Notes on Differential Calculus

Contents

1. Differentiability

Definition 1.1: Let $f : (a, b) \to \mathbb{R}^n$, and let $f_i = \pi_i \circ f$ be its components. Then, f is differentiable at $t_0 \in (a, b)$ if the following limit exists.

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

Remark: The vector $f'(t_0)$ represents the tangent to the curve f at the point $f(t_0)$. The full tangent line is the parametric curve $f(t) + f'(t_0)(t - t_0)$.

Definition 1.2: Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^m$. Then, f is differentiable at $x \in U$ if there exists a linear transformation $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - \lambda h}{\|h\|} = 0.$$

The derivative of f at x is denoted by $\lambda = Df(x)$.

Remark: In a neighbourhood of x, we may approximate

$$f(x + h) \approx f(x) + Df(x)(h).$$

Remark: The statement that this quantity goes to zero means that each of the m components must also go to zero. For each of these limits, there are n axes along which we can let $h \to 0$. As a result, we obtain $m \times n$ limits, which allow us to identify the $m \times n$ components of the matrix representing the linear transformation λ (in the standard basis). These are the partial derivatives of f, and the matrix of λ is the Jacobian matrix of f evaluated at x.

Example: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. By choosing $\lambda = T$, we see that T is differentiable everywhere, with $DT(x) = T$ for every choice of $x \in \mathbb{R}^n$. This is made obvious by the fact that the best linear approximation of a linear map at some point is the map itself; indeed, the 'approximation' is exact.

Lemma 1.1: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, with derivative $Df(x)$, then
1. f is continuous at x.
2. The linear transformation $Df(x)$ is unique.

Proof: We prove the second part. Suppose that λ, μ satisfy the requirements for $Df(x)$; it can be shown that $\lim_{h\to 0}(\lambda - \mu)h \,/\, \|h\| = 0$. Now, if $\lambda v \neq \mu v$ for some non-zero vector $v \in \mathbb{R}^n$, then

$$\lambda v - \mu v = \frac{\lambda(tv) - \mu(tv)}{\|tv\|} \cdot \|v\| \to 0,$$

a contradiction. □

2. Chain rule

Exercise I: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists $M > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\|T\boldsymbol{x}\| \leq M\|\boldsymbol{x}\|.$$

Solution: Set $\boldsymbol{v}_i = T(\boldsymbol{e}_i)$ where $\boldsymbol{e}_i$ are the standard unit basis vectors of $\mathbb{R}^n$. Then,

$$\|T\boldsymbol{x}\| = \left\|\sum_i x_i \boldsymbol{v}_i\right\| \leq \sum_i \|x_i \boldsymbol{v}_i\| \leq \max\|\boldsymbol{v}_i\| \sum_i |x_i|.$$

Since each $|x_i| \leq \|\boldsymbol{x}\|$, set $M = n \max\|\boldsymbol{v}_i\|$ and write

$$\|T\boldsymbol{x}\| \leq \max\|\boldsymbol{v}_i\| \sum_i |x_i| \leq \max\|\boldsymbol{v}_i\| \cdot n\|\boldsymbol{x}\| = M\|\boldsymbol{x}\|.$$

Theorem 2.1 (Chain Rule): Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^m \to \mathbb{R}^k$ where f is differentiable at $a \in \mathbb{R}^n$ and g is differentiable at $f(a) \in \mathbb{R}^m$. Then, $g \circ f$ is differentiable, with $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$. Note that this means that the Jacobian matrices simply multiply.

Proof: Set $b = f(a) \in \mathbb{R}^m$, $\lambda = Df(a)$, $\mu = Dg(f(a))$. Define

$$\begin{aligned} \varphi &: \mathbb{R}^n \to \mathbb{R}^m, \quad \varphi(x) = f(x) - f(a) - \lambda(x - a), \\ \psi &: \mathbb{R}^m \to \mathbb{R}^k, \quad \psi(y) = g(y) - g(b) - \mu(y - b). \end{aligned}$$

We claim that

$$\lim_{x \to a} \frac{g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x - a)}{\|x - a\|} = 0.$$

Write the numerator as

$$g \circ f(x) - g \circ f(a) - \mu \circ \lambda(x - a) = \psi(f(x)) + \mu(\varphi(x)).$$

Note that

$$\lim_{x \to a} \frac{\varphi(x)}{\|x - a\|} = 0, \quad \lim_{y \to b} \frac{\psi(y)}{\|y - b\|} = 0.$$

Thus, find $M > 0$ such that

$$\|\mu(\varphi(x))\| \leq \|\varphi(x)\|$$

for all $x \in \mathbb{R}^n$, hence

$$\lim_{x \to a} \frac{\|\mu(\varphi(x))\|}{\|x - a\|} \leq \lim_{\{x \to a\}} \frac{M\|\varphi(x)\|}{\|x - a\|} = 0.$$

Now write

$$\lim_{f(x) \to b} \frac{\psi(f(x))}{\|f(x) - b\|} = 0,$$

hence for any $\varepsilon > 0$, there is a neighbourhood of b on which

$$\|\psi(f(x))\| \leq \varepsilon\|f(x) - b\| = \varepsilon\|\varphi(x) + \lambda(x - a)\|.$$

Apply the triangle inequality and find $M' > 0$ such that

$$\|\psi(f(x))\| \leq \varepsilon\|\varphi(x)\| + \varepsilon M'\|x - a\|.$$

Thus,

$$\lim_{x \to a} \frac{\|\psi(f(x))\|}{\|x - a\|} \leq \lim_{x \to a} \frac{\varepsilon\|\varphi(x)\|}{\|x - a\|} + \varepsilon M' = \varepsilon M'.$$

Since $\varepsilon > 0$ was arbitrary, this limit is zero, completing the proof. □

3. Partial derivatives

Definition 3.1: Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$. The partial derivative of f with respect to the coordinate x_j at some $a \in U$ is defined by the following limit, if it exists.

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f\left(a + h e_j\right) - f(a)}{h}.$$

Lemma 3.1: If $f : U \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}^n$, then

$$Df(a)(x_1, ..., x_n) = x_1, \frac{\partial f}{\partial x_1}(a) + ... + x_n, \frac{\partial f}{\partial x_n}(a).$$

Example: Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} (xy) \,/\, (x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$; it is not even continuous there. However, both partial derivatives of f exist at $(0, 0)$.

Lemma 3.2: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then the matrix representation of $Df(a)$ in the standard basis is given by

$$[Df(a)] = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{ij}.$$

Lemma 3.3: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$, and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable at $f(a) \in \mathbb{R}^m$. Then, the matrix representation of $D(g \circ f)(a)$ in the standard basis is the product

$$[D(g \circ f)(a)] = [Dg(f(a))][Df(a)] = \left[\sum_{\ell=1}^{m} \frac{\partial g_i}{\partial y_\ell} \frac{\partial f_\ell}{\partial x_j}\right]_{ij}.$$

In other words,

$$\frac{\partial}{\partial x_j}(g \circ f)_i(a) = \sum_{\ell=1}^{m} \frac{\partial g_i}{\partial y_\ell}(f(a)) \frac{\partial f_\ell}{\partial x_j}(a).$$

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and let $\Gamma(f) = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\}$ be the graph of f. Now, let $\gamma : [-1, 1] \to \Gamma(f)$ be a differentiable curve, represented by

$$\gamma(t) = (g(t), h(t), f(g(t), h(t))).$$

Then, we can compute the derivative

$$\gamma'(a) = \left(g'(a), h'(a), g'(a) \frac{\partial f}{\partial x} + h'(a) \frac{\partial f}{\partial y} \Big|_{(g(a), h(a))} \right)$$

Exercise II: Consider the inner product map, $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. What is its derivative?

Solution: We treat the inner product as a map $g : \mathbb{R}^{2n} \to \mathbb{R}$, which acts as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle := g\left(x_1, ..., x_n, y_1, ..., y_n\right) = x_1 y_1 + ... + x_n y_n.$$

Now, note that

$$\frac{\partial g}{\partial x_i} = y_i, \quad \frac{\partial g}{\partial y_i} = x_i.$$

Thus,

$$\begin{aligned} Dg(\boldsymbol{a}, \boldsymbol{b})(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{i=1}^{n} x_i \frac{\partial g}{\partial x_i}(\boldsymbol{a}, \boldsymbol{b}) + \sum_{i=1}^{n} y_i \frac{\partial g}{\partial y_i}(\boldsymbol{a}, \boldsymbol{b}) \\ &= \sum_{i=1}^{n} x_i b_i + \sum_{i=1}^{n} y_i a_i \\ &= \langle \boldsymbol{x}, \boldsymbol{b} \rangle + \langle \boldsymbol{y}, \boldsymbol{a} \rangle. \end{aligned}$$

In other words, the matrix representation of the derivative of the inner product map at the point $(\boldsymbol{a}, \boldsymbol{b})$ is given by $\left[\boldsymbol{b}^\top \boldsymbol{a}^\top\right]$.

Exercise III: Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a differentiable curve. What is the derivative of the real map $t \mapsto \|\gamma(t)\|^2$?

Solution: We write this map as $t \mapsto \langle \gamma(t), \gamma(t) \rangle$. Consider the scheme

$$\mathbb{R} \to \mathbb{R}^{2n} \to \mathbb{R}, \quad t \mapsto \begin{pmatrix} \gamma(t) \\ \gamma(t) \end{pmatrix} \mapsto \langle \gamma(t), \gamma(t) \rangle.$$

Pick a point $t \in \mathbb{R}$, whence the derivative of the map at t is

$$\begin{pmatrix} \gamma(t)^\top & \gamma(t)^\top \end{pmatrix} \begin{pmatrix} \gamma'(t) \\ \gamma'(t) \end{pmatrix} = 2\langle \gamma(t), \gamma'(t) \rangle.$$

Remark: Consider the surface $S^{n-1} \subset \mathbb{R}^n$, and pick an arbitrary differentiable curve $\gamma : \mathbb{R} \to S^{n-1}$. Now, the tangent vector $\gamma'(t)$ is tangent to the sphere S^{n-1} at any point $\gamma(t)$. We claim that this tangent drawn at $\gamma(t)$ is always perpendicular to the position vector $\gamma(t)$. This is made trivial by our exercise: the map $t \mapsto \|\gamma(t)\|^2 = 1$ is a constant map since γ is a curve on the unit sphere. This means that it has zero derivative, forcing $\langle \gamma(t), \gamma'(t) \rangle = 0$.

3.1. Directional derivatives

Definition 3.1.1: Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$. The directional derivative of f along a direction $\boldsymbol{v} \in \mathbb{R}^n$ at a point $a \in U$ is defined by the following limit, if it exists.

$$\nabla_v f(a) = \lim_{h \to 0} \frac{f(a + h\boldsymbol{v}) - f(a)}{h}.$$

Example: Consider

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto \begin{cases} x^3 \,/\, (x^2 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$. However, all directional derivatives derivatives of f exist at $(0, 0)$. Indeed, consider a direction $(\cos\theta, \sin\theta)$, and examine the limit

$$\lim_{t \to 0} \frac{1}{t}[f(t\cos\theta, t\sin\theta) - f(0, 0)] = \cos^3\theta.$$

Definition 3.1.2: Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. The gradient of f is defined as the map

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \left[\frac{\partial f}{\partial x_i}(x)\right]_i.$$

Remark: The gradient at a point $x \in \mathbb{R}^n$ is thought of as a vector. In contrast, the derivative is thought of as a linear transformation. Otherwise, we see that $\nabla f(x) = [Df(x)]$.

Definition 3.1.3: Let $C^1(\mathbb{R}^n)$ be the set of real-valued differentiable functions on $\mathbb{R}^n$. Fix a point $a \in \mathbb{R}^n$, then fix a tangent vector $v \in \mathbb{R}^n$. Then, the map

$$\nabla_v : C^1(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto Df(a)(v)$$

is a linear functional. The quantity $\nabla_v f$ is called the directional derivative of f in the direction v at the point a.

Remark: We can represent ∇_v as the operator

$$\nabla_v(\cdot) = D(\cdot)(a)(v) = \sum_i v_i \frac{\partial}{\partial x_i}\bigg|_a = v \cdot \nabla(\cdot).$$

Lemma 3.1.1: The directional derivatives ∇_v form a vector space called the tangent space, attached to the point $a \in \mathbb{R}^n$. This can be identified with the vector space $\mathbb{R}^n$ by the natural map $\nabla_v \mapsto v$. The standard basis can be informally denoted by the vectors

$$\nabla_{e_1} := \frac{\partial}{\partial x_1}, \quad \dots \quad, \nabla_{e_n} := \frac{\partial}{\partial x_n}.$$

3.2. Differentiation on manifolds *

Definition 3.2.1: A homeomorphism is a continuous, bijective map whose inverse is also continuous.

Lemma 3.2.1: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Denote the graph of f as

$$\Gamma(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}.$$

Then, $\Gamma(f)$ is a smooth manifold.

Proof: Consider the homeomorphism

$$\varphi : \Gamma(f) \to \mathbb{R}^n, \quad (x, f(x)) \mapsto x.$$

This is clearly bijective, continuous (restriction of a projection map), with a continuous inverse (from the continuity of f). Call this homeomorphism φ a coordinate map on $\Gamma(f)$. □

Definition 3.2.2: Let $f : M \to \mathbb{R}$ where M is a smooth manifold, with a coordinate map $\varphi : M \to \mathbb{R}^n$. We say that f is differentiable at a point $a \in M$ if $f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(a)$.

Definition 3.2.3: Let $f : M \to \mathbb{R}$ where M is a smooth manifold, let $\varphi : M \to \mathbb{R}^n$ be a coordinate map, and let $a \in M$. Let $\gamma : \mathbb{R} \to M$ be a curve such that $\gamma(0) = a$, and further let γ be differentiable in the sense that $\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$ is differentiable. The directional derivative of f at a along γ is defined as

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \lim_{h \to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}\Bigg|_{t=0}.$$

Note that we are taking the derivative of $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ in the conventional sense.

Lemma 3.2.2: Let γ_1 and γ_2 be two curves in M such that $\gamma_1(0) = \gamma_2(0) = a$, and

$$\frac{d}{dt} \varphi \circ \gamma_1(t)\Big|_{t=0} = \frac{d}{dt} \varphi \circ \gamma_2(t)\Big|_{t=0}.$$

In other words, γ_1 and γ_2 pass through the same point a at $t = 0$, and have the same velocities there. Then, the directional derivatives of f at a along γ_1 and γ_2 are the same.

Definition 3.2.4: Let M be a smooth manifold, and let $a \in M$. Consider the following equivalence relation on the set of all curves γ in M such that $\gamma(0) = a$.

$$\gamma_1 \sim \gamma_2 \quad \Longrightarrow \quad \frac{d}{dt}\varphi \circ \gamma_1(t)\Big|_{t=0} = \frac{d}{dt}\varphi \circ \gamma_2(t)\Big|_{t=0}.$$

Each resultant equivalence class of curves is called a tangent vector at $a \in M$. Note that all these curves in a particular equivalence class pass through a with the same velocity vector.

The collection of all such tangent vectors, i.e. the space of all curves through a modulo the equivalence relation which identifies curves with the same velocity vector through a, is called the tangent space to M at a, denoted $T_a M$.

Remark: Each tangent vector $v \in T_a M$ acts on a differentiable function $f : M \to \mathbb{R}$ yielding a (well-defined) directional derivative at a.

$$v : C^1(M) \to \mathbb{R}, \quad f \mapsto \frac{d}{dt} f\big(\gamma_v(t)\big)\Big|_{t=0}.$$

Thus, the tangent space represents all the directions in which taking a derivative of f makes sense.

Remark: The tangent space $T_a M$ is a vector space. Upon fixing f, the map $Df(a) : T_a M \to \mathbb{R}$, $v \mapsto vf(a)$ is a linear functional on the tangent space.

Remark: Given a tangent vector $v \in T_a M$, it can be identified with its corresponding velocity vector in $\mathbb{R}^n$. Thus, the tangent space $T_a M$ can be identified with the geometric tangent plane drawn to the manifold M at the point a.

4. Mean value theorem

Consider a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, and fix $a \in \mathbb{R}^n$. Define the functions

$$g_i : \mathbb{R} \to \mathbb{R}, \quad g_i(x) = f(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n).$$

Then, each g_i is differentiable, with

$$g_i{}'(x) = \frac{\partial f}{\partial x_i}(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n).$$

By applying the Mean Value Theorem on some interval $[c, d]$, we can find $\alpha \in (c, d)$ such that $g_i(d) - g_i(c) = g_i{}'(\alpha)(d - c)$. In other words,

$$f(..., d, ...) - f(..., c, ...) = \frac{\partial f}{\partial x_i}(..., \alpha, ...)(d - c).$$

Theorem 4.1: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$. Then, f is differentiable at a if all the partial derivatives $\partial f \,/\, \partial x_j$ exist in a neighbourhood of a and are continuous at a.

Proof: Without loss of generality, let $m = 1$. We claim that

$$\lim_{h \to 0} \frac{1}{\|h\|} \left\| f(a+h) - f(a) - \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a) h_i \right\| = 0.$$

Examine

$$\begin{aligned} f(a+h) - f(a) &= f(a_1 + h_1, ..., a_n + h_n) - f(a_1, ..., a_n) \\ &= f(a_1 + h_1, ..., a_n + h_n) - f(a_1 + h_1, ..., a_{n-1} + h_{n-1}, a_n) + \\ &\quad f(a_1 + h_1, ..., a_{n-1} + h_{n-1}, a_n) - f(a_1 + h_1, ..., a_{n-1}, a_n) + \\ &\quad ... \\ &\quad f(a_1 + h_1, a_2, ..., a_n) - f(a_1, ..., a_n) \\ &= \frac{\partial f}{\partial x_n}(c_n) h_n + ... + \frac{\partial f}{\partial x_1}(c_1) h_1. \end{aligned}$$

The last step follows from the Mean Value Theorem. As $h \to 0$, each $c_i \to a$. Thus,

$$\begin{aligned} \frac{1}{\|h\|} \left\| f(a+h) - f(a) - \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a) h_i \right\| &= \frac{1}{\|h\|} \left\| \sum_{i=0}^{n} \left(\frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right) h_i \right\| \\ &\leq \sum_{i=0}^{n} \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right| \frac{|h_i|}{\|h\|} \\ &\leq \sum_{i=0}^{n} \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right|. \end{aligned}$$

Taking the limit $h \to 0$, observe that $(\partial f / \partial x_i)(c_i) \to (\partial f / \partial x_i)(a)$ by the continuity of the partial derivatives, completing the proof. □

Corollary 4.1.1: All polynomial functions on $\mathbb{R}^n$ are differentiable.

Theorem 4.2: Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable with continuous partial derivatives, and let $a \in \mathbb{R}^n$ be a point of local maximum. Then, $Df(a) = 0$.

Proof: We need only show that each

$$\frac{\partial f}{\partial x_i}(a) = 0.$$

This must be true, since a is also a local maximum of each of the restrictions g_i as defined earlier.
□

5. Inverse and implicit function theorems

Theorem 5.1 (Inverse function theorem): Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on a neighbourhood of $a \in \mathbb{R}^n$, and let $\det\left(Df(a)\right) \neq 0$. Then, there exist neighbourhoods U of a and W of $f(a)$ such that the restriction $f : U \to W$ is invertible. Furthermore, f^{-1} is continuous on U and differentiable on U.

Lemma 5.2: Consider a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, and let M denote the surface defined by the zero set of f. Then, M can be represented as the graph of a differentiable function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ at those points where $Df \neq 0$.

Proof: Without loss of generality, suppose that $\partial f \,/\, \partial x_n \neq 0$ at some point $a \in M$. It can be shown that the map

$$F : \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto (x_1, x_2, ..., x_{n-1}, f(x))$$

is invertible in a neighbourhood W of a, with a continuous and differentiable inverse of the form

$$G : \mathbb{R}^n \to \mathbb{R}^n, \quad u \mapsto (u_1, u_2, ..., u_{n-1}, g(u)).$$

Since $F \circ G$ must be the identity map on W, we demand

$$(x_1, x_2, ..., x_{n-1}, f(x_1, x_2, ..., x_{n-1}, g(x))) = (x_1, x_2, ..., x_{n-1}, x_n).$$

Thus, the zero set of f in this neighbourhood of a satisfies $x_n = 0$, hence

$$f(x_1, x_2, ..., x_{n-1}, g(x_1, x_2, ..., x_{n-1}, 0)) = 0.$$

In other words, the part of the surface M in the neighbourhood of a is precisely the set of points

$$(x_1, x_2, ..., x_{n-1}, g(x_1, x_2, ..., x_{n-1}, 0)).$$

Simply set

$$h : \mathbb{R}^{n-1} \to \mathbb{R}, \quad x \mapsto g(x_1, x_2, ..., x_{n-1}, 0),$$

whence the surface M is locally represented by the graph of h. $\square$

Remark: Note that by using

$$f(x_1, ..., x_{n-1}, h(x_1, ..., x_{n-1})) = 0$$

on the surface, we can use the chain rule to conclude that for all $1 \leq i < n$, we have

$$\frac{\partial f}{\partial x_i}(a) + \frac{\partial f}{\partial x_n}(a) \frac{\partial h}{\partial x_i}(a_1, ..., a_{n-1}) = 0.$$

Theorem 5.3 (Implicit function theorem): Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable in an open set containing (a, b), with $f(a, b) = 0$. Let $\det\left(\partial f^j \,/\, \partial x_{n+k}(a, b)\right) \neq 0$. Then, there exists an open set $U \subset \mathbb{R}^n$ containing a, an open set $V \subset \mathbb{R}^m$ containing b, and a differentiable function $g : U \to V$ such that $f(x, g(x)) = 0$.

Remark: The condition on the determinant can be rephrased as rank $Df(a, b) = m$.

Theorem 5.4: Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and let M be the surface defined by its zero set. Furthermore, let $\nabla f(a) \neq 0$ for some $a \in M$; thus, M can be locally represented by a graph on $\mathbb{R}^{n-1}$. Then, $\nabla f(a)$ is normal to the tangent vectors drawn at a to M; in fact, the perpendicular space of $\nabla f(a)$ is precisely the tangent space $T_a M$.

Proof: Consider a tangent vector drawn at a to M, represented by the differentiable curve $\gamma : \mathbb{R} \to M$, $\gamma(0) = a$; note that we use the identification $\gamma'(0) = v \in \mathbb{R}^n$. Then, calculate

$$\frac{d}{dt} f(\gamma(t))\big|_{t=0} = Df(\gamma(0))(\gamma'(0)) = Df(a)(v).$$

On the other hand, we have $f(\gamma(t)) = 0$ identically. Thus,

$$v \cdot \nabla f(a) = Df(a)(v) = 0$$

as claimed. □

6. Taylor's theorem

Theorem 6.1 (Clairaut): Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives. Then,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Theorem 6.2 (Taylor): Let $f : \mathbb{R}^2 \to \mathbb{R}$ have continuous second order partial derivatives, and let $\left(x_0, y_0\right) \in \mathbb{R}^2$. Then, there exists $\varepsilon > 0$ such that for all $\left\|\left(x - x_0, y - y_0\right)\right\| < \varepsilon$,

$$\begin{aligned} f(x,y) = f\left(x_0, y_0\right) &+ \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}\left(y - y_0\right) \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\left(y - y_0\right)^2 \\ &+ \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - x_0) + R(x,y), \end{aligned}$$

where as $(x,y) \to \left(x_0, y_0\right)$, the remainder term vanishes as

$$\frac{|R(x,y)|}{\left\|\left(x - x_0, y - y_0\right)\right\|^2} \to 0.$$

All partial derivatives here are evaluated at $\left(x_0, y_0\right)$.

Proof: This follows from applying the Taylor's Theorem in one variable to the real function $g : \mathbb{R} \to \mathbb{R}, t \mapsto f\left((1-t)\left(x_0, y_0\right) + t(x,y)\right)$. □

7. Critical points and extrema

Definition 7.1: We say that $a \in \mathbb{R}^n$ is a critical point of $f : \mathbb{R}^n \to \mathbb{R}$ if all $\partial f \,/\, \partial x^j = 0$ there.

Lemma 7.1: All points of extrema of a differentiable function are critical points.

Proof: We already know that $Df(a) = 0$ where a is either a point of maximum or minimum. □

Example: In order to find a point of extrema of a C^2-smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, we first identify a critical point $\left(x_0, y_0\right)$. Next, we must find a neighbourhood of $\left(x_0, y_0\right)$ which contains no other critical points – to do this, apply Taylor's Theorem. Indeed, we see that

$$f(x,y) = f\left(x_0, y_0\right) + A(x - x_0)^2 + 2B(x - x_0)\left(y - y_0\right) + C\left(y - y_0\right)^2 + R_2.$$

For non-degeneracy of solutions, we demand $AC - B^2 \neq 0$, i.e. at $\left(x_0, y_0\right)$, we want

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 \neq \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}.$$

If $AC - B^2 > 0$ and $\partial^2 f \,/\, \partial x^2 > 0$, then we have found a point of minima; if $\partial^2 f \,/\, \partial x^2 < 0$, then we have found a point of maximum. If $AC - B^2 < 0$, then we have found a saddle point.

Example: Suppose that we wish to maximize the function $f : \mathbb{R}^2 \to \mathbb{R}$, given an equation of constraint $g = 0$, where $g : \mathbb{R}^2 \to \mathbb{R}$. Using the method of Lagrange multipliers, we look for solutions of the system

$$\begin{cases} \nabla f(x, y) + \lambda \nabla g(x, y) = 0 \\ g(x, y) = 0. \end{cases}$$