Errata and Some Notes for Topics in Algebraic Geometry by Luc Illusie

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Abstract

These notes correct a few typos, errors and some notes in *Topics in Algebraic Geometry* by Prof. Luc Illusie. The original book is [Illusie].

Contents

1 Errata

♦ 1. (Page 10, line -6) Actually L, M are considered as two bicomplexes centered at 0-th column instead of mapping cones;

♦ 2. (Page 18, line 2) Replace
$$\begin{tikzcd} & N \\ L \arrow[ur, "+1"] \arrow[r, "u"] & M \arrow[u, "v"'] \end{tikzcd} \quad \text{by} \quad \begin{tikzcd} & N \arrow[dl, "+1"'] \\ L \arrow[r, "u"] & M \arrow[u, "v"'] \end{tikzcd};$$

♦ 3. (Page 20, line 5) Replace $L \xrightarrow{u} M \to C(u) \xrightarrow{-pr} L[1]$ by $L \xrightarrow{u} M \xrightarrow{i} C(u) \xrightarrow{-pr} L[1]$;

♦ 4. (Page 21, line -5) Replace $u\tilde{f} = 0$ by $u\tilde{f} = f$;

♦ 5. (Page 24, line 12) Replace $\mathrm{Hom}_{\mathcal{C}(S^{-1})} = H(X,Y)/\sim$ by $\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X,Y) = H(X,Y)/\sim$;

♦ 6. (Page 25, line -3) Replace
$$\begin{tikzcd} M[-1] \arrow[r] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd} \quad \text{by} \quad \begin{tikzcd} M[-1] \arrow[r, "t'"] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd};$$

♦ 7. (Page 27, the first paragraph) Replace I^Y to I_Y twice and replace $(I_X)^\circ$ to $(I^X)^\circ$;

♦ 8. (Page 27, line 9) Replace $\mathcal{A}$ to $\mathcal{C}$;

♦ 9. (Page 27, line 10) Replace $\varinjlim_{X' \xrightarrow{s} X \in (I_X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ to $\varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$;

♦ 10. (Page 27, line 12) Replace (X', t, f) to (X', s, f);

♦ 11. (Page 27, line -4) Replace $u \in (I_X)^\circ$ to $u \in (I^X)^\circ$;

- ⧫ 12. (Page 28, line 6) Replace $(C\left(S^{-1}\ ,Q\right)$ to $(C(S^{-1}),Q)$;
- ⧫ 13. (Page 28, line -6) Replace $C(\mathcal{A})(S^{-1})$ to $C(\mathcal{A})(\mathrm{qis}^{-1})$;
- ⧫ 14. (Page 32, line 3) Replace $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1} \to$ to $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1}K \to$ three times;
- ⧫ 15. (Page 36, the second paragraph) Replace all $\tau_{[a,b]}L$ to $\tau_{[a+1,b]}L$ and replace $\tau_{[b-1,b]}L$ to $\tau_{[b,b]}L$;
- ⧫ 16. (Page 40, line 18) Replace $K^+(\mathcal{J})(\mathrm{qis}^{-1})$ to $K^+(\mathcal{I})(\mathrm{qis}^{-1})$;
- ⧫ 17. (Page 40, line -7) Replace (3.8) to (3.10);
- ⧫ 18. (Page 41, line 1) Replace $\{M \to M''$, where $M'' \in K^+(\mathcal{A})$ to $\{M \to M''$, where $M'' \in K^+(\mathcal{A})\}$;
- ⧫ 19. (Page 41, line 2) Replace (e.g. 4.13) to (4.18);
- ⧫ 20. (Page 41, second paragraph) This proof probably has a mistake that pashout may not preserve monomorphism, see [Ka];
- ⧫ 21. (Page 42, lemma 4.29) This proof probably has a mistake that pashout may not preserve monomorphism;
- ⧫ 22. (Page 43, line -1) Replace $E' \in \mathcal{A}$ to $E' \in \mathcal{A}'$;
- ⧫ 23. (Page 45, line 4) Replace (4.18) to (4.27);
- ⧫ 24. (Page 45, line -4) Replace $\eta : FQ \to QG$ to $\eta : QF \to GQ$;
- ⧫ 25. (Page 46, line 2,3) Replace $F(\varepsilon(L'))$ to $\varepsilon(L')$;
- ⧫ 26. (Page 58, line 11) Replace Lemma 6.7 to Proposition 6.7;
- ⧫ 27. (Page 60, line -3,-5) Replace zero to trivial;
- ⧫ 28. (Page 64, line 6) Replace 6.8 to 6.7;
- ⧫ 29. (Page 68, line -4) Replace $C^n(\mathcal{U} \cap V, F)$ to $\check{C}^n(\mathcal{U} \cap V, F)$;
- ⧫ 30. (Page 71, line 4) The proof is same as Theorem 8.3 which reduce to the case of Lemma 8.4, so here we use the same homotopy operator k in 8.4;
- ⧫ 31. (Page 86, line 9) Replace $(-1)^j$ to $(-1)^{j+1}$;
- ⧫ 32. (Page 87, line 13) Replace 1.2 to 2.2;
- ⧫ 33. (Page 88, line 5) Replace $M/(f_1,\cdots,f_r)M$ to $M/(f_1,\cdots,f_{r-1})M$ twice;
- ⧫ 34. (Page 88, line -12) Replace K^{n+1} to $K^{n+1}(v)$;
- ⧫ 35. (Page 88, line -4,-5) Replace $\bigwedge^1 A$ to $\bigwedge^1 A^r$ and replace $\bigwedge^{r-1} A$ to $\bigwedge^{r-1} A^r$;
- ⧫ 36. (Page 89, line -11) Replace $\mathrm{Hom}(K.(f)^{-r}, N)$ to $\mathrm{Hom}(K.(f)^{-r}, A)$;

- ⧫ 37. (Page 90, line -6) Replace canormal to conormal;
- ⧫ 38. (Page 91, line 3) Replace $A[\frac{t_0}{t_i}, ..., \frac{t_r}{t_{i-1}}, \frac{t_r}{t_{i+1}}, ..., \frac{t_r}{t_i}]$ to $A[\frac{t_0}{t_i}, ..., \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, ..., \frac{t_r}{t_i}]$;
- ⧫ 39. (Page 92, line 1) Replace $\check{H}(\mathcal{U}, \mathcal{O}(n))$ to $\check{H}^q(\mathcal{U}, \mathcal{O}(n))$;
- ⧫ 40. (Page 92, line 1) Replace $\bigcup_{i=1}^p U_{i_j}$ to $\bigcup_{i=0}^p U_{i_j}$;
- ⧫ 41. (Page 92, line -9) Replace $\check{C}_{-n} = (0 \to \bigoplus_i t_i^{-n} B \to \cdots)$ to $\check{C}_{-n} = (\bigoplus_i t_i^{-n} B \to \cdots)$;
- ⧫ 42. (Page 93, line -12) Replace $H^r K^{\cdot}(t_0^n, ..., t_r^n, B)$ to $H^{r+1} K^{\cdot}(t_0^n, ..., t_r^n, B)$;
- ⧫ 43. (Page 94, line -6) Replace $k \bigotimes_{\mathcal{O}_{X,x}} L$ to $L_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$;
- ⧫ 44. (Page 95, line -13) Replace $U_i f$ to $(U_i)_f$;
- ⧫ 45. (Page 95, line -8) Replace 3.1 to 4.1;
- ⧫ 46. (Page 96, line -14) Replace $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes m}$ to $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes n}$;
- ⧫ 47. (Page 100, line -10) Replace X_0 to X_s;
- ⧫ 48. (Page 105, line 3) Replace ia to is;
- ⧫ 49. (Page 106, line 1) Delete the sentence "associated to L_1 and L_2 repectively";
- ⧫ 50. (Page 106, line 2) Replace $i = 1, 2$ to $i = 0, 1$;
- ⧫ 51. (Page 106, line -7) Replace R_n to B_n;
- ⧫ 52. (Page 107, line -11) Replace X_0 to X;
- ⧫ 53. (Page 107, line -9) In this place, $Z = Ass(\mathscr{F})$;

2 Some Notes

♣(Page 72, Theorem 8.12) **Theorem of Leray.** Let $(X, \mathcal{O}_X)$ be a ringed space and F be an $\mathcal{O}_X$-module. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of it. If for every nonempty finite subset $J \subset I$ and every $q > 0$ such that $H^q(U_J, F) = 0$ where $U_J = \bigcap_{j \in J} U_j$, then $\check{H}^n(\mathfrak{U}, F) \cong H^n(X, F)$.

The first proof. Consider $\mathscr{H}^q(X, F)$ be a presheaf with $U \mapsto H^q(U, F)$. By Grothendieck spectral sequence, there exists a spectral sequence such that $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) \Rightarrow H^{p+q}(X, F)$ and $\check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) = 0$ for $p > 0$ in this situation. Then the E_2 page is

$$\begin{tikzcd}
\cdots & 0 \arrow[dr] & 0 \arrow[dr] & 0 & \cdots \\
\cdots & \check{H}^{p-1}(\mathfrak{U}, F) & \check{H}^p(\mathfrak{U}, F) & \check{H}^{p+1}(\mathfrak{U}, F) & \cdots
\end{tikzcd}$$

Since it converge to $H^p(X, F)$ and for now $E_2 = E_\infty$, then we win. Here we use the fact that $\check{H}^p(\mathfrak{U}, -)$ as the right derived functor of $\check{H}^0(\mathfrak{U}, -)$, see St 01EN in [St]. □

The second proof. See St 01EV in [St]. □

♣(Page 84, Corollary 1.4) Here we need to show that $R^q f_* F$ is a sheaf associated to the presheaf $V \mapsto H^q(f^{-1}(V), F)$. For now we assume $f : X \to Y$ be the morphism between ringed spaces and F is any $\mathcal{O}_X$-module.

Proof. Let $F[0]$ quasi-isomorphic to I^* where I^k are injective $\mathcal{O}_Y$-modules. So $R^q f_* F = H^q(Rf_* F) = H^q(f_* I^*)$. We find that $H^i(f_* I^*)$ is a sheaf associated to the presheaf

$$\begin{aligned}
V \mapsto & \frac{\ker(f_* I^i(V) \to f_* I^{i+1}(V))}{\operatorname{Im}(f_* I^{i-1}(V) \to f_* I^i(V))} \\
&= \frac{\ker(I^i(f^{-1}V) \to I^{i+1}(f^{-1}V))}{\operatorname{Im}(I^{i-1}(f^{-1}V) \to I^i(f^{-1}V))} = H^i(f^{-1}(V), F)
\end{aligned}$$

and we win. □

♣(Page 85, Corollary 1.6) Actually we can show that if f is qcqs morphism and $F \in Qcoh(X)$, then $R^q f_* F \in Qcoh(Y)$ for all $q \geq 0$. For $q = 0$, see [UT1] 10.27. For $q > 0$ and f qcqs, see St 01XJ in [St].

♣(Page 89, line -11) The reason of the first euqality is that if we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \bigwedge^{r-1} A^r \arrow[r] \arrow[d] \arrow[dl, dashed] & \cdots \arrow[r] & \bigwedge^1 A^r \arrow[r] \arrow[d] & A \arrow[d] \\
0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So $\operatorname{Hom}_{K(A)}(K.(f), A[r]) = \operatorname{Hom}(K.(f)^{-r}, A)/(\text{homotopical equivenlence})$. Since the homotopical equivenlence are determind by $\bigwedge^{r-1} A^r \to A$, so

$$\begin{aligned}
\operatorname{Hom}_{K(A)}(K.(f), A[r]) &= \operatorname{Hom}(K.(f)^{-r}, A)/(\bigwedge^{r-1} A^r \to A) \\
&= \operatorname{coker}(\operatorname{Hom}(K.(f)^{-r}, A) \to \operatorname{Hom}(K.(f)^{-r+1}, A))
\end{aligned}$$

and we win.
♣(Page 90) Actually in the definition we defined $i : Y \to X$ is Koszul-regular immersion. We say $i : Y \to X$ is a regular immersion if locally we have $I|_U = (f_1, ..., f_r)$ where $f_1, ..., f_r$ is regular. Similarly, one can define H_1-regular as in the Theorem 2.2(3). All of these are equivalence if X is locally noetherian, see St 063I.

In the remark $N_{Y/X} = I/I^2$ is locally free, see St 063C and St 063H. Let $i : X \to Y$ be a closed immersion with regular of codimension r, then we have the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \cong \omega_{Y/X}[-r], \omega_{Y/X} = \left(\bigwedge^r N_{Y/X}\right)^\vee .$$

Actually one can assume X be a ringed space and I is Koszul-regular. The proof see St 0BQZ.
♣(Page 94) In general case, we say F is coherent if for all open U and all $n \geq 0$, $\ker(\mathcal{O}_X^n|_U \to F|_U)$ is finite type. But in the locally noetherian case, this is the same as finitely presentation or quasi-coherent+finite type.

In the hypothesis of Lemma 4.2, we can just let X is qcqs and E is quasi-coherent. For the proof is easy, I will omit it, see [UT1] Theorem 7.22.
♣(Page 97) In the proof of 4.6, we have $i_*\mathcal{F} \otimes \mathcal{O}_P(n) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}_P(n))$. This isomorphism we use the projection formula, as follows.
Theorem.(Projection Formula) Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect (See St 08CM), then

$$Rf_*E \otimes^L_{\mathcal{O}_Y} K = Rf_*(E \otimes^L_{\mathcal{O}_Y} Lf^*K)$$

in $D(\mathcal{O}_Y)$.

In St 0B55 we find that if f is a homeomorphism onto a closed subset, then this is an isomorphism always.
♣(Page 101) In the proof of remark, we find that $R^qf_*F = H^q(X, F)^\sim$. The reason as follows.

Let $f : X \to S$ is qcqs and we let S affine and $F \in Qcoh(X)$. Then $Rf_*F \in Qcoh(X)$, see St 01XJ. By Leray spectral sequence, we have $E_2^{p,q} = H^p(S, R^qf_*F) \Rightarrow H^{p+q}(X, F)$. Since $Rf_*F \in Qcoh(X)$, we have $E_2^{p,q} = 0$ for all $p > 0$, then $E_2 = E_\infty$, then $H^0(S, R^qf_*F) = H^q(X, F)$. Since S affine, we have $R^qf_*F = H^q(X, F)^\sim$.
♣(Page 107) In the fact (1), we claim that if $s \in \Gamma(X, \mathcal{O}_X)$ such that $s(x) \neq 0$ for all $x \in Ass(\mathcal{F})$, then $s : (F) \to (F)$ is injective where X is affine noetherian and $\mathcal{F}$ is of finite type. Actually we can use the following conclusion of commutative algebra:
Theorem. If R is Noetherian ring and $f : M \to N$ be a map of R-modules. Assume that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ at least one of the following happens: (i) $M_\mathfrak{p} \to N_\mathfrak{p}$ is injective; (ii) $\mathfrak{p} \notin Ass(M)$. Then f is injective.

Proof of the Theorem. Now we claim that $Ass(\ker f) = \emptyset$, hence $\ker f = 0$. Since in the case of $\mathfrak{p}$ finitely generated (this is right since R Noetherian), then $\mathfrak{p} \in Ass(M)$ iff $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$. So there exists $x \in \ker(M_\mathfrak{p} \to N_\mathfrak{p})$ with $Ann_{R_\mathfrak{p}}(x) = \mathfrak{p}R_\mathfrak{p}$. This is impossible in both above case. □

In the fact (2), we have the classical conclusion: M is a finitely generated A-module, then $\mathfrak{p} \in Supp(M)$ iff $\mathfrak{p} \in V(Ann(M))$. Actually, we let $M = (t_1, ..., t_n)_A$, then

$$\mathfrak{p} \in Supp(M) \Leftrightarrow M_\mathfrak{p} \neq 0 \Leftrightarrow \mathfrak{p} \supset \bigcap_i Ann(t_i) \Leftrightarrow \mathfrak{p} \in V(Ann(M)),$$

well done.

3 Remarks

♠ 1. **Here we assume that a single commutative diagram occupies one line;**

♠ 2. **I omitted the section (4.14) about** Ext **and extensions of groups;**

♠ 3. **I omitted some proofs if I have read before, such as the proof of Theorem II.4.7 (2)⇒(1);**

♠ 4. **If you find errors in my errata, please send to my email: 1225046792@qq.com.**

References

[Illusie] Luc Illusie, *Topics in Algebraic Goemetry*, Université de Paris-Sud Département de Mathématiques, `http://staff.ustc.edu.cn/~yiouyang/Illusie.pdf`.

[Ka] Masaki Kashiwara, Pierre Schapira, *Sheaves on Manifolds*, Springer, 1994.

[St] Stacks project collaborators, *Stacks project*, `https://stacks.math.columbia.edu/`.

[UT1] Ulrich Görtz, Torsten Wedhorn, *Algebraic Goemetry I: Schemes, 2ed edition*, Springer Spektrum, 2020.