변환 영역 처리
Image Processing at Frequency Space

담당교수: 김민기

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- ❖공간 주파수의 이해
- ❖이산 퓨리에 변환 (Discrete Fourier Transform)
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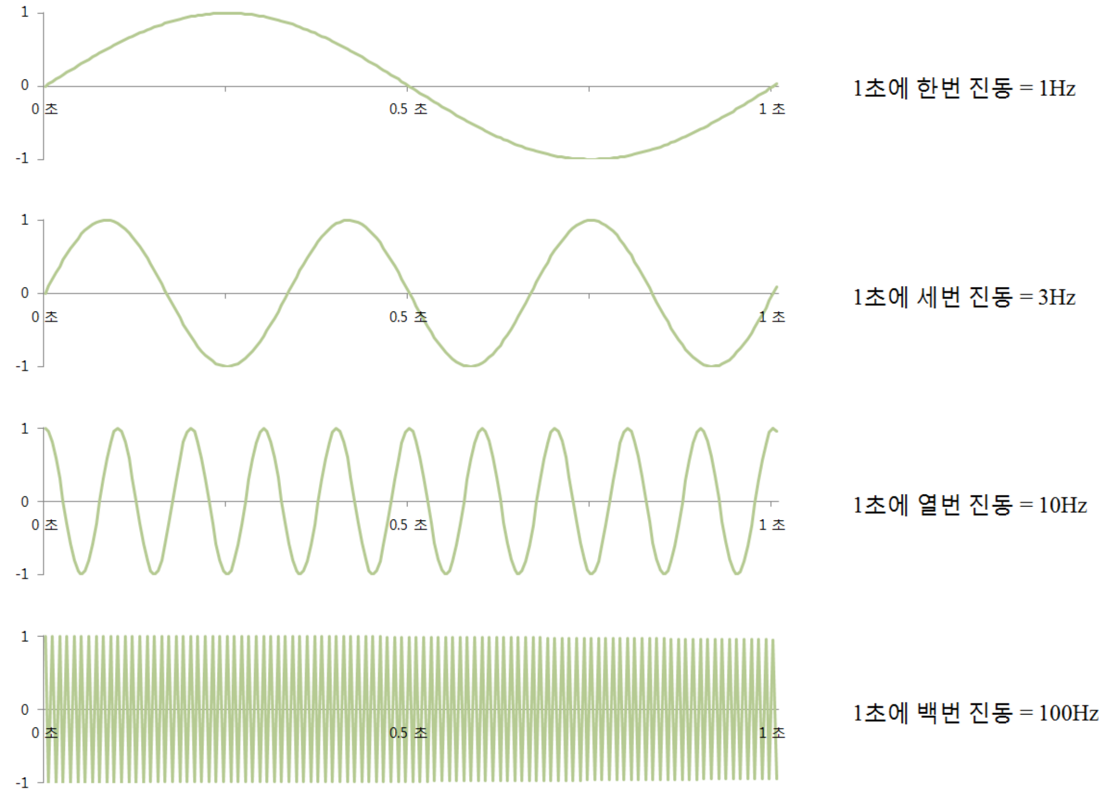
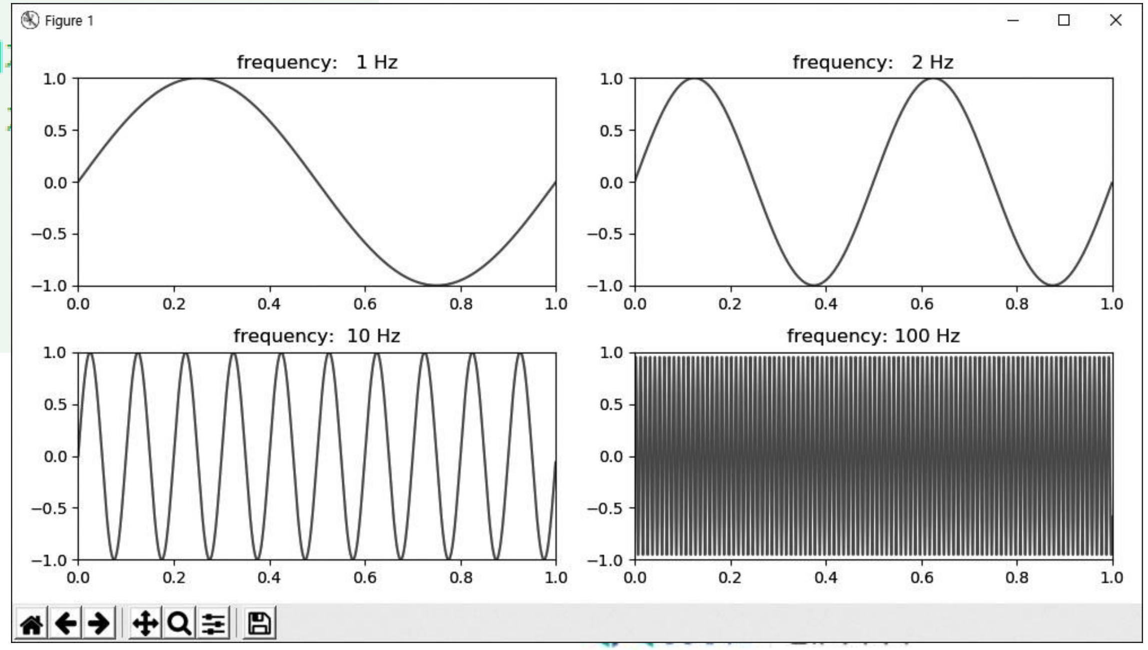
공간 주파수의 이해

❖ 주파수 (frequency)
- 주파수의 단위: 헤르츠(Hz) 1초 동안에 진동하는 횟수



예제 9.1.1 주파수 그리기 |– 01.frequence.py

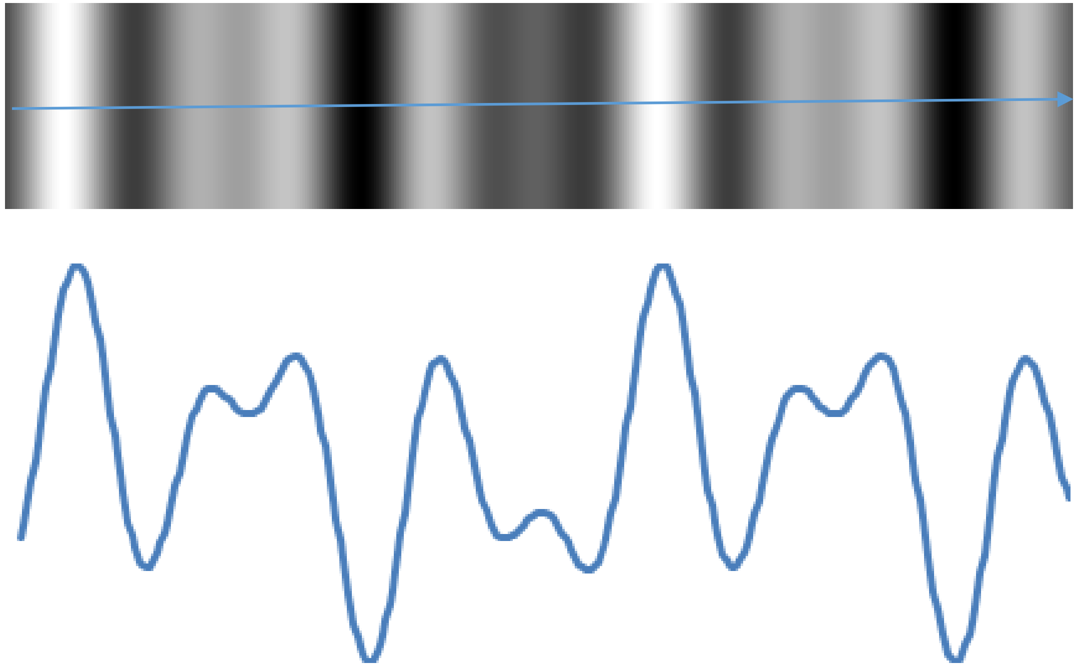
```
import matplotlib.pyplot as plt                       # 그래프그리기 라이브러리 임포트
import numpy as np

t = np.arange(0, 1, 0.001)                            # 샘플링 범위 및 개수
Hz = [1, 2, 10, 100]                                  # 주파수 예시
gs = [np.sin(2 * np.pi * t * h) for h in Hz]          # sin 함수 계산

plt.figure(figsize=(10, 5))
for i, g in enumerate(gs):
        plt.subplot(2, 2, i+1), plt.plot(t, g)        # 그래프 그리
        plt.xlim(0, 1), plt.ylim(-1, 1 )              # x, y축 범위
        plt.title(“frequency: %3d Hz” % Hz[i])
plt.tight_layout()
plt.show()
```



gs = [2 * np.sin(2 * np.pi * f * t - np.pi/4) for f in Hz]

영상에서의 주파수

- 공간상에서 화소 밝기의 변화율
- 이런 의미에서 공간 주파수라는 표현을 사용



영상에서의 주파수

❖ 저주파 공간 영역
- 화소 밝기가 거의 변화가 없거나 점진적으로 변화하는 것
- 영상에서 배경 부분이나 객체의 내부에 많이 있음

❖ 고주파 공간 영역
- 화소 밝기가 급변하는 것
- 경계부분이나 객체의 모서리 부분

영상에서의 주파수

❖영상을 주파수 영역별로 분리할 수 있다면?

Fourier Transform

❖ 퓨리에 변환

- 임의의 신호는 다양한 주파수를 갖는 주기 함수의 합으로 표현될 수 있다.
- 주기 함수: 정현파(sin), 여현파(cos)

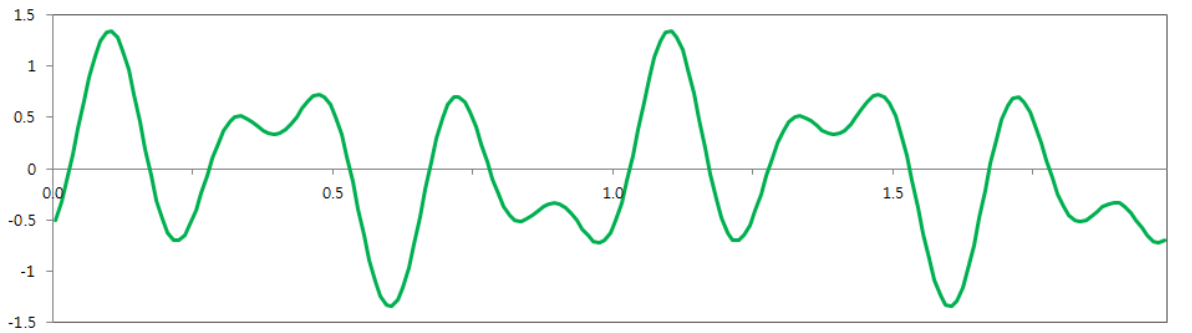
$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot g_f(t)\,df$$

$g_f(t)$: 주파수가 f인 기저함수
$G(f)$: 기저함수의 계수

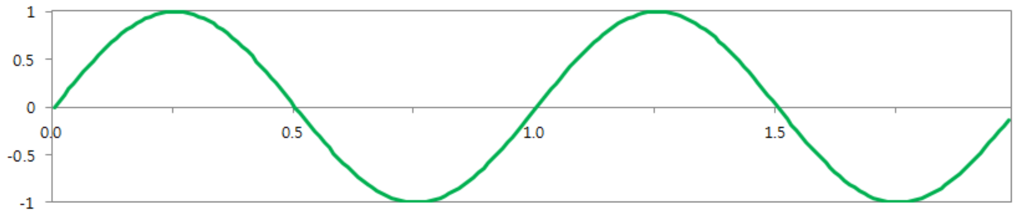
퓨리에 변환

$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot g_f(t)\,df$$

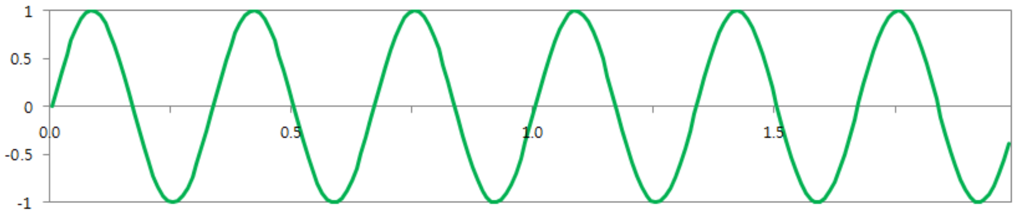
$g_f(t)$: 주파수가 f인 기저함수
$G(f)$: 기저함수의 계수



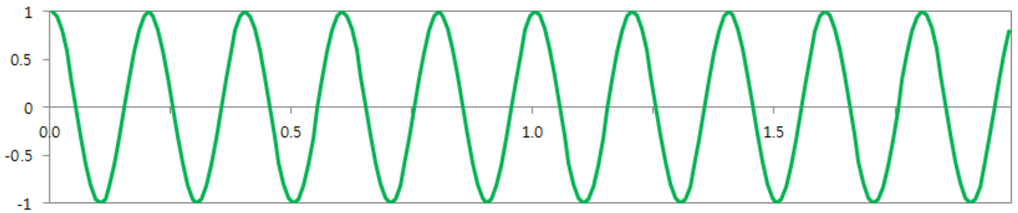
$g(t) = 0.3 * g_1(t)+0.7 * g_2(t)-0.5 * g_3(t)$



$g_1(t) = \sin(2\pi 1t)$

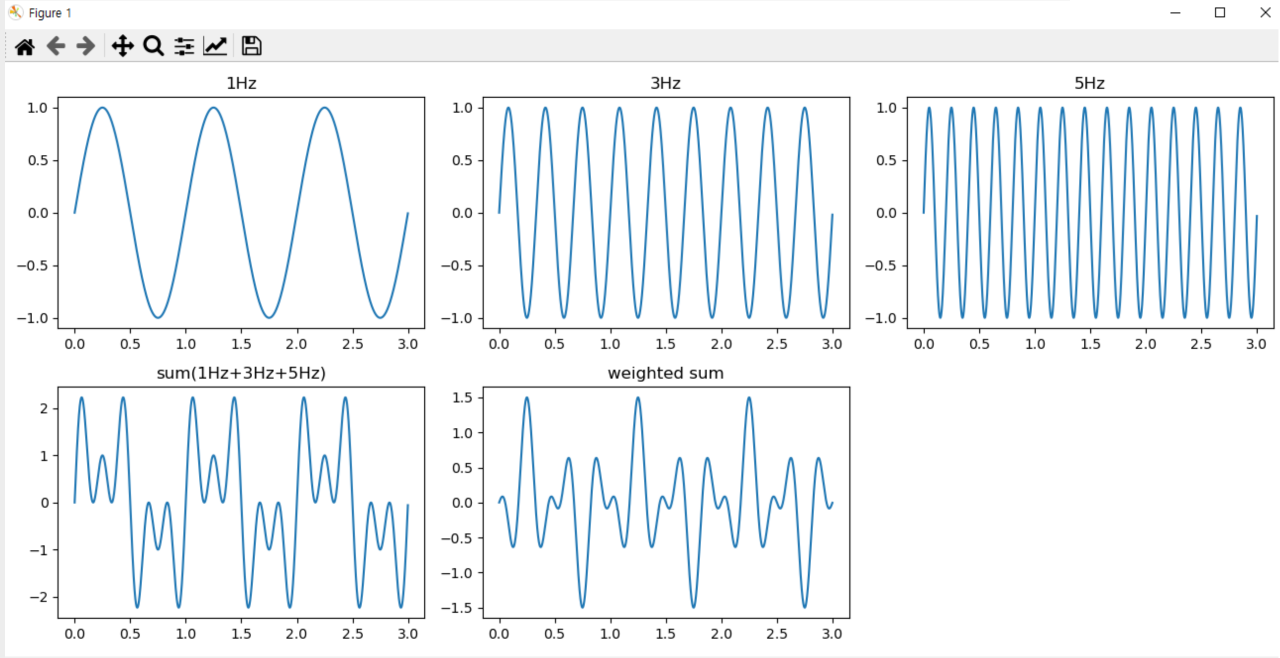


$g_2(t) = \sin(2\pi 3t)$



$g_3(t) = \cos(2\pi 5t)$

```python
import matplotlib.pyplot as plt
import numpy as np

t = np.arange(0, 3, 0.001)     # Time vector
g = [0] * 5
g[0] = np.sin(2 * np.pi * 1 * t)
g[1] = np.sin(2 * np.pi * 3 * t)
g[2] = np.sin(2 * np.pi * 5 * t)
g[3] = g[0] + g[1] + g[2]
g[4] = 0.3 * g[0] - 0.7 * g[1] + 0.5 * g[2]

titles = ['1Hz', '3Hz', '5Hz', 'sum(1Hz+3Hz
plt.figure(figsize=(13, 6))
for i, title in enumerate(titles):
      plt.subplot(2, 3, i+1)
      plt.plot(t, g[i])
      plt.title(title)
plt.tight_layout()
plt.show()
```



Sine & Cosine 함수

- Periodic functions
- General form of sine and cosine functions:

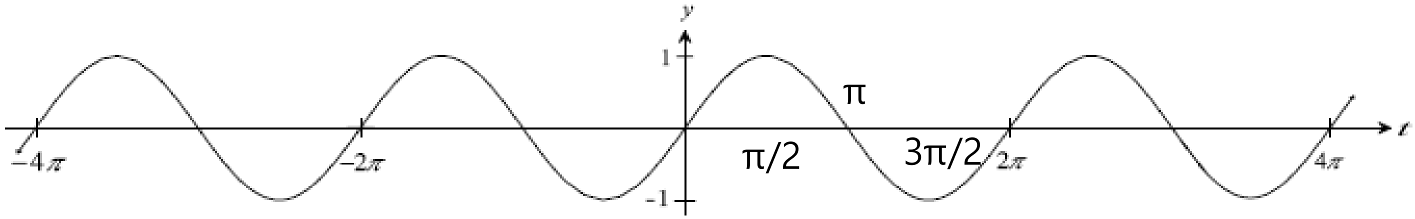
$$y\,(t) = A\sin[a(t+b)] \qquad y\,(t) = A\cos[a(t+b)]$$

$\|A\|$	amplitude
$\frac{2\pi}{\|a\|}$	period
b	phase shift

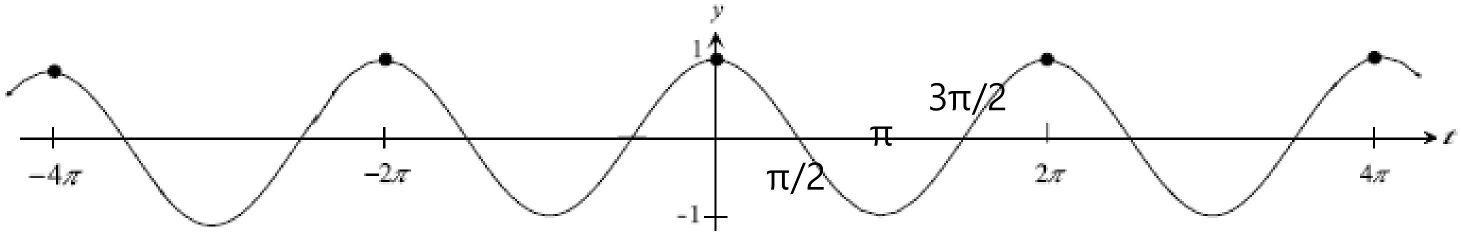
Sine & Cosine 함수

- case: A=1, b=0, a=1

$$y(t) = A\sin[a(t+b)]$$



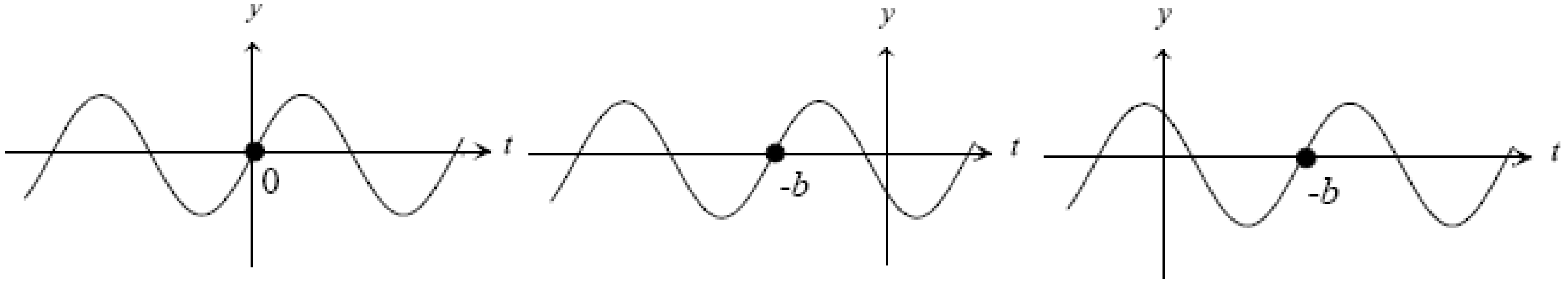
$$y(t) = A\cos[a(t+b)]$$



Sine & Cosine 함수

- Shifting or translating the sine function by a const b

$$y(t) = A\sin[a(t+b)]$$



(a) $y = \sin t$ (b) $y = \sin(t+b)$, $b > 0$ (c) $y = \sin(t+b)$, $b < 0$

Note: cosine is a shifted sine function:

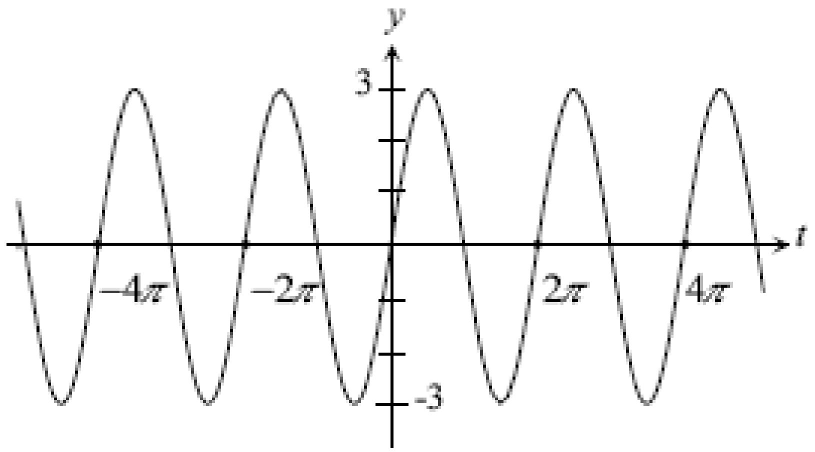
$$\cos(t) = \sin(t + \frac{\pi}{2})$$

Sine & Cosine 함수

- Changing the amplitude A

$$y(t) = A\sin[a(t+b)]$$

$$y = 3\sin t.$$

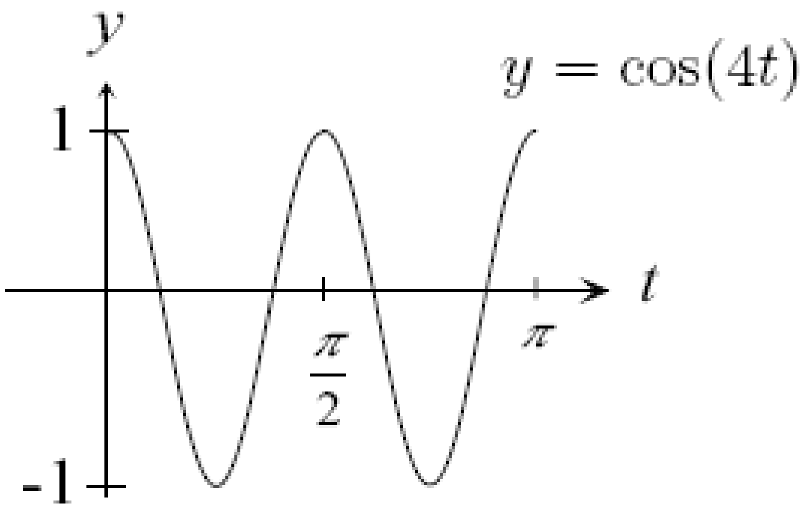


Sine & Cosine 함수

- Changing the period $T=2\pi/|a|$

  consider $A=1$, $b=0$: $y=\cos(at)$

$$y(t) = A\cos[a(t+b)]$$



a=4 → period $2\pi/4 = \pi/2$

shorter period
higher frequency
(i.e., oscillates faster)

Frequency is defined as $f=1/T$

Alternative notation: $\sin(at)=\sin(2\pi t/T)=\sin(2\pi ft)$

(if $a > 0$, $T=2\pi/a$ → $a=2\pi/T$)

Complex Number

❖ A complex number **x** is of the form:

$$x = a + bi, \qquad \text{where} \quad i = \sqrt{-1}$$

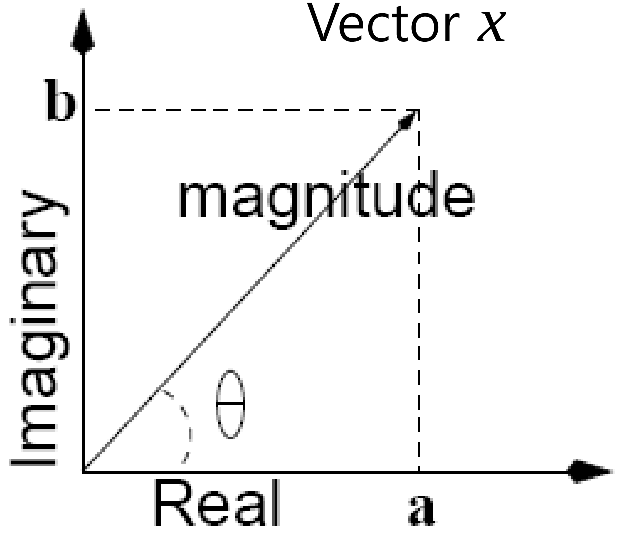
❖ Addition: $(a + bi) + (\mathrm{c} + \mathrm{d}i) = (\mathrm{a} + \mathrm{c}) + (\mathrm{b} + \mathrm{d})i$

❖ Multiplication: $(a + bi)(\mathrm{c} + \mathrm{d}i) = (\mathrm{ac} - \mathrm{bd}) + (\mathrm{ad} + \mathrm{bc})i$

Complex Number

❖Magnitude-Phase (i.e., vector) representation

$$x = a + bi$$



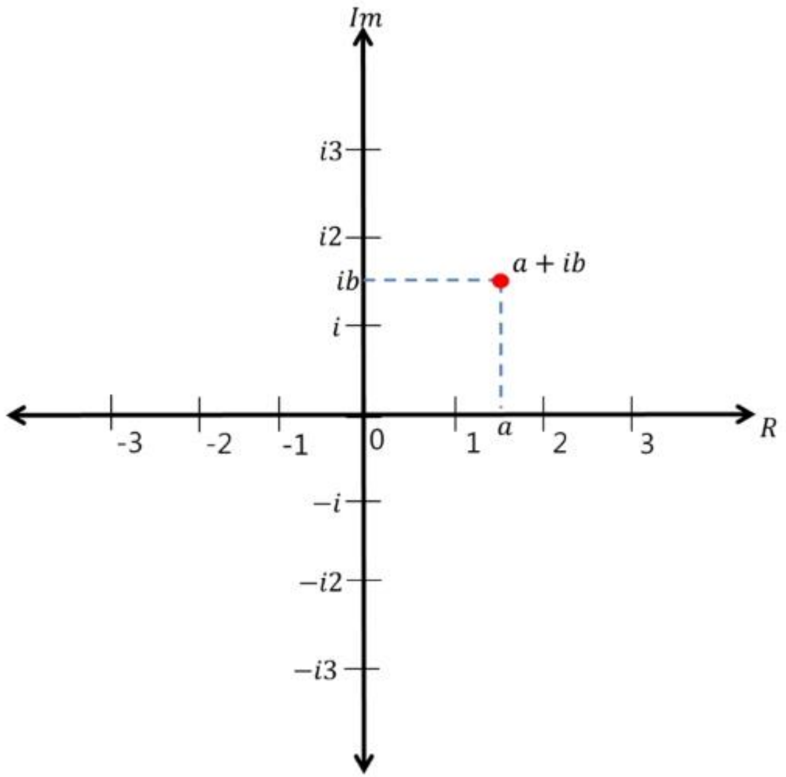
$$|x| = \sqrt{a^2 + b^2}$$

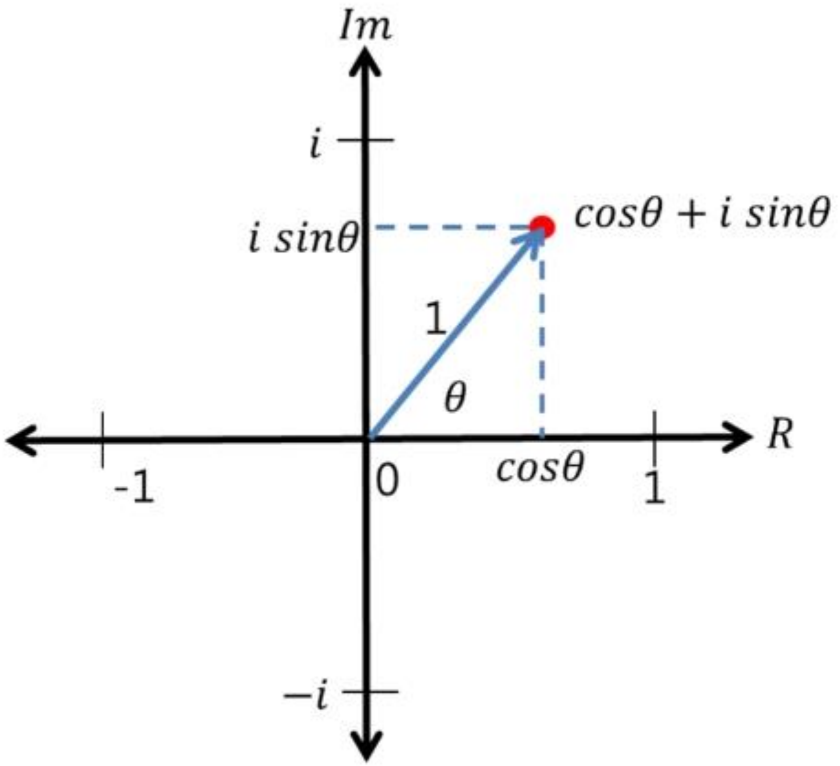
$$\theta(x) = tan^{-1}\left(\frac{b}{a}\right)$$

Magnitude-Phase notation:

$$x = |x|(cos\theta + isin\theta)$$

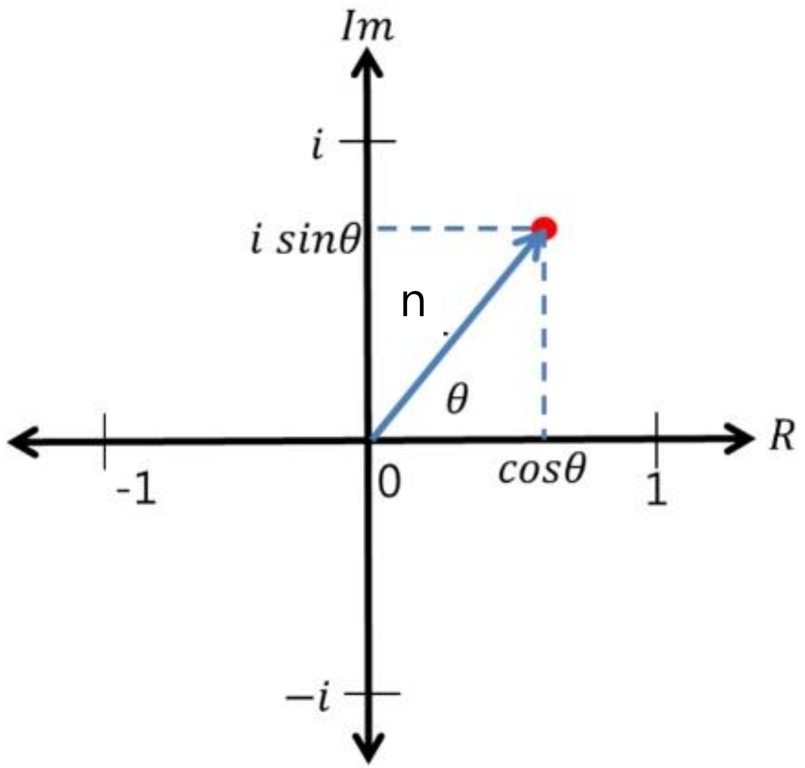
Complex Number → Complex Plane



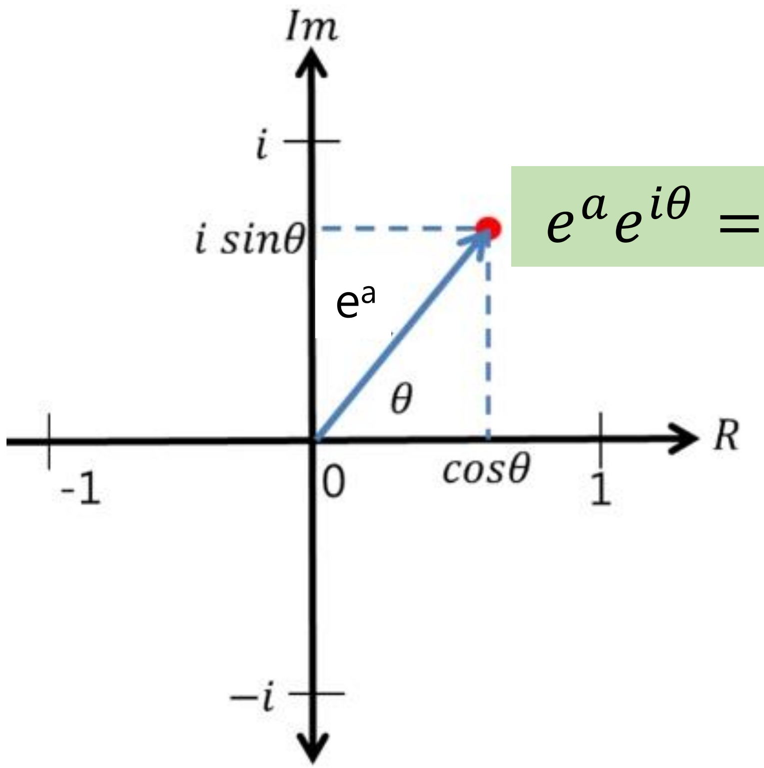


$$e^{i\theta} = cos\theta + isin\theta$$

Complex Number → Complex Plane



$$n(cos\theta + isin\theta) = ne^{i\theta}$$



$$e^{a}e^{i\theta} = e^{a+i\theta}$$

Fourier Transform

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \quad \text{where } j = \sqrt{-1}$$

$$e^{i\theta} = cos\theta + isin\theta$$

  - Inverse Fourier transform:

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

❖ u : the frequency variables

❖ x : the spatial variables

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (continuous case)

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

  - Inverse Fourier transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

1차원 Discrete Fourier Transform

- The one-dimensional Fourier transform and its inverse
  - Fourier transform (discrete case)

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-\frac{j2\pi ux}{N}}, where\ u = 0, 1, 2, \dots N-1$$

DFT의 계산량은 입력신호의 길이(N)의 제곱에 비례

  - Inverse Fourier transform:

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{\frac{j2\pi ux}{N}}, where\ x = 0, 1, 2, \dots N-1$$

Alternative notation: sin(at)=sin(2πt/T)=sin(2πft)

(if a > 0, T=2π/a → a=2π/T)

2차원 Discrete Fourier Transform

- The two-dimensional Fourier transform and its inverse
  - Fourier transform (discrete case)

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$, where \;\; u = 0, 1, 2, \dots, M-1, v = 0, 1, 2, \dots, N-1$$

  - Inverse Fourier transform:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$, where \;\; x = 0, 1, 2, \dots, M-1, y = 0, 1, 2, \dots, N-1$$

Discrete Fourier Transform

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-\frac{j2\pi ux}{N}}, where \;\; u = 0, 1, 2, \dots N-1$$

$$Re\{F(u)\} = \sum_{x=0}^{N-1} \left(Re\{f(x)\} cos\left(-\frac{2\pi}{N}ux\right) - Im\{f(x)\} \sin\left(-\frac{2\pi}{N}ux\right) \right)$$

$$Im\{F(u)\} = \sum_{x=0}^{N-1} \left(Im\{f(x)\} cos\left(-\frac{2\pi}{N}ux\right) + Re\{f(x)\} \sin\left(-\frac{2\pi}{N}ux\right) \right)$$

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{\frac{j2\pi ux}{N}}, where \;\; x = 0, 1, 2, \dots N-1$$

$$Re\{f(x)\} = \frac{1}{N} \sum_{x=0}^{N-1} \left(Re\{F(u)\} cos\left(\frac{2\pi}{N}ux\right) - Im\{F(u)\} \sin\left(\frac{2\pi}{N}ux\right) \right)$$

$$Im\{f(x)\} = \frac{1}{N} \sum_{x=0}^{N-1} \left(Im\{F(u)\} cos\left(\frac{2\pi}{N}ux\right) + Re\{F(u)\} \sin\left(\frac{2\pi}{N}ux\right) \right)$$

퓨리에 변환 Summary

❖복소수로 표현되는 기저 함수를 사용

$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot g_f(t)\,df$$

$g_f(t)$: f 주파수에 대한 기저함수
$G(f)$: 기저함수의 계수

$$g_f(t)=\cos(2\pi f t)+j\cdot\sin(2\pi f t)=e^{j2\pi ft}$$

$$G(f)=\int_{-\infty}^{\infty} g(t)\cdot e^{-j2\pi ft}\,dt$$

원본 신호로부터 기저함수의 계수를 얻는 것
→ 푸리에 변환

$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot e^{j2\pi ft}\,df$$

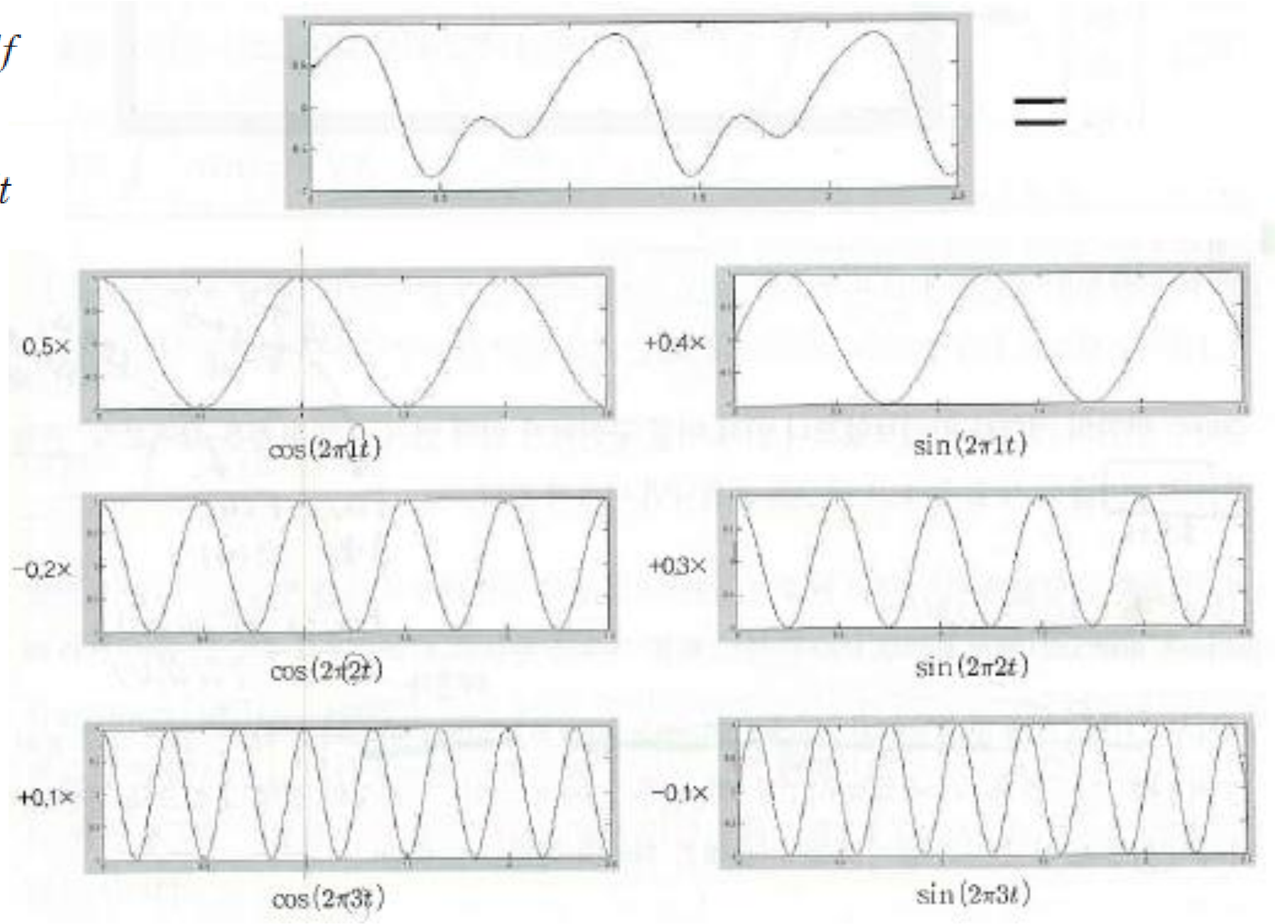
기저함수의 계수로부터 원본 신호를 얻는 것
→ 푸리에 역변환

퓨리에 변환

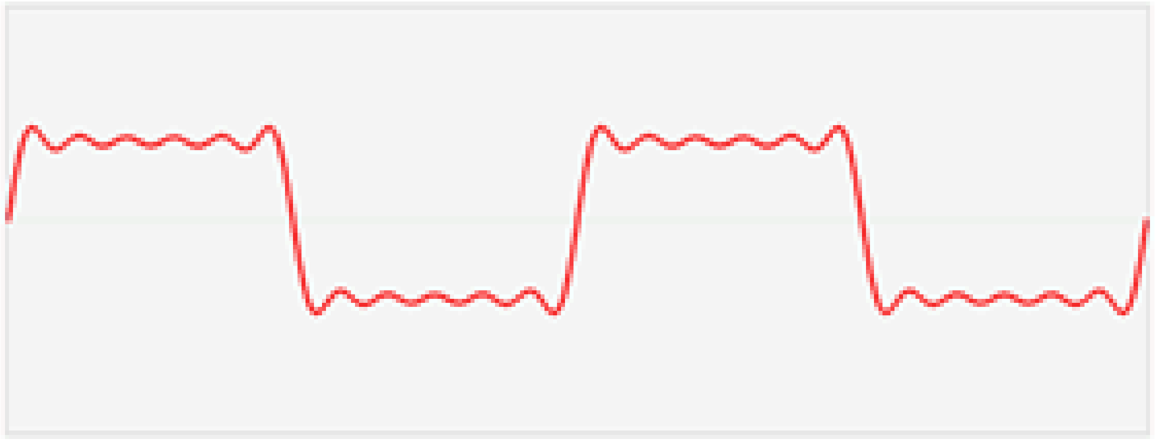
$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot g_f(t)df \qquad g_f(t)=\cos(2\pi ft)+j\cdot\sin(2\pi ft)=e^{j2\pi ft}$$

$$g(t)=\int_{-\infty}^{\infty} G(f)\cdot e^{j2\pi ft}df$$

$$G(f)=\int_{-\infty}^{\infty} g(t)\cdot e^{-j2\pi ft}dt$$



퓨리에 변환



https://commons.wikimedia.org/wiki/File:Fourier_series_and_transform.gif

퓨리에 변환의 표현

- $F(u)$ can be expressed in polar coordinates:

$$F(u) = |F(u)|e^{j\theta}$$

$$where \ \ |F(u)| = \sqrt{R^2(u) + I^2(u)}$$ (magnitude or spectrum)

$$\theta(u) = tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$ (phase angle or phase spectrum)

- Power spectrum:

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

Fourier Transform

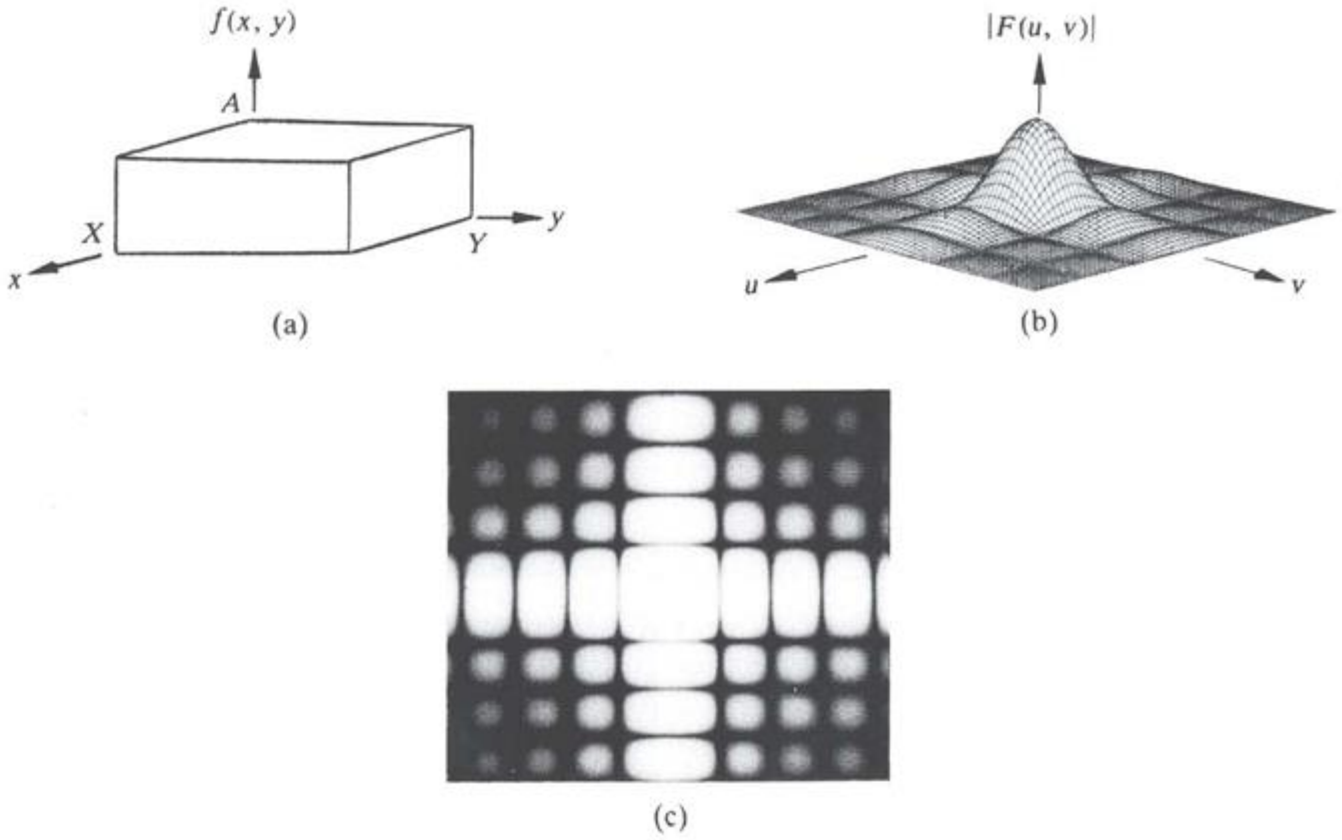
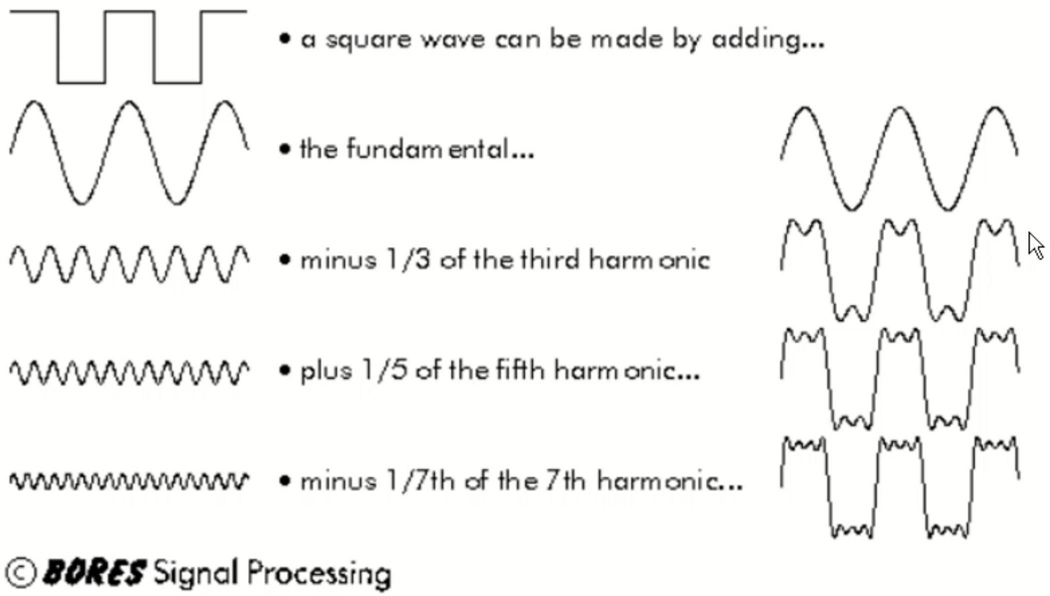
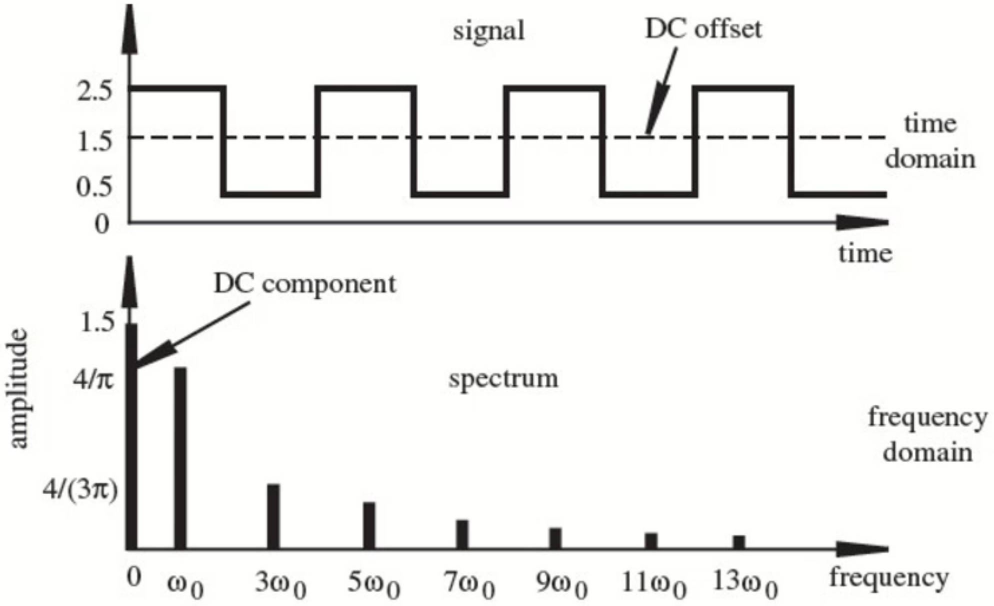


Figure 3.2 *(a) A 2-D function; (b) its Fourier spectrum; and (c) the spectrum displayed as an intensity function.*

Fourier Series

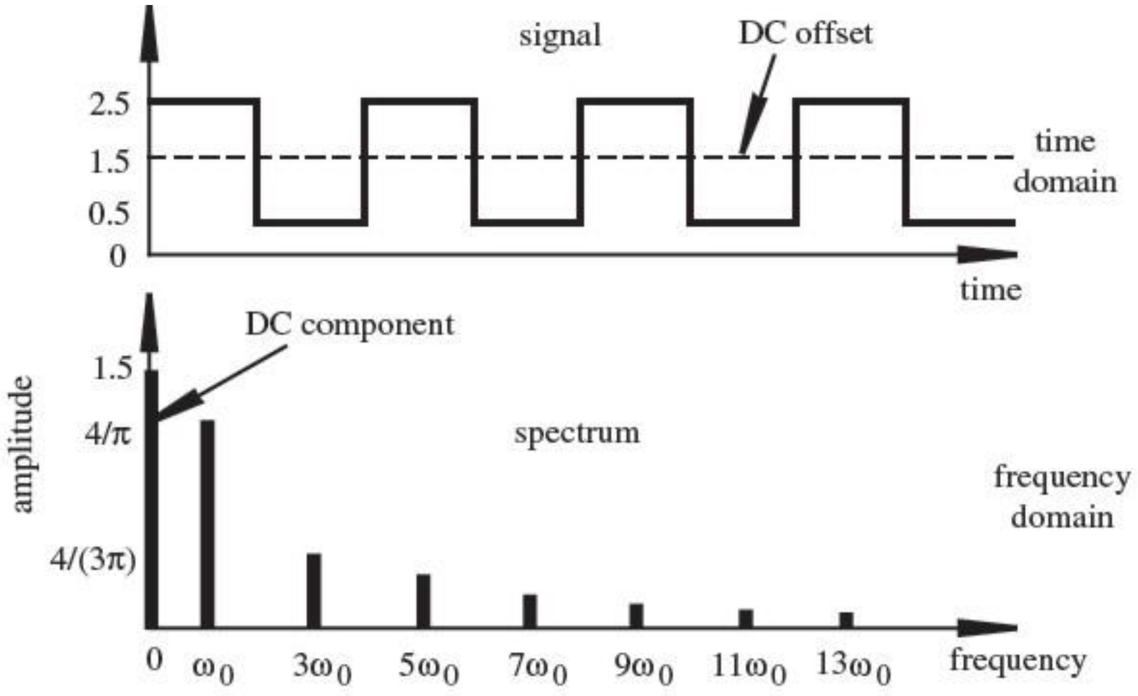


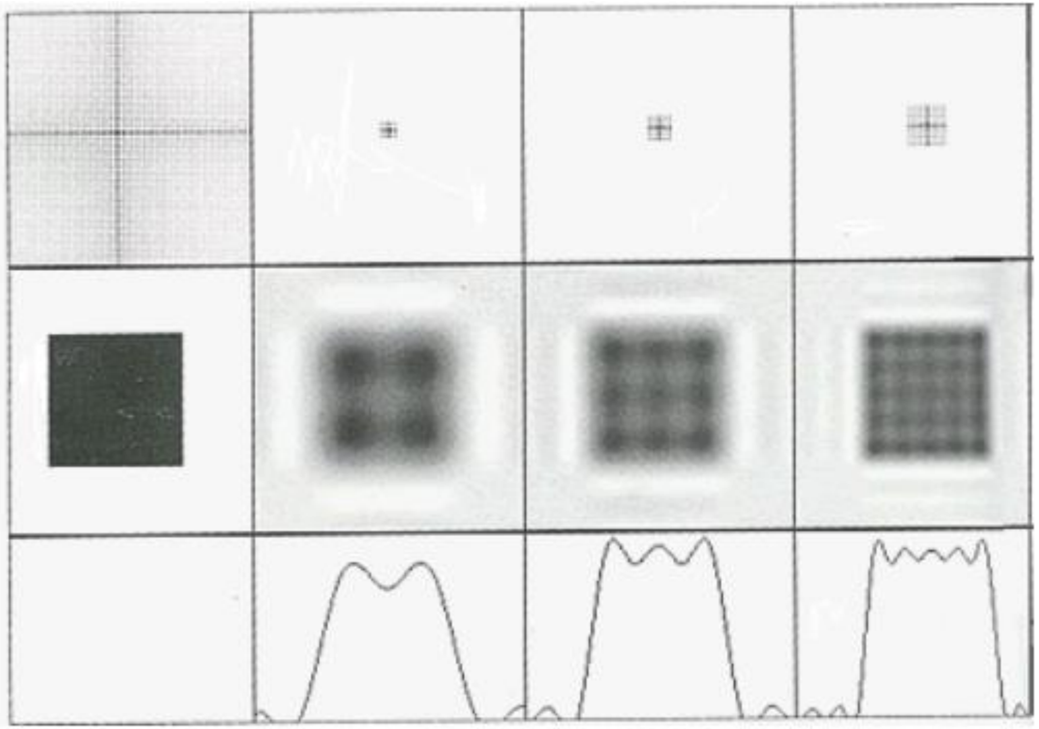


Square Wave> http://www.falstad.com/fourier/e-square.html

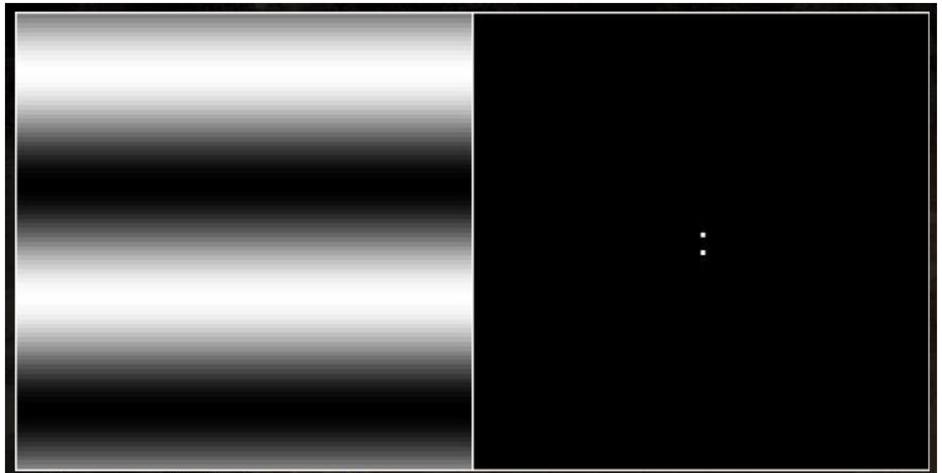
Magnitude & Phase View> http://www.falstad.com/fourier/e-phase.html

Fourier Transform

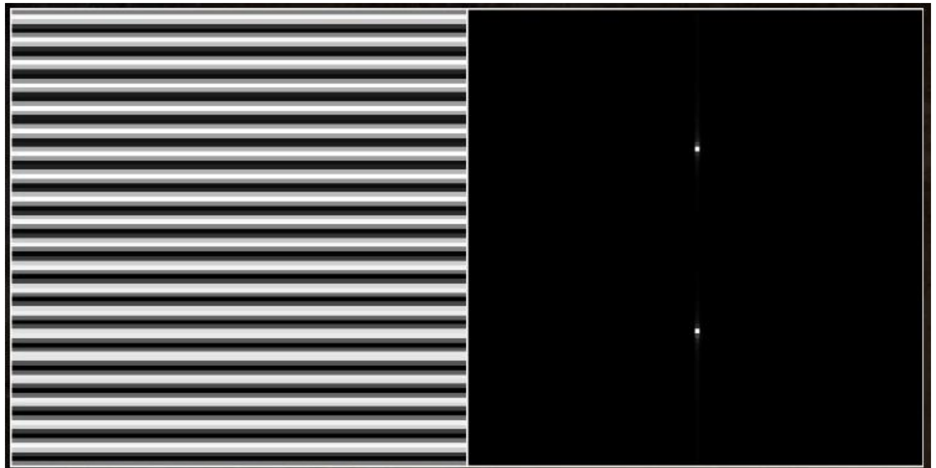




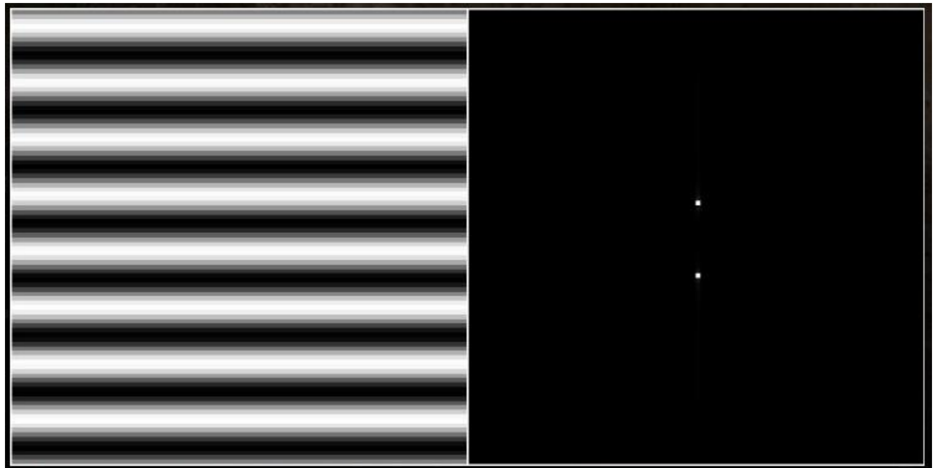
Fourier Transform



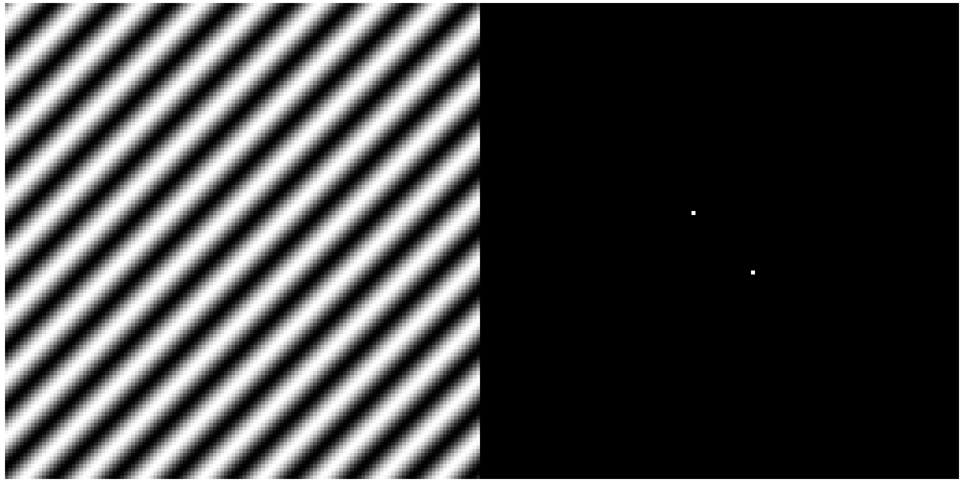
FT of a sinusoid with frequency of 1.



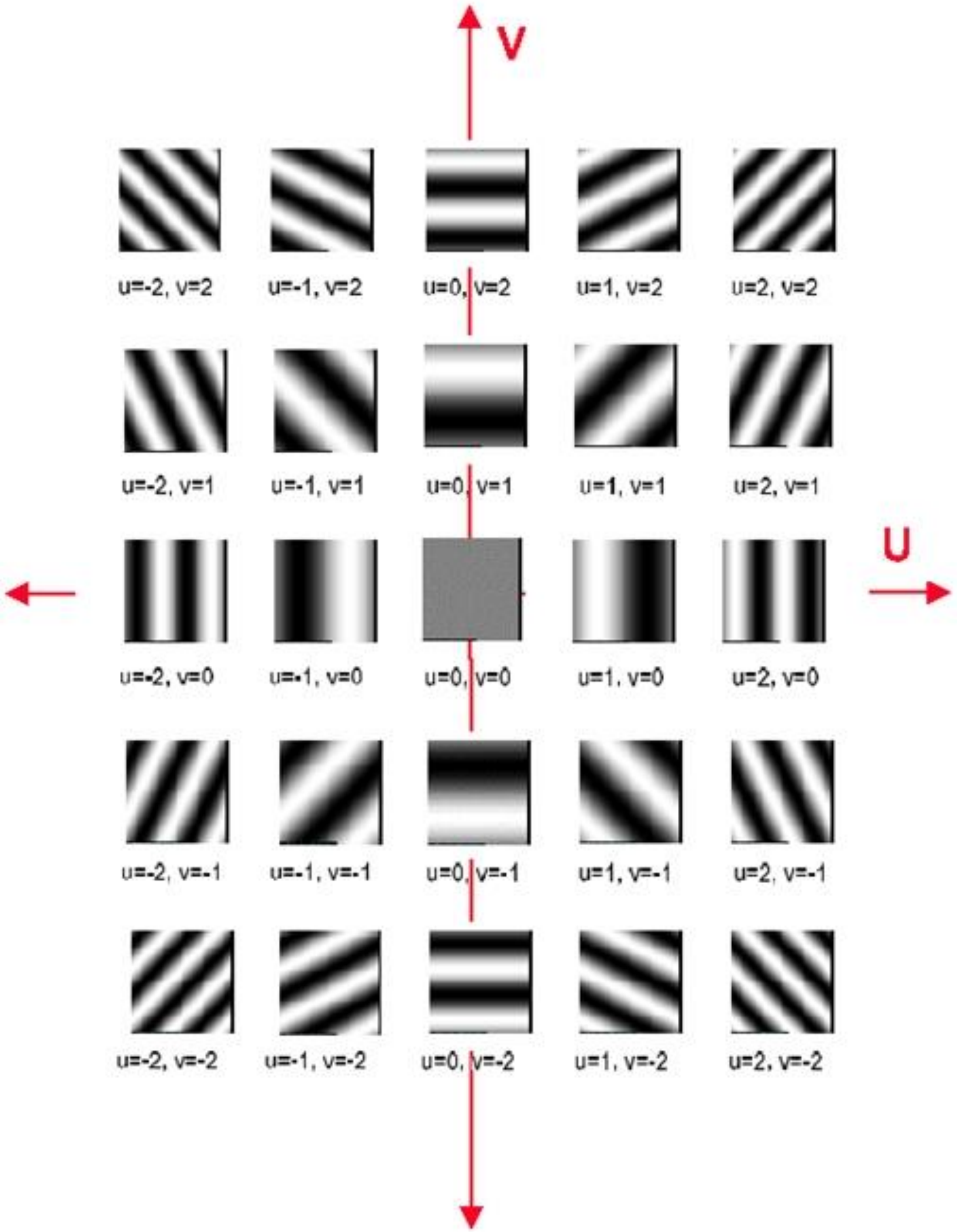
FT of a sinusoid with frequency of 10.



FT of a sinusoid with frequency of 4.

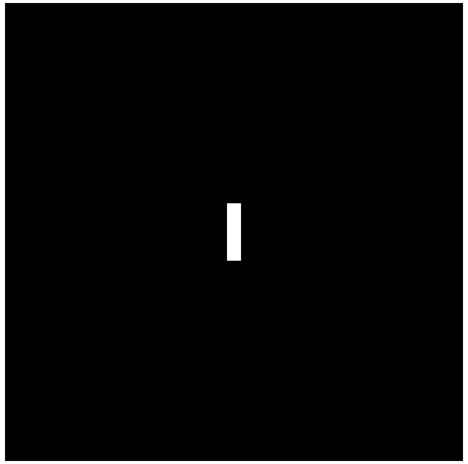


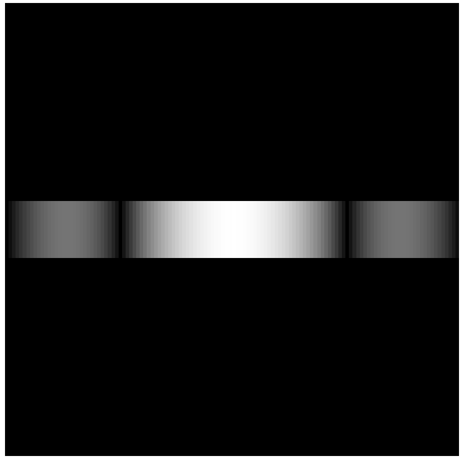
Fourier Transform

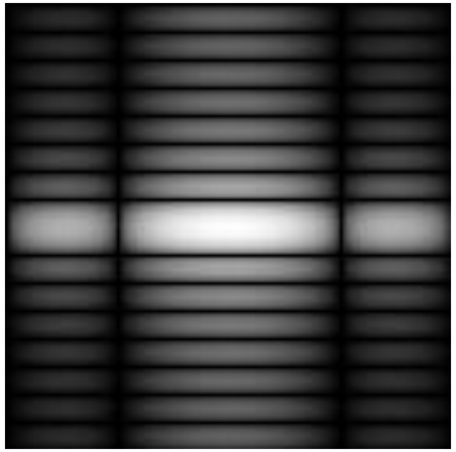


Fourier Transform

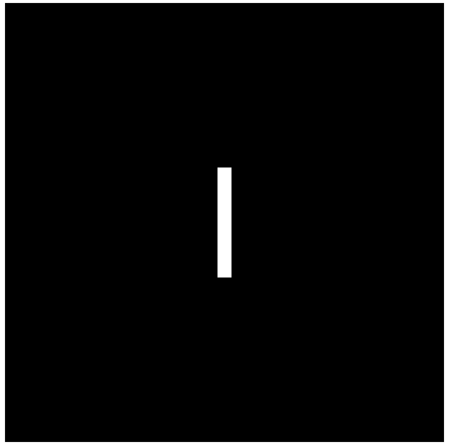
The 2D DFT **F(u,v)** can be obtained by
1. taking the 1D DFT of every row of image **f(x,y)** → F(u, y)
2. taking the 1D DFT of every column of **F(u,y)** → F(u, v)


(a) f(x,y)

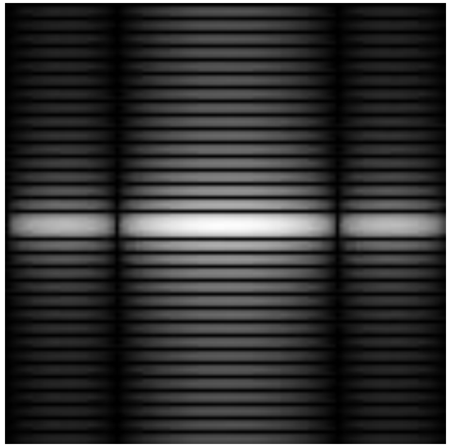

(b) F(u,y)

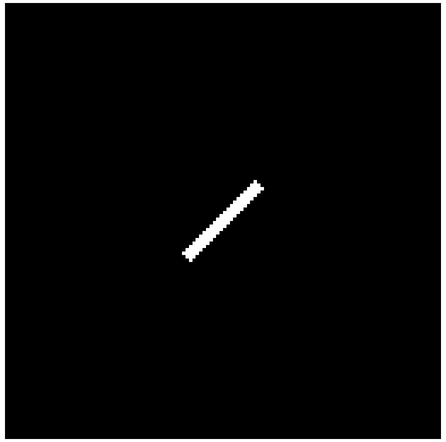

(c) F(u,v)

Fourier Transform

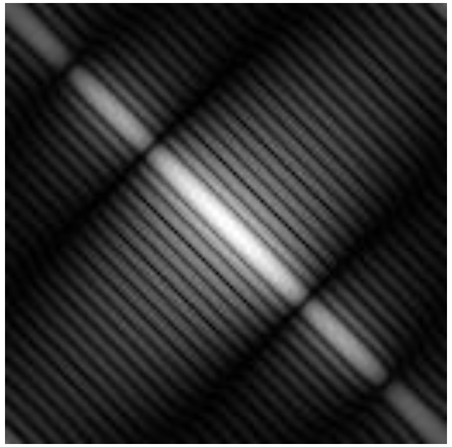


DFT



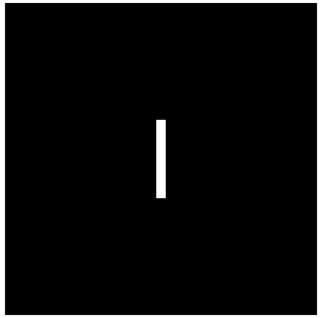


DFT

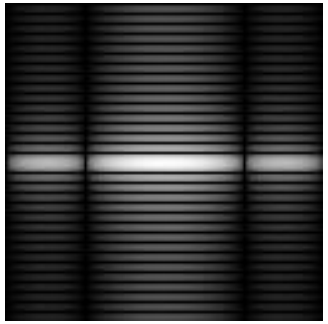


Fourier Transform

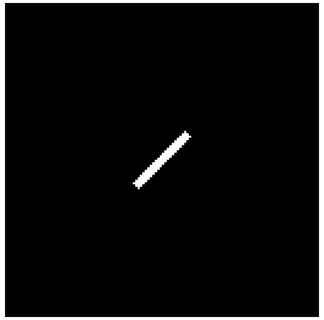
A



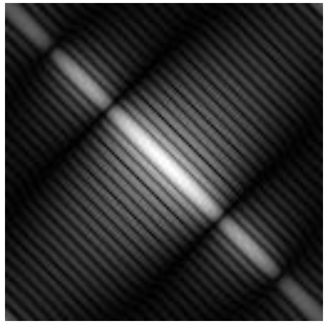
DFT



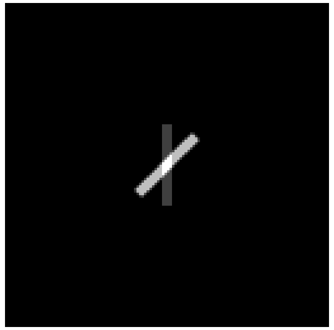
B



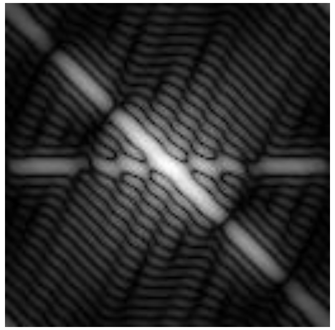
DFT



0.25 * A
+ 0.75 * B

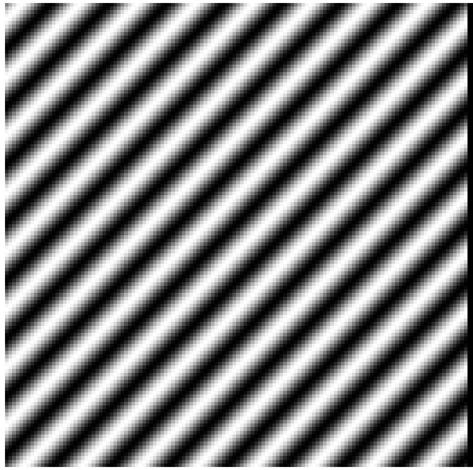


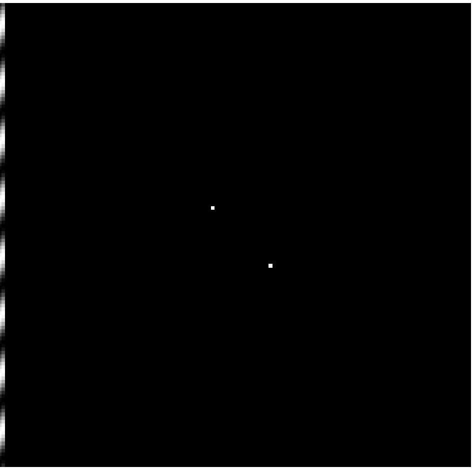
DFT



Fourier Transform

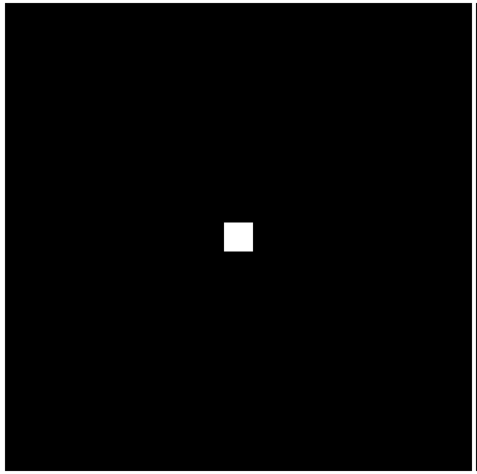
Sine wave

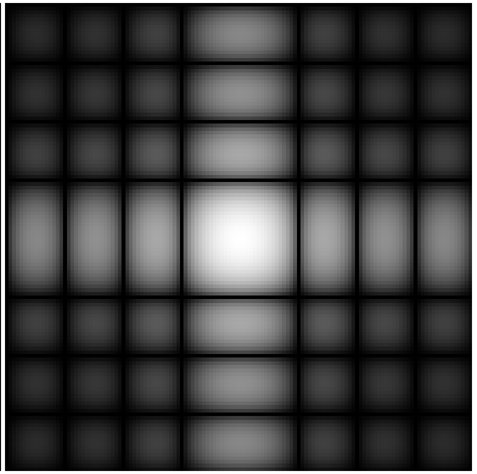




Its DFT

Rectangle





Its DFT