18.330: Introduction to Numerical Analysis

Problem Set #2: Direct & Iterative Methods for Solving Linear Systems

due October 12^{th} @ 9:30 am

Instructions: There are eight problems listed below, six are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. You may either submit your problem set solutions via Canvas, or hand them in at the beginning of class. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 LU Factorization of Tridiagonal Matrices (M)

An $n \times n$ matrix A is said to be *tridiagonal* if $A_{i,j} = 0$ for all $|i-j| > 1$. Prove that if

$$|A_{i,i}| \geq |A_{i,i-1}| + |A_{i,i+1}| \quad \text{and} \quad A_{i,i-1}A_{i,i+1} \neq 0 \quad \text{for } i = 2, ..., n-1,$$

and $|A_{1,1}| > |A_{1,2}| > 0$ and $|A_{n,n}| > |A_{n,n-1}| > 0$, then A is non-singular. What does the pattern of the non-zero entries of the LU factorization (without pivoting) of a tridiagonal matrix look like and how many multiplications/divisions are required to produce it?

2 Cholesky Factorization of a Special Matrix (M)

Produce an LU factorization of the $n \times n$ matrix A with entries $A_{i,j} = \binom{i+j}{j}$ for $i, j = 0, 1, ..., n-1$ (note: indexing starts at zero), and verify your solution (e.g., show that $A = LU$).

3 Hands on with $PA = LU$ Factorization (M)

(a) Produce a $PA = LU$ factorization (using partial pivoting) of:

$$(1) \quad \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \qquad (2) \quad \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{pmatrix} \qquad (3) \quad \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \end{pmatrix}$$

(b) Write down the 4×4 matrix P such that multiplying on the left by P causes the second and fourth rows of a matrix to be exchanged. What is the effect of multiplying on the right by P?

(c) Change four entries of the leftmost matrix to make the following equation valid:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

4 Growth Factor for Partial Pivoting (M)

Let A be a non-singular $n \times n$ matrix, and $PA = LU$ under partial pivoting (e.g., PA is a partially pivoted matrix). Prove that $|L|_\infty \leq 1$ and $|U|_\infty \leq 2^{n-1}|A|_\infty$, where $|X|_\infty = \max_{i,j} |X_{i,j}|$.

5 Concrete Error Analysis (C)

Consider the $n \times n$ matrix A with entries $A_{i,j} = 5/(i+2j-1)$. Let $x = (1, ..., 1)^T \in \mathbb{R}^n$, and $b = Ax$. For $n = 5, 10, 15$, use the backslash command "\" to compute x in double precision. Report the forward error $\|x - \hat{x}\|_\infty$ and condition number $\|A\|_\infty \|A^{-1}\|_\infty$. How does the error magnification (e.g., how much larger the forward error is than the unit roundoff u) compare to the condition number?

Note: if using Julia, you may need to use the LinearAlgebra package for some functions (e.g., perform the "using LinearAlgebra" command).

6 From Complex to Real (M)

Let $\Re(\cdot)$ and $\Im(\cdot)$ be the real and imaginary parts of a matrix or vector. Let $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$, and A, $\Re A$, and $\Im A$ all be non-singular. Show that the $n \times n$ complex linear equation $Ax = b$ is equivalent to the two $n \times n$ real linear equations

$$\begin{aligned} \left[(\Im A)^{-1}\Re A + (\Re A)^{-1}\Im A\right]\Re x &= (\Im A)^{-1}\Re b + (\Re A)^{-1}\Im b \\ \left[(\Im A)^{-1}\Re A + (\Re A)^{-1}\Im A\right]\Im x &= (\Im A)^{-1}\Im b - (\Re A)^{-1}\Re b. \end{aligned}$$

7 Convergence of the Gauss-Seidel Method (M)

Consider the Gauss-Seidel iterative method presented in class for solving $Ax = b$:

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1}).$$

Prove that Gauss-Seidel converges for any strictly (row) diagonally dominant matrix A.

8 Computing with Iterative Methods (C)

Consider the sparse linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ has entries $A_{i,i} = 3$, $i = 1, ..., n$, $A_{i,i+1} = A_{i+1,i} = -1$, $i = 1, ..., n-1$, and $A_{i,j} = 0$ otherwise, $x = (1, ..., 1)^T \in \mathbb{R}^n$, and $b = (2, 1, 1, ..., 1, 2)^T \in \mathbb{R}^n$. Use the Jacobi Method and Gauss-Seidel method to solve this system to six correct decimal places (forward error in infinity norm) for $n = 100, 1000, 10000$. Report the number of steps needed for each technique and the backward error (with respect to b). What do you notice about the results?

Student Feedback

Please let me know how you're finding the course and the first two problem sets. Did you feel that the first problem set was graded fairly? How is the pace of lecture? Please rate the difficulty and volume of the second problem set.