Errata and Some Notes for Topics in Algebraic Geometry by Luc Illusie

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Abstract

These notes correct a few typos, errors and some notes in *Topics in Algebraic Geometry* by Prof. Luc Illusie. The original book is [Illusie].

Contents

1 Errata

♦ 1. (Page 10, line -6) Actually L, M are considered as two bicomplexes centered at 0-th column instead of mapping cones;

♦ 2. (Page 18, line 2) Replace
$$\begin{tikzcd} & N \\ L \arrow[ur, "+1"] \arrow[r, "u"] & M \arrow[u, "v"'] \end{tikzcd}$$
by
$$\begin{tikzcd} & N \arrow[dl, "+1"'] \\ L \arrow[r, "u"] & M \arrow[u, "v"'] \end{tikzcd}$$
;

♦ 3. (Page 20, line 5) Replace $L \xrightarrow{u} M \to C(u) \xrightarrow{-pr} L[1]$ by $L \xrightarrow{u} M \xrightarrow{i} C(u) \xrightarrow{-pr} L[1]$;

♦ 4. (Page 21, line -5) Replace $u\tilde{f} = 0$ by $u\tilde{f} = f$;

♦ 5. (Page 24, line 12) Replace $\mathrm{Hom}_{\mathcal{C}(S^{-1})} = H(X,Y)/\sim$ by $\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X,Y) = H(X,Y)/\sim$;

♦ 6. (Page 25, line -3) Replace
$$\begin{tikzcd} M[-1] \arrow[r] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$
by
$$\begin{tikzcd} M[-1] \arrow[r, "t'"] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$$
;

♦ 7. (Page 27, the first paragraph) Replace I^Y by I_Y twice and replace $(I_X)^\circ$ by $(I^X)^\circ$;

♦ 8. (Page 27, line 9) Replace $\mathcal{A}$ by $\mathcal{C}$;

♦ 9. (Page 27, line 10) Replace $\varinjlim_{X' \xrightarrow{s} X \in (I_X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ by $\varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$;

♦ 10. (Page 27, line 12) Replace (X', t, f) by (X', s, f);

♦ 11. (Page 27, line -4) Replace $u \in (I_X)^\circ$ by $u \in (I^X)^\circ$;

- ♦ 12. (Page 28, line 6) Replace $(C\left(S^{-1}\right., Q)$ by $(C(S^{-1}), Q)$;
- ♦ 13. (Page 28, line -6) Replace $C(\mathcal{A})(S^{-1})$ by $C(\mathcal{A})(\mathrm{qis}^{-1})$;
- ♦ 14. (Page 32, line 3) Replace $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1} \to$ by $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1}K \to$ three times;
- ♦ 15. (Page 36, the second paragraph) Replace all $\tau_{[a,b]}L$ by $\tau_{[a+1,b]}L$ and replace $\tau_{[b-1,b]}L$ by $\tau_{[b,b]}L$;
- ♦ 16. (Page 40, line 18) Replace $K^+(\mathcal{J})(\mathrm{qis}^{-1})$ by $K^+(\mathcal{I})(\mathrm{qis}^{-1})$;
- ♦ 17. (Page 40, line -7) Replace (3.8) by (3.10);
- ♦ 18. (Page 41, line 1) Replace $\{M \to M''$, where $M'' \in K^+(\mathcal{A})$ by $\{M \to M''$, where $M'' \in K^+(\mathcal{A})\}$;
- ♦ 19. (Page 41, line 2) Replace (e.g. 4.13) by (4.18);
- ♦ 20. (Page 41, second paragraph) This proof probably has a mistake that pashout may not preserve monomorphism, see [Ka];
- ♦ 21. (Page 42, lemma 4.29) This proof probably has a mistake that pashout may not preserve monomorphism;
- ♦ 22. (Page 43, line -1) Replace $E' \in \mathcal{A}$ by $E' \in \mathcal{A}'$;
- ♦ 23. (Page 45, line 4) Replace (4.18) by (4.27);
- ♦ 24. (Page 45, line -4) Replace $\eta : FQ \to QG$ by $\eta : QF \to GQ$;
- ♦ 25. (Page 46, line 2,3) Replace $F(\varepsilon(L'))$ by $\varepsilon(L')$;
- ♦ 26. (Page 58, line 11) Replace Lemma 6.7 by Proposition 6.7;
- ♦ 27. (Page 60, line -3,-5) Replace zero by trivial;
- ♦ 28. (Page 64, line 6) Replace 6.8 by 6.7;
- ♦ 29. (Page 68, line -4) Replace $C^n(\mathcal{U} \cap V, F)$ by $\check{C}^n(\mathcal{U} \cap V, F)$;
- ♦ 30. (Page 71, line 4) The proof is same as Theorem 8.3 which reduce to the case of Lemma 8.4, so here we use the same homotopy operator k in 8.4;
- ♦ 31. (Page 87, line 13) Replace 1.2 by 2.2;
- ♦ 32. (Page 88, line 5) Replace $M/(f_1, \cdots, f_r)M$ by $M/(f_1, \cdots, f_{r-1})M$ twice;
- ♦ 33. (Page 88, line -12) Replace K^{n+1} by $K^{n+1}(v)$;
- ♦ 34. (Page 88, line -4,-5) Replace $\bigwedge^1 A$ by $\bigwedge^1 A^r$ and replace $\bigwedge^{r-1} A$ by $\bigwedge^{r-1} A^r$;
- ♦ 35. (Page 89, line -11) Replace $\mathrm{Hom}(K.(f)^{-r}, N)$ by $\mathrm{Hom}(K.(f)^{-r}, A)$;
- ♦ 36. (Page 90, line -6) Replace canormal by conormal;
- ♦ 37. (Page 91, line 3) Replace $A[\frac{t_0}{t_i}, ..., \frac{t_r}{t_{i-1}}, \frac{t_r}{t_{i+1}}, ..., \frac{t_r}{t_i}]$ by $A[\frac{t_0}{t_i}, ..., \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, ..., \frac{t_r}{t_i}]$;

2 Some Notes

♣(Page 72, Theorem 8.12) **Theorem of Leray.** Let $(X, \mathcal{O}_X)$ be a ringed space and F be an $\mathcal{O}_X$-module. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of it. If for every nonempty finite subset $J \subset I$ and every $q > 0$ such that $H^q(U_J, F) = 0$ where $U_J = \bigcap_{j \in J} U_j$, then $\check{H}^n(\mathfrak{U}, F) \cong H^n(X, F)$.

The first proof. Consider $\mathscr{H}^q(X, F)$ be a presheaf with $U \mapsto H^q(U, F)$. By Grothendieck spectral sequence, there exists a spectral sequence such that $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) \Rightarrow H^{p+q}(X, F)$ and $\check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) = 0$ for $p > 0$ in this situation. Then the E_2 page is

$$\begin{tikzcd}
\cdots & 0 \arrow[dr] & 0 \arrow[dr] & 0 & \cdots \\
\cdots & \check{H}^{p-1}(\mathfrak{U}, F) & \check{H}^p(\mathfrak{U}, F) & \check{H}^{p+1}(\mathfrak{U}, F) & \cdots
\end{tikzcd}$$

Since it converge to $H^p(X, F)$ and for now $E_2 = E_\infty$, then we win. Here we use the fact that $\check{H}^p(\mathfrak{U}, -)$ as the right derived functor of $\check{H}^0(\mathfrak{U}, -)$, see St 01EN in [St]. □

The second proof. See St 01EV in [St]. □

♣(Page 84, Corollary 1.4) Here we need to show that $R^q f_* F$ is a sheaf associated to the presheaf $V \mapsto H^q(f^{-1}(V), F)$. For now we assume $f : X \to Y$ be the morphism between ringed spaces and F is any $\mathcal{O}_X$-module.

Proof. Let $F[0]$ quasi-isomorphic to I^* where I^k are injective $\mathcal{O}_Y$-modules. So $R^q f_* F = H^q(Rf_* F) = H^q(f_* I^*)$. We find that $H^i(f_* I^*)$ is a sheaf associated to the presheaf

$$\begin{aligned}
V \mapsto & \frac{\ker(f_* I^i(V) \to f_* I^{i+1}(V))}{\operatorname{Im}(f_* I^{i-1}(V) \to f_* I^i(V))} \\
& = \frac{\ker(I^i(f^{-1}V) \to I^{i+1}(f^{-1}V))}{\operatorname{Im}(I^{i-1}(f^{-1}V) \to I^i(f^{-1}V))} = H^i(f^{-1}(V), F)
\end{aligned}$$

and we win. □

♣(Page 85, Corollary 1.6) Actually we can show that if f is qcqs morphism and $F \in Qcoh(X)$, then $R^q f_* F \in Qcoh(Y)$ for all $q \geq 0$. For $q = 0$, see [UT1] 10.27. For $q > 0$ and f qcqs, see St 01XJ in [St].
♣(Page 89, line -11) The reason of the first euqality is that if we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \bigwedge^{r-1} A^r \arrow[r] \arrow[d] \arrow[dl, dashed] & \cdots \arrow[r] & \bigwedge^1 A^r \arrow[r] \arrow[d] & A \arrow[d] \\
0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So $\operatorname{Hom}_{K(A)}(K.(f), A[r]) = \operatorname{Hom}(K.(f)^{-r}, A)/(\text{homotopical equivenlence})$. Since the homotopical equivenlence are determind by $\bigwedge^{r-1} A^r \to A$, so

$$\begin{aligned}
\operatorname{Hom}_{K(A)}(K.(f), A[r]) &= \operatorname{Hom}(K.(f)^{-r}, A)/(\bigwedge^{r-1} A^r \to A) \\
&= \operatorname{coker}(\operatorname{Hom}(K.(f)^{-r}, A) \to \operatorname{Hom}(K.(f)^{-r+1}, A))
\end{aligned}$$

and we win.

♣(Page 90) Actually in the definition we defined $i : Y \to X$ is Koszul-regular immersion. We say $i : Y \to X$ is a regular immersion if locally we have $I|_U = (f_1, ..., f_r)$ where $f_1, ..., f_r$ is regular. Similarly, one can define H_1-regular as in the Theorem 2.2(3). All of these are equivalence if X is locally noetherian, see St 063I.

In the remark $N_{Y/X} = I/I^2$ is locally free, see St 063C and St 063H. Let $i : X \to Y$ be a closed immersion with regular of codimension r, then we have the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \cong \omega_{Y/X}[-r], \omega_{Y/X} = \left(\bigwedge^r N_{Y/X}\right)^\vee.$$

3 Remarks

♠ 1. **Here we assume that a single commutative diagram occupies one line;**

♠ 2. **I omitted the section (4.14) about** Ext **and extensions of groups;**

♠ 3. **If you find errors in my errata, please send to my email: 1225046792@qq.com.**

References

[Illusie] Luc Illusie, *Topics in Algebraic Goemetry*, Université de Paris-Sud Département de Mathématiques, `http://staff.ustc.edu.cn/~yiouyang/Illusie.pdf`.

[Ka] Masaki Kashiwara, Pierre Schapira, *Sheaves on Manifolds*, Springer, 1994.

[St] Stacks project collaborators, *Stacks project*, `https://stacks.math.columbia.edu/`.

[UT1] Ulrich Görtz, Torsten Wedhorn, *Algebraic Goemetry I: Schemes, 2ed edition*, Springer Spektrum, 2020.