Slido: **#ADA2022**

CSIE 2136 Algorithm Design and Analysis, Fall 2022

Graph Algorithms - II

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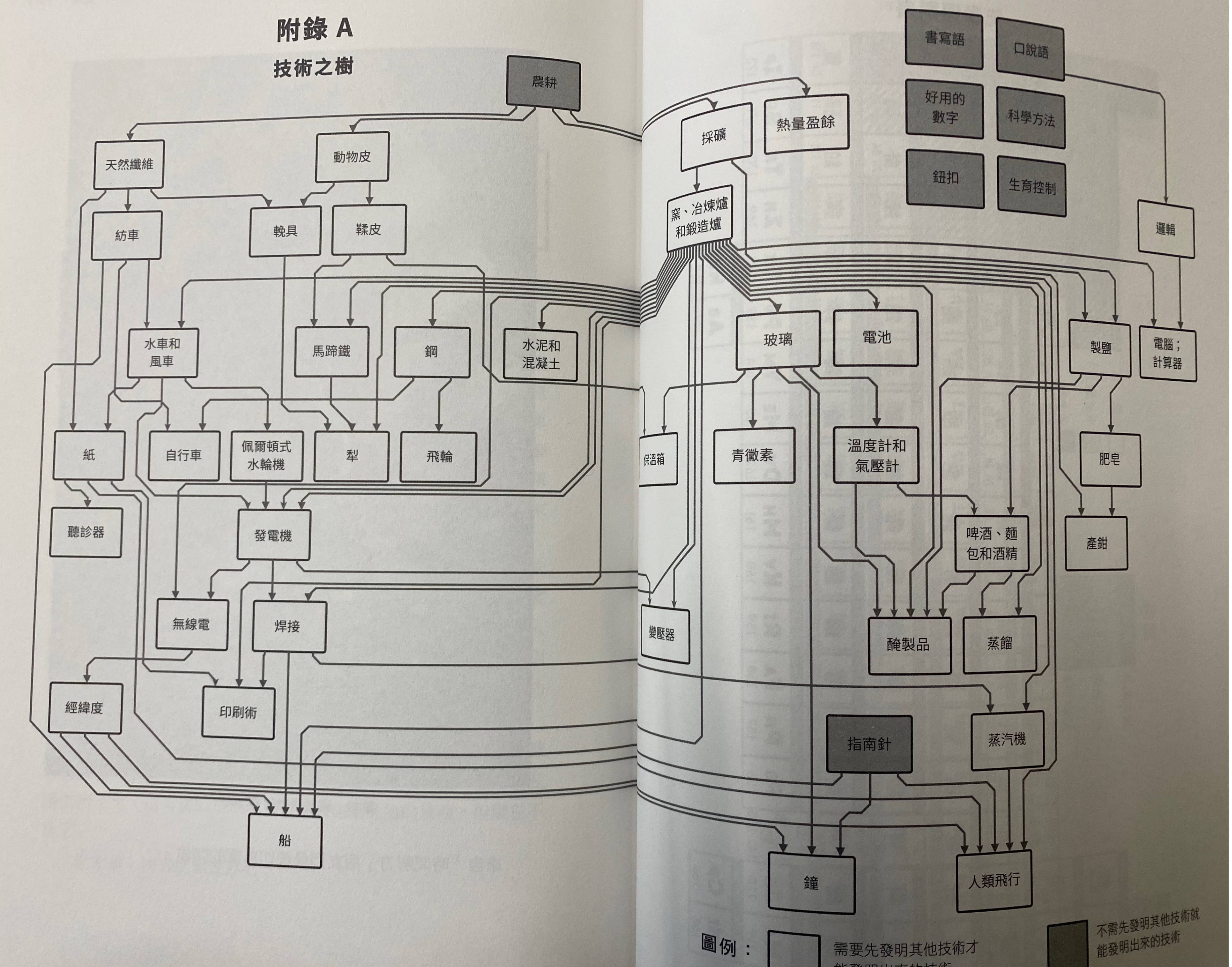
Today’s Agenda

- Finish last week’s slides⋯
- DFS applications
  - Topological sort [Ch. 22.4]
  - Strongly-connected components [Ch. 22.5]
- Minimum spanning trees [Ch. 23]
  - Kruskal's algorithm
  - Prim's algorithm

Application of DFS: Topological Sort

Textbook chapter 22.4

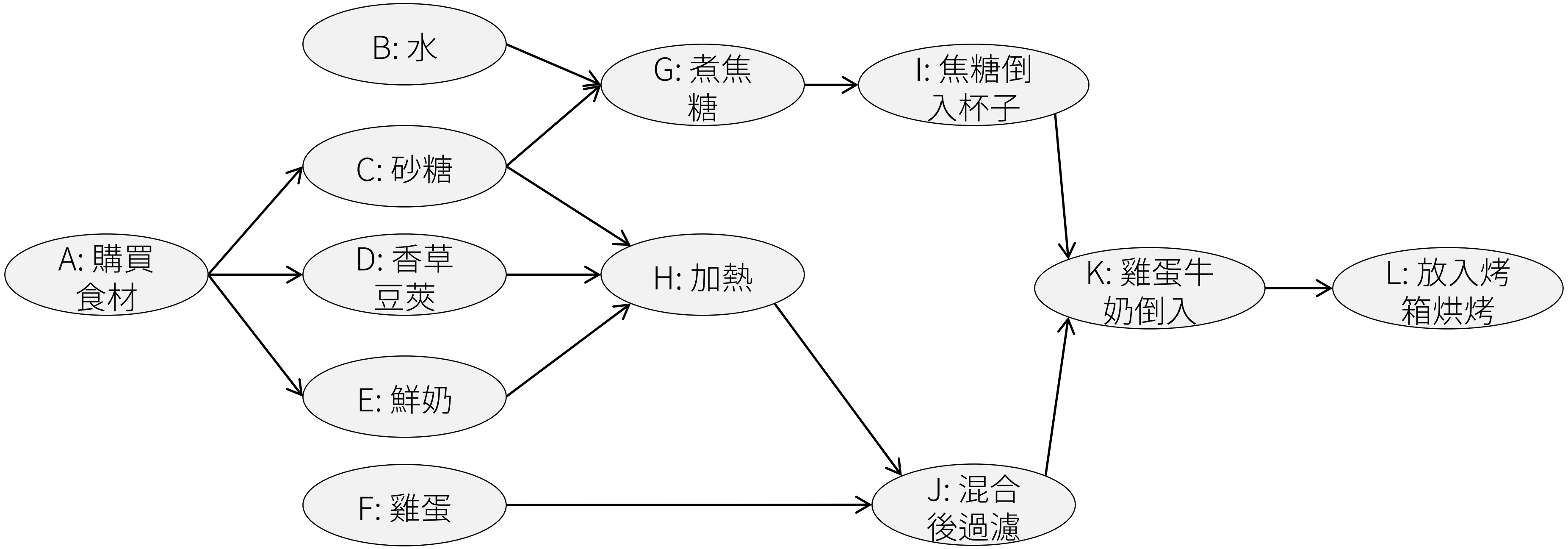


製造文明：不管落在地球歷史的哪段時期，都能保全性命、發展技術、創造歷史，成為新世界的神
How to Invent Everything: A Survival Guide for the Stranded Time Traveler. Ryan North. 2019.

Q: 新手一次只能做一件事，用什麼順序才能順利做出布丁？

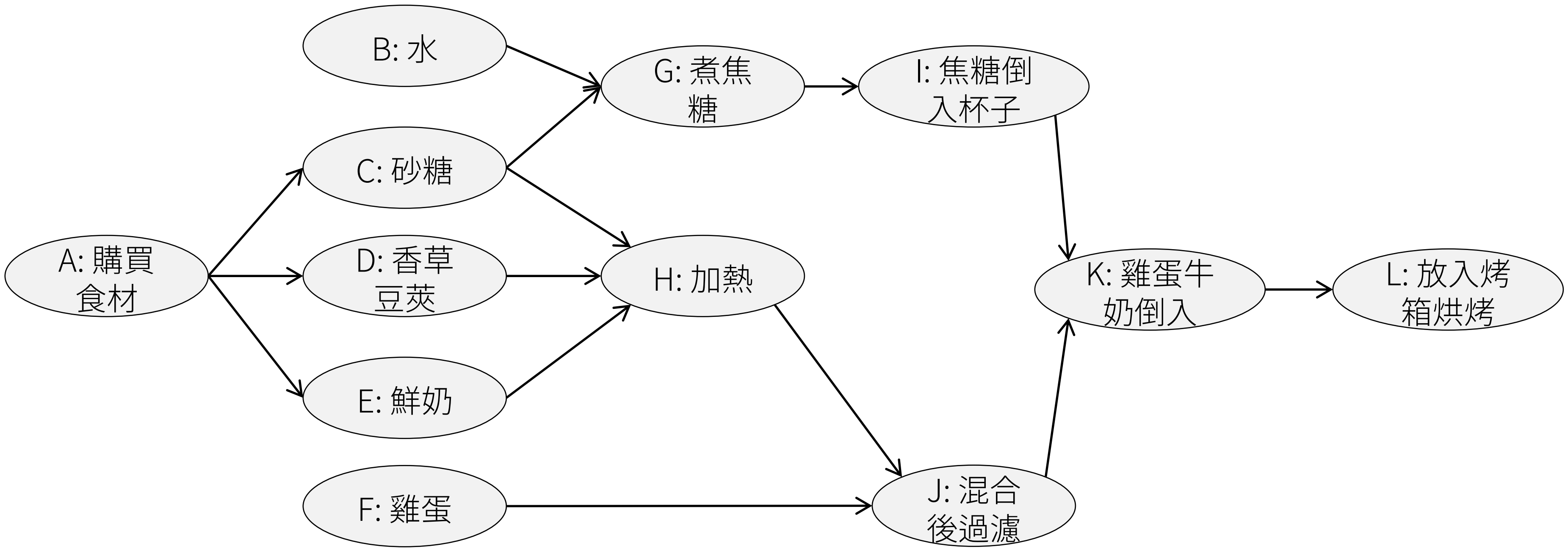
A->B: 要先處理完 A 才能處理 B

Intuition: 前置作業要先完成，才能做後面的步驟



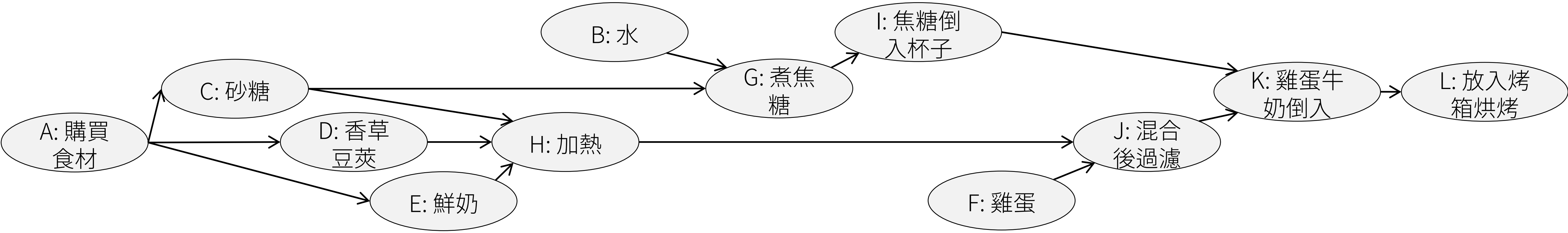
Topological Sort

- Input: a directed acyclic graph (DAG) $G = (V, E)$
  - Often indicates precedence among events (X must happen before Y)
- Output: a linear ordering of all its vertices such that for all edges (u, v) in E, u precedes v in the ordering



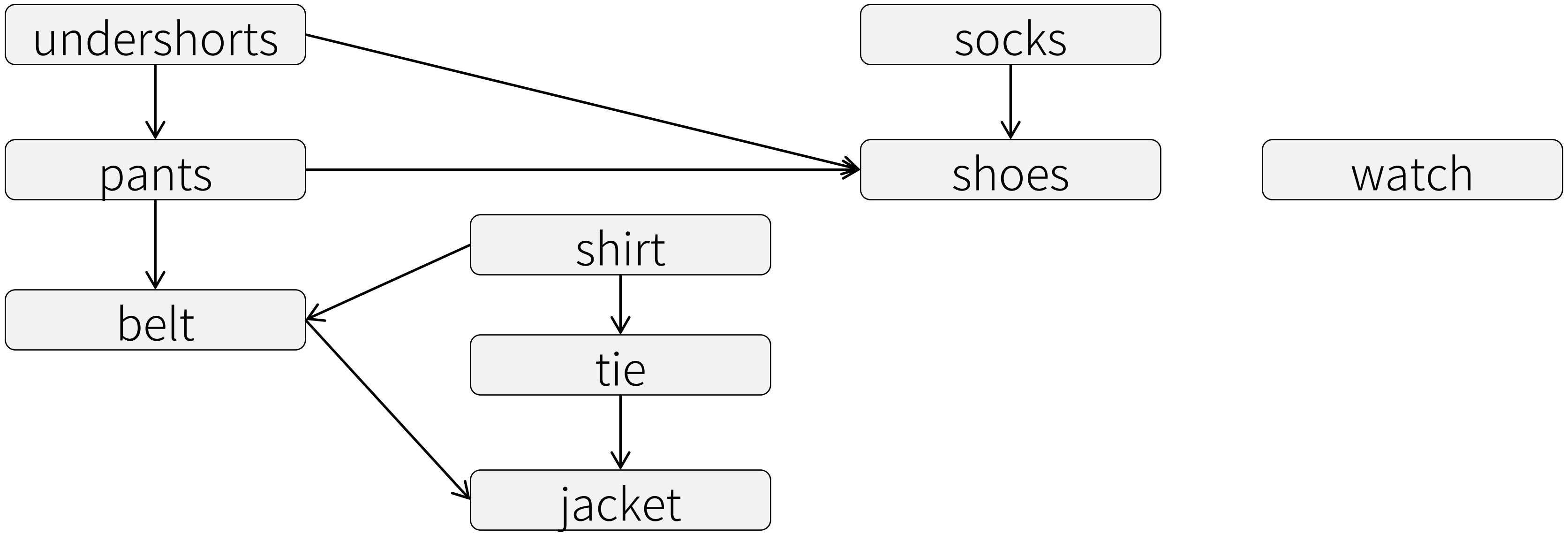
Topological Sort

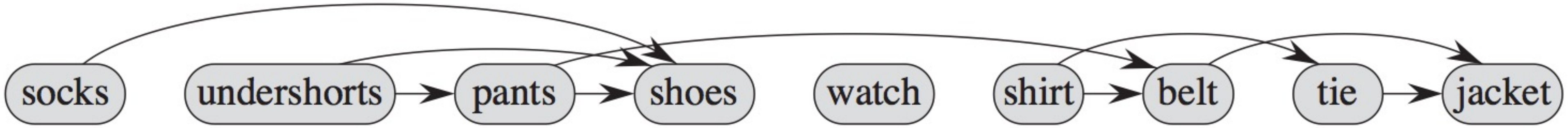
- <u>Input</u>: a directed acyclic graph (DAG) $G = (V, E)$
  - Often indicates precedence among events (X must happen before Y)
- <u>Output</u>: a linear ordering of all its vertices such that for all edges (u, v) in E, u precedes v in the ordering
- <u>Alternative view</u>: a vertex ordering along a horizontal line so that all directed edges go from left to right



Topological Sort

- Alternative view: a vertex ordering along a horizontal line so that all directed edges go from left to right



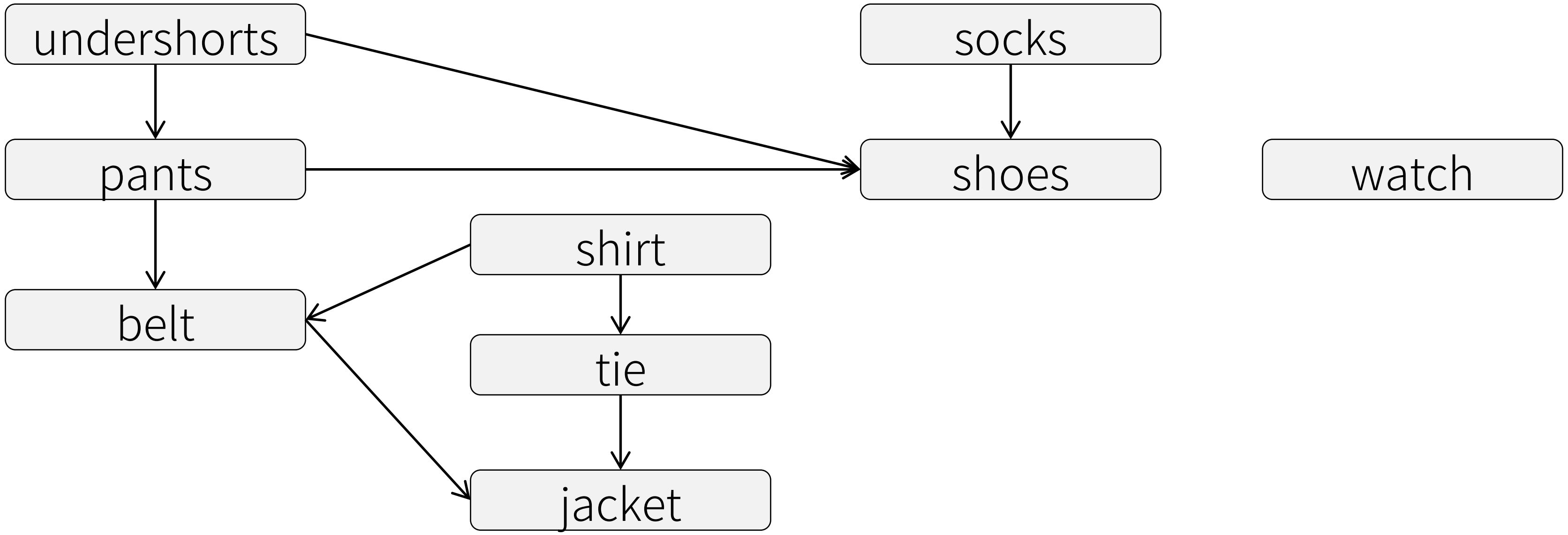


Topological sort algorithm

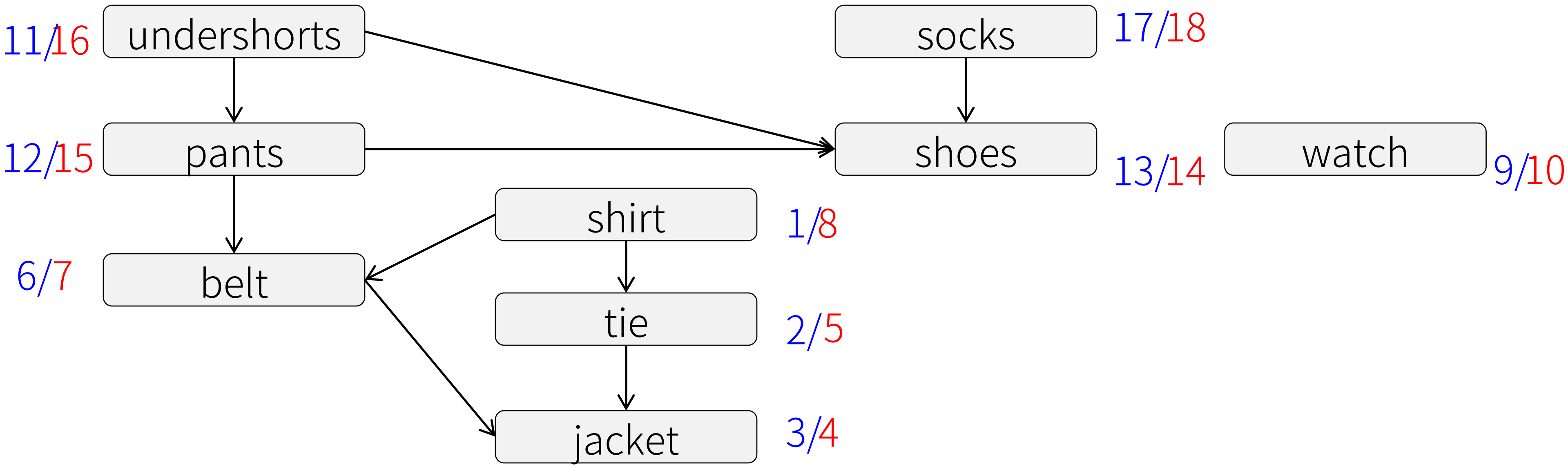
```
TOPOLOGICAL-SORT(G) //G is a DAG
    Call DFS(G) to compute finishing times v.f for each vertex v
    As each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
```

- Vertices are ordered by their DFS finishing times (in a descending order)
- We will prove this linked list comprises a topological ordering

Topological sort using DFS



Topological sort using DFS



socks	undershorts	pants	shoes	watch	shirt	belt	tie	jacket

Running time analysis

```
TOPOLOGICAL-SORT(G) //G is a DAG
    Call DFS(G) to compute finishing times v.f for each vertex v
    As each vertex is finished, insert it onto the front of a linked list
    return the linked list of vertices
```

- DFS with adjacency lists: $\Theta(V+E)$ time
- Insert each vertex to the linked list: $\Theta(V)$ time
- => total running time is $\Theta(V+E)$

Theorem 22.12 Correctness of topological sort algorithm

The algorithm produces a topological sort of the input DAG

請證明：若存在 edge (u, v)，在 topological sort 生成的 vertex list 中，u 一定在 v 前面（也就是 $u.f > v.f$ 成立）

Proof

- When (u, v) is explored, u is gray.
- Consider three cases of v: gray, white, black

Theorem 22.12 Correctness of topological sort algorithm

The algorithm produces a topological sort of the input DAG

Proof (cont.)

- v = gray
  => (u, v) = back edge
  => G is cyclic (by Lemma 22.11)
  => Contradiction, so v cannot be gray
- v = white
  => v becomes descendant of u (by white-path theorem)
  => v will be finished before u (by parenthesis theorem)
  => $v.f < u.f$
- v = black
  => v is already finished
  => $v.f < u.f$

Q: Is there a topological order for a cyclic graph?

True/False: A DAG must contain a vertex whose in-degree is 0.

Another topological sort algorithm: Kahn's algorithm

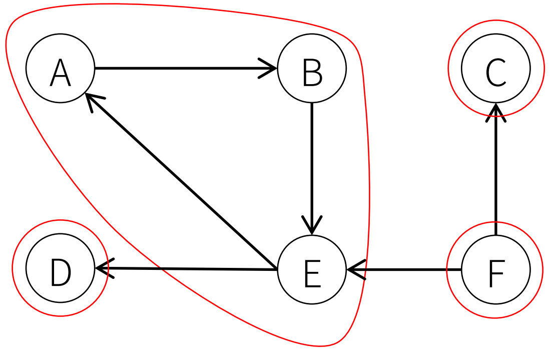
- Intuition: removing "source vertices" one by one and updating in-degree values
  - Source vertices: vertices with in-degree = 0

- Correctness: why is there always a vertex with zero in-degree?
- Running time is $\Theta(V + E)$
  - Need to maintain in-degree values and a queue of current source vertices

Strongly Connected Components (SCC)

Strongly connected components of a directed graph

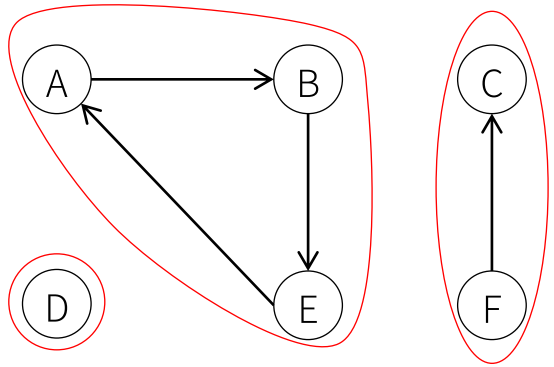
The strongly connected components of a directed graph are the equivalence classes of vertices under the "mutually reachable" relation. That is, a strongly connected component is a maximal subset of mutually reachable nodes.



4 strongly connected components: {A,B,E}, {C}, {D}, {F}

Weakly connected components of a directed graph

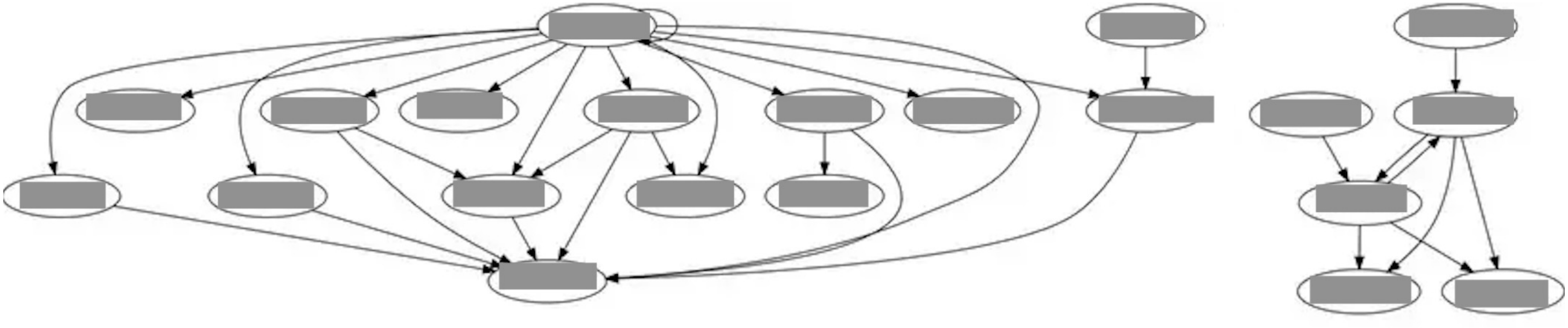
The weakly connected components of a directed graph are the equivalence classes of vertices under the "is reachable from" relation if all directed edges are replaced by undirected ones.



3 weakly connected components: {A,B,E}, {C,F}, {D}

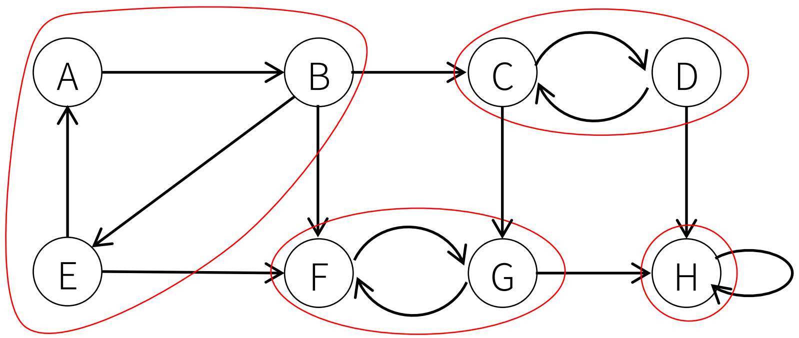
Example: Homework-reference graph

- A directed graph in which vertices are students and edges represent “acknowledgments”
- To ease the grading process, the TAs want to cluster the answers by identifying weakly and strongly connected components



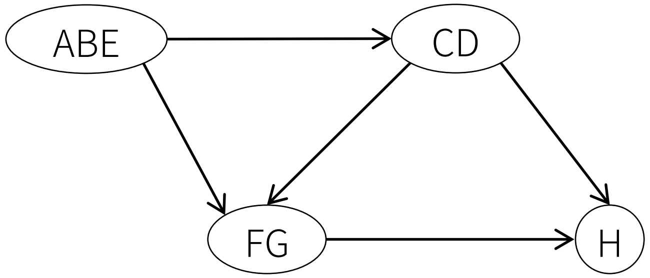
Decomposing a directed graph

- A directed graph is a DAG of its SCC



$G = (V, E)$

Contract each SCC into one vertex



Component graph $G^{scc} = (V^{scc}, E^{scc})$

Q: Show that a component graph must be a DAG

Q: Does the following algorithm **determine** whether a graph G is strongly connected in $O(V + E)$ time?

```
Run BFS in G from any node s
Run BFS in the transpose of G, from the same source node s
If both BFS executions found all nodes, return true; otherwise, return false
```

Note: we denote a transpose or reverse graph of a directed graph $G = (V, E)$ as G^T, and $G^T = (V, E^T)$ where $E^T = \{(v, u) \mid (u, v) \in E\}$

Finding SCC: the Kosaraju-Sharir algorithm

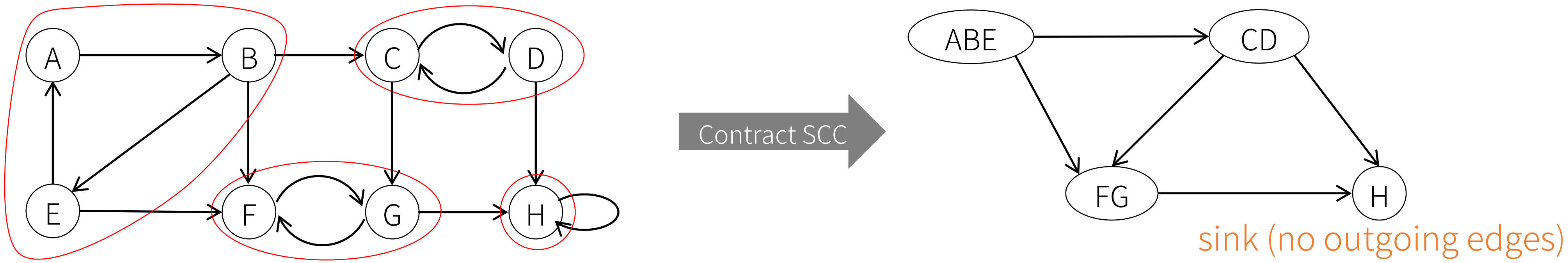
Strongly-Connected-Components(G)

1 call $DFS(G)$ to compute finishing times $u.f$ for each vertex u
2 compute G^T
3 call $DFS(G^T)$, but in the main loop of DFS, consider the vertices in order of decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the DFS forest formed in line 3 as a separate strongly connected component

- Input: a directed graph $G = (V, E)$
- Output: strongly connected components
- Time complexity
  - 2 DFS executions
  - $\Theta(V + E)$ using adjacency lists

Finding SCC

- Observation 1: Starting from s, DFS finds all reachable nodes from s. Hence, if we can select a vertex in a sink SCC as the starting vertex for DFS, then DFS will discover all (and only) vertices in the sink SCC.
  - => we can find SCCs one by one in a reverse topological order of G^{scc}!
  - However, how to identify a vertex in a sink SCC?

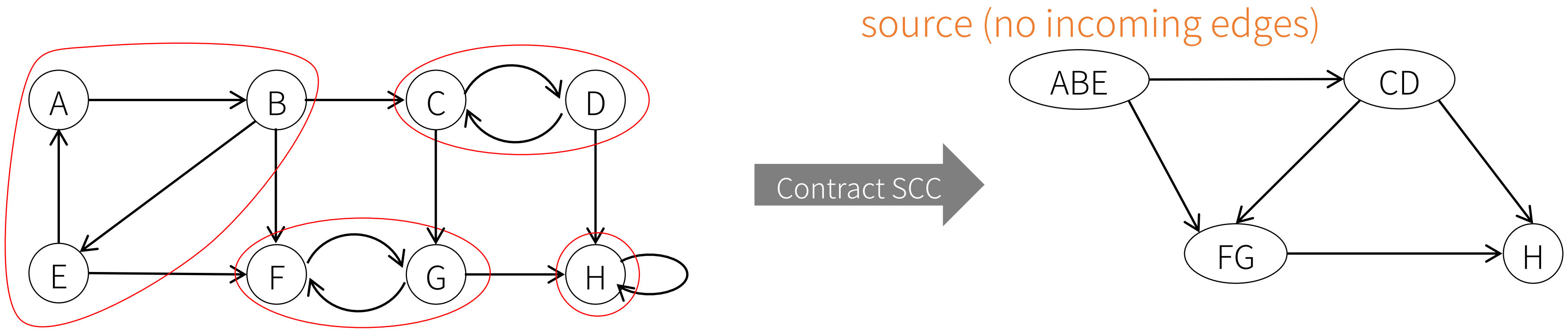


$G = (V, E)$

Component graph $G^{scc} = (V^{scc}, E^{scc})$

Finding SCC

- Observation 2 (Exercises 22.5-4): An SCC in G is also an SCC in G^T. Also, a source SCC in G is a sink SCC in G^T.
- Observation 3: Finding a vertex in a source SCC is easy. The vertex with the highest finishing time (found by running DFS in G) must be in a source SCC.
  - Implied by Lemma 22.14 (will prove it in a few slides)



$G = (V, E)$

Component graph $G^{scc} = (V^{scc}, E^{scc})$

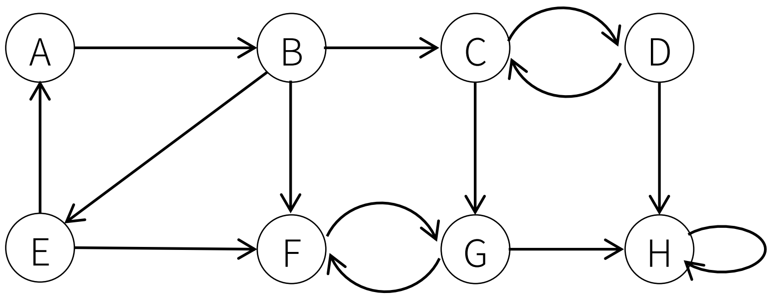
Finding SCC: the Kosaraju-Sharir algorithm

```
Strongly-Connected-Components(G)
1   call DFS(G) to compute finishing times u.f for each vertex u
2   compute G^T
3   call DFS(G^T), but in the main loop of DFS, consider the vertices in order of
        decreasing u.f (as computed in line 1)
4   output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component
```

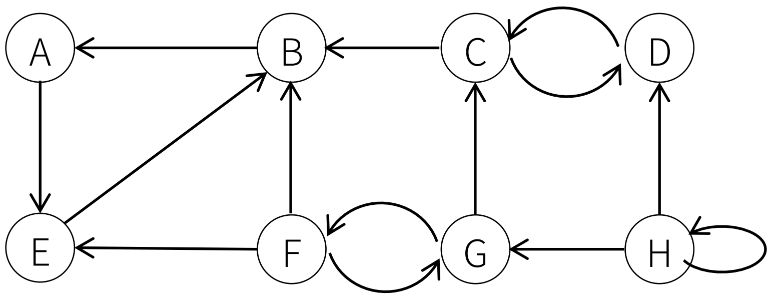
- Observation 1: Starting from s, DFS finds all reachable nodes from s. Hence, if we can select a vertex in a sink SCC as the starting vertex for DFS, then DFS will discover all (and only) vertices in the sink SCC.
- Observation 2 (Exercises 22.5-4): An SCC in G is also an SCC in G^T. Also, a source SCC in G is a sink SCC in G^T.
- Observation 3: Finding a vertex in a source SCC is easy. The vertex with the highest finishing time (found by running DFS in G) must be in a source SCC.

Let’s try it!

```
call DFS(G) to compute u.f
```




```
compute G^T
call DFS(G^T), in decreasing
    order of u.f
```



Lemma 22.14

Let C and C' be distinct strongly connected components in directed graph $G = (V, E)$. Suppose that there is an edge (u, v) where u in C and v in C'.
Then $f(C) > f(C')$.
Here we define $f(U) = max_{u \in U}\{u.f\}$, and $d(U) = min_{u \in U}\{u.d\}$

Proof: Consider two cases: $d(C) < d(C')$ and $d(C) > d(C')$

- If $d(C) < d(C')$:
  - Let x be the first vertex discovered in C
  - => At $t = x.d$, all vertices in C and C' are WHITE
  - => At $t = x.d$, there is a white path from x to every vertex in C and C'
  - => By the white-path theorem, they are all x's decendants in the DFS tree
  - => By the parenthesis theorem, $x.f$ is the largest
  - => $f(C) = x.f > f(C')$

Lemma 22.14

Let C and C' be distinct strongly connected components in directed graph $G = (V, E)$. Suppose that there is an edge (u, v) where u in C and v in C'. Then $f(C) > f(C')$.
Here we define $f(U) = max_{u \in U}\{u.f\}$, and $d(U) = min_{u \in U}\{u.d\}$

Proof (cont'd) If $d(C) > d(C')$:

- Let y be the first vertex discovered in C'

=> At $t = y.d$, all vertices in C' are white

=> At $t = y.d$, there is a white path from y to every vertex in C'

=> By the white-path theorem and the parenthesis theorem, all other vertices in C' are y's descendants and $y.f$ is the largest among them

=> $f(C') = y.f$

- Also, since there is no path from C' to C (why?), no vertex in C is reachable from y

=> At $t = y.f$, all vertices in C are still WHITE; that is, $d(C) > y.f$

- Combining that $f(C) > d(C)$, we have $f(C) > d(C) > y.f = f(C')$

Q: Can the following algorithms correctly find SCCs?

```
Strongly-Connected-Components-1(G)
1   compute G^T
2   call DFS(G^T) to compute finishing times u.f for each vertex u
3   call DFS(G), but in the main loop of DFS, consider the vertices in order of
          decreasing u.f (as computed in line 2)
4   output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component
```

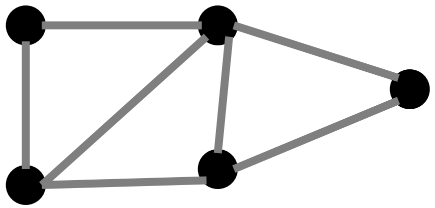
```
Strongly-Connected-Components-2(G)
1   call DFS(G) to compute finishing times u.f for each vertex u
2   call DFS(G), but in the main loop of DFS, consider the vertices in order of
          increasing u.f (as computed in line 1)
3   output the vertices of each tree in the DFS forest formed in line 3 as a
    separate strongly connected component
```

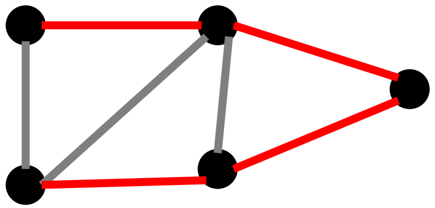
Minimum Spanning Trees

Textbook Chapter 23

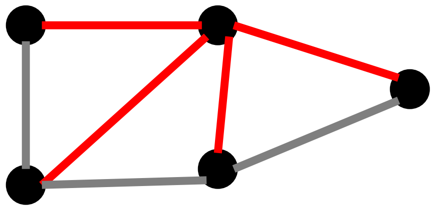
Spanning tree

- Spanning tree of a connected undirected graph G = a subgraph that is a tree and connects all the vertices
  - Exactly $|V| - 1$ edges
  - Acyclic
- There can be many spanning trees of a graph





Spanning tree 1

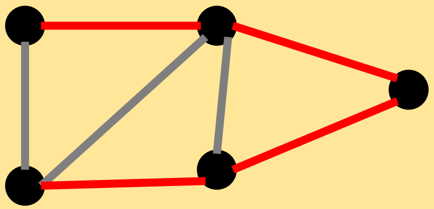


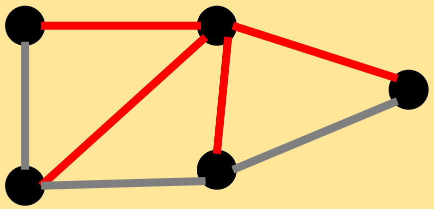
Spanning tree 2

Spanning tree

- BFS and DFS also generate spanning trees
  - BFS tree is typically “short and bushy”
  - DFS tree is typically “long and stringy”

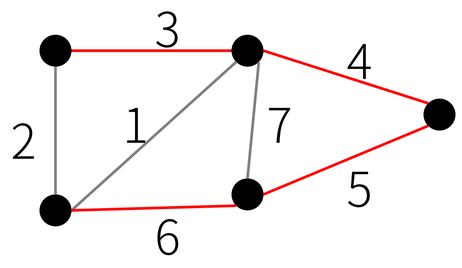
Q: Can these spanning trees be generated from BFS or DFS?



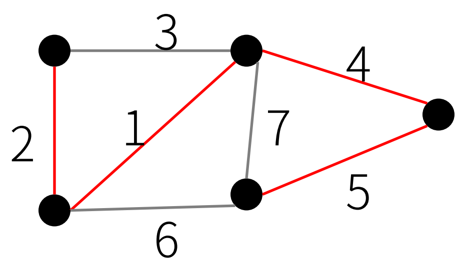


Minimum spanning tree (MST)

- A minimum spanning tree of a graph G is a spanning tree with minimal weight
- Weight of a tree T = the sum of weights of all edges in T



Weight = 18



Weight = 12, MST

Q: How to find an MST in an unweighted graph (i.e., edges have equal weights)?

Q: Given a weighted graph G, can there be more than one MST?

Q: If the edge weights of G are all increased by the same constant, does an MST of the old graph remain an MST in the re-weighted graph?

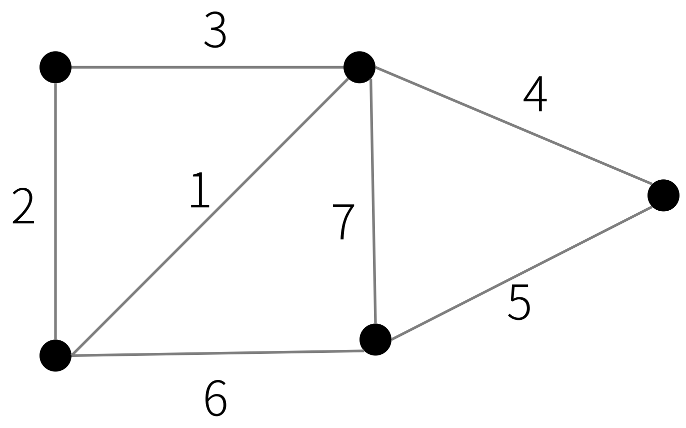
Minimum spanning tree (MST)

- Finding an MST is an optimization problem

- Two greedy algorithms compute an MST:
  - Kruskal's algorithm: consider edges in ascending order of weight. At each step, select the next edge as long as it does not create cycle.
  - Prim's algorithm: start with any vertex s and greedily grow a tree from s. At each step, add the edge of the least weight to connect an isolated vertex.

Kruskal's algorithm

```
Kruskal(G)
    start with T = V (no edges)
    for each edge in increasing order by weight
        if adding edge to T does not create a cycle
            then add edge to T
```



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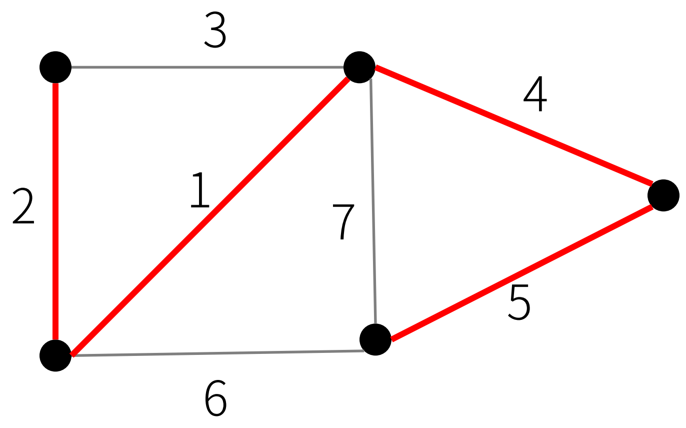


Weight = 12
MST

Implementation of Kruskal’s algorithm

```
MST-KRUSKAL(G,w)  // w = weights
    A = empty // edge set of MST
    for v in G.V
        MAKE-SET(v)
    sort the edges of G.E into non-decreasing order by weight
    for (u,v) in G.E, taken in non-decreasing order by weight
        if FIND-SET(u) ≠ FIND-SET(v) // cycle test
            A = A ∪ (u, v)
            UNION(u,v)
    return A
```

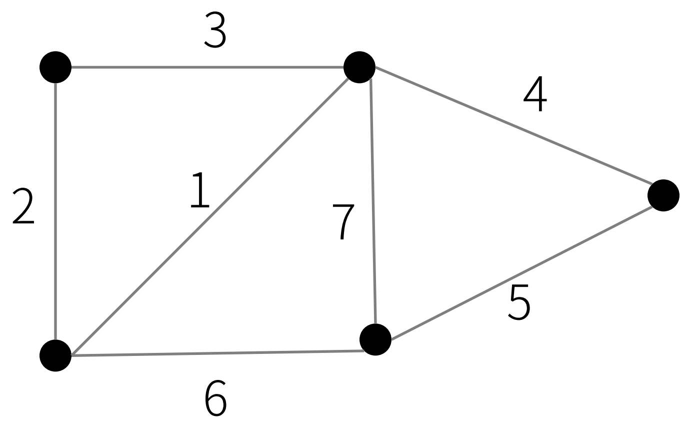
- Disjoint-set data structure: MAKE-SET, FIND-SET, UNION
- Each set contains the vertices in one tree of the current forest

Running time analysis

- Using disjoint-set-forest implementation with union-by-rank and path compression, Kruskal's algorithm can run in $O(E \log V)$
- [Ch. 21] The disjoint-set-forest implementation with union-by-rank and path compression for m operations on n elements is $O\big(m\,\alpha(n)\big)$, where $\alpha(n)$ is a very slow growing function.
- Kruskal's running time = sorting edge + disjoint-set operations
  - Sorting edge = $O(E \log E)$ = $O(E \log V)$
  - Disjoint-set operations = $O\big(m\,\alpha(n)\big) = O\big((2V + 2E - 1)\,\alpha(V)\big) = O(E\,\alpha(V))$
    - $m = 2V + E - 1, n = V$
  - Note that $V^2 \geq E \geq V - 1$ on a connected graph without multi-edges

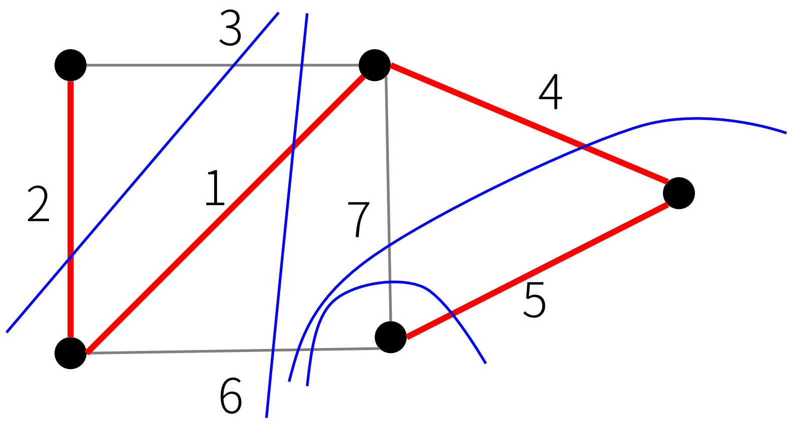
Prim's Algorithm

```
Prim(G)
    Start with a tree T with one vertex (any vertex)
    while T is not a spanning tree
        Find least-weight edge that connects T to a new vertex
        Add this edge to T
```



Prim's Algorithm

```
Prim(G)
    Start with a tree T with one vertex (any vertex)
    while T is not a spanning tree
        Find least-weight edge that connects T to a new vertex
        Add this edge to T
```



Weight = 12
MST

Implementation of Prim's algorithm

```
MST-PRIM(G, w, r) //w = weights, r = root
  for u in G.V
      u.key = ∞
      u.π = NIL
  r.key = 0
  Q = G.V //BUILD-MIN-QUEUE
  while Q ≠ empty
      u = EXTRACT-MIN(Q)
      for v in G.adj[u]
          if v ∈ Q and w(u,v) < v.key
              v.π = u
              v.key = w(u,v) //DECREASE-KEY
```

- Q = min-priority queue, containing vertices not yet in the tree
- $v.key$ = minimum weight of any edge connecting v to the tree
- $v.\pi$ = the parent of v in the tree

Running time analysis

- Binary min-heap [Ch. 6]
  - BUILD-MIN-HEAP = $O(V)$
  - EXTRACT-MIN = $O(\log V)$
  - DECREASE-KEY = $O(\log V)$
- Running time of Prim = $O(V \log V + E \log V)$
  = $O(E \lg V)$, because $V = O(E)$ in a connected graph

- Can be improved to $O(E + V \log V)$ using Fibonacci heaps [Ch. 19]

MST properties

MST Uniqueness

MST is unique if all edge weights are distinct. More generally, MST is unique if we apply a unique edge order.

Cycle property

For simplicity, apply a unique edge order, thus a unique MST.
Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.

Cut property

For simplicity, apply a unique edge order, thus a unique MST.
Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.

MST Uniqueness

MST is unique if all edge weights are distinct

Proof by contradiction

- Suppose there are two MSTs T_A and T_B on the same graph
- Let e be the least-weight edge in $T_A \cup T_B$ and e is not in both
- WLOG, assume e is in T_A
- Add e to T_B

=> $\{e\} \cup \ T_B$ contains a cycle C

=> C includes at least one edge e’ that is not in T_A

=> In T_B, replacing e’ with e yields a MST with less cost

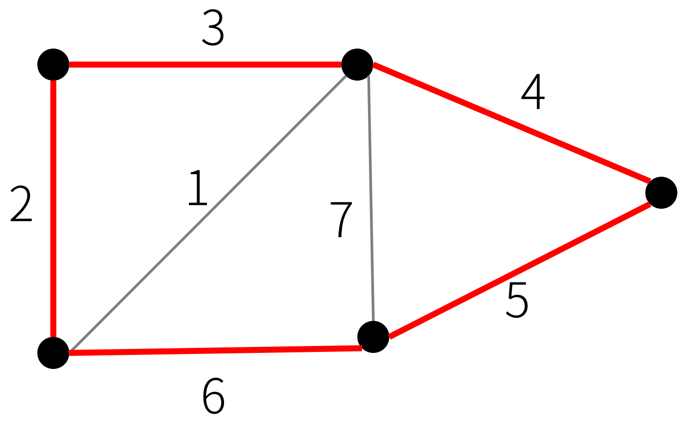
=> Contradiction!

MST uniqueness when edge weights are not distinct

- We can still break tie and ensure a unique MST by applying a lexicographical order of edges
- Let's define a new weight function w' over edges such that
  - $w'(e_i) < w'(e_j)$ if $w(e_i) < w(e_j)$ or $(w(e_i) = w(e_j)$ and $i < j)$
  - $w'(S_a) < w'(S_b)$ if $w(S_a) < w(S_b)$ or $(w(S_a) = w(S_b)$ and $S_a \backslash S_b$ has a lower indexed edge than $S_b \backslash S_a)$
- Hence, there is a unique MST w.r.t. to this new weight function w'
- Note: Having a unique edge order (and a unique MST) is useful for proving the correctness of Prim's and Kruskal's algorithms. However, the two algorithms **DO NOT** require the weights to be distinct.

Cycle property

For simplicity, apply a unique edge order and thus a unique MST.
Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.



MST does not contain the edge of cost 6

Cycle property

For simplicity, apply a unique edge order and thus a unique MST.
Let C be any cycle in the graph G, and let e be an edge with the maximum weight on C. Then the MST does not contain e.

Proof by contradiction

- Suppose e is in the MST

=> Removing e disconnects the MST T into two components T_1 and T_2

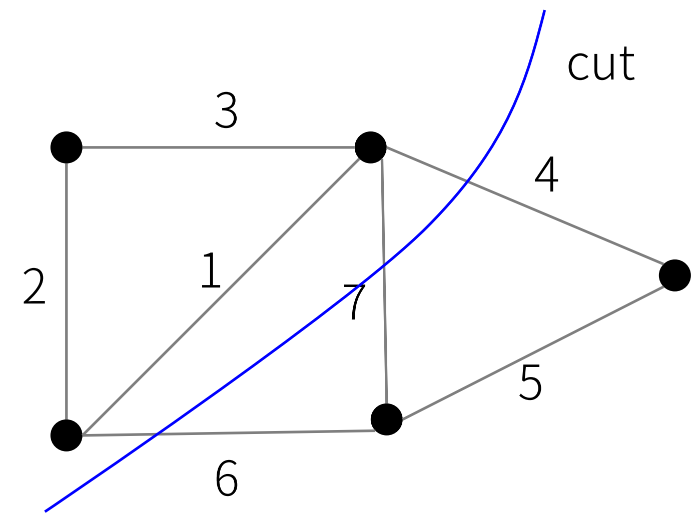
=> There exists another edge e' in C that can reconnect T_1&T_2 into T'

=> Since $w(e') < w(e)$, the new tree T' has a lower weight than T

=> Contradiction!

Cut property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.



MST containing the edge of cost 4

Cut property

For simplicity, apply a unique edge order and thus an unique MST.
Let C be a cut (i.e., a partition of the vertices) in the graph, and let e be the edge with the minimum cost across C. Then the MST contains e.

Proof by contradiction

- Suppose e is not in the current MST T

=> Adding e creates a cycle in the MST T

=> There exists another edge e' in the cut C that can break the cycle; removing e' to generate a new tree T'

=> Since $w(e') > w(e)$, the new tree has a lower weight

=> Contradiction!

Kruskal's algorithm computes the MST

Proof

- Consider whether adding e creates a cycle:

1. If adding e to T creates a cycle C
   - Then e is the max weight edge in C
   - The cycle property ensures that e is not in the MST
2. If adding $e = (u, v)$ to T does not create a cycle
   - Before adding e, the current set contains at least two trees T_1 and T_2 such that u in T_1 and v in T_2
   - e is the minimum cost edge on the cut of T_1 and $V \backslash T_1$
   - The cut property ensures that e is in the MST

Correctness of Prim's algorithm

Prim's algorithm computes the MST

Proof

1. Prove that all edges found by Prim's are in the MST:
   - Prim's algorithm adds the cheapest edge e with exactly one endpoint in the current tree T
   - The cut property ensures that e is in the MST
2. Because Prim's outputs a spanning tree, |edges found by Prim's| = $V - 1$

- => Edges found by Prim's = edges on the MST

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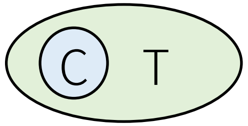
Appendix: Correctness of the Kosaraju-Sharir algorithm

Theorem 22.16 Correctness of the Kosaraju-Sharir algorithm

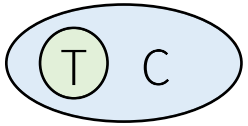
The Kosaraju-Sharir algorithm correctly computes the strongly connected components of the directed graph G provided as its input

Proof by induction on the number of DFS trees in line 3

- Inductive hypothesis: the first k trees produced are SCC
  - Base case: when $k = 0$, trivially correct
- Inductive step: assume the first k trees are SCC, consider the $(k+1)$th tree T
  - Let u be the first vertex of T, and let u be in SCC C
  - We will show that the vertices of T are the same as vertices in C



All vertices in C are in T:



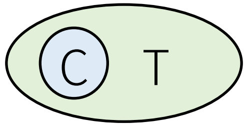
All vertices in T are in C:

Theorem 22.16 Correctness of the Kosaraju-Sharir algorithm

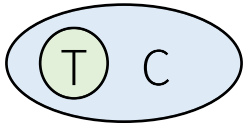
The Kosaraju-Sharir algorithm correctly computes the strongly connected components of the directed graph G provided as its input

Proof by induction (cont'd)

- Inductive step: assume the first k trees are SCC, consider the $(k+1)$th tree T
  - Let u be the first vertex of T, and let u be in SCC C
  - We will show that the vertices of T are the same as vertices in C
  - All vertices in C are in T:
    By the inductive hypothesis, at $t = u.d$, all other vertices of C are white.
    By the white-path theorem, all vertices in C are descendants of u in T.



  - All vertices in T are in C:
    By construction, $u.f$ is the largest among vertices that have yet to be visited in line 3.
    That is, $u.f = f(C) > f(C')$, where C' is any SCC other than C that has yet to be visited.
    Lemma 22.4 implies that there is no edge from C' to C in G (thus no edge from C to C' in G^T), so T will not contain any vertices in any C'.



Appendix:
Proof Techniques

Statement	Ways to Prove it	Ways to Use it	How to Negate it
p	• Prove that p is true. • Assume p is false, and derive a contradiction.	• p is true. • If p is false, you have a contradiction.	not p
p and q	• Prove p, and then prove q.	• p is true. • q is true.	(not p) or (not q)
p or q	• Assume p is false, and deduce that q is true. • Assume q is false, and deduce that p is true. • Prove that p is true. • Prove that q is true.	• If $p \Rightarrow r$ and $q \Rightarrow r$ then r is true. • If p is false, then q is true. • If q is false, then p is true.	(not p) and (not q)
$p \Rightarrow q$	• Assume p is true, and deduce that q is true. • Assume q is false, and deduce that p is false.	• If p is true, then q is true. • If q is false, then p is false.	p and (not q)
$p \iff q$	• Prove $p \Rightarrow q$, and then prove $q \Rightarrow p$. • Prove p and q. • Prove (not p) and (not q).	• Statements p and q are interchangeable.	(p and (not q)) or ((not p) and q)
$(\exists x \in S)\ P(x)$	• Find an x in S for which $P(x)$ is true.	• Say "let x be an element of S such that $P(x)$ is true."	$(\forall x \in S)$ not $P(x)$
$(\forall x \in S)\ P(x)$	• Say "let x be any element of S." Prove that $P(x)$ is true.	• If $x \in S$, then $P(x)$ is true. • If $P(x)$ is false, then $x \notin S$.	$(\exists x \in S)$ not $P(x)$

https://math.berkeley.edu/~hutching/teach/proofs.pdf

Table 1: Logic in a nutshell.

Common Proof Techniques

- Proof by contradiction
- Proof by induction
- Proof by exhaustion (Enumerate all cases)