Errata and Some Notes for Topics in Algebraic Geometry by Luc Illusie

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Abstract

These notes correct a few typos, errors and some notes in *Topics in Algebraic Geometry* by Prof. Luc Illusie. The original book is [Illusie].

Contents

1 Remarks

♠ 1. **Here we assume that a single commutative diagram occupies one line;**

♠ 2. **I omitted the subsection I.4.14, the section III.4 and the chapter IV;**

♠ 3. **I omitted some proofs if I have read before, such as the proof of Theorem II.4.7 (2)⇒(1);**

♠ 4. **If you find errors in my errata, please send them to me. My homepage:** `https://dvlxlwz.github.io/`

♠ 5. **The note [Psi1] is written by my friend Psi and I have learned lot from him. If you need to read this file, you could send me an email and I will ask him;**

2 Errata

♦ 1. (Page 10, line -6) Actually L, M are considered as two bicomplexes centered at 0-th column instead of mapping cones;

♦ 2. (Page 18, line 2) Replace $\begin{tikzcd} & N \\ L \arrow[ur, "+1"] \arrow[r, "u"] & M \arrow[u, "v"] \end{tikzcd}$ by $\begin{tikzcd} & N \arrow[dl, "+1"'] \\ L \arrow[r, "u"] & M \arrow[u, "v"] \end{tikzcd}$;

♦ 3. (Page 20, line 5) Replace $L \xrightarrow{u} M \to C(u) \xrightarrow{-pr} L[1]$ by $L \xrightarrow{u} M \xrightarrow{i} C(u) \xrightarrow{-pr} L[1]$;

♦ 4. (Page 21, line -5) Replace $u\tilde{f} = 0$ by $u\tilde{f} = f$;

♦ 5. (Page 24, line 12) Replace $\mathrm{Hom}_{\mathcal{C}(S^{-1})} = H(X,Y)/\sim$ by $\mathrm{Hom}_{\mathcal{C}(S^{-1})}(X,Y) = H(X,Y)/\sim$;

- ⧫ 6. (Page 25, line -3) Replace $\begin{tikzcd} M[-1] \arrow[r] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$ by $\begin{tikzcd} M[-1] \arrow[r, "t'"] & Y \\ X \arrow[u, "f'"] \arrow[ur, "f"'] & \end{tikzcd}$;
- ⧫ 7. (Page 27, the first paragraph) Replace I^Y to I_Y twice and replace $(I_X)^\circ$ to $(I^X)^\circ$;
- ⧫ 8. (Page 27, line 9) Replace $\mathcal{A}$ to $\mathcal{C}$;
- ⧫ 9. (Page 27, line 10) Replace $\varinjlim_{X' \xrightarrow{s} X \in (I_X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$ to $\varinjlim_{X' \xrightarrow{s} X \in (I^X)^\circ} \mathrm{Hom}_{\mathcal{C}}(X', Y)$;
- ⧫ 10. (Page 27, line 12) Replace (X', t, f) to (X', s, f);
- ⧫ 11. (Page 27, line -4) Replace $u \in (I_X)^\circ$ to $u \in (I^X)^\circ$;
- ⧫ 12. (Page 28, line 6) Replace $(C\big(S^{-1}\ ,Q)$ to $(C(S^{-1}), Q)$;
- ⧫ 13. (Page 28, line -6) Replace $C(\mathcal{A})(S^{-1})$ to $C(\mathcal{A})(\mathrm{qis}^{-1})$;
- ⧫ 14. (Page 32, line 3) Replace $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1} \to$ to $\tau_{\leq a}K \xrightarrow{f} K \xrightarrow{g} \tau_{\geq a+1}K \to$ three times;
- ⧫ 15. (Page 36, the second paragraph) Replace all $\tau_{[a,b]}L$ to $\tau_{[a+1,b]}L$ and replace $\tau_{[b-1,b]}L$ to $\tau_{[b,b]}L$;
- ⧫ 16. (Page 40, line 18) Replace $K^+(\mathcal{J})(\mathrm{qis}^{-1})$ to $K^+(\mathcal{I})(\mathrm{qis}^{-1})$;
- ⧫ 17. (Page 40, line -7) Replace (3.8) to (3.10);
- ⧫ 18. (Page 41, line 1) Replace $\{M \to M''$, where $M'' \in K^+(\mathcal{A})$ to $\{M \to M''$, where $M'' \in K^+(\mathcal{A})\}$;
- ⧫ 19. (Page 41, line 2) Replace (e.g. 4.13) to (4.18);
- ⧫ 20. (Page 41, second paragraph) This proof probably has a mistake that pashout may not preserve monomorphism, see [Ka];
- ⧫ 21. (Page 42, lemma 4.29) This proof probably has a mistake that pashout may not preserve monomorphism;
- ⧫ 22. (Page 43, line -1) Replace $E' \in \mathcal{A}$ to $E' \in \mathcal{A}'$;
- ⧫ 23. (Page 45, line 4) Replace (4.18) to (4.27);
- ⧫ 24. (Page 45, line -4) Replace $\eta : FQ \to QG$ to $\eta : QF \to GQ$;
- ⧫ 25. (Page 46, line 2,3) Replace $F(\varepsilon(L'))$ to $\varepsilon(L')$;
- ⧫ 26. (Page 58, line 11) Replace Lemma 6.7 to Proposition 6.7;
- ⧫ 27. (Page 60, line -3,-5) Replace zero to trivial;
- ⧫ 28. (Page 64, line 6) Replace 6.8 to 6.7;

♦ 29. (Page 68, line -4) Replace $C^n(\mathcal{U} \cap V, F)$ to $\check{C}^n(\mathcal{U} \cap V, F)$;

♦ 30. (Page 71, line 4) The proof is same as Theorem 8.3 which reduce to the case of Lemma 8.4, so here we use the same homotopy operator k in 8.4;

♦ 31. (Page 86, line 9) Replace $(-1)^j$ to $(-1)^{j+1}$;

♦ 32. (Page 87, line 13) Replace 1.2 to 2.2;

♦ 33. (Page 88, line 5) Replace $M/(f_1, \cdots, f_r)M$ to $M/(f_1, \cdots, f_{r-1})M$ twice;

♦ 34. (Page 88, line -12) Replace K^{n+1} to $K^{n+1}(v)$;

♦ 35. (Page 88, line -4,-5) Replace $\bigwedge^1 A$ to $\bigwedge^1 A^r$ and replace $\bigwedge^{r-1} A$ to $\bigwedge^{r-1} A^r$;

♦ 36. (Page 89, line -11) Replace $\mathrm{Hom}(K.(f)^{-r}, N)$ to $\mathrm{Hom}(K.(f)^{-r}, A)$;

♦ 37. (Page 90, line -6) Replace canormal to conormal;

♦ 38. (Page 91, line 3) Replace $A[\frac{t_0}{t_i}, ..., \frac{t_r}{t_{i-1}}, \frac{t_r}{t_{i+1}}, ..., \frac{t_r}{t_i}]$ to $A[\frac{t_0}{t_i}, ..., \frac{t_{i-1}}{t_i}, \frac{t_{i+1}}{t_i}, ..., \frac{t_r}{t_i}]$;

♦ 39. (Page 92, line 1) Replace $\check{H}(\mathcal{U}, \mathcal{O}(n))$ to $\check{H}^q(\mathcal{U}, \mathcal{O}(n))$;

♦ 40. (Page 92, line 1) Replace $\bigcup_{i=1}^p U_{i_j}$ to $\bigcup_{i=0}^p U_{i_j}$;

♦ 41. (Page 92, line -9) Replace $\check{C}_{-n} = (0 \to \bigoplus_i t_i^{-n} B \to \cdots)$ to $\check{C}_{-n} = (\bigoplus_i t_i^{-n} B \to \cdots)$;

♦ 42. (Page 93, line -12) Replace $H^r K^{\cdot}(t_0^n, ..., t_r^n, B)$ to $H^{r+1} K^{\cdot}(t_0^n, ..., t_r^n, B)$;

♦ 43. (Page 94, line -6) Replace $k \bigotimes_{\mathcal{O}_{X,x}} L$ to $L_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$;

♦ 44. (Page 95, line -13) Replace $U_i f$ to $(U_i)_f$;

♦ 45. (Page 95, line -8) Replace 3.1 to 4.1;

♦ 46. (Page 96, line -14) Replace $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes m}$ to $(F \otimes L^{\otimes r}) \otimes (L')^{\otimes n}$;

♦ 47. (Page 100, line -10) Replace X_0 to X_s;

♦ 48. (Page 105, line 3) Replace ia to is;

♦ 49. (Page 106, line 1) Delete the sentence "associated to L_1 and L_2 repectively";

♦ 50. (Page 106, line 2) Replace $i = 1, 2$ to $i = 0, 1$;

♦ 51. (Page 106, line -7) Replace R_n to B_n;

♦ 52. (Page 107, line -11) Replace X_0 to X;

♦ 53. (Page 107, line -9) In this place, $Z = Ass(\mathscr{F})$;

♦ 54. (Page 117, line 10,12) Replace $A' \otimes I^2/I^2$ to $A' \otimes (I/I^2)$ twice;

♦ 55. (Page 117, line -8) Replace $Z = \mathrm{Spec}(C)$ to $X = \mathrm{Spec}(C)$;

♦ 56. (Page 119, line 4) Replace $\begin{array}{ccc} X & \xhookrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & \nearrow_{g} & \\ Y & & \end{array}$ to $\begin{array}{ccc} X & \xhookrightarrow{i} & Z \\ {\scriptstyle f}\downarrow & \swarrow_{g} & \\ Y & & \end{array}$;

♦ 57. (Page 119, line -1) Replace $\mathrm{Hom}_B(I \otimes_C B, M)$ to $\mathrm{Hom}_B(J \otimes_C B, M)$;

♦ 58. (Page 120, line 2) Replace $0 \to \mathrm{Der}_A(B, M) \to \mathrm{Hom}_A(C, M) \to \mathrm{Der}_C(I, M)$ to $0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Hom}_B(J/J^2, M)$;

♦ 59. (Page 121, line -6) Replace $\{t \in X(k[\varepsilon]) : xi = x\} \simeq (m_x/m_x^2)^\wedge$ to $\{t \in X(k[\varepsilon]) : ti = x\} \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$;

♦ 60. (Page 121, line -6) Replace $\mathcal{T}_x = \cdots$ by

$$\mathcal{T}_x = \{h \in \mathrm{Hom}_k(\mathcal{O}_{X,x}, k[\varepsilon]) : \pi h = p\} = \mathrm{Der}_k(\mathcal{O}_{X,x}, k[\varepsilon])$$
$$= \mathrm{Hom}_{\mathcal{O}_{X,x}}(\Omega^1_{X/k,x} \otimes_{\mathcal{O}_{X,x}} k(x), k) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee;$$

♦ 61. (Page 129, line 8) Replace $\mathcal{O}'_X$ to $\mathcal{O}_{X'}$;

♦ 62. (Page 130, line 9) Replace $\mathcal{I}$ to $\mathcal{I}^2$;

♦ 63. (Page 131, Lemma 2.8) The condition $\mathscr{E}xt^1_{\mathcal{O}_X}(E, F) = 0$ should be replaced. See notes Below;

♦ 64. (Page 132, line -5) Replace (f^*, D) to (f_*, D);

♦ 65. (Page 132, line -1) Replace $f^{-1}(\mathcal{O}_S)$ to $g^{-1}(\mathcal{O}_S)$;

♦ 66. (Page 134, line -4) Replace $\mathrm{Ext}_S(Y, f_*\mathcal{I})$ to $\mathrm{Ext}_S(X, \mathcal{I})$;

♦ 67. (Page 143, line 6) Replace $k(x) = k$ to $k(x) = \overline{k}$;

♦ 68. (Page 143, line -3) Replace $d_{Z/k} \otimes k(x)$ to $\{d_{Z/k}(f_i) \otimes k(x)\}_{1 \le i \le r}$;

♦ 69. (Page 143, line -1) Replace $\begin{array}{ccc} \mathcal{I}/\mathcal{I}^2 & \xhookrightarrow{d_{Z/k}} & \Omega^1_{Z/k} \otimes k(x) \\ & \searrow^{\varphi} & \uparrow{\scriptstyle d_{Z/k}} \\ & & \mathfrak{m}/\mathfrak{m}^2 \end{array}$ to $\begin{array}{ccc} \mathcal{I}/\mathcal{I}^2 & \xhookrightarrow{d_{Z/k} \otimes k(x)} & \Omega^1_{Z/k} \otimes k(x) \\ & \searrow^{\varphi} & \uparrow{\scriptstyle d_{Z/k} \otimes k(x)} \\ & & \mathfrak{m}/\mathfrak{m}^2 \end{array}$;

♦ 70. (Page 143, Theorem 3.7(2)) See my notes below;

♦ 71. (Page 145, line 5) Replace $\Omega_{\mathbb{A}^n_{k'}/k'} \otimes k(x')$ to $i'^*\Omega^1_{\mathbb{A}^n_{k'}/k'} \otimes k(x')$;

♦ 72. (Page 145, line 7) Replace $\mathcal{N}' \otimes k(x)$ to $\mathcal{N}' \otimes k(x')$;

♦ 73. (Page 145, line 9) Replace the diagram to $\begin{array}{ccc} \mathcal{N}' \otimes k(x') & \longrightarrow & i'^*\Omega^1_{\mathbb{A}^n_{k'}/k'} \otimes k(x') \\ \uparrow & & \uparrow \\ \mathcal{N} \otimes k(x) & \longrightarrow & i^*\Omega^1_{\mathbb{A}^n_k/k} \otimes k(x) \end{array}$;

♦ 74. (Page 145, line 11) Replace 2.10(4)(b) to 2.3(4)(b);

♦ 75. (Page 146, line -3) Replace injective to exact;

♦ 76. (Page 146, line -1) Replace $\varinjlim$ to $\varprojlim$ twice;

♦ 77. (Page 147, line -4) This is not from 3.11, see my notes below;

♦ 78. (Page 149, line 7) Delete the whole santence since (2) implies (1) has been proved below;

♦ 79. (Page 149, line -7) Replace $df_i(x)$ to $df_i(x) \otimes k(x)$;

♦ 80. (Page 149, line -1) Replace $\mathrm{rk}_{k(x)}\Omega^1_{X_y/y}$ to $\mathrm{rk}_{k(x)}\Omega^1_{X_y/y} \otimes k(x)$;

♦ 81. (Page 150, line 8) Replace $\Omega^1_{X/Y} \otimes k(y)$ to $\Omega^1_{X/Y} \otimes \mathcal{O}_{X_y}$;

♦ 82. (Page 161, line -2) Replace $j^! i!$ to $j^! i^!$;

♦ 83. (Page 162, line 4) Replace $D^+(X)$ to $D^+(Y)$;

♦ 84. (Page 162, line 8) Replace $D^+(X), D^+(Y)$ to $D^+(Y), D_+(X)$;

♦ 85. (Page 163, line -2) Replace (3)(a) to (3)(b);

♦ 86. (Page 165, line -8) Replace $\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$ to $\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$;

♦ 87. (Page 165, line -2) Replace $i'_* i^! f^* F$ to $i'_* i'^! f^* F$;

♦ 88. (Page 172, line 5) Replace 1.24 to 1.23;

♦ 89. (Page 175, line 11) Replace

$$Rf_* R\mathcal{H}om(E, F) \cong \mathcal{H}om^\bullet(E', F') \to \mathcal{H}om^\bullet(E', F')$$

to

$$Rf_* R\mathcal{H}om(E, F) \cong f_*\mathcal{H}om^\bullet(E', F') \to \mathcal{H}om^\bullet(f_*E', f_*F');$$

♦ 90. (Page 180, line 2) Replace $\text{proj.dim}_{\mathcal{O}_{P,x}} = \cdots$ to $\text{proj.dim}_{\mathcal{O}_{P,x}} \mathcal{O}_{X,x} = \cdots$;

♦ 91. (Page 180, line 10) Replace $f^! \mathcal{O}_Y \simeq i^! g^! \mathcal{O}_Y = \cdots$ to $f^! \mathcal{O}_S \cong i^! g^! \mathcal{O}_S = \cdots$;

3 Some Notes

♣(Page 72, Theorem 8.12) **Theorem of Leray.** Let $(X, \mathcal{O}_X)$ be a ringed space and F be an $\mathcal{O}_X$-module. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of it. If for every nonempty finite subset $J \subset I$ and every $q > 0$ such that $H^q(U_J, F) = 0$ where $U_J = \bigcap_{j \in J} U_j$, then $\check{H}^n(\mathfrak{U}, F) \cong H^n(X, F)$.

The first proof. Consider $\mathscr{H}^q(X, F)$ be a presheaf with $U \mapsto H^q(U, F)$. By Grothendieck spectral sequence, there exists a spectral sequence such that $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) \Rightarrow H^{p+q}(X, F)$ and $\check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, F)) = 0$ for $p > 0$ in this situation. Then the E_2 page is

$$\begin{tikzcd}
\cdots & 0 \arrow[dr] & 0 \arrow[dr] & 0 & \cdots \\
\cdots & \check{H}^{p-1}(\mathfrak{U}, F) & \check{H}^p(\mathfrak{U}, F) & \check{H}^{p+1}(\mathfrak{U}, F) & \cdots
\end{tikzcd}$$

Since it converge to $H^p(X, F)$ and for now $E_2 = E_\infty$, then we win. Here we use the fact that $\check{H}^p(\mathfrak{U}, -)$ as the right derived functor of $\check{H}^0(\mathfrak{U}, -)$, see St 01EN in [St]. $\square$

The second proof. See St 01EV in [St]. $\square$

♣(Page 84, Corollary 1.4) Here we need to show that $R^q f_* F$ is a sheaf associated to the presheaf $V \mapsto H^q(f^{-1}(V), F)$. For now we assume $f : X \to Y$ be the morphism between ringed spaces and F is any $\mathcal{O}_X$-module.

Proof. Let $F[0]$ quasi-isomorphic to I^* where I^k are injective $\mathcal{O}_Y$-modules. So $R^q f_* F = H^q(Rf_* F) = H^q(f_* I^*)$. We find that $H^i(f_* I^*)$ is a sheaf associated to the presheaf

$$\begin{aligned}
V \mapsto & \frac{\ker(f_* I^i(V) \to f_* I^{i+1}(V))}{\operatorname{Im}(f_* I^{i-1}(V) \to f_* I^i(V))} \\
&= \frac{\ker(I^i(f^{-1}V) \to I^{i+1}(f^{-1}V))}{\operatorname{Im}(I^{i-1}(f^{-1}V) \to I^i(f^{-1}V))} = H^i(f^{-1}(V), F)
\end{aligned}$$

and we win. $\square$

♣(Page 85, Corollary 1.6) Actually we can show that if f is qcqs morphism and $F \in Qcoh(X)$, then $R^q f_* F \in Qcoh(Y)$ for all $q \geq 0$. For $q = 0$, see [UT1] 10.27. For $q > 0$ and f qcqs, see St 01XJ in [St].

♣(Page 89, line -11) The reason of the first euqality is that if we consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \bigwedge^{r-1} A^r \arrow[r] \arrow[d] \arrow[dl, dashed] & \cdots \arrow[r] & \bigwedge^1 A^r \arrow[r] \arrow[d] & A \arrow[d] \\
0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So $\operatorname{Hom}_{K(A)}(K.(f), A[r]) = \operatorname{Hom}(K.(f)^{-r}, A)/(\text{homotopical equivenlence})$. Since the homotopical equivenlence are determind by $\bigwedge^{r-1} A^r \to A$, so

$$\begin{aligned}
\operatorname{Hom}_{K(A)}(K.(f), A[r]) &= \operatorname{Hom}(K.(f)^{-r}, A)/(\bigwedge^{r-1} A^r \to A) \\
&= \operatorname{coker}(\operatorname{Hom}(K.(f)^{-r}, A) \to \operatorname{Hom}(K.(f)^{-r+1}, A))
\end{aligned}$$

and we win.

♣(Page 90) Actually in the definition we defined $i : Y \to X$ is Koszul-regular immersion. We say $i : Y \to X$ is a regular immersion if locally we have $I|_U = (f_1, ..., f_r)$ where $f_1, ..., f_r$ is regular. Similarly, one can define H_1-regular as in the Theorem 2.2(3). All of these are equivalence if X is locally noetherian, see St 063I.

In the remark $N_{Y/X} = I/I^2$ is locally free, see St 063C and St 063H. Let $i : X \to Y$ be a closed immersion with regular of codimension r, then we have the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X) \cong \omega_{Y/X}[-r], \omega_{Y/X} = \left(\bigwedge^r N_{Y/X}\right)^\vee .$$

Actually one can assume X be a ringed space and I is Koszul-regular. The proof see St 0BQZ.

♣(Page 94) In general case, we say F is coherent if for all open U and all $n \geq 0$, $\ker(\mathcal{O}_X^n|_U \to F|_U)$ is finite type. But in the locally noetherian case, this is the same as finitely presentation or quasi-coherent+finite type.

In the hypothesis of Lemma 4.2, we can just let X is qcqs and E is quasi-coherent. For the proof is easy, I will omit it, see [UT1] Theorem 7.22.

♣(Page 97) In the proof of 4.6, we have $i_*\mathcal{F} \otimes \mathcal{O}_P(n) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}_P(n))$. This isomorphism we use the projection formula, as follows.

Theorem.(Projection Formula) Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect (See St 08CM), then

$$Rf_*E \otimes^L_{\mathcal{O}_Y} K = Rf_*(E \otimes^L_{\mathcal{O}_Y} Lf^*K)$$

in $D(\mathcal{O}_Y)$.

In St 0B55 we find that if f is a homeomorphism onto a closed subset, then this is an isomorphism always.

♣(Page 101) In the proof of remark, we find that $R^qf_*F = H^q(X, F)^\sim$. The reason as follows.

Let $f : X \to S$ is qcqs and we let S affine and $F \in Qcoh(X)$. Then $Rf_*F \in Qcoh(X)$, see St 01XJ. By Leray spectral sequence, we have $E_2^{p,q} = H^p(S, R^qf_*F) \Rightarrow H^{p+q}(X, F)$. Since $Rf_*F \in Qcoh(X)$, we have $E_2^{p,q} = 0$ for all $p > 0$, then $E_2 = E_\infty$, then $H^0(S, R^qf_*F) = H^q(X, F)$. Since S affine, we have $R^qf_*F = H^q(X, F)^\sim$.

♣(Page 107) In the fact (1), we claim that if $s \in \Gamma(X, \mathcal{O}_X)$ such that $s(x) \neq 0$ for all $x \in Ass(\mathcal{F})$, then $s : (F) \to (F)$ is injective where X is affine noetherian and $\mathcal{F}$ is of finite type. Actually we can use the following conclusion of commutative algebra:

Theorem. If R is Noetherian ring and $f : M \to N$ be a map of R-modules. Assume that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ at least one of the following happens: (i) $M_\mathfrak{p} \to N_\mathfrak{p}$ is injective; (ii) $\mathfrak{p} \notin Ass(M)$. Then f is injective.

Proof of the Theorem. Now we claim that $Ass(\ker f) = \emptyset$, hence $\ker f = 0$. Since in the case of $\mathfrak{p}$ finitely generated (this is right since R Noetherian), then $\mathfrak{p} \in Ass(M)$ iff $\mathfrak{p}R_\mathfrak{p} \in Ass(M_\mathfrak{p})$. So there exists $x \in \ker(M_\mathfrak{p} \to N_\mathfrak{p})$ with $Ann_{R_\mathfrak{p}}(x) = \mathfrak{p}R_\mathfrak{p}$. This is impossible in both above case. □

In the fact (2), we have the classical conclusion: M is a finitely generated A-module, then $\mathfrak{p} \in Supp(M)$ iff $\mathfrak{p} \in V(Ann(M))$. Actually, we let $M = (t_1, ..., t_n)_A$, then

$$\mathfrak{p} \in Supp(M) \Leftrightarrow M_\mathfrak{p} \neq 0 \Leftrightarrow \mathfrak{p} \supset \bigcap_i Ann(t_i) \Leftrightarrow \mathfrak{p} \in V(Ann(M)),$$

well done.
♣(Page 111) In the definition of A-derivation, we should claim a basic property: for $D \in \mathrm{Der}_A(B, M)$ we have $D(a) = 0$ for all $a \in A$. This is because $D(a) = aD(1)$ and $D(1) = 1 \cdot D(1) + D(1) \cdot 1$. This is easy but important and we will prove some exact sequence by using this such as $C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$.
♣(Page 121) In the proof of Corollary 1.22, we will not use the equalities in the original proof. Actually, we have $\mathcal{T}_x = \{h \in \mathrm{Hom}_k(\mathcal{O}_{X,x}, k[\varepsilon]) : \pi h = p\}$ apparently, as the following diagram, since we have a bijective correspondence between $\mathrm{Spec}(R) \to X$ and $\mathcal{O}_{X,x} \to R$ where R is local.

$$\begin{tikzcd} k[\varepsilon]/(\varepsilon^2) \arrow[rr, "\pi"] & & k = \mathcal{O}_{X,x}/\mathfrak{m}_x \\ & \mathcal{O}_{X,x} \arrow[ul, dashed, "h"] \arrow[ur, "p"'] & \end{tikzcd}$$

Next these $h : \mathcal{O}_{X,x} \to k[\varepsilon]$ iff maps $m \in \mathfrak{m}_x$ to a linear object $a\varepsilon$. So we get a morphism $H : \mathfrak{m}_x \to k, m \mapsto a$ which induce $h' : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k, [m] \mapsto a$ and we get $\mathcal{T}_x \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee, h \mapsto h'$ and it's easy to see that this is an isomorphism, well done.
♣(Page 122) In the proof of the Euler exact sequence, we first claim that $\ker u = M$ generated by $e_i t_j - e_j t_i, j \neq i$. Consider the Koszul complex $K.(u) : \cdots \to \bigwedge^2 B(-1)^{r+1} \to B(-1)^{r+1} \to B \to 0$. So we know that $K.(u) \simeq B[0]$. So $\ker u = \mathrm{Im}(\bigwedge^2 B(-1)^{r+1} \to B(-1)^{r+1})$, so $\ker u = M$ generated by $e_i t_j - e_j t_i, j \neq i$. Note that e_i is degree 1 in $B(-1)$.

Finally we claim that $\phi_i : \Omega^1_{P/S}|_{U_i} \to \widetilde{M}|_{U_i}$ satisfies $\phi_i = \phi_j$ in $U_i \cap U_j$. Since $\frac{t_k}{t_i} = \frac{t_k}{t_j}\frac{t_j}{t_i}$, we have $d(\frac{t_k}{t_i}) = \frac{t_k}{t_j} d(\frac{t_j}{t_i}) + d(\frac{t_k}{t_j})\frac{t_j}{t_i}$, so

$$d\left(\frac{t_k}{t_i}\right) - \frac{t_k}{t_j} d\left(\frac{t_j}{t_i}\right) = d\left(\frac{t_k}{t_j}\right)\frac{t_j}{t_i}.$$

Apply ϕ_i, ϕ_j to left, right side, repectively, we get the same thing $\frac{t_j e_k - t_k e_j}{t_i t_j}$, so we can glue.

More generally, we have more general Euler exact sequence. For the proof see Theorem 4.5.13 in [MB].

Theorem. Let E be a quasi-coherent module on a scheme S. Let $p : \mathbb{P}(E) \to S$ be the associated projective scheme. Then there is an exact sequence of quasi-coherent modules on $\mathbb{P}(E)$

$$0 \to \Omega^1_{\mathbb{P}(E)/S} \to p^*(E)(-1) \to \mathcal{O}_{\mathbb{P}(E)} \to 0$$

The epimorphism is dual to the canonical one $p^*(E) \to \mathcal{O}_{\mathbb{P}(E)}(1)$.
♣(Page 124) Note that this definition of formally smooth (resp. unramified or étale) is **local** but different with [St] and EGA which set T_0, T here are any affine schemes with thickening of order 1. But they are equivalent but not trivial. To verify this, we need to show that in affine case it can be check on zariski local. We need the following conclusion.

Theorem.(Projectivity on the Ring is Zariski Local) For a ring A an A-module M being projective iff there exists an affine open cover $\{U_i\}$ with $\widetilde{M}(U_i)$ is projective U_i-module.

This is a not trivial result and can be derived by Michel Raynaud, Laurent Gruson's work [RG]. But unfortunately, there are some mistakes in their paper (see BCnrd's words in Is projectiveness a Zariski-local property of modules?) and (fortunately) be fixed in Alexander Perry's paper [AP].

Let's back to the main result. The arguments follows from [Psi1] and EGA-IV.

Now we consider the following diagram:

$$\begin{tikzcd} T_0 \arrow[r, "g_0"] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

where T_0, T are any affine schemes with thickening of order 1 and if g_0 can be lifting on local, then we need to show that there exists an global lifting g. For now we need a lemma.
Lemma. If $f : X \to Y$ is formally smooth (as in [Illusie]), then $\Omega^1_{X/Y}$ is locally projective.

Proof. By the **Theorem** we can let $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(A)$ are affine. Let $Z = \mathrm{Spec}(P) = \mathbb{A}^I_A$ with surjective $P \twoheadrightarrow B$. So we have a closed immersion $j : Z \to X$ with ideal I. Consider

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(P/I^2) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

has local lifting. So the extension $0 \to I/I^2 \to P/I^2 \to B \to 0$ is locally split, so is

$$0 \to I/I^2 \to \Omega^1_{P/A} \otimes B \to \Omega^1_{B/A} \to 0$$

see Lemma II.2.7. Since $\Omega^1_{P/A} \otimes B$ is free, well done. $\square$

Now take any open $U \subset T$ with U_0 as its thickening of order 1 in T_0. Let

$$\mathscr{P}(U) = \{g : U \to X : g \text{ lifting } g_0|_{U_0}\}$$

and $\mathscr{P}$ is a $\mathscr{G} = \mathscr{H}om_{\mathcal{O}_{T_0}}(g_0^*\Omega^1_{X/Y}, \mathcal{I})$-torsor by EGA-IV-16.5.14. For now we have
Lemma. Let $\mathscr{F}$ is locally projective over affine scheme S, then for any $\mathscr{G} \in Qcoh(S)$ we have $H^1(S, \mathscr{H}om_{\mathcal{O}_S}(\mathscr{F}, \mathscr{G})) = 0$. (Proof see [Psi1])

Now since $g_0^*\Omega^1_{X/Y}$ is locally projective, we have $H^1(T_0, \mathscr{G}) = 0$ which told us $\mathscr{P}$ is a trivial torsor. So $\mathscr{P}(T_0) \neq \emptyset$, well done!
♣(Page 129) Now we focus on the remark. Consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I} \arrow[r, "j"] & \mathcal{A} \arrow[r, "p"] & \mathcal{O}_X \arrow[r] & 0 \\ & & f^{-1}\mathcal{O}_X \arrow[u] \arrow[ur] & & \end{tikzcd}$$

with exact row and $\mathcal{A}$ is an $f^{-1}(\mathcal{O}_X)$-algebra. p is an $f^{-1}(\mathcal{O}_X)$-algebra map. Let $\mathcal{A}$ satisfies $j(p(x)z) = xj(z)$ for all $x \in \Gamma(U, \mathcal{A}), z \in \Gamma(U, \mathcal{I})$. **This is very important: For now we consider $\mathcal{I}$ as ideal of $\mathcal{A}$ instead of $\mathcal{O}_X$-module!!!** So for $z, z' \in \Gamma(U, \mathcal{I})$, we have $j(z)j(z') = j(p(j(z))z') = 0$, in this case $\mathcal{I}^2 = 0$. We see $(|X|, \mathcal{A})$ is a scheme. For the proof see [Psi1].
♣(Page 131) The details of the prove in Lemma 2.7 we refer to the notes [Psi1].

In Lemma 2.8, we should rewrite it as follows (See also [Psi1]):
Lemma 2.8. Let X be a scheme and let $E \in Qcoh(X)$ be of finite type. Assume that there exists a basis $\mathfrak{B}$ of X such that for all $U \in \mathfrak{B}$ and all $F \in Qcoh(U)$ such that $\mathscr{E}xt^1_{\mathcal{O}_U}(E|_U, F) = 0$, then E is locally free.

Proof. For any $x \in X$ we let $x \in U \in \mathfrak{B}$ with

$$0 \to F \to \mathcal{O}_U^n \to E|_U \to 0.$$

So $F \in Qcoh(U)$ and $\mathscr{E}xt^1_{\mathcal{O}_U}(E|_U, F) = 0$. Let $e \in \mathrm{Ext}^1_{\mathcal{O}_U}(E|_U, F)$ be the extension presented by the above exact sequence on U. Then there exists $x \in V \subset U$ such that $e|_V = 0$, that is, it splits on V. So E_x is finitely generated projective, that is, E_x is free. It's easy to see that there exists $x \in W \subset V$ such that $E|_W$ is free, well done. □

Now if $f : X \to Y$ is smooth, then for any local Y-extension of X by any $I \in Qcoh(U)$ is locally trivial. That is, $\mathscr{E}xt_Y(U, I) \cong \mathscr{E}xt^1_{\mathcal{O}_U}(\Omega^1_{X/Y}|_U, I) = 0$. By Lemma 2.8 we have $\Omega_{X/Y}$ locally free. In this place $\mathscr{E}xt_Y(U, I)$ is the sheaf associated to $V \mapsto \mathrm{Ext}^1_Y(V, I|_V)$.

♣(Page 142) A regular local ring $(R, \mathfrak{m})$ is always a domain. Indeed, by Krull's intersection theorem we have $\bigcap \mathfrak{m}^n = \{0\}$. Now let $f, g \in R$ with $fg = 0$. Let $f \in \mathfrak{m}^a, g \in \mathfrak{m}^b$ where a, b maximal. Then $fg = 0 \in \mathfrak{m}^{a+b+1}$. Since $\mathrm{gr}_{\mathfrak{m}}(R) \cong R[x_1, ..., x_d]$, we have either $f \in \mathfrak{m}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. So we have $f = 0$ or $g = 0$.

Further more, we have the following famous result, see the proof in St 00OC:

Theorem. Let $(R, \mathfrak{m}, \kappa)$ be a Noetherian local ring. The following are equivalent:

(1) κ has finite projective dimension as an R-module;

(2) R has finite global dimension, $\mathrm{gl.dim}(R) < \infty$;

(3) R is a regular local ring.

Moreover, in this case $\mathrm{gl.dim}(R) = \dim(R) = \dim_\kappa(\mathfrak{m}/\mathfrak{m}^2)$.

♣(Page 143) I will rewrite the proof of **Theorem 3.7.**(2) because I don't know how the (*) and (**) to deduce the equalities of dimensions (I believe there are something wrong here but I'm disinclined to correct them...)

Theorem. Let k be a field and X/k be of finite type. If k is prefect, then X is regular if and only if X/k is smooth.

We just need to prove the following property in commutative algebra:

Proposition. Let k be a field and S be a finite type k-algebra. Let $\mathfrak{p} \in \mathrm{Spec}(S)$ where $\kappa(\mathfrak{p})/k$ is separated. Then S/k is smooth at $\mathfrak{p}$ if and only if $S_\mathfrak{p}$ is regular.

Proof of the Proposition. Let $R = S_\mathfrak{p}$ and let $\mathfrak{m} = \mathfrak{p}S_\mathfrak{p}, \kappa = \kappa(\mathfrak{p})$. First we claim that $d : \mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{R/k} \otimes_R \kappa$ is injective, see St 00TU. Then we have the exact sequence $0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{R/k} \otimes_R \kappa \to \Omega^1_{\kappa/k} \to 0$. Since κ/k is separated, we have $\dim_\kappa \Omega^1_{\kappa/k} = trdeg_k(\kappa)$, then

$$\dim_\kappa \Omega^1_{R/k} \otimes_R \kappa = \dim_\kappa + trdeg_k(\kappa) \geq \dim R + trdeg_k(\kappa) = \dim_\mathfrak{q} S.$$

Then it is a equality if and only if $R = S_\mathfrak{q}$ is regular. We use St 00TT and well done. □

♣(Page 147) In the proof of Propisiton 3.14, A is not artinian. So we can not use 3.11 to deduce the freeness by flatness directly. Actually we have an easy proof where $(A, \mathfrak{m})$ is Noetherian local ring. Let M is finitely generated flat A-module. Consider a free A-module F with surjection $F \to M$ such that $F/\mathfrak{m}F \cong M/\mathfrak{m}M$. Consider $0 \to K \to F \to M \to 0$ which is purely exact since M flat. Then $0 \to K/\mathfrak{m}K \to F/\mathfrak{m}F \to M/\mathfrak{m}M \to 0$ is exact. So $K = \mathfrak{m}K$. Use Nakayama we have $K = 0$. Well done!

♣(Page 173) We claim that $Rf_*(\mathcal{O}_X^q(-i))=0$ for $1\leq i\leq r$ for all q. The reason see ♣(Page 176, line -9) below.
♣(Page 176) When we prove the case of projection, we let $L=\omega(-d)$ and we have

$$Rf_*R\mathcal{H}om(\omega(-d),f^*M\otimes\omega)[r]\cong Rf_*(f^*M)(d)[r].$$

Here we use the following conclusion which you could find it at 08DQ:
Theorem. Let X be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\vee=R\mathcal{H}om(K,\mathcal{O}_X)$ is a perfect object too and $(K^\vee)^\vee\cong K$. There is a functorial isomorphism

$$M\otimes^L_{\mathcal{O}_X}K^\vee\cong R\mathcal{H}om(K,M).$$

So now we have:

$$\begin{aligned}Rf_*R\mathcal{H}om(\omega(-d),f^*M\otimes\omega)[r]&=Rf_*R((\omega(-d))^\vee\otimes^L f^*M\otimes\omega[r])\\&=Rf_*(f^*M(d))[r].\end{aligned}$$

♣(Page 176, line -9) Now we have $R\mathcal{H}om(Rf_*\omega(-d),M)\cong\mathcal{H}om^\bullet(R^rf_*\omega(-d)[-r],M)$. Acutually in [EGAIII] III.1.4.11, we have $H^q(u^{-1}(V),\mathscr{F})\cong\Gamma(V,R^qu_*\mathscr{F})$ where u is quasicompact and separated and V is affine open. Then we have [EGAIII] III.2.1.15:
Theorem. Let Y be a scheme and let $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module with rank $r+1$ and $X=\mathbb{P}(\mathscr{E})$ with $f:X\to Y$. Then $R^qf_*(\mathscr{O}_X(n))\neq 0$ if and only if $(q=0,n\geq 0)$ or $(q=r,n\leq -r-1)$.

So for now $\omega(-d)=\mathscr{O}_X(-r-1-d)$ where $d\geq 0$. Let $\mathscr{F}=\omega(-d)$ then $R^qf_*\mathscr{F}=0$ if and only if $q=r$. So we use $\tau_{\leq r}$ we have $Rf_*\mathscr{F}$ is quasi-isomorphic to $R^rf_*\mathscr{F}[-r]$.
♣(Page 180) We use the following conclusion in commutative algebra, the proof is at St 090V:
Theorem. Let R be a Noetherian local ring and M be a nonzero finite R-module with finite projective dimension. Then we have

$$\mathrm{depth}(R)=\mathrm{proj.dim}(M)+\mathrm{depth}(M).$$

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