

# ERRATA AND SOME NOTES FOR TOPICS IN ALGEBRAIC GEOMETRY BY LUC ILLUSIE

XIAOLONG LIU

## Abstract

These notes correct a few typos, errors and some notes in *Topics in Algebraic Geometry* by Prof. Luc Illusie. The original book is [Illusie].

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## 1 Errata

- ◆ 1. (Page 10, line -6) Actually  $L, M$  are considered as two bicomplexes centered at 0-th column instead of mapping cones;

◆ 2. (Page 18, line 2) Replace 
$$\begin{array}{ccc} & N & \\ +1 \nearrow & \uparrow v & \\ L & \xrightarrow{u} & M \end{array}$$
 by 
$$\begin{array}{ccc} & N & \\ +1 \nwarrow & \uparrow v & \\ L & \xrightarrow{u} & M \end{array};$$

- ◆ 3. (Page 20, line 5) Replace  $L \xrightarrow{u} M \rightarrow C(u) \xrightarrow{-pr} L[1]$  by  $L \xrightarrow{u} M \xrightarrow{i} C(u) \xrightarrow{-pr} L[1]$ ;

- ◆ 4. (Page 21, line -5) Replace  $u\tilde{f} = 0$  by  $u\tilde{f} = f$ ;

- ◆ 5. (Page 24, line 12) Replace  $\mathrm{Hom}_{\mathcal{C}(S^{-1})} = H(X, Y)/\sim$  by  $\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X, Y) = H(X, Y)/\sim$ ;

◆ 6. (Page 25, line -3) Replace 
$$\begin{array}{ccc} M[-1] & \longrightarrow & Y \\ f' \uparrow & \nearrow f & \\ X & & \end{array}$$
 by 
$$\begin{array}{ccc} M[-1] & \xrightarrow{t'} & Y \\ f' \uparrow & \nearrow f & \\ X & & \end{array};$$

- ◆ 7. (Page 27, the first paragraph) Replace  $I^Y$  to  $I_Y$  twice and replace  $(I_X)^\circ$  to  $(I^X)^\circ$ ;

- ◆ 8. (Page 27, line 9) Replace  $\mathcal{A}$  to  $\mathcal{C}$ ;

- ◆ 9. (Page 27, line 10) Replace  $\varinjlim_{X' \xrightarrow{s} X \in (I_X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$  to  $\varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ ;

- ◆ 10. (Page 27, line 12) Replace  $(X', t, f)$  to  $(X', s, f)$ ;

- ◆ 11. (Page 27, line -4) Replace  $u \in (I_X)^\circ$  to  $u \in (I^X)^\circ$ ;

- ◆ 12. (Page 28, line 6) Replace  $(C(S^{-1}, Q)$  to  $(C(S^{-1}), Q)$ ;
- ◆ 13. (Page 28, line -6) Replace  $C(\mathcal{A})(S^{-1})$  to  $C(\mathcal{A})(\text{qis}^{-1})$ ;
- ◆ 14. (Page 32, line 3) Replace  $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1}K \rightarrow$  to  $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1}K \rightarrow$  three times;
- ◆ 15. (Page 36, the second paragraph) Replace all  $\tau_{[a,b]}L$  to  $\tau_{[a+1,b]}L$  and replace  $\tau_{[b-1,b]}L$  to  $\tau_{[b,b]}L$ ;
- ◆ 16. (Page 40, line 18) Replace  $K^+(\mathcal{J})(\text{qis}^{-1})$  to  $K^+(\mathcal{I})(\text{qis}^{-1})$ ;
- ◆ 17. (Page 40, line -7) Replace (3.8) to (3.10);
- ◆ 18. (Page 41, line 1) Replace  $\{M \rightarrow M'', \text{ where } M'' \in K^+(\mathcal{A})\}$  to  $\{M \rightarrow M'', \text{ where } M'' \in K^+(\mathcal{A})\}$ ;
- ◆ 19. (Page 41, line 2) Replace (e.g. 4.13) to (4.18);
- ◆ 20. (Page 41, second paragraph) This proof probably has a mistake that pushout may not preserve monomorphism, see [Ka];
- ◆ 21. (Page 42, lemma 4.29) This proof probably has a mistake that pushout may not preserve monomorphism;
- ◆ 22. (Page 43, line -1) Replace  $E' \in \mathcal{A}$  to  $E' \in \mathcal{A}'$ ;
- ◆ 23. (Page 45, line 4) Replace (4.18) to (4.27);
- ◆ 24. (Page 45, line -4) Replace  $\eta : FQ \rightarrow QG$  to  $\eta : QF \rightarrow GQ$ ;
- ◆ 25. (Page 46, line 2,3) Replace  $F(\varepsilon(L'))$  to  $\varepsilon(L')$ ;
- ◆ 26. (Page 58, line 11) Replace Lemma 6.7 to Proposition 6.7;
- ◆ 27. (Page 60, line -3,-5) Replace zero to trivial;
- ◆ 28. (Page 64, line 6) Replace 6.8 to 6.7;
- ◆ 29. (Page 68, line -4) Replace  $C^n(\mathcal{U} \cap V, F)$  to  $\check{C}^n(\mathcal{U} \cap V, F)$ ;
- ◆ 30. (Page 71, line 4) The proof is same as Theorem 8.3 which reduce to the case of Lemma 8.4, so here we use the same homotopy operator  $k$  in 8.4;
- ◆ 31. (Page 87, line 13) Replace 1.2 to 2.2;
- ◆ 32. (Page 88, line 5) Replace  $M/(f_1, \dots, f_r)M$  to  $M/(f_1, \dots, f_{r-1})M$  twice;
- ◆ 33. (Page 88, line -12) Replace  $K^{n+1}$  to  $K^{n+1}(v)$ ;
- ◆ 34. (Page 88, line -4,-5) Replace  $\bigwedge^1 A$  to  $\bigwedge^1 A^r$  and replace  $\bigwedge^{r-1} A$  to  $\bigwedge^{r-1} A^r$ ;
- ◆ 35. (Page 89, line -11) Replace  $\text{Hom}(K.(f)^{-r}, N)$  to  $\text{Hom}(K.(f)^{-r}, A)$ ;
- ◆ 36. (Page 90, line -6) Replace canormal to conormal;

- ◆ 37. (Page 91, line 3) Replace  $A[\frac{t_0}{t_i}, \dots, \frac{t_r}{t_{i-1}}, \frac{t_r}{t_{i+1}}, \dots, \frac{t_r}{t_i}]$  to  $A[\frac{t_0}{t_i}, \dots, \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, \dots, \frac{t_r}{t_i}]$ ;
- ◆ 38. (Page 92, line 1) Replace  $\check{H}(\mathcal{U}, \mathcal{O}(n))$  to  $\check{H}^q(\mathcal{U}, \mathcal{O}(n))$ ;
- ◆ 39. (Page 92, line 1) Replace  $\bigcup_{i=1}^p U_{i_j}$  to  $\bigcup_{i=0}^p U_{i_j}$ ;
- ◆ 40. (Page 92, line -9) Replace  $\check{C}_{-n} = (0 \rightarrow \bigoplus_i t_i^{-n} B \rightarrow \dots)$  to  $\check{C}_{-n} = (\bigoplus_i t_i^{-n} B \rightarrow \dots)$ ;
- ◆ 41. (Page 93, line -12) Replace  $H^r K^\cdot(t_0^n, \dots, t_r^n, B)$  to  $H^{r+1} K^\cdot(t_0^n, \dots, t_r^n, B)$ ;
- ◆ 42. (Page 94, line -6) Replace  $k \bigotimes_{\mathcal{O}_{X,x}} L$  to  $L_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ ;
- ◆ 43. (Page 95, line -13) Replace  $U_i f$  to  $(U_i)_f$ ;
- ◆ 44. (Page 95, line -8) Replace 3.1 to 4.1;
- ◆ 45. (Page 96, line -14) Replace  $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes m}$  to  $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes n}$ ;

## 2 Some Notes

♣(Page 72, Theorem 8.12) **THEOREM OF LERAY.** Let  $(X, \mathcal{O}_X)$  be a ringed space and  $F$  be an  $\mathcal{O}_X$ -module. Let  $\mathfrak{U} = \{U_i\}_{i \in I}$  be an open covering of it. If for every nonempty finite subset  $J \subset I$  and every  $q > 0$  such that  $H^q(U_J, F) = 0$  where  $U_J = \bigcap_{j \in J} U_j$ , then  $\check{H}^n(\mathfrak{U}, F) \cong H^n(X, F)$ .

*The first proof.* Consider  $\mathcal{H}^q(X, F)$  be a presheaf with  $U \mapsto H^q(U, F)$ . By Grothendieck spectral sequence, there exists a spectral sequence such that  $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(X, F)) \Rightarrow H^{p+q}(X, F)$  and  $\check{H}^p(\mathfrak{U}, \mathcal{H}^q(X, F)) = 0$  for  $p > 0$  in this situation. Then the  $E_2$  page is

$$\begin{array}{ccccccc} \cdots & & 0 & & 0 & & 0 & & \cdots \\ & & \searrow & & \searrow & & & & \\ \cdots & & \check{H}^{p-1}(\mathfrak{U}, F) & & \check{H}^p(\mathfrak{U}, F) & & \check{H}^{p+1}(\mathfrak{U}, F) & & \cdots \end{array}$$

Since it converge to  $H^p(X, F)$  and for now  $E_2 = E_\infty$ , then we win. Here we use the fact that  $\check{H}^p(\mathfrak{U}, -)$  as the right derived functor of  $\check{H}^0(\mathfrak{U}, -)$ , see St 01EN in [St].  $\square$

*The second proof.* See St 01EV in [St].  $\square$

♣(Page 84, Corollary 1.4) Here we need to show that  $R^q f_* F$  is a sheaf associated to the presheaf  $V \mapsto H^q(f^{-1}(V), F)$ . For now we assume  $f : X \rightarrow Y$  be the morphism between ringed spaces and  $F$  is any  $\mathcal{O}_X$ -module.

*Proof.* Let  $F[0]$  quasi-isomorphic to  $I^*$  where  $I^k$  are injective  $\mathcal{O}_Y$ -modules. So  $R^q f_* F = H^q(Rf_* F) = H^q(f_* I^*)$ . We find that  $H^i(f_* I^*)$  is a sheaf associated to the presheaf

$$\begin{aligned} V &\mapsto \frac{\ker(f_* I^i(V) \rightarrow f_* I^{i+1}(V))}{\text{Im}(f_* I^{i-1}(V) \rightarrow f_* I^i(V))} \\ &= \frac{\ker(I^i(f^{-1}V) \rightarrow I^{i+1}(f^{-1}V))}{\text{Im}(I^{i-1}(f^{-1}V) \rightarrow I^i(f^{-1}V))} = H^i(f^{-1}(V), F) \end{aligned}$$

and we win.  $\square$

♣(Page 85, Corollary 1.6) Actually we can show that if  $f$  is qcqs morphism and  $F \in Qcoh(X)$ , then  $R^q f_* F \in Qcoh(Y)$  for all  $q \geq 0$ . For  $q = 0$ , see [UT1] 10.27. For  $q > 0$  and  $f$  qcqs, see St 01XJ in [St].

♣(Page 89, line -11) The reason of the first euqality is that if we consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \bigwedge^{r-1} A^r & \longrightarrow & \cdots & \longrightarrow & \bigwedge^1 A^r & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

(A dashed arrow points from  $A$  to  $0$  in the second row.)

So  $\text{Hom}_{K(A)}(K(f), A[r]) = \text{Hom}(K(f)^{-r}, A)/(\text{homotopical equivalence})$ . Since the homotopical equivalence are determined by  $\bigwedge^{r-1} A^r \rightarrow A$ , so

$$\begin{aligned} \text{Hom}_{K(A)}(K(f), A[r]) &= \text{Hom}(K(f)^{-r}, A) / \left( \bigwedge^{r-1} A^r \rightarrow A \right) \\ &= \text{coker}(\text{Hom}(K(f)^{-r}, A) \rightarrow \text{Hom}(K(f)^{-r+1}, A)) \end{aligned}$$

and we win.

♣(Page 90) Actually in the definition we defined  $i : Y \rightarrow X$  is Koszul-regular immersion. We say  $i : Y \rightarrow X$  is a regular immersion if locally we have  $I|_U = (f_1, \dots, f_r)$  where  $f_1, \dots, f_r$  is regular. Similarly, one can define  $H_1$ -regular as in the Theorem 2.2(3). All of these are equivalence if  $X$  is locally noetherian, see St 063I.

In the remark  $N_{Y/X} = I/I^2$  is locally free, see St 063C and St 063H. Let  $i : X \rightarrow Y$  be a closed immersion with regular of codimension  $r$ , then we have the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \cong \omega_{Y/X}[-r], \omega_{Y/X} = \left( \bigwedge^r N_{Y/X} \right)^\vee.$$

Actually one can assume  $X$  be a ringed space and  $I$  is Koszul-regular. The proof see St 0BQZ.

♣(Page 94) In general case, we say  $F$  is coherent if for all open  $U$  and all  $n \geq 0$ ,  $\ker(\mathcal{O}_X^n|_U \rightarrow F|_U)$  is finite type. But in the locally noetherian case, this is the same as finitely presentation or quasi-coherent+finite type.

In the hypothesis of Lemma 4.2, we can just let  $X$  is qcqs and  $E$  is quasi-coherent. For the proof is easy, I will omit it, see [UT1] Theorem 7.22.

♣(Page 97) In the proof of 4.6, we have  $i_*\mathcal{F} \otimes_{\mathcal{O}_P}(n) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}_P(n))$ . This isomorphism we use the projection formula, as follows.

**Theorem.**(Projection Formula) Let  $f : X \rightarrow Y$  be a morphism of ringed spaces. Let  $E \in D(\mathcal{O}_X)$  and  $K \in D(\mathcal{O}_Y)$ . If  $K$  is perfect (See St 08CM), then

$$Rf_*E \otimes_{\mathcal{O}_Y}^L K = Rf_*(E \otimes_{\mathcal{O}_Y}^L Lf^*K)$$

in  $D(\mathcal{O}_Y)$ .

In St 0B55 we find that if  $f$  is a homeomorphism onto a closed subset, then this is an isomorphism always.

### 3 Remarks

- ♠ 1. Here we assume that a single commutative diagram occupies one line;
- ♠ 2. I omitted the section (4.14) about Ext and extensions of groups;
- ♠ 3. If you find errors in my errata, please send to my email: 1225046792@qq.com.

### References

- [Illusie] Luc Illusie, *Topics in Algebraic Geometry*, Université de Paris-Sud Département de Mathématiques, <http://staff.ustc.edu.cn/~yiouyang/Illusie.pdf>.
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- [UT1] Ulrich Görtz, Torsten Wedhorn, *Algebraic Geometry I: Schemes, 2ed edition*, Springer Spektrum, 2020.