

SOME ALGEBRAIC TOPOLOGY

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1 The Fundamental Group and Covering Space

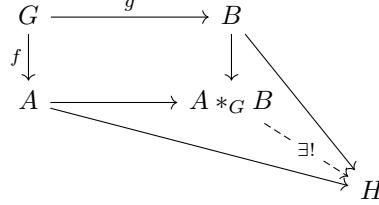
Theorem 1.1 (van Kampen). *Let $X = \bigcup_{\alpha} A_{\alpha}$ where A_{α} are path-connected open sets with a basepoint x_0 . Let all $A_{\alpha} \cap A_{\beta}$ are path-connected, then consider*

$$\begin{array}{ccc}
 \pi_1(A_{\alpha} \cap A_{\beta}) & \xrightarrow{i_{\alpha\beta}} & \pi_1(A_{\alpha}) \\
 i_{\beta\alpha} \downarrow & & \downarrow j_{\alpha} \\
 \pi_1(A_{\beta}) & \xrightarrow{j_{\beta}} & \pi_1(X)
 \end{array}$$

where all maps induced by inclusions. Then j_{α} induce $\Phi : *_{\alpha}\pi_1(A_{\alpha}) \rightarrow \pi_1(X)$ is surjective. If $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ are path-connected, then $\ker \Phi$ is a normal subgroup generated by all elements of form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_{\alpha} \cap A_{\beta})$.

Remark 1.2. *In the case of two open sets U, V with $U \cap V$ path-connected, we have the following.*

In the category of groups \mathfrak{Grp} , we can describe pushout of $f : G \rightarrow A$ and $g : G \rightarrow B$. We let $A *_G B$ as $A * B / (f(a)g(a)^{-1})_{a \in G}$, then we have the following universal property in \mathfrak{Grp} :



We call it the amalgamated product of A and B with amalgam G . So in the van Kampen theorem with U, V , we have

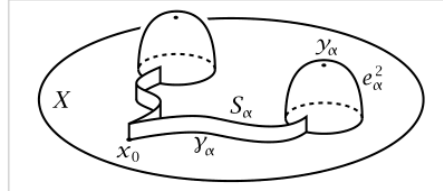
$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V).$$

Next step we need to calculate the fundamental group of a CW complex. Obviously we just need to consider X_2 .

Consider 1-skeleton X , then 2-cell e_α^2 attach on X via $\phi_\alpha : S^1 \rightarrow X$. Then we get Y . Note that if we fix a base point $s_0 \in S^1$, then π_α is a loop. Consider the original base point $x_0 \in X$, let γ_α is a path on X from x_0 to $\phi_\alpha(s_0)$. Then $\gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha$ is a loop which is null-homotopy in Y when the 2-cell attaching. Let $N \subset \pi_1(X, x_0)$ be a normal subgroup generated by $\gamma_\alpha \phi_\alpha \bar{\gamma}_\alpha$.

Theorem 1.3. Inclusion $X \hookrightarrow Y$ induce surjection $\pi_1(X, x_0) \twoheadrightarrow \pi_1(Y, x_0)$ with kernel N , that is, $\pi_1(Y) \cong \pi_1(X)/N$.

Proof. Consider a larger space Z with $Z \simeq Y$:



Pick one point y_α on each 2-cells, respectively, as in the diagram. Then $A = Z - \bigcup_\alpha \{y_\alpha\}$ can be deformation retracted to X . Let $B = Z - X$ which is null-homotopic. Use van Kampen theorem to $\{A, B\}$ and we get $\pi_1(Z) \cong \pi_1(A) * \pi_1(B) / (\cdot)$ where (\cdot) is the image of $\pi_1(A \cap B) \rightarrow \pi_1(A)$, which is N . \square

Proposition 1.4. Let M, N are two orientable closed surfaces. Then there exists $f : M \rightarrow N$ to be a covering space iff $g(M) = mn + 1$ and $g(N) = m + 1$ for some $m, n \geq 0$.

Proof. Trivial. \square

2 Homology

2.1 Singular Homology

Theorem 2.1 (Excision Theorem). Let $Z \subset A \subset X$ where $\text{cl}(Z) \subset \text{int}(A)$, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induce $H_n(X - Z, A - Z) \cong H_n(X, A)$.

If now we let $B = X - Z$ we have $H_n(B, A \cap B) \cong H_n(X, A)$.

Proposition 2.2. *For good pairs (X, A) , map $q : (X, A) \rightarrow (X/A, A/A)$ induce $q_* : H_n(X, A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$.*

Proof. Let V be the open set deformation retracts into A , consider

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{f} & H_n(X, V) & \xleftarrow{g} & H_n(X - A, V - A) \\ q_* \downarrow & & q_* \downarrow & & q_* \downarrow \\ H_n(X/A, A/A) & \xrightarrow{u} & H_n(X/A, V/A) & \xleftarrow{v} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

f, u are isomorphisms by the long exact sequences of triples (X, V, A) and $(X/A, V/A, A/A)$. And g, v are isomorphisms directly by excision. The right hand q_* is isomorphism. So is the left. \square

2.2 Cellular Homology

Proposition 2.3. *If $f : S^n \rightarrow S^n$ has no fixed points, then $\deg f = (-1)^{n+1}$.*

Proof. Consider $f_t(x) = ((1-t)f(x) - tx)/|(1-t)f(x) - tx|$ and well done. \square

Corollary 2.4. $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely on S^n is n is even.

Proof. Let G be such non-trivial group. The degree map give us homomorphism $G \rightarrow \{\pm 1\}$. Since action is free, this map sends all non-trivial elements of G to $(-1)^{n+1}$. When n even, kernel is trivial. Well done. \square

Proposition 2.5. *Let $M \in O(n+1)$ which induce $f_M : S^n \rightarrow S^n$ by $x \mapsto Mx$, then $\deg f_M = \det M$.*

Proof. This follows from the fact that any orthogonal matrix can be decomposed into the reflection matrixes and rotation matrixes. \square

Theorem 2.6 (Hairly Ball). S^n has a continuous field of nonzero tangent vectors iff n is odd.

Proof. Consider such vector field $v(x)$ and view it as centering at origin. Let $|v(x)| = 1$ via $v(x)/|v(x)|$. Consider $f_t(x) = (\cos t)x + (\sin t)v(x)$. Then $\deg(-\text{id}) = \deg(\text{id}) = 1$, so $(-1)^{n+1} = 1$, so n is odd.

Conversely if $n = 2k - 1$, then let $v(x_1, \dots, x_{2k}) = (-x_2, -x_1, \dots, -x_{2k}, -x_{2k-1})$. \square

Proposition 2.7 (Euler Characteristic). *We have:*

- (a) *For finite CW complexes X, Y , we have $\chi(X \times Y) = \chi(X)\chi(Y)$.*
- (b) *If a finite CW complex $X = A \cup B$ of two subcomplexes A, B , then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.*
- (c) *For an n -sheeted covering space of finite CW complexes $p : \tilde{X} \rightarrow X$, we have $\chi(\tilde{X}) = n\chi(X)$.*

Proof. (a)(b) are trivial. For (c), given an m -dimensional CW-complex X , one can lift the CW-structure to a CW-structure on \tilde{X} by lifting the characteristic maps $\phi_\alpha : e_\alpha^k \rightarrow X$, which can be done since $\pi_1(D^k) = 0$. There are exactly n lifts of ϕ_α to \tilde{X} . So for each k -cell e^k in X , there exists n k -cells in the lifted CW-structure on \tilde{X} which are mapped homeomorphically onto e^k . Hence well done. \square

Now we consider CW complex X with k -skeleton X_k . We have the following elementary conclusion:

(b) $H_k(\dot{X}^n) = 0$ for $k > n$;

(c) Inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \cong H_k(X)$ for $k < n$.

Example 2.11. Let M_g be the closed orientable surface of genus g , then the cellular complex is $0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \rightarrow 0$.

Example 2.12. Let N_g be the closed non-orientable surface of genus g , then the cellular complex is $0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \rightarrow 0$ where $f : 1 \mapsto (2, \dots, 2)$.

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

when n is even and

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0$$

when n is odd.

Example 2.14 (Acyclic Space). Let X obtained from $S^1 \vee S^1$ by attaching two 2-cells by words a^5b^{-3} and $b^2(ab)^{-2}$. Then the cellular complex is $0 \rightarrow \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \rightarrow 0$ where $f = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$.

2.3 Mayer-Vietoris

Theorem 2.15 (Mayer-Vietoris Sequence). *Let $A, B \subset X$ with $X = \text{int}(A) \cup \text{int}(B)$. Then we have*

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{x \mapsto (x, -x)} C_n(A) \oplus C_n(B) \xrightarrow{(x, y) \mapsto x+y} C_n(A+B) \longrightarrow 0$$

Then induce the long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A \cap B) & \xrightarrow{(i_{1*}, -i_{2*})} & H_n(A) \oplus H_n(B) & \xrightarrow{g_* + j_*} & H_n(X) \\ & & & & & & \downarrow \partial \\ & & & & \cdots & \longleftarrow & H_{n-1}(A \cap B) \end{array}$$

where $i_1 : A \cap B \rightarrow A, i_2 : A \cap B \rightarrow B$ and $g : A \rightarrow X, j : B \rightarrow X$.

Theorem 2.16 (Mapping Torus and Mayer-Vietoris Sequence). *Let $f, g : X \rightarrow Y$ and let $Z = X \times I / ((x, 0) \sim f(x), (x, 1) \sim g(x))$ be the mapping torus, then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(X) & \xrightarrow{f_*-g_*} & H_n(Y) & \xrightarrow{i_*} & H_n(Z) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H_{n-1}(X) \end{array}$$

More special case, we let $f : A \cap B \rightarrow A, g : A \cap B \rightarrow B$, then we can get the traditional Mayer-Vietoris sequence.

Theorem 2.17 (Relative Mayer-Vietoris Sequence). *Let $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A, D \subset B$. Then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A \cap B, C \cap D) & \longrightarrow & H_n(A, C) \oplus H_n(B, D) & \longrightarrow & H_n(X, Y) \\ & & & & & & \downarrow \\ & & & & & & H_{n-1}(A \cap B, C \cap D) \\ & & & & \longleftarrow & & \end{array}$$

derived by nine lemma and long exact sequence.

2.4 More Applications

2.4.1 Embedding and Homology

Theorem 2.18 (Invariance of Domain). *Let M and N are both n -dimensional topological manifolds and $f : M \rightarrow N$ is one-one and continuous, then f is open.*

Proof. See [1] page 235. □

Corollary 2.19. *If $f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous injective map where U is open, then $m \leq n$.*

Proof. If not, we let $m > n$. Consider $g : U \rightarrow \mathbb{R}^n \times \mathbb{R}^{m-n}$ with $x \mapsto (f(x), 0)$. By invariance of domain, the image of g , which is $f(U) \times \{0\}$, is open in \mathbb{R}^m which is impossible. □

Remark 2.20. *But unfortunately, for any $m, n > 0$, there is a continuous surjective map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. See [Existence of a continuous surjective function].*

2.4.2 Borsuk-Ulam Type Theorem

For any two-sheeted covering space $p : X' \rightarrow X$, we have exact sequence

$$0 \rightarrow C_n(X, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} C_n(X', \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} C_n(X, \mathbb{Z}/2\mathbb{Z}) \rightarrow 0$$

as p_* is surjective follows from homotopy lifting property and as each $\sigma : \Delta^n \rightarrow X$ has precisely two lifts σ'_1, σ'_2 , then τ maps σ to $\sigma'_1 + \sigma'_2$ holds as the coefficient is $\mathbb{Z}/2\mathbb{Z}$. Hence from this we have the long exact sequence

$$\cdots \rightarrow H_n(X, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} H_n(X', \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(X, \mathbb{Z}/2\mathbb{Z}) \rightarrow \cdots$$

This is a special case of Gysin sequence.

Theorem 2.21 (Borsuk). *A map $f : S^n \rightarrow S^n$ with $f(-x) = -f(x)$ must have odd degree.*

Proof. Consider the covering space $p : S^n \rightarrow \mathbb{R}P^n$. As $f(-x) = -f(x)$, we have

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ \downarrow p & & \downarrow p \\ \mathbb{R}P^n & \xrightarrow{\bar{f}} & \mathbb{R}P^n \end{array}$$

We claim that the following diagram commute:

$$\begin{array}{ccccccc}
0 & \longrightarrow & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\tau} & C_i(S^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{P_\sharp} & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \\
& & \downarrow \bar{f}_\sharp & & \downarrow f_\sharp & & \downarrow \bar{f}_\sharp \\
0 & \longrightarrow & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\tau} & C_i(S^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{P_\sharp} & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0
\end{array}$$

The right square is trivial. The left square commutes since for $\sigma : \Delta^i \rightarrow \mathbb{R}P^n$ with lifts σ'_1, σ'_2 , the two lifts of $\bar{f}\sigma$ are $f\sigma'_1, f\sigma'_2$ since $f(-x) = -f(x)$.

Finally taking long exact sequence we can find that $f_* : H_n(S^n, \mathbb{Z}/2\mathbb{Z}) \rightarrow H_n(S^n, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism by induction on dimension using the trivial fact that they are isomorphisms in dimension 0. So f must have odd degree. \square

Corollary 2.22 (Borsuk-Ulam). *Every map $g : S^n \rightarrow \mathbb{R}^n$, there exists a point $x \in S^n$ with $g(x) = g(-x)$.*

Proof. Let $f(x) = g(x) - g(-x)$, then f is odd. If f is nowhere vanish, we replace f by $f/|f|$ and get a morphism $f : S^n \rightarrow S^{n-1}$ which is still odd. Restrict it on the equator, which is still odd, has odd degree by the theorem of Borsuk. But this restriction is nullhomotopic as it is a restriction of $f|_{D^n}$ in the hemisphere. \square

Corollary 2.23. *Whenever S^n is expressed as the union of $n+1$ closed sets A_0, \dots, A_n , then at least one of these sets must contain a pair of antipodal points.*

Proof. We define $d_i : S^n \rightarrow \mathbb{R}, x \mapsto \inf_{y \in A_i} |x - y|$. Let $g : S^n \rightarrow \mathbb{R}^n, x \mapsto (d_1(x), \dots, d_n(x))$. By Borsuk-Ulam theorem, it obtaining a pair of antipodal points $x, -x$ with $d_i(x) = d_i(-x), i = 1, \dots, n$. If either of these distances is 0, then well done. If not, $x, -x \in A_0$, well done. \square

2.4.3 The Lefschetz Fixed Point Theorem

Theorem 2.24 (Lefschetz). *If X is a finite simplicial complex, or more generally a retract of a finite simplicial complex and $f : X \rightarrow X$ is a map with $\tau(f) = \sum_n (-1)^n \text{tr}(f_* : H_n(X) \rightarrow H_n(X)) \neq 0$, then f has a fixed point.*

3 Cohomology

3.1 Universal Coefficient Theorem and Künneth Formula

Theorem 3.1 (Universal Coefficient Spectral Sequence). *For cohomology we have*

$$E_2^{p,q} = \text{Ext}_R^q(H_p(C_*), G) \Rightarrow H^{p+q}(C_*; G)$$

where R is a ring with unit, C_* is a chain complex of free modules over R , G is any (R, S) -bimodule for some ring with a unit S . The differential d^r has degree $(1-r, r)$.

Similarly for homology

$$E_{p,q}^2 = \text{Tor}_q^R(H_p(C_*), G) \Rightarrow H_*(C_*; G)$$

and the differential d_r having degree $(r-1, -r)$.

Theorem 3.2 (Universal Coefficient Theorem for Homology). *Let R be a PID and let C_* a chain complex of R -modules such that C_n is free for all n and let M be an R -module. Then there is a natural short exact sequence of R -modules*

$$0 \rightarrow H_n(C_*) \otimes_R M \rightarrow H_n(C_* \otimes_R M) \rightarrow \text{Tor}_1^R(H_{n-1}(C_*), M) \rightarrow 0$$

which is split non-naturally.

Proof. As $0 \rightarrow Z_n(C_*) \rightarrow C_n \rightarrow B_{n-1}(C_*) \rightarrow 0$ is exact with $B_{n-1}(C_*)$ free since R is PID, then this sequence split. Hence $Z_n(C_*) \otimes_R M \rightarrow C_n \otimes_R M$ is also injective. As we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} C_{n-1} \otimes_R M & \longrightarrow & Z_n(C_*) \otimes_R M & \longrightarrow & H_n(C_*) \otimes_R M & \longrightarrow & 0 \\ \downarrow = & & \downarrow & & \downarrow \alpha & & \\ C_{n-1} \otimes_R M & \longrightarrow & Z_n(C_* \otimes_R M) & \longrightarrow & H_n(C_* \otimes_R M) & \longrightarrow & 0 \end{array}$$

by some easy diagram chase we find that $\alpha : H_n(C_*) \otimes_R M \rightarrow H_n(C_* \otimes_R M)$ injective. Let's consider its cokernel.

Pick any free resolution $0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$. As C_i free, we have $0 \rightarrow F_1 \otimes_R C_* \rightarrow F_0 \otimes_R C_* \rightarrow M \otimes_R C_* \rightarrow 0$ which give us the long exact sequence. Split it into short exact sequences

$$\begin{array}{c} 0 \longrightarrow \text{coker}(H_n(C_* \otimes_R F_1) \rightarrow H_n(C_* \otimes_R F_0)) \\ \downarrow \\ H_n(C_* \otimes_R M) \\ \downarrow \\ \text{ker}(H_{n-1}(C_* \otimes_R F_1) \rightarrow H_{n-1}(C_* \otimes_R F_0)) \longrightarrow 0 \end{array}$$

Actually α is trivially an isomorphisms when we consider the free module. As $\text{coker}(H_n(C_*) \otimes_R F_1 \rightarrow H_n(C_*) \otimes_R F_0) \cong H_n(C_*) \otimes_R M$ and $\text{ker}(H_{n-1}(C_*) \otimes_R F_1 \rightarrow H_{n-1}(C_*) \otimes_R F_0) \cong \text{Tor}_1^R(H_{n-1}(C_*), M)$, we get the theorem. \square

Theorem 3.3 (Universal Coefficient Theorem for Cohomology). *Let R be a PID and let C_* a chain complex of R -modules such that C_n is free for all n and let M be an R -module. Then there is a natural short exact sequence of R -modules*

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(C_*), M) \rightarrow H^n(\text{Hom}(C_*, M)) \rightarrow \text{Hom}(H_n(C_*), M) \rightarrow 0$$

which is split non-naturally.

Proof. Similar as the version of homology. \square

Theorem 3.4 (Algebraic Künneth Formula). *Let R be a PID and let C_*, C'_* a chain complex of R -modules such that C_n is free for all n . Then there is a natural short exact sequence of R -modules*

$$0 \rightarrow \bigoplus_{p+q=n} (H_p(C_*) \otimes_R H_q(C'_*)) \rightarrow H_n(C_* \otimes_R C'_*) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(C_*) \otimes_R H_q(C'_*)) \rightarrow 0.$$

Theorem 3.5 (Topological Künneth Formula). *Let R be a PID and let X, Y are two CW complexes. Then there is a natural short exact sequence of R -modules*

$$0 \rightarrow \bigoplus_{p+q=n} (H_p(X; R) \otimes_R H_q(Y; R)) \rightarrow H_n(X \times Y; R) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1^R(H_p(X; R) \otimes_R H_q(Y; R)) \rightarrow 0.$$

Example 3.6. Let R be a commutative ring with ideal I, J , then $\text{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}$ and $\text{Ext}_R^1(R/I, M) \cong \text{Hom}_R(I, M)/M_I$ where $M_I := \{g_m : i \mapsto im\} \subset \text{Hom}_R(I, M)$.

Proof. Follows from $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$. \square

3.2 Cup and Cap Products

Definition 3.7 (Cross Product). Let R be a commutative ring with unit and let X, Y be spaces. We define morphism of chain complexes

$$C^*(X; R) \otimes_R C^*(Y; R) \rightarrow \text{Hom}(C_*(X) \otimes C_*(Y), R) \rightarrow C^*(X \times Y; R)$$

where the first one is the natural map and the second is dual of the following Alexander-Whitney map:

$$C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y), \sigma \mapsto \sum_{p+q=n} p(\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma)_q$$

where $p\sigma := \sigma|_{[v_0, \dots, v_p]}$ and $\sigma_q := \sigma|_{[v_{n-q}, \dots, v_n]}$ when $\sigma \in C_n(-)$. This induce the map

$$\bigoplus_{p+q=n} H^p(X; R) \otimes_R H^q(Y; R) \xrightarrow{\times} H^n(X \times Y; R)$$

which is the cross product.

Definition 3.8 (Cup Product). Let R be a commutative ring with unit and let X be a space. For $\Delta : X \rightarrow X \times X$ be the diagonal, we define

$$\begin{array}{ccc} H^p(X; R) \otimes_R H^q(X; R) & \xrightarrow{\times} & H^{p+q}(X \times X; R) \\ & \searrow \cup & \downarrow \Delta^* \\ & & H^{p+q}(X; R) \end{array}$$

to be the cup product.

Proposition 3.9. Let R be a commutative ring and let X be a space.

(a) Alexander-Whitney map gives an explicit product formula:

$$(\alpha \cup \beta)(\sigma) = \alpha_p(\sigma) \cdot \beta_q(\sigma_q), \quad \forall \alpha \in C^p(X; R), \beta \in C^q(X; R), \sigma : \Delta^{p+q} \rightarrow X.$$

(b) $H^*(X; R)$ is a graded commutative ring with unit:

- Let $1 \in H^0(X; R)$ be represented by the cocycle which takes every singular 0-simplex to $1 \in R$. Then $1 \cup \alpha = \alpha \cup 1 = \alpha$ for any $\alpha \in H^*(X; R)$.
- $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.
- $\alpha \cup \beta = (-1)^{pq} \beta \cup \alpha$ for all $\alpha \in H^p(X; R)$ and $\beta \in H^q(X; R)$.

(c) Let $f : X \rightarrow Y$ be a continuous map. Then

$$f^* : H^*(Y; R) \rightarrow H^*(X; R)$$

is a morphism of graded commutative rings, i.e. $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.

Remark 3.10. Actually if we define the cup product in the level of chain complex by (a), then $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta(\beta)$ for $\alpha \in C^k(X; R)$. This is coincident to the original definition since the coboundary map of complex $C^*(X; R) \otimes_R C^*(Y; R)$ has the similar formula.

Theorem 3.11 (Künneth Formula). Assume R is a PID, if $H^*(X; R)$ or $H^*(Y; R)$ are finitely generated free R -modules, we have an isomorphism of graded commutative rings

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\sim} H^*(X \times Y; R)$$

where the first one we define $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$.

Definition 3.12 (Cap Product). We define

$$C^p(X; R) \times C_{p+q}(X; R) \xrightarrow{\cap} C_q(X; R), \phi \cap \sigma = \phi(p\sigma)\sigma_q.$$

Then one can check that $\partial(\phi \cap \sigma) = (-1)^p(\phi \cap \partial\sigma - \delta\phi \cap \sigma)$. This induce the cap product:

$$H^p(X; R) \times H_{p+q}(X; R) \xrightarrow{\cap} H_q(X; R).$$

Proposition 3.13. The cap product extends naturally to the relative case: for any good pair (X, A) , we have

$$(a) \quad H^p(X, A; R) \otimes_R H_{p+q}(X, A; R) \xrightarrow{\cap} H_q(X; R);$$

$$(b) \quad H^p(X; R) \otimes_R H_{p+q}(X, A; R) \xrightarrow{\cap} H_q(X, A; R).$$

More generally, we have

$$H^p(X, A; R) \otimes_R H_{p+q}(X, A \cup B; R) \xrightarrow{\cap} H_q(X, B; R).$$

Sketch. Just need to check \cap induce $C^*(X, A; R) \times C_*(A \cup B; R) \rightarrow C_*(B)$. □

Proposition 3.14. We have the following:

(a) If $f : X \rightarrow Y$ continuous, then we have

$$f_*(\sigma) \cap \phi = f_*(\sigma \cap f^*\phi).$$

(b) For any $\sigma \in C_{k+l}(X; R)$, $\phi \in C^k(X; R)$ and $\psi \in C^l(X; R)$, we have

$$\psi(\sigma \cap \phi) = (\phi \cup \psi)(\sigma).$$

Proof. Directly check. □

3.3 Orientations

We consider the n -manifold is T2 with locally homeomorphic to \mathbb{R}^n . Here we let $H_n(M|A; R) := H_n(M, M - A; R)$. We consider a sheaf both as a functor and as a topological space by the trivial choice of topological basis.

Fix any commutative ring with unit R .

Definition 3.15. We define \mathbb{O}_R be a locally constant sheaf of R -modules on M whose stalk at a point is $H_n(X|x; R)$. Of course, $\mathbb{O}_R = R \otimes_{\mathbb{Z}} \mathbb{O}_{\mathbb{Z}}$.

Actually there is a associated (framed) bundle of \mathbb{O}_R as follows: we define a principal R^\times -bundle $\tilde{\mathbb{O}}_R \rightarrow M$ which send the open set $U \subset M$ to $\{\text{trivializations } \alpha : \underline{R}_U \cong \mathbb{O}_R|_U\}$.

Definition 3.16. An R -orientation of M is a global section of the associated principal R^\times -bundle $\tilde{\mathbb{O}}_R \rightarrow M$. In this case we have $\mathbb{O}_R \cong \underline{R}_M$ and we say M is R -orientable.

Remark 3.17. When $\mathbb{Z} = R$, then we will ignore R .

Remark 3.18. As $H_n(M|x; R) \cong H_n(M|x; \mathbb{Z})_{\mathbb{Z}} R$, we consider a subsheaf $\mathbb{O}_r \subset \mathbb{O}_R$ for each $r \in R$ consist of $\pm \mu_x \otimes r \in H_n(M|x; R)$ where μ_x is a generator of $H_n(M|x; \mathbb{Z}) \cong \mathbb{Z}$. Then as a topological space, if $r = -r$, then $\mathbb{O}_r = M$; if not, then $\mathbb{O}_r \cong \tilde{\mathbb{O}}_{\mathbb{Z}}$.

Hence if M is orientable, then it is R -orientable for all R . Any manifold is \mathbb{F}_2 -orientable.

Proposition 3.19. Consider the principal R^\times -bundle $\tilde{\mathbb{O}}_R \rightarrow M$, then $\tilde{\mathbb{O}}_R$ is always R -orientable.

Proof. Follows from construction and $H_n(\tilde{\mathbb{O}}_R|\mu_x; R) \cong H_n(U(\mu_B)|\mu_x; R) \cong H_n(B|x; R) \cong H_n(M|x; R)$. Well done. \square

Proposition 3.20. Let M connected. Then M is orientable if and only if $\tilde{\mathbb{O}}_{\mathbb{Z}}$ is connected. In particular, if $\pi_1(M)$ has no subgroup of index 2, then it is orientable.

Proof. In this case $R = \mathbb{Z}$ and $\tilde{\mathbb{O}}_{\mathbb{Z}}$ principal $\mathbb{Z}/2\mathbb{Z}$ -bundle which is indeed a two-sheeted covering space. Now the result follows directly from the following fact:

- If $p : E \rightarrow X$ is a covering space with a section $s : X \rightarrow E$, then $s(X) \subset E$ is both open and closed. Hence, it is a union of connected components of E .

Well done. \square

Remark 3.21. We can generalize this into R but I do not care about them.

Theorem 3.22. Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then for any section $(x \mapsto \alpha_x) \in \Gamma(M, \mathbb{O}_R)$ there exists a unique class $\alpha_A \in H_n(M|A; R)$ whose image in $H_n(M|x; R)$ is α_x for all $x \in A$. Moreover, $H_i(M|A; R) = 0, i > n$.

Sketch of the Proof. More details see [2]. Our method is to reduce the case in to simple one.

(i) **If this hold for $A, B, A \cap B$, then this is also hold of $A \cup B$.** Use the MV-principle, we have:

$$0 = H_{n+1}(M|A \cap B) \rightarrow H_n(M|A \cup B) \rightarrow H_n(M|A) \oplus H_n(M|B) \rightarrow H_n(M|A \cap B)$$

then this is easy to see;

(ii) **Reduce to the case $M = \mathbb{R}^n$.** Actually we can let $A = \bigcup_{i=1}^m A_i$ where A_i in some \mathbb{R}^n . Then use MV-principle and induction, well done;

(iii) **Consider the case $M = \mathbb{R}^n$ and $A = \bigcup_{i=1}^m A_i$ where A_i is convex.** Use the MV-principle as (ii) we can let A is convex. Then the result is trivial by $H_i(\mathbb{R}^n|A) \cong H_i(\mathbb{R}^n|x)$ naturally;

(iv) **Consider the case $M = \mathbb{R}^n$ and A be any compact.** Let $\alpha \in H_i(\mathbb{R}^n|A)$ represented by z and let $C \subset \mathbb{R}^n - A$ be the union of the images of the singular simplices in ∂z . Then one can cover some closed balls over A outside of C . Let K be the union of these balls and we see that the relative cycle z defines an element $\alpha_K \in H_i(\mathbb{R}^n|K)$ mapping to the given $\alpha \in H_i(\mathbb{R}^n|A)$. Use (iii) to $H_i(\mathbb{R}^n|K)$, well done. \square

Theorem 3.23. *Let M be a closed connected n -manifold. Then*

(a) *If M is R -orientable, then the map $H_n(M; R) \rightarrow H_n(M|x; R) \cong R$ is an isomorphism for all $x \in M$;*

(b) *If M is not R -orientable, then the map $H_n(M; R) \rightarrow H_n(M|x; R) \cong R$ is injective for all $x \in M$ with image $\{r \in R : 2r = 0\}$.*

By the isomorphism $H_n(M; R) \rightarrow H_n(M|x; R) \cong R$, the element in $H_n(M; R)$ is called **fundamental class** if its image in any $H_n(M|x; R) \cong R$ is a generator.

Proof. By Theorem 3.22 for $A = M$, we have $H_n(M; R) \cong \Gamma(M, \mathbb{O}_R)$.

For (a), if M is R -orientable, then the map $H_n(M; R) \rightarrow H_n(M|x; R) \cong R$, which is just the evaluation map $e_x : \Gamma(M, \mathbb{O}_R) \rightarrow H_n(M|x; R)$, is isomorphism since $\mathbb{O}_R \cong \underline{R}_M$ canonically.

For (b), M is not R -orientable then it is not \mathbb{Z} -orientable. By Remark 3.18 we have $\mathbb{O}_R = \bigoplus_{r \in U} \mathbb{O}_r$ where $U = R/\{\pm 1\}$ which is well defined since $\mathbb{O}_r = \mathbb{O}_{-r}$. We know that if $r = -r$, then $\mathbb{O}_r = M$; if not, then $\mathbb{O}_r \cong \tilde{\mathbb{O}}_{\mathbb{Z}}$. Hence as it is not \mathbb{Z} -orientable, there is no global section of $\tilde{\mathbb{O}}_{\mathbb{Z}}$. As when $r = -r$ there is the trivial section of $\mathbb{O}_r = M$. Hence we get the result. \square

Corollary 3.24. *Let M be a closed connected n -manifold. If M is closed and orientable, then $H_{n-1}(M - \{x\}) \cong H_{n-1}(M)$.*

Proof. Indeed, from the pair $(M, M - \{x\})$ we have

$$\cdots \rightarrow H_n(M) \rightarrow H_n(M, M - \{x\}) \rightarrow H_{n-1}(M - \{x\}) \rightarrow H_{n-1}(M) \rightarrow H_{n-1}(M, M - \{x\}) \rightarrow \cdots$$

As M is orientable, then $H_n(M) \cong H_n(M, M - \{x\})$ by Theorem 3.23(a). Since $H_{n-1}(M, M - \{x\}) = 0$, we have $H_{n-1}(M - \{x\}) \cong H_{n-1}(M)$. \square

Corollary 3.25. *Let M be a closed connected n -manifold. The torsion subgroup of $H_{n-1}(M; \mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ if M is nonorientable.*

Proof. If M is orientable and if $H_{n-1}(M; \mathbb{Z})$ contained torsion, then for some prime p and universal coefficient, we have

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow H_n(M; \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(M), \mathbb{Z}/p\mathbb{Z}) \rightarrow 0$$

Then $H_n(M; \mathbb{Z}/p\mathbb{Z})$ is bigger than $\mathbb{Z}/p\mathbb{Z}$ which is impossible.

If M is nonorientable, we let $H_{n-1}(M) = F \oplus \bigoplus_j \mathbb{Z}/p_j\mathbb{Z}$, then we have

$$\begin{aligned} 0 \longrightarrow 0 \longrightarrow H_n(M; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_j \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_j\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0 \\ \parallel \\ \bigoplus_j \frac{p_j\mathbb{Z} \cap 2\mathbb{Z}}{2p_j\mathbb{Z}} \end{aligned}$$

then we have $H_{n-1}(M)_{\text{tor}} = \mathbb{Z}/2\mathbb{Z}$. \square

Proposition 3.26. *If M is a connected noncompact n -manifold, then $H_i(M; R) = 0$ for all $i \geq n$.*

Proof. Let z be a cycle represent an element of $H_i(M; R)$. It has a compact image and we let U be an open set cover it with compact closure. Let $V = M - \text{cl}(U)$ and consider $(M, U \cup V, V)$ we have

$$\begin{array}{ccccc} 0 = H_{i+1}(M, U \cup V; R) & \longrightarrow & H_i(U \cup V, V; R) & \longrightarrow & H_i(M, V; R) = 0 \\ & & \uparrow \cong & & \uparrow \\ & & H_i(U; R) & \longrightarrow & H_i(M; R) \end{array}$$

When $i > n$ we have $H_i(U; R) = 0$ so z is a boundary in U and so in M , so $H_i(M; R) = 0$.

When $i = n$, class $[z] \in H_n(M; R)$ defines a section $x \mapsto [z]_x$ of M_R . This section determined by the value in single point since M is connected. Also consider

$$\begin{array}{ccccc} 0 = H_{n+1}(M, U \cup V; R) & \longrightarrow & H_n(U \cup V, V; R) & \longrightarrow & H_n(M, V; R) \\ & & \uparrow \cong & & \uparrow \\ & & H_n(U; R) & \longrightarrow & H_n(M; R) \end{array}$$

Then since M is noncompact and z has a compact image, there must have some point x such that $[z]_x = 0$, so $[z]_x = 0$ for all $x \in M$. Then $[z] = 0$ in $H_n(M, V; R)$, so is in $H_n(U; R)$ and then in $H_n(M; R)$. We win. \square

Example 3.27. Let M, N are both closed connected n -manifolds. Show that $M \sharp N$ is orientable if and only if both M, N are orientable. What is $H_i(M \sharp N)$?

Analysis. If M, N are orientable, then consider pair $(M \sharp N, S^{n-1})$ with quotient $M \sharp N / S^{n-1} \cong M \vee N$. If $M \sharp N$ is not orientable, then we have injection of \mathbb{Z} -modules $\mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z}$ which is impossible.

If one of them is not orientable, we say N , then we claim that $M \sharp N$ is not orientable. Consider the pair $(M \sharp N, M - \{p\})$, we have

$$\cdots \rightarrow H_n(M - \{p\}) \rightarrow H_n(M \sharp N) \rightarrow H_n(M \sharp N, M - \{p\}) \rightarrow \cdots$$

By Proposition 3.26 we have $H_n(M - \{p\}) = 0$. As $H_n(M \sharp N, M - \{p\}) = H_n(M \sharp N / (M - \{p\})) = H_n(N) = 0$, we find that $H_n(M \sharp N) = 0$. Hence $M \sharp N$ is not orientable.

Now we will compute $H_i(M \sharp N)$. Consider pair $(M \sharp N, S^{n-1})$ with quotient $M \sharp N / S^{n-1} \cong M \vee N$ again. We have

$$\cdots \rightarrow \tilde{H}_i(S^{n-1}) \rightarrow \tilde{H}_i(M \sharp N) \rightarrow \tilde{H}_i(M \vee N) \rightarrow \tilde{H}_{i-1}(S^{n-1}) \rightarrow \cdots$$

Hence if $i \neq n-1, n$, then $\tilde{H}_i(M \sharp N) \cong \tilde{H}_i(M \vee N) \cong \tilde{H}_i(M) \oplus \tilde{H}_i(N)$. We consider $i = n-1, n$ and we need to consider

$$0 \rightarrow \tilde{H}_n(M \sharp N) \rightarrow \tilde{H}_n(M \vee N) \rightarrow \tilde{H}_{n-1}(S^{n-1}) \rightarrow \tilde{H}_{n-1}(M \sharp N) \rightarrow \tilde{H}_{n-1}(M \vee N) \rightarrow 0.$$

Three cases:

If both M, N are orientable, then so is $M \sharp N$. Hence $\tilde{H}_n(M \sharp N) \cong \mathbb{Z}$ and we have

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \tilde{H}_{n-1}(M \sharp N) \rightarrow \tilde{H}_{n-1}(M \vee N) \rightarrow 0.$$

By some analysis of topology we find that the first map is $1 \mapsto (1, 1)$ and the second one is $(a, b) \mapsto a - b$. Hence $\tilde{H}_{n-1}(M \sharp N) \cong \tilde{H}_{n-1}(M \vee N)$.

If M is orientable but N is not, then $M \sharp N$ is not and we have $\tilde{H}_n(M \sharp N) = 0$ and

$$0 \rightarrow 0 \rightarrow \mathbb{Z} \oplus 0 \rightarrow \mathbb{Z} \rightarrow \tilde{H}_{n-1}(M \sharp N) \rightarrow \tilde{H}_{n-1}(M \vee N) \rightarrow 0.$$

We know that $\mathbb{Z} \oplus 0 \rightarrow \mathbb{Z}$ induced by $(1, 0) \mapsto 1$ by trivial reason. Hence $\tilde{H}_{n-1}(M \sharp N) \cong \tilde{H}_{n-1}(M \vee N)$.

If both M, N are not orientable, so is $M \sharp N$ and we have $\tilde{H}_n(M \sharp N) = 0$ and

$$0 \rightarrow 0 \rightarrow 0 \rightarrow \mathbb{Z} \rightarrow \tilde{H}_{n-1}(M \sharp N) \rightarrow \tilde{H}_{n-1}(M \vee N) \rightarrow 0.$$

Here we need more information of these manifolds. □

Example 3.28. Let M is a closed connected m -manifold and N is a closed connected n -manifold. Show that $M \times N$ is orientable if and only if both M, N are orientable.

Proof. By Topological Künneth Formula, we have

$$0 \rightarrow H_m(M) \otimes H_n(N) \rightarrow H_{m+n}(M \times N) \rightarrow \text{Tor}_1^R(H_m(M) \otimes H_{n-1}(N)) \oplus \text{Tor}_1^R(H_{m-1}(M) \otimes H_n(N)) \rightarrow 0.$$

By Theorem 3.23 we have $H_m(M) \otimes H_n(N) \cong H_{m+n}(M \times N)$ and the result follows by Theorem 3.23 again. □

3.4 Poincaré Duality

First consider cohomology with compact supports.

Definition 3.29. Let $C_c^i(X; G)$ be the subgroup of $C^i(X; G)$ consisting of cochains $\phi : C^i(X) \rightarrow G$ for which there exists a compact set $K = K_\phi \subset X$ such that ϕ is zero on all chains in $X - K$. Note that $\delta\phi$ is then also zero on chains in $X - K$, so $\delta\phi$ lies in $C_c^{i+1}(X; G)$ and the $C_c^i(X; G)$'s for varying i form a subcomplex of the singular cochain complex of X . The cohomology groups $H_c^i(X; G)$ of this subcomplex are the cohomology groups with compact supports.

Another way we let compact $K \hookrightarrow L$ induce $(X, X - L) \hookrightarrow (X, X - K)$, then we have $C^i(X, X - K; G) \hookrightarrow C^i(X, X - L; G)$ and $H^i(X, X - K; G) \rightarrow H^i(X, X - L; G)$.

Proposition 3.30. Since $K \subset X$ are compact sets form a direct system via inclusions. Then we have

$$\varinjlim H^i(X, X - K; G) \cong H_c^i(X; G).$$

Theorem 3.31 (Poincaré Duality). Let M be a R -oriented n -manifold. First we define a map $D_M : H_c^k(M; R) \rightarrow H_{n-k}(M; R)$. Consider compact sets $K \subset L \subset M$, we have

$$\begin{array}{ccccc} H_n(M|L; R) & \xrightarrow{\times} & H^k(M|L; R) & \xrightarrow{\quad \cap \quad} & H_{n-k}(M; R) \\ i_* \downarrow & & i^* \uparrow & \nearrow \cap & \\ H_n(M|K; R) & \xrightarrow{\times} & H^k(M|K; R) & & \end{array}$$

By previous theorem we can find unique elements $\mu_K \in H_n(M|K; R), \mu_L \in H_n(M|L; R)$ restricting to a given orientation of M at each point of K and L , respectively.

So we have $i_*(\mu_L) = \mu_K$ and $\mu_K \cap x = i_*(\mu_L) \cap x = \mu_L \cap i^*(x)$ for all $x \in H^k(M|K; R)$.

So when K vary, we also have $H^k(M|K; R) \xrightarrow{\mu_K \cap (-)} H_{n-k}(M; R)$ which induce

$$D_M : H_c^k(M; R) = \varinjlim H^i(X|K; G) \cong H_{n-k}(M; R).$$

Remark 3.32. When M is a closed R -oriented n -manifold, if $[M]$ is the fundamental class, we have isomorphism

$$D_M : H^k(M; R) \xrightarrow{[M] \frown (-), \cong} H_{n-k}(M; R).$$

Proposition 3.33. A closed manifold of odd dimension has Euler characteristic zero.

Proof. If M is orientable, then $\text{rank}(H_i(M; \mathbb{Z})) = \text{rank}(H^{n-i}(M; \mathbb{Z})) = \text{rank}(H_{n-i}(M; \mathbb{Z}))$ by Poincaré duality and universal coefficient theorem. If n is odd, well done.

If M is not orientable, the similar argument we have $\sum_i (-1)^i \dim H_i(M; \mathbb{Z}/2\mathbb{Z}) = 0$. Now we claim that $\sum_i (-1)^i \dim H_i(M; \mathbb{Z}/2\mathbb{Z}) = \sum_i (-1)^i \text{rank}(H_i(M; \mathbb{Z}))$. Each \mathbb{Z} summand of $H_i(M; \mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summand of $H_i(M; \mathbb{Z}/2\mathbb{Z})$; each $\mathbb{Z}/m\mathbb{Z}$ (where m even) of $H_i(M; \mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summands of $H_i(M; \mathbb{Z}/2\mathbb{Z})$ and $H_{i+1}(M; \mathbb{Z}/2\mathbb{Z})$ which canceled; each $\mathbb{Z}/m\mathbb{Z}$ (where m odd) of $H_i(M; \mathbb{Z})$ contribute nothing. Well done. \square

3.5 Other Duality

Example 3.34 (Euler Characteristic of Boundaries). Let W be a compact $(2m+1)$ -dimensional manifold, then $\chi(\partial W) = 2\chi(W)$.

Proof. Consider $W \times I$ as a $(2m+2)$ -manifold with $\partial(W \times I) = (W \times \{0\}) \cup (M \times I) \cup (W \times \{1\})$. Let $M = \partial W$. Let $U = \partial(W \times I) - (W \times \{1\})$ and $V = \partial(W \times I) - (W \times \{0\})$. Then U, V are open in $\partial(W \times I)$. Both U, V are open in $\partial(W \times I)$. Moreover $U, V \simeq W, U \cap V \simeq M$. So by MV sequence

$$\begin{array}{ccccccc} H_{i+1}(U \cup V) & \longrightarrow & H_i(U \cap V) & \longrightarrow & H_i(U) \oplus H_i(V) & \longrightarrow & H_i(U \cup V) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{i+1}(\partial(W \times I)) & \longrightarrow & H_i(M) & \longrightarrow & H_i(W) \oplus H_i(W) & \longrightarrow & H_i(\partial(W \times I)) \end{array}$$

Since $\chi(\partial(W \times I)) = 0$ since $\dim \partial(W \times I)$ is odd. So

$$2\chi(W) = \chi(M) + \chi(\partial(W \times I)) = \chi(M),$$

well done. \square

Corollary 3.35. If $M = \partial W$ for some compact manifold W , then $\chi(M)$ is even.

Example 3.36 (Boundary of Orientable Manifold is Orientable). Let M be a R -orientable n -manifold with boundary ∂M , then ∂M is R -orientable.

Proof. See [3]. Consider a coordinate $U \cong \mathbb{H}^n$ of $x \in \partial M$. Let $V = \partial U = u \cap \partial M$, and choose $y \in \text{int}(U) = U - V$. We consider R -coefficient homology group, then we have

$$\begin{aligned} H_n(\text{int}(M), \text{int}(M) - \text{int}(U)) &\xrightarrow{R\text{-orientable}, \cong} H_n(\text{int}(M), \text{int}(M) - y) \\ &\xrightarrow{\text{Homotopy by boundary collar}, \cong} H_n(M, M - y) \\ &\xrightarrow{R\text{-orientable}, \cong} H_n(M, M - \text{int}(U)) \\ &\xrightarrow{\partial, \cong} H_n(M - \text{int}(U), M - U) \\ &\xrightarrow{\text{Homotopy by boundary collar}, \cong} H_n(M - \text{int}(U), M - \text{int}(U) - x) \\ &\xrightarrow{\text{Excision of } \text{int}(M) - \text{int}(U), \cong} H_n(\partial M, \partial M - x) \\ &\xrightarrow{R\text{-orientable}, \cong} H_n(\partial M, \partial M - V). \end{aligned}$$

Well done. □

Remark 3.37. In smooth case, we can calculate the transition function. See Theorem 1.3 in <http://staff.ustc.edu.cn/~wangzuoq/Courses/21F-Manifolds/Notes/Lec24.pdf>.

Theorem 3.38 (Poincaré Duality with Boundaries). Suppose M is a compact R -orientable n -manifold whose boundary ∂M is decomposed as the union of two compact $(n-1)$ dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism $D_M : H^k(M, A; R) \xrightarrow{[M] \frown (-), \cong} H_{n-k}(M, B; R)$.

Corollary 3.39 (Lefschetz Duality). Suppose M is a compact R -orientable n -manifold and take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism $D_M : H^k(M, \partial M; R) \xrightarrow{[M] \frown (-), \cong} H_{n-k}(M; R)$ and $D_M : H^k(M; R) \xrightarrow{[M] \frown (-), \cong} H_{n-k}(M, \partial M; R)$.

Theorem 3.40 (Generalized Local Homology Groups). Let K be a compact, locally contractible subspace of a closed orientable n -manifold M , then

$$H_i(M, M - K; \mathbb{Z}) \cong H^{n-i}(K; \mathbb{Z}).$$

Theorem 3.41 (Alexander Duality). If K is a compact, locally contractible subspace of S^n , then for all i and any abelian group G , we have

$$\tilde{H}_i(S^n - K; G) \cong \tilde{H}^{n-i-1}(K; G).$$

Theorem 3.42 (Poincaré-Alexander-Lefschetz Duality). Let M be an n -manifold R -oriented by ϑ where R is any commutative ring with an identity element. For any R -module G , and let $L \subset K$ be compact subsets of M . Then the cap product induce the isomorphism

$$\cap[\vartheta] : \lim_{(U,V) \supset (K,L) \text{ open}} H^p(U, V; G) \xrightarrow{\cong} H_{n-p}(M - L, M - K; G).$$

Proof. See Theorem VI.8.3 in [1]. More corollary we refer book Homology, Cohomology, and Sheaf Cohomology for Algebraic Topology, Algebraic Geometry, and Differential Geometry section 14.5. □

Corollary 3.43. Let M be an n -manifold R -oriented by ϑ where R is any commutative ring with an identity element. For any R -module G , and let $L \subset K$ be compact subsets of M . If both of them are good pair, then the cap product induce the isomorphism

$$H^p(K, L; G) \cong \lim_{(U,V) \supset (K,L) \text{ open}} H^p(U, V; G) \xrightarrow{\cap[\vartheta]} H_{n-p}(M - L, M - K; G).$$

Proof. From Lemma 9 and Theorem 10 in chapter 6.1 in [4]. □

3.6 Cohomology Rings

As before we have

$$\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$$

where $\alpha \in C_{k+l}(X; R)$, $\phi \in C^k(X; R)$, $\psi \in C^l(X; R)$. So we have

$$\begin{array}{ccc} H^l(X; R) & \xrightarrow{h} & \text{Hom}_R(H_l(X; R), R) \\ \phi \smile \downarrow & & (\frown \phi)^* \downarrow \\ H^{k+l}(X; R) & \xrightarrow{h} & \text{Hom}_R(H_{k+l}(X; R), R) \end{array}$$

For closed R -orientable n -manifold M , consider an important pair:

$$\begin{aligned} H^k(M; R) \times H^{n-k}(M; R) &\longrightarrow R \\ (\phi, \psi) &\longmapsto (\phi \smile \psi)[M] \end{aligned}$$

Proposition 3.44. *This pair is nonsingular for closed R -orientable manifolds when R is a field or when $R = \mathbb{Z}$ and torsion in $H^*(M; \mathbb{Z})$ is factored out.*

Proof. By the universal coefficient theorem and the Poincaré duality, we have an isomorphism

$$H^{n-k}(M; R) \xrightarrow{h} \text{Hom}_R(H_{n-k}(M; R), R) \xrightarrow{D_M^*} \text{Hom}_R(H^k(M; R), R).$$

Well done. □

Corollary 3.45. *If M is a connected closed orientable n -manifold, then for each element $\alpha \in H^k(M; \mathbb{Z})$ of infinite order that is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M; \mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$. With coefficients in a field the same conclusion holds for any $\alpha \neq 0$.*

Proof. Follows directly from the nonsingular pair. □

Proposition 3.46. *If $R \rightarrow S$ be a ring map, then so is $H^*(X, A; R) \rightarrow H^*(X, A; S)$.*

Proof. Trivial. □

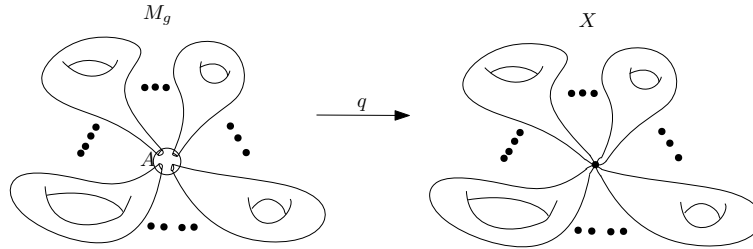
Example 3.47 (Moore Spaces). *Let $M(G, n)$ is a Moore space, let G generated by g_i with relations F_j . then $H^*(M(G, n); \mathbb{Z}) \cong \mathbb{Z}[g_i]/(g_i^2, F_j)$.*

Example 3.48 (Oriented Closed Surfaces). *Let $g \geq 0$ and M_g be the genus g oriented closed surface. Then*

$$H^*(M_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i, y_i, z]_{i=1}^g}{(x_i y_i - z, x_i x_j, y_i y_j, z^2, x_i z, y_i z, \{x_i y_k : i \neq k\})}$$

for $\deg x_i = \deg y_j = 1$ and $\deg z = 2$.

Proof. Let $X := \bigvee_{i=1}^g T_i$ where T_i are tori. Let $i : A \hookrightarrow M_g$ be the inclusion where $A = S^2 \setminus \bigcup_{j=1}^g D^2 \simeq \bigvee_{j=1}^{g-1} S^1$ as following diagram:



where $q : M_g \rightarrow M_g/A \cong X$ be the quotient map since $M_g \cong T_1 \# T_2 \# \dots \# T_g$. Hence we get a ring map $q^* : H^*(X) \rightarrow H^*(M_g)$.

First we need to find the relation of cohomology classes via $q^* : H^*(X) \rightarrow H^*(M_g)$. The only non-trivial cases are $* = 1, 2$. For $* = 1$, we consider the exact sequence

$$H_1(A) \xrightarrow{i_*} H_1(M_g) \xrightarrow{q_*} H_1(X) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(M_g).$$

We find that $i_* : H_1(A) \rightarrow H_1(M_g)$ is zero since the loops in A will be in the commutator of $\pi_1(M_g)$ and $H_1(M_g) = \text{Abel}(\pi_1(M_g))$. Moreover, $i_* : H_0(A) \rightarrow H_0(M_g)$ is injective by the definition, so $\delta = 0$. Hence $q_* : H_1(M_g) \cong H_1(X)$ which induce $q^* : H^1(X) \cong H^1(M_g)$ by universal coefficient theorem.

For $* = 2$, we let $p_i : X \rightarrow T_i$ are projects in to the i -th torus. Then we have

$$H_2(M_g) \cong \mathbb{Z} \xrightarrow{q_*} H_2(X) \cong \mathbb{Z}^g \xrightarrow{(p_i)_*} H_2(T_i) \cong \mathbb{Z}$$

As the top cell e^2 of M_g send via q, p_i is also a top cell T_i , by cellular chain complex and its homology we know that $(p_i)_* \circ q_* : 1 \mapsto 1$. Hence $q_* : 1 \mapsto (1, \dots, 1)$. So $q^* : H^2(X) \cong \mathbb{Z}^g \rightarrow H^2(M_g)$ given by $\gamma_i \mapsto 1$ for the generators of every summands.

Finally we know the work of q^* . Let $H^1(X)$ generated by $\{\alpha_i, \beta_i\}_{i=1}^g$ and $H^2(X)$ generated by $\{\gamma_i\}_{i=1}^g$. We know that $\alpha_i \cup \beta_i = \gamma_i$ and all other cup product between them will be zero by the case of tori. Hence if we let $x_i = q^*\alpha_i, y_i = q^*\beta_i$ and $z = q^*\gamma_1 = \dots = q^*\gamma_g$, we have

$$H^*(M_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i, y_i, z]_{i=1}^g}{(x_i y_i - z, x_i x_j, y_i y_j, z^2, x_i z, y_i z, \{x_i y_k : i \neq k\})}$$

for $\deg x_i = \deg y_j = 1$ and $\deg z = 2$. Well done. \square

Example 3.49 (Non-oriented Closed Surfaces). *Let $g \geq 0$ and N_g be the genus g non-oriented closed surface. Then*

$$H^*(N_g; \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[x_i, y]_{i=1}^g}{(x_i^2 = y, y^2, \{x_i x_j : i \neq j\})}$$

for $\deg x_i = 1$ and $\deg y = 2$.

Proof. Here all the homology and cohomology groups are of $\mathbb{Z}/2\mathbb{Z}$ -coefficients.

Let $X := \bigvee_{i=1}^g \mathbb{R}P^2$ and let $i : A \hookrightarrow N_g$ be the inclusion where $A = S^2 \setminus \coprod_{j=1}^g D^2 \simeq \bigvee_{j=1}^{g-1} S^1$ as the orinetable case. Hence we have $q : N_g \rightarrow M_g/A \cong X$ be the qoutient map since $N_g \cong \underbrace{\mathbb{R}P^2 \sharp \mathbb{R}P^2 \sharp \dots \sharp \mathbb{R}P^2}_g$ and get the ring map $q^* : H^*(X) \rightarrow H^*(N_g)$.

First we need to find the relation of cohomology classes via $q^* : H^*(X) \rightarrow H^*(N_g)$. The only non-trivial cases are $* = 1, 2$. For $* = 1$, we consider the exact sequence

$$H_1(A) \xrightarrow{i_*} H_1(N_g) \xrightarrow{q_*} H_1(X) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(N_g).$$

We find that $i_* : H_1(A) \rightarrow H_1(N_g)$ is zero since the loops in A will be two times of the loops in $\pi_1(N_g)$ and $H_1(N_g) = \text{Abel}(\pi_1(N_g)) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Moreover, $i_* : H_0(A) \rightarrow H_0(N_g)$ is injective by the definition, so $\delta = 0$. Hence $q_* : H_1(N_g) \cong H_1(X)$ which induce $q^* : H^1(X) \cong H^1(N_g)$ by universal coefficient theorem.

For $* = 2$, we let $p_i : X \rightarrow \mathbb{R}P^2$ are projects in to the i -th space. Then we have

$$H_2(N_g) \cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{q_*} H_2(X) \cong \mathbb{Z}/2\mathbb{Z}^g \xrightarrow{(p_i)_*} H_2(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$$

As the top cell e^2 of N_g send via q, p_i is also a top cell $\mathbb{R}P^2$, by cellular chain complex and its homology we know that $(p_i)_* \circ q_* : 1 \mapsto 1$. Hence $q_* : 1 \mapsto (1, \dots, 1)$. So $q^* : H^2(X) \cong \mathbb{Z}/2\mathbb{Z}^g \rightarrow H^2(N_g)$ given by $\gamma_i \mapsto 1$ for the generators of every summands.

Finally we know the work of q^* . Let $H^1(X)$ generated by $\{\alpha_i\}_{i=1}^g$ and $H^2(X)$ generated by $\{\beta_i\}_{i=1}^g$. We know that $\alpha_i^2 = \beta$ and all other cup product between them will be zero by the case of tori. Hence if we let $x_i = q^*\alpha_i$ and $y = q^*\beta_1 = \dots = q^*\beta_g$, we have

$$H^*(N_g; \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[x_i, y]_{i=1}^g}{(x_i^2 = y, y^2, \{x_i x_j : i \neq j\})}$$

for $\deg x_i = 1$ and $\deg y = 2$. Well done. \square

Proposition 3.50. *Let M_1 and M_2 be closed oriented manifolds of dimension n . If $f : M_1 \rightarrow M_2$ is a continuous map of non-zero degree, and $x \in H^k(M_2; \mathbb{Z})$ is non-torsion, then $f^*x \neq 0$.*

Proof. Let x_i denote the oriented generator of $H^n(M_i; \mathbb{Z})$, then $f^*x_2 = \deg(f)x_1$. As $x \in H^k(M_2; \mathbb{Z})$ is non-torsion, by Poincaré duality there is $y \in H^{n-k}(M_2; \mathbb{Z})$ with $x \cup y \neq 0$, so $x \cup y = rx_2$ for $r \neq 0$. Now $f^*x \cup f^*y = f^*(x \cup y) = r \deg(f)x_1 \neq 0$. Hence $f^*x \neq 0$. \square

Corollary 3.51. *If there exists a map $M_1 \rightarrow M_2$ of non-zero degree, then for all k we have $\text{rank} H^k(M_1) \geq \text{rank} H^k(M_2)$. In particular, if $M_g \rightarrow M_h$ of non-zero degree, then $g \geq h$.*

Proof. Directly from the Proposition. \square

Example 3.52 (Complex Projective Spaces). *We have the ring isomorphisms*

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha], \quad |\alpha| = 2.$$

Proof. Inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induce the same cohomology group of degree less than $2n-2$, so by induction on n we have $H^{2i}(\mathbb{C}P^n; \mathbb{Z})$ is generated by α^i for $i < n$. By the Corollary 3.45 we can find $m\alpha^{i-1}$ such that $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n; \mathbb{Z})$, so $m = \pm 1$, well done. For the case of $\mathbb{C}P^\infty$, this follows from the cellular cohomology. \square

Remark 3.53. *Similarly, we have $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$ for $|\alpha| = 4$.*

Example 3.54 (Real Projective Spaces). *We have the ring isomorphisms*

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha], \quad |\alpha| = 1.$$

Furthermore, we have $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha)$ for $|\alpha| = 2$ and

$$H^*(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}), |\alpha| = 2, & n = 2k; \\ \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta), |\alpha| = 2, |\beta| = n, & n = 2k+1. \end{cases}$$

Here β is a generator of $H^{2k+1}(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}$.

Proof. For the coefficient of $\mathbb{Z}/2\mathbb{Z}$ this is similar as complex projective space.

By Proposition 3.46 we have a ring map $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \rightarrow H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$. Consider the cellular cochain as follows:

$$\begin{array}{ccccccc} \cdots & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{\quad} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{\quad} & 0 \end{array}$$

Hence the ring map $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}[\alpha]$ is injective in the positive dimension with the image in the even part of $\mathbb{Z}/2\mathbb{Z}[\alpha]$. Hence $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]/(2\alpha)$ for $|\alpha| = 2$.

For $n = 2k$ this is the same, hence $H^*(\mathbb{R}P^{2k}; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$ for $|\alpha| = 2$. For $n = 2k + 1$, note that the top cohomology is \mathbb{Z} . We let a generator of it is β , hence α, β generated by α, β . But in this case $\beta^2 = 0$ and $\alpha\beta = 0$ by dimension reason. Hence $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$ for $|\alpha| = 2, |\beta| = 2k + 1$. \square

Example 3.55 (Torus). *We have*

$$H^*(T^n; R) \cong \bigwedge_R [\alpha_1, \dots, \alpha_n]$$

for generators $\alpha_i \in H^1(S^1; R)$.

More generally, we have

$$H^*(S^{k_1} \times \dots \times S^{k_n}; R) \cong \bigwedge_R [\alpha_1, \dots, \alpha_n]$$

when all k_i odd for generators $\alpha_i \in H^1(S^{k_i}; R)$.

Proof. Follows directly from Künneth formula. \square

Example 3.56 (Same Cohomology Group with Different Ring). *The spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$ has the same cohomology groups but with the different ring structure.*

Proof. We have $\tilde{H}^*(S^2 \vee S^4; \mathbb{Z}) \cong \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^4; \mathbb{Z})$. Now as $\alpha^2 = 0$ for $\alpha \in H^2(S^2; \mathbb{Z})$ which is impossible for $\mathbb{C}P^2$, then well done. \square

Example 3.57 (Same Cohomology with Additive Structure but Different Cup Product). *The spaces $\mathbb{C}P^3$ and $S^2 \times S^4$ has the same cohomology groups but with the different ring structure.*

Proof. By Künneth formula we have $H^*(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^4)$ for $|\alpha| = 2$ and $|\beta| = 4$ with signature-different product. Now as $\alpha^2 = 0$ for $|\alpha| = 2$ which is impossible for $\mathbb{C}P^3$, then well done. \square

Remark 3.58. *Another example is $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. To compute $H^*(\mathbb{C}P^2 \sharp \mathbb{C}P^2; \mathbb{Z})$, you may consider the map $\mathbb{C}P^2 \sharp \mathbb{C}P^2 \rightarrow \mathbb{C}P^2 \vee \mathbb{C}P^2$.*

Example 3.59 (Same \mathbb{Z} -Cohomology Ring with Different $\mathbb{Z}/2\mathbb{Z}$ -Ones). *We have ring isomorphism $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong H^*(\mathbb{R}P^{2k} \vee S^{2k+1}; \mathbb{Z})$ but this is not true for $\mathbb{Z}/2\mathbb{Z}$ -coefficient rings.*

Proof. The isomorphism $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong H^*(\mathbb{R}P^{2k} \vee S^{2k+1}; \mathbb{Z})$ is trivial. But for the generator $\alpha \in H^1(\mathbb{R}P^{2k+1}; \mathbb{Z}/2\mathbb{Z})$ we have $\alpha^{2k+1} \neq 0$. This is impossible for $\mathbb{R}P^{2k} \vee S^{2k+1}$. \square

4 Applications in the Classical Results

Example 4.1 (Jordan Curve). *Actually we view $S^1 \subset \mathbb{R}^2$ as one-point compactification $S^1 \subset S^2$, then we use Alexander duality as*

$$\tilde{H}_0(S^2 - S^1; \mathbb{Z}) \cong \tilde{H}^1(S^1; \mathbb{Z}) \cong \mathbb{Z},$$

so $H_0(S^2 - S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, well done.

Example 4.2 (Jordan-Brouwer Separation Theorem). *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then $\mathbb{R}^n - S$ has two components.*

Proof. Also we let it as in one-point compactification $S \subset S^n$. Now we didn't know whether S is orientable or not, we consider $\mathbb{Z}/2\mathbb{Z}$ as coefficient, then we use Alexander duality and Poincaré duality

$$\tilde{H}_0(S^n - S; \mathbb{Z}/2\mathbb{Z}) \cong \tilde{H}^{n-1}(S; \mathbb{Z}/2\mathbb{Z}) \cong H_0(S; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

well done. \square

Example 4.3 (Compact Hypersurface as Boundary). *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then S be the boundary of some domain in \mathbb{R}^n .*

Proof. Trivial by Jordan-Brouwer Separation Theorem. \square

Proposition 4.4. *Let $X \subset \mathbb{R}^n$ be a compact and locally contractible, then $H_i(X; \mathbb{Z}) = 0$ for $i \geq n$ and torsion-free for $i = n - 1, n - 2$.*

Proof. View $X \subset S^n$ by one-point compactification. Hence by Alexander duality we have $\tilde{H}_{n-i-1}(S^n - X; \mathbb{Z}) \cong \tilde{H}^i(X; \mathbb{Z})$. Then using universal coefficient theorem as in the following examples we can get the result. \square

Example 4.5 (Compact Hypersurface in \mathbb{R}^n is Orientable). *If S be a connected compact hypersurface S in \mathbb{R}^n is orientable.*

Proof 1. Follows from Proposition 4.4 directly. But here we will go through the proof of that proposition.

Since $\dim S = n - 1$, we have to calculate $H_{n-2}(S; \mathbb{Z})$. Also we let it as in one-point compactification $S \subset S^n$. WLOG we let $n > 1$. If S is not orientable, we have $H_{n-1}(S; \mathbb{Z}) = 0$ and $H_{n-2}(S; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, then we have

$$\begin{aligned} \mathbb{Z} &\cong \tilde{H}_0(S^n - S; \mathbb{Z}) \cong H^{n-1}(S) \\ &\cong \text{Hom}_{\mathbb{Z}}(H_{n-1}(S; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Ext}_{\mathbb{Z}}^1(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

which is impossible. Well done. \square

Proof 2. Here we give another method. Take $x \in S$ and $u \in N_x(\mathbb{R}^n/S)$ with $\|u\| = 1$. By Jordan-Brouwer separation theorem we may let u always in the same component when x is varying on S . Consider a non-trivial vector field $X(x) = u(x)$. Now $i_X(\text{vol})$ restricted to S is a volume form on S where vol is the canonical volume form on \mathbb{R}^n . \square

Proof 3. Moreover we could prove that the normal bundle of S is trivial. See <https://math.stackexchange.com/questions/863960/orientation-of-hypersurface>. \square

Example 4.6. *We show that $\mathbb{R}P^n$ can be embedded in \mathbb{R}^{n+1} if and only if $n = 1$ and $\mathbb{C}P^n$ can be embedded in \mathbb{R}^{2n+1} if and only if $n = 1$.*

Proof. The same proof holds for $\mathbb{C}P^n$ and we only consider $\mathbb{R}P^n$. Here we only consider the $\mathbb{Z}/2\mathbb{Z}$ -coefficient groups.

If $n = 1$, then $\mathbb{R}P^1 \cong S^1$. Hence it can be embedded in \mathbb{R}^2 .

Conversely we let $n \geq 2$. If $\mathbb{R}P^n$ can be embedded in \mathbb{R}^{n+1} , then we have embedding $\mathbb{R}P^n \hookrightarrow S^{n+1}$. Let $H^*(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$. Consider the good pair $(S^{n+1}, \mathbb{R}P^n)$ we have $H^{n+1}(S^{n+1}, \mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}^2$. By Corollary 3.43 we have $H^{n+1}(S^{n+1}, \mathbb{R}P^n) \cong H_0(S^{n+1} - \mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}^2$. Hence $S^{n+1} - \mathbb{R}P^n$ has two connected components. Let A, B are closure of these components in S^{n+1} , hence $A \cup B = S^{n+1}$ and $A \cap B = \mathbb{R}P^n$. Hence we get $H^{n+1}(A) \oplus$

$H^{n+1}(B) \cong H^{n+1}(S^{n+1} - \mathbb{R}P^n)$. As by Corollary 3.43 again we have $H^{n+1}(S^{n+1} - \mathbb{R}P^n) \cong H_0(S^{n+1}, \mathbb{R}P^n) = 0$, hence $H^{n+1}(A) = H^{n+1}(B) = 0$.

Now since $n \geq 2$, we have $H^1(S^{n+1}) = H^2(S^{n+1}) = 0$. By MV-sequence for (A, B) we have $H^1(A) \oplus H^1(B) \cong H^1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ via canonical pullback. WLOG we let $H^1(B) = 0$. Let $i : \mathbb{R}P^n \subset A$ and $i^*\alpha = x$, then $i^*\alpha^n = x^n$ generates $H^n(\mathbb{R}P^n)$. MV-sequence again:

$$0 \rightarrow H^n(A) \oplus H^n(B) \rightarrow H^n(\mathbb{R}P^n) \rightarrow H^{n+1}(S^{n+1}) \rightarrow H^{n+1}(A) = H^{n+1}(B) = 0.$$

This is impossible since $H^n(A) \rightarrow H^n(\mathbb{R}P^n)$ surjective but $H^{n+1}(S^{n+1}) \cong \mathbb{Z}/2\mathbb{Z}$. \square

Remark 4.7. *Actually you can show this using Stiefel-Whitney class for \mathbb{R}^n and total Pontryagin class for $\mathbb{C}P^n$, see `EmbeddedProj`.*

5 Some Other Things

Proposition 5.1. *The infinity sphere S^∞ is contractible.*

Proof. Step 1. Consider $f_t : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ as $(x_1, x_2, \dots) \mapsto (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$.

Step 2. Consider $g_t : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ as $(x_1, x_2, \dots) \mapsto (1-t)(0, x_1, \dots) + t(1, 0, \dots)$.

Step 3. Consider $\frac{g_t}{|g_t|} \circ \frac{f_t}{|f_t|}$ which give us $\mathbf{1}_{S^\infty} \simeq (x \mapsto (1, 0, \dots))$. \square

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