18.330: Introduction to Numerical Analysis

Problem Set #6: Numerical Differentiation & Integration

due November 27^{th} @ 11:59 pm

Instructions: There are seven problems listed below, five are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. Please submit your problem set solutions via Canvas. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 Numerical Differentiation: Theory vs Practice (C)

Make a table of the error of the three-point centered-difference formula for $f'(0)$, where $f(x) = \sin x - \cos x$, with $h = 10^{-i}$, $i = 1, ..., 12$. Draw a plot of the results. Does the minimum error correspond to theoretical expectations in exact arithmetic, and, if not, why is this the case?

2 Building Higher Order Formulae (M)

Derive an $O(h^4)$ five point formula to approximate $f'(x_0)$ that uses $f(x_0 - h)$, $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, and $f(x_0 + 3h)$.

3 Quadrature with Interpolating Polynomials (M)

Show that there exists no polynomial interpolatory quadrature using n points that integrates polynomials of degree $2n$ exactly, i.e., for any fixed interval $[a, b]$ there does not exist $\{(\alpha_i, x_i)\}_{i=1}^n$ such that $\int_a^b P(x)dx = \sum_{i=1}^n \alpha_i P(x_i)$ for all polynomials P of degree at most $2n$.

4 Error of Newton-Cotes Integration (M)*

For the remainder E_n of the Newton-Cotes formula of order n in the interval $[-1, 1]$ applied to the Chebyshev polynomial $T_{n+1}(x)$, show that

$$E_n(T_{n+1}) = \frac{(n+1)!4^{n+1}}{n^{n+2}} \int_0^n \binom{z}{n+1} dz$$

for all $n \in \mathbb{N}$. From this conclude that if n is odd, then $|E_n(T_{n+1})| \geq \gamma_n$, where

$$\gamma_n = \frac{(n-1)!4^{n+1}}{3n^{n+2}} \to \infty \text{ as } n \to \infty.$$

5 Variants of Gauss Quadrature (M)

(a) Find the quadrature weights a_0, a_1, a_2, a_3 and the remaining quadrature points x_1, x_2 of a quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(1)$$

that is exact for all polynomials of degree at most five.

(b) Find the quadrature weights a_0, a_1, a_2 and the remaining quadrature points x_1, x_2 of a quadrature formula of the form

$$\int_{-1}^{1} f(x)\,dx \approx a_0 f(-1) + a_1 f(x_1) + a_2 f(x_2)$$

that is exact for all polynomials of degree at most four.

6 Comparing Composite Rules (C)

Determine the value of n required to approximate

$$(a) \quad \int_0^2 e^{2x} \sin 3x\,dx \quad \text{to within } 10^{-4} \qquad\qquad (b) \quad \int_1^2 x \ln x\,dx \quad \text{to within } 10^{-5}$$

using the composite trapezoid rule, composite Simpson's rule, and composite midpoint rule.

7 Quadrature with Equal Weights (M)

For each $n = 1, 2, 3$, produce a quadrature rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

with equal weights that integrates polynomials of degree at most n exactly.

Student Feedback

Please rate the difficulty and volume of the sixth problem set.