18.330: Introduction to Numerical Analysis

Problem Set #5: Function Approximation

due November 13^{th} @ 11:59 pm

Instructions: There are seven problems listed below, five are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. Please submit your problem set solutions via Canvas. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 Vandermonde Matrix (M)

Let $x_0, ..., x_n$ be $n+1$ distinct points. Show that the Vandermonde matrix $V \in \mathbb{R}^{(n+1)\times(n+1)}$ with $V_{ij} = x_i^j$, $i, j = 0, 1, ..., n$, has determinant

$$\det V = \prod_{0\leq i<j\leq n} (x_j - x_i).$$

Hint: The determinant of a matrix is unchanged by subtracting a multiple of one column from another column.

2 Rational Function Approximation (M)*

Given $2(n+1)$ distinct points $x_0, x_1, ..., x_n, z_0, z_1, ..., z_n$, and $n+1$ values $y_0, y_1, ..., y_n \in \mathbb{R}$, show that there exists a unique function of the form

$$u(x) = \sum_{k=0}^{n} \frac{a_k}{x - z_k}$$

such that $u(x_i) = y_i$ for $i = 0, 1, ..., n$.

3 Interpolation in the Plane (M)

Consider polynomial interpolation of functions $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ using six points on the boundary of an equilateral triangle. We consider two different point configurations (points displayed by red dots), the first with six points all equidistant from the vertices of the triangle, and the second with three points at the vertices and three points at the midpoints of the sides:

In each of these cases, do the six points exactly interpolate (e.g., interpolate with remainder zero) quadratic polynomials (e.g., polynomials of the form $a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$)? Please provide a proof of your conclusions.

4 Runge Phenomena (C)

Perform polynomial interpolation for $f(x) = 1/(1+25x^2)$ on $[-1,1]$ using 25 equally spaced points $-1 = x_1 < ... < x_{25} = 1$. Plot $f(x)$ and the interpolating polynomial, and report the largest error between $f(x)$ and the computed polynomial on $[-1,1]$. Perform the same interpolation, now using Chebyshev points. Plot the result and report the error.

5 Practice with Cubic Splines (C)

Find the equations and plot the natural cubic spline that interpolates the data points:

$$(a) \quad (0,3),\ (1,5),\ (2,4),\ (3,1) \qquad\qquad (b) \quad (-1,3),\ (0,5),\ (3,1),\ (4,1),\ (5,1).$$

6 Best Linear Approximation (M)

Find the linear least squares polynomial approximation on the interval $[-1,1]$ for the following functions:

$$(a) \quad x^2 - 2x + 3 \qquad (b) \quad x^3 \qquad (c) \quad \frac{1}{x+2} \qquad (d) \quad e^x \qquad (e) \quad \ln(x+2)$$

7 Approximating an Unbounded Function (M)

The function $f(t) = \ln(1/t)$ becomes infinite as $t \to 0$, but can still be approximated arbitrarily well on $[0,1]$ by polynomials in the least squares sense. Illustrate this by showing that

$$\min_{p \in \mathcal{P}_n} \left(\int_0^1 |f(t) - p(t)|^2 \, dt \right)^{1/2} = \frac{1}{n+1},$$

where $\mathcal{P}_n$ is the set of polynomials of degree at most n.

Hint: The following two facts may prove useful. Let $\pi_j(t)$ denote the j^{th} degree orthogonal polynomial on the interval $[0,1]$ with weight function $w = 1$, normalized so that $\pi_j(1) = 1$. Then

$$\int_0^1 \pi_j^2(t)\,dt = \frac{1}{2j+1} \quad \text{for } j \geq 0 \qquad \text{and} \qquad \int \pi_j(t)\ln(1/t)\,dt = \begin{cases} 1 & \text{if } j = 0 \\ \frac{(-1)^j}{j(j+1)} & \text{if } j > 0 \end{cases}.$$

Student Feedback

Please rate the difficulty and volume of the fifth problem set. Are there any additional topics you would especially like to see covered at the end of the course?