Some Notes for Algebraic Spaces and Stacks by Martin Olsson

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Abstract

I take some notes about the book *Algebraic Spaces and Stacks* written by Prof. Martin Olsson [1], aiming to study some basic theory of algebraic stacks.

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1 Some Notes

♣(Page 54, 2.3.14) The aim is that we will introduce: Consider ringed topos (C, T, Λ), if $F \in \mathrm{Mod}_\Lambda$ and let $\underline{\mathscr{H}^i}(F) \in \mathrm{PMod}_\Lambda$ sending $X \in C$ to $H^i(C/X, F)$. Then why $\underline{\mathscr{H}^i}(-)$ is the i-th derived functor of inclusion functor $\mathrm{Mod}_\Lambda \hookrightarrow \mathrm{PMod}_\Lambda$?

Actually this is nearly trivial when we show some fundamental results.

First, giving a ringed topos (C, T, Λ) and let $X \in C$, we need to know something about the localization of ringed sites (follows from St 03DH). Consider forgetful functor $j_X : C/X \to C$, which trivially induce a morphism between topoi $j_X = (j_X^*, j_{X,*}) : (C, T) \to (C/X, T/X)$. Let $\Lambda_X = j_{X,*}\Lambda$ such that making $(C/X, T/X, \Lambda_X)$ to be a ringed topos (Note that there are some difference between this book and the stacks project, actually f^*, f_* in this book coorespond to the $u_!, u^{-1}$ in the stacks project).

Second, as we defined $H^i(C, -) = R^i\Gamma(C, -) := R^i\mathrm{Hom}(\Lambda, -)$, we again defined for $X \in C$, $H^i(X, -) = R^i\Gamma(X, -)$.

Lemma.(St 03F3) (1) If I is an injective Λ-module, then $I|_U = j_{X,*}I$ is an injective Λ_X-module;

(2) For any $F \in \mathrm{Mod}_\Lambda$, we have $H^p(X, F) = H^p(C/X, F)$.

Proof of Lemma. Trivial as j_X^* exact. □

Proposition.(St 06YK) Consider ringed topos (C, T, Λ), if $F \in \mathrm{Mod}_\Lambda$ and let $\underline{\mathscr{H}^i}(F) \in \mathrm{PMod}_\Lambda$ sending $X \in C$ to $H^i(C/X, F)$. Then $\underline{\mathscr{H}^i}(-)$ is the i-th derived functor of inclusion functor $i : \mathrm{Mod}_\Lambda \hookrightarrow \mathrm{PMod}_\Lambda$.

Proof. Easy to see that i is left exact, choose injective resolution $F \to I^*$. So $R^p i = H^p(I^*)$. Hence the section of $R^p i(F)$ over $X \in C$ is given by

$$\frac{\ker(I^n(X) \to I^{n+1}(X))}{\mathrm{Im}(I^{n-1}(X) \to I^n(X))},$$

which is just $H^p(X, F) = H^p(C/X, F)$. Well done. □

♣(Page 54, PROPOSITION 2.3.15)
LEMMA.(St 01FW) Let ringed site (C,Λ). Let $F\in \mathrm{Mod}_\Lambda$ and let $X\in C$. Let $p>0$ and we take $\xi\in H^p(C/X,F)$. Then there exists a covering $\{X_i\to X\}$ such that $\xi|_{U_i}=0$ for all i.

Proof of Lemma. Easy to see, just as St 01FW. □

So now in the proof of the PROPOSITION 2.3.15, α is zero in $\check{H}^0(\mathscr{X},\underline{\mathscr{H}^{i_0}}(F))$. But it is not zero in $H^{i_0}(C/X,F)$! And we find that if $s+t=i_0, 0<s<i_0$, then $\check{H}^s(\mathscr{X},\underline{\mathscr{H}^t}(F))=0$. Together with $\check{H}^{i_0}(\mathscr{X},\underline{\mathscr{H}^0}(F))=\check{H}^{i_0}(\mathscr{X},F)$ and the spectral sequence

$$E_2^{s,t}=\check{H}^s(\mathscr{X},\underline{\mathscr{H}^t}(F))\Rightarrow H^{s+t}(C/X,F),$$

we get that α is not zero in $\check{H}^{i_0}(\mathscr{X},F)$, well done.

References

[1] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.