18.330: Introduction to Numerical Analysis

Problem Set #4: Eigenvalue Problems

due November 2^{nd} @ 9:30 am

Instructions: There are eight problems listed below, six are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. Please submit your problem set solutions via Canvas. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 Facts from Linear Algebra (M)

(a) For an $n \times n$ matrix A with eigenvalues $\lambda_1, ..., \lambda_n$, prove that $\det(A) = \prod_{i=1}^{n} \lambda_i$ and $\mathrm{trace}(A) = \sum_{i=1}^{n} \lambda_i$, where $\mathrm{trace}(A) := \sum_{i=1}^{n} A_{ii}$.

(b) Prove that if λ is an eigenvalue of a matrix A, then λ is also an eigenvalue of A^T.

(c) Is the spectral radius $\rho(\cdot)$ a matrix norm? Please provide rigorous justification for your conclusion.

2 Eigenvalue Perturbation (M)

Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$, B an $n \times n$ matrix, and λ an eigenvalue of $A + B$. Prove that

$$\min_{i=1,\ldots,n} |\lambda - \lambda_i| \leq \|C\|_p \|C^{-1}\|_p \|B\|_p,$$

where C is a non-singular matrix such that $C^{-1}AC$ is diagonal, $p = 1, 2, \infty$, and $\|\cdot\|_p$ is the natural matrix norm induced by the vector p-norm.

3 Gershgorin Discs and Diagonalization (M)

For each of the following matrices show whether the matrix is diagonalizable, and if so, produce D and S with $A = SDS^{-1}$. Furthermore, in each case compare the spectral radius $\rho(A)$ with the best estimate for $\rho(A)$ produced by the Gershgorin Disc Theorem.

$$(a) \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{pmatrix}.$$

4 Newton's Method (M)*

Consider Newton's method applied to the eigenvalue problem $Ax = \lambda x$, $x^T x = 1$. Produce an explicit formula for the iterates $\{(\lambda_k, x_k)\}_{k=1}^{\infty}$ of Newton's method of the form

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = F\left[\begin{pmatrix} x_k \\ \lambda_k \end{pmatrix}\right],$$

for an explicit function F depending only on x_k, λ_k and $(A - \lambda_k I)^{-1}$.

5 Power Method and Aitkin's Δ^2 Acceleration (C)

Build a random orthogonal matrix $Q \in \mathbb{R}^{1000\times 1000}$ by generating a matrix B with independent standard normal entries (e.g., use the randn function) and computing a QR factorization $B = QR$ of it (using the qr command). Let $D \in \mathbb{R}^{1000\times 1000}$ be a diagonal matrix with entries 1 to 1000 on the diagonal, and set $A = QDQ^T$. Perform 50 iterations of the power method, reporting the Rayleigh quotient $x^T Ax/x^T x$ at each step. Apply Aitkin's Δ^2 method to the sequence of Rayleigh quotients and report the results. How does Aitkin's approximation to the spectral radius of A compare to that of the power method?

6 Householder's Method (M)

Use Householder's method to place the following matrices in tridiagonal form:

$$(a) \quad \begin{pmatrix} 10 & -4 & -3 \\ -4 & -3 & -4 \\ -3 & -4 & 5 \end{pmatrix} \qquad\qquad (b) \quad \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}$$

7 QR Applied to Symmetric Tridiagonal Matrix (M)

Let $T \in \mathbb{R}^{n\times n}$ be a symmetric tridiagonal matrix. Prove that if $T = QR$, where Q is orthogonal and R is upper triangular, then $R_{ij} = 0$ for $j > i + 2$ and RQ is a symmetric tridiagonal matrix.

8 Unshifted QR Algorithm in Practice (C)

Consider the $n \times n$ tridiagonal matrix T with $T_{ii} = 2$ for $i = 1, ..., n$ and $T_{i,i+1} = T_{i+1,i} = -1$ for $i = 1, ..., n - 1$. Perform 100 iterations of the unshifted QR algorithm for $n = 5, 50, 500$. Report the largest magnitude off-diagonal entry at the end in each case. To what extent can you estimate the eigenvalues in each case?

Student Feedback

Please rate the difficulty and volume of the fourth problem set.