The Planar/Hyper-Planar Rotated Polar Coordinate System and Its Mathematical Solid Vector Addition, Multiplication, Division, Dot Product and Cross Product Operations

Jalal Al-Anssari
Department of Electrical Engineering and Computer Science
University of Cincinnati, Cincinnati, Ohio, United States
University of Baghdad, Baghdad, Iraq
Email: raoufjb@mail.uc.edu
Email: jalal_bassim@hotmail.com

Inam Naser
Department of Electrical Engineering and Computer Science
University of Cincinnati, Cincinnati, Ohio, United States
University of Technology, Baghdad, Iraq
Email: naseris@mail.uc.edu
Email: inam_naser@yahoo.com

Anca Ralescu
Department of Electrical Engineering and Computer Science
University of Cincinnati, Cincinnati, Ohio, United States
Email: ralescal@ucmail.uc.edu
Email: ancaralescu@gmail.com

***Abstract*—In geometry, the curvilinear coordinate systems have curved coordinate lines. They are used to localize physical or virtual quantities in image processing computer science, space science, earth science, cartography, quantum mechanics, relativity, and other engineering sciences. Their Mathematical operations are useful for defining the motion of these quantities and objects under the influence of central forces, and solving equation of curved boundary conditions. The typical ones are the correct 2D Polar mathematical operations; and the wrong 3D Spherical mathematical operations. The problem statement is that there is no available definition of curvilinear one that is 3D and correct, except the Solid Vector Subtraction operation which was recently proposed and used to develop 3D spatial filters of field of vectors for geometrical edge detection in 3D image processing science. In this research, this Solid Vector Subtraction is extended by proposing its 3D and correct definitions of: (1) Rotated Polar coordinate system; (2) 2D, 3D, and hyper dimensional space; and (3) Other whole complementary set of Solid Vector Addition, Multiplication, Division, Dot Product, and Cross Product operations. Because their justification has a long derivation, it is for future work. These Solid Vector operations are 3D and correct.**

***Keywords*-Solid Vector; 3d Gradient; 3d Laplacian; 3d Unsharp; 3d Gradient-Based Laplacian; Solid Vector Subtraction; Solid Vector Addition; Solid Vector Multiplication; Solid Vector Division; Solid Vector Dot Product; Solid Vector Cross Product; Rotated Polar coordinate system; Jalal coordinate system**

I. Introduction

In geometry, the curvilinear coordinate systems such as the Polar and Spherical have curved coordinate lines. They are used to localize physical or virtual quantities (e.g. scalars, vectors, or tensors) or objects (e.g. space ship, satellite, or CAD model) in image processing computer science, space science, earth science, cartography, quantum mechanics, relativity, and other engineering sciences. Mathematical operations of curvilinear coordinate system are more useful than the Cartesian mathematical operations for some application such as defining the motion of these quantities and objects under the influence of central forces, and solving equation of curved boundary conditions. In space science, as was mentioned in [1], they show significant superior accuracy for relative motion in circular reference orbit applications–such as: (1) formation flying, (2) spacecraft rendezvous, (3) terminal guidance, (4) space debris evolution, and (5) collision avoidance.

In image processing computer science, mathematical operations of curvilinear coordinate systems that are applicable to three and hyper-dimensional space are of special interest because they act as a foundation for a wide range of 3D image processing and computer vision applications–such as: (1) developing 3D and hyper dimensional artificial intelligence and machine learning descriptors and algorithms such as geometrical edge magnitude and direction, (2) detecting imaging similarities and dissimilarities, (3) controlling of 3D objects rotations in the 3D space–such as drones, or CAD virtual reality objects, (4) sculpturing 3D objects using CAD applications (such as personalizing biomedical equipment like artificial bones using Scada machines), (5) developing indoor or outdoor navigation using vision-based GPS system such as 3D SLAM, (6) developing vision-based obstacles avoiding applications (such as for UAVs, and elderly falling alerting system). The curvilinear Solid Vector Subtraction operation was recently proposed and implemented by Al-Anssar J. Naser I. and Ralescu A. [2], [3] and also implemented by Naser I. Al-Anssar J. and Ralescu A. [4], [5] in order to define the 3D Gradient, Laplacian, Unsharp, and Gradient-Based Laplacian spatial filters of a field of vectors geometrical edge detectors in 3D images (point clouds).

They have called it Solid vector because the vector direction is unpartitioned and its length is unpartitioned as well. Therefore, this Solid vector has only two components which are: (1) only one directional (angular) (θ) component; and (2) only one magnitude (length) (r) component.

Accordingly, the typical curvilinear Polar vector is a Solid Vector because it has only one angular component and only one length component; while the typical curvilinear Spherical vector is not a Solid Vector because it has two angular components (θ and ϕ).

This Polar coordinate system, is hereafter refereed as the Fixed Polar coordinate system because its y and x axes lay on a fixed two-dimensional still plane, the x, y plane that never moves or rotates. Where its y axis is defined as its directional Zero vector, $PoleY$, which is the lowest starting point of its θ angular component.

Noting that the justification is out of the scope of this research due to its long derivation and it will be left for future work, while the 2D Polar is correct and 3D Spherical is wrong; the **problem statement** is that there is necessity of 3D and correct curvilinear mathematical operations. Although, the 3D and correct Solid Vector Subtraction operation was recently proposed in [2]; however it still lack definitions of its complementary set of mathematical operations and coordinate system.

The Solid Vector subtraction can be used to develop robust 3D SLAM and rotation control of objects because they are used to propose 3D spatial filters that detects geometrical edge magnitude and direction that is salient geometrical feature that can be used to detect exact correspondences between consecutive video frames which can be used to exactly measure the rotation and translation in 3D SLAM algorithm.

The drawback of the problem of the unavailability of 3D and correct curvilinear mathematical operations is that the state of the art 3D SLAM fails after only a short time of running. Because when they were unavailable, there applications such as 3D spatial filters were unavailable and hence geometrical edge magnitude and directions salient features were unavailable. Therefore, the state of the art 3D SLAM algorithms used random and approximate geometrical features by picking up three random pixels in each 3D video frame and using approximation methods to find their correspondences in its consecutive 3D video frame in order to compute the rotation and translation that happened between them. Randomness and approximations caused drifting errors which caused their fast failure.

The **objective** is to extend this Solid Vector Subtraction and propose 3D and correct definitions of its: (1) coordinate system; (2) hyper dimensional space; and (3) other whole complementary set of Solid Vector Addition, Multiplication, Division, Dot Product, and Cross Product operations.

This proposed curvilinear Solid Vector coordinate system, is hereafter refereed as the Rotated Polar coordinate system (Jalal coordinate system) because it is a rotated version of the Fixed Polar coordinate system in the 2D, 3D and hyper dimensional space. And its mathematical operations, hereafter refereed as Solid Vector mathematical operations.

The **methodologies** are:

1) In order to create the Rotated Polar coordinate system, which is a rotated version of the Fixed Polar, that is implemented in the 3D space on the contrary of being fixed to the still x, y plane: a) we make its first axis, the Directional Zero vector, that replaces the $PoleY$ of the Fixed Polar, and that is the lowest starting point of the angular component θ as one of the operands vectors of the Solid Vector Subtraction operation and b) we make its second axis that replaces the x axis of the Fixed Polar as the Directional Sign Axis of the Solid Vector Subtraction operation
2) In order to extend its definition to the hyper dimensional space, we specify a pair of vectors as components of a Hyper Rotated Polar when the Solid Vector Subtraction operations between them and their unified Directional Zero vector produce a unified Directional Sign Axis (including both of its positive and negative directions)
3) In order to propose the whole complementary set of the Solid Vector: 1) Addition, 2) Multiplication, 3) Division, 4) Dot Product and 5) Cross Product operations., we use the triangulation operations

From this point on, this chapter is organized as follows: section III, the literature review; section IV, the rotated polar coordinate system; section V, the rotated polar mathematical operations; section VI, the conclusions.

II. Contributions

We propose novel definitions of:

1) a Rotated Polar coordinate system based on two novel axis: 1) first axis is the dynamic Directional Pole Zero Vector and 2) second axis is the Directional Sign Axis of the Solid Vector Subtraction operation
2) a 2D, 3D, and hyper dimensional Rotated Polar coordinate system space
3) a Solid Vector Addition
4) a Solid Vector Multiplication
5) a Solid Vector Division
6) a Solid Vector Dot Product
7) a Solid Vector Cross Product

III. The Literature Review

This section presents the typical three coordinate systems and their mathematical operations that are: (1) the Cartesian; (2) the 2D Polar; (3) the 3D Spherical; (4) the Cartesian Magnitude Dot Product operation; (5) the Cartesian Magnitude Cross Product operation; and (6) the Solid Vector Subtraction operation [6]. The Hyper-Plane concept is also presented at the end of this section.

A. The Cartesian Coordinate System and Its Component-Wise Cartesian Mathematical Vector Operations

1) The Cartesian Coordinate System: The Cartesian coordinate system is a three-dimensional system. Its unit vectors point to the direction of the increasing coordinates. A point P can be represented as a three numbers $P(x, y, z)$, where x, y, z are the coordinate of P. These coordinates range from

$-\infty$ to $+\infty$. The Cartesian vector can be represented by three components $V(V_1, V_2, V_3)$ and can be decomposed into a sum of its components [7]. Some other papers about the Cartesian coordinate system can also be found in [8]–[12].

2) The Component-Wise Cartesian Mathematical Vector Operations: the Component-Wise Cartesian Mathematical Vector Operations are implemented in the Cartesian coordinate system and are component-wise operations because they handle the operands and resulting vectors as a separated layers of Cartesian axes (e.g. $x, y, z \ldots$) [13].

As Larson R. *et al.*presented in their book [14], let $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2, v_3, \ldots, v_n)$ be vectors in R^n and let c be a real number. Then the sum of u and v is defined as the vector

$$u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \ldots, u_n + v_n), \quad (1)$$

So on for the Subtraction.

While the scalar multiple of u by c is defined as the vector

$$cu = (cu_1, cu_2, cu_3, \ldots, cu_n), \quad (2)$$

So on for the scalar division.

3) The Cartesian Magnitude Dot Product: The Dot product operation (scalar product, or inner product), hereafter refereed as Magnitude Dot Product, is a linearity measure of the two vectors.

The Magnitude Dot Product between $V1$ and $V2$ is defined by equation 3.

$$V_1.V_2 =< V_1, V_2 >= ||V1||||V2|| \cos(\theta) \quad (3)$$

Its result is a scalar quantity. When it is applied on two perpendicular vectors, their result is zero. The order of its vector operands in its multiplication does not matter [7].

4) The Cartesian Magnitude Cross Product: The Cross product (or Vector product), here after refereed as the Magnitude Cross Product, produces the resulting vector that is perpendicular on the vector operands and its length is the positive area of the parallelogram having the two vector operands as sides.

The Magnitude Cross product operation between V_1 and V_2 results in a vector VC which has a magnitude and direction, and is defined by equation 4.

$$VC = (V_1 \times V_2)_{Magnitude\,Cross\,product} \quad (4)$$

The magnitude of the produced vector VC is defined by equation 5.

$$VC_{Magnitude\,Cross\,Product} = ||V1||\,||V2|| \sin(\theta) \quad (5)$$

The direction of the produced vector VC is computed using the right hand rule. When the Magnitude Cross Product is applied on two parallel vectors, the result is zero vector, and the order of the vector operands in the multiplication does matter [7].

B. The 2D Polar Coordinate System and Its Mathematical Vector Operations

1) The 2D Fixed Polar Coordinate System: It is a two-dimensional system. Its unit vectors denoted as a_θ and a_r are not drawn at the origin but at a convenient point in 2d plane. They point to the direction of the increasing coordinates variables and are orthogonal to each other. A point P can be represented as a two numbers $P(\theta, r)$. Where θ, r are the coordinates of P. The θ coordinate range from 0 to 2π. The r coordinate ranges from $zero$ to $+\infty$. The Polar vector can be represented by two components $A(A_\theta, A_r)$, where A_θ and A_r are called the components of A. A Polar vector can be decomposed into a sum of its components $A = A_\theta + A_r$ [15]. Some other research that tackled applications for the Polar coordinate system can be found in [16].

2) The 2D Fixed Polar Mathematical Vector Operations: The features of the two-dimensional Fixed Polar mathematical vector operations are specified as:

1) They are implemented in the Polar coordinate system
2) They handle the vector as only two separated layers: (1) the first consists of only one angular layer (e.g. θ) that starts from Pole Y; and (2) the second consists of only one magnitude layer (e.g. r)
3) Their Polar Addition and Subtraction operations are defined as that: let $u = (u_\theta, u_r)$ and $v = (v_\theta, v_r)$ be vectors in R^2 and let c be a real number. Then the sum of u and v is defined as the vector

$$u + v = (u_\theta + v_\theta, u_r + v_r) \quad (6)$$

So on for the Subtraction.

While the Polar scalar multiplication and division of u by (c_θ, c_r) is defined as the vector

$$(c_\theta, c_r)u = (c_\theta u_\theta, c_r u_r) \quad (7)$$

So on for the scalar division.

4) Their Dot product operation is defined as that: if A and B are Polar vectors represented in terms of components $A(A_\theta, A_r)$ and $B(B_\theta, B_r)$, their Scalar product is:

$$A.B = A_\theta B_\theta + A_r B_r \quad (8)$$

5) The Polar vector magnitude is

$$|A| = \sqrt{A_\theta^2 + A_r^2} \quad (9)$$

C. The 3D Spherical Coordinate System and Its Mathematical Vector Operations

1) The Three-Dimensional Spherical Coordinate System: It is a three-dimensional system. Its unit vectors denoted as a_θ, a_ϕ and a_r are not drawn at the origin but at a convenient point in 3d space. They point to the direction of the increasing coordinates variables and are orthogonal to each other. A point P can be represented as a three numbers $P(\theta, \phi, r)$, where θ, ϕ, r are the coordinates of P. The θ coordinate range from 0 to π. The ϕ coordinate range from 0 to 2π. The r coordinate ranges from $zero$ to $+\infty$. The Spherical vector can be represented by three components $A(A_\theta, A_\phi, A_r)$, where A_θ, A_ϕ and A_r are called the component of A. A Spherical vector can be decomposed into a sum of its components $A = A_\theta + A_\phi + A_r$ [7]. In their research [1], Chao H. *et al.*established a linear model with spherical coordinates for relative motion in an elliptical reference orbit. Wang J. *et al.*in [17] proposed an effective method to detect the recompression in the color images by using the conversion error, rounding error, and truncation error on the pixel in the spherical coordinate system.

2) The 3D Spherical Mathematical Vector Operations: The features of the 3d Spherical mathematical vector operations are:

1) They are implemented in the Spherical coordinate system.
2) They handle the vector as three separated layers: (1) the first consists of one angular layer (e.g. θ) that starts from Pole Y; (2) the second consists of one angular layer (e.g. ϕ) that starts from Pole X; and (3) the third consists of one magnitude layer (e.g. r).
3) Their Spherical Addition and Subtraction operations are defined such that: let $u = (u_\theta, u_\phi, u_r)$ and $v = (v_\theta, v_\phi, v_r)$ be vectors in R^3 and let (c_θ, c_ϕ, c_r) be a real multipliers numbers. Then the sum of u and v is defined as the vector

$$u + v = (u_\theta + v_\theta, u_\phi + v_\phi, u_r + v_r) \tag{10}$$

So on for the Subtraction.
While the Spherical scalar multiplication and division of u by (c_θ, c_ϕ, c_r) is defined as the vector

$$(c_\theta, c_r)u = (c_\theta u_\theta, c_\phi u_\theta, c_r u_r) \tag{11}$$

So on for the scalar division.
4) Their Dot product operation is defined in [7] as that if A and B are Spherical vectors represented in terms of components $A(A_\theta, A_\phi, A_r)$ and $B(B_\theta, B_\phi, B_r)$, their Scalar product is:

$$A.B = A_\theta B_\theta + A_\phi B_\phi + A_r B_r \tag{12}$$

5) Their Cross product operation is defined in [7] as that if A and B are Spherical vectors represented in terms of components $A(A_\theta, A_\phi, A_r)$ and $B(B_\theta, A_\phi, B_r)$, their Cross product can be obtained by evaluating the following determinant:

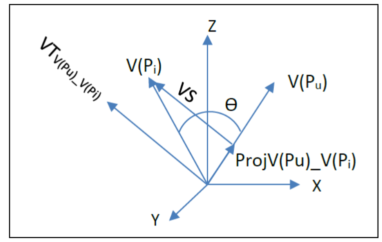

Fig. 1. Cited from [2], this figure shows the Solid Vector Subtraction operation.

$$A \times B = \begin{vmatrix} a_\theta & a_\phi & a_r \\ A_\theta & A_\phi & A_r \\ B_\theta & B_\phi & B_r \end{vmatrix} \tag{13}$$

6) The Spherical vector magnitude is

$$|A| = \sqrt{A_\theta^2 + A_\phi^2 + A_r^2} \tag{14}$$

D. The Mathematical Solid Vector Subtraction Operation

In the Solid Vector Subtraction operation, that was proposed by Al-Anssari J., Naser I. and Ralescu A in [2], and is illustrated in figure 1, the given inputs are the two vector operands $V(P_u)$ and $V(P_i)$, where the first vector operand $V(P_u)$ is made as the Directional Pole Zero Axis, and the produced outputs are: (1) the directional Solid Vector difference vector VT that consists of: (a) the angular displacement θ that is in the direction of (b) the Directional Sign Axis VS, where the Directional Sign Axis is the vector that is produced from the Cartesian vector subtraction of the second vector operand $V(P_i)$ from the projection of $V(P_i)$ on the Directional Zero Axis $V(P_u)$; and (2) the length difference M.

E. The Hyper-Plane

A hyper-plane, in geometry, is a subspace that its dimension is one less than that of its ambient space. For a 3d space, its hyper-planes are the 2d planes. For a 2d space, its hyper-planes are the 1d lines. Hyper-planes are useful for applications related to machine learning especially to create support vector machines for such tasks as computer vision and natural language processing [18], [19].

IV. The Rotated Polar Coordinate System

In this section, the Rotated Polar coordinate system is proposed which is a rotated version of the Fixed Polar coordinate system from fixed two-dimensional x, y plane to a freely rotating plane/ hyper-plane in the Cartesian space around the origin point Null vector, where all of its vectors are confined in this same Plane/ Hyper-Plane.

The Rotated Polar coordinate system that is shown in figure 2, is very convenient whenever problems that having Spherical symmetry are being dealt with (e.g. edge detection

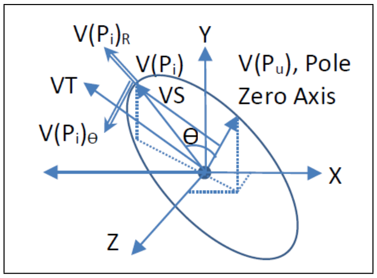

Fig. 2. This figure shows the Rotated Polar coordinate system.

in point clouds surfaces that is based on the Solid Vector Subtraction operation). More specifically, it is used to implement this Solid Vector Subtraction and its complementary set of the Solid Vector mathematical operations presented in section V; while its corresponding Cartesian coordinate system is used to localize its vectors in the space. In the following subsections, its center, polar axes, its plane/ hyper-plane and its unit vectors are defined.

A. The Rotated Polar Coordinate System Center

The center of the Rotated Polar coordinate system falls on the $NullVector$ origin point of its Cartesian coordinate system.

B. The Rotated Polar Coordinate System Axes

The Rotated Polar coordinate system has only two axes that are: (1) The Directional Pole Zero Axis, and (2) the Directional Sign Axis, as described below.

1) The Directional Pole Zero Axis: It is the first axis of the Rotated Polar coordinate system denoted by V_{zero} and has the following properties:

1) It is operation-based that is specific for its Solid Vector mathematical operation, dynamically change with it, and can be different from the others.
2) It does not have a fixed orientation that fall on the Cartesian axes, but it freely rotates around the origin point Null Vector and can be pointing anywhere in the 3-d/ hyper-dimensional space.
3) Has a direction that is set up to describe the “no quantity”, “lowest starting”, “empty” direction with respect to the other vectors operands, that are involved in the Solid Vector mathematical operation, and can be one of them (e.g. in the Solid Vector Subtraction, the subtrahend is the Directional Pole Zero Axis; and as will be proposed later in the current research, the addend is the Directional Pole Zero Axes of the Solid Vector Addition).

The above properties of the first axis of the Rotated Polar enable the Solid Vector mathematical operations to comply with the Solid Vector term definition (V.1).

They are on the contrary to the following properties of the first axis of Spherical and Fixed Polar coordinate systems (Pole Y) that make the Spherical and Fixed Polar not compliant to the Solid Vector definition:

1) It is coordinate system-based that is static (universal), the same for all its Spherical and Fixed Polar mathematical operations.
2) It has a fixed orientation that falls on the Cartesian y axes, Pole Y.
3) Its Cartesian y axis direction describes the “no quantity”, “lowest starting”, “empty” direction with respect to the other vectors operands, that are involved in the Spherical and Fixed Polar mathematical operations.

2) The Directional Sign Axis: Denoted by VS, it is the second axis of the Rotated Polar coordinate system which is produced by the Directional Solid Vector Subtraction operation between the Directional Pole Zero Axis and the other vector operand that is involved in the mathematical Solid Vector Subtraction operation. It is perpendicular to the Directional Pole Zero Axis and describes the directional sign of the orientation divergence between their Zero Vector and operand vectors.

C. The Rotated Polar Coordinate System Plane/ Hyper-Plane

The Rotated Polar plane/ hyper-plane is defined by the two axes; the Directional Pole Zero Axis and the Directional Sign Axis of its Rotated Polar coordinate system, and according with their rotation, it freely rotates around its corresponding Cartesian coordinate system origin point Null Vector. It also contains all of the other Rotated Polar coordinate system vectors.

The definition IV.1 of the theory of vector spaces of the Rotated Polar Coordinate System 2D, 3D Plane and Hyper-Plane of a set of vectors is proposed here, that is:

Definition IV.1. A set of vectors is said to be planar/ hyper-planar if a plane/ hyper-plane in the space contains the origin point Null Vector, the Directional Pole Zero vector, and all the other vectors in this set with at least one of them non parallel with the Directional Pole Zero Vector; if no plane/ hyper-plane in the space can be written in this way, then the vectors are said to be non-planar/ hyper-planar.

Also, its mathematical definition IV.2 is proposed, that is:

Definition IV.2. A set of vectors is said to be planar/ hyper-planar if the Solid Vector subtraction operations between each of its vectors and their shared same Directional Pole Zero Vector produces the same Directional Sign Axis (including both of its positive and negative signs). If any of the Solid Vector subtraction operations between any one of its vectors and their shared same Directional Pole Zero Vector produces a different Directional Sign Axis (including both of its positive and negative signs), the set of vectors is said to be non-planar/ hyper-planar.

D. The Rotated Polar Coordinate System Unit Vectors

In the Rotated Polar coordinate system, where $V(P_u)$ is the Directional Pole Zero Axis, and $V(P_i)$ is a vector in this coordinate system, VS is the Directional Sign Axis that is produced from the Solid Vector Subtraction operation between $V(P_u)$ and $V(P_i)$. Unit vectors of $V(P_i)$ in this system, denoted $V(P_i)_R$, $V(P_i)_\theta$ are usually not drawn at the origin point Null Vector, but at a convenient point in space, that is the end point of $V(P_i)$. They are orthogonal to each other. $V(P_i)_\theta$ points in the direction of the increasing Absolute Solid Vector Directional Difference between $V(P_i)$ and the Directional Pole Zero axis $V(P_u)$ on the plane/ hyper-plane that contains $V(P_u)$ and VS and on the side of the Directional Sign Axis VS. $V(P_i)_R$ points in the direction of the increasing vector length.

$$V(P_i) : (R, \theta@VS) \tag{15}$$

Where R, $\theta@VS$ are called $V(P_i)$ Rotated Polar vector components; θ is the angular displacement, @ means "belongs to", and VS is the Directional Sign Axis that θ belongs to.

The ranges of these components variables are

$$0 \leq V(P_i)_R < \infty \tag{16}$$

$$0 \leq V(P_i)_\theta \leq 1 \tag{17}$$

Where the θ Zero number starts from Directional Pole Zero vector $V(P_u)$, measured by the radian angle unit.

V. The Rotated Polar Mathematical Operations

In this section, the novel Solid Vector mathematical Addition, Multiplication, Division, Dot Product and Cross Product operations are proposed that are a whole complementary set to the Solid Vector Subtraction operation that Al-Anssari J. proposed in [2]. They are called Solid Vector because they handle the vector as a one solid quantity unit where they handle the only two Rotated Polar vector components $\theta@VS$ and R, where the Zero reference for their $\theta@VS$ directional polar component is the Directional Pole Zero Axis V_{Zero}, that is dynamic specific for each operation and change with it.

The Solid Vector term is defined as:

Definition V.1. The vector that has only two polar components that are: (1) only one directional $\theta@VS$ component; and (2) only one length r component.

Note that, the symbol VT refer to $\theta@VS$, as equations 18 shows, therefore, sometimes the $\theta@VS$ symbol is used instead of VT or vice versa. So on for M and R, as equation 19 shows.

$$VT = \theta@VS \tag{18}$$

$$M = R \tag{19}$$

Their corresponding Cartesian vector components are used to localize their vectors in the space. The given inputs and produced output of each of the Solid Vector operation are listed as follows:

A. Solid Vector Addition Operation

In the Solid Vector Addition operation, that is illustrated in figure 3, the given inputs are the vector operand $V(P_u)$ that is, in this operation, considered the Rotated Pole Zero Axis, and the Rotated Polar components $VT = \theta@VS$ and $M = R$; the produced output is the Cartesian result vector $V(P_i)$.

The mathematical Solid Vector addition operation of the vector operand $V(P_u)$ with VT and M consists of two sub-operations: (1) the Directional addition operation; (2) and the Magnitude addition operation, as equation 20 shows.

$$V(P_i) = (V(P_u) + \{VT_{V(P_i)\_V(P_u)}, M_{V(P_i)\_V(P_u)}\})_{SV} \tag{20}$$

1) First, to compute the Directional Solid Vector Addition operation, $V(P_u)$ is added to $VT_{V(P_i)\_V(P_u)}$, as the following steps show:-
   a) it is supposed that the length of the prime output vector, $V'(P_i)$ is equal to the length of $V(P_u)$ as shown in equation 21.

$$||V'(P_i)|| = ||V(P_u)||, \tag{21}$$

   b) Therefore, the vector $V'(P_i)$ is computed by the Cartesian vector addition operation of the two vectors: (1) $Proj_{V(P_u)\_V'(P_i)}$, the projection of $V'(P_i)$ on $V(P_u)$; (2) and $Proj_{VT_{V(P_i)\_V(P_u)}\_V'(P_i)}$, the projection of $V'(P_i)$ on $VT_{V(P_i)\_V(P_u)}$, as equation 22 shows.

$$V'(P_i) = Proj_{V(P_u)\_V'(P_i)} + Proj_{VT_{V(P_i)\_V(P_u)}\_V'(P_i)} \tag{22}$$

   Where in equation 22, the vector $Proj_{V(P_u)\_V'(P_i)}$ is computed by using equation 23, and the vector $Proj_{VT_{V(P_i)\_V(P_u)}\_V'(P_i)}$ is computed by using equation 24.

$$Proj_{V(P_u)} V'(P_i) = \frac{||V'(P_i)|| \cos(\theta)}{||V(P_u)||} V(P_u), \tag{23}$$

$$Proj_{VT_{V(P_i)\_V(P_u)}} V'(P_i) = \frac{||V'(P_i)|| \sin(\theta)}{||VT_{V(P_i)\_V(P_u)}||} VT_{V(P_i)\_V(P_u)}, \tag{24}$$

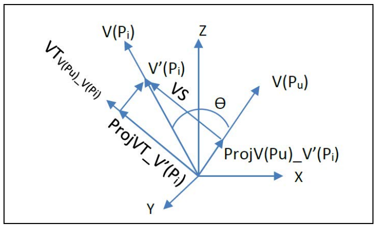

Fig. 3. shows the Solid Vector addition operation.

Where in equation 23 and equation 24, the radian angle θ between $V(P_u)$ and $V'(P_i)$ is equal to the length of the vector $VT_{V(P_i)\_V(P_u)}$, as equation 25 shows; and the name of the vector VT is a shortcut name for the same vector $VT_{V(P_i)\_V(P_u)}$.

$$\theta = ||VT_{V(P_i)\_V(P_u)}||, \tag{25}$$

2) Second, the magnitude difference, $M_{V(P_i)\_V(P_u)}$, is added to $V'(P_i)$ to get the result outcome of the *Solid Vector* addition operation, $V(P_i)$, as shown by equation 26.

$$V(P_i) = \frac{||V'(P_i)|| + M_{V(P_i)\_V(P_u)}}{||V'(P_i)||} V'(P_i) \tag{26}$$

The produced result vector $V(P_i)$ is represented in the Cartesian coordinate system.

B. Solid Vector Multiplication Operation

In the Solid Vector Multiplication operation shown in figure 4, the given inputs are: the Directional Pole Zero Axis V_{zero}, the vector operand $V(P_2)$, and the Rotated Polar angular θ multiplier and magnitude M multiplier. The Directional Sign Axis is computed by the Directional Solid Vector Subtraction operation between the vector operand $V(P_2)$ and the Directional Pole Zero Axis V_{zero}. And the produced outputs are: the angular component $\theta@VS$ and the magnitude component M combined in the produced Cartesian vector $V(P_1)$ result.

There are two types of the multiplication process between a vector and a scalar: (1) multiplication of the vector $V(P_2)$ direction by an angular scalar multiplier ($\theta_{multiplier}$); (2) multiplication of the vector $V(P_2)$ length by a magnitude scalar multiplier ($M_{multiplier}$), as equation 27 shows.

$$V(P_1) = \left(V(P_2) * \left\{ \begin{array}{c} \theta_{multiplier} \\ M_{multiplier} \end{array} \right\}\right)_{SV\ with\ respect\ to\ V_{zero}} \tag{27}$$

To perform the multiplication process of equation 27, first $V(P_2)$ is multiplied by the angular multiplier ($\theta_{multiplier}$), then the produced vector is multiplied by the magnitude multiplier ($M_{multiplier}$) as the following two sub-processes.

1) Multiplication of The Vector Direction by an Angular Scalar Multiplier: In order to multiply the vector direction by the angular scalar, $\theta_{multiplier}$, it is needed to get the Directional Pole Zero Axis, V_{zero}, of this multiplication process, then to perform the following steps sequentially:

1) Getting the input Directional Pole Zero vector V_{zero}.
2) Solid Vector Subtracting $V(P_2)$ from V_{zero} in order to compute Absolute Directional Solid Vector difference ($\theta_{V(P_2)\_V_{zero}}$) and the Directional Sign Axis (VS) according to the Solid Vector Subtraction operation presented in [2].
3) Multiplying the Absolute Directional Solid Vector difference ($\theta_{V(P_2)\_V_{zero}}$) by the radian angular multiplier ($\theta_{multiplier}$) to produce the prime output vector ($V'(P_1)$) Rotated Polar angular component ($\theta_{V'(P_1)\_V_{zero}}$).

$$\theta_{V'(P_1)\_V_{zero}} = \theta_{V(P_2)\_V_{zero}} * \theta_{multiplier} \tag{28}$$

4) The length of the prime output vector ($V'(P_1)$) is supposed to be equal to the length of the input vector ($V(P_2)$) as equation 29 shows.

$$||V'(P_1)|| = ||V(P_2)|| \tag{29}$$

5) Computing the projection of the prime output vector ($V'(P_1)$) on both the Directional Pole Zero vector (V_{zero}) and the Directional Sign Axis (VS).

$$Proj_{V_{zero}} V'(P_1) = \frac{||V'(P_1)|| \cos(\theta_{V'(P_1)\_V_{zero}})}{||V_{zero}||} V_{zero} \tag{30}$$

$$Proj_{VS} V'(P_1) = \frac{||V'(P_1)|| \sin(\theta_{V'(P_1)\_V_{zero}})}{||VS||} VS \tag{31}$$

6) Computing the prime output vector ($V'(P_1)$) by adding the above produced two vectors ($Proj_{V_{zero}} V'(P_1)$) and ($Proj_{VS} V'(P_1)$) using the Cartesian vectors addition operation as shown in equation (32).

$$V'(P_1) = Proj_{V_{zero}} V'(P_1) + Proj_{VS} V'(P_1) \tag{32}$$

2) Multiplication of The Vector Length by a Magnitude Scalar Multiplier: To multiply the vector length by a magnitude multiplier, the length of the vector is just multiplied by the magnitude multiplier and the direction of the vector is kept the same.

$$V(P_1) = M_{multiplier} * V'(P_1); \tag{33}$$

Where in equation 33 the multiplication here is a Cartesian scalar by a Vector multiplication.

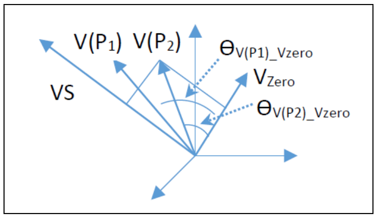

Fig. 4. Solid Vector Directional Multiplication of a vector by an angular scalar.

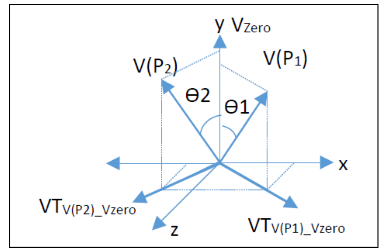

Fig. 5. Directional dot product and cross product.

C. Solid Vector Division Operation

In the Solid Vector Division operation, the given inputs are: the Directional Pole Zero Axis V_{zero}, the vector operand $V(P_2)$, and the Rotated Polar angular θ divisor and magnitude M divisor. The Directional Sign Axis VS is computed by the Directional Solid Vector Subtraction operation between the vector operand $V(P_2)$ and the Directional Pole Zero Axis V_{zero}. And the produced results are: the angular component $\theta@VS$ and the length component M combined in a Cartesian vector $V(P_1)$ result.

There are two types of the division operations between a vector and a scalar: (1) dividing a vector direction by the Rotated Polar angular divisor ($\theta_{divisor}$); (2) dividing a vector length by a magnitude divisor ($M_{divisor}$), as equation 34 shows.

$$V(P_1) = \left(V(P_2) * \left\{ \begin{array}{c} \frac{1}{\theta_{Divisor}} \\ \frac{1}{M_{Divisor}} \end{array} \right\}\right)_{SV\ with\ respect\ to\ V_{zero}} \tag{34}$$

To perform the division process of equation 34, first $V(P_2)$ is multiplied by the angular divisor $(\frac{1}{\theta_{divisor}})$, then the produced vector is multiplied by the magnitude divisor $(\frac{1}{M_{divisor}})$ in a similar way to the Solid Vector Multiplication process described above.

D. Solid Vector Directional Dot Product Operation

In the Solid Vector Directional Dot Product operation shown in figure 5, the given inputs are the vector operands $V(P_1)$ and $V(P_2)$, and the Directional Pole Zero Axis (V_{zero}); the produced result is a scalar quantity; This operation is performed by the following steps:

1) The vector $V(P_1)$ is Solid Vector subtracted from V_{zero} in order to get $VT_{V(P_1)\_V_{zero}}$ as shown in equation 35.

$$VT_{V(P_1)\_V_{zero}} = (V(P_1) - V_{zero})_{SV} \tag{35}$$

2) The vector $V(P_2)$ is Solid Vector subtracted from V_{zero} in order to get $VT_{V(P_2)\_V_{zero}}$ as shown in equation 36.

$$VT_{V(P_2)\_V_{zero}} = (V(P_2) - V_{zero})_{SV} \tag{36}$$

3) The Solid Vector Directional Dot Product operation between $V(P_1)$ and $V(P_2)$ is equal to the Cartesian Magnitude Dot Product operation, that is presented in the section III-A3, between the two vectors computed above $(VT_{V(P_1)\_V_{zero}})$ and $(VT_{V(P_2)\_V_{zero}})$ as shown in equation 37.

$$\begin{aligned} &(V(P_1).V(P_2))_{SV\ Directional\ Dot\ Product} \\ &= (VT_{V(P_1)\_V_{zero}}.VT_{V(P_2)\_V_{zero}})_{Cartesian\ Dot\ Product} \\ &= ||VT_{V(P_1)\_V_{zero}}||||VT_{V(P_2)\_V_{zero}}|| \\ &\cos(\theta_{VT_{V(P_1)\_V_{zero}}\_VT_{V(P_2)\_V_{zero}}}) \end{aligned} \tag{37}$$

E. Solid Vector Directional Cross Product Operation

In the Solid Vector Directional Cross Product operation shown in figure 5, the given inputs are the vector operands $V(P_1)$ and $V(P_2)$, and the Directional Pole Zero Axis (V_{zero}); the produced output is the Cartesian result vector $V(P_3)$; This operation is performed by the following steps:

1) The vector $V(P_1)$ is Solid Vector subtracted from V_{zero} in order to get $VT_{V(P_1)\_V_{zero}}$ as shown in equation 38.

$$VT_{V(P_1)\_V_{zero}} = (V(P_1) - V_{zero})_{SV} \tag{38}$$

2) The vector $V(P_2)$ is Solid Vector subtracted from V_{zero} in order to get $VT_{V(P_2)\_V_{zero}}$ as shown in equation 39.

$$VT_{V(P_2)\_V_{zero}} = (V(P_2) - V_{zero})_{SV} \tag{39}$$

3) The Solid Vector Directional Cross Product operation between $V(P_1)$ and $V(P_2)$ produces a vector $V(P_3)$ which is equal to the Cartesian Magnitude Cross Product operation between the two vectors computed above $(VT_{V(P_1)\_V_{zero}})$ and $(VT_{V(P_2)\_V_{zero}})$ as shown in equation 40.

$$\begin{aligned} &V(P_3) = (V(P_1) \times V(P_2))_{SV\ Directional\ Cross\ Product} \\ &= (VT_{V(P_1)\_V_{zero}} \times \\ &VT_{V(P_2)\_V_{zero}})_{Cartesian\ Cross\ Product} \end{aligned} \tag{40}$$

Thus, the produced vector $V(P_3)$ magnitude is computed using equation 41.

$$\begin{aligned} &V(P_3)_{Magnitude} = ||VT_{V(P_1)\_V_{zero}}||\,||VT_{V(P_2)\_V_{zero}}|| \\ &\sin(\theta_{VT_{V(P_1)\_V_{zero}}\_VT_{V(P_2)\_V_{zero}}}) \end{aligned} \tag{41}$$

The direction of the produced vector $V(P_3)$ is computed using the right hand rule that is implemented on the two vectors $(VT_{V(P_1)\_V_{zero}})$ and $(VT_{V(P_2)\_V_{zero}})$.

VI. The Conclusions

This research extended the recently proposed 3D Solid Vector Subtraction operation by proposing novel definitions of its Rotated Polar coordinate system, 3D and hyperspace, and its other whole complementary set of Solid Vector operations which are applicable to 2d, 3d and hyperdimensional space. The effect of introducing their definitions is converting the correct 2D Fixed Polar to a 3D correct Rotated Polar which opens the door to the future work that is, in addition to justifying these Solid Vector operations, to invent robust 3D SLAM, rotation control of objects, 3d object recognition and detection, and 3d virtual sculpturing.

Acknowledgments

The authors would like to thank The Higher Committee of Education Development in Iraq (HCED) and the Iraqi Ministry of Higher Education for funding this research.

References

[1] C. Han, H. Chen, G. Alonso, Y. Rao, J. Cubas, J. Yin, and X. Wang, "A linear model for relative motion in an elliptical orbit based on a spherical coordinate system," *Acta Astronautica*, vol. 157, pp. 465 – 476, 2019. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S009457651830910X 1, 4

[2] J. Al-Anssari, I. Naser, and A. Ralescu, "Three-dimensional gradient spatial filter of a field of vectors for geometrical edges magnitude detection in point cloud surfaces," *2019 Joint 8th International Conference on Informatics, Electronics & Vision (ICIEV) and 2019 3rd International Conference on Imaging, Vision & Pattern Recognition (icIVPR)*, pp. 362–370, 2019. 1, 2, 4, 6, 7

[3] ——, "Three-dimensional laplacian spatial filter of a field of vectors for geometrical edges magnitude and direction detection in point cloud surfaces," *2019 IEEE International Conference on Humanized Computing and Communication (HCC)*, pp. 83–93, 2019. 1

[4] I. Naser, J. Al-Anssari, and A. Ralescu, "Three-dimensional gradient-based laplacian spatial filter of a field of vectors for geometrical edges magnitude detection in point cloud surfaces," *2019 Joint 8th International Conference on Informatics, Electronics & Vision (ICIEV) and 2019 3rd International Conference on Imaging, Vision & Pattern Recognition (icIVPR)*, pp. 354–361, 2019. 1

[5] ——, "Three-dimensional unsharp masking spatial filter of a field of vectors for geometrical edges magnitude and direction detection in point cloud surfaces," *2019 IEEE International Conference on Humanized Computing and Communication (HCC)*, pp. 68–76, 2019. 1

[6] G. Collins, *The Foundations of Celestial Mechanics (Astronomy and Astrophysics Series)*. Pachart Pub House, 1989. 2

[7] B. Adamczyk, *Foundations of Electromagnetic Compatibility: With Practical Applications*, 1st ed. Wiley Publishing, 2017. 3, 4

[8] M. Hosseini, H. Hassanabadi, and S. Hassanabadi, "Solutions of the dirac-weyl equation in graphene under magnetic fields in the cartesian coordinate system," *The European Physical Journal Plus*, vol. 134, no. 1, pp. 1–6, 2019. 3

[9] G. V. Garca and t. lu, "Parameterization of the ellipse based on the valencia's sphere, without to use a cartesian coordinate system," *Annals of the Faculty of Engineering Hunedoara*, vol. 16, no. 4, pp. 59–67, 2018. 3

[10] Y. A. Grigoriev and A. V. Tsiganov, "On superintegrable systems separable in Cartesian coordinates," *Physics Letters A*, vol. 382, no. 32, pp. 2092–2096, Aug 2018. 3

[11] B. . Kim, J. . Sun, S. . Kim, M. . Kang, and S. . Ko, "Cnn-based ugs method using cartesian-to-polar coordinate transformation," *Electronics Letters*, vol. 54, no. 23, pp. 1321–1322, 2018. 3

[12] Y. Luo, F. Zhao, N. Li, and H. Zhang, "A modified cartesian factorized back-projection algorithm for highly squint spotlight synthetic aperture radar imaging," *IEEE Geoscience and Remote Sensing Letters*, vol. 16, no. 6, pp. 902–906, June 2019. 3

[13] G. Hay, *Vector and tensor analysis*, ser. Dover Books on Science. Dover Publication, 1953. [Online]. Available: https://books.google.com/books?id=ar3wswEACAAJ 3

[14] R. Larson, *Elementary Linear Algebra*. Cengage Learning, 2012. [Online]. Available: https://books.google.com/books?id=rqWCVMYk5mEC 3

[15] H. D. Tagare, "Polar coordinates: What they are and how to use them," accessed in 2019, notes. [Online]. Available: http://noodle.med.yale.edu/hdtag/notes/coord.pdf 3

[16] S. Gai, F. Da, and X. Fang, "A novel camera calibration method based on polar coordinate," *PloS one*, vol. 11, no. 10, pp. e0 165 487–e0 165 487, 2016. 3

[17] J. Wang, H. Wanga, J. Li, X. Luo, Y. Shi, and S. K. Jha, "Detecting double jpeg compressed color images with the same quantization matrix in spherical coordinates," *IEEE Transactions on Circuits and Systems for Video Technology*, pp. 1–1, 2019. 4

[18] "Hyperplane," accessed in July 2019, wikipedia page. [Online]. Available: https://en.wikipedia.org/wiki/Hyperplane 4

[19] S. Abbott, "Geometry, by v. v. prasolov and v. m. tikhomirov (trans. o. v. sipacheva). translations of mathematical monographs 200. pp. 257. 69.50. 2001. isbn 0 8218 2038 9 (american mathematical society)." *The Mathematical Gazette*, vol. 86, no. 507, p. 563564, 2002. 4