18.330: Introduction to Numerical Analysis

Problem Set #3: Solving Non-Linear Equations

due October 24^{th} @ 9:30 am

Instructions: There are eight problems listed below, six are theoretical and two are computational, denoted by an (M) and (C) respectively. You may complete as many problems as you want, but please **submit only four problems** for grading. Particularly challenging problems have been marked with a star next to them. For computational problems, please include a copy of your source code and output. You may collaborate with classmates and reference outside material, but you must write your own solutions and note your collaborators and sources. Please submit your problem set solutions via Canvas. **Late submissions will not be accepted** – to be granted an exception (due to illness or extenuating circumstance) please contact Student Support Services and have them contact me before the assignment deadline.

1 Infinitely Many Square Roots (M)

Prove that $\lim_{n\to\infty} \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n \text{ square roots}} = 2$.

2 Fixed Point Iteration (M)

The function $f(x) = x^4 + 2x^2 - x - 3$ has exactly one positive real root p. Consider the functions

$$(a) \quad g_1(x) = (3 + x - 2x^2)^{1/4} \qquad (b) \quad g_2(x) = \sqrt{\frac{x+3-x^4}{2}}$$

$$(c) \quad g_3(x) = \sqrt{\frac{x+3}{x^2+2}} \qquad (d) \quad g_4(x) = \frac{3x^4+2x^2+3}{4x^3+4x-1}.$$

Show that each function $g_i(x)$ has a fixed point at p, perform four fixed point iterations, if possible, with $x_0 = 1$, and report $|x_4 - p|$ in each case. Which function gives the best approximation to the solution, and what method was used to produce it?

3 Convergence Rates (M)

The following sequences all converge to zero as $n \to \infty$:

$$v_n = 10^{-n}, \qquad x_n = 3^{-n} n^{10}, \qquad y_n = 10^{-3\cdot 2^n}.$$

For each, please show whether the order of convergence is linear ($\alpha = 1$, $0 < \lambda < 1$) or quadratic ($\alpha = 2$). The sequence $z_n = e^{-e^n}$ also converges to zero as $n \to \infty$. What is the exact order α of convergence?

4 Dividing Without Division (M)

Create a fixed point iteration $g(x) = a_3x^3 + a_2x^2 + a_1x$ for computing $1/\alpha$ that converges cubically, where a_1, a_2, a_3 are constants depending only on α and involve only addition, subtraction, and multiplication. Prove that your choice of $g(x)$ is a cubically convergent fixed point iteration (e.g., $\alpha = 3$) and characterize exactly the region of convergence (i.e., which initial guesses converge to $1/\alpha$).

5 Inverting a Matrix (M)

Show that, for a non-singular $n \times n$ matrix A, the iteration $A_{n+1} = A_n(2I - AA_n)$, $n = 0, 1, ...,$ converges quadratically to the inverse A^{-1} if $\|I - AA_0\| < 1$, for any sub-multiplicative matrix norm $\|\cdot\|$.

6 Testing Out Our Methods (C)

Consider the equation $\frac{1}{2}x - \sin x = 0$. Show that there is only one positive root and that is in the interval $[\pi/2, \pi]$. Compute the root to 7 and 15 decimal places using (a) bisection method on the interval $[\pi/2, \pi]$ and (b) Newton's method for the starting value $x_0 = 3\pi/4$. Report the number of iterations required in each case.

7 Aitkin's Δ^2 Method (M)

Given a sequence $\{x_n\}_{n=0}^{\infty}$, $x_n \neq p$ for all n, for which $\lim_{n\to\infty}(x_{n+1} - p)/(x_n - p) = \lambda$, $|\lambda| < 1$, prove that the quantity

$$\hat{x}_n := x_n - \frac{(\Delta x_n)^2}{\Delta^2 x_n},$$

where $\Delta x_n := x_{n+1} - x_n$ and $\Delta^2 x_n := x_n - 2x_{n+1} + x_{n+2}$, is well-defined for n sufficiently large and that

$$\lim_{n\to\infty} \frac{\hat{x}_n - p}{x_n - p} = 0.$$

8 Evaluating Expanded Polynomials (C)

(a) Compute the expanded form $p(x) = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ of the polynomial $(x-1)^5$ for 400 equally spaced points between .998 and 1.002 in double precision, and plot your computed values divided by the unit roundoff. What do you observe? By computing $p(x)$ in expanded form, how accurately can you hope to estimate the root $p = 1$?

(b) Compute the expanded form $p(x) = x^5 - 100x^4 + 3995x^3 - 79700x^2 + 794004x - 3160080$ of the polynomial $(x-18)(x-19)(x-20)(x-21)(x-22)$ for 400 equally spaced points between 21.8 and 22.2 in single precision, and plot your computed values divided by the unit roundoff. What do you observe? By computing $p(x)$ in expanded form, how accurately can you hope to estimate the root $p = 22$ in single precision?

Student Feedback

Please rate the difficulty and volume of the third problem set.