

SOME NOTES FOR ALGEBRAIC SPACES AND STACKS BY MARTIN OLSSON

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Abstract

I take some notes about the book *Algebraic Spaces and Stacks* written by Prof. Martin Olsson [1], aiming to study some basic theory of algebraic stacks.

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1 Some Notes

♣(Page 54, 2.3.14) The aim is that we will introduce: Consider ringed topos (C, T, Λ) , if $F \in \text{Mod}_\Lambda$ and let $\underline{\mathcal{H}}^i(F) \in \text{PMod}_\Lambda$ sending $X \in C$ to $H^i(C/X, F)$. Then why $\underline{\mathcal{H}}^i(-)$ is the i -th derived functor of inclusion functor $\text{Mod}_\Lambda \hookrightarrow \text{PMod}_\Lambda$?

Actually this is nearly trivial when we show some fundamental results.

First, giving a ringed topos (C, T, Λ) and let $X \in C$, we need to know something about the localization of ringed sites (follows from St 03DH). Consider forgetful functor $j_X : C/X \rightarrow C$, which trivially induce a morphism between topoi $j_X = (j_X^*, j_{X,*}) : (C, T) \rightarrow (C/X, T/X)$. Let $\Lambda_X = j_{X,*}\Lambda$ such that making $(C/X, T/X, \Lambda_X)$ to be a ringed topos (Note that there are some difference between this book and the stacks project, actually f^*, f_* in this book coorespond to the $u_!, u^{-1}$ in the stacks project).

Second, as we defined $H^i(C, -) = R^i\Gamma(C, -) := R^i\text{Hom}(\Lambda, -)$, we again defined for $X \in C$, $H^i(X, -) = R^i\Gamma(X, -)$.

LEMMA.(St 03F3) (1) If I is an injective Λ -module, then $I|_U = j_{X,*}I$ is an injective Λ_X -module;

(2) For any $F \in \text{Mod}_\Lambda$, we have $H^p(X, F) = H^p(C/X, F)$.

Proof of Lemma. Trivial as j_X^* exact. □

PROPOSITION.(St 03AY) Consider ringed topos (C, T, Λ) , if $F \in \text{Mod}_\Lambda$ and let $\underline{\mathcal{H}}^i(F) \in \text{PMod}_\Lambda$ sending $X \in C$ to $H^i(C/X, F)$. Then $\underline{\mathcal{H}}^i(-)$ is the i -th derived functor of inclusion functor $i : \text{Mod}_\Lambda \hookrightarrow \text{PMod}_\Lambda$.

Proof. Easy to see that i is left exact, choose injective resolution $F \rightarrow I^*$. So $R^pi = H^p(I^*)$. Hence the section of $R^pi(F)$ over $X \in C$ is given by

$$\frac{\ker(I^n(X) \rightarrow I^{n+1}(X))}{\text{Im}(I^{n-1}(X) \rightarrow I^n(X))},$$

which is just $H^p(X, F) = H^p(C/X, F)$. Well done. □

References

- [1] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.