Some Algebraic Topology

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1 The Fundamental Group and Covering Space

Theorem 1.1 (van Kampen). *Let $X = \bigcup_\alpha A_\alpha$ where A_α are path-connected open sets with a basepoint x_0. Let all $A_\alpha \cap A_\beta$ are path-connected, then consider*

$$\begin{tikzcd}
\pi_1(A_\alpha \cap A_\beta) \arrow[r, "i_{\alpha\beta}"] \arrow[d, "i_{\beta\alpha}"'] & \pi_1(A_\alpha) \arrow[d, "j_\alpha"] \\
\pi_1(A_\beta) \arrow[r, "j_\beta"'] & \pi_1(X)
\end{tikzcd}$$

where all maps induced by inclusions. Then j_α induce $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected, then $\ker \Phi$ is a normal subgroup generated by all elements of form $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$.

Remark 1.1. *In the case of two open sets U, V with $U \cap V$ path-connected, we have the following.*

In the category of groups $\mathfrak{Grp}$, we can describe pushout of $f : G \to A$ and $g : G \to B$. We let $A *_G B$ as $A * B/(f(a)g(a)^{-1})_{a\in G}$, then we have the following universal property in $\mathfrak{Grp}$:

$$\begin{tikzcd}
G \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d] \arrow[ddr] & \\
A \arrow[r] \arrow[drr] & A *_G B \arrow[dr, dashed, "\exists!"] & \\
 & & H
\end{tikzcd}$$

We call it the amalgamated product of A and B with amalgam G. So in the van Kampen theorem with U, V, we have

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V).$$

2 Homology

2.1 Singular Homology

Theorem 2.1 (Excision Theorem). *Let $Z \subset A \subset X$ where $\mathrm{cl}(Z) \subset \mathrm{int}(A)$, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induce $H_n(X - Z, A - Z) \cong H_n(X, A)$.*

If now we let $B = X - Z$ we have $H_n(B, A \cap B) \cong H_n(X, A)$.

Proposition 2.1. *For good pairs (X, A), map $q : (X, A) \to (X/A, A/A)$ induce $q_* : H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$.*

Proof. Let V be the open set deformation retracts into A, consider

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "f"] \arrow[d, "q_*"'] & H_n(X, V) \arrow[d, "q_*"'] & H_n(X - A, V - A) \arrow[l, "g"'] \arrow[d, "q_*"'] \\
H_n(X/A, A/A) \arrow[r, "u"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "v"']
\end{tikzcd}$$

f, u are isomorphisms by the long exact sequences of triples (X, V, A) and $(X/A, V/A, A/A)$. And g, v are isomorphisms directly by excision. The right hand q_* is isomorphism. So is the left. $\square$

2.2 Cellular Homology

Theorem 2.2 (Hairly Ball)**.** *S^n has a continuous field of nonzero tangent vectors iff n is odd.*

Proof. Consider such vector field $v(x)$ and view it as centering at origin. Let $|v(x)| = 1$ via $v(x)/|v(x)|$. Consider $f_t(x) = (\cos t)x + (\sin t)v(x)$. Then $\deg(-\mathrm{id}) = \deg(\mathrm{id}) = 1$, so $(-1)^{n+1} = 1$, so n is odd.

Conversely if $n = 2k-1$, then let $v(x_1, ..., x_{2k}) = (-x_2, -x_1, ..., -x_{2k}, -x_{2k-1})$. $\square$

Now we consider CW complex X with k-skeleton X_k. We have the following elementary conclusion:

Lemma 2.1. *(a) $H^k(X_n, X_{n-1})$ is zero when $k \neq n$ and free abelian with basis of n-cells of X when $k = n$;*
(b) $H_k(X^n) = 0$ for $k > n$;
(c) Inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \cong H_k(X)$ for $k < n$.

$$\begin{tikzcd}
 & & & & 0 \\
 & & 0 \arrow[d] & H_n(X^{n+1}) = H_n(X) \arrow[ur] & \\
 & & H_n(X^n) \arrow[ur] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[r, "d_{n+1}"] \arrow[ur, "\partial_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

Theorem 2.3 (Cellular Boundary Formula)**.** *The map d_n in above diagram we have $d_n(e^n_\alpha) = \sum_\beta \deg(S^{n-1}_\alpha = \partial e^n_\alpha \to X^{n-1} \to S^{n-1}_\beta) e^{n-1}_\beta$ where the map is the attaching map of e^n_α with the quotient map collapsing $X^{n-1} - e^{n-1}_\beta$ to a point.*

2.3 Mayer-Vietoris

Theorem 2.4 (Mayer-Vietoris Sequence)**.** *Let $A, B \subset X$ with $X = \mathrm{int}(A) \cap \mathrm{int}(B)$. Then we have*

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A+B) \longrightarrow 0$$

Then induce the long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A \cap B) \arrow[r, "(i_{1*}, -i_{2*})"] & H_n(A) \oplus H_n(B) \arrow[r, "g_* + j_*"] & H_n(X) \arrow[d, "\partial"] \\
 & & \cdots & H_{n-1}(A \cap B) \arrow[l]
\end{tikzcd}$$

where $i_1 : A \cap B \to A, i_2 : A \cap B \to B$ *and* $g : A \to X, j : B \to X$.

Theorem 2.5 (Mapping Torus and Mayer-Vietoris Sequence)**.** *Let* $f, g : X \to Y$ *and let* $Z = X \times I/((x,0) \sim f(x), (x,1) \sim g(x))$ *be the mapping torus, then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(X) & \xrightarrow{f_* - g_*} & H_n(Y) & \xrightarrow{i_*} & H_n(Z) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H_{n-1}(X) \end{array}$$

More special case, we let $f : A \cap B \to A, g : A \cap B \to B$*, then we can get the traditional Mayer-Vietoris sequence.*

Theorem 2.6 (Relative Mayer-Vietoris Sequence)**.** *Let* $(X, Y) = (A \cup B, C \cup D)$ *with* $C \subset A, D \subset B$*. Then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A \cap B, C \cap D) & \longrightarrow & H_n(A, C) \oplus H_n(B, D) & \longrightarrow & H_n(X, Y) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H_{n-1}(A \cap B, C \cap D) \end{array}$$

derived by nine lemma and long exact sequence.

2.4 More Applications

2.4.1 Embedding and Homology

Theorem 2.7 (Invariance of Domain)**.** *If* U *is open in* $\mathbb{R}^n$*, then for any embedding (homeomorphic to image)* $h : U \to \mathbb{R}^n$ *the image* $h(U)$ *is open in* $\mathbb{R}^n$*.*

Proof. We will work on S^n. We just need to show that $h(D^n - \partial D^n)$ is open in S^n. Omitted. □

2.4.2 Borsuk-Ulam Type Theorem

Theorem 2.8 (Borsuk)**.** *A map* $f : S^n \to S^n$ *with* $f(-x) = -f(x)$ *must have odd degree.*

Corollary 2.1 (Borsuk-Ulam)**.** *Every map* $g : S^n \to \mathbb{R}^n$*, there exists a point* $x \in S^n$ *with* $g(x) = g(-x)$*.*

Corollary 2.2. *Whenever* S^n *is expressed as the union of* $n + 1$ *closed sets* $A_0, ..., A_n$*, then at least one of these sets must contain a pair of antipodal points.*

Proof. We define $d_i : S^n \to \mathbb{R}, x \mapsto \inf_{y \in A_i} |x - y|$. Let $g : S^n \to \mathbb{R}^n, x \mapsto (d_1(x), ..., d_n(x))$. By Borsuk-Ulam theorem, it obtaining a pair of antipodal points $x, -x$ with $d_i(x) = d_i(-x), i = 1, ..., n$. If either of these distances is 0, then well done. If not, $x, -x \in A_0$, well done. □

2.4.3 The Lefschetz Fixed Point Theorem

Theorem 2.9 (Lefschetz)**.** *If* X *is a finite simplicial complex, or more generallya retract of a finite simplicial complex and* $f : X \to X$ *is a map with* $\tau(f) = \sum_n (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X)) \neq 0$*, then* f *has a fixed point.*

3 Cohomology

3.1 Orientations

Theorem 3.1. *Let M be a closed connected n-manifold. Then*

(a) If M is R-orientable, then the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is an isomorphism for all $x \in M$;

(b) If M is not R-orientable, then the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is injective for all $x \in M$ with image $\{r :\in R : 2r = 0\}$.

By the isomorphism $H_n(M;R) \to H_n(M|x;R) \cong R$, the element in $H_n(M;R)$ is called fundamental class if its image in any $H_n(M|x;R) \cong R$ is a generator.

Theorem 3.2. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then for any section $(x \mapsto \alpha_x) \in \Gamma(M, M_R)$ there exists a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$. Moreover, $H_i(M|A;R) = 0, i > n$.*

Sketch of the Proof. Our method is to reduce the case in to simple one.

(i) **If this hold for $A, B, A \cap B$, then this is also hold of $A \cup B$.** Use the MV-principle, we have:

$$0 = H_{n+1}(M|A \cap B) \longrightarrow H_n(M|A \cup B) \longrightarrow H_n(M|A) \oplus H_n(M|B) \longrightarrow H_n(M|A \cap B)$$

then this is easy to see;

(ii) **Reduce to the case $M = \mathbb{R}^n$.** Actually we can let $A = \bigcup_{i=1}^m A_i$ where A_i in some $\mathbb{R}^n$. Then use MV-principle and induction, well done;

(iii) **Consider the case $M = \mathbb{R}^n$ and $A = \bigcup_{i=1}^m A_i$ where A_i is convex.** Use the MV-principle as (ii) we can let A is convex. Then the result is trivial by $H_i(\mathbb{R}^n|A) \cong H_i(\mathbb{R}^n|x)$ naturally;

(iv) **Consider the case $M = \mathbb{R}^n$ and A be any compact.** Let $\alpha \in H_i(\mathbb{R}^n|A)$ represented by z and let $C \subset \mathbb{R}^n - A$ be the union of the images of the singular simplices in ∂z. Then one can cover some closed balls over A outside of C. Let K be the union of these balls and we see that the relative cycle z defines an element $\alpha_K \in H_i(\mathbb{R}^n|K)$ mapping to the given $\alpha \in H_i(\mathbb{R}^n|A)$. Use (iii) to $H_i(\mathbb{R}^n|K)$, well done. $\square$

Corollary 3.1. *Let M be a closed connected n-manifold. The torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ if M is nonorientable.*

Proof. If M is orientable and if $H_{n-1}(M;\mathbb{Z})$ contained torsion, then by universal coefficient, we have

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow H_n(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(M), \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

Then $H_n(M;\mathbb{Z}/2\mathbb{Z})$ is bigger than $\mathbb{Z}/2\mathbb{Z}$ which is impossible.

If M is nonorientable, we let $H_{n-1}(M) = F \oplus \bigoplus_j \mathbb{Z}/p_j\mathbb{Z}$, then we have

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & H_n(M;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & \bigoplus_j \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_j\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\
 & & & \bigoplus_j \frac{p_j\mathbb{Z} \cap 2\mathbb{Z}}{2p_j\mathbb{Z}} &
\end{tikzcd}$$

then we have $H_{n-1}(M) = \mathbb{Z}/2\mathbb{Z}$. $\square$

Proposition 3.1. *If M is a connected noncompact n-manifold, then $H_i(M;R)=0$ for all $i \geq n$.*

Proof. Let z be a cycle represent an element of $H_i(M;R)$. It has a compact image and we let U be an open set cover it with compact closure. Let $V = M - \mathrm{cl}(U)$ and consider $(M, U \cup V, V)$ we have

$$\begin{tikzcd}
0 = H_{i+1}(M, U \cup V; R) \arrow[r] & H_i(U \cup V, V; R) \arrow[r] & H_i(M, V; R) = 0 \\
 & H_i(U;R) \arrow[u, "\cong"'] \arrow[r] & H_i(M;R) \arrow[u]
\end{tikzcd}$$

When $i > n$ we have $H_i(U;R) = 0$ so z is a boundary in U and so in M, so $H_i(M;R) = 0$.

When $i = n$, class $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ of M_R. This section determined by the value in single point since M is connected. Also consider

$$\begin{tikzcd}
0 = H_{n+1}(M, U \cup V; R) \arrow[r] & H_n(U \cup V, V; R) \arrow[r] & H_n(M, V; R) \\
 & H_n(U;R) \arrow[u, "\cong"'] \arrow[r] & H_n(M;R) \arrow[u]
\end{tikzcd}$$

Then since M is noncompact and z has a compact image, there must have some point x such that $[z]_x = 0$, so $[z]_x = 0$ for all $x \in M$. Then $[z] = 0$ in $H_n(M, V; R)$, so is in $H_n(U;R)$ and then in $H_n(M;R)$. We win. ☐

3.2 Cap product and the Duality Theorem

First consider cohomology with compact supports.

Definition 1. *Let $C_c^i(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\phi : C^i(X) \to G$ for which there exists a compact set $K = K_\phi \subset X$ such that ϕ is zero on all chains in $X - K$. Note that $\delta\phi$ is then also zero on chains in $X - K$, so $\delta\phi$ lies in $C_c^{i+1}(X;G)$ and the $C_c^i(X;G)$'s for varying i form a subcomplex of the singular cochain complex of X . The cohomology groups $H_c^i(X;G)$ of this subcomplex are the cohomology groups with compact supports.*

Another way we let compact $K \hookrightarrow L$ induce $(X, X-L) \hookrightarrow (X, X-K)$, then we have $C^i(X, X-K; G) \hookrightarrow C^i(X, X-L; G)$ and $H^i(X, X-K; G) \to H^i(X, X-L; G)$.

Proposition 3.2. *Since $K \subset X$ are compact sets form a direct system via inclusions. Then we have*

$$\varinjlim H^i(X, X-K; G) \cong H_c^i(X;G).$$

Theorem 3.3 (Poincaré Duality)**.** *Let M be a R-oriented n-manifold. First we define a map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$. Consider compact sets $K \subset L \subset M$, we have*

$$\begin{tikzcd}
H_n(M|L;R) \arrow[d, "i_*"'] \arrow[r, dash, "\times"] & H^K(M|L;R) \arrow[r, "\frown"] & H_{n-k}(M;R) \\
H_n(M|K;R) \arrow[r, dash, "\times"] & H^k(M|K;R) \arrow[u, "i^*"] \arrow[ur, "\frown"'] &
\end{tikzcd}$$

By previous theorem we can find unique elements $\mu_K \in H_n(M|K;R), \mu_L \in H_n(M|L;R)$ restricting to a given orientation of M at each point of K and L, respectively.

So we have $i_*(\mu_L) = \mu_K$ and $\mu_K \frown x = i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$ for all $x \in H^k(M|K;R)$. So when K vary, we also have $H^k(M|K;R) \xrightarrow{\mu_K \frown (-)} H_{n-k}(M;R)$ which induce

$$D_M : H_c^k(M;R) = \varinjlim H^i(X|K;G) \cong H_{n-k}(M;R).$$

Remark 3.1. *When M is a closed R-oriented n-manifold, if $[M]$ is the fundamental class, we have isomorphism*

$$D_M : H^k(M;R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M;R).$$

Proposition 3.3. *A closed manifold of odd dimension has Euler characteristic zero.*

Proof. If M is orientable, then $\mathrm{rank}(H_i(M;\mathbb{Z})) = \mathrm{rank}(H^{n-i}(M;\mathbb{Z})) = \mathrm{rank}(H_{n-i}(M;\mathbb{Z}))$ by Poincaré duality and universal coefficient theorem. If n is odd, well done.

If M is not orientable, the similar argument we have $\sum_i(-1)^i \dim H_i(M;\mathbb{Z}/2\mathbb{Z}) = 0$. Now we claim that $\sum_i(-1)^i \dim H_i(M;\mathbb{Z}/2\mathbb{Z}) = \sum_i(-1)^i \mathrm{rank}(H_i(M;\mathbb{Z}))$. Each $\mathbb{Z}$ summand of $H_i(M;\mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summand of $H_i(M;\mathbb{Z}/2\mathbb{Z})$; each $\mathbb{Z}/m\mathbb{Z}$ (where m even) of $H_i(M;\mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summands of $H_i(M;\mathbb{Z}/2\mathbb{Z})$ and $H_{i+1}(M;\mathbb{Z}/2\mathbb{Z})$ which canceled; each $\mathbb{Z}/m\mathbb{Z}$ (where m odd) of $H_i(M;\mathbb{Z})$ contribute nothing. Well done. □

3.3 Connection with Cup Product

By some calculation we have

$$\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$$

where $\alpha \in C_{k+l}(X;R), \phi \in C^k(X;R), \psi \in C^l(X;R)$. So we have

$$\begin{array}{ccc} H^l(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(X;R),R) \\ {\scriptstyle \phi\smile}\downarrow & & \downarrow{\scriptstyle (\frown\phi)^*} \\ H^{k+l}(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(X;R),R) \end{array}$$

For closed R-orientable n-manifold M, consider an important pair:

$$\begin{array}{ccc} H^k(M;R) \times H^{n-k}(M;R) & \longrightarrow & R \\ (\phi,\psi) & \longmapsto & (\phi \smile \psi)[M] \end{array}$$

Proposition 3.4. *This pair is nonsingular for closed R-orientable manifolds when R is a field or when $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.*

Corollary 3.2. *If M is a connected closed R-orientable n-manifold, then for each element $\alpha \in H^k(M;\mathbb{Z})$ of infinite order that is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. With coefficients in a field the same conclusion holds for any $\alpha \neq 0$.*

Example 3.1 (Cohomology Ring of Projective Space)**.** *We will show that $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), |\alpha| = 2$. Similar we have $H^*(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}), |\alpha| = 1$. Inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induce the same cohomology group of degree less than $2n-2$, so by induction on n we have $H^{2i}(\mathbb{C}P^n;\mathbb{Z})$ is generated by α^i for $i < n$. By the corollary we can find $m\alpha^{i-1}$ such that $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n;\mathbb{Z})$, so $m = \pm 1$, well done.*

3.4 Other Duality

Example 3.2 (Euler Charactristic of Boundaries)**.** *Let W be a compact $(2m+1)$-dimensional manifold and $M = \partial W$, then $\chi(M) = 2\chi(W)$.*

Proof. Consider $W \times I$ as a $(2m+2)$-manifold with $\partial(W \times I) = (W \times \{0\}) \cup (M \times I) \cup (W \times \{1\})$. Let $U = \partial(W \times I) - (W \times \{1\})$ and $V = \partial(W \times I) - (W \times \{0\})$. Then U, V are open in $\partial(W \times I)$. Both U, V are open in $\partial(W \times I)$. Moreover $U, V \simeq W, U \cap V \simeq M$. So by MV sequence

$$\begin{array}{ccccccc} H_{i+1}(U \cup V) & \longrightarrow & H_i(U \cap V) & \longrightarrow & H_i(U) \oplus H_i(V) & \longrightarrow & H_i(U \cup V) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{i+1}(\partial(W \times I)) & \longrightarrow & H_i(M) & \longrightarrow & H_i(W) \oplus H_i(W) & \longrightarrow & H_i(\partial(W \times I)) \end{array}$$

Since $\chi(\partial(W \times I)) = 0$ since $\dim \partial(W \times I)$ is odd. So

$$2\chi(W) = \chi(M) + \chi(\partial(W \times I)) = \chi(M),$$

well done. $\square$

Corollary 3.3. *If $M = \partial W$ for some compact manifold W, then $\chi(M)$ is even.*

Example 3.3 (Boundary of Orientable Manifold is Orientable)**.** *Let M be a R-orientable n-manifold with boundary ∂M, then ∂M is R-orientable.*

Proof. Consider a coordinate $U \cong \mathbb{H}^n$ of $x \in \partial M$. Let $V = \partial U = u \cap \partial M$, and choose $y \in \mathrm{int}(U) = U - V$. We consider R-coefficient homology group, then we have

$$\begin{aligned} H_n(\mathrm{int}(M), \mathrm{int}(M) - \mathrm{int}(U)) &\xrightarrow{R-\text{orientable},\cong} H_n(\mathrm{int}(M), \mathrm{int}(M) - y) \\ &\xrightarrow{\text{Homotopy by boundary collar},\cong} H_n(M, M - y) \\ &\xrightarrow{R-\text{orientable},\cong} H_n(M, M - \mathrm{int}(U)) \\ &\xrightarrow{\partial,\cong} H_n(M - \mathrm{int}(U), M - U) \\ &\xrightarrow{\text{Homotopy by boundary collar},\cong} H_n(M - \mathrm{int}(U), M - \mathrm{int}(U) - x) \\ &\xrightarrow{\text{Excision of } \mathrm{int}(M)-\mathrm{int}(U),\cong} H_n(\partial M, \partial M - x) \\ &\xrightarrow{R-\text{orientable},\cong} H_n(\partial M, \partial M - V). \end{aligned}$$

Well done. $\square$

Remark 3.2. *In smooth case, we can calculate the transition function. See Theorem 1.3 in* `http://staff.ustc.edu.cn/~wangzuoq/Courses/21F-Manifolds/Notes/Lec24.pdf`.

Theorem 3.4 (Poincaré Duality with Boundaries)**.** *Suppose M is a compact R-orientable n-manifold whose boundary ∂M s decomposed as the union of two compact $(n-1)$ dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism $D_M : H^k(M, A; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M, B; R)$.*

Corollary 3.4 (Lefschetz Duality)**.** *Suppose M is a compact R-orientable n-manifold and take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism $D_M : H^k(M, \partial M; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M; R)$ and $D_M : H^k(M; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M, \partial M; R)$.*

Theorem 3.5 (Alexander Duality)**.** *If K is a compact, locally contractible subspace of S^n, then for all i and any abelian group G, we have*

$$\widetilde{H}_i(S^n - K; G) \cong \widetilde{H}^{n-i-1}(K; G).$$

Example 3.4 (Jordan Curve)**.** *Actually we view $S^1 \subset \mathbb{R}^2$ as one-point compactification $S^1 \subset S^2$, then we use Alexander duality as*

$$\widetilde{H}_0(S^2 - S^1; \mathbb{Z}) \cong \widetilde{H}^1(S^1; \mathbb{Z}) \cong \mathbb{Z},$$

so $H_0(S^2 - S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, well done.

Example 3.5 (Jordan-Brouwer Separation Theorem)**.** *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then $\mathbb{R}^n - S$ has two components.*

Proof. Also we let it as in one-point compactification $S \subset S^n$. Now we didn't know whether S is orientable or not, we consider $\mathbb{Z}/2\mathbb{Z}$ as coefficient, then we use Alexander duality and Poincaré duality

$$\widetilde{H}_0(S^n - S; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H}^{n-1}(S; \mathbb{Z}/2\mathbb{Z}) \cong H_0(S; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

well done. □

Example 3.6 (Compact Hypersurface as Boundary)**.** *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then S be the boundary of some domain in $\mathbb{R}^n$.*

Proof. Trivial by Jordan-Brouwer. □

Example 3.7 (Compact Hypersurface in $\mathbb{R}^n$ is Orientable)**.** *If S be a connected compact hypersurface S in $\mathbb{R}^n$ is orientable.*

Proof 1. Since $\dim S = n-1$, we have to calculate $H_{n-2}(S; \mathbb{Z})$. Also we let it as in one-point compactification $S \subset S^n$. WLOG we let $n > 1$. If S is not orientable, we have $H_{n-1}(S; \mathbb{Z}) = 0$ and $H_{n-2}(S; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, then we have

$$\begin{aligned}
\mathbb{Z} &\cong \widetilde{H}_0(S^n - S; \mathbb{Z}) \cong H^{n-1}(S) \\
&\cong \mathrm{Hom}_{\mathbb{Z}}(H_{n-1}(S; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \\
&\cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

which is impossible. Well done. □

Proof 2. Here we give another method. Take $x \in S$ and $u \in N_x(\mathbb{R}^n/S)$ with $\|u\| = 1$. By Jordan-Brouwer separation theorem we may let u always in the same component when x is varying on S. Consider a non-trivial vector field $X(x) = u(x)$. Now $i_X(\mathrm{vol})$ restricted to S is a volume form on S where vol is the canonical volume form on $\mathbb{R}^n$. □

Proof 3. Moreover we could prove that the normal bundle of S is trivial. See `https://math.stackexchange.com/questions/863960/orientation-of-hypersurface`. □

References

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