18.330: Introduction to Numerical Analysis

Problem Set #7: Numerical Solution to Differential Equations

due December 6^{th} @ 11:59 pm

Instructions: There are seven problems listed below, five are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. Please submit your problem set solutions via Canvas. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 Finite Differences for $q(x) < 0$ (C)

In the below boundary value problems, $q(x) < 0$, but unique solutions still exist (and are given below). Use the finite differences algorithm to approximate the solutions and compare your results to the exact solution for each (given below) at each grid point. For each problem, please plot the exact and numerically computed solution together, as well as their difference at each grid point.

$$(a) \quad \text{use } h = \frac{\pi}{20} \text{ to solve } \begin{cases} y'' + y = 0, \quad t \in [0, \pi/4] \\ y(0) = 1 \\ y(\frac{\pi}{4}) = 1 \end{cases},$$

exact solution: $y(t) = \cos t + (\sqrt{2} - 1)\sin t$.

$$(b) \quad \text{use } h = \frac{\pi}{20} \text{ to solve } \begin{cases} y'' + 4y = \cos t, \quad t \in [0, \pi/4] \\ y(0) = 0 \\ y(\frac{\pi}{4}) = 0 \end{cases},$$

exact solution: $y(t) = -\dfrac{1}{3}\cos 2t - \dfrac{1}{3\sqrt{2}}\sin 2t + \dfrac{1}{3}\cos t$.

$$(c) \quad \text{use } h = \frac{1}{20} \text{ to solve } \begin{cases} y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{2\ln t}{t^2}, & t \in [1,2] \\ y(1) = \frac{1}{2} \\ y(2) = \ln 2 \end{cases},$$

exact solution: $y(t) = \dfrac{4}{t} - \dfrac{2}{t^2} + \ln t - \dfrac{3}{2}$.

$$(d) \quad \text{use } h = \frac{1}{5} \text{ to solve } \begin{cases} y'' - 2y' + y = te^t - t, & t \in [0,2] \\ y(0) = 0 \\ y(2) = -4 \end{cases},$$

exact solution: $y(t) = \dfrac{1}{6}t^3e^t - \dfrac{5}{3}te^t + 2e^t - t - 2$.

2 Approximating Parabolic PDEs (C)

Approximate the solution to the following parabolic partial differential equations using the forward difference method, and compare your results to the exact solutions at time $t = \frac{2}{5}$.

$$(a) \quad \text{use } h = \frac{1}{5}, k = \frac{1}{25} \text{ to solve } \begin{cases} u_t - \frac{4}{\pi^2}u_{xx} = 0, & x \in (0,4),\ t > 0 \\ u(0,t) = u(4,t) = 0 & t > 0 \\ u(x,0) = \sin(\frac{\pi}{4}x)\big(1 + 2\cos(\frac{\pi}{4}x)\big) & x \in [0,4] \end{cases},$$

exact solution: $u(x,t) = e^{-t}\sin(\frac{\pi}{2}x) + e^{-t/4}\sin(\frac{\pi}{4}x)$.

$$(b) \quad \text{use } h = \frac{1}{10}, k = \frac{1}{25} \text{ to solve } \begin{cases} u_t - \frac{1}{\pi^2}u_{xx} = 0, & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = \cos\big(\pi(x - \frac{1}{2})\big) & x \in [0,1] \end{cases},$$

exact solution: $u(x,t) = e^{-t}\cos\big(\pi(x - \frac{1}{2})\big)$.

3 Picard Iteration (M)

Find the exact solution to the initial value problem $y' = -y^2$, $y(0) = 1$, and compare it to the approximations obtained by Picard iteration. Compute the third iterate y_3 of Picard iteration and compare the error $y - y_3$ to the theoretical estimates from class.

4 Failure of Euler's Method (M)

Show that Euler's method fails to approximate the solution $y(t) = (\frac{2}{3}t)^{3/2}$ of the initial value problem $y' = y^{1/3}$, $y(0) = 0$. Explain this failure.

5 An Improved Single-Step Method (M)

Prove that the single step method

$$y_{j+1} = y_j + hf\left(t_j + \frac{h}{2}, y_j + \frac{h}{2}f(t_j, y_j)\right)$$

has order two if f is twice continuously differentiable.

6 Making a Boundary Value Problem Homogenous (M)

Show that the boundary value problem

$$y'' = f(t, y, y'), \quad t \in [a, b]$$

with inhomogenous boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ can be equivalently transformed into a boundary value problem with homogenous boundary conditions.

7 An Infinite Boundary Condition (M)

Consider the non-linear boundary value problem

$$\begin{cases} y''' + \frac{1}{2}yy'' = 0, \quad t \in [0, \infty) \\ y(0) = y'(0) = 0 \\ y'(\infty) = 1. \end{cases}$$

Letting $y''(0) = \lambda$ and $z(t) = \lambda^{-1/3}y(\lambda^{-1/3}t)$ (assuming $\lambda > 0$), derive an initial value problem for z on $0 \le x < \infty$. How can the solution of the initial value problem for z be used to obtain the solution $y(t)$ of the given boundary value problem?