# Project 1B: Two-Dimensional Distributions, Marginals and Covariance Structure

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**Abstract**

In this article, we will discuss some special properties about the two-dimensional distributions, such as how to construct a bivariant distribution from two single disttibutions, what difference between dimension two and higher dimension and some special bivariant distributions. For the basic, we refer [4], [8], [6], [1], [7].

## Contents



## 1 Introduction

In this article, we will discuss some special properties about the two-dimensional distributions, such as how to construct a bivariant distribution from two single disttibutions and what difference between dim = 2 and higher dimension and some special bivariant distributions.

For the first question, we will introduce the FGM copula and derive some properties about it. For the second, we will learn something about the random walk and prove that the simple symmetric random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d \leq 2$ and transient in dimensions $d > 2$. For the third, we will explore the bivariant normal distribution.

(I will write some history here in after version.)

## 2 Preliminaries: Random Variables, Distributions and Covariance

For now we will introduce some basic definitions which we will use.

**Difinition 2.1.** *Consider random vector $\xi(\omega) = (\xi_1(\omega), \cdots, \xi_n(\omega))$ whose components are random variables in some probability space $(\Omega, \mathscr{A}, P)$. The function $F(x_1, \cdots, x_n) = \mathsf{P}(\xi_1(\omega) < x_1, \cdots, \xi_n(\omega) < x_n)$ is called* ***distribution function*** *of random vector $\xi$.*

*In the continuous case, there exists a non-negative function $p(x_1, \cdots, x_n)$ such that*

$$F(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p(y_1, \cdots, y_n) dy_1 \cdots dy_n,$$

*where $\int_{\mathbb{R}^n} p(y_1, \cdots, y_n) dy_1 \cdots dy_n = 1$ and it is called* ***probability density function****.*

**Difinition 2.2.** *The random variables $\xi_1, \cdots, \xi_n$ are said to be* ***(mutually) independent*** *if*

$$\mathsf{P}(\xi_1(\omega) = x_1, \cdots, \xi_n(\omega) = x_n) = \mathsf{P}(\xi_1(\omega) = x_1) \cdots \mathsf{P}(\xi_n(\omega) = x_n))$$

*for all $x_1, \cdots, x_n$.*

**Example 1** (Multinomial distribution)**.** *We let the possibly results of the test are $A_1, ..., A_r$ where $P(A_i) = p_i$ and $p_1 + ... + p_r = 1$. We repeat it $n$ times and assume the results are independent. If we let $\xi_i$ be the number of occurrences of $A_i$, then*

$$\mathsf{P}(\xi_i = k_i) = \frac{n!}{k_1! \cdots k_r!} p_1^{k_1} \cdots p_r^{k_r};$$

From now we just discuss marginal distribution in the case of two-dimensional distributions and the higher dimension are similar.

**Difinition 2.3** (Marginal)**.** *We consider two-dimensional distributions $(\xi, \eta)$.*

*(1)[General] Let its distribution function is $F(x, y)$ and let $F_1(x) = \mathsf{P}(\xi < x) = F(x, +\infty)$ and $F_2(y) = \mathsf{P}(\eta < y) = F(+\infty, y)$. These are called the* ***marginal distribution functions*** *of $F(x, y)$.*

*(2)[Discrete] If $\xi$ take values at $x_1, x_2, ...$ and $\eta$ take values at $y_1, y_2, ...$, and if we let $\mathsf{P}(\xi = x_i, \eta = y_j) = p(x_i, y_j)$ and $\mathsf{P}(\xi = x_i) = p_1(x_i), P(\eta = y_j) = p_2(y_j)$, then*

$$\sum_j p(x_i, y_j) = p_1(x_i), \sum_i p(x_i, y_j) = p_2(y_j),$$

*and these are marginal distribution functions;*

*(3)[Continuous] Let its probability density function is $p(x, y)$,then*

$$F_1(x) = \int_{-\infty}^{x} du \int_{-\infty}^{\infty} p(u, y) dy, F_2(y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{y} p(x, v) dv,$$

*so the probability density functions of $F_1(x), F_2(y)$ are*

$$p_1(x) = \int_{\mathbb{R}} p(x, y) dy, p_2(y) = \int_{\mathbb{R}} p(x, y) dx,$$

*which are called marginal distribution functions, respectively.*

**Difinition 2.4** (Expectation)**.** *In general case, we will use a special integral, called Riemann-Stieltjes integral, to define it.*

*Let the distribution function of the random vector* $(\xi_1, ..., \xi_r)$ *is* $F(x_1, ..., x_r)$*. The expectation of* $(\xi_1, ..., \xi_r)$ *is* $(\mathsf{E}\xi_1, ..., \mathsf{E}\xi_r)$*, where*

$$\mathsf{E}\xi_t = \int_{\mathbb{R}^r} x_i dF(x_1, ..., x_n) = \int_{\mathbb{R}} x_i dF_i(x_i),$$

*where* $F_i$ *is the distribution function of* $\xi_i$*.*

*So in the case of discrete,* $\mathsf{E}(\xi_i) = \sum_i x_i p(x_i)$*; in the continuous case,* $\mathsf{E}(\xi_i) = \int_{\mathbb{R}} x_i p_i(x_i) dx_i$*.*

**Remark 2.5.** *This definition is actually from the law of the unconscious statistician:*

$$\mathsf{E}g(\xi_1, ..., \xi_d) = \int_{\mathbb{R}^d} g(x_1, ..., x_d) dF(x_1, ..., x_d).$$

**Difinition 2.6** (Variance and Covariance)**.** *Consider a random vector* $\boldsymbol{\xi} = (\xi_1, ..., \xi_r)$*, we define its Covariance matrix as*

$$\mathsf{Cov}(\boldsymbol{\xi}) = \begin{pmatrix} \mathsf{Var}\xi_1 & \mathsf{Cov}(\xi_1, \xi_2) & \cdots & \mathsf{Cov}(\xi_1, \xi_r) \\ \mathsf{Cov}(\xi_2, \xi_1) & \mathsf{Var}\xi_2 & \cdots & \mathsf{Cov}(\xi_2, \xi_r) \\ \vdots & \vdots & & \vdots \\ \mathsf{Cov}(\xi_r, \xi_1) & \mathsf{Cov}(\xi_r, \xi_2) & \cdots & \mathsf{Var}\xi_r \end{pmatrix},$$

*where* $\mathsf{Var}\xi_i = \mathsf{E}(\xi_i - \mathsf{E}\xi_i)^2$ *is the variance of* $\xi_i$ *and* $\mathsf{Cov}(\xi_i, \xi_j) = \mathsf{E}(\xi_i - \mathsf{E}\xi_i)(\xi_j - \mathsf{E}\xi_j)$ *is the covariance of* $\xi_i, \xi_j$*. So* $\mathsf{Cov}(\xi_i, \xi_i) = \mathsf{Var}\xi_i$ *and* $\mathsf{Cov}(\boldsymbol{\xi}) = (\mathsf{Cov}(\xi_i, \xi_j))_{r \times r}$*.*

**Remark 2.7.** *(1) So we can use the law of the unconscious statistician to calculate them;*

*(2) The matrix* $\mathsf{Cov}(\boldsymbol{\xi})$ *is non-negative definited since for all* $t_j \in \mathbb{R}$ *we have*

$$\sum_{j,k} \mathsf{Cov}(\xi_j, \xi_k) = \mathsf{E}\left(\sum_j t_j(\xi_j - \mathsf{E}\xi_j)\right)^2 \geq 0.$$

# 3 Main results and their proofs

## 3.1 Farlie-Gumbel-Morgenstern (FGM) copula

We will discover how to construct from two single distributions to a bivariant distribution and discover its properties such as FGM copula in [1],[5] and so on.

**Difinition 3.1** (Farlie-Gumbel-Morgenstern (FGM) copula)**.** *For two distributions* $U \sim F_1(u), V \sim F_2(v)$*, we can get the FGM copula as*

$$C(U, V) = UV(1 + \alpha(1 - U)(1 - V)),$$

*where* $\alpha \in [-1, 1]$*. if* $\alpha$ *is greater than zero, the FGM copula provides positive dependence; if* $\alpha$ *is smaller than zero, it returns negative dependence; when* $\alpha$ *is zero, it reduces to the independence copula.*

**Remark 3.2.** *We can see that the marginals of FGM copula is* $F_1, F_2$*, respectively. Moreover, in [2], we have the more generalized version of FGM copula.*

Now we start. Let $G(x,y) = F_1(x)F_2(y)(1+\alpha(1-F_1(x))(1-F_2(y)))$, then its probability density function is $g(x,y) = \partial^2_{x,y}G = f_1(x)f_2(y)(1+\alpha(2F_1(x)-1)(2F_2(y)-1))$. So we have

$$
\begin{aligned}
\mathsf{Cov}(x,y) &= \int_{\mathbb{R}^2} (x-\mathsf{E}x)(y-\mathsf{E}y)g(x,y)dxdy \\
&= \left(\int_{\mathbb{R}} (x-\mathsf{E}x)f_1(x)dx\right)\left(\int_{\mathbb{R}} (y-\mathsf{E}y)f_2(y)dy\right) \\
&+ \alpha\left(\int_{\mathbb{R}} (x-\mathsf{E}x)f_1(x)(2F_1(x)-1)dx\right)\left(\int_{\mathbb{R}} (y-\mathsf{E}y)f_2(y)(2F_2(y)-1)dx\right) \\
&= \alpha\left(\int_{\mathbb{R}} (x-\mathsf{E}x)f_1(x)(2F_1(x)-1)dx\right)\left(\int_{\mathbb{R}} (y-\mathsf{E}y)f_2(y)(2F_2(y)-1)dx\right) \\
&= \alpha\left(\int_{\mathbb{R}} xf_1(x)(2F_1(x)-1)dx\right)\left(\int_{\mathbb{R}} yf_2(y)(2F_2(y)-1)dx\right)
\end{aligned}
$$

where the last step is because $\int_{\mathbb{R}}(2F_i(z)-1)f_i(z)dz = \int_0^1(2u-1)du = 0$. Also we have

$$
\begin{aligned}
\rho(x,y) = \frac{\mathsf{Cov}(x,y)}{\mathsf{Var}(x)\mathsf{Var}(y)} &= \frac{\alpha}{\mathsf{Var}(x)\mathsf{Var}(y)} \\
&\times \left(\int_{\mathbb{R}} xf_1(x)(2F_1(x)-1)dx\right)\left(\int_{\mathbb{R}} yf_2(y)(2F_2(y)-1)dx\right).
\end{aligned}
$$

Moreover we can find the upper bound of its correlation coefficient. We have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}} x(2F(x)-1)f(x)dx\right)^2 = \left(\int_{\mathbb{R}} (x-\mathsf{E}x)(2F(x)-1)f(x)dx\right)^2 \\
\le &\left(\int_{\mathbb{R}} (x-\mathsf{E}x)^2f(x)dx\right)\left(\int_{\mathbb{R}} (2F(x)-1)^2f(x)dx\right) = \frac{\mathsf{Var}(x)^2}{3},
\end{aligned}
$$

where the last step is because $\int_{\mathbb{R}}(2F(z)-1)^2f(z)dz = \int_0^1(2u-1)^2du = \frac{1}{3}$. So

$$
\begin{aligned}
\rho(x,y) = &\frac{\alpha}{\mathsf{Var}(x)\mathsf{Var}(y)}\left(\int_{\mathbb{R}} xf_1(x)(2F_1(x)-1)dx\right) \\
&\times \left(\int_{\mathbb{R}} yf_2(y)(2F_2(y)-1)dx\right) \le \frac{\alpha}{3},
\end{aligned}
$$

well done.

## 3.2 Recurrence & Transience in Random Walk

**Difinition 3.3** (Random Walk on $\mathbb{Z}^d$)**.** *Let $X_1, ...$ be a sequence of $\mathbb{Z}^d$-valued independent and identically distributed random variables. A **random walk** started at $z \in \mathbb{Z}^d$ is the sequence $(S_n)_{n\ge 0}$ on $\mathbb{Z}^d$ where $S_0 = z$ and $S_n = S_{n-1} + X_n, n \ge 1$*

**Difinition 3.4** (Recurrence & transience)**.** *We say that a random walk is **recurrent** if it visits its starting position infinitely often with probability one and **transient** if it visits its starting position finitely often with probability one.*

This we refer [3]. Now we let $N = \sum_{n\geq 0} \mathbb{1}_{\{S_n=S_0\}}$. So recurrent means $\mathsf{P}(N=\infty)=1$ and transience means $\mathsf{P}(N<\infty)=1$. Let $\tau = \inf\{n\geq 1 : X_1+\cdots+X_n=0\}$ denote the first time the walk is back to the starting point. For now we will state some conclusion without proofs.

**Lemma 3.5** (Either Recurrent or Transient)**.** *For each $n \geq 1$,*

$$\mathsf{P}(N=n) = \mathsf{P}(\tau=\infty)\mathsf{P}(\tau<\infty)^{n-1}.$$

*In particular, every random walk is either recurrent or transient.*

**Lemma 3.6.** *A random walk is transient if $\mathsf{E}N<\infty$ and recurrent if $\mathsf{E}N=\infty$.*

*Proof.* We can prove that $\mathsf{E}N = \frac{1}{\mathsf{P}(\tau=\infty)}$. □

**Difinition 3.7** (Simple Symmetric Random Walk)**.** *A random walk over $\mathbb{Z}^d$ is called Simple Symmetric Random Walk (SSRW) if all $X_k$ take values over $\{\pm e_1, ..., \pm e_d\}$ and $P(X_k=(\pm)e_i)=\frac{1}{2d}$.*

**Lemma 3.8** (Expectation Formula)**.** *Let $\phi(k) = \mathsf{E}(e^{ikX_1}) = \mathsf{E}(e^{ikX_i})$, then*

$$\mathsf{E}N = \lim_{t\uparrow 1}\int_{[-\pi,\pi]^d} \frac{1}{1-t\phi(k)}\frac{dk}{(2\pi)^d}.$$

**Theorem 3.9** (Recurrence/transience of SSRW)**.** *The simple symmetric random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d\leq 2$ and transient in dimensions $d>2$.*

*Proof.* We have

$$\phi(k) = \frac{1}{2d}\sum_{j=1}^{d}(e^{ik_j}+e^{-ik_j}) = \frac{1}{d}\sum_{j=1}^{d}\cos(k_j).$$

By $1-\cos x = 2\sin^2(x/2)$ inequalities $\frac{2x}{\pi} \le \sin x \le x$, we have

$$1 - t + 2t\frac{|k|^2}{\pi^2 d} \le 1 - t\phi(k) \le 1 - t + \frac{|k|^2}{\pi^2 d}.$$

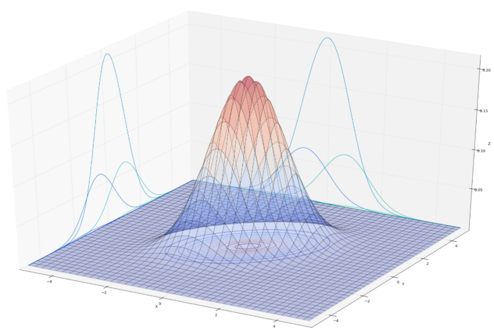
Taking the limit we find that the function $k \mapsto 1 - t\phi(k)$ is uniformly integrable around $k = 0$ if and only if the function $k \mapsto |k|^2$ is integrable, i.e.,

$$\mathsf{E}N < \infty \text{ if and only if } \int_{|k|<1} \frac{dk}{|k|^2} < \infty,$$

Since we find that the integral is finite if and only if $d > 2$, we win! □

## 3.3 Bivariant Normal Distribution

Bivariant normal distribution is one of the most important examples in probability and statistics and we will discover it here.



The probability density function of $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ is

$$p(x,y) = \frac{1}{2\pi\sigma_2\sigma_2(1-\rho^2)} \exp\left[-\frac{1}{2(1-\rho^2)} \times \left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right]$$

where $\sigma_i > 0 (i = 1, 2), |\rho| < 1$. If we let

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

where $\sigma_{12} = \sigma_{21} = \rho\sigma_1\sigma_2, \sigma_{ii} = \sigma_i^2$. Then we can rewrite it as

$$p(\boldsymbol{x}) = \frac{1}{2\pi(\det \boldsymbol{C})^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{C}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right].$$

**Theorem 3.10** (Canonical Decomposition of Bivariant Normal Distribution)**.** *We have*

$$p(x,y) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{(y-(\mu_2+\rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)))^2}{2\sigma_1^2(1-\rho^2)}\right).$$

*Proof.* This is just a directly calculation and I will omit it. ☐

In this theorem we actually can rewrite it as:

$$N_{x,y}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = N_x(\mu_1, \sigma_1^2) \times N_y(\mu_1 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1), \sigma_2^2(1 - \rho^2))$$

**Proposition 3.11.** *The marginals of bivariant normal distribution $N_{x,y}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ is $N_x(\mu_1, \sigma_1^2)$ and $N_y(\mu_2, \sigma_2^2)$.*

*Proof.* Just use the theorem, we have

$$\begin{aligned} p_1(x) = \int_{\mathbb{R}} p(x, y)dy &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x - \mu_1)^2}{2\sigma_1^2}\right) \\ &\times \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1 - \rho^2}} \exp\left(-\frac{(y - (\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x - \mu_1)))^2}{2\sigma_1^2(1 - \rho^2)}\right) dy \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(x - \mu_1)^2}{2\sigma_1^2}\right) \end{aligned}$$

and another is similar. ☐

**Remark 3.12.** *So we see that the marginals of bivariant normal distribution are normal distributions, but the converse is not true. Actually, we let $\phi(x) = \frac{1}{2\pi}e^{-x^2/2}, -\infty < x < \infty$ and let*

$$g(x) = \begin{cases} \cos x, & |x| < \pi; \\ 0, & |x| \geq \pi. \end{cases}$$

*Now we let $p(x, y) = \phi(x)\phi(y) + \frac{1}{2\pi}e^{-\pi^2}g(x)g(y)$.*

For now we explore the meanings of $\mu_i, \sigma_i, \rho$. By the proposition of marginals, we know that $\mu_i, \sigma_i^2$ are expectations, repectively variances of its marginals. Then we will see what the $\rho$ are.

Actually let $z = \frac{1}{\sqrt{1-\rho^2}}(\frac{x-\mu_1}{\sigma_1} - \rho\frac{y-\mu_2}{\sigma_2}), t = \frac{y-\mu_2}{\sigma_2}$, then we have

$$\begin{aligned} \sigma_{12} &= \int_{\mathbb{R}^2} (x - \mu_1)(y - \mu_2)p(x, y)dxdy \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} (\sigma_1\sigma_2\sqrt{1 - \rho^2}tz + \rho\sigma_1\sigma_2 t^2)e^{-(t^2 - z^2)/2}dzdt = \rho\sigma_1\sigma_2, \end{aligned}$$

so $\sigma_{12} = \mathsf{Cov}(x, y) = \rho\sigma_1\sigma_2$ and $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$ is the correlation coefficient. So $\boldsymbol{C} = \mathsf{Cov}(\boldsymbol{\xi})$ where $\boldsymbol{\xi} \sim N(\mu_i, \sigma_i, \rho)$.

**Corollary 1.** *At the case of bivariant normal distribution, uncorrelation if and only if independence.*

*Proof.* By the discuss above, independence if and only if $\rho = 0$ which is of course if and only if uncorrelation. ☐

Finally, we will consider how the distribution changes after act a linear transform.

**Proposition 3.13.** *Consider a matrix* $\boldsymbol{M}$ *and* $\boldsymbol{\xi} \sim N(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$*. For convenience we let* $\boldsymbol{\mu} = (\mu_1, \mu_2)^T, \boldsymbol{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$*. Then* $\boldsymbol{\eta} = \boldsymbol{M\xi} \sim N(\boldsymbol{M\mu}, \boldsymbol{MCM}^T)$*.*

*Proof.* Consider the characteristic function, for any real vector $\boldsymbol{t}$, we have

$$\begin{aligned} f_{\boldsymbol{\eta}}(\boldsymbol{t}) &= \mathsf{E}e^{i\boldsymbol{t}^T\boldsymbol{M\xi}} = \mathsf{E}e^{i(\boldsymbol{M}^T\boldsymbol{t})^T\boldsymbol{\xi}} \\ &= \exp\left(i(\boldsymbol{M\mu})^T\boldsymbol{t} - \frac{1}{2}\boldsymbol{t}^T(\boldsymbol{MCM}^T)\boldsymbol{t}\right), \end{aligned}$$

so $\boldsymbol{\eta} = \boldsymbol{M\xi} \sim N(\boldsymbol{M\mu}, \boldsymbol{MCM}^T)$. □

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