^2}{2} + B_0 \end{pmatrix} \quad (30)$$

Where $k = \sqrt{k_x^2 + k_y^2}$. Then exact eigenvalues and exact eigenstates can be obtained,

$$|u_1\rangle = |u_+\rangle = N\begin{pmatrix} \sqrt{1+x^2} - x \\ \frac{k_y - ik_x}{k} \end{pmatrix}$$

$$|u_0\rangle = |u_-\rangle = N\begin{pmatrix} \frac{k_y + ik_x}{k} \\ -(\sqrt{1+x^2} - x) \end{pmatrix} \quad (31)$$

Eigenvalues are $E_\pm = \frac{k^2}{2} \pm \alpha_R k\sqrt{1+x^2}$, where $x = \frac{B_0}{\alpha_R k}$ and normalization factor is given by $N = \frac{1}{\sqrt{2(1+x^2 - x\sqrt{1+x^2}}}$. Now QGT can be obtained considering the ground state of the perturbed system. Noticing that $\partial_k \hat{H}' = \partial_k \hat{H}$, QGT can be obtained using the equation (19). Detailed derivation can be founded in ***Appendix C***

$$Q_{k_x k_y} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y k_x(1+2\Delta) + i(\Delta^2 + \Delta)k^2\right) \quad (i)$$

$$Q_{k_y k_x} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y k_x(1+2\Delta) - i(\Delta^2 + \Delta)k^2\right) \quad (j)$$

$$Q_{k_x k_x} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2 + 2k_y^2\Delta + k^2\Delta^2\right) \quad (k)$$

$$Q_{k_y k_y} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_x^2 + 2k_x^2\Delta + k^2\Delta^2\right) \quad (l)$$

Now Tensor terms given by the equations (i), (j), (k) & (l) can be used to write the QGT, when $\mu, \nu \leftrightarrow k_x, k_y$,

$$Q_{\mu\nu} = \begin{pmatrix} Q_{k_x k_x} & Q_{k_x k_y} \\ Q_{k_y k_x} & Q_{k_y k_y} \end{pmatrix}$$

$$\equiv g_{\mu\nu} - \frac{i}{2}\Omega_{\mu\nu}$$

Hence, we $g_{\mu\nu}$ and $\Omega_{\mu\nu}$ can be defined and then metric components of the QGT can be obtained as follows.

$$g_{k_x k_x} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2 + 2k_y^2\Delta + k^2\Delta^2\right) \quad (32)$$

$$g_{k_y k_y} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_x^2 + 2k_x^2\Delta + k^2\Delta^2\right) \quad (33)$$

$$g_{k_x k_y} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y k_x(1+2\Delta)\right) \quad (34)$$

$$\Omega_{k_x k_y} = \frac{8\alpha_R^2 N^4(\Delta^2 + \Delta)}{(\alpha_R^2 k^2 + B_0^2)} \quad (35)$$

Where $N = \frac{1}{\sqrt{2(1+x^2 - x\sqrt{1+x^2}}}$, $\Delta = x\left(x - \sqrt{1+x^2}\right)$, $x = \frac{B_0}{\alpha_R k}$ and $k = \sqrt{k_x^2 + k_y^2}$. Expressions obtained for $g_{k_x k_x}$ and $g_{k_y k_y}$ metric components and the Berry curvature are pretty much consistent with a previous study[10]. QGT metric components obtained above can be plotted.

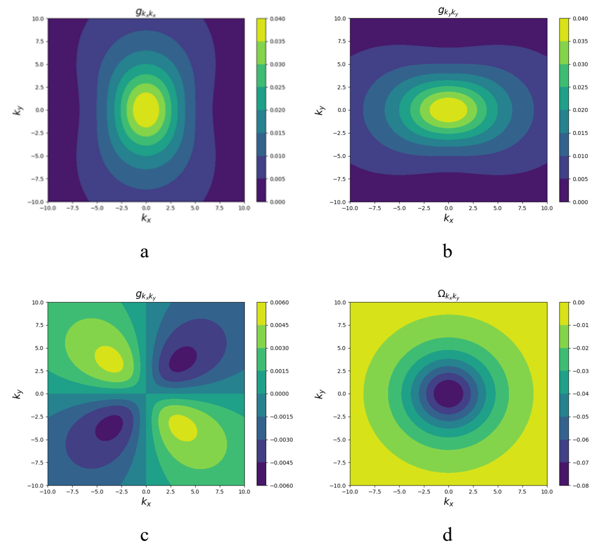

Figure 3 Contour plots of Riemannian metric components and Berry curvature vs k_x and k_y. There is a maximum at Gamma point in (a) contour plot of $g_{k_x k_x}$ (b) contour plot of $g_{k_y k_y}$. (c) contour of $g_{k_x k_y}$ shows bit different pattern, there are two maxima on main diagonal and two minima on off diagonal. Finally, (d) contour plot of Berry curvature, at Gamma point it's minimum, radially, increases up to some finite. All the contours have been plotted taking $\alpha_R = 5$ and $B_0 = 25$.

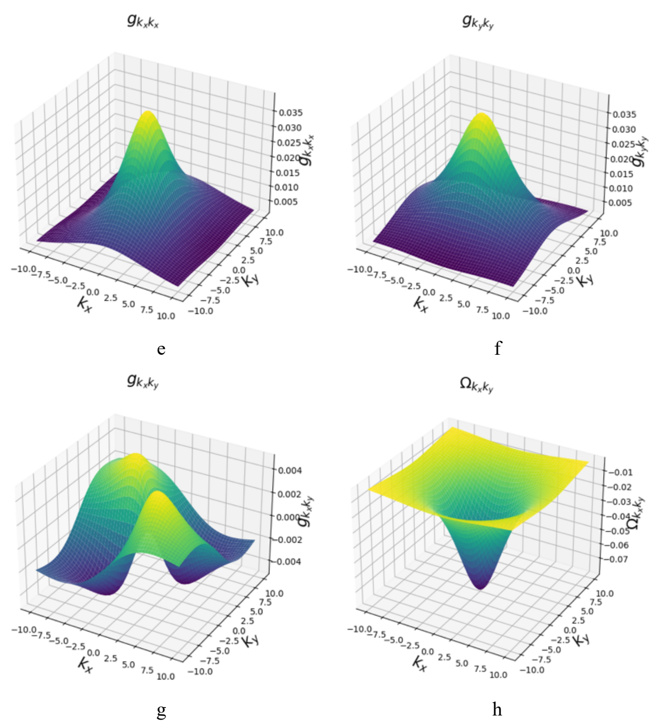

Figure 4 3D plots of Riemannian metric and Berry curvature vs k_x and k_y. (e), (f) and (g) are 3D plot of $g_{k_x k_x}$, $g_{k_y k_y}$, and $g_{k_x k_y}$ respectively while (h) represents the Berry curvature. The maxima and minima can be clearly visualized using 3D plots. Depending on the parameters that are being changed the Quantum distance (Amplitude distance and Phase distance) behaves like somewhat curved and symmetric space. All 3D plots have been plotted taking $\alpha_R = 5$ and $B_0 = 25$ like the contour plots in Figure 3.

The Berry curvature is consistent with [11]. Presence of an external magnetic field was a necessary condition to bring non-zero Berry curvature to the system. These results explain how the QGT changes with and without Zeeman effect (Broken time-reversal symmetry). Most importantly, metric components of the Rashba Hamiltonian with Time-Reversal symmetry can be recovered setting $B_0 = 0$ which means $x = 0$ and thus $\Delta = 0$. As well as one can easily see that $\Delta^2 + \Delta = -x\sqrt{x^2+1}(2x^2+1)$ which guarantee that Berry curvature is trivial only if the external magnetic field is zero.

Conclusion

The study shows that the importance of studying quantum geometry which allows to interpret interesting phenomena related to the eigenstate of the system. Expression for QGT was obtained considering parameter depend on Hilbert space and adiabatic evaluation. Adiabatic approximation confirmed that the eigenstate would not be changed throughout the process. Hence, all the calculations were done considering the ground state as the instantaneous eigenstate in which same throughout the process. Rashba Hamiltonian without an external magnetic field gives zero Berry curvature because of combined inversion and spin-flip and time reversal symmetry which was trivial quantum geometry. However, applying external magnetic field along the z direction, changed trivial quantum geometry to non-trivial quantum geometry. Finally, Quantum Geometry can be used study sophisticated and novel quantum mechanical concepts which is associate with the eigenstates of the system.

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Appendices

- *<u>Appendix A</u> - Derivation of the QGT for Rashbha Hamiltonian with Time-Reversal Symmetry*

$$\hat{H}(k_x, k_y) = \begin{pmatrix} \frac{k^2}{2} & \alpha_R(k_y + ik_x) \\ \alpha_R(k_y - ik_x) & \frac{k^2}{2} \end{pmatrix} \tag{36}$$

Where $k = \sqrt{k_x^2 + k_y^2}$. It’s straightforward to get exact eigenvalues and exact eigenstates,

$$|u_1\rangle = |u_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \frac{k_y - ik_x}{k} \end{pmatrix}, \qquad |u_0\rangle = |u_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{k_y + ik_x}{k} \\ -1 \end{pmatrix} \tag{37}$$

Eigenvalues are $E_\pm = \frac{k^2}{2} \pm \alpha_R k$

$$\hat{H}(k_x, k_y) = \begin{pmatrix} \frac{k^2}{2} & \alpha_R(k_y + ik_x) \\ \alpha_R(k_y - ik_x) & \frac{k^2}{2} \end{pmatrix}$$

$$\partial_{k_x}\hat{H} = \begin{pmatrix} k_x & i\alpha_R \\ -i\alpha_R & k_x \end{pmatrix}, \qquad \partial_{k_y}\hat{H}(k_x, k_y) = \begin{pmatrix} k_y & \alpha_R \\ \alpha_R & k_y \end{pmatrix}$$

Thus $\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle$ and $\langle u_1|\partial_{k_y}\hat{H}|u_0\rangle$ can be obtained to construct QGT which is defined by the equation (19),

$$\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle = \frac{1}{2}\begin{pmatrix}\frac{k_y - ik_x}{k} & -1\end{pmatrix}\begin{pmatrix}k_x & i\alpha_R \\ -i\alpha_R & k_x\end{pmatrix}\begin{pmatrix}1 \\ \frac{k_y - ik_x}{k}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}\frac{k_y - ik_x}{k} & -1\end{pmatrix}\begin{pmatrix}k_x + i\alpha_R\left(\frac{k_y - ik_x}{k}\right) \\ -i\alpha_R + k_x\left(\frac{k_y - ik_x}{k}\right)\end{pmatrix}$$

$$= \frac{1}{2}\left(\frac{i\alpha_R\left(k_y - ik_x\right)^2}{k^2} + i\alpha_R\right)$$

$$= \frac{\alpha_R}{k^2}\left(k_x k_y + ik_y^2\right)$$

And,

$$\langle u_1|\partial_{k_y}\hat{H}|u_0\rangle = \frac{1}{2}\begin{pmatrix}1 & \frac{k_y + ik_x}{k}\end{pmatrix}\begin{pmatrix}k_y & \alpha_R \\ \alpha_R & k_y\end{pmatrix}\begin{pmatrix}\frac{k_y + ik_x}{k} \\ -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & \frac{k_y + ik_x}{k}\end{pmatrix}\begin{pmatrix}k_y\left(\frac{k_y + ik_x}{k}\right) - \alpha_R \\ \alpha_R\left(\frac{k_y + ik_x}{k}\right) - k_y\end{pmatrix}$$

$$= \frac{1}{2}\left(-\alpha_R + \frac{\alpha_R\left(k_y + ik_x\right)^2}{k^2}\right)$$

$$= \frac{\alpha_R}{k^2}\left(-k_x^2 + ik_x k_y\right)$$

Computing $Q_{k_x k_y}$, $Q_{k_y k_x}$, $Q_{k_x k_x}$, and $Q_{k_y k_y}$

$$Q_{k_x k_y} = \frac{\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle\langle u_1|\partial_{k_y}\hat{H}|u_0\rangle}{(-2\alpha_R k)^2}$$

$$= \frac{\alpha_R^2}{k^4}\frac{\left(k_x k_y + ik_y^2\right)\left(-k_x^2 + ik_x k_y\right)}{4\alpha_R^2 k^2}$$

$$= \frac{1}{4k^6}\left(-k_x^3 k_y - k_y^3 k_x\right)$$

$$Q_{k_x k_y} = \frac{-k_x k_y}{4k^4} \tag{e}$$

$$Q_{k_y k_x} = \frac{\langle u_0|\partial_{k_y}\hat{H}|u_1\rangle\langle u_1|\partial_{k_x}\hat{H}|u_0\rangle}{(-2\alpha_R k)^2}$$

$$= \frac{1}{4k^6}\left(-k_x^2 - ik_x k_y\right)\left(k_x k_y - ik_y^2\right)$$

$$Q_{k_y k_x} = \frac{-k_x k_y}{4k^4} \tag{f}$$

$$Q_{k_x k_x} = \frac{\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle\langle u_1|\partial_{k_x}\hat{H}|u_0\rangle}{(-2\alpha_R k)^2}$$

$$= \frac{1}{4k^6}\left(k_x k_y + ik_y^2\right)\left(k_x k_y - ik_y^2\right)$$

$$= \frac{1}{4k^6}\left(k_x^2 k_y^2 + k_y^4\right)$$

$$Q_{k_x k_x} = \frac{k_y^2}{4k^4} \tag{g}$$

$$Q_{k_yk_y} = \frac{\langle u_0|\partial_{k_y}\widehat{H}|u_1\rangle\langle u_1|\partial_{k_y}\widehat{H}|u_0\rangle}{(-2\alpha_R k)^2}$$

$$= \frac{1}{4k^6}\left(-k_x^2 - ik_xk_y\right)\left(-k_x^2 + ik_xk_y\right)$$

$$= \frac{1}{4k^6}\left(k_x^2k_y^2 + k_x^4\right)$$

$$Q_{k_yk_y} = \frac{k_x^2}{4k^4} \tag{h}$$

Now Tensor terms given by the equations (e), (f), (g) & (h) can be used to write the QGT

When $\mu, \nu \leftrightarrow k_x, k_y$,

$$Q_{\mu\nu} = \begin{pmatrix} Q_{k_xk_x} & Q_{k_xk_y} \\ Q_{k_yk_x} & Q_{k_yk_y} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{k_y^2}{4k^4} & \frac{-k_xk_y}{4k^4} \\ \frac{-k_xk_y}{4k^4} & \frac{k_x^2}{4k^4} \end{pmatrix}$$

$$= \frac{1}{4k^4}\begin{pmatrix} k_y^2 & -k_xk_y \\ -k_xk_y & k_x^2 \end{pmatrix} - \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\equiv g_{\mu\nu} - \frac{i}{2}\Omega_{\mu\nu}$$

$$g_{\mu\nu} = \frac{1}{4k^4}\begin{pmatrix} k_y^2 & -k_xk_y \\ -k_xk_y & k_x^2 \end{pmatrix}, \qquad \Omega_{\mu\nu} = 0 \tag{38}$$

Considering the ground state of the Hamiltonian (24), the components of the QGT are,

$$g_{k_xk_y} = g_{k_yk_x} = \frac{-k_xk_y}{4k^4}$$

$$g_{k_xk_x} = \frac{k_y^2}{4k^4}$$

$$g_{k_yk_y} = \frac{k_x^2}{4k^4}$$

$$\Omega_{k_xk_y} = 0$$

- <u>Appendix B</u> – *Derivation of the QGT for Rashbha Hamiltonian after breaking Time-Reversal Symmetry by applying Magnetic Field along z direction.*

$$\widehat{H'} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{2\alpha_R}{\hbar}\left(S_xp_y - S_yp_x\right) - \frac{2}{\hbar}B_0S_z \tag{39}$$

Assuming $m = 1$ and $\hbar = 1$, then the Hamiltonian can be written as,

$$\widehat{H}'\left(k_x, k_y\right) = \begin{pmatrix} \frac{k^2}{2} - B_0 & \alpha_R\left(k_y + ik_x\right) \\ \alpha_R\left(k_y - ik_x\right) & \frac{k^2}{2} + B_0 \end{pmatrix} \tag{40}$$

Where $k = \sqrt{k_x^2 + k_y^2}$. Then exact eigenvalues and exact eigenstates can be obtained,

$$|u_1\rangle = |u_+\rangle = N\begin{pmatrix} \sqrt{1+x^2}-x \\ \frac{k_y - ik_x}{k} \end{pmatrix}$$

$$|u_0\rangle = |u_-\rangle = N\begin{pmatrix} \frac{k_y + ik_x}{k} \\ -\left(\sqrt{1+x^2}-x\right) \end{pmatrix} \tag{41}$$

Eigenvalues are $E_\pm = \frac{k^2}{2} \pm \alpha_R k\sqrt{1+x^2}$, where $x = \frac{B_0}{\alpha_R k}$ and normalization factor is given by $N = \frac{1}{\sqrt{2(1+x^2-x\sqrt{1+x^2}}}$. Now QGT can be obtained considering the ground state of the perturbed system. Noticing that $\partial_k \widehat{H}' = \partial_k \widehat{H}$, QGT can be obtained using the equation (19).

$$\langle u_0|\partial_{k_x}\widehat{H}|u_1\rangle = N^2\begin{pmatrix} \frac{k_y - ik_x}{k} & -\left(\sqrt{1+x^2}-x\right) \end{pmatrix}\begin{pmatrix} k_x & i\alpha_R \\ -i\alpha_R & k_x \end{pmatrix}\begin{pmatrix} \sqrt{1+x^2}-x \\ \frac{k_y - ik_x}{k} \end{pmatrix}$$

$$= N^2\begin{pmatrix} \frac{k_y - ik_x}{k} & -\left(\sqrt{1+x^2}-x\right) \end{pmatrix}\begin{pmatrix} k_x\left(\sqrt{1+x^2}-x\right) + \frac{i\alpha_R}{k}\left(k_y - ik_x\right) \\ -i\alpha_R\left(\sqrt{1+x^2}-x\right) + \frac{k_x\left(k_y - ik_x\right)}{k} \end{pmatrix}$$

$$= N^2\left[\frac{k_x\left(k_y - ik_x\right)}{k}\left(\sqrt{1+x^2}-x\right) + \frac{i\alpha_R}{k^2}\left(k_y - ik_x\right)^2 + i\alpha_R\left(\sqrt{1+x^2}-x\right)^2 - \frac{k_x\left(k_y - ik_x\right)}{k}\left(\sqrt{1+x^2}-x\right)\right]$$

$$= \frac{i\alpha_R N^2}{k^2}\left[\left(k_y - ik_x\right)^2 + k^2\left(\sqrt{1+x^2}-x\right)^2\right]$$

$$= \frac{2\alpha_R N^2}{k^2}\left\{k_x k_y + i\left[k_y^2 + x\left(x - \sqrt{1+x^2}\right)k^2\right]\right\}$$

Similarly,

$$\langle u_1|\partial_{k_y}\widehat{H}|u_0\rangle = N^2\begin{pmatrix} \left(\sqrt{1+x^2}-x\right) & \frac{k_y + ik_x}{k} \end{pmatrix}\begin{pmatrix} k_y & \alpha_R \\ \alpha_R & k_y \end{pmatrix}\begin{pmatrix} \frac{k_y + ik_x}{k} \\ -\left(\sqrt{1+x^2}-x\right) \end{pmatrix}$$

$$= N^2\begin{pmatrix} \left(\sqrt{1+x^2}-x\right) & \frac{k_y + ik_x}{k} \end{pmatrix}\begin{pmatrix} \frac{k_y(k_y + ik_x)}{k} - \alpha_R\left(\sqrt{1+x^2}-x\right) \\ \frac{\alpha_R(k_y + ik_x)}{k} + k_y\left(\sqrt{1+x^2}-x\right) \end{pmatrix}$$

$$= N^2\left[\frac{k_y\left(k_y + ik_x\right)}{k}\left(\sqrt{1+x^2}-x\right) + \frac{\alpha_R}{k^2}\left(k_y + ik_x\right)^2 - \alpha_R\left(\sqrt{1+x^2}-x\right)^2 - \frac{k_y\left(k_y + ik_x\right)}{k}\left(\sqrt{1+x^2}-x\right)\right]$$

$$= \frac{\alpha_R N^2}{k^2}\left[\left(k_y + ik_x\right)^2 - k^2\left(\sqrt{1+x^2}-x\right)^2\right]$$

$$= \frac{2\alpha_R N^2}{k^2}\left\{-k_x^2 - x\left(x - \sqrt{1+x^2}\right)k^2 + ik_x k_y\right\}$$

For the simplicity let's redefine the $\langle u_0|\partial_{k_x}\widehat{H}|u_1\rangle$ and $\langle u_1|\partial_{k_y}\widehat{H}|u_0\rangle$ as follows,

$$\langle u_0|\partial_{k_x}\widehat{H}|u_1\rangle = \frac{2\alpha_R N^2}{k^2}\left(k_x k_y + i\left(k_y^2 + \Delta k^2\right)\right)$$

$$\langle u_1|\partial_{k_y}\widehat{H}|u_0\rangle = \frac{2\alpha_R N^2}{k^2}\left(-k_x^2 - \Delta k^2 + ik_x k_y\right)$$

Where $\Delta = x\left(x - \sqrt{1+x^2}\right)$ and now computing $Q_{k_x k_y}$, $Q_{k_y k_x}$, $Q_{k_x k_x}$, and $Q_{k_y k_y}$. Notice that $(E_0 - E_1)^2 = \alpha_R^2 k^2(1+x^2) = \alpha_R^2 k^2 + B_0^2$

$$Q_{k_xk_y} = \frac{\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle\langle u_1|\partial_{k_y}\hat{H}|u_0\rangle}{(E_0 - E_1)^2}$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(k_xk_y + i\left(k_y^2 + \Delta k^2\right)\right) \times \left(-k_x^2 - \Delta k^2 + ik_xk_y\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(-k_yk_xk^2 - 2k_xk_yk^2\Delta + i(-\Delta k^4 - \Delta^2 k^4)\right)$$

$$= \frac{-4\alpha_R^2 N^4}{(\alpha_R^2 k^2 + B_0^2)}\left(\frac{k_yk_x(1+2\Delta)}{k^2} + i(\Delta^2 + \Delta)\right)$$

$$Q_{k_yk_x} = \frac{\langle u_0|\partial_{k_y}\hat{H}|u_1\rangle\langle u_1|\partial_{k_x}\hat{H}|u_0\rangle}{(E_0 - E_1)^2}$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(k_xk_y - i\left(k_y^2 + \Delta k^2\right)\right) \times \left(-k_x^2 - \Delta k^2 - ik_xk_y\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(-k_yk_xk^2 - 2k_xk_yk^2\Delta + i(\Delta k^4 + \Delta^2 k^4)\right)$$

$$= \frac{-4\alpha_R^2 N^4}{(\alpha_R^2 k^2 + B_0^2)}\left(\frac{k_yk_x(1+2\Delta)}{k^2} - i(\Delta^2 + \Delta)\right)$$

$$Q_{k_xk_x} = \frac{\langle u_0|\partial_{k_x}\hat{H}|u_1\rangle\langle u_1|\partial_{k_x}\hat{H}|u_0\rangle}{(E_0 - E_1)^2}$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(k_xk_y + i\left(k_y^2 + \Delta k^2\right)\right) \times \left(k_xk_y - i\left(k_y^2 + \Delta k^2\right)\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2k_x^2 + k_y^4 + 2\Delta k_y^2 k^2 + \Delta^2 k^4\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2 + 2k_y^2\Delta + k^2\Delta^2\right)$$

$$Q_{k_yk_y} = \frac{\langle u_0|\partial_{k_y}\hat{H}|u_1\rangle\langle u_1|\partial_{k_y}\hat{H}|u_0\rangle}{(E_0 - E_1)^2}$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(-k_x^2 - \Delta k^2 - ik_xk_y\right) \times \left(-k_x^2 - \Delta k^2 + ik_xk_y\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^4(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2k_x^2 + k_x^4 + 2\Delta k_x^2 k^2 + \Delta^2 k^4\right)$$

$$= \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_x^2 + 2k_x^2\Delta + k^2\Delta^2\right)$$

Hence,

$$Q_{k_xk_y} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_yk_x(1+2\Delta) + i(\Delta^2 + \Delta)k^2\right) \tag{i}$$

$$Q_{k_yk_x} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_yk_x(1+2\Delta) - i(\Delta^2 + \Delta)k^2\right) \tag{j}$$

$$Q_{k_xk_x} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_y^2 + 2k_y^2\Delta + k^2\Delta^2\right) \tag{k}$$

$$Q_{k_yk_y} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)}\left(k_x^2 + 2k_x^2\Delta + k^2\Delta^2\right) \tag{l}$$

Where $\Delta = x\left(x - \sqrt{1+x^2}\right)$ and $x = \frac{B_0}{\alpha_R k}$. Now Tensor terms given by the equations (i), (j), (k) & (l) can be used to write the QGT, when $\mu, \nu \leftrightarrow k_x, k_y$,

$$Q_{\mu\nu} = \begin{pmatrix} Q_{k_x k_x} & Q_{k_x k_y} \\ Q_{k_y k_x} & Q_{k_y k_y} \end{pmatrix}$$

$$= \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \left\{ \begin{pmatrix} k_y^2 + 2k_y^2\Delta + k^2\Delta^2 & -k_y k_x(1+2\Delta) \\ -k_y k_x(1+2\Delta) & k_x^2 + 2k_x^2\Delta + k^2\Delta^2 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} 0 & 2(\Delta^2+\Delta)k^2 \\ -2(\Delta^2+\Delta)k^2 & 0 \end{pmatrix} \right\}$$

$$\equiv g_{\mu\nu} - \frac{i}{2}\Omega_{\mu\nu}$$

Hence, we $g_{\mu\nu}$ and $\Omega_{\mu\nu}$ can be defined as,

$$g_{\mu\nu} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \begin{pmatrix} k_y^2 + 2k_y^2\Delta + k^2\Delta^2 & -k_y k_x(1+2\Delta) \\ -k_y k_x(1+2\Delta) & k_x^2 + 2k_x^2\Delta + k^2\Delta^2 \end{pmatrix}$$

$$\Omega_{\mu\nu} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \begin{pmatrix} 0 & 2(\Delta^2+\Delta)k^2 \\ -2(\Delta^2+\Delta)k^2 & 0 \end{pmatrix}$$

Therefore, components of the QGT are,

$$g_{k_x k_x} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \left(k_y^2 + 2k_y^2\Delta + k^2\Delta^2\right)$$

$$g_{k_y k_y} = \frac{4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \left(k_x^2 + 2k_x^2\Delta + k^2\Delta^2\right)$$

$$g_{k_x k_y} = \frac{-4\alpha_R^2 N^4}{k^2(\alpha_R^2 k^2 + B_0^2)} \left(k_y k_x(1+2\Delta)\right)$$

$$\Omega_{k_x k_y} = \frac{8\alpha_R^2 N^4(\Delta^2+\Delta)}{(\alpha_R^2 k^2 + B_0^2)}$$