

# **Reading Notes: Moduli Spaces of Curves**

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## **Part I**

# **The basic facts of curves**





# Chapter 1

## Basic facts of general curves

### 1.1 Standard results

**Definition 1.1.1.** A curve over  $k$  is a pure one-dimensional scheme  $C$  of finite type over  $k$ . If  $C$  is proper, we define the arithmetic genus (simply the genus) of  $C$  as  $g(C) := g_a(C) = 1 - \chi(C, \mathcal{O}_C)$ . By Review A.3.1, if  $C$  is geometrically connected and geometrically reduced, this is equal to  $h^1(C, \mathcal{O}_C)$ .

**Theorem 1.1.2** (St 0B5Y). Let  $k$  be a field. Let  $C$  be a proper scheme of dimension  $\leq 1$  over  $k$ . Let  $L$  be an invertible  $\mathcal{O}_X$ -module. Let  $C_i$  be the irreducible components of dimension 1. Then  $L$  is ample if and only if  $\deg(L|_{C_i}) > 0$  for all  $i$ .

**Theorem 1.1.3** (Serre duality of smooth curves). Let  $C$  be a smooth projective curve over  $k$  with canonical bundle  $\omega_C = \Omega_C$ , then for any vector bundle  $F$  we get

$$H^0(C, F^\vee \otimes \omega_C) \cong H^1(C, F)^\vee.$$

If we define the geometrical genus  $g_e(C) = h^0(C, \omega_C)$  and if  $C$  is smooth projective curve which is geometrically connected and geometrically reduced, then  $h^0(C, \mathcal{O}_C) = 1$ . Hence by serre-duality we get  $g_e(C) = g_a(C)$ .

**Theorem 1.1.4** (Riemann-Roch for smooth curves). Let  $C$  be a smooth projective curve over  $k$  with a line bundle  $L$ , then

$$\chi(C, L) = h^0(C, L) - h^0(C, \omega_C \otimes L^\vee) = \deg L + 1 - g.$$

**Theorem 1.1.5** (Positivity of divisors on smooth curves). Let  $C$  be a smooth projective curve over  $k$  of genus  $g$  with a line bundle  $L$ , then

- (a) if  $\deg L \geq 2g$ , then  $L$  is base-point-free;
- (b) if  $\deg L \geq 2g + 1$ , then  $L$  is very ample;
- (c) if  $\deg L > 0$ , then  $L$  is ample.
- (d) if  $\deg L < 0$ , then  $h^0(C, L) = 0$ .

*Proof.* See the section IV.3 of [25] for the proof when  $k$  is algebraic closed. This is also right when  $k$  is not algebraic closed, see section 20.2 in [36].

Here we use another method to show (d) as a special case of [25] Ex.III.7.1. We just consider the case  $C$  is integral. If  $\deg L < 0$ , then  $L^{-1}$  is ample. Let  $h^0(C, L) > 0$  and take a nonzero

$s \in H^0(C, L)$ . As  $H^0(C, L) = \text{Hom}(\mathcal{O}_C, L)$ , we can get  $- \times s : \mathcal{O}_C \rightarrow L$ . As  $C$  integral,  $s$  must nonzero at the generic point, hence  $- \times s : \mathcal{O}_C \rightarrow L$  is injective. Hence we get  $L^{-1} \subset \mathcal{O}_C$ . Let  $n$  such that  $L^{-n}$  generated by global sections, we get  $L^{-n} \subset \mathcal{O}_C$ . Hence  $H^0(C, L^{-n}) \subset H^0(C, \mathcal{O}_C)$ . Consider hilbert polynomial  $\chi(L^{-n}) = \alpha n + \beta$  as  $\deg \chi(L^{-n}) = \dim \text{supp}(L^{-1}) = \dim C = 1$ . By Serre's vanishing theorem, we get for  $n \rightarrow \infty$ , we have  $\chi(L^{-n}) = h^0(C, L^{-n}) \rightarrow \infty$ . This is impossible since  $h^0(C, \mathcal{O}_C) < \infty$ .  $\square$

**Theorem 1.1.6** (Riemann-Hurwitz Theorem, St 0C1B). *Let  $f : X \rightarrow Y$  be a separable morphism of smooth proper curves over a field  $k$  and if  $k = H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$  and  $X$  and  $Y$  have genus  $g_X$  and  $g_Y$ , then*

$$2g_X - 2 = (2g_Y - 2) \deg(f) + \deg R$$

where  $R$  be the ramified divisor. Moreover,  $\deg R = \sum_x d_x [\kappa(x) : k]$  where  $d_x = \text{length}_{\mathcal{O}_{X,x}} \Omega_{X/Y,x}$ . Of course if  $\mathcal{O}_{X,x}$  is tamely ramified over  $\mathcal{O}_{Y,f(x)}$  then  $d_x = e_x - 1$ . If not, we only have  $d_x > e_x - 1$  where  $e_x$  is the ramification index.

## 1.2 Automorphisms of curves

Here we only consider smooth connected projective curves of genus  $g$  over an algebraically closed field  $k$ .

**Proposition 1.2.1.** *For  $g = 0$ , we get  $\text{Aut}(\mathbb{P}_k^1) \cong \text{PGL}_2$ . Moreover, if we consider all automorphisms fixed  $n$  points, then this group is finite if and only if  $n \geq 3$ .*

*Proof.* See [25] Example II.7.1.1, we get  $\text{Aut}(\mathbb{P}_k^1) \cong \text{PGL}_2$ . Moreover, all automorphisms fixed  $n$  points is finite if and only if  $n \geq 3$  by easy linear algebra.  $\square$

**Proposition 1.2.2.** *For curve  $C$  with  $g = 1$ , we get  $\text{Aut}(C)$  is infinite group. Moreover, if we consider all automorphisms fixed  $n$  points, then this group is finite if and only if  $n \geq 1$ .*

*Proof.* In this case  $C$  is actually a group scheme of dimension 1 (by Picard varieties) and  $C$  can then act on  $C$ . Hence  $C \subset \text{Aut}(C)$ , hence infinite. Moreover, by [25] Corollary IV.4.7, if we fixed one point  $P_0$ , then  $\text{Aut}(C; P_0)$  is finite.  $\square$

**Proposition 1.2.3** (Hurwitz). *For curve  $C$  with  $g \geq 2$ , the group  $\text{Aut}(C)$  is finite. Moreover, if  $k$  has characteristic 0, we have  $\#(\text{Aut}(C)) \leq 84g - 84$ .*

*Proof.* See [25] Ex.V.1.11 and Hurwitz's Automorphism Theorem.  $\square$

**Lemma 1.2.4** (St 0E67). *Let  $X$  be a smooth, proper, connected curve over  $k$  of genus  $g$ .*

- (a) *If  $g \geq 2$ , then  $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$ ;*
- (b) *If  $g = 1$  and  $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  is nonzero, then  $D$  does not fix any closed point of  $X$ ;*
- (c) *If  $g = 0$  and  $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  is nonzero, then  $D$  can fix at most 2 closed points of  $X$ .*

**Remark 1.2.5.** *We will say an element  $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  fixes  $x$  if  $D(\mathcal{I}) \subset \mathcal{I}$  where  $\mathcal{I}$  is the ideal sheaf of  $x$ .*

*Sketch.* As we have the canonical derivation  $d : \mathcal{O}_X \rightarrow \Omega_{X/k}$ , taking any  $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$  we get  $D = f \circ d$  where  $f \in \text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$  and  $\deg(\Omega_{X/k}) = 2g - 2$ .

- (a) *If  $g \geq 2$ , then  $\deg(\Omega_{X/k}) > 0$ . Hence*

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, T_{X/k}) = \Gamma(X, T_{X/k}) = 0,$$

hence  $f = 0$ ;

(b)(c) We claim that the vanishing of  $f$  at  $x \in X$  is equivalent to the statement that  $D$  fixes  $x$ . Indeed, by St 0C1E we get for the uniformizer  $z \in \mathcal{O}_{X,x}$ ,  $dz$  is a basis of  $\Omega_{X,x}$ . Since  $D(z) = f(dz)$ , we conclude the claim.

If  $g = 1$ , then a nonzero  $f$  does not vanish anywhere. Hence by the claim,  $D$  does not fix any closed point of  $X$ . If  $g = 0$ , then a nonzero  $f$  vanishes in a divisor of degree 2. Hence by the claim,  $D$  can fix at most 2 closed points of  $X$ .  $\square$

**Lemma 1.2.6.** *Let  $X$  be a proper scheme over a field  $k$  of dimension  $\leq 1$ , then the following are equivalent*

- (i)  $\underline{\text{Aut}}(X)$  is geometrically reduced over  $k$  and has dimension 0;
- (ii)  $\underline{\text{Aut}}(X) \rightarrow \text{Spec}(k)$  is unramified;
- (iii)  $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$ .

*Proof.* See St 0DSW and St 0E6G. Note that these two lemmas can also give the results about automorphism groups of smooth connected curves.  $\square$

**Proposition 1.2.7.** *Let  $C$  be a curve of genus  $g$  over a field  $k$  of characteristic 0, then for any non-trivial automorphism of  $C$  fixed at most  $2g + 2$  points.*

*Proof.* See I.F-4 in [3] for now. To add.  $\square$



# Chapter 2

## Families of curves

### 2.1 Families of smooth curves

**Lemma 2.1.1** (Families of smooth curves). *Let  $(S, f : C \rightarrow S)$  in  $\mathcal{M}_g$  for  $g \geq 2$ .*

- (i)  $f_*\mathcal{O}_C = \mathcal{O}_S$ ;
- (ii) For  $k > 1$  the sheaf  $f_*(\Omega_{C/S}^1)^{\otimes k}$  is locally free of rank  $(2k-1)(g-1)$  on  $S$ , and for any  $g : S' \rightarrow S$ , we get an isomorphism  $g^*f_*(\Omega_{C/S}^1)^{\otimes k} \cong f'_*(\Omega_{C'/S'}^1)^{\otimes k}$ . Moreover,  $R^i f_*(\Omega_{C/S}^1)^{\otimes k} = 0, i > 0$ ;
- (iii) The sheaf  $f_*\Omega_{C/S}^1$  is locally free of rank  $g$  on  $S$ , and for any  $g : S' \rightarrow S$ , we get an isomorphism  $g^*f_*\Omega_{C/S}^1 \cong f'_*\Omega_{C'/S'}^1$ . Moreover,  $R^1 f_*(\Omega_{C/S}^1) = \mathcal{O}_S$  and  $R^i f_*(\Omega_{C/S}^1) = 0, i > 1$ ;
- (iv) For  $k \geq 3$ ,  $(\Omega_{C/S}^1)^{\otimes k}$  is relative very ample.

*Proof.* (i) By definition, for all  $s \in S$  the  $C_s$  is proper geometrically connected and geometrically reduced, then by Review A.3.1 we get  $H^0(C_s, \mathcal{O}_s) = \kappa(s)$ , hence  $\phi_s^0 : f_*\mathcal{O}_C \otimes \kappa(s) \rightarrow H^0(C_s, \mathcal{O}_s)$  is surjective. By Review A.1.1 with  $i = 0$ , we get  $\phi_s^0$  is an isomorphism and  $f_*\mathcal{O}_C$  is a line bundle. Now consider the natural map  $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$  induce a surjective fiber map  $\kappa(s) \rightarrow f_*\mathcal{O}_C \otimes \kappa(s)$  by seen

$$\kappa(s) \rightarrow f_*\mathcal{O}_C \otimes \kappa(s) \rightarrow H^0(C_s, \mathcal{O}_s) = \kappa(s).$$

Thus  $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$  is surjective, hence an isomorphism.

(ii) For all  $s \in S$  and  $k > 1$  we get  $H^1(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = H^0(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes(1-k)})^\vee = 0$  as  $(\Omega_{C_s/\kappa(s)}^1)^{\otimes(1-k)}$  is anti-ample. Hence  $H^i(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = 0$  for  $i > 0$ . Now use Review A.1.1 we get  $R^i f_*(\Omega_{C/S}^1)^{\otimes k} = 0, i > 0$ .

On the other hand, by Riemann-Roch theorem, we get

$$h^0(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = \deg((\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) + 1 - g = (2k-1)(g-1).$$

Use Review A.1.1 again, we get  $f_*(\Omega_{C/S}^1)^{\otimes k}$  is locally free of rank  $(2k-1)(g-1)$  on  $S$ .

(iii) By Review A.1.1 and the fact  $H^i(C_s, \Omega_{C/S}^1 \otimes \kappa(s)) = 0, i > 1$  implies  $R^i f_*\Omega_{C/S}^1 = 0, i > 1$ . Now we use the duality  $f_*\mathcal{H}om(F, \Omega_{C/S}^1) \cong \mathcal{H}om(R^1 f_*F, \mathcal{O}_S)$ , then let  $F = \Omega_{C/S}^1$ . We get  $f_*\mathcal{O}_C \cong (R^1 f_*\Omega_{C/S}^1)^*$ . Hence  $R^1 f_*\Omega_{C/S}^1 \cong (f_*\mathcal{O}_C)^* = \mathcal{O}_S^* \cong \mathcal{O}_S$ .

By Review A.1.1(ii) with  $i = 1$ , we get  $\phi_s^0 : f_*\Omega_{C/S}^1 \otimes \kappa(s) \rightarrow H^0(C_s, \Omega_{C_s/\kappa(s)}^1)$  is surjective, hence an isomorphism. Then apply Review A.1.1(i)-(ii) with  $i = 0$  to imply  $f_*\Omega_{C/S}^1$  is locally free of rank  $h^0(C_s, \Omega_{C_s/\kappa(s)}^1) = g$ .

(iv) Easy to see for any  $s \in S$  the fiber  $(\Omega_{C_s/\kappa(s)}^1)^{\otimes k}$  is very ample as  $\deg(\Omega_{C_s/\kappa(s)}^1)^{\otimes k} = k(2g - 2) \geq 2g + 1$ . Using noetherian approximation, we may let  $S$  is noetherian. Then use Review A.3.2 and well done.  $\square$

**Proposition 2.1.2** (Flatness Criterion over Smooth Curves). *Let  $C$  be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field) and  $X \rightarrow C$  a qcqs morphism of schemes. A quasi-coherent  $\mathcal{O}_X$ -module  $F$  is flat over  $C$  if and only if every associated point of  $F$  maps to the generic point of  $C$ .*

## 2.2 Families of elliptic curves

This section are some preliminaries of the coarse moduli space of  $\mathcal{M}_{1,1}$ . Here we follows [34] 13.1 and for the basic theory of single elliptic curves, we refer [25] IV.4.

# Chapter 3

## Singularities of curves

### 3.1 $\delta$ -invariant

The the more details, see St 0C3Q and St 0C3Z.

**Lemma 3.1.1** (St 0C3S). *Let  $(A, \mathfrak{m})$  be a reduced 1-dimensional local ring of finite type over a field  $k$ . Let  $A'$  be the integral closure of  $A$  in the total ring of fractions of  $A$ . Then  $A'$  is a normal with  $A \rightarrow A'$  is finite, and  $A'/A$  has finite length as an  $A$ -module.*

**Definition 3.1.2.** *Let  $A$  be a reduced 1-dimensional local ring of finite type over a field  $k$ . The  $\delta$ -invariant of  $A$  is  $\text{length}_A(A'/A)$  where  $A'$  is as in Lemma.*

*Let  $X$  be a scheme locally of finite type over  $k$ . Let  $x \in X$  such that  $\mathcal{O}_{X,x}$  is reduced with dimension 1. The  $\delta$ -invariant of  $X$  at  $x$  is the  $\delta$ -invariant of  $\mathcal{O}_{X,x}$ .*

**Proposition 3.1.3** (St 0C3V). *Let  $A$  be a reduced 1-dimensional local ring of finite type over a field  $k$ . Then  $\hat{A}$  has the same  $\delta$ -invariant as  $A$  and  $A' \otimes_A \hat{A}$  is the integral closure of  $\hat{A}$  in its total ring of fractions.*

**Proposition 3.1.4** (St 0C1R). *Let  $X$  be a reduced scheme locally finite type over a field of dimension 1 with normalization  $f : \tilde{X} \rightarrow X$ . Then  $\mathcal{O}_X \subset f_*\mathcal{O}_{\tilde{X}}$  and  $f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$  is a direct sum of skyscraper sheaves  $\mathcal{Q}_x$  in the singular points  $x$  and  $\mathcal{Q}_x = (f_*\mathcal{O}_{\tilde{X}})_x/\mathcal{O}_{X,x}$  has finite length equal to the  $\delta$ -invariant of  $X$  at  $x$ .*





# Chapter 4

## The varieties associated to curves

In most of cases of this chapter we focus on the proper reduced curves over an algebraically closed field  $k$  (may not irreducible and smooth). But when we let  $C$  smooth, we will automatically let  $C$  irreducible. When we consider  $\mathbb{C}$  we can use the language of Riemann surfaces via Serre's GAGA-principle.

### 4.1 Jacobian variety of curves

#### 4.1.1 Analytic approach

We let  $C$  a smooth projective curve of genus  $g$  over  $\mathbb{C}$ . We follows [3].

► **Approach 1.** If  $\omega, \omega'$  are holomorphic forms on  $C$ , then checking locally we get  $\omega \wedge \omega' = 0$  and  $\int_C \sqrt{-1} \omega \wedge \bar{\omega} > 0$  ( $\omega \neq 0$ ). Moreover we have  $d\omega = 0$ , hence  $[\omega] \in H_{DR}^1(C) \cong H^1(C, \mathbb{C})$ .

Choose a basis  $\omega_1, \dots, \omega_g \in H^0(C, K)$  and  $\gamma_1, \dots, \gamma_{2g} \in H_1(C, \mathbb{Z})$ . We can define the period matrix

$$\Omega = (\Omega_1, \dots, \Omega_{2g})_{g \times 2g}, \Omega_i = \begin{pmatrix} \int_{\gamma_i} \omega_1 \\ \vdots \\ \int_{\gamma_i} \omega_g \end{pmatrix}.$$

Hence by construction we can see that  $\Omega_1, \dots, \Omega_{2g}$  generates a lattice  $\Lambda$  in  $\mathbb{C}^g$ . Hence we define  $J(C) := \mathbb{C}^g / \Lambda$  as the Jacobian variety of  $C$ .

► **Approach 2.** Or equivalently, consider  $H_1(C, \mathbb{Z}) \hookrightarrow H^0(C, K)^\vee$  by  $\gamma \mapsto \int_\gamma$ , then  $J(C) = H^0(C, K)^\vee / H_1(C, \mathbb{Z})$ . Sometimes we call this Albanese torus when  $X$  be a compact Kähler manifold of higher dimension (See [26], for example).

► **Approach 3.** Let  $\text{Pic}^0(C)$  be the degree 0 line bundles (or the kernel of the first Chern class in higher dimension). We can consider the exponential sequence and since  $C$  is compact, then  $H^1(C, \mathbb{Z}) \rightarrow H^1(C, \mathcal{O}_C)$  is injective, hence  $\text{Pic}^0(C) \cong H^1(C, \mathcal{O}_C) / H^1(C, \mathbb{Z})$ . By some easy argument of Hodge theory (as [26] Corollary 3.3.6), it is a complex torus.

**Proposition 4.1.1.** *These three approaches defined the same variety associated to  $C$ .*

*Proof.* The first two approaches are the same trivially.

Now we consider the second and the third approaches. Actually this is by Serre duality  $H^0(C, K)^\vee \cong H^1(C, \mathcal{O}_C)$  and Poincaré duality  $H^1(C, \mathbb{Z}) \cong H_1(C, \mathbb{Z})$  which is compatible by trivial reasons.  $\square$

### 4.1.2 Algebraic approach

Here we follow [25] and let  $C$  a smooth projective curve over an algebraically closed field  $k$ . Actually this is just the special case of the Picard variety.

Let  $T$  be a scheme over  $k$  and we let  $\text{Pic}^0(C \times_k T)$  be the subgroup of  $\text{Pic}(C \times_k T)$  consisting of invertible sheaves whose restrict to any fibers  $C_t$  for  $t \in T$  has degree 0. Hence we can define  $\text{Pic}^0(C/T) := \text{Pic}^0(C \times_k T)/p^*\text{Pic}T$  where  $p : C \times_k T \rightarrow T$ .

We can show that (omitted here) the functor

$$J(C) : (\text{Sch}/\text{Spec}k)^{\text{opp}} \rightarrow (\text{Sets}), T \mapsto \text{Pic}^0(C/T)$$

is represented by a  $k$ -scheme, we also denoted  $J(C)$ . Here we find that for any points  $x \in C(k)$ , we find that  $x : \text{Spec}k \rightarrow J(C)$  correspond to an element of  $\text{Pic}^0(C)$ . Hence this notation make sense.

**Proposition 4.1.2.** *The Jacobian variety  $J(C)$  is a group variety over  $k$ .*

*Proof.* We define  $e : \text{Spec}k \rightarrow J(C)$  correspond to  $0 \in \text{Pic}^0(C/k)$  be the identity. Let  $i : J(C) \rightarrow J(C)$  correspond to  $\mathcal{L}_{\text{univ}}^{-1} \in \text{Pic}^0(C/J(C))$  be the inverse. Let  $\mu : J(C) \times_k J(C) \rightarrow J(C)$  correspond to  $p_1^*\mathcal{L}_{\text{univ}} \otimes p_2^*\mathcal{L}_{\text{univ}} \in \text{Pic}^0(C/J(C) \times_k J(C))$  be the multiple. Then the axiom of the group varieties is easy to check.  $\square$

**Proposition 4.1.3.** *The Zariski tangent space  $T_{J(C),0} \cong H^1(C, \mathcal{O}_C)$ .*

*Proof.* Consider the dual number  $T = k[\varepsilon]/(\varepsilon^2) \rightarrow J(C)$  by sending  $\text{Spec}k$  to 0. By [25] Ex.III.4.6, we get

$$0 \rightarrow H^1(C, \mathcal{O}_C) \rightarrow \text{Pic}C[\varepsilon] \rightarrow \text{Pic}C \rightarrow 0.$$

Hence we win.  $\square$

**Proposition 4.1.4.** *The Jacobian variety  $J(C)$  is proper and nonsingular over  $k$ .*

*Proof.* Using valuation criterion, we need to extend the line bundle at a codimension 2 point of  $C \times \text{Spec}R$ . This is trivial. It is nonsingular by [25] Remark IV.4.10.9.  $\square$

**Proposition 4.1.5.** *When  $C$  is a elliptic curve, then  $J(C) \cong C$ . In particular,  $C$  has a group structure.*

*Proof.* Omitted, see [25] Theorem IV.4.11.  $\square$

## 4.2 Picard varieties of curves

For more detail about the general Picard scheme, we refer [30]. Here we focus on the theory on curves over an algebraically closed field  $k$ . We follow [5] in St 0B92.

**Definition 4.2.1** (Picard functor). *Let  $f : X \rightarrow S$  be a morphism in the big fppf-site  $(Sch)_{fppf}$ . Consider the functor*

$$\mathrm{Pic}_{X/S}^{Psh} : (Sch/S)_{fppf} \rightarrow (Sets), T \mapsto \mathrm{Pic}(X_T).$$

*Let  $\mathrm{Pic}_{X/S} := (\mathrm{Pic}_{X/S}^{Psh})^{fppf}$  the sheafification in fppf-topology.*

**Proposition 4.2.2.** *(St 0B9N) Let  $f : X \rightarrow S$  as in the definition which admits a section  $\sigma$ . Assume that  $\mathcal{O}_T \cong f_{T,*}\mathcal{O}_{X_T}$  for all  $T \in \mathrm{Ob}((Sch/S)_{fppf})$ , then we have*

$$0 \longrightarrow \mathrm{Pic}T \longrightarrow \mathrm{Pic}X_T \xrightarrow{\sigma_T^*} \mathrm{Pic}_{X/S}(T) \longrightarrow 0$$

*is exact and split by  $\sigma_T^*$ .*

*sketch.* • The left-exactness don't need the  $\sigma$ : WLOG we let  $S = T$ . If  $f^*N \cong \mathcal{O}_X$ , then  $f_*f^*N \cong f_*\mathcal{O}_X \cong \mathcal{O}_S$  by assumption. Since  $N$  is locally trivial, we see that the canonical map  $N \rightarrow f_*f^*N$  is locally an isomorphism (because  $\mathcal{O}_S \rightarrow f_*f^*\mathcal{O}_S$  is an isomorphism by assumption). Hence we conclude that  $N \rightarrow f_*f^*N \rightarrow \mathcal{O}_S$  is an isomorphism and we see that  $N$  is trivial. This proves the first arrow is injective.

The exactness in the middle is easy by fppf-descent of quasi-coherent sheaves.

• The right-exactness need the  $\sigma$ : Let  $K(T) := \ker(\sigma_T^*)$ , hence  $\mathrm{Pic}(X_T) \cong \mathrm{Pic}T \oplus K(T)$  and  $K(T) \subset \mathrm{Pic}_{X/S}(T)$ . As  $\mathrm{Pic}_{X/S}$  is the sheafification of  $K$ , we just need to show that  $K$  is a fppf-sheaf. I omitted here.  $\square$

**Lemma 4.2.3.** *If  $C$  be a smooth projective curve over an algebraically closed field  $k$ , then the hypotheses of the previous Proposition are satisfied.*

*Proof.* We of course have a  $k$ -rational point (hence a section). Moreover, as  $H^0(C, \mathcal{O}_C) = k$ , by cohomology and base change we get  $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{C_T}$  is an isomorphism.  $\square$

If  $C$  be a smooth projective curve over an algebraically closed field  $k$  with a closed point  $\sigma$ . Consider the functor

$$\mathrm{Pic}_{C/k,\sigma} : (Sch/k)^{opp} \rightarrow (Sets), T \mapsto \ker(\sigma_T^* : \mathrm{Pic}(C_T) \rightarrow \mathrm{Pic}T),$$

which is isomorphic to  $\mathrm{Pic}_{C/k}$  before by the previous propositions. Hence we denote it by  $\mathrm{Pic}_{C/k}$ .

**Theorem 4.2.4.** *(St 0B9Z, St 0BA0) Let  $C$  be a smooth projective curve of genus  $g$  over an algebraically closed field  $k$ .*

- (i) *The functor  $\mathrm{Pic}_{C/k}$  is representable by a group scheme, denote it also by  $\mathrm{Pic}_{C/k}$ ;*
- (ii) *There is the disjoint decomposition of  $g$ -dimensional smooth proper varieties*

$$\mathrm{Pic}_{C/k} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}_{C/k}^d;$$

- (iii) *The closed points of  $\mathrm{Pic}_{C/k}^d$  correspond to invertible  $\mathcal{O}_C$ -modules of degree  $d$ ;*
- (iv)  *$\mathrm{Pic}_{C/k}^0$  is an open and closed subgroup scheme.*

*Sketch.* (iv) follows from the fact (ii). (iii) is trivial by definition. For the disjoint decomposition in (ii), by St 0B9T (locally constant of Euler characteristic) that for all  $d \in \mathbb{Z}$  there is an open subfunctor  $\mathrm{Pic}_{C/k}^d \subset \mathrm{Pic}_{C/k}$  whose value on a scheme  $T$  over  $k$  consists of those  $L \in \mathrm{Pic}_{C/k,\sigma}(T)$  such that  $\chi(C_t, L_t) = d + 1 - g$  and moreover we have  $\mathrm{Pic}_{C/k,\sigma} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}_{C/k,\sigma}^d$ . For (i) and the smoothness and properness of  $\mathrm{Pic}_{C/k}$  we omitted, we refer St 0BA0.  $\square$

### 4.3 Basic fact of the determinantal varieties

We refer [3] Chapter II.

Let  $M = M(m, n) := \mathbb{A}_{\mathbb{C}}^{mn}$  be the variety of  $m \times n$  matrix. Let  $M_k \subset M$  be a subvariety consist of matrixes at most rank  $k$ . This is called the generic determinantal variety.

We also can let  $\widetilde{M}_k = \{(A, W) \in M \times \text{Gr}(n - k, n) : AW = 0\}$  be a smooth connected subvariety of  $M \times \text{Gr}(n - k, n)$ . If  $\pi : \text{Gr}(n - k, n) \times M \rightarrow M$ , then  $\pi(\widetilde{M}_k) = M_k$ . Hence we can get  $M_k$  is irreducible of codimension  $(m - k)(n - k)$ .

**Proposition 4.3.1.** *We have  $\text{Sing}(M_k) = M_{k-1}$ .*

*Proof.* This need some calculation, I omit it here.  $\square$

**Theorem 4.3.2** (The Second Fundamental Theorem of Invariant Theory). *The ideal of  $M_k$  in  $M$  generated by all  $(k + 1) \times (k + 1)$  minors and is radical.*

*Proof.* The proof of this fact relies on a detailed analysis of the homogeneous coordinate ring of the Grassmannian  $\text{Gr}(k, N)$  under the Plücker embedding. See [3] Page 71-76.  $\square$

**Theorem 4.3.3** (The First Fundamental Theorem of Invariant Theory). *Let  $G = \text{GL}(n, \mathbb{C})$  and act on  $M(m, k) \times M(k, n)$  as  $g(A, B) = (Ag^{-1}, gB)$ . Let the multiplication  $\mu : M(m, k) \times M(k, n) \rightarrow M(m, n)$ , hence  $\text{Im} \mu = M_k$ . If we let  $M(m, k) \times M(k, n) = \text{Spec} S$ , then  $M_k = \text{Spec} S^G$ . Moreover,  $M_k$  is normal.*

*Proof.* Just some linear algebra, see [3] Page 77-79.  $\square$

**Theorem 4.3.4.**  *$M_k$  is Cohen-Macaulay.*

*Proof.* This proof is much complicated by showing the cone over a Schubert variety is Cohen-Macaulay. See [3] Page 80-82.  $\square$

Now we consider the general case of the determinantal variety. If  $X$  be a scheme over  $\mathbb{C}$  and  $\phi : E \rightarrow F$  be a morphism of vector bundles of rank  $n, m$ . Let open  $U$  be the trivialization, hence  $\phi|_U$  be a  $m \times n$  matrix. Hence this induce  $f : U \rightarrow M(m, n)$ . Let  $U_k = f^{-1}(M_k)$  and glue it together, we get  $X_k(\phi) \subset X$ . We call this the  $k$ -th determinantal variety. (Similarly we can get  $\widetilde{X}_k(\phi)$ . Also we have an intrinsical definition, see [3] Page 84.)

Hence  $\text{codim}_X X_k(\phi) \leq (m - k)(n - k)$ . Combining the previous local theorem and some algebra, we can get:

**Proposition 4.3.5.** *Let  $X$  be a smooth projective variety over  $\mathbb{C}$  with  $\phi : E \rightarrow F$  be a morphism of vector bundles of rank  $n, m$ . If  $\text{codim}_X X_k(\phi) = (m - k)(n - k)$ , then  $X_k(\phi)$  is Cohen-Macaulay.*

### 4.4 The varieties of special linear series on a curve

Let  $C$  be a smooth projective curve of genus  $g$  over  $\mathbb{C}$ . Let  $C_d := C^d / S_d$  and easy to see that  $C_d$  be the set of effective divisors of degree  $d$  on  $C$ .

Fixed  $p_0 \in C$  we first consider  $C \rightarrow J(C)$  as  $p \mapsto \int_{p_0}^p$ , then this can extend to  $\text{Div}(C) \rightarrow J(C)$  as

$$\sum_i p_i - \sum_j q_j \mapsto \sum_i \int_{p_0}^{p_i} - \sum_j \int_{p_0}^{q_j}.$$

Hence we have  $\nu : \text{Div}^d(C) \rightarrow J(C)$ . Hence we also can restrict to  $\mu : C_d \rightarrow J(C)$ .

Now using the Abel's theorem (see [3] Page 18), we have the following factorization

$$\begin{array}{ccccc} C_d & \hookrightarrow & \text{Div}^d(C) & \xrightarrow{-/\sim} & \text{Pic}^d(C) \\ & \searrow \mu & \downarrow \nu & \swarrow u & \\ & & J(C) & & \end{array}$$

► **The variety  $C_d^r$ :** Now let

$$C_d^r = \{D \in C_d : \dim |D| \geq r\}$$

with the variety-structure by using Brill-Noether matrix (see [3] IV.1, omitted here).

► **The variety  $W_d^r(C)$ :** Roughly speaking, we can let

$$W_d^r(C) = \{\text{parametrizing complete series } |D| : \deg D = d, h^0(C, \mathcal{O}_C(D)) \geq r+1\} \subset \text{Pic}_C^d.$$

Hence if we consider  $f : C_d \rightarrow \text{Pic}_C^d$ , then  $f(C_d^r) = W_d^r(C)$ .

The precise argument coming from the representability of the Picard variety  $\text{Pic}_C^d$  as follows. Let the degree  $d$  universal line bundle (or they called Poincaré line bundle)  $\mathcal{L} = \mathcal{L}_{\text{univ}}$  on  $C \times \text{Pic}_C^d$ . Let  $v : \text{Pic}_C^d \times C \rightarrow \text{Pic}_C^d$  be the projection.

Take  $E$  be an effective divisor on  $C$  such that  $m := \deg E \geq 2g - d - 1$ . Let  $\Gamma = E \times \text{Pic}_C^d$  be a divisor on  $C \times \text{Pic}_C^d$ , by some sheaf-theoric argument (kind of flat base-change, see [3] IV.2.6), we have  $R^1 v_* \mathcal{L}(\Gamma) = 0$  and  $v_* \mathcal{L}(\Gamma)$  is locally free of rank  $n = d + m - g + 1$ . Hence we have

$$0 \rightarrow v_* \mathcal{L} \rightarrow K^0 := v_* \mathcal{L}(\Gamma) \xrightarrow{\gamma} K^1 := v_*(\mathcal{L}(\Gamma)/\mathcal{L}) \rightarrow R^1 v_* \mathcal{L} \rightarrow 0.$$

Hence  $\ker \gamma = v_* \mathcal{L}$ ,  $\text{coker} \gamma = R^1 v_* \mathcal{L}$  and  $\text{rank} K^0 = n$ ,  $\text{rank} K^1 = m$ . Now we let

$$W_d^r(C) := X_{m+d-g-r}(\gamma) \text{ where } X = \text{Pic}_C^d.$$

Note that  $W_d^r(C)$  is independent of the choice of  $E$ , see [3] Page 179.

By [3] Lemma IV.3.1, we get

**Proposition 4.4.1.** *The variety  $W_d^r(C)$  represented the functor*

$$S \mapsto \left\{ L \in \text{Pic}^d(C \times S) \text{ such that the fitting rank of } R^1 \phi_* L \text{ is at least } g - d + r \right\}.$$

(For  $\mathcal{F} \in \text{Coh}(X)$  with presentation  $\mathcal{O}_X^n \xrightarrow{\gamma} \mathcal{O}_X^m \rightarrow \mathcal{F}$  the fitting rank of  $\mathcal{F}$  is the largest integer  $h$  such that the ideal in  $\mathcal{O}_X$  generated by the  $(m - h + 1) \times (m - h + 1)$  minors of  $A$  vanishes.)

Also by [3] Lemma IV.3.1, we get the set of all  $\mathbb{C}$ -valued points of  $W_d^r(C)$  is just

$$\{L \in \text{Pic}^d(C) : h^0(C, L) \geq r+1\}.$$

**Proposition 4.4.2.** *For  $r \geq d - g$ , the each component of  $W_d^r(C)$  has dimension greater or equal to the Brill-Noether number*

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

*Proof.* Trivial by the local analysis above. □

**Proposition 4.4.3.** For  $f : C_d \rightarrow \text{Pic}_C^d$ , we get  $f^{-1}(W_d^r(C)) = C_d^r$ .

*Proof.* See [3] Proposition IV.3.4. □

► **The variety  $G_d^r(C)$ :** Roughly speaking, we can let

$$G_d^r(C) = \{\text{parametrizing } \mathfrak{g}_d^r \text{ of degree } d \text{ and dimension } r\}.$$

For precise definition, we let

$$G_d^r(C) := \tilde{X}_{m+d-g-r}(\gamma)$$

as previous construction. For any  $\mathbb{C}$ -valued point  $(L, V)$  where  $L \in \text{Pic}^d(C)$  and  $V$  be a  $(r+1)$ -dimensional subspace of  $\ker \gamma_L$ . By [3] Lemma IV.3.1 we get  $\ker \gamma_L = H^0(C, L)$ , we get

$$G_d^r(C)(\mathbb{C}) = \{(L, V) : L \in \text{Pic}^d(C), V \in \text{Gr}(r+1, H^0(C, L))\}.$$

Hence parametrizing all  $\mathfrak{g}_d^r$ . Actually  $G_d^r(C)$  can also defined as a representable functor, we refer [3] page 182-183 and omit it here.

Now we collect some conclusions of these three varieties in [3] section IV.4.

**Proposition 4.4.4.** (i) Every component of  $G_d^r(C)$  has dimension at least equal to the Brill-Noether number  $\rho = g - (r+1)(g-d+r)$ ;

(ii) Let  $w = (L, W \subset H^0(C, L))$  be a point in  $G_d^r(C)$  and consider the cup product

$$\mu_{0,W} : W \otimes H^0(C, K \otimes L^{-1}) \rightarrow H^0(C, K) = H^1(C, \mathcal{O}_C)^\vee,$$

then  $\dim T_w G_d^r(C) = \rho + \dim \ker \mu_{0,W}$ . In particular,  $G_d^r(C)$  is smooth at  $w$  of dimension  $\rho$  if and only if  $\ker \mu_{0,W} = 0$ ;

(iii) Let  $L \in W_d^r(C) \setminus W_d^{r+1}(C)$  (hence  $r \geq d-g$ ), then  $T_L W_d^r(C) \cong (\text{Im } \mu_0)^\perp$  where  $\mu_0 : H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \rightarrow H^0(C, K)$  be the cup product;

(iv) Let  $L \in W_d^{r+1}(C)$ , then  $T_L W_d^r(C) = T_L \text{Pic}_C^d$ ;

(v) If  $G_d^r(C)$  is smooth of dimension  $\rho$ , then  $W_d^r(C)$  is Cohn-Macaulay, reduced and normal. If  $d < g+r$  then  $\text{Sing}(W_d^r(C)) = W_d^{r+1}(C)$ .

*Proof.* For (i),(ii), we refer [3] IV.4.1; for (iii),(iv), we refer [3] IV.4.2; for (v) we refer [3] IV.4.4. □

## 4.5 The basic results of the Brill-Noether theory

## **Part II**

# **The basic theory of moduli space of curves**





## Chapter 5

# $\mathcal{M}_g$ be a Deligne-Mumford Stack for $g \neq 1$

Here we mainly consider the  $g \neq 1$  curves.

**Definition 5.0.1.** Let  $\mathcal{M}_g$  be the fibered category over schemes with objects of form  $(S, f : C \rightarrow S)$  where  $S$  be a scheme and  $f$  be a proper smooth morphism such that every geometric fiber of  $S$  is a connected genus  $g$  curve. The morphisms are base-change.

Our main result of this section is to prove that  $\mathcal{M}_g$  is a Deligne-Mumford stack for  $g \geq 2$ . For  $g = 0$  we can run the same argument and when we just consider the stack over  $Sch/k$  where  $k$  be algebraically closed, we can get  $\mathcal{M}_0 \cong BPG L_2$ . Here we follows [1].

### 5.1 $\mathcal{M}_g$ be a stack for $g \neq 1$

**Lemma 5.1.1** (Descent for polarized schemes). *Let  $\mathcal{P}ol$  be the category whose objects are pairs  $(f : X \rightarrow Y, L)$  where  $f$  is a proper flat morphism and  $L$  is a relatively ample invertible sheaf. The morphism are diagrams of cartesian with isomorphic pullback of line bundles. Consider the fibered category  $\mathcal{P}ol \rightarrow (Sch)$ , then it has effective fppf-descent.*

*Proof.* See [34] 4.4.10. □

**Theorem 5.1.2.** *For  $g \neq 1$ , the fibered category  $\mathcal{M}_g$  is a stack.*

*Proof.* Consider  $\mathcal{M}_g \rightarrow \mathcal{P}ol$  sends  $C \rightarrow S$  to  $(C \rightarrow S, \Omega_{C/S}^1)$  when  $g \geq 2$  and  $(C \rightarrow S, \Omega_{C/S}^{1, \otimes -1})$  when  $g = 0$ .

For a fppf covering  $S' \rightarrow S$ , then we get

$$\begin{array}{ccc} \mathcal{M}_g(S) & \longrightarrow & \mathcal{P}ol(S) \\ \downarrow & & \downarrow \cong \\ \mathcal{M}_g(S' \rightarrow S) & \longrightarrow & \mathcal{P}ol(S' \rightarrow S) \end{array}$$

Hence every object of  $\mathcal{M}_g(S' \rightarrow S)$  is in the essential image of  $\mathcal{M}_g(S)$ . By the descent of sheaves used in  $h_- \rightarrow h_+$  making it fully faithful. □

## 5.2 For $g \geq 2$ , $\mathcal{M}_g$ be a Deligne-Mumford stack

Now let  $L_{C/S} = (\Omega_{C/S}^1)^{\otimes 3}$ . By Lemma 2.1.1 (iv), for any family of smooth curves  $p : D \rightarrow S$  we get a closed immersion  $D \hookrightarrow \mathbb{P}(p_* L_{D/S})$  where  $p_* L_{D/S}$  is locally free of rank  $5g - 5$ . Let  $H = \text{Hilb}_{\mathbb{P}^{5g-6}}^P$  where  $P(t) = \deg(L_{C/S}^{\otimes t}) + 1 - g = (6g - 6)t + 1 - g$  be the Hilbert polynomial of  $D_s \hookrightarrow \mathbb{P}_{\kappa(s)}^{5g-6}$ . Let the universal closed subscheme:

$$\begin{array}{ccc} C & \hookrightarrow & H \times \mathbb{P}^{5g-6} \\ & \searrow \pi & \downarrow \\ & & H \end{array}$$

► **Claim 1.** There is a unique subscheme  $H' \subset H$  consist of  $h \in H$  such that

- (a)  $C_h \rightarrow \text{Spec}(\kappa(h))$  is smooth and geometrically connected;
- (b)  $C_h \hookrightarrow \mathbb{P}_{\kappa(h)}^{5g-6}$  is embedded by complete linear system  $|L_{C_h/\kappa(h)}|$ ;
- (c) the line bundles  $L_{C_{H'}/H'}$  and  $\mathcal{O}_{C_{H'}}(1)$  differ by a pullback of a line bundle from  $H'$  (that is, there exists a line bundle  $N$  over  $H'$  such that  $L_{C_{H'}/H'} \otimes p^* N = \mathcal{O}_{C_{H'}}(1)$ ).

Moreover, if  $T \rightarrow H$  be a morphism such that (a)-(c) hold for the family  $C_T \rightarrow T$ , then  $T \rightarrow H$  factors through  $H'$ .

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target, the condition on  $H$  that  $C_h$  is smooth is open. Consider the Stein factorization (St 03H0)  $C \rightarrow \tilde{H} := \text{Spec}_H \pi_* \mathcal{O}_C \rightarrow H$  where  $C \rightarrow \tilde{H}$  is proper with geometrically connected fibres and  $\tilde{H} \rightarrow H$  is finite. As  $\mathcal{O}_H \rightarrow \pi_* \mathcal{O}_C$  is a morphism between coherent sheaves, then the kernel and cokernel of it have closed supports. Hence  $\tilde{H} \rightarrow H$  is an isomorphism over an open subscheme of  $H$ , which is precisely where the fibers of  $C \rightarrow H$  are geometrically connected. Hence the points satiefies (a) be a open subscheme of  $H$ , denoted by  $H_1 \subset H$ .

By Review A.1.2, there exists a locally closed subscheme  $H_2 \subset H_1$  such that a morphism  $T \rightarrow H_1$  factor through  $H_2$  if and only if  $L_{C_T/T}$  and  $\mathcal{O}_{C_T}(1)$  differ by a pullback of a line bundle from  $T$ . In particular, (c) holds and for all  $h \in H_2$ ,  $L_{C_h/\kappa(h)} \cong \mathcal{O}_{C_h}(1)$ .

For (b), let  $\pi_2 : C_2 := C_{H_2} \rightarrow H_2$ . Consider  $\alpha : H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \rightarrow \pi_{2,*} \mathcal{O}_{C_2}(1)$  of vector bundles of rank  $5g - 5$  on  $H_2$  with fiber  $\alpha_h : H^0(\mathbb{P}_{\kappa(h)}^{5g-6}, \mathcal{O}(1)) \rightarrow H^0(C_h, \mathcal{O}_{C_h}(1)) \cong H^0(C_h, L_{C_h/\kappa(h)})$ . As they have the same rank,  $\alpha_h$  is an isomorphism if and only if  $h$  is not in  $\text{supp}(\text{coker}(\alpha))$ . Let  $H' = H_2 \setminus (\text{supp}(\text{coker}(\alpha)))$  and it satisfies (a)-(c) with that universal property.

► **Claim 2.** The group scheme  $\text{PGL}_{5g-5} = \underline{\text{Aut}}(\mathbb{P}_{\mathbb{Z}}^{5g-6})$  act on  $H$  as: for  $g \in \text{Aut}(\mathbb{P}_S^{5g-6})$  and  $[D \subset \mathbb{P}_S^{5g-6}] \in H(S)$ , we let  $g \cdot [D \subset \mathbb{P}_S^{5g-6}] = [g(D) \subset \mathbb{P}_S^{5g-6}]$ . As  $H'$  is  $\text{PGL}_{5g-5}$ -invariant, we claim that  $\mathcal{M}_g \cong [H'/\text{PGL}_{5g-5}]$  be an algebraic stack. (See St 044O, St 04UV for quot stacks)

Consider  $H' \rightarrow \mathcal{M}_g$  as  $[D \subset \mathbb{P}_S^{5g-6}] \mapsto (D \rightarrow \mathbb{P}_S^{5g-6} \rightarrow S)$  is well defined by **Claim 1**. This morphism is  $\text{PGL}_{5g-5}$ -invariant, hence descends to  $[H'/\text{PGL}_{5g-5}]^{pre} \rightarrow \mathcal{M}_g$  (**Why?**). We claim that this map is fully faithful. Indeed, for a family  $p : D \rightarrow S$  in  $H'$  given by  $D \subset \mathbb{P}_S^{5g-6}$ , we get  $\mathcal{O}_D(1) \cong L_{D/S} \otimes p^* M$  for some line bundle  $M$  on  $S$ . Use (b) we get

$$H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_S \rightarrow p_* \mathcal{O}_D(1) \cong p_*(L_{D/S} \otimes p^* M) \cong p_* L_{D/S} \otimes M$$

be an isomorphism. Then any automorphism of  $D \rightarrow S$  induces an automorphism of  $L_{D/S}$  and thus an automorphism of  $p_* L_{D/S} \otimes M$ , which induce an automorphism of  $\mathbb{P}_S^{5g-6}$  preserving  $D$ . By Theorem 5.1.2,  $\mathcal{M}_g$  be a stack, hence induce  $[H'/\text{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$  which is fully faithful since stackification is fully faithful. Finally we check that  $[H'/\text{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$  is essentially

surjective. As these are all stacks, then they satisfied effective descent of étale covering. Hence we just need to show that for any  $p : D \rightarrow S$ , there exists an étale covering  $\{S_i \rightarrow S\}$  such that each  $D_{S_i}$  is in the image of  $H'(S_i) \rightarrow \mathcal{M}_g(S_i)$ . Actually since  $L_{D/S}$  defined  $D \hookrightarrow \mathbb{P}(p_*L_{D/S})$  and  $p_*L_{D/S}$  is locally free of rank  $5g - 5$ , we let  $\{S_i\}$  be open (zariski, hence étale) covering of  $S$  such that  $(p_*L_{D/S})|_{S_i}$  are all free. Well done.

► **Claim 3.** The algebraic stack  $\mathcal{M}_g$  is a Deligne-Mumford stack.

By Theorem C.1.1, we just need to show for any smooth connected proper curve  $C$  over a algebraic closed field  $k$ , the group scheme  $G := \underline{\text{Aut}}_k(C) = \text{Aut}(C)$  is finite and reduced. We find that  $T_{G,e}$  can be identified with the automorphism group of the trivial first order deformation of  $C$ . Hence by Proposition 8.1.6, we get  $T_{G,e} = H^0(C, T_C) = 0$ , well done.

### 5.3 First properties of $\mathcal{M}_g$ for $g \geq 2$

**Proposition 5.3.1.** As  $\mathcal{M}_g \cong [H'/\text{PGL}_{5g-5}]$  and  $H'$  is locally of finite type, then  $\mathcal{M}_g$  is locally of finite type over  $\mathbb{Z}$ . As  $H'$  is noetherian, so is  $\mathcal{M}_g$ . So it is finite type over  $\mathbb{Z}$ .

**Proposition 5.3.2.**  $\mathcal{M}_g$  have affine diagonal. Indeed, since we have  $\mathcal{M}_g \cong [H'/\text{PGL}_{5g-5}]$  which is an algebraic stack, then we have cartesian square

$$\begin{array}{ccc} H' \times \text{PGL}_{5g-5} & \longrightarrow & H' \times H' \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \longrightarrow & \mathcal{M}_g \times \mathcal{M}_g \end{array}$$

As  $\text{PGL}_{5g-5}$  affine, then  $\text{PGL}_{5g-5} \times H' \rightarrow H' \times H' \rightarrow H'$  affine, so is  $\text{PGL}_{5g-5} \times H' \rightarrow H' \times H'$ .

### 5.4 Smoothness and dimension of $\mathcal{M}_g$ for $g \geq 2$

**Proposition 5.4.1.** If  $C$  is a smooth connected projective curve of genus  $g \geq 2$  over  $k$ , then  $\dim T_{\mathcal{M}_g, [C]} = 3g - 3$ .

*Proof.* By Proposition 8.1.2, we get  $T_{\mathcal{M}_g, [C]} = H^1(C, T_C)$ . As  $\deg T_C < 0$ , we get  $H^0(C, T_C) = 0$ . So by Riemann-Roch we get

$$\dim T_{\mathcal{M}_g, [C]} = \dim H^1(C, T_C) = -\chi(T_C) = -\deg T_C + g - 1 = 3g - 3,$$

well done. □

**Theorem 5.4.2.** For  $g \geq 2$ , the Deligne-Mumford stack  $\mathcal{M}_g$  is smooth over  $\mathbb{Z}$  of relative dimension  $3g - 3$ .

*Proof.* Let a field  $k$  and a smooth projective connected curve  $C \rightarrow \text{Spec}(k)$ . Consider the following 2-diagram:

$$\begin{array}{ccccc} & & [C] & & \\ & \curvearrowright & & \searrow & \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A_0) & \longrightarrow & \mathcal{M}_g \\ & & \downarrow & & \downarrow f \\ & & \text{Spec}(A) & \longrightarrow & \text{Spec}(\mathbb{Z}) \end{array}$$

where  $A \rightarrow A_0$  be a surjective maps of artinian local rings with residue field  $k$  with  $k = \ker(A \rightarrow A_0)$ . The map  $\mathrm{Spec}(A_0) \rightarrow \mathcal{M}_g$  corresponds to a family of curves  $C_0 \rightarrow \mathrm{Spec}(A_0)$  and a cartesian:

$$\begin{array}{ccccc} C & \longrightarrow & C_0 & \dashrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_0) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

of solid arrows. So to find the lifting, we just need to find the dashed arrows, that is, deformation of  $C$  along  $A$ . By Proposition 8.1.6(iii), there exists a cohomology class  $ob_C \in H^2(C, T_C)$  such that this happens if and only if  $ob_C = 0$ . Hence this is right as  $C$  be a curve. Hence  $\mathcal{M}_g$  is smooth. By Theorem C.1.2, we get

$$\dim_{[C]} \mathcal{M}_g = \dim T_{\mathcal{M}_g, [C]} - \dim \mathrm{Aut}(C).$$

By the final step of the proof of the DM-ness of  $\mathcal{M}_g$ , we get  $\dim \mathrm{Aut}(C) = 0$ . Hence  $\dim_{[C]} \mathcal{M}_g = \dim T_{\mathcal{M}_g, [C]} = 3g - 3$ , well done.  $\square$

## 5.5 For $g = 0$

By the same argument we can get  $\mathcal{M}_0$  be a Deligne-Mumford stack.

**Proposition 5.5.1.**  $\mathcal{M}_0$  may (not) be a stack over  $\mathbb{Z}$  isomorphic to  $BPGL_2$ .

*Analysis.* We should repeat the proof of the case  $g \geq 2$ . But when we consider  $L_{C/S} = (\Omega_{C/S}^1)^{\otimes(-1)}$  for  $f : C \rightarrow S$ , we get  $\deg(L_{C_s/\kappa(s)}) = 2$  and  $f_* L_{C/S}$  is locally free of rank 3, which induce  $C \hookrightarrow \mathbb{P}_S^2$ . And with the Hilbert polynomial  $p(t) = 2t + 1$ . So we will use  $PGL_3$  and get  $\mathcal{M}_0 \cong [H'/PGL_3]$  for some subscheme  $H' \subset \underline{\mathrm{Hilb}}_{\mathbb{P}^2}^{p(t)}$ !

Actually by St 0C6U, we can not identify  $C_s$  over some  $\kappa(s)$  with  $\mathbb{P}_{\kappa(s)}^1$  since we may have **no invertible sheaves of odd degree!** If we can assume this, we have  $M$  of degree 1. The Hilbert polynomial  $p(t) = t + 1$  in  $\mathbb{P}^1$  and can show that it is  $\mathbb{P}^1$  by some tricks (actually this is right for all linear subspaces). Hence

$$\mathcal{M}_0 \cong [\underline{\mathrm{Hilb}}_{\mathbb{P}^1}^{p(t)}/PGL_2] = [\mathrm{Grass}(2, 2)/PGL_2] = BPGL_2,$$

well done.  $\square$

**Corollary 5.5.2.** When we consider the stack over  $\mathrm{Sch}/k$  for any algebraically closed field  $k$ , we have  $\mathcal{M}_0 \cong BPGL_2$ .

*Proof.* Actually here we have invertible sheaves of degree 1. Hence any proper smooth curve of genus 0 be  $\mathbb{P}_k^1$ . The Hilbert polynomial  $p(t) = t + 1$  in  $\mathbb{P}_k^1$  and can show that it is  $\mathbb{P}_k^1$  by some tricks. Hence

$$\mathcal{M}_0 \cong [\underline{\mathrm{Hilb}}_{\mathbb{P}_k^1}^{p(t)}/PGL_2] = [\mathrm{Grass}_k(2, 2)/PGL_2] = BPGL_2,$$

well done.  $\square$

# Chapter 6

## Nodal curves

### 6.1 Basic facts of nodal curves

The the more details, see St 0C46.

**Definition 6.1.1** (Nodes). *Let  $C$  be a curve over  $k$ .*

- (a) *If  $k$  algebraically closed, we say that  $p \in C(k)$  is a node is we have  $\widehat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(xy)$ ;*
- (b) *If  $k$  need not be algebraically closed, we say a closed point  $p \in C$  is a node if there exists a node  $p' \in C_{\bar{k}}$  over  $p$ .*

*We say  $C$  be a nodal curve if every closed point is either smooth or nodal.*

**Proposition 6.1.2.** *Let  $C$  be a curve over  $k$ . Consider the following statments.*

- (a)  *$p \in C$  is a node;*
- (b)  *$\kappa(p)/k$  is separable,  $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$  as a  $k$ -algebra where  $ax^2 + bxy + cy^2$  is a nondegenerate quadratic form over  $\kappa(p)$ ;*
- (c)  *$\kappa(p)/k$  is separable and  $\mathcal{O}_{C,p}$  is reduced, has  $\delta$ -invariant 1.*

*Then we have (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c).*

*Proof.* See St 0C49 and St 0C4D.

We assume (a) $\Leftrightarrow$ (b). Here by Lemma 3.1.3, we just need to consider the case  $\mathcal{O}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$  where  $Q = ax^2 + bxy + cy^2$  is a nondegenerate quadratic form.

Case (I): If  $Q$  is split, we may let  $\mathcal{O}_{C,p} \cong \kappa(p)[[x, y]]/(xy)$  after some coordinate transformation. Then we get

$$\widetilde{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(x) \times \kappa(p)[[x, y]]/(y);$$

Case (II): If not, in this case  $c \neq 0$  and nondegenerate means  $b^2 - 4ac \neq 0$ . Hence  $\kappa' = \kappa(p)[t]/(a + bt + ct^2)$  is a degree 2 separable extension of  $\kappa(p)$ . Then  $t = y/x$  is integral over  $\mathcal{O}_{C,p}$ . and we conclude that

$$\widetilde{\mathcal{O}}_{C,p} = \kappa'[[x]]$$

with  $y$  mapping to  $tx$  on the right hand side.

In both cases one verifies by hand that the  $\delta$ -invariant is 1, well done. □

**Remark 6.1.3.** (i) *As for a node  $p \in C$  in a nodal curve  $C$ , we have  $\kappa(p)/k$  is separable. As the two cases above, if  $p$  is of case (I), then  $f^{-1}(p)$  has two points with residue fields  $\kappa(p)$ . If  $p$  is*

of case (II), then  $f^{-1}(p)$  has only one point with residue field  $\kappa'$ , a degree 2 separable extension of  $\kappa(p)$ ;

(ii) As in (i), all closed points of  $\tilde{C}$  is regular with separable residue fields over  $k$ . Hence  $\tilde{C}$  is smooth over  $k$  by St 00TV.

**Proposition 6.1.4.** *If  $C$  is a curve over  $k$  and  $p \in C$  be a node. Then exists a finite separable field extension  $K/k$ , a point  $P \in C_K$  over  $p$  and  $\hat{\mathcal{O}}_{C_K, P} \cong K[[x, y]]/(xy)$ .*

*Proof.* By Proposition 6.1.2(b), we get  $\kappa(p)/k$  is separable,  $\hat{\mathcal{O}}_{C, p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$  as a  $k$ -algebra where  $Q = ax^2 + bxy + cy^2$  is a nondegenerate quadratic form over  $\kappa(p)$ . If  $Q$  is split, well done. If not, let  $K = k[t]/(at^2 + bt + c)$  be a separable extension over  $k$  with  $Q$  split, well done.  $\square$

## 6.2 Genus fomula

Let  $k$  be algebraically closed field now. Let  $C$  be a connected nodal projective curve over  $k$ . Let  $z_1, \dots, z_s$  be its nodes and  $C_1, \dots, C_t$  be its irreducible components.

By Proposition A.3.3(1) and (4), we get  $\tilde{C} = \coprod_{i=1}^t \tilde{C}_i$  where  $\tilde{C}, \tilde{C}_i$  are normalizations. Let  $f : \tilde{C} \rightarrow C$ . By Proposition 3.1.4, we get a exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow f_* \mathcal{O}_{\tilde{C}} \rightarrow \bigoplus_{i=1}^s \mathcal{Q}_i \rightarrow 0$$

where  $\mathcal{Q}_i$  supported over  $z_i$ . Since by Proposition 6.1.2(c), we get  $\mathcal{Q}_i = \kappa(z_i)$  as the  $\delta$ -invariant are all 1, hence we get

$$0 \rightarrow \mathcal{O}_C \rightarrow f_* \mathcal{O}_{\tilde{C}} \rightarrow \bigoplus_{i=1}^s \kappa(z_i) \rightarrow 0.$$

Hence we get long exact sequence

$$0 \rightarrow \underbrace{H^0(C, \mathcal{O}_C)}_1 \rightarrow \underbrace{H^0(\tilde{C}, \mathcal{O}_{\tilde{C}})}_t \rightarrow \underbrace{\bigoplus_{i=1}^s \kappa(z_i)}_s \rightarrow \underbrace{H^1(C, \mathcal{O}_C)}_{g(C)} \rightarrow \underbrace{H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})}_{\sum_{i=1}^t g(\tilde{C}_i)} \rightarrow 0$$

with the labels underneath indicating the dimensions.

**Theorem 6.2.1** (Genus fomula). *With the situation as above, we get*

$$g(C) = \sum_{i=1}^t g(\tilde{C}_i) + s - t + 1.$$

*Proof.* Trivial by the argument above.  $\square$

## 6.3 The dualizing sheaf

We have three way to see this. Consider  $C$  be a fixed nodal curve over  $k$ .

### 6.3.1 The first way

We find that  $C$  is locally complete intersection as we can check locally. As for a node  $p \in C$ , we have  $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$  for some nondegenerate quadratic form. By [33] Theorem 21.2(iii), we get  $\mathcal{O}_{C,p}$  is a complete intersection over  $k$ . Hence by [25] Theorem III.7.11 (adjunction formula for l.c.i), if we embed it into  $\mathbb{P}^N$ , then we have  $\omega_C \cong \omega_{\mathbb{P}^N} \otimes \bigwedge^{N-1}(\mathcal{I}/\mathcal{I}^2)$  where  $\mathcal{I}$  be the ideal sheaf. As this is locally complete intersection, this is a line bundle.

### 6.3.2 The second way

This is an abstract way of duality theory, see St 0E31 for more details. As  $C$  is locally complete intersection, then by St 0BVA we get  $C$  is Gorenstein. By St 0BS2,  $C$  must have a dualizing complex  $\omega_C^*$ . By 0BFQ, as  $C$  is Gorenstein,  $\omega_C^*$  is invertible. By  $C$  Cohen-Macaulay,  $\omega_C^* = \omega_C[0]$ . Hence we win.

### 6.3.3 The third way

We can explicit  $\omega_C$  precisely. Let  $\Sigma$  be the set of nodes of  $C$  and let  $U = C \setminus \Sigma$ . Let the normalization  $f : \widetilde{C} \rightarrow C$  and  $\widetilde{\Sigma} := f^{-1}(\Sigma)$ ,  $\widetilde{U} := f^{-1}(U)$ . Now  $\widetilde{C}$  is smooth, then we have the dualizing sheaf (line bundle)  $\Omega_{\widetilde{C}}$ . We get

$$0 \rightarrow \Omega_{\widetilde{C}} \rightarrow \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \rightarrow \mathcal{O}_{\widetilde{\Sigma}} \rightarrow 0.$$

Actually the sections of  $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$  is the rational sections of  $\Omega_{\widetilde{C}}$  with at worst simple poles in  $\widetilde{\Sigma}$ . Hence for any open  $V \subset \widetilde{C}$  and  $y \in V \cap \widetilde{\Sigma}$  we have the residue  $\text{res}_y : \Gamma(V, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \rightarrow \kappa(y)$ .

**Definition 6.3.1.** We define the subsheaf  $\omega_C \subset f_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$  as for any open  $V \subset C$  we have

$$\Gamma(V, \omega_C) = \left\{ s \in \Gamma(f^{-1}(V), \Omega_{\widetilde{C}}(\widetilde{\Sigma})) : \begin{array}{l} \text{for any } z_i \in V \cap \Sigma \\ \text{and } f^{-1}(z_i) = \{p_i, q_i\} \text{ with } \text{res}_{p_i}(s) + \text{res}_{q_i}(s) = 0 \end{array} \right\}.$$

Hence we get two exact sequences

$$\begin{aligned} 0 \rightarrow \omega_C \rightarrow f_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) &\longrightarrow \bigoplus_{z_i \in \Sigma} \kappa(z_i) \longrightarrow 0 \\ s &\longmapsto (\text{res}_{p_i}(s) - \text{res}_{q_i}(s)) \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow f_*\Omega_{\widetilde{C}} \rightarrow \omega_C &\rightarrow \bigoplus_{z_i \in \Sigma} \kappa(z_i) \rightarrow 0 \\ s &\longmapsto (\text{res}_{p_i}(s)) \end{aligned}$$

**Proposition 6.3.2.** Let  $C$  be a nodal curve  $C$  over  $k$ .

- (a) If  $g : C' \rightarrow C$  be an étale morphism, then  $g^*\omega_C \cong \omega_{C'}$ ;
- (b) Conclude that  $\omega_C$  be a line bundle.

*Proof.* (a) As the normalization commutes with étale base change (see St 03GE), we have the cartesian with normalizations

$$\begin{array}{ccc} \widetilde{C}' & \xrightarrow{g'} & \widetilde{C} \\ \downarrow f' & & \downarrow f \\ C' & \xrightarrow{g} & C \end{array}$$

By flat base change, we have the process  $g^*\omega_C \subset g^*f_*\Omega_{\tilde{C}}(\tilde{\Sigma}) \cong f'_*(g')^*\Omega_{\tilde{C}}(\tilde{\Sigma}) = f'_*\Omega_{\tilde{C}'}(\tilde{\Sigma}')$ . By definition and this process, we get  $g^*\omega_C \cong \omega_{C'}$ .

(b) Use Corollary A.2.3 and Proposition 6.1.4, there exists a separable extension  $K/k$  such that we get the common étale neighborhood as

$$\begin{array}{ccc} & (U, u) & \\ F \swarrow & & \searrow G \\ (C, p) & & (\text{Spec}K[x, y]/(xy), 0) \end{array}$$

Let  $D = \text{Spec}K[x, y]/(xy)$  and normalization  $\tilde{D} \cong \mathbb{A}_K^1 \sqcup \mathbb{A}_K^1$ . Then  $\Gamma(\tilde{D}, \Omega_{\tilde{D}}) = \Gamma(\mathbb{A}_K^1, \omega_{\mathbb{A}_K^1}) \times \Gamma(\mathbb{A}_K^1, \omega_{\mathbb{A}_K^1})$  and  $(\frac{dx}{x}, -\frac{dy}{y})$  be a section of  $\omega_D$ . As any section is of form  $(f(x)\frac{dx}{x}, -g(y)\frac{dy}{y})$  where  $f(0) = g(0)$ , which is precisely the condition for  $(f, g) \in \Gamma(\tilde{D}, \mathcal{O}_{\tilde{D}})$  to descend to a global function on  $D$ . In other words,  $\omega_D \cong \mathcal{O}_D$  with generator  $(\frac{dx}{x}, -\frac{dy}{y})$ . By (a), we get  $\omega_U = G^*\omega_D$ , hence  $\omega_U$  is a line bundle. As  $F^*\omega_C = \omega_U$  be a line bundle, we use the descent theory and we win.  $\square$

**Proposition 6.3.3.** *Let  $C$  be a proper nodal curve  $C$  over  $k$ , then  $\omega_C$  be the dualizing line bundle of  $C$ .*

*Proof.* (See [2]) We may assume that  $k$  is algebraic closed. Choose a divisor  $D = r_1 + \dots + r_h$  consisting of distinct smooth points of  $C$ , with the property that any component of  $C$  contains at least one of the  $r_i$ 's. We first claim that  $H^1(\omega_C(D)) = 0$ . Indeed, we get an exact sequence

$$0 \rightarrow (f_*\omega_{\tilde{C}}) \otimes \mathcal{O}_C(D) = f_*(\omega_{\tilde{C}}(D)) \rightarrow \omega_C(D) \rightarrow \bigoplus_{\text{nodes}} k \rightarrow 0.$$

Hence deduce a surjection

$$H^1(C, f_*(\omega_{\tilde{C}}(D))) = H^1(\tilde{C}, \omega_{\tilde{C}}(D)) \rightarrow H^1(C, \omega_C(D)).$$

As  $\tilde{C}$  is smooth, we get for any irreducible components and Serre duality in smooth case, we get  $H^1(\tilde{C}, \omega_{\tilde{C}}(D)) = 0$  as  $D$  meets every irreducible components. Hence  $H^1(\omega_C(D)) = 0$ .

Next we deduce an exact sequence by using the claim as

$$H^0(C, \omega_C(D)) \rightarrow H^0(C, \omega_C(D)/\omega_C) \rightarrow H^1(C, \omega_C) \rightarrow H^1(C, \omega_C(D)) = 0.$$

For any  $\phi \in H^1(C, \omega_C)$  we have some lifts  $\phi' \in H^0(C, \omega_C(D)/\omega_C)$ . We define the trace map as

$$\text{tr}_C : H^1(C, \omega_C(D)) \rightarrow k, \phi \mapsto 2\pi\sqrt{-1} \sum_{i=1}^l \text{res}_{r_i} \phi'$$

and this is well defined by using residue theorem (of definition). Perfect pairing is omitted.  $\square$

**Proposition 6.3.4.** *Let  $C$  be a nodal curve  $C$  over  $k$  and  $T \subset C$  be an irreducible component and  $D_T$  be the union of the intersections of  $T$  and another irreducible components, then  $\omega_C|_T = \omega_T(D_T)$ .*

*Proof.* Trivial by definition of the dualizing sheaves.  $\square$



## 6.4 Local structure of nodes

**Theorem 6.4.1** (Local structure of nodes). *Let  $\pi : C \rightarrow S$  be a flat and finitely presented morphism such that every geometric fiber is a curve. Let  $p \in C$  be a node in  $C_s$ . Then we have a following diagram*

$$\begin{array}{ccccc} (C, p) & \xleftarrow{\text{\acute{e}t}} & (U, u) & \xrightarrow{\text{\acute{e}t}} & (\text{Spec} A[x, y]/(xy - f), 0) \\ \downarrow & & \downarrow & \swarrow & \\ (S, s) & \xleftarrow{\text{\acute{e}t}} & (\text{Spec} A, s') & & \end{array}$$

where each horizontal arrow is a residually-trivial pointed étale morphism and  $f \in A$  is a function vanishing at  $s'$ .

*Sketch.* See [1] 5.2.12 or St 0CBY for more details.

**Step 1. Reduce to  $S$  of finite type over  $\mathbb{Z}$ .** Using noetherian approximation.

**Step 2. Reduce to the case where  $\widehat{\mathcal{O}}_{C_s, p} \cong \kappa(s)[[x, y]]/(xy)$ .** Just need to use Proposition 6.1.4 and since separable, we can find a étale neighborhood  $(S', s')$  such that  $\kappa(s') = K$ .

**Step 3. Show that  $\widehat{\mathcal{O}}_{C, p} \cong \widehat{\mathcal{O}}_{S, s}[[x, y]]/(xy - f)$  where  $f \in \widehat{\mathfrak{m}}_s$ .** Using the Schlessinger's theorem in formal deformation theory to deduce a diagram similar as what we want at the completion level.

**Step 4. Apply Artin approximation (Theorem A.2.2).** Using Artin approximation to deduce our diagram from the completion level.  $\square$

**Corollary 6.4.2.** *Let  $\pi : C \rightarrow S$  be a flat and finitely presented morphism such that every geometric fiber is a curve, then the locus  $C^{\leq \text{nod}} = \{p \in C : p \in C_{\pi(p)} \text{ either smooth or node}\} \subset C$  is open.*

*Proof.* First we know the smooth locus is open. If  $p \in C_{\pi(p)}$  is a node, then by Theorem 6.4.1 we get an étale morphism  $g : (U, u) \rightarrow (C, p)$ . Then  $p \in g(U) \subset C^{\leq \text{nod}}$  is open.  $\square$

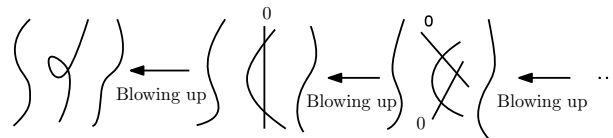
**Corollary 6.4.3.** *Let  $\pi : C \rightarrow S$  be a proper flat and finitely presented morphism such that every geometric fiber is a curve, then the locus  $S^{\leq \text{nod}} = \{s \in S : C_s \text{ is nodal}\} \subset S$  is open.*

*Proof.* As we find that

$$S^{\leq \text{nod}} = S \setminus \pi(C \setminus C^{\leq \text{nod}}).$$

By the previous Corollary and  $\pi$  is proper, then  $S^{\leq \text{nod}}$  is open.  $\square$

**Remark 6.4.4.** *Actually later we can prove that the stack  $\mathcal{M}_g^{\leq \text{nod}}$  is a algebraic stack. But the main problem of  $\mathcal{M}_g^{\leq \text{nod}}$  is that it is not separated and not of finite type. We can see the figure below for intuitive understanding:*



**Corollary 6.4.5** (Comparison). *Let's compare  $\omega_{\mathcal{C}/Y}$  and  $\Omega_{\mathcal{C}/Y}^1$  where  $\phi : \mathcal{C} \rightarrow Y$  are a family of complex nodal curves. We will follow [2] X.2 and more general we can see [32] 6.4.2. Also, we will work on the complex topology.*

*Pick a node  $p$  in some fiber, then by Theorem 6.4.1 we get near  $p$  we have the composition  $\phi|_U : U \hookrightarrow \mathbb{C}^2 \times Y \rightarrow Y$  where  $U$  defined by  $F := xy - f$ . By adjunction formula we get the local generator of  $\omega_{\mathcal{C}/Y}$  is  $F^{-1}dx \wedge dy \pmod{F}$ . Using [32] Lemma 6.4.12, we get a homomorphism*

$$\rho : \Omega_{\mathcal{C}/Y}^1 \rightarrow \omega_{\mathcal{C}/Y}$$

*given by id if it near smooth points and  $\rho(\alpha) = F^{-1}\alpha' \wedge dF \pmod{F}$  if near the nodes where  $\alpha'$  is on  $\mathbb{C}^2 \times Y \rightarrow Y$  restriction is  $\alpha$ . Actually near nodes we have  $\rho(dx) = xF^{-1}dx \wedge dy$  and  $\rho(dy) = -yF^{-1}dx \wedge dy$ . Now we consider*

$$0 \rightarrow \ker \rho \rightarrow \Omega_{\mathcal{C}/Y}^1 \xrightarrow{\rho} \omega_{\mathcal{C}/Y} \rightarrow \text{coker} \rho \rightarrow 0.$$

• **Claim 1.**  $\rho(\Omega_{\mathcal{C}/Y}^1) = \mathcal{I}\omega_{\mathcal{C}/Y}$  where  $\mathcal{I}$  be the ideal locally generated by  $x, y$  (locally ideal of that node).

*Let  $S$  be the subspace correspond to  $\mathcal{I}$ , then for now  $\text{coker} \rho = \omega_{\mathcal{C}/Y} \otimes \mathcal{O}_S$ . As locally near nodes we get  $xy = f$  and  $\rho(\Omega_{\mathcal{C}/Y}^1)$  generated by  $xF^{-1}dx \wedge dy$  and  $yF^{-1}dx \wedge dy$ , then  $\mathcal{I}$  be the ideal locally generated by  $x, y$ .*

• **Claim 2.** *When  $Y$  be a single point, then  $\ker \rho$  is the one-dimensional complex vector space generated by the class of  $xdy = -ydx$ .*

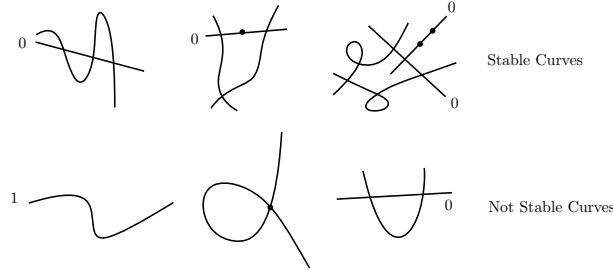
*This is trivial by this construction.*

# Chapter 7

## Stable curves

### 7.1 Basic facts of stable curves

An  $n$ -pointed curve is a curve  $C$  over a field  $k$  together with an ordered collection of  $k$ -rational points  $p_1, \dots, p_n \in C$  which we call the marked points. A point  $q \in C$  of an  $n$ -pointed curve is called special if  $q$  is a node or a marked point.



**Definition 7.1.1.** A  $n$ -pointed curve  $(C, p_1, \dots, p_n)$  over  $k$  is *prestable* if it is a geometrically connected, nodal and projective curve, and  $p_1, \dots, p_n \in C(k)$  are distinct smooth points.

A  $n$ -pointed curve  $(C, p_1, \dots, p_n)$  over  $k$  is *semistable* if

- (a) it is prestable;
- (b) every smooth rational subcurve  $\mathbb{P}^1 \subset C$  contains at least 2 special points;
- (c)  $C$  is not of genus 1 without marked points.

A  $n$ -pointed curve  $(C, p_1, \dots, p_n)$  over  $k$  is *stable* if

- (a) it is semistable;
- (b) every smooth rational subcurve  $\mathbb{P}^1 \subset C$  contains at least 3 special points.

**Remark 7.1.2.** (1) Note that there are no  $n$ -pointed stable curve of genus  $g$  if  $2g - 2 + n \leq 0$  by Proposition 7.1.4. We will often impose the condition that  $2g - 2 + n > 0$  in order to exclude these special cases;

(2) An automorphism of a stable curve  $(C, p_1, \dots, p_n)$  is an automorphism  $\alpha : C \rightarrow C$  such that  $\alpha(p_i) = p_i$ . We denote by  $\text{Aut}(C, p_1, \dots, p_n)$  the group of automorphisms;

(3) For some general Riemann Roch theorem (such as OBS6) and the fact that the prestable curves are proper geometrically connected and geometrically reduced, then  $\deg(\omega_C) = 2g - 2$ .

**Proposition 7.1.3.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed nodal projective curve such that the points  $p_i$  are distinct and smooth. Let  $\pi : \tilde{C} \rightarrow C$  be the normalization and  $\tilde{p}_i \in \tilde{C}$  be the unique preimage of  $p_i$  and  $\tilde{q}_1, \dots, \tilde{q}_m \in \tilde{C}$  be an ordering of the preimages of nodes. Then*

- (a)  *$(C, p_1, \dots, p_n)$  is stable if and only if every connected component of  $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$  is stable.*
- (b) *The group scheme  $\underline{\text{Aut}}(C, \{p_i\})$  is an algebraic group.*
- (c)  *$\underline{\text{Aut}}(C, \{p_i\})$  is naturally a closed scheme of  $\underline{\text{Aut}}(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$  with the same connected component of identity.*

*Proof.* (a) Easy to see that we just need to verify that every smooth rational subcurve  $\mathbb{P}^1 \subset C$  contains at least 3 special points if and only if every connected component of  $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$  have the same property. This is also trivial as we just need to consider the rational component of  $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$  and using the genus formula.

- (b)
- (c)

□

**Proposition 7.1.4.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed prestable curve. The following are equivalent*

- (i)  *$(C, p_1, \dots, p_n)$  is stable;*
- (ii)  *$\text{Aut}(C, p_1, \dots, p_n)$  is finite; and*
- (iii)  *$\omega_C(p_1 + \dots + p_n)$  is ample.*

*Proof.* (i)  $\Leftrightarrow$  (ii). By the results in Section 1.2 we get for smooth connected projective curve, its automorphism group is finite if and only if  $\Leftrightarrow 2g - 2 + n > 0$  for  $(g, n)$ . Now consider the normalization  $f : (\tilde{C}, \{\tilde{p}_i\}_{i=1}^n, \{\tilde{q}_j\}_{j=1}^{2s}) \rightarrow (C, p_1, \dots, p_n)$  with  $\tilde{C} = \coprod_{j=1}^t \tilde{C}_j$ . By Proposition 7.1.3 (a), we have (i)  $\Leftrightarrow$  for all  $j$ ,  $(\tilde{C}_j, \{\tilde{p}_i \in \tilde{C}_j\}_{i=1}^n, \{\tilde{q}_k \in \tilde{C}_j\}_{k=1}^{2s})$  is stable. As all  $\tilde{C}_j$  have marked points and use Proposition 7.1.3 (c), (ii)  $\Leftrightarrow$  for all  $j$ ,  $\text{Aut}(\tilde{C}_j, \{\tilde{p}_i \in \tilde{C}_j\}_{i=1}^n, \{\tilde{q}_k \in \tilde{C}_j\}_{k=1}^{2s})$  are finite. Hence by the case of smooth case, we win.

(i)  $\Leftrightarrow$  (iii). By Proposition A.3.4, 6.3.4 and consider the normalization  $\pi : \tilde{C} \rightarrow C$ , we get  $\omega_C(p_1 + \dots + p_n)$  is ample if and only if  $\pi^*\omega_C(p_1 + \dots + p_n)$  is ample if and only if for any irreducible components  $T \subset \tilde{C}$ ,  $\omega_C(p_1 + \dots + p_n)|_T = \omega_T(\sum_{p_i \in T} p_i + D_T)$  is ample. This latter condition holds precisely if each  $\mathbb{P}^1 \subset \tilde{C}$  contains at least three points that lie over nodes or marked points (using Theorem 1.1.2) and we win. □

## 7.2 Positivity of the dualizing sheaf

**Theorem 7.2.1.** *For any  $n$ -pointed stable curve  $(C, p_1, \dots, p_n)$ , the bundle  $(\omega_C(p_1 + \dots + p_n))^{\otimes k}$  is very ample for  $k \geq 3$ .*

*Proof.* We just prove the case of  $k$  is algebraically closed and no marked points. In this case we just need to show that its sections separates points and tangent vectors. We just need to show

- (i) for all closed points  $x \neq y$ , we have surjection

$$H^0(C, \omega_C^{\otimes k}) \twoheadrightarrow H^0(C, (\omega_C^{\otimes k} \otimes \kappa(x)) \oplus (\omega_C^{\otimes k} \otimes \kappa(y)));$$

- (ii) for all closed point  $x$ , we have surjection

$$H^0(C, \omega_C^{\otimes k}) \twoheadrightarrow H^0(C, \omega_C^{\otimes k} \otimes \mathcal{O}_C/I_x^2).$$

Hence for  $x, y \in C(k)$  (maybe the same points) and their ideal  $I_x, I_y$ , we have

$$0 \rightarrow \omega_C^{\otimes k} \otimes I_x I_y \rightarrow \omega_C^{\otimes k} \rightarrow \omega_C^{\otimes k} \otimes \mathcal{O}_C / I_x I_y \rightarrow 0.$$

So we just need to show that  $H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) = 0$ . By Serre duality, we need to show

$$\begin{aligned} H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) &= H^0(C, (\omega_C^{\otimes k} \otimes I_x I_y)^\vee \otimes \omega_C) \\ &= H^0(C, \mathcal{H}om(\omega_C^{\otimes k} \otimes I_x I_y, \omega_C)) = \text{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = 0. \end{aligned}$$

We need a case analysis on whether  $x, y$  are smooth or nodal.

If  $x \in C$  is smooth, then  $I_x = \mathcal{O}_C(-x)$  is invertible. If  $x \in C$  is a node, consider the blowing up  $\pi : C' \rightarrow C$  along  $x$  with  $\pi^{-1}(x) = \{x_1, x_2\}$ . Then for any line bundle  $L$  on  $C$  we claim that

$$\text{Hom}(I_x, L) \cong H^0(C', \pi^* L), \text{Hom}(I_x^2, L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We just prove the first statement, the second is similar. First we have

$$0 \rightarrow \mathcal{O}_C \rightarrow \pi_* \mathcal{O}_{C'} \rightarrow \kappa(x) \rightarrow 0,$$

tensoring  $L$  we get

$$0 \rightarrow L \rightarrow \pi_* \mathcal{O}_{C'} \otimes L = \pi_* \pi^* L \rightarrow L(x) \rightarrow 0.$$

Hence we have

$$0 \rightarrow \text{Hom}(I_x, L) \rightarrow \text{Hom}(\pi^* I_x, \pi^* L) \xrightarrow{f} \text{Hom}(I_x, L(x)) = \text{Hom}(I_x / I_x^2, L(x)).$$

On the other hand, we have a short exact sequence

$$0 \rightarrow \pi^* L \rightarrow \pi^* L(x_1 + x_2) \rightarrow \pi^* L(x_1) \oplus \pi^* L(x_2) \rightarrow 0$$

inducing

$$0 \rightarrow H^0(C', \pi^* L) \rightarrow H^0(C', \pi^* L(x_1 + x_2)) \xrightarrow{g} \pi^* L(x_1) \oplus \pi^* L(x_2).$$

Let  $J = \mathcal{O}_{C'}(-x_1 - x_2) \subset \mathcal{O}_{C'}$ , we get

$$0 \rightarrow K \rightarrow \pi^* I_x \rightarrow J \rightarrow 0$$

where  $\text{supp}(K) = \{x_1, x_2\}$  by checking locally. Since  $\pi^* L$  is torsion free at  $x_1, x_2$ , we have  $\text{Hom}(K, \pi^* L) = 0$ , so this defines an isomorphism

$$\text{Hom}(\pi^* I_x, \pi^* L) \cong \text{Hom}(J, \pi^* L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We also have isomorphisms  $I_x / I_x^2 \cong \pi_*(J / J^2)$  with  $\text{Hom}(I_x / I_x^2, L(x)) \cong \text{Hom}(\pi_*(J / J^2), L(x))$ . Actually  $\text{Hom}(\pi_*(J / J^2), L(x)) \cong \pi^* L(x_1) \oplus \pi^* L(x_2)$  (**Why?**). Hence conclude these, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(I_x, L) & \longrightarrow & \text{Hom}(\pi^* I_x, \pi^* L) & \longrightarrow & \text{Hom}(I_x / I_x^2, L(x)) \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & H^0(C', \pi^* L) & \longrightarrow & H^0(C', \pi^* L(x_1 + x_2)) & \longrightarrow & \pi^* L(x_1) \oplus \pi^* L(x_2) \end{array}$$

Hence the claim is right.

**Case (I).** If  $x, y$  are all smooth points, then  $\deg(\omega_C^{\otimes(1-k)}(x+y)) = (1-k)(2g-2) + 2 < 0$  for  $k \geq 3$ . Hence

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = H^0(C, \omega_C^{\otimes(1-k)}(x+y)) = 0.$$

**Case (II).** If  $x$  is a node and  $y$  is a smooth point, then by the claim, we win.

**Case (III.1).** If  $x = y$  is a node, then by the claim we get

$$\mathrm{Hom}(I_x^2, \omega_C^{\otimes(1-k)}) \cong H^0(C', \pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)).$$

Consider the normalization  $\tilde{C}$  of  $C$  (and  $C'$ , also), we consider an irreducible component  $E \subset \tilde{C}$ . Then  $\pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)$  restrict to  $E$  has degree

$$(1-k)(2g_E - 2 + \#\{E \cap \tilde{\Sigma}\}) + \#(\{x_1, x_2\} \cap E)$$

is negative unless  $k = 3$ ,  $\{x_1, x_2\} \subset E$ ,  $E$  is a rational curve meeting the other components of  $C$  in exactly one other point. In this case the degree on  $E$  is zero. So this global section is determined by its value at the point of  $E$  meeting the other components of  $C$ . Since not every component of  $\tilde{C}$ , we win.

**Case (III.2).** If  $x \neq y$  are all nodes, the blowing up  $\varpi : C'' \rightarrow C$  along  $\{x, y\}$ . We can get similar conclusion

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) \cong H^0(C', \varpi^* \omega_C^{\otimes(1-k)}).$$

This is zero since in any irreducible of normalization has negative degree.  $\square$

### 7.3 Families of stable curves

**Definition 7.3.1.** (1) A family of  $n$ -pointed nodal curves is a flat, proper and finitely presented morphism  $C \rightarrow S$  of schemes with  $n$  sections  $\sigma_1, \dots, \sigma_n : S \rightarrow C$  such that every geometric fiber  $C_s$  is a (reduced) connected nodal curve.

(2) A family of  $n$ -pointed stable curves (resp. semistable curves, prestable curves) is a family  $C \rightarrow S$  of  $n$ -pointed nodal curves such that every geometric fiber  $(C_s, \sigma_1(s), \dots, \sigma_n(s))$  is stable (resp. semistable, prestable).

**Remark 7.3.2.** (1) We can define the fibered category of groupoid  $\overline{\mathcal{M}}_{g,n}$  as for any scheme  $S$ , define  $\overline{\mathcal{M}}_{g,n}(S) = \{(C, \sigma_1, \dots, \sigma_n) \rightarrow S : \text{is a family of stable curves of genus } g\}$ . Note also that since the geometric fibers are stable curves, the image of each  $\sigma_i$  is a divisor contained in the smooth locus and we can form the line bundle  $\omega_{C/S}(\sum_i \sigma_i)$ .

(2) We can define relative dualizing line bundle  $\omega_{C/S}$  as  $C \rightarrow S$  is l.c.i. By [24], we can get the following properties: (2.a)  $\omega_{C/S}|_{C_s} = \omega_{C_s}$ ; (2.b) for any  $f : T \rightarrow S$  we have  $f^* \omega_{C/S} = \omega_{C \times_S T/T}$ .

**Proposition 7.3.3.** Let  $\pi : (C, \sigma_1, \dots, \sigma_n) \rightarrow S$  be a family of  $n$ -pointed stable curves of genus  $g$ . Let  $L = \omega_{C/S}(\sum_i \sigma_i)$ . If  $k \geq 3$ , then  $L^{\otimes k}$  is relatively very ample and  $\pi_* L^{\otimes k}$  is a vector bundle of rank  $(2k-1)(g-1) + kn$ .

*Proof.* Similar as the smooth case by using Riemann Roch and cohomology and base change. Omitted here.  $\square$

**Proposition 7.3.4** (Openness of stability). Let  $\pi : (C, \sigma_1, \dots, \sigma_n) \rightarrow S$  be a family of  $n$ -pointed nodal curves. The locus of  $S$  such that  $(C_s, \sigma_i(s))$  is stable is open.

*Proof.* As the locus such that  $\sigma_i(s)$  is smooth is open, we just need to let this family is prestable. Using 7.1.4, we have two arguments:

**Argument 1.** Group scheme  $\underline{\text{Aut}}(C/S, \sigma_i) \rightarrow S$  has upper semicontinuous dimension of fibers, then as stable locus is the locus such that it is dimension 0 locus. Hence open.

**Argument 2.** Using the openness of ample locus.  $\square$

**Proposition 7.3.5** (Openness of being nodal). *Let  $f : X \rightarrow S$  be a flat proper morphism of  $\mathbb{C}$ -schemes. Then the set of all  $s \in S$  such that  $X_s = f^{-1}(s)$  is not a connected nodal curve is closed in  $S$ . If, in addition,  $n$  sections  $\sigma_i$  of  $f$  are given, then the set of all  $s \in S$  such that  $(X_s; \sigma_i(s))$  is not a connected  $n$ -pointed nodal curve is closed in  $S$ .*

*Sketch.* We will give a sketch and for the detailed proof see [2] Proposition XI.5.1. First we need to let the fibers of  $f$  has dimension 1 by flatness and properness.

• **Step 1. Reduce to the case that fibers are connected and have no embedded components.** Easy to see that  $\dim H^0(X_s, \mathcal{O}_{X_s}) = 1$  for all  $s \in S$  if  $X_s$  is connected and reduced. As this is the stalks of  $f_* \mathcal{O}_X$ , we consider the free resolution  $K^0 \xrightarrow{\alpha} K^1 \rightarrow \dots$  at some open subset. Hence the locus of  $\dim H^0(X_s, \mathcal{O}_{X_s}) > 1$  is the locus of  $\text{rank}(\alpha) \leq \text{rank}(K^0) - 2$ . Hence is closed.

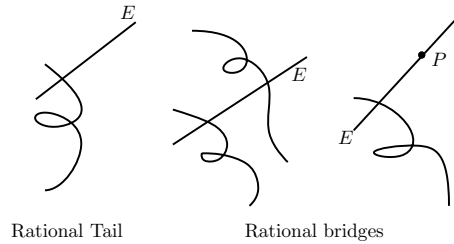
• **Step 2. Show that being neither nodal nor smooth is closed.** Here we need to represent nodes by some functions. Then we use some equivalent conditions (see [2] Lemma X.2.3) that if  $f$  be a function over 0 and  $f(0) = 0$ , then  $f$  defines the smooth point 0 if and only if the first-order partials of  $f$  not vanish at the origin;  $f$  defines the node 0 if and only if the first-order partials of  $f$  vanish and the Hessian not vanish.  $\square$

## 7.4 Rational tails and bridges

**Definition 7.4.1.** *Let  $(C, p_1, \dots, p_n)$  be a  $n$ -pointed prestable curve. We say a smooth rational subcurve  $E \cong \mathbb{P}^1 \subset C$  is*

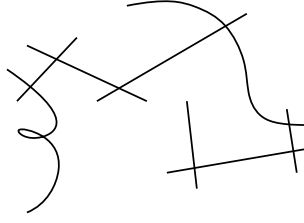
(i) *a rational tail if  $E$  meets other irreducible components at exactly 1 time, and  $E$  contains no marked points;*

(ii) *a rational bridge if either  $E$  meets other irreducible components at exactly 2 time and contains no marked points, or  $E$  meets other irreducible components at exactly 1 time and contains exactly 1 marked point.*



**Remark 7.4.2.** (1)  $C$  is stable if and only if it is prestable and has no rational tails and bridges;  
 (2)  $C$  is semistable if and only if it is prestable and has no rational tails.

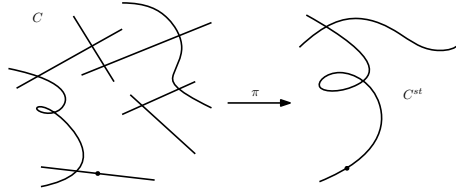
Or we can also have chain of rational tails and bridges like:



## 7.5 The stable model

### 7.5.1 The stable model of a single curve

Let  $(C, p_1, \dots, p_n)$  be a  $n$ -pointed prestable curve. Let  $E_i \subset C$  be its rational tails and bridges. We let  $C^{st} := \overline{C \setminus \bigcup_i E_i}$  and let  $\pi : C \rightarrow C^{st}$  be the induced map. Let  $p'_i = \pi(p_i)$ , then  $(C^{st}, \{p'_i\})$  is a stable curve, which we call the stable model of  $(C, \{p_i\})$  and  $\pi : C \rightarrow C^{st}$  the stabilization morphism. Like this:



For the serious argument of contraction to the stable curves, we refer [5]: contracting rational tails (St 0E3G), contracting rational bridges (St 0E7M), contracting to a stable curve (St 0E7N). We omitted here.

### 7.5.2 The stable model of a family of curves

For a family of nodal curves, we also have the following conclusion.

**Proposition 7.5.1.** *Let  $(C \rightarrow S, \sigma_i)$  be a family of  $n$ -pointed prestable curves. Then there exists a unique (up to isomorphism) morphism  $\pi : C \rightarrow C^{st}$  such that*

- (a)  $(C^{st} \rightarrow S, \{\sigma'_i\})$  is a  $n$ -pointed family of stable curves where  $\sigma'_i = \pi \circ \sigma_i$ ;
- (b) for each  $s \in S$ ,  $(C_s, \{\sigma_i(s)\}) \rightarrow (C_s^{st}, \{\sigma'_i(s)\})$  is the stable model;
- (c)  $\mathcal{O}_{C^{st}} = \pi_* \mathcal{O}_C$  and  $R^1 \pi_* \mathcal{O}_C = 0$  and this remains true after base change by a morphism  $S' \rightarrow S$  of schemes;
- (d) If  $C \rightarrow S$  is a family of semistable curves, then  $\omega_{C/S}(\sum_i \sigma_i)$  is the pullback of the relatively ample line bundle  $\omega_{C^{st}/S}(\sum_i \sigma'_i)$ .

*Proof.* See St 0E7B. □



## Chapter 8

# Deformation theory of nodal and stable curves

After some basic results over arbitrary fields, we will focus on the curves over  $\mathbb{C}$ . We mainly follows [2] chapter XI (all results over  $\mathbb{C}$ ) and some results over arbitrary fields we follows [1]. Some basic result and proofs we follows [35]. Here we let  $k[\varepsilon] := k[x]/(x^2)$ .

### 8.1 Elementary deformation theory and smooth objects

**Definition 8.1.1.** *Let  $X$  be a scheme over  $k$ . A first order deformation of  $X$  is a scheme  $\mathcal{X}$  flat over  $k[\varepsilon] = k[\varepsilon]/(\varepsilon^2)$  with  $X \cong \mathcal{X} \times_{k[\varepsilon]} k$ .*

*We say  $\mathcal{X}$  is trivial if  $\mathcal{X}$  is isomorphic as first deformations to  $\mathcal{X} \times_k k[\varepsilon]$ , and locally trivial if there exists a Zariski-cover  $X = \bigcup_i U_i$  such that  $\mathcal{X}|_{U_i}$  is a trivial first order deformation of  $U_i$ , that is,  $U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$  where  $\mathcal{X}|_{U_i} \subset \mathcal{X}$  be a open subscheme with the same topology of  $U_i$ .*

We let  $\text{Def}(X)$  be the isomorphism classes of first order deformations of  $X$  and  $\text{Def}^{lt}(X)$  be the isomorphism classes of locally trivial first order deformations of  $X$ .

**Proposition 8.1.2** (See [1] D.1.11). *For a scheme  $X$  of finite type over  $k$  with affine diagonal, there is a bijection*

$$\text{Def}^{lt}(X) \leftrightarrow H^1(X, T_X).$$

*In particular, if  $X_0$  is smooth, then we have bijection*

$$\text{Def}(X) \leftrightarrow H^1(X, T_X),$$

*as every first order deformations of smooth affine schemes is trivial.*

*Sketch.* For a locally trivial first order deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec } k & \hookrightarrow & \text{Spec } k[\varepsilon] \end{array}$$

let affine covering  $\{U_i\}$  of  $X$  such that  $\mathcal{X}|_{U_i}$  be a trivial first order deformation. Hence we get isomorphisms  $\phi_i : U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$ . Let  $\phi_{ij} := \phi_j^{-1}|_{U_{ij} \times_k k[\varepsilon]} \circ \phi_i|_{U_{ij} \times_k k[\varepsilon]}$  are automorphisms of first order defs, hence we get  $\phi_{ij} \in \text{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/k}, \mathcal{O}_{U_{ij}})$ . As they satisfies cocycle condition, we get  $\{\phi_{ij}\} \in H^1(X, T_X)$  by Čech theory (this is independent on the choice of covering, see [35] Proposition 1.2.9). Converse is trivial.  $\square$

**Remark 8.1.3.** For a locally trivial first order deformations  $\xi$  of  $X$ , we gives a class  $\kappa(\xi) \in H^1(X, T_X)$  is called the Kodaira-Spencer class of  $\xi$ .

**Definition 8.1.4.** Consider a family of deformation of a smooth algebraic variety  $X$  over  $k$

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_f \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } k & \xleftarrow{s} & S & \xleftarrow{f} & \text{Spec } k[\varepsilon] \end{array}$$

$\curvearrowright$

hence we get

$$\kappa_{\mathcal{X}/S, s} : T_{S, s} \rightarrow H^1(X, T_X),$$

we called it Kodaira-Spencer map.

**Definition 8.1.5.** Let  $A' \twoheadrightarrow A$  has square-free kernel and  $X \rightarrow \text{Spec}(A)$  is flat. A deformation of  $X \rightarrow \text{Spec}(A)$  over  $A'$  is  $X' \rightarrow \text{Spec}(A')$  with  $X' \times_{A'} A \cong X$ . A morphism of deformations over  $A'$  is a morphism of schemes over  $A'$  restricting to the identity on  $X$ .

**Proposition 8.1.6.** Let  $A' \twoheadrightarrow A$  has square-free kernel  $J$ . If  $X \rightarrow \text{Spec}(A)$  is a smooth morphism of schemes where  $X$  has affine diagonal, then

- (a) the group of automorphisms of a deformation  $X' \rightarrow \text{Spec}(A')$  of  $X \rightarrow \text{Spec}(A)$  over  $A'$  is bijective to  $H^0(X, T_{X/A} \otimes_A J)$ ;
- (b) If there exists a deformation of  $X \rightarrow \text{Spec}(A)$  over  $A'$ , then the set of isomorphism classes of all such deformations is a torsor under  $H^1(X, T_{X/A} \otimes_A J)$ ;
- (c) There is an element  $ob_X \in H^2(X, T_{X/A} \otimes_A J)$  with the property that there exists a deformation of  $X \rightarrow \text{Spec}(A)$  over  $A'$  if and only if  $ob_X = 0$ .

*Proof.* See [1] Proposition D.2.6. To add.  $\square$

Back to smooth curves over  $\mathbb{C}$  pf genus  $g$ .

**Theorem 8.1.7.** Let  $(C; q_1, \dots, q_n)$  be a  $n$ -pointed smooth  $n$ -pointed genus  $g$  curve over  $\mathbb{C}$ .

(i) We have

$$\text{Def}(C; q_1, \dots, q_n) \leftrightarrow H^1(C, T_C(-\sum_{i=1}^n q_i));$$

(ii) There exists a deformation

$$\phi : C \rightarrow (B, b_0), \sigma_i : B \rightarrow C \text{ such that } \chi : (C; q_1, \dots, q_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of  $(C; q_1, \dots, q_n)$  such that the Kodaira-Spencer map

$$\kappa : T_{b_0} B \rightarrow H^1(C, T_C(-\sum_{i=1}^n q_i))$$

is an isomorphism and  $B$  is a polydisc of dimension  $3g - 3 + n + h^0(C, T_C(-\sum_{i=1}^n q_i))$ .

*Proof.* See [2] Theorem XI.2.12. To add.  $\square$

## 8.2 Elementary deformations of nodal and stable curves

**Lemma 8.2.1.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed nodal, connected and projective curve over  $k$  with each  $p_i \in C$  smooth. Let  $\{q_1, \dots, q_s\}$  be the nodes of  $C$ . Let  $(\tilde{C}, p_i, q'_j, q''_j)$  be the pointed normalization  $\pi : \tilde{C} \rightarrow C$  and  $\pi^{-1}(q_j) = \{q'_j, q''_j\}$ . Then we have the spectral sequence*

$$E_2^{p,q} = H^p(C, \mathcal{E}xt_{\mathcal{O}_C}^q(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \Rightarrow \text{Ext}_{\mathcal{O}_C}^{p+q}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$$

such that induce the following exact sequence

$$\begin{array}{ccc} 0 & \longrightarrow & H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \\ & & \downarrow \\ 0 & \longleftarrow & \bigoplus_j \text{Ext}_{\mathcal{O}_{C,q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}) \longleftarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) \end{array}$$

Moreover,  $\forall j$  we have  $\text{Ext}_{\hat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}) = k$  and  $\text{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) = 0$ .

*Proof.* By Grothendieck spectral sequence, we have

$$E_2^{p,q} = H^p(C, \mathcal{E}xt_{\mathcal{O}_C}^q(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \Rightarrow \text{Ext}_{\mathcal{O}_C}^{p+q}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C).$$

As  $C$  is a curve,  $E_2^{p,q} = 0$  for  $p \geq 2$ .

By [14] Propostion 2.3, we get an exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow 0.$$

As  $\Omega_C$  is locally free away from nodes,  $\mathcal{E}xt_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$  is zero-dimensional sheaf supported only at nodes. Hence  $E_2^{1,1} = 0$  and

$$\begin{aligned} E_2^{0,1} &= H^0(C, \mathcal{E}xt_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \\ &= \bigoplus_j \text{Ext}_{\mathcal{O}_{C,q_j}}^1(\Omega_{C,q_j}, \mathcal{O}_{C,q_j}) = \bigoplus_j \text{Ext}_{\hat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}). \end{aligned}$$

where  $\hat{\Omega}_{C,q_j} = \Omega_{\hat{\mathcal{O}}_{C,q_j}}$ . Hence we get that exact sequence.

Similarly  $\mathcal{E}xt_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$  is zero-dimensional sheaf supported only at nodes, then

$$E_2^{0,2} = H^0(C, \mathcal{E}xt_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) = \bigoplus_j \text{Ext}_{\hat{\mathcal{O}}_{C,q_j}}^2(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}).$$

Write  $\hat{\mathcal{O}}_{C,q_j} = k[[x, y]]/(xy)$  and consider the locally free resolution

$$0 \longrightarrow \hat{\mathcal{O}}_{C,q_j} \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} \hat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(dx, dy)} \Omega_{\hat{\mathcal{O}}_{C,q_j}} \longrightarrow 0$$

Hence we get  $\text{Ext}_{\hat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}) = k$  and  $\text{Ext}_{\hat{\mathcal{O}}_{C,q_j}}^2(\Omega_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}) = 0$ . Hence  $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$  and  $\text{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) = 0$ .  $\square$

**Proposition 8.2.2.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed nodal, connected and projective curve over  $k$  with each  $p_i \in C$  smooth. Let  $\{q_1, \dots, q_s\}$  be the nodes of  $C$ . Let  $(\tilde{C}, p_i, q'_j, q''_j)$  be the*

pointed normalization  $\pi : \tilde{C} \rightarrow C$  and  $\pi^{-1}(q_j) = \{q'_j, q''_j\}$ . Then we have the following exact sequence

$$0 \rightarrow \text{Def}^{dt}(C) \rightarrow \text{Def}(C) \rightarrow \bigoplus_j \text{Def}(\hat{\mathcal{O}}_{C,q_j}) \rightarrow 0$$

and

$$\text{Def}^{dt}(C) \cong \text{Def}(\tilde{C}, p_i, q'_j, q''_j) \cong H^1(\tilde{C}, T_{\tilde{C}}(-\sum_i p_i - \sum_j (q'_j + q''_j))),$$

$$\text{Def}(C) \cong \text{Ext}^1_{\mathcal{O}_C}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C),$$

$$\text{Def}(\hat{\mathcal{O}}_{C,q_j}) \cong \text{Ext}^1_{\hat{\mathcal{O}}_{C,q_j}}(\Omega^1_{\hat{\mathcal{O}}_{C,q_j}}, \hat{\mathcal{O}}_{C,q_j}) \cong k.$$

Under these identifications, this exact sequence corresponds to the exact sequence in the Lemma.

*Sketch.* WLOG again we let  $n = 0$ . If  $\mathcal{C} \rightarrow \text{Spec}k[\varepsilon]$  is a locally trivial first order deformation of  $C$ , each node  $q_j$  extend to a section  $\tilde{q}_j : \text{Spec}k[\varepsilon] \rightarrow \mathcal{C}$ . The pointed normalization of  $\mathcal{C}$  along the sections  $\tilde{q}_j$  is a first order deformation of the (possible disconnected) pointed normalization  $(\tilde{C}, p_i, q'_j, q''_j)$ . This gives a map  $\text{Def}^{dt}(C) \rightarrow \text{Def}(\tilde{C}, p_i, q'_j, q''_j)$ . The inverse is provided by gluing the sections of a first order deformation of  $(\tilde{C}, p_i, q'_j, q''_j)$  along nodes.

If  $\mathcal{C} \rightarrow \text{Spec}k[\varepsilon]$  is a first order deformation of  $C$ , then ideal sheaf  $I$  of  $C \rightarrow \mathcal{C}$  is  $I = I/I^2 \cong \mathcal{O}_C$ . The right exact sequence

$$I/I^2 \rightarrow \Omega_{\mathcal{C}/k} \rightarrow \Omega_{C/k} \rightarrow 0$$

is left exact at every smooth point of  $C$ . As  $C \rightarrow \text{Spec}k$  is generically smooth and it follows that  $\mathcal{O}_C \cong I/I^2 \rightarrow \Omega_{\mathcal{C}/k}$  is generically injective, hence injective. Hence this defines  $\text{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$ . This is bijective (one can see [2] section XI.3).  $\square$

**Remark 8.2.3.** Hence we also have the Kodaira-Spencer map for some  $\mathcal{C} \rightarrow (S, s)$  as

$$\kappa_{S,s} : T_{S,s} \rightarrow \text{Ext}^1_{\mathcal{O}_C}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C).$$

**Remark 8.2.4.** Let  $C$  be a nodal curve over  $\mathbb{C}$  and let  $p \in C$  be a node with normalization  $N$  and preimages  $\{p_1, p_2\}$ .

• **Claim 1.**  $\text{Ext}^1(\Omega_{C,p}, \mathcal{O}_{C,p}) \cong \bigwedge^2(\mathfrak{m}_p/\mathfrak{m}_p^2) \otimes_{\mu_2} \tau$  where  $\mu_2 = \{\pm 1\}$  and  $\tau$  be the set consisting of the two possible orderings of the branches of  $C$  at  $p$ .

I omit this, see [2] page 180.

• **Claim 2.**  $\text{Ext}^1(\Omega_{C,p}, \mathcal{O}_{C,p}) \cong T_{N,p_1} \otimes T_{N,p_2}$ .

Trivial by Claim 1 and  $\mathfrak{m}_p/\mathfrak{m}_p^2 = T_{C,p} = T_{N,p_1} \oplus T_{N,p_2}$  and  $\bigwedge^2 T_{N,p_1} \oplus T_{N,p_2}$  identify with  $T_{N,p_1} \otimes T_{N,p_2}$  depends on the choice of an ordering of the two summands, we win.

**Remark 8.2.5.** Let  $(C; p_1, \dots, p_n)$  be an  $n$ -pointed nodal curve over  $\mathbb{C}$  for simplicity, and let  $W = \{w_1, \dots, w_l\}$  be some set of nodes of  $C$ . Let  $f : N \rightarrow C$  be the partial normalization at these nodes with  $f^{-1}(w_i) = \{r_i, q_i\}$ . Let  $D = \sum_i p_i$  with inverse  $\tilde{D}$  and  $E = \sum(r_i + q_i)$ .

• **Claim 1.**  $\mathcal{H}om(\Omega^1_C, \mathcal{O}_C(-D)) \cong f_* \mathcal{H}om(\Omega^1_N, \mathcal{O}_N(-\tilde{D} - E))$ .

This is trivially true at points away from  $W$ , so we just need to consider the points in  $W$ . Pick any  $w_i \in W$ , we get  $\text{Hom}(\Omega^1_{C,w_i}, \mathcal{O}_{C,w_i}) = \text{Hom}(\mathcal{I}_{w_i} \omega_{C,w_i}, \mathcal{O}_{C,w_i})$  by Corollary 6.4.5. As  $\mathcal{I}_{w_i} \omega_{C,w_i} = \omega_{N,r_i} \oplus \omega_{N,q_i}$  and  $\mathcal{I}_{w_i} = \mathcal{O}_{N,r_i}(-r_i) \oplus \mathcal{O}_{N,q_i}(-q_i)$  (Why?), we get

$$\text{Hom}(\Omega^1_{C,w_i}, \mathcal{O}_{C,w_i}) = \bigoplus_{p=r_i, q_i} \text{Hom}(\omega_{N,p}, \mathcal{O}_{N,p}(-p)),$$

hence  $\mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D)) \cong f_* \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E))$ .

•**Claim 2.** We have

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) &\rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \\ &\bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C, w_i}^1, \mathcal{O}_{C, w_i}) \rightarrow 0. \end{aligned}$$

By Claim 1, we get  $H^1(N, \mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D))) \cong H^1(N, \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)))$ . Hence by Lemma 8.2.1, we get

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(N, \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E))) & \longrightarrow & \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) \\ & & \parallel & & \downarrow \\ 0 & \longrightarrow & H^1(N, \mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D))) & \longrightarrow & \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \\ & & & & \downarrow \\ & \longrightarrow & \bigoplus_{w \in \text{Sing}(C), w \notin W} \text{Ext}^1(\Omega_{C, w}^1, \mathcal{O}_{C, w}) & \longrightarrow & 0 \\ & & \downarrow & & \\ & \longrightarrow & \bigoplus_{w \in \text{Sing}(C)} \text{Ext}^1(\Omega_{C, w}^1, \mathcal{O}_{C, w}) & \longrightarrow & 0 \end{array}$$

hence we get

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) &\rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \\ &\bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C, w_i}^1, \mathcal{O}_{C, w_i}) \rightarrow 0. \end{aligned}$$

Note that the term on the left classifies first-order deformations which are locally trivial at the nodes belonging to  $W$ , and the one on the right classifies first-order smoothings of these nodes.

•**Claim 3.**  $\bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C, w_i}^1, \mathcal{O}_{C, w_i}) = \bigoplus_{i=1}^l T_{N, r_i} \otimes T_{N, q_i}$ .

By claims in Remark 8.2.4, this is trivial.

Here is a similar result as before over  $\mathbb{C}$  via analytic GAGA.

**Theorem 8.2.6.** Let  $(C; p_1, \dots, p_n)$  be an  $n$ -pointed nodal curve of genus  $g$  over  $\mathbb{C}$ . There exists a deformation

$$\phi : \mathcal{C} \rightarrow (B, b_0), \sigma_i : B \rightarrow \mathcal{C} \text{ such that } \chi : (C; p_1, \dots, p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of  $(C; p_1, \dots, p_n)$  such that the Kodaira-Spencer map

$$\kappa : T_{b_0} B \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$$

is an isomorphism and  $B$  is a polydisc of dimension  $3g - 3 + n + \dim \text{Hom}(\Omega_C, \mathcal{O}_C)$ .

Finally, if  $s$  is the number of nodes of  $C$ , one can choose coordinates  $t_1, \dots, t_s, \dots$  on  $B$ , vanishing at  $b_0$ , in such a way that the locus parameterizing deformations which are locally trivial at the  $i$ -th node is  $t_i = 0$ ; in particular, the locus parameterizing singular curves is  $t_1 \cdots t_s = 0$ .

*Proof.* See [2] Theorem XI.3.17. To add. □

Back to the general case.

**Proposition 8.2.7.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed nodal, connected and projective curve over  $k$  with each  $p_i \in C$  smooth. Let  $A' \rightarrow A$  be a surjection of artinian local  $k$ -algebras with residue field  $k$  such that  $J = \ker(A' \rightarrow A)$  satisfies  $\mathfrak{m}_{A'} J = 0$ . If  $C_A \rightarrow \operatorname{Spec}(A)$  be a family of nodal curves such that  $C \cong C_A \times_A k$ , then*

(a) *The group of automorphisms of a deformation  $C_{A'} \rightarrow \operatorname{Spec}(A')$  of  $C_A \rightarrow \operatorname{Spec}(A)$  over  $A'$  is bijective to  $\operatorname{Ext}_{\mathcal{O}_C}^0(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$ ;*

(b) *If there exists a deformation of  $C_A \rightarrow \operatorname{Spec}(A)$  over  $A'$ , then the set of isomorphism classes of all such deformations is a torsor under  $\operatorname{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$ ;*

(c) *There is an element  $ob_{C_A} \in \operatorname{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$  with the property that there exists a deformation of  $C_A \rightarrow \operatorname{Spec}(A)$  over  $A'$  if and only if  $ob_{C_A} = 0$ .*

*Proof.* To add. □

**Lemma 8.2.8** (St 0E68). *Let  $k$  be an algebraically closed field. Let  $X$  be an at-worst-nodal, proper, connected 1-dimensional scheme over  $k$ . Let  $f : \tilde{X} \rightarrow X$  be the normalization. Let  $S \subset \tilde{X}$  be the set of points where  $f$  is not an isomorphism.*

$$\operatorname{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \{D' \in \operatorname{Der}_k(\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) : D' \text{ fixed every } x' \in S\}.$$

*Proof.* Let  $x \in X$  be a node with the preimage  $x', x'' \in \tilde{X}$ . Pick two uniformizers  $u, v$  in  $\mathcal{O}_{\tilde{X}, x'}$  and  $\mathcal{O}_{\tilde{X}, x''}$ , respectively. Hence we have

$$0 \rightarrow \mathcal{O}_{X, x} \rightarrow \mathcal{O}_{\tilde{X}, x'} \times \mathcal{O}_{\tilde{X}, x''} \rightarrow k \rightarrow 0,$$

thus we can view  $u, v$  as elements of  $\mathcal{O}_{X, x}$  with  $uv = 0$ .

Since  $(u)$  is annihilator of  $v$  in  $\mathcal{O}_{C, x}$  and vice versa, we see that  $D(u) \in (u)$  and  $D(v) \in (v)$ . As  $\mathcal{O}_{\tilde{C}, x'} = k + (u)$  we conclude that we can extend  $D$  to  $\mathcal{O}_{\tilde{C}, x'}$  and moreover the extension fixes  $x'$ . This produces a  $D'$  in the right hand side of the equality. Conversely, given a  $D'$  fixing  $x'$  and  $x''$  we find that  $D'$  preserves the subring  $\mathcal{O}_{C, x} \subset \mathcal{O}_{\tilde{C}, x'} \times \mathcal{O}_{\tilde{C}, x''}$  and this is how we go from right to left in the equality. □

**Proposition 8.2.9.** *Let  $(C, p_1, \dots, p_n)$  be an  $n$ -pointed stable curve of genus  $g$  over  $k$ . Then*

$$\dim_k \operatorname{Ext}_{\mathcal{O}_C}^i \left( \Omega_C \left( \sum_i p_i \right), \mathcal{O}_C \right) = \begin{cases} 0, & i = 0, 2; \\ 3g - 3 + n, & i = 1. \end{cases}$$

*Proof.* We let  $k$  is algebraically closed and has no marked point. Let  $\pi : \tilde{C} \rightarrow C$  be a normalization and let  $\Sigma \subset C$  be the set of nodes. Let  $\tilde{\Sigma} = \pi^{-1}(\Sigma) \subset \tilde{C}$ .

By Lemma 8.2.1, we get  $\dim_k \operatorname{Ext}_{\mathcal{O}_C}^2(\Omega_C, \mathcal{O}_C) = 0$ . For  $\operatorname{Ext}^0$ , we first claim that

$$\operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}(\tilde{\Sigma}), \mathcal{O}_{\tilde{C}}) \cong \operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

This is equivalent to show

$$\operatorname{Der}_k(\mathcal{O}_C, \mathcal{O}_C) \cong \{D' \in \operatorname{Der}_k(\mathcal{O}_{\tilde{C}}, \mathcal{O}_{\tilde{C}}) : D' \text{ fixes every points in } \tilde{\Sigma}\}.$$

Actually this is just Lemma 8.2.8. This finish the claim. Hence we get

$$\operatorname{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \cong \operatorname{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}(\tilde{\Sigma}), \mathcal{O}_{\tilde{C}}) \cong H^0(\tilde{C}, T_{\tilde{C}}(-\tilde{\Sigma})) = 0.$$

For  $\text{Ext}^1$ , by Lemma 8.2.1 we have

$$\begin{array}{ccc} 0 \rightarrow H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) \rightarrow \bigoplus_j \text{Ext}_{\widehat{\mathcal{O}}_{C, q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C, q_j}}, \widehat{\mathcal{O}}_{C, q_j}) \\ \parallel \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ H^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 0 \end{array}$$

and  $\text{Ext}_{\widehat{\mathcal{O}}_{C, q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C, q_j}}, \widehat{\mathcal{O}}_{C, q_j}) = k$ . This equality in this exact sequence is because

$$H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$

be the set of locally trivial first order deformation of  $C$  preserving nodes and this is equivalent to the set of locally trivial first order deformation of  $\widetilde{C}$  fixed  $\widetilde{\Sigma}$ , which is  $H^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma}))$ .

Now let  $\widetilde{C} = \coprod_{i=1}^t \widetilde{C}_i$  are connected components and  $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$ . First we have

$$h^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) = h^0(\widetilde{C}_i, \Omega_{\widetilde{C}_i}^{\otimes 2}(\widetilde{\Sigma}_i)) = 3g(\widetilde{C}_i) - 3 + \#(\widetilde{\Sigma}_i),$$

hence

$$\begin{aligned} \dim_k \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) &= h^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) + \#(\Sigma) \\ &= \sum_{i=1}^t (3g(\widetilde{C}_i) - 3 + \#(\widetilde{\Sigma}_i)) + \#(\Sigma) = 3 \sum_{i=1}^t g(\widetilde{C}_i) - 3t + 3\#(\Sigma) = 3g - 3 \end{aligned}$$

by the genus formula, we win.  $\square$

### 8.3 Basic concept of Kuranishi family

We will work on analytic category over  $\mathbb{C}$  via Serre's GAGA.

**Definition 8.3.1.** Let  $(C; p_1, \dots, p_n)$  be an  $n$ -pointed connected nodal curve of genus  $g$  over  $\mathbb{C}$ . A deformation  $\phi : \mathcal{C} \rightarrow (B, b_0), \sigma_i : B \rightarrow \mathcal{C}$  such that  $\chi : (C; p_1, \dots, p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$  of  $(C; p_1, \dots, p_n)$  is said to be a Kuranishi family for  $(C; p_1, \dots, p_n)$  if it satisfies the following condition:

**(Condition K).** For any deformation  $\psi : \mathcal{D} \rightarrow (E, e_0)$  of  $(C; p_1, \dots, p_n)$  and for any sufficiently small connected neighborhood  $U$  of  $e_0$ , there is a unique morphism of deformations of  $n$ -pointed curves

$$\begin{array}{ccc} \mathcal{D}|_U & \xrightarrow{F} & \mathcal{C} \\ \downarrow \psi|_U & & \downarrow \phi \\ (U, e_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

**Remark 8.3.2.** In the algebraic case, the neighborhood  $U$  of  $e_0$  taken étale locally. Actually this is the same as all analytic local and étale local here.

**Remark 8.3.3** (Versal). If we just let the deformation satisfies (condition K) except for uniqueness and the Kodaira-Spencer map at the central fiber be an isomorphism, then we call it a versal deformation.

We will show, the Kuranishi family for  $(C; p_1, \dots, p_n)$  exists if and only if  $(C; p_1, \dots, p_n)$  is stable, at next two sections.

**Corollary 8.3.4.** *The Kodaira-Spencer map of a Kuranishi family at the base point is an isomorphism.*

*Proof.* This is trivial as the family  $\mathcal{C}_\varepsilon \rightarrow \text{Spec} \mathbb{C}[\varepsilon]$  just has and has unique map to  $\phi : \mathcal{C} \rightarrow (B, b_0)$ . This defines a bijection between  $T_{B, b_0}$  and  $\text{Def}(C) = \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$  via  $\kappa_{B, b_0}$ .  $\square$

**Corollary 8.3.5.** *Let there be given a deformation of a stable  $n$ -pointed curve  $(C; p_1, \dots, p_n)$  over the pointed analytic space  $(E, e_0)$ . Suppose that its Kodaira-Spencer map at  $e_0$  is an isomorphism and that  $E$  is smooth at  $e_0$ . Then the deformation is a Kuranishi family for  $(C; p_1, \dots, p_n)$ .*

*Proof.* To add.  $\square$

**Corollary 8.3.6.** *The base of the Kuranishi family of a stable  $n$ -pointed curve  $(C; p_1, \dots, p_n)$  of genus  $g$  is smooth of dimension  $3g - 3 + n$ .*

*Proof.* By the previous corollary and Theorem 8.2.6 and the uniqueness of the Kuranishi family by the universal property.  $\square$

**Corollary 8.3.7.** *Let  $X \rightarrow S$  be a family of stable  $n$ -pointed curves, and  $s_0$  a point of  $S$ . If  $X \rightarrow S$  is a Kuranishi family for  $X_{s_0}$ , then it is a Kuranishi family for  $X_s$ , for all  $s$  in an open neighborhood  $U$  of  $s_0$ .*

*Proof.* From the previous results that  $X \rightarrow S$  is Kuranishi for  $X_s$  if and only if  $s$  is a smooth point of  $S$  and the Kodaira-Spencer map at  $s$  is an isomorphism. The first of these conditions is clearly open.

Since the dimension of  $\text{Ext}^1(\Omega_{X_s}^1, \mathcal{O}_{X_s}(-\sum \sigma_i(s)))$  is independent of  $s$ , the second condition translates into a rank condition for a map between vector bundles and hence is open and We win.  $\square$

## 8.4 The Hilbert scheme of $\nu$ -canonical curves

For any stable  $n$ -pointed genus  $g$  curve  $(C; p_i)$ , if we let  $D = \sum_i p_i$ , then by Proposition 7.3.3 that for all  $\nu \geq 3$ , the  $\nu$ -log-canonical bundle  $\omega_C(D)^{\otimes \nu}$  is very ample and embeds  $C$  into  $\mathbb{P}^{N-1}$  where  $N = (2\nu - 1)(g - 1) + \nu n$ . Let  $P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$ , we consider the Hilbert scheme  $\underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$ .

By Proposition 7.3.5, we get the nonempty subset  $U \subset \underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$  parameterizing connected  $n$ -pointed nodal curves is open. Let the  $(\pi : \mathcal{Y} \rightarrow U, \sigma_i)$  be the restriction of the universal family. As the general points of  $U$  does not correspond to an  $n$ -pointed curve embedded by the  $\nu$ -fold log-canonical sheaf, we need to define a new subscheme.

**Definition 8.4.1.** *Let  $F = (\pi^* \mathcal{O}_{\mathbb{P}^{N-1}}(1))^{-1} \otimes \omega_\pi(\sum_i \sigma_i)^{\otimes \nu}$ . We define  $H_{\nu, g, n} \subset U \subset \underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$  as a subscheme by*

$$H_{\nu, g, n}(X) := \left\{ \alpha : X \rightarrow U \left| \begin{array}{c} \text{correspond to} \\ \begin{array}{ccc} \mathcal{Y} \times_U X & \xrightarrow{\beta} & \mathcal{Y} \\ \downarrow \eta & & \downarrow \pi \\ X & \xrightarrow{\alpha} & U \end{array} \\ \text{such} \\ \text{that } \beta^* F \cong \eta^* G \text{ for some } G \in \text{Pic}(X) \end{array} \right. \right\}.$$



We call  $H_{\nu,g,n}$  as the Hilbert scheme of  $\nu$ -log-canonically embedded, stable,  $n$ -pointed, genus  $g$  curves.

**Lemma 8.4.2.** *Let  $h = (C \subset \mathbb{P}^m; p_1, \dots, p_n)$  be a nodal curve where  $p_1, \dots, p_n$  be distinct smooth points of  $C$  and  $D = \sum_i p_i$ . Let  $H$  be the Hilbert scheme parameterizing the  $(n+1)$ -tuples  $(Y; q_1, \dots, q_n)$ , where  $Y$  is a subscheme of  $C \subset \mathbb{P}^m$  and  $q_1, \dots, q_n$  points on it, then we have the exact sequence*

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}_C}(\Omega_C^1, \mathcal{O}_C(-D)) &\rightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}^m}}(\Omega_{\mathbb{P}^m}^1, \mathcal{O}_C) \\ &\rightarrow T_h H \xrightarrow{\beta} \text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-D)) \end{aligned}$$

where  $\beta$  is just the Kodaira-Spencer map at  $h$  associated to the universal family over  $H$ .

*Proof.* Consider

$$\begin{array}{ccccccc} \mathcal{D}^* & & \mathcal{C}^* & & \mathcal{B}^* & & \mathcal{A}^* \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathcal{O}_C & \longrightarrow & \mathcal{I}_C / \mathcal{I}_C^2 & \longrightarrow & \Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_C & \longrightarrow & \Omega_C^1 \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_D & \longrightarrow & \mathcal{I}_D / \mathcal{I}_D^2 & \longrightarrow & \Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_D & \longrightarrow & 0 \end{array}$$

Hence we get

$$0 \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*) \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{B}^*, \mathcal{D}^*) \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \rightarrow \text{Ext}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*).$$

As  $\mathcal{A}^* \rightarrow \mathcal{D}^*$  is equivalent to  $\Omega_C^1 \rightarrow \ker(\mathcal{O}_C \rightarrow \mathcal{O}_D) = \mathcal{O}_C(-D)$ , hence  $\text{Hom}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*) = \text{Hom}(\Omega_C^1, \mathcal{O}_C(-D))$ . As  $\mathcal{B}^* \rightarrow \mathcal{D}^*$  determined by  $\mathcal{B}^0 \rightarrow \mathcal{D}^0$ , we get

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{B}^*, \mathcal{D}^*) = \text{Hom}_{\mathcal{O}_C}(\Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_C, \mathcal{O}_C) = \text{Hom}_{\mathcal{O}_{\mathbb{P}^m}}(\Omega_{\mathbb{P}^m}^1, \mathcal{O}_C).$$

It is trivial that  $\text{Hom}_{\mathcal{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \cong T_h H$ . The final term is actually the isomorphism classes of first-order deformations of  $h$ , hence is  $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-D))$  (see [2] XI.(3.11)). Hence we win.  $\square$

**Theorem 8.4.3.** *Let  $2g - 2 + n > 0$  and  $\nu \geq 3$  and  $N = (2\nu - 1)(g - 1) + \nu n$ . Then  $H_{\nu,g,n}$  defined as above satisfied the following statements.*

(i) *Let  $h = (C; p_1, \dots, p_n)$  be a stable curve in  $\mathbb{P}^{N-1}$  embedded by the  $\nu$ -fold log-canonical system and  $D = \sum_i p_i$ . Then we have the exact sequence*

$$0 \rightarrow H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C) \rightarrow T_h(H_{\nu,g,n}) \xrightarrow{\lambda} \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow 0$$

where  $\lambda$  is the Kodaira-Spencer map at  $h$  of the universal family on  $H_{\nu,g,n}$ . In particular,

$$\dim T_h H_{\nu,g,n} = 3g - 3 + n + N^2 - 1;$$

(ii)  $H_{\nu,g,n}$  is smooth and quasi-projective of dimension  $3g - 3 + n + N^2 - 1$ .

*Sketch.* (i) By Euler sequence and Lemma 8.4.2, we have

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(\Omega_C^1, \mathcal{O}_C(-D)) & & & & \\
 & & \downarrow & & & & \\
 0 & \rightarrow & H^0(C, \mathcal{O}_C) \rightarrow H^0(C, \mathcal{O}_C(1))^{\oplus N} \rightarrow & \text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C) & \xrightarrow{\delta} & H^1(C, \mathcal{O}_C) & \\
 & & & \downarrow \gamma & & & \\
 & & & T_h H & \xrightarrow{\beta} & \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) & 
 \end{array}$$

Now we will analyze several groups and morphisms above.

- **The map  $\beta$  associates to every first-order embedded deformation of  $h$ .** (Trivial)
- **The elements of  $\text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C)$  correspond to fiber space maps  $j : C \times \text{Spec} \mathbb{C}[\varepsilon] \rightarrow \mathbb{P}^{N-1} \times \text{Spec} \mathbb{C}[\varepsilon]$ .** (Omitted, see [2] page 200)
- **The map  $\delta$  associates to any such object the infinitesimal deformation of line bundles on  $C$  given by  $j^*(\mathcal{O}_{\mathbb{P}^{N-1}}^1(1) \otimes \mathcal{O}_{\text{Spec} \mathbb{C}[\varepsilon]}) \otimes (\mathcal{O}_C^1(1) \otimes \mathcal{O}_{\text{Spec} \mathbb{C}[\varepsilon]})^{-1}$ .** (Omitted, see [2] page 201)
- **The elements of  $H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$  is the tangent space to  $\text{PGL}(N)$ .** (Omitted)

Let  $v = \Gamma(\alpha) \in T_h H$  tangent to  $H_{\nu, g, n}$  where  $\alpha \in \text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C)$ . Hence  $v$  from a fiber space map  $j : C \times \text{Spec} \mathbb{C}[\varepsilon] \rightarrow \mathbb{P}^{N-1} \times \text{Spec} \mathbb{C}[\varepsilon]$  such that

$$j^*(\mathcal{O}_{\mathbb{P}^{N-1}}^1(1) \otimes \mathcal{O}_{\text{Spec} \mathbb{C}[\varepsilon]}) = \omega_C(D)^{\otimes \nu} \otimes \mathcal{O}_{\text{Spec} \mathbb{C}[\varepsilon]}.$$

Then  $\delta(\alpha) = 0$  and hence  $\alpha \in H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$ . Conversely we find that the image of  $H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$  in  $T_h H$  contained in  $T_h H_{\nu, g, n}$ . By Proposition 8.2.9 we get  $\text{Hom}(\Omega_C^1, \mathcal{O}_C(-D)) = 0$ , hence we have

$$0 \rightarrow H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C) \rightarrow T_h(H_{\nu, g, n}) \xrightarrow{\lambda} \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)).$$

Actually  $\lambda$  is surjective since any infinitesimal deformations of  $h$  can be embedded via the  $\nu$ -fold log-canonical system. Hence we win.

(ii) By the basic theory of Hilbert schemes,  $H_{\nu, g, n}$  is quasi-projective by the trivial reason. We now will show that  $H_{\nu, g, n}$  is smooth of dimension  $3g - 3 + n + N^2 - 1$ . By (i) we get  $\dim T_h H_{\nu, g, n} = 3g - 3 + n + N^2 - 1$ , hence  $\dim H_{\nu, g, n} \leq 3g - 3 + n + N^2 - 1$ . If we have showed that  $\dim H_{\nu, g, n} \geq 3g - 3 + n + N^2 - 1$ , then well done.

Here we just give a sketch, details see [2] Proposition XI.5.12. By Theorem 8.2.6, we get a  $(3g - 3 + n)$ -dimensional deformation  $\phi : \mathcal{C} \rightarrow (B, b_0)$ . Let  $\mathcal{C}_b = \phi^{-1}(b)$  and  $D_b = \sum_i \sigma_i(b)$ . Consider a principle  $\text{PGL}(N)$ -bundle over  $B$  as

$$\mathcal{B} := \left\{ (b, F) \left| \begin{array}{l} b \in B \text{ and } F \text{ a basis of } H^0(\mathcal{C}_b, \omega_{\mathcal{C}_b}(D_b)^{\otimes \nu}), \\ \text{modulo homotheties} \end{array} \right. \right\}.$$

Take  $F_0$  correspond to  $C \subset \mathbb{P}^{N-1}$  and consider the family

$$\mathcal{X} := \mathcal{B} \times_B \mathcal{C} \xrightarrow{\psi} \mathcal{B}, \tau_i : \mathcal{B} \rightarrow \mathcal{X}.$$

Via some projective frame of  $\psi_*(\omega_{\mathcal{X}/\mathcal{B}}(\sum \tau_i)^{\otimes \nu})$ , we have  $\mathcal{X} \rightarrow \mathbb{P}^{N-1} \times \mathcal{B}$ , which induce  $\xi : \mathcal{B} \rightarrow$

$H_{\nu,g,n}$ . Hence we have

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_e(G) & \longrightarrow & T_{(b_0, F_0)}\mathcal{B} & \longrightarrow & T_{b_0}B \longrightarrow 0 \\
& & \parallel & & \downarrow d\xi & & \downarrow \rho \\
0 & \longrightarrow & H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C) & \longrightarrow & T_h(H_{\nu,g,n}) & \longrightarrow & \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \longrightarrow 0
\end{array}$$

where  $\rho$  is Kodaira-Spencer map. As  $\rho$  is an isomorphism, we have  $d\xi$  is also an isomorphism. Hence locally  $\xi$  is a local isomorphism at  $(b_0, F_0)$ . As  $\dim \mathcal{B} = 3g - 3 + n + N^2 - 1$ , well done.  $\square$

## 8.5 Construction of Kuranishi families

Let  $\nu \geq 3$  and  $(C; p_1, \dots, p_n) \subset \mathbb{P}^{N-1}$  be a stable  $n$ -pointed genus  $g$  curve where  $N = (2\nu - 1)(g - 1) + \nu n$ , via  $\nu$ -fold log-canonical system. We consider it as  $x_0 \in H_{\nu,g,n}$ . Fix the universal family  $\mathcal{Y} \rightarrow H_{\nu,g,n}$  with sections  $\sigma_i : H \rightarrow \mathcal{Y}$ . Let

$$H_{\nu,g,n} \subset \text{Hilb}_{\mathbb{P}^{N-1}}^{P_\nu} \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^M \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^K$$

acted by  $\mathbb{G} = \text{PGL}(N) \subset \text{PGL}(K + 1)$  and let  $\text{Aut}(C; p_i) = \mathbb{G}_{x_0} \subset \mathbb{G} = \text{PGL}(N)$  be the stabilizer of  $x_0$ .

Let the orbit  $O(x_0) \subset H_{\nu,g,n}$  of  $x_0$  under  $\mathbb{G}$ , which is a smooth subvariety of dimension  $N^2 - 1$ . (Here is not important. But we need to read here, and to add.)

**Lemma 8.5.1.**

*Proof.*

$\square$

**Theorem 8.5.2.** *There is a locally closed  $(3g - 3 + n)$ -dimensional smooth subscheme  $X \subset H_{\nu,g,n}$  including  $x_0$  such that the restriction of the universal family of  $H_{\nu,g,n}$  over  $X$  is a Kuranishi family for all of its fibers.*

*In addition, one can choose an  $X$  with the following properties:*

- (i)  $X$  is affine and  $\mathbb{G}_{x_0}$ -invariant;
- (ii) For any  $y \in X$ , we have  $\mathbb{G}_y \subset \mathbb{G}_{x_0}$ ;
- (iii) For any  $y \in X$ , there is a  $\mathbb{G}_y$ -invariant neighborhood  $U \subset X$  of  $y$  such that  $\mathbb{G}_y = \{\gamma \in \mathbb{G} : \gamma(U) \cap U \neq \emptyset\}$  in the analytic topology.

*Proof.* See [2] Theorem XI.6.5. To add.

$\square$

Hence we get a Kuranishi family  $(\pi : \mathcal{C} \rightarrow (X, x_0), \sigma_i)$ .

**Definition 8.5.3** (Standard algebraic Kuranishi family). *Let  $(C; p_1, \dots, p_n)$  be a stable  $n$ -pointed genus  $g$  curve with  $G = \text{Aut}(C; p_i)$ . Let  $(\pi : \mathcal{C} \rightarrow (X, x_0), \sigma_i)$  be the Kuranishi family in Theorem 8.5.2 and it is called a standard algebraic Kuranishi family if the following conditions are satisfied.*

- (a)  $X$  is affine and the family is a Kuranishi family for all of its fibers;
- (b) The action of  $G_{x_0}$  on the central fiber extends to compatible actions on  $\mathcal{C}$  and  $X$ ;
- (c) For any  $y \in X$  we have  $G_y := \text{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \text{stab}_{G_{x_0}}(y)$ ;
- (d) For any  $y \in X$ , there is a  $G_y$ -invariant analytic neighborhood  $U$  of  $y$  in  $X$  such that any isomorphism (of  $n$ -pointed curves) between fibers over  $U$  is induced by an element of  $G_y$ .

**Definition 8.5.4** (Standard Kuranishi family). *Let  $(C; p_1, \dots, p_n)$  be a stable  $n$ -pointed genus  $g$  curve with  $G = \text{Aut}(C; p_i)$ . We will say a Kuranishi family  $\mathcal{X} \rightarrow (B, b_0), \tau_i : B \rightarrow \mathcal{X}$  of  $(C; p_1, \dots, p_n)$  is called a standard Kuranishi family if the following conditions are satisfied.*

- (a)  *$B$  is a connected complex manifold and the family is a Kuranishi family at every points of  $B$ ;*
- (b) *The action of  $G$  on the central fiber extends to compatible actions on  $\mathcal{X}$  and  $B$ ;*
- (c) *Any isomorphism (of  $n$ -pointed curves) between fibers is induced by an element of  $G$ .*

**Remark 8.5.5.** *In fact, given any Kuranishi family, there is a neighborhood of the base point such that the restriction is standard. By the uniqueness of the Kuranishi family, it suffices to notice that this is true for a standard algebraic Kuranishi family. By Theorem 8.5.2 and we win.*

**Corollary 8.5.6.** *Any (maybe not stable) nodal curves  $(C; p_1, \dots, p_n)$  has a versal deformation (is unique up to an isomorphism, which however need not be unique).*

*Proof.* Adding some smooth marked points such that it becomes a stabel curve. Then taking the Kuranishi family of it and ignore the added marked points.  $\square$

**Corollary 8.5.7.** *Any family of nodal curves can be locally embedded in a family of nodal curves with a reduced, or even smooth, base.*

*Proof.* For any family of nodal curves  $\eta : X \rightarrow S$  and let  $s_0 \in S$ . By the previous corollary we can get a versal deformation  $\pi : \mathcal{X} \rightarrow (B, b_0)$  of  $\eta^{-1}(s_0)$ . After shrinking  $S$  (étale locally), we have a closed immersion  $S \hookrightarrow T$  where  $T$  is smooth and a cartesian

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow \eta & & \downarrow \pi \\ S & \xrightarrow{\alpha} & B \end{array}$$

with  $b_0 = \alpha(s_0)$ . Hence we get cartesians

$$\begin{array}{ccccc} X & \xrightarrow{(\eta, \beta)} & S \times \mathcal{X} & \longrightarrow & T \times \mathcal{X} \\ \downarrow \eta & & \downarrow (\text{id}_S, \pi) & & \downarrow (\text{id}_T, \pi) \\ S & \xrightarrow{(\text{id}_S, \alpha)} & S \times B & \longrightarrow & T \times B \end{array}$$

Clearly,  $T \times B$  is smooth, and  $S \rightarrow T \times B$  is a closed immersion.  $\square$

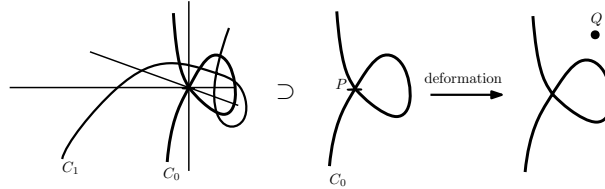
# Chapter 9

## The stack of all curves

### 9.1 Families of all arbitrary curves

**Definition 9.1.1.** Here we redefine a curve over  $k$  is a scheme  $C$  of finite type over  $k$  of dimension 1 (rather than pure dimension 1). The genus of  $C$  is defined as  $g(C) = 1 - \chi(C, \mathcal{O}_C)$ .

**Remark 9.1.2.** Why we not allow pure dimension 1? Since they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For example in [25] Example III.9.8.4, a flat family of rational curves defined by  $\mathbb{P}^1 \rightarrow \mathbb{P}^3$  via  $[x : y] \mapsto [x^3 : x^2y : xy^2 : ty^3]$  for any  $t \neq 0$ . As  $t \rightarrow 0$ , we may get a singular non-reduced curve  $C_0$  with an embedded point at that node, but  $C_0$  can deform to the disjoint union of a plane nodal curve and a point in  $\mathbb{P}^3$ .



**Definition 9.1.3.** (i) A family of curves over a scheme  $S$  is a flat, proper and finitely presented morphism  $C \rightarrow S$  of algebraic spaces such that every fiber is a curve.

(ii) A family of  $n$ -pointed curves is a family of curves  $C \rightarrow S$  together with  $n$  sections  $\sigma_1, \dots, \sigma_n : S \rightarrow C$  (with no condition on whether they are distinct or land in the relative smooth locus of  $C$  over  $S$ ).

**Remark 9.1.4.** (i) When we consider a family of stable curve, since the relative dualizing sheaf is ample, we can get it is projective, hence must be a scheme;

(ii) There are some examples such that  $C$  are not a scheme.

**Proposition 9.1.5.** If  $C \rightarrow S$  is a family of curves over  $S$ , there exists an étale cover  $S' \rightarrow S$  such that  $C_{S'} \rightarrow S'$  is projective.

*Sketch-Local to global.* Consider cartesian

$$\begin{array}{ccccccc}
 C_0 := C_s & \hookrightarrow & C_1 & \hookrightarrow & \cdots & \hookrightarrow & \widehat{C} \hookrightarrow C \\
 \downarrow & & \downarrow & & & & \downarrow \\
 S_0 := \mathrm{Spec} \kappa(s) & \hookrightarrow & S_1 := \mathrm{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^2 & \hookrightarrow & \cdots & \hookrightarrow & \widehat{S} := \mathrm{Spec} \widehat{\mathcal{O}}_{S,s} \longrightarrow S
 \end{array}$$

**Step 1.**  $C_0 \rightarrow \mathrm{Spec} \kappa(s)$ . By St 0ADD, every separated algebraic space of dimension one is a scheme. Any one-dimensional proper  $\kappa(s)$ -scheme is projective by St 0A26. In particular we get a ample line bundle  $L_0$  on  $C_0$ .

**Step 2.**  $C_n \rightarrow S_n$ . The obstruction to deforming a line bundle  $L_n$  on  $C_n$  to  $L_{n+1}$  on  $C_{n+1}$  lives in  $H^2(C_0, \mathcal{O}_{C_0})$  and thus vanishes as  $\dim C_0 = 1$ . Thus there exists a compatible sequence of line bundles  $L_n$  on  $C_n$ . Since ampleness is an open condition in families and  $L_0$  is ample,  $L_n$  is also ample.

**Step 3.**  $\widehat{C} \rightarrow \widehat{S}$  with  $\widehat{S}$  noetherian. Use Grothendieck's Existence Theorem we get an equivalence  $\mathrm{Coh}(\widehat{C}) \rightarrow \varprojlim \mathrm{Coh}(C_n)$ . As  $\widehat{C} \rightarrow \widehat{S}$  is proper, then by Chow's lemma there exists a projective birational morphism  $C' \rightarrow \widehat{C}$  of algebraic spaces such that  $C' \rightarrow S$  is projective. This allows one to reduce Grothendieck's Existence Theorem for  $\widehat{C} \rightarrow \widehat{S}$  to  $C' \rightarrow \widehat{S}$  using devissage. As a result, using again that ampleness is an open condition in families we can extend the sequence of line bundle  $L_n$  to a line bundle  $\widehat{L}$  on  $\widehat{C}$  which is ample.

**Step 4.**  $S$  is of finite type over  $\mathbb{Z}$ . For every closed point  $s \in S$ , apply Artin Approximation to the functor

$$(Sch/S) \rightarrow (Sets), (T \rightarrow S) \mapsto \mathrm{Pic}(C_T)$$

to obtain an étale neighborhood  $(S', s') \rightarrow (S, s)$  of  $s$  and a line bundle  $L'$  on  $C_{S'}$  extending  $L_0$ . By openness of ampleness, we can replace  $S'$  with an open neighborhood of  $s'$  such that  $L'$  is relatively ample over  $S'$ .

**Step 5.** General  $S$ . Use noetherian approximation. □

## 9.2 Algebraicity of the stack of all curves

**Definition 9.2.1.** Let  $\mathcal{M}_{g,n}^{\mathrm{all}}$  denote the category over  $Sch_{\mathrm{et}}$  whose objects over  $S$  consists of families of curves  $C \rightarrow S$  and  $n$  sections  $\sigma_i : S \rightarrow C$ . A morphism  $(C' \rightarrow S', \sigma'_i) \rightarrow (C \rightarrow S, \sigma_i)$  is the data of the cartesian

$$\begin{array}{ccc}
 C' & \xrightarrow{g} & C \\
 \sigma'_i \uparrow \downarrow & & \sigma_i \uparrow \downarrow \\
 S' & \xrightarrow{f} & S
 \end{array}$$

with  $g \circ \sigma'_i \rightarrow \sigma_i \circ f$ .

**Lemma 9.2.2.**  $\mathcal{M}_{g,n}^{\mathrm{all}}$  is a stack over  $Sch_{\mathrm{et}}$ .

*Proof.* Handle  $n = 0$ . Let  $S' \rightarrow S$  be an étale cover with  $C' \rightarrow S'$ . And  $\alpha : p_1^* C' \rightarrow p_2^* C'$  is an isomorphism over  $S' \times_S S'$  satisfying the cocycle condition. The quotient of the étale equivalence relation

$$\begin{array}{ccccc}
 R := p_1^* C' & \xrightarrow[p_2 \circ \alpha]{p_1} & C' & \dashrightarrow & C := C'/R \\
 \downarrow & & \downarrow & & \downarrow \\
 S' \times_S S' & \xrightarrow[p_2]{p_1} & S' & \longrightarrow & S
 \end{array}$$

Well done.  $\square$

**Lemma 9.2.3.**  $\Delta : \mathcal{M}_{g,n}^{\text{all}} \rightarrow \mathcal{M}_{g,n}^{\text{all}} \times \mathcal{M}_{g,n}^{\text{all}}$  is representable.

*Proof.* Handle  $n = 0$ . Consider the cartesian

$$\begin{array}{ccc} \text{Isom}_T(C_1, C_2) & \longrightarrow & T \\ \downarrow & & \downarrow (C_1, C_2) \\ \mathcal{M}_{g,n}^{\text{all}} & \xrightarrow{\Delta} & \mathcal{M}_{g,n}^{\text{all}} \times \mathcal{M}_{g,n}^{\text{all}} \end{array}$$

We need to show  $\text{Isom}_T(C_1, C_2)$  is an algebraic space. By Proposition 9.1.5, there exists an étale cover  $T' \rightarrow T$  such that  $C_{i,T'} \rightarrow T'$  is projective. Hence we may let  $C_1, C_2$  are projective over  $T$ . Indeed, as

$$\text{Isom}_T(C_1, C_2) \times_T T' = \text{Isom}_{T'}(C_{1,T'}, C_{2,T'}),$$

we get  $\text{Isom}_{T'}(C_{1,T'}, C_{2,T'}) \rightarrow \text{Isom}_T(C_1, C_2)$  is representable, surjective and étale. Hence if  $\text{Isom}_{T'}(C_{1,T'}, C_{2,T'})$  is an algebraic space, so is  $\text{Isom}_T(C_1, C_2)$ .

**Fact.** (St 05XD) If  $f : X \rightarrow Y$  are  $T$ -morphism such that  $X, Y$  are proper, flat and locally of finite presentation over  $T$ , then for any  $U \rightarrow T$  such that  $X_U \cong Y_U$  if and only if  $U \rightarrow T$  factor through an open subscheme  $T_0 \subset T$ .

Now we get the inclusions

$$\text{Isom}_T(C_1, C_2) \subset \text{Mor}_T(C_1, C_2) \subset \text{Hilb}(C_1 \times_T C_2 / T)$$

where the second inclusion is  $(g : C_1 \rightarrow C_2) \mapsto (\Gamma_g : C_1 \rightarrow C_1 \times_T C_2)$ . The first inclusion is representable open immersion by the above fact. The second inclusion, we find that a subspace  $[Z \subset C_1 \times_T C_2] \in \text{Hilb}(C_1 \times_T C_2 / T)$  is in the image of the inclusion if and only if  $Z \rightarrow C_1 \times_T C_2 \rightarrow C_1$  is an isomorphism (and similarly for other valued points). Therefore by the above fact we win.  $\square$

**Theorem 9.2.4.**  $\mathcal{M}_{g,n}^{\text{all}}$  is an algebraic stack locally of finite type over  $\mathbb{Z}$ .

*Sketch. Step 1. Reduce to  $n = 0$ .* Since  $\mathcal{M}_{g,n+1}^{\text{all}}$  is the universal family over  $\mathcal{M}_{g,n}^{\text{all}}$ , we can prove the conclusion at the case  $\mathcal{M}_g^{\text{all}}$ . (Why?)

**Step 2. Look for possible smooth cover  $H'$  of  $\mathcal{M}_g^{\text{all}}$ .** Let  $C_0$  be any projective curves  $C_0$  over  $k$ . Choosing an embedding  $C_0 \subset \mathbb{P}_k^N$  such that  $h^1(C_0, \mathcal{O}(1)) = 0$  by Serre's vanishing theorem. Let  $P(t)$  be its Hilbert polynomial. Let  $H := \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^N / \mathbb{Z}}^P$  be the Hilbert scheme which is projective over  $\mathbb{Z}$ . Consider the universal family

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}_H^N \\ \downarrow & \swarrow & \\ H & & \end{array}$$

there is a point  $h_0 \in H(k)$  such that  $\mathcal{C}_{h_0} = C_0$ . By Review A.1.1 we can find an open neighborhood  $H' \subset H$  of  $h_0$  such that for any  $s \in H'$  we have  $h^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$ . Now consider

$$H' \rightarrow \mathcal{M}_g^{\text{all}}, [C \hookrightarrow \mathbb{P}^N] \mapsto [C],$$

and by the representability of the diagonal, this map is representable as  $H'$  is a scheme.

**Step 3. Show that  $H' \rightarrow \mathcal{M}_g^{all}$  is smooth.** Using Infinitesimal Lifting Criterion such that for all surjections  $A \rightarrow A_0$  of artinian local rings with residue field  $k$  such that  $k = \ker(A \rightarrow A_0)$  and for all diagrams

$$\begin{array}{ccc}
 & \text{Spec } k & \\
 \swarrow & & \searrow \\
 \text{Spec } A_0 & \xrightarrow{[\mathcal{C}_0 \subset \mathbb{P}_{A_0}^N]} & H' \\
 \downarrow & \nearrow [\mathcal{C} \subset \mathbb{P}_A^N] & \downarrow \\
 \text{Spec } A & \xrightarrow{\mathcal{C}} & \mathcal{M}_g^{all}
 \end{array}$$

(Note: The top arrow from Spec k to Spec A\_0 is labeled  $[C \subset \mathbb{P}_k^N]$  in the original image.)

We need to find that dotted arrow. This diagram is equivalent to

$$\begin{array}{ccccc}
 & & \mathbb{P}_k^N & & \mathbb{P}_{A_0}^N & & \mathbb{P}_A^N \\
 & \nearrow & & \nearrow & & \nearrow & \\
 C & \xrightarrow{\quad} & C_0 & \xrightarrow{\quad} & C & \xrightarrow{\quad} & C \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spec } k & \hookrightarrow & \text{Spec } A_0 & \hookrightarrow & \text{Spec } A & & 
 \end{array}$$

For simplifying, we let  $C$  is of locally complete intersection (general case see [21] and [20]). Let  $\mathcal{J}$  be the ideal sheaf of  $C \rightarrow \mathbb{P}_k^N$  generated by regular sequence locally and that  $\mathcal{J}/\mathcal{J}^2$  is a vector bundle on  $C$  with

$$0 \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \Omega_{\mathbb{P}_k^N}|_C \rightarrow \Omega_C \rightarrow 0.$$

By long exact sequence we get

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C) \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_{\mathbb{P}_k^N}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}_k^N}|_C).$$

Consider the canonical sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(1)^{\oplus N+1} \rightarrow T_{\mathbb{P}^N}|_C \rightarrow 0.$$

Since  $H^2(C, \mathcal{O}_C) = 0$  and  $H^1(C, \mathcal{O}_C(1)) = 0$  by  $[C] \in H'$  we get  $H^1(C, T_{\mathbb{P}_k^N}|_C) = 0$ . Hence we get a surjection

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C) \twoheadrightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C).$$

Use some deformation theory (**But I don't know! May be use something about cotangent complex which is the reason we let  $C$  is of locally complete intersection!**) we get  $\text{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C)$  classifies embedded deformations of  $\mathcal{C}_0 \rightarrow \mathbb{P}_{A_0}^N$  to  $\mathcal{C}' \rightarrow \mathbb{P}_A^N$  and  $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C)$  classifies deformations of  $\mathcal{C}_0$  over  $A_0$  to  $\mathcal{C}'$  over  $A$ . As the map is  $[\mathcal{C}' \rightarrow \mathbb{P}_A^N] \mapsto \mathcal{C}'$  and is surjective, we win.  $\square$

### 9.3 Algebraicity of several stacks and boundedness of stable curves

**Proposition 9.3.1** (Several stacks). *We have inclusions*

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{ss} \subset \mathcal{M}_{g,n}^{\text{pre}} \subset \mathcal{M}_{g,n}^{\leq \text{nodal}} \subset \mathcal{M}_{g,n}^{all}$$

of prestacks. Then all of these are open substacks, hence all of these are algebraic stacks locally of finite type over  $\mathbb{Z}$ .



*Proof.* • By Theorem 9.2.4,  $\mathcal{M}_{g,n}^{all}$  is an algebraic stack locally of finite type over  $\mathbb{Z}$ .

- $\mathcal{M}_{g,n}^{\leq nodal} \subset \mathcal{M}_{g,n}^{all}$  is an open substack. Actually by Corollary 6.4.3 we get the nodal locus is open when  $C$  is a scheme. In general for an étale cover  $g : C' \rightarrow C$  by a scheme, we find that a point  $p \in C'$  is a node in its fiber if and only if  $g(p)$  is a node in its fiber. We win.
- $\mathcal{M}_{g,n}^{pre} \subset \mathcal{M}_{g,n}^{\leq nodal}$  is an open substack. This is because for a family  $(C \rightarrow S, \{\sigma_i\})$  of nodal curves, the locus  $\{s \in S : \sigma_i(s) \text{ are disjoint and smooth}\}$  is open.
- $\mathcal{M}_{g,n}^{ss} \subset \mathcal{M}_{g,n}^{pre}$  is an open substack. This is because the nef locus is open (**But I don't know the relation between nefness and semistable!**).
- $\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{ss}$  is an open substack. Indeed the stable locus is open by Proposition 7.3.4.
- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$  is an open substack. Indeed this is by the fact that smooth locus is open.  $\square$

**Proposition 9.3.2.**  $\overline{\mathcal{M}}_{g,n}$  is a quasi-compact smooth Deligne-Mumford stack of dimension  $3g - 3 + n$  over  $\mathbb{Z}$ .

*Proof.* •  $\overline{\mathcal{M}}_{g,n}$  is quasi-compact. Let  $(C, p_1, \dots, p_n)$  be a  $n$ -pointed stable curve. By Theorem 7.2.1, we get  $(\omega_C(p_1 + \dots + p_n))^{\otimes 3}$  is very ample, we get  $C \hookrightarrow \mathbb{P}^N$  with Hilbert polynomial  $P(t)$ . This is independent of  $C$ . Hence consider closed subscheme  $H \subset \underline{\text{Hilb}}_{\mathbb{P}^N/\mathbb{Z}}^P \times (\mathbb{P}^N)^n$  of embedded curve and  $n$  points  $(C \hookrightarrow \mathbb{P}^N, p_i \in C)$ . Consider a forgetful functor

$$H \rightarrow \mathcal{M}_{g,n}^{all}, (C \hookrightarrow \mathbb{P}^N, p_i \in C) \mapsto (C, \{p_i\}).$$

Then the image of  $|H| \rightarrow |\mathcal{M}_{g,n}^{all}|$  contains  $\overline{\mathcal{M}}_{g,n}$ . As  $\underline{\text{Hilb}}_{\mathbb{P}^N/\mathbb{Z}}^P$  is projective, then  $H$  is quasi-compact. Hence  $\overline{\mathcal{M}}_{g,n}$  is quasi-compact.

- $\overline{\mathcal{M}}_{g,n}$  is Deligne-Mumford stack. By Proposition 8.2.9 for  $i = 0$  and Proposition 8.2.7 (a), we get a  $n$ -pointed stable curve  $(C, p_1, \dots, p_n)$  has no infinitesimal automorphisms, i.e. that the Lie algebra  $T_e \text{Aut}(C, p_1, \dots, p_n)$  is trivial. Since the automorphism group scheme  $\text{Aut}(C, p_1, \dots, p_n)$  is of finite type, this implies that  $\text{Aut}(C, p_1, \dots, p_n)$  is finite and discrete, hence  $\overline{\mathcal{M}}_{g,n}$  is a quasi-compact Deligne-Mumford stack.
- $\overline{\mathcal{M}}_{g,n}$  is smooth over  $\text{Spec} \mathbb{Z}$ . Proposition 8.2.9 for  $i = 2$  and Proposition 8.2.7 (c) implies that there are no obstructions to deforming  $C$ . As the algebraicity of  $\overline{\mathcal{M}}_{g,n}$ , this will allow us to invoke the Infinitesimal Lifting Criterion to establish that  $\overline{\mathcal{M}}_{g,n}$  is smooth over  $\text{Spec} \mathbb{Z}$ .
- $\overline{\mathcal{M}}_{g,n}$  has relative dimension  $3g - 3 + n$  over  $\text{Spec} \mathbb{Z}$ . Proposition 8.2.9 for  $i = 1$  and Proposition 8.2.7 (b) implies that isomorphism classes of deformations of  $(C, p_1, \dots, p_n)$ , it is identified with the Zariski tangent space of  $\overline{\mathcal{M}}_{g,n}$  at the point corresponding to  $(C, p_1, \dots, p_n)$ . This will allow us to conclude that  $\overline{\mathcal{M}}_{g,n}$  has relative dimension  $3g - 3 + n$  over  $\mathbb{Z}$ .  $\square$

## 9.4 The family of elliptic curves $\mathcal{M}_{1,1}$

**Proposition 9.4.1.**  $\mathcal{M}_1$  is not a stack.

*Proof.* See [34] Remark 8.4.15 for the References.  $\square$

**Remark 9.4.2.** If we let  $\mathcal{M}'_1$  as morphisms of algebraic spaces, then this will be a stack. This follows the Picard functor and the stack  $\mathcal{M}_{1,1}$ . In fact if we consider the universal elliptic curve  $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$ , then Picard functor gives  $\mathcal{M}'_1 \rightarrow \mathcal{M}_{1,1}$  which induce  $\mathcal{M}'_1 \cong B\mathcal{E}$ . We omitted here.

**Proposition 9.4.3.**  $\mathcal{M}_{1,1}$  is a smooth Deligne-Mumford stack.

*Proof.* By Proposition 9.3.1 we get that  $\mathcal{M}_{1,1}$  is an open substack of  $\overline{\mathcal{M}}_{1,1}$ . Hence it is a smooth Deligne-Mumford stack.  $\square$

**Proposition 9.4.4.**  $\mathcal{M}_{1,1}$  has a coarse moduli space  $M_{1,1} \cong \mathbb{A}^1$ .

*Proof.*  $\square$

## Chapter 10

# Stable reduction: why $\overline{\mathcal{M}}_{g,n}$ is proper?

In this section we will use the Valuative Criterion (Theorem C.1.3 (1)) to show that  $\overline{\mathcal{M}}_{g,n}$  is proper. The existence of extension is called stable reduction, which is our main theorem:

**Lemma 10.0.1.** *The diagonal of the stack  $\mathcal{M}_{g,n}^{all}$  is separated. In particular,  $\mathcal{M}_{g,n}^{all} \rightarrow \text{Spec}(\mathbb{Z})$  is quasi-separated.*

*Proof.* Omitted. See St 0DSQ. □

**Theorem 10.0.2** (Stable reduction). *Let  $R$  be a DVR with fraction field  $K$  and  $\Delta = \text{Spec}(R)$ ,  $\Delta^* = \text{Spec}(K)$ . If  $(\mathcal{C}^* \rightarrow \Delta^*, s_1^*, \dots, s_n^*)$  is a family of  $n$ -pointed stable curves of genus  $g$ , then there exists a finite cover  $\Delta' \rightarrow \Delta$  of spectrums of DVRs and a family  $(\mathcal{C}' \rightarrow \Delta', s_1', \dots, s_n')$  of stable curves extending  $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \rightarrow \Delta'^*$ . As*

given by

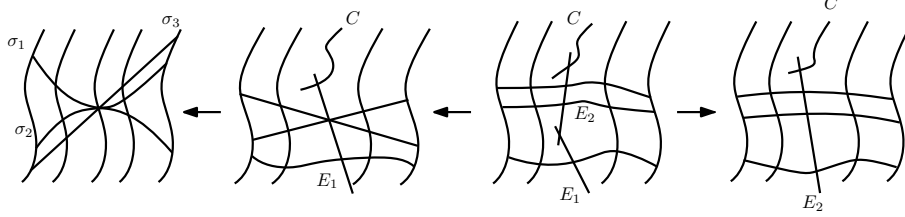
After proving this and the uniqueness, we can get the following conclusion:

**Theorem 10.0.3.** *If  $2g - 2 + n > 0$ , then  $\overline{\mathcal{M}}_{g,n}$  is a proper smooth Deligne-Mumford stack of dimension  $3g - 3 + n$  over  $\mathbb{Z}$ .*

By using the Keel-Mori Theorem, we get

**Corollary 10.0.4.** *If  $2g - 2 + n > 0$ , there exists a coarse moduli space  $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$  where  $\overline{M}_{g,n}$  is a proper algebraic space over  $\mathbb{Z}$ .*

**Example 10.0.5.** *Let  $\Delta = \text{Spec}(R)$  where  $R$  be a DVR with uniformizer  $t$ . Let  $C$  be a smooth curve and consider  $\mathcal{C} = C \times \Delta$  with sections  $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$  as following diagram. The first two arrows are blowing up and the third is contracting  $E_1$ .*



**Remark 10.0.6.** *Actually there are several methods to prove this. The first proof due to the original paper [9] by consider the Jacobians of curves and reduce the case into the semistable reduction of abelian varieties. Our method follows [22] by using some birational geometry of surfaces to prove the case of characteristic 0. There is another method can deal the positive or mixed characteristic case by [4], for this we refer St 0E8C.*

## 10.1 Proof of stable reduction in characteristic 0

**Lemma 10.1.1.** *Let  $R$  be a DVR with uniforming  $t$  and  $0 := (t)$ . Let  $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$  be a flat, proper and finitely presented morphisms such that each geometric fiber is a curve. Assume that  $\mathcal{C}$  is regular. Let  $p \in \mathcal{C}_0$ .*

(a) *If  $p$  is a smooth point in the reduced fiber  $(\mathcal{C}_0)_{\text{red}}$ . Show that after possibly an extension of DVRs, there exists an étale neighborhood of  $p$  (defined over  $R$ )*

$$\text{Spec}R[x, y]/(x^a - t) \rightarrow \mathcal{C}.$$

(b) *If  $p$  is a node in the reduced fiber  $(\mathcal{C}_0)_{\text{red}}$ . Show that there exists an étale neighborhood of  $p$  (defined over  $R$ )*

$$\text{Spec}R[x, y]/(x^a y^b - t) \rightarrow \mathcal{C}.$$

*Proof.*

□

**Lemma 10.1.2.** *Let  $a, b, m$  be positive integers such that both  $a$  and  $b$  divide  $m$ .*

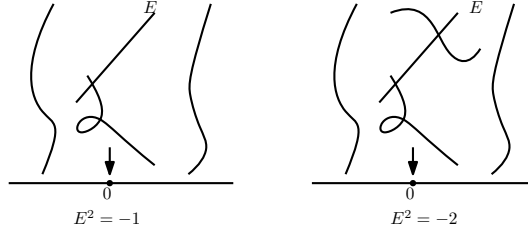
(a) *Let  $X = \text{Spec}k[x, t]/(t^m - x^a)$  and normalization  $\tilde{X} \rightarrow X$ . Then each preimage of the origin is locally defined by  $x = t^k$  for some  $k$ .*

(b) *Let  $X = \text{Spec}k[x, y, t]/(t^m - x^a y^b)$  and normalization  $\tilde{X} \rightarrow X$ . Then each preimage of the origin is locally defined by  $t^k = xy$ . In particular is a reduced and nodal point in the fiber over  $t = 0$ .*

*Proof.* (a) We have  $x^a - t^m = \prod_{i=0}^{a-1} (x - \zeta^i t^{m/a})$  where  $\zeta$  be a primitive  $a$ -th root of unity. Hence the origin has  $a$  preimages in  $\tilde{X}$  locally defined by  $x - \zeta^i t^{m/a}$ , respectively.

(b)

□



**Lemma 10.1.3.** *Let  $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$  be a family of nodal curves where  $R$  be a DVR such that the general fiber  $\mathcal{C}^*$  is smooth. Then if  $E$  is a rational tail (rational bridge with out marked points) of  $\mathcal{C}_0$ , then  $E^2 = -1$  ( $E^2 = -2$ ). As*

*Proof.* For any  $E \cong \mathbb{P}^1 \subset \mathcal{C}_0$ , then  $0 = E \cdot \mathcal{C}_0 = E^2 + E \cdot E^c$ . We win. (Actually I'm not familiar with surface and the intersection numbers!)  $\square$

For simplicity of notation, we assume that there are no marked points, i.e.  $n = 0$ . Fix a spectrum of DVR  $\Delta = \text{Spec}(R)$ ,  $\Delta^* = \text{Spec}(K)$  and  $t \in R$  is the uniformizer, and  $0 = (t) \in \text{Spec} R$  the unique closed point. Consider  $\mathcal{C}^* \rightarrow \Delta^*$  be a family of stable curve.

**STEP 1. Reduce to the case where  $\mathcal{C}^* \rightarrow \Delta^*$  is smooth.** If  $\mathcal{C}^*$  has  $k$  nodes, then after a finite extension of  $K$  we can arrange that each node is given by  $K$ -points  $p_i \in \mathcal{C}^*(K)$ . Let the pointed normalization  $(\tilde{\mathcal{C}}^*, q_1, \dots, q_{2k})$  of it. By induction on the genus  $g$ , we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of  $K$  (and  $R$ ), this produces a family of curves  $\mathcal{C} \rightarrow \Delta$  extending  $\mathcal{C}^* \rightarrow \Delta^*$ .

**STEP 2. Find some flat extension  $\mathcal{C} \rightarrow \Delta$ .** As  $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$  is very ample, we can get an embedding as follows

$$\begin{array}{ccc}
 \mathbb{P}^{5g-6} \times \Delta^* & \hookrightarrow & \mathbb{P}^{5g-6} \times \Delta \\
 \uparrow |\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}| & & \uparrow \\
 \mathcal{C}^* & \hookrightarrow & \mathcal{C} := \overline{\mathcal{C}^*} \\
 \downarrow & & \downarrow f \\
 \Delta^* & \hookrightarrow & \Delta
 \end{array}$$

where  $\mathcal{C} := \overline{\mathcal{C}}$  be the scheme-theoretic image of  $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$ . Now we focus on  $f$ . Actually the scheme-theoretic closure does not bring more embedded points. Hence by Proposition 2.1.2 we get  $f$  is flat.

**STEP 3. Use embedded resolutions to find a resolution of singularities  $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$  so that the reduced central fiber  $(\tilde{\mathcal{C}}_0)_{red}$  is nodal.** By Theorem B.0.2, there exists a finite sequence of blow-ups at closed points of  $\mathcal{C}_0$  yielding a projective birational morphism

$$\begin{array}{ccc}
 \tilde{\mathcal{C}}_0 \subset \tilde{\mathcal{C}} & \longrightarrow \cdots \longrightarrow & \mathcal{C} \supset \mathcal{C}_0 \\
 & \searrow & \downarrow \\
 & & \Delta
 \end{array}$$

such that  $\tilde{\mathcal{C}}$  is regular flat family of curves and such that the reduced central fiber  $(\tilde{\mathcal{C}}_0)_{red}$  is nodal. Now replace  $\mathcal{C}$  by  $\tilde{\mathcal{C}}$ .

**STEP 4. Perform a base change  $\Delta' \rightarrow \Delta$  such that the normalization of the total family  $\mathcal{C} \times_{\Delta} \Delta'$  has a reduced nodal central fiber with many rational tails and bridges.**

By Lemma 10.1.1, we choose local coordinates  $x, y$  around  $p \in \mathcal{C}_0$  (étale locally and formally locally) such that  $\mathcal{C} \rightarrow \Delta$  can be described as follows:

- (i) If  $p \in (\mathcal{C}_0)_{red}$  is a smooth point, then  $(x, y) \mapsto x^a$  and the multiplicity of the irreducible component of  $\mathcal{C}_0$  containing  $p$  is  $a$ ;
- (ii) If  $p \in (\mathcal{C}_0)_{red}$  is a (separated) node, then  $(x, y) \mapsto x^a y^b$  and the two components of  $\mathcal{C}_0$  containing  $p$  have multiplicities  $a$  and  $b$ .

Let  $m$  be the least common multiple of the multiplicities of the irreducible components of  $\mathcal{C}_0$ . Let the ramified morphism  $\Delta' = \text{Spec}(R) \rightarrow \Delta$  given by  $t \mapsto t^m$ . Hence we get

$$\begin{array}{ccccc} \tilde{\mathcal{C}}' & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ & & \downarrow & & \downarrow \\ & & \Delta' & \longrightarrow & \Delta \end{array}$$

where  $\mathcal{C}' = \mathcal{C} \times_{\Delta} \Delta'$  and  $\tilde{\mathcal{C}}' \rightarrow \mathcal{C}'$  be the normalization. Consider  $p \in (\mathcal{C}_0)_{red}$ .

(a) If  $p$  is a smooth point, then the unique preimage of  $p$  in  $\mathcal{C}'$  defined locally by  $x^a - t^m$ . By Lemma 10.1.2 (a), we get each preimage of  $p$  in  $\tilde{\mathcal{C}}'$  is locally defined by  $x = t^k$  which are the smooth points in  $\tilde{\mathcal{C}}'_0$ ;

(b) If  $p$  is a node, then the unique preimage of  $p$  in  $\mathcal{C}'$  defined locally by  $x^a y^b - t^m$ . By Lemma 10.1.2 (a), we get each preimage of  $p$  in  $\tilde{\mathcal{C}}'$  is locally defined by  $xy = t^k$  which are reduced and nodal points in  $\tilde{\mathcal{C}}'_0$ . If  $k > 1$ ,  $\tilde{\mathcal{C}}'$  have  $A_{k-1}$ -singularity.

Hence now we replace  $\mathcal{C}$  by  $\tilde{\mathcal{C}}'$ , which has a reduced central fiber with many rational tails and bridges.

**STEP 5. After taking the minimal model, contract all rational tails and bridges in the central fiber.** Using Theorem B.0.1 we get a minimal resolution  $\mathcal{C}' \rightarrow \mathcal{C}$  and we get a family of prestable curves  $\mathcal{C}' \rightarrow \Delta$  where  $\mathcal{C}'$  is regular. By Lemma 10.1.3 and Corollary B.0.4, we can get a projective birational map  $\mathcal{C}' \rightarrow \mathcal{C}'_{min}$  where  $\mathcal{C}'_{min}$  is semistable. So we replace  $\mathcal{C}$  by  $\mathcal{C}'_{min}$ . (This is the semistable reduction!) Using Proposition 7.5.1, we can get a relative canonical stabel model  $\mathcal{C}' \rightarrow \mathcal{C}^{st}$ .

## 10.2 Explicit stable reduction

**Proposition 10.2.1.** *Let  $\mathcal{C} \rightarrow \Delta$  be a generically smooth, proper and flat family such that  $(\mathcal{C}_0)_{red}$  is nodal. Let  $\mathcal{C}_0 = \sum_i a_i D_i$  where  $a_i$  is the multiplicity of  $D_i$ . Let  $\Delta' \rightarrow \Delta$  defined by  $t \mapsto t^p$  where  $p$  prime and set  $\mathcal{C}' = \mathcal{C} \times_{\Delta} \Delta'$ . Then after taking normalization  $\tilde{\mathcal{C}}' \rightarrow \mathcal{C}$  is branched cover ramified over  $\sum_i (a_i \pmod{p}) D_i$ .*

**Example 10.2.2** (Stable reduction of  $A_{2k+1}$ -singularity). *Let  $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$  be a generically smooth family degenerating to a  $A_{2k+1}$ -singularity in the central fiber where have local equation around the singular point is  $y^2 = x^{2k+1} + t$ . Now we will work through the steps in the proof of stable reduction. The first two steps have already finished, now we start at step 3.*

► **STEP 3. Use embedded resolutions to find a resolution of singularities  $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$  so that the reduced central fiber  $(\tilde{\mathcal{C}}_0)_{red}$  is nodal.** We consider two charts in blowing up with coordinates  $x', y'$  where the original coordinates are  $x, y$ , as:

$$\begin{array}{ccc} E|_{U_1} = V(x') \hookrightarrow U_1 \hookrightarrow \tilde{\mathcal{C}} = \text{Bl}_p \mathcal{C} & (x', y') & E|_{U_2} = V(y') \hookrightarrow U_2 \hookrightarrow \tilde{\mathcal{C}} = \text{Bl}_p \mathcal{C} & (x', y') \\ \downarrow \swarrow & \downarrow & \downarrow \swarrow & \downarrow \\ \mathcal{C} & (x', x'y') & \mathcal{C} & (x'y', y') \end{array}$$

•**The first blowing up.** In the first chart, the preimage of  $y^2 - x^{2k+1}$  is  $x'^2 y'^2 - x'^{2k+1} = x'^2(y'^2 - x'^{2k-1})$ ; in the second chart it is  $y'^2 - (x'y')^{2k+1} = y'^2(1 - x'^{2k+1}y'^{2k-1})$ . Hence the exceptional divisor  $E_1$  has multiplicity 2.

•**The second blowing up.** In the first chart, the preimage of  $x^2(y^2 - x^{2k-1})$  is  $x'^4(y'^2 - x'^{2k-3})$ ; in the second chart it is  $x'^2 y'^4(1 - x'^{2k-1}y'^{2k-3})$ . Hence the exceptional divisor  $E_2$  has multiplicity 4.

•**After  $k$  blowing ups.** We get  $x^{2k}(y^2 - x)$  with the exceptional divisors  $E_i$  has multiplicity  $2i$ .

•**One more blowing up.** We get the preimage of  $x^{2k}(y^2 - x)$  in the second chart is

$$x'^{2k} y'^{2k+1} (y' - x')$$

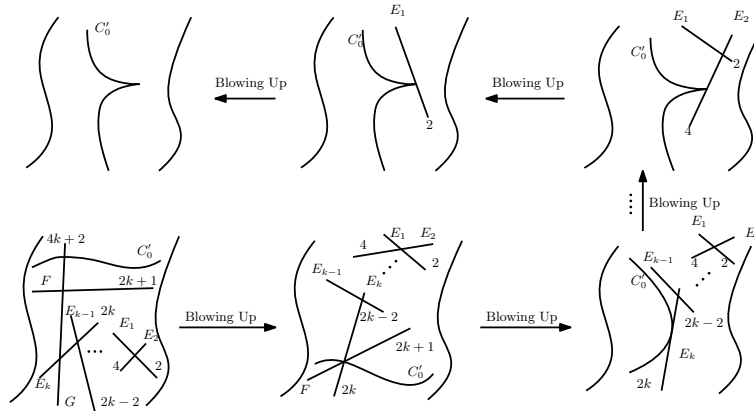
with the exceptional divisor  $F$  has multiplicity  $2k + 1$ .

•**The final blowing up.** We get the preimage of  $x^{2k} y^{2k+1} (y - x)$  in the first chart is

$$x'^{4k+2} y'^{2k+1} (y' - 1)$$

with the exceptional divisor  $G$  has multiplicity  $4k + 2$ .

The process as follows:

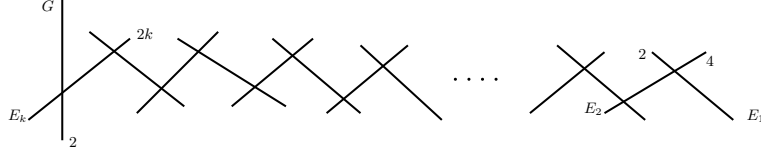


►**STEP 4. Perform a base change  $\Delta' \rightarrow \Delta$  such that the normalization of the total family  $\mathcal{C} \times_{\Delta} \Delta'$  has a reduced nodal central fiber with many rational tails and bridges.**

•**The first base change.** First consider  $\Delta' \rightarrow \Delta, t \mapsto t^{2k+1}$  and normalizing. After inductively apply to the prime factorization  $2k + 1$  and normalization, we will use the proposition to analyze the preimage of these irreducible component. Actually we get this  $2k + 1$ -degree cover ramified over  $C'_0 + \sum_i E_i$  and we just need to consider  $F, G$ . For  $G$ , its preimage  $G'$  ramified at two points (intersects  $E_k, C'_0$ ) with index  $2k$ . By Riemann-Hurwitz Theorem, we get  $2g(G') - 2 = (2k + 1)(2g(G) - 2) + 4k = -2$ . Hence  $g(G') = 0$  and  $G' \cong \mathbb{P}^1$ . For  $F$ , its preimage  $F'$  is unramified at all points, hence  $F' = \coprod_{j=1}^{2k+1} F_j$  are copies of  $F$ . Hence replace  $\Delta$  by  $\Delta'$ , we get the central fiber as  $C_0 = C'_0 + 2G' + \sum_j F_j + \sum_i 2iE_i$ .

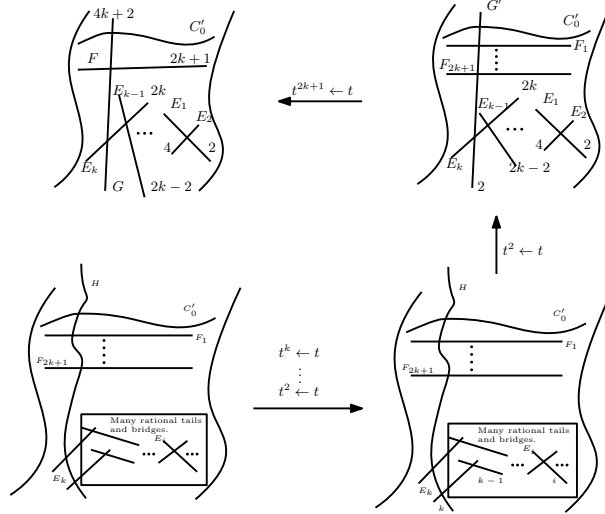
•**The second base change.** Consider  $\Delta' \rightarrow \Delta, t \mapsto t^2$  and normalizing. Actually we get this 2-degree cover ramified over  $C'_0 + \sum_j F_j$  and we just need to consider  $G', E_i$ . For  $G'$ , the preimage  $H$  ramified at  $2k + 2$  points ( $C'_0, F_j$ ). Hence we get  $g(H) = k$  by Riemann-Hurwitz Theorem. For  $E_i$ , the things become more complicated as follows:

But we can easy to see that after this process, these things are just plenty of rational bridges and rational tails.

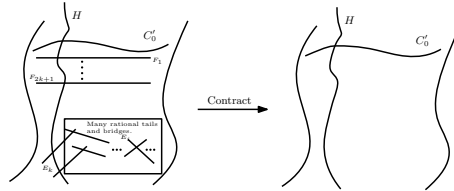


•**The final base changes.** Here we just need to consider  $E_i$  and these have multiplicity  $i$ . Consider  $t \mapsto t^k$ , then (many)  $E_k$  has two ramified points, hence by Riemann-Hurwitz Theorem  $g(E'_k) = 0$ , hence rational. Then consider  $t \mapsto t^{k-1}, \dots, t \mapsto t^2$ , we have the same results. Hence we also get plenty of rational tails and bridges, which are all multiplicity 1.

The whole process as follows:



►**STEP 5. Contract all rational tails and bridges in the central fiber.** Now we kill all  $-1$ -curves (many  $E_1$  and all  $F_j$ ), and then (every)  $E_2$  become  $-1$ -curves. Inductively, we kill all  $E_i$  and  $F_j$  and get a stable central fiber as follows and we win.



### 10.3 Separatedness of $\overline{\mathcal{M}}_{g,n}$

**Proposition 10.3.1.** Let  $R$  be a DVR with fraction field  $K$  with  $\Delta = \text{Spec}(R)$ ,  $\Delta^* = \text{Spec}(K)$ . Let  $(\mathcal{C} \rightarrow \Delta, \sigma_1^*, \dots, \sigma_n^*)$  and  $(\mathcal{D} \rightarrow \Delta, \tau_1^*, \dots, \tau_n^*)$  are  $n$ -pointed stable curves, then for any  $\alpha^* : \mathcal{C}^* \rightarrow \mathcal{D}^*$  with  $\tau_i^* = \alpha^* \circ \sigma_i^*$  over generic fiber can extends to a unique isomorphism  $\alpha : \mathcal{C} \rightarrow \mathcal{D}$



with  $\tau_i = \alpha \circ \sigma_i$ .

$$\begin{array}{ccc}
 \mathcal{C}^* & \xrightarrow{\alpha^*} & \mathcal{D}^* \\
 \searrow & & \downarrow \\
 & & \Delta^*
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{C} & \dashrightarrow^{\alpha} & \mathcal{D} \\
 \searrow & & \downarrow \\
 & & \Delta
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \Delta^* \\
 & \swarrow & \\
 & & \Delta
 \end{array}$$

*Proof.* We only prove the case of  $n = 0$  generically smooth curves. Let  $\mathcal{C}' \rightarrow \mathcal{C}, \mathcal{D}' \rightarrow \mathcal{D}$  be the minimal resolutions and let  $\Gamma \subset \mathcal{C}' \times_{\Delta} \mathcal{D}'$  be the clpsure of the graph of  $\text{id} \times \alpha^* : \mathcal{C}^* \rightarrow \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$ . Let  $\Gamma' \rightarrow \Gamma$  be the minimal resolution. Hence we get birational projective maps  $\Gamma' \rightarrow \mathcal{C}'$  and  $\Gamma' \rightarrow \mathcal{D}'$ . By the same proof of [25] Theorem II.8.19, we get

$$\Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \Gamma(\Gamma', \omega_{\Gamma'/\Delta}^{\otimes k}) \cong \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k})$$

for all  $k \geq 0$ . As the canonical bundle are ample, we get

$$\mathcal{C}' \cong \text{Proj} \bigoplus_k \Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \text{Proj} \bigoplus_k \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k}) \cong \mathcal{D}'.$$

Furthermore, we know that  $\mathcal{C}, \mathcal{D}$  are stable models of  $\mathcal{C}', \mathcal{D}'$ , respectively. By the uniqueness of stable models, we get  $\alpha : \mathcal{C} \cong \mathcal{D}$  extending  $\alpha^*$ .  $\square$



# Chapter 11

## Gluing and forgetful morphisms

We follow [12].

### 11.1 Gluing morphisms

**Proposition 11.1.1.** *There are finite morphisms of algebraic stacks*

$$F : \overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} \rightarrow \overline{\mathcal{M}}_{g,n+m-2}$$

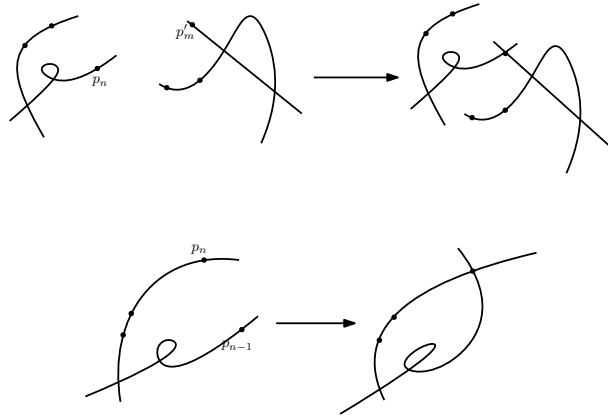
$$((C, p_1, \dots, p_n), (C', p'_1, \dots, p'_m)) \mapsto (C \cap C', p_1, \dots, p_{n-1}, p'_1, \dots, p'_m),$$

and

$$G : \overline{\mathcal{M}}_{g-1,n} \rightarrow \overline{\mathcal{M}}_{g,n-2}$$

$$(C, p_1, \dots, p_n) \mapsto (C/(p_{n-1} \sim p_n), p_1, \dots, p_{n-2}).$$

As follows:



*Sketch-using pushout.* By the stable reduction, these maps are of course representable and proper. As they have the finite fibers, these maps are now finite. Now for  $F$  we let  $n = m = 1$  and for  $G$  we let  $n = 2$ .

**For  $F$ :** Let  $(\pi : \mathcal{C} \rightarrow S, \sigma), (\pi' : \mathcal{C}' \rightarrow S, \sigma')$  are stable curves. As  $\sigma, \sigma'$  are closed immersions, we get the pushout exists by the theory of Ferrand (St 0ECH) and as we have the finite cover  $\mathcal{C} \sqcup \mathcal{C}' \rightarrow \mathcal{C}$ , we get this pushout is proper and flat (omitted):

$$\begin{array}{ccccc}
 & \text{Spec}(A) & \xrightarrow{\quad} & \text{Spec}A[y] & \\
 & \swarrow & & \swarrow & \\
 \text{Spec}A[x] & \xrightarrow{\quad} & \text{Spec}A[x, y]/(xy) & \xrightarrow{\quad} & \text{Spec}A[y] \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \mathcal{C}' & \xrightarrow{\sigma'} & S & \xrightarrow{\sigma} & \mathcal{C} \\
 & \searrow & & \searrow & \\
 & \mathcal{C} & \xrightarrow{\quad} & \mathcal{C} & 
 \end{array}$$

where  $\text{Spec}A[x]$  is an étale neighborhood of  $\sigma(s)$  which is the pulback of étale neighborhood  $\text{Spec}(A)$  of any  $s \in S$ . Since an étale morphism from an affine scheme extend over closed immersions, there is an étale neighborhood  $\text{Spec}A[y]$  is an étale neighborhood of  $\sigma'(s)$ . Then the pushout can be easy to compute as  $\text{Spec}(A[x] \times_A A[x]) \cong \text{Spec}A[x, y]/(xy)$ . By some results of pushout (in St 0D2G), we get  $\text{Spec}A[x, y]/(xy) \rightarrow \mathcal{C}$  is an étale neighborhood of  $s$ . Hence  $\tilde{\mathcal{C}} \rightarrow S$  is nodal along  $S$ . Checking fibers we get  $\tilde{\mathcal{C}}_s$  is stable.

**For  $G$ :** Let  $(\mathcal{C} \rightarrow S, \sigma_1, \sigma_2)$  are stable curve. Here we consider the pushout:

$$\begin{array}{ccc}
 S \sqcup S & \xrightarrow{\sigma_1 \sqcup \sigma_2} & \mathcal{C} \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & \tilde{\mathcal{C}}
 \end{array}$$

which is étale locally like

$$\begin{array}{ccc}
 \text{Spec}(A \times A) & \xrightarrow{(0,1)} & \text{Spec}A[t] \\
 \downarrow & & \downarrow \\
 \text{Spec}A & \longrightarrow & \text{Spec}A[x, y]/(y^2 - x^2(x+1))
 \end{array}$$

where we find that  $x := t^2 - 1, y = t^3 - t$  generate  $A \times_{A \times A} A[t]$ , then well done.  $\square$

## 11.2 Boundary divisors of $\overline{\mathcal{M}}_g$

Consider the closed substacks

$$\begin{aligned}
 \delta_0 &= \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g) \\
 \delta_i &= \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g)
 \end{aligned}$$

where  $i = 1, \dots, \lfloor g/2 \rfloor$ .

As these maps are finite, we get  $\dim \delta_0 = \dim \overline{\mathcal{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$  and similar  $\dim \delta_i = 3g - 4$ . Hence these are divisors of  $\overline{\mathcal{M}}_g$ .

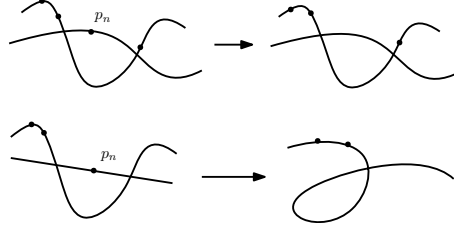
(By analyzing the formal deformation space of a stable curve, one can show that  $\delta = \bigcup_{j=0}^{\lfloor g/2 \rfloor} \delta_j$  is a normal crossings divisor.)

## 11.3 Forgetful morphisms

**Proposition 11.3.1.** *By Proposition 7.5.1, there is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n-1}, (C, p_1, \dots, p_n) \mapsto (C^{st}, p_1, \dots, p_{n-1}).$$

As



## 11.4 Universal family $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$

This section we follow [12] and [13]. We consider the universal family  $\mathcal{U}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$  of  $\overline{\mathcal{M}}_{g,n}$ . Actually the definition of universal family as for any family of stable curves  $(\mathcal{C} \rightarrow S, \{\sigma_i\})$ , we have the following universal property of cartesian

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & \mathcal{U}_{g,n} \\ \downarrow & & \downarrow \\ S & \dashrightarrow^{\exists!} & \overline{\mathcal{M}}_{g,n} \end{array}$$

The existence given by 2-Yoneda's Lemma and some descent theory (omitted). Here we express this family as follows:

**Lemma 11.4.1** (See [13]).  $\mathcal{U}_{g,n}(S)$  to be the set of families of curves  $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n, \sigma)$  where  $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n) \in \overline{\mathcal{M}}_{g,n}(S)$  and  $\sigma$  is an extra section without smooth condition.

*Proof.* Fix  $\pi : \mathcal{C} \rightarrow S$ . We first let  $\Sigma(S) : \mathcal{U}_{g,n}(S) \rightarrow \overline{\mathcal{M}}_{g,n}(S)$  as  $(\pi, \sigma_i, \sigma) \mapsto (\pi, \sigma_i)$  be the canonical map and  $\Sigma_i(S) : \overline{\mathcal{M}}_{g,n}(S) \rightarrow \mathcal{U}_{g,n}(S)$  as  $(\pi, \sigma_1, \dots, \sigma_n) \mapsto (\pi, \sigma_1, \dots, \sigma_n; \sigma_i)$ . Finally we need to define  $\mathcal{C} \rightarrow \mathcal{U}_{g,n}$  as  $(\text{pr}_2 : \mathcal{C} \times_S \mathcal{C} \rightarrow \mathcal{C}, s_i, \Delta)$  where  $s_i = (\sigma_i \circ \pi, \text{id}_{\mathcal{C}})$  and  $\Delta = (\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}})$ . Hence we get the following cartesian diagram of fibered categories

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{U}_{g,n} \\ \sigma_1, \dots, \sigma_n \uparrow \left( \downarrow \pi \right) & & \Sigma \downarrow \left( \uparrow \Sigma_1, \dots, \Sigma_n \right) \\ S & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

Well done. □

Now we consider

$$\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}, (\mathcal{C} \rightarrow S) \mapsto (\mathcal{C}^{st} \rightarrow S, \sigma'_1, \dots, \sigma'_n, \sigma')$$

where this stabilization aiming to make  $(\mathcal{C}^{st} \rightarrow S, \sigma'_1, \dots, \sigma'_n)$  in  $\overline{\mathcal{M}}_{g,n}(S)$ .

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1} & \longrightarrow & \mathcal{U}_{g,n} \\ & \searrow & \downarrow \\ & & \overline{\mathcal{M}}_{g,n} \end{array}$$

**Remark 11.4.2** (More explicit construction). Fix  $f : X \rightarrow S$  in  $\overline{\mathcal{M}}_{g,n+1}(S)$  and hence we get

$$X = \underline{\text{Proj}}_S \left( \bigoplus_{m \geq 0} f_* \omega_{X/S} \left( \sum_{i=1}^{n+1} \sigma_i \right)^{\otimes m} \right).$$

Now we let

$$c(X) := \underline{\text{Proj}}_S \left( \bigoplus_{m \geq 0} f_* \omega_{X/S} \left( \sum_{i=1}^n \sigma_i \right)^{\otimes m} \right).$$

with  $\sigma'_i : S \xrightarrow{\sigma_i} X \rightarrow c(X)$ . Hence  $(c(X); \sigma'_1, \dots, \sigma'_n)$  be a family of  $n$ -pointed stable curves. Hence we get  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}$ . Here we follows the proof before chapter 8 in [7].

**Proposition 11.4.3.** The morphism  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}$  is an isomorphism over  $\overline{\mathcal{M}}_{g,n}$ .

*Sketch.* Now we construct an inverse map  $\mathcal{U}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$ .

**Step 1. Construct that family of curves.** Let  $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n, \sigma)$  be an element in  $\mathcal{U}_{g,n}(S)$ . As  $\sigma$  is a closed immersion, it defined by an ideal sheaf  $i : \mathcal{I}_\sigma \hookrightarrow \mathcal{O}_\mathcal{C}$ . Define the coherent sheaf  $K$  by the exact sequence

$$0 \rightarrow \mathcal{O}_\mathcal{C} \xrightarrow{\delta} \mathcal{I}_\sigma^\vee \oplus \mathcal{O}_\mathcal{C}(\sigma_1 + \dots + \sigma_n) \rightarrow K \rightarrow 0$$

where  $\delta = (i^\vee, j)$  where  $j$  is also an embedding. Now consider  $\mathcal{C}' = \underline{\text{Proj}} \text{Sym}(K) \xrightarrow{p} \mathcal{C} \rightarrow S$ .

**Step 2. Construct the section.** In [12], Knudsen introduce a notion called stably reflexive module. Knudsen separate the two cases of  $\sigma$  as this is local on  $S$ : (I)  $\sigma$  meets a non-smooth point in the fiber; (II)  $\sigma$  is a divisor meets one of these sections  $\sigma_i$ .

In both cases we find the surjections form as  $\sigma^* K \rightarrow \sigma^*(-)$  or  $\sigma_i^* K \rightarrow \sigma_i^*(-)$  to getting lifts where showed that all  $\sigma^*(-)$  are line bundles (may using stably reflexive module). The picture when  $S = \text{Spec}(k)$  as follows:



Hence we omitted all details and get  $(\mathcal{C}' \rightarrow S, \sigma'_1, \dots, \sigma'_n, \sigma') \in \overline{\mathcal{M}}_{g,n+1}(S)$ . For this detailed proof, we refer the original paper [12] Theorem 2.4 or the new paper [13]. One can also see [2] X.8 for more detailed proof over  $\mathbb{C}$ .  $\square$

# Chapter 12

## Irreducibility

As  $\overline{\mathcal{M}}_{g,n}$  is a smooth Deligne-Mumford stack, its irreducibility if and only if connectedness. As  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  be a universal family, it has connected fibers. Hence by induction, we can reduce the case of  $\overline{\mathcal{M}}_g$ . Moreover, by Keel-Mori theorem we get the coarse moduli space  $\overline{\mathcal{M}}_g \rightarrow \overline{M}_g$  which induce the homeomorphism  $|\overline{\mathcal{M}}_g| \cong |\overline{M}_g|$ . Hence we can reduce the case of  $\overline{M}_g$ . Hence we have the following relations:

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible} &\Leftrightarrow \overline{\mathcal{M}}_{g,n} \text{ connected} \Leftrightarrow \overline{\mathcal{M}}_g \text{ connected (or irreducible)} \\ &(\Leftrightarrow \mathcal{M}_g \text{ connected and dense in } \overline{\mathcal{M}}_g) \Leftrightarrow \overline{M}_g \text{ connected.} \end{aligned}$$

Here the denseness of  $\mathcal{M}_g$  in the proper Deligne-Mumford stack  $\overline{\mathcal{M}}_g$  is called **Deligne-Mumford compactification**.

**Remark 12.0.1** (Some historical remarks). (i) In 19th century, Clebsch and Hurwitz establishing irreducibility of  $M_g$  in characteristic 0 by using the classical topological argument;

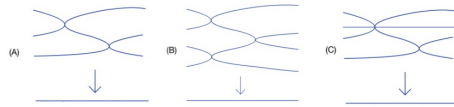
(ii) In the appendix of the paper [23] by Fulton in [16] gives a completely algebraic proof for this in characteristic 0 in 1982;

(iii) In the paper [9], Deligne and Mumford give two arguments of irreducibility of  $\overline{M}_{g,n}$  in characteristic  $p$  (by reduction to characteristic 0) in 1969;

(iv) In paper [15], Fulton established the irreducibility of  $\overline{M}_{g,n}$  in characteristic  $p$  where  $p > g + 1$  in 1969.

### 12.1 Preliminaries–Branched coverings

**Definition 12.1.1.** Let  $C$  be a connected smooth curve on  $k$ . A branched covering of  $\mathbb{P}_k^1$  is a separable finite morphism  $f : C \rightarrow \mathbb{P}_k^1$ . We say  $f$  is simply branched if for any branched point  $x \in \mathbb{P}_k^1$ , there is at most one ramification point in the fiber  $f^{-1}(x)$  and such a point has index 2.



Here (A) is simply branched but (B),(C) are not.

**Lemma 12.1.2.** *Let  $C$  be a smooth, connected and projective curve of genus  $g$  over an algebraically closed field  $k$  of characteristic 0. If  $L$  is a line bundle of degree  $d \geq g + 1$  (*I think we may let  $d \gg 0$* ), then for a subspace  $V \subset H^0(C, L)$  of dimension 2 we get  $C \rightarrow \mathbb{P}^1$  a simply branched.*

*Proof.* (*This proof need to re-think.*) As  $h^0(C, L) = d + 1 - g$ , we get  $\dim \text{Gr}(2, H^0(C, L)) = 2(d - g - 1)$ . Here  $\text{char}(k) = 0$ , the map  $C \rightarrow \mathbb{P}^1$  is finite separable. So  $C \rightarrow \mathbb{P}^1$  is not a simply branched covering if and only if one of the following conditions holds

- (a)  $V$  has a base point;
- (b) there exists a ramification point with index  $> 2$ ;
- (c) there exists 2 ramification points in the same fiber.

**For (a)**, then there exists  $p \in C$  such that for all  $s \in V$  vanishing at  $p$ , that is,  $s \in H^0(C, L(-p))$ . The dimension of  $V \in \text{Gr}(2, H^0(C, L))$  have this property is

$$\dim \text{Gr}(2, H^0(C, L(-p))) = 2d - 2g - 4.$$

**For (b)**, then there exists  $s \in V$  vanishing to order 3 at a point  $p$ , that is,  $s \in H^0(C, L(-3p))$ . The dimension of  $V \in \text{Gr}(2, H^0(C, L))$  have this property is

$$\dim \mathbb{P}H^0(C, L(-3p)) + \dim \mathbb{P}(H^0(C, L)/(s)) = 2d - 2g - 4.$$

Hence varying  $p \in C$ , the locus of  $\text{Gr}(2, H^0(C, L))$  failing (b) has dimension  $\dim \text{Gr}(2, H^0(C, L)) - 1$ ;

**For (c)**, then there exists independent  $s_1, s_2 \in V$  vanishing to order 2 at a point  $p$ , that is,  $s_1, s_2 \in H^0(C, L(-2p))$ . The dimension of  $V \in \text{Gr}(2, H^0(C, L))$  have this property is

$$\dim \text{Gr}(2, H^0(C, L(-2p))) = 2d - 2g - 6.$$

Hence varying  $p \in C$ , the locus of  $\text{Gr}(2, H^0(C, L))$  failing (b) has dimension  $- - -$  □

**Lemma 12.1.3.** *If  $C \rightarrow \mathbb{P}^1$  is a simply branched cover of degree  $d > 2$  in characteristic 0, then  $\text{Aut}(C/\mathbb{P}^1)$  is trivial.*

*Proof.* Any  $\alpha \in \text{Aut}(C/\mathbb{P}^1)$  must fix the  $2g + 2d - 2$  branched points by Riemann-Hurwitz Theorem and simplyness. By Proposition 1.2.7, there are no non-trivial automorphisms of a smooth curve fixing more than  $2g + 2$  points. Hence as  $d > 2$ ,  $\text{Aut}(C/\mathbb{P}^1)$  is trivial. □

**Remark 12.1.4.** *Here we give some notes for the proof of Clebsch and Hurwitz in 19th to show that  $M_g$  is connected over  $\mathbb{C}$ . We define*

$$H_{d,b} = \{C \rightarrow \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

where  $b = 2g + 2d - 2$ . By the previous lemma,  $H_{d,b}$  is an algebraic space or a topological space (if  $k = \mathbb{C}$ , *why?*). Let  $\text{Sym}^b \mathbb{P}^1 \setminus \Delta$  as the variety of  $b$  unordered distinct points in  $\mathbb{P}^1$  (which can also be written as the complement  $\mathbb{P}^b \setminus \Delta$  of the discriminant hypersurface), we have a diagram

$$\begin{array}{ccc} & H_{d,b} & \\ \swarrow & & \searrow \\ \mathcal{M}_g & & \text{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array}$$



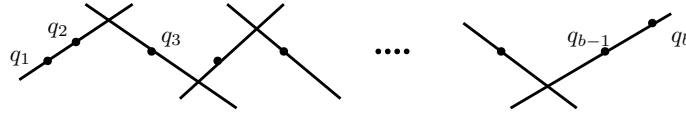
with the canonical maps. Then they showed that  $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$  is finite étale (actually this can be showed by using deformation theory pure algebraically, see [1] Lemma 5.7.9. We omitted here). By Lemma 12.1.2, we get  $H_{d,b} \rightarrow \mathcal{M}_g$  is surjective. Hence we need to show that  $H_{d,b}$  is connected. Combining these and some properties of monodromy theory, they proved this. For more detail, see [1] subsection 5.7.2.

## 12.2 Irreducibility over characteristic 0 using admissible covers

In this section we will use the method of admissible covers to gives a completely algebraic proof for the irreducibility in characteristic 0, which appears in the appendix of the paper [23] by Fulton in [16].

**Proposition 12.2.1.** *Let  $C$  be a smooth, connected and projective curve of genus  $g$  over an algebraically closed field  $k$  of characteristic 0. There exists a connected curve  $T$  with points  $t_1, t_2 \in T$  and a family  $\mathcal{C} \rightarrow T$  of stable curves such that  $\mathcal{C}_{t_1} \cong C$  and  $\mathcal{C}_{t_2}$  is a singular stable curve.*

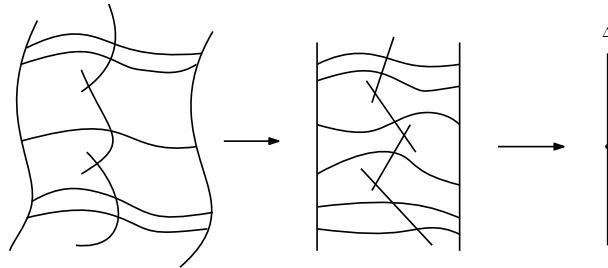
*Proof.* By Lemma 12.1.2 we get for  $d \gg 0$  there exists a finite cover  $C \rightarrow \mathbb{P}^1$  of degree  $d$  simply branched over  $b = 2g + 2d - 2$  distinct points  $p_1, \dots, p_b$  in  $\mathbb{P}^1$ . This gives a  $b$ -pointed stable curve  $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,b}$ . By Remark 12.1.4 ( $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$  is finite étale), we get  $G \in M_{0,b}$  in general (WTF?). Then  $G$  can degenerates to  $(D_0, q_1, \dots, q_b)$  as the following picture



In the other words, there is a DVR  $R$  and fraction field  $K$  with  $\Delta = \text{Spec}(R) \rightarrow \overline{M}_{0,n}$  be a stable curve  $(\mathcal{D} \rightarrow \Delta, \sigma_i)$  with generic fiber  $(\mathbb{P}^1, p_i)$  and special fiber  $(D_0, q_1, \dots, q_b)$ . Hence we have a simply branched covering  $\mathcal{C}^* \rightarrow \Delta^*$  and extend to  $\mathcal{C} \rightarrow \mathcal{D}$  by taking  $\mathcal{C}$  as the integral closure of  $\mathcal{O}_{\mathcal{D}}$  in  $K(\mathcal{C}^*)$  as

$$\begin{array}{ccccc} \mathcal{C} & \dashrightarrow & \mathcal{D} & \longrightarrow & \Delta \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{C}^* & \longrightarrow & \mathcal{D}^* & \longrightarrow & \Delta^* \end{array}$$

Hence we get this diagram:



Now we just need to make  $\mathcal{C}$  be a singular stable curve. Purity of the branch locus (What's this?) implies the central fiber  $\mathcal{C}_0 \rightarrow \mathcal{D}_0$  is ramified at  $\sigma_i(0)$ . By  $\Delta' \rightarrow \Delta, t \mapsto t^m$  we can replace

$\mathcal{C}$  such that  $\mathcal{C}_0 \rightarrow \mathcal{D}_0$  is ramified only over  $\sigma_i(0)$  and possibly over nodes of  $\mathcal{D}_0$ . By an analysis of possible extensions  $\mathcal{C} \rightarrow \mathcal{D}$ , one can show that  $\mathcal{C}_0$  is a nodal curve (missing details). Therefore  $\mathcal{C} \rightarrow \Delta$  is a family of nodal curves.

Now we take  $\mathcal{C} \rightarrow \mathcal{C}^{st}$  and just need to check  $\mathcal{C}_0^{st}$  is singular. For any irreducible component  $T \subset \mathcal{C}_0^{st}$ , apply Riemann-Hurwitz to  $T \rightarrow \mathbb{P}^1 \subset \mathcal{D}_0$  we get  $2g(T) - 2 = -2d + R$ . If  $\mathbb{P}^1$  is the middle one, we get  $R \leq 2 + d - 1$ ; if  $\mathbb{P}^1$  is the boundary one, we get  $R \leq 1 + 2d - 2$ . Hence  $R \leq 2d - 1$  and  $g(T) = 0$ . Hence  $T$  is rational. Hence  $\mathcal{C}_0^{st}$  is singular.  $\square$

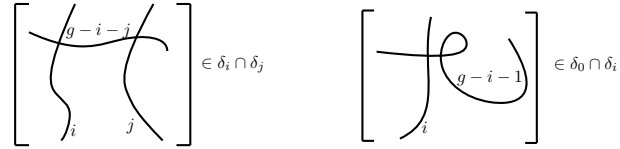
**Proposition 12.2.2.** *If  $\overline{\mathcal{M}}_{g',n'}$  is irreducible for all  $g' < g$ , then  $\delta = \overline{\mathcal{M}}_{g,2} \setminus \mathcal{M}_{g,2}$  is connected.*

*Proof.* Let  $\delta = \delta_0 \cup \delta_1 \cup \dots \cup \delta_{\lfloor g/2 \rfloor}$  where

$$\delta_0 = \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g)$$

$$\delta_i = \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g)$$

where  $i = 1, \dots, \lfloor g/2 \rfloor$ . Hence  $\delta_0, \delta_i$  are connected by hypotheses. Easy to see that these divisors intersect as the points of  $|\overline{\mathcal{M}}_g|$ :  $\square$



**Theorem 12.2.3.**  *$\overline{\mathcal{M}}_{g,n}$  is irreducible.*

*Proof.* By the argument at beginning, we just need to show  $\overline{\mathcal{M}}_g$  is connected. By Proposition 12.2.1 every smooth curve degenerates to a stable singular curve in the boundary  $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ . By induction on  $g$  and Proposition 12.2.2 we get  $\delta$  is connected, so is  $\overline{\mathcal{M}}_g$ .  $\square$

**Remark 12.2.4.** *For the irreducibility in positive characteristic, we omitted and we refer the original [9], [15]. For the sketch, we refer subsection 5.7.4 in [1].*

# Chapter 13

## Projectivity

We will prove the coarse moduli space  $\overline{M}_{g,n}$  is projective follows [28] and [37].

**Remark 13.0.1.** *Some generalizations of the projectivities:*

- (a) In [27] shows the moduli of stable varieties in any dimension is projective;
- (b) In [6] and [38] shows the moduli of  $K$ -polystable Fano varieties is projective.

Let the universal family  $\pi : \mathcal{U}_g \rightarrow \overline{\mathcal{M}}_g$  and we define  $k$ -th pluri-canonical bundle as the vector bundle  $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ . Indeed,  $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$  is a coherent sheaf on the stack  $\overline{\mathcal{M}}_g$  by the coherence theorem. We need to check that it is a vector bundle. By definition of the vector bundle over Deligne-Mumford stack, we need to show for any  $S \rightarrow \overline{\mathcal{M}}_g$  the sheaf  $(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}))|_S$  is a vector bundle over  $S$ . As  $S \rightarrow \overline{\mathcal{M}}_g$  correspond to  $\pi_S : \mathcal{C} \rightarrow S$ , we get

$$(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}))|_S \cong \pi_{S,*}(\omega_{\mathcal{C}/S}^{\otimes k}).$$

By some argument with Review A.1.1 we can show that  $\pi_{S,*}(\omega_{\mathcal{C}/S}^{\otimes k})$  is a vector bundle.

Moreover, we get use the Riemann-Roch Theorem to deduce that

$$\text{rank}(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})) = \begin{cases} g, & k = 1; \\ (2k-1)(g-1), & k > 1. \end{cases}$$

Now we consider the line bundle over  $\overline{\mathcal{M}}_g$

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}).$$

We will show that for  $k \gg 0$ , the line bundle  $\lambda_k$  descends to an ample line bundle on  $\overline{M}_g$ , then we get  $\overline{M}_g$  is a projective scheme.

### 13.1 Kollár's Criteria

**Lemma 13.1.1.** *Let  $\mathcal{X}$  be a proper Deligne-Mumford stack with coarse moduli space  $\mathcal{X} \rightarrow X$ . Suppose  $L$  line bundle over  $\mathcal{X}$  with*

- (a)  $L$  is semiample (i.e.  $L^N$  is basepoint-free for some  $N > 0$ );
- (b) for every proper integral curve  $T$  and map  $f : T \rightarrow \mathcal{X}$  such that  $f(T) \subset |\mathcal{X}|$  is not a single point,  $\deg L|_T > 0$ .

*Then for some  $N > 0$ ,  $L^{\otimes N}$  descends to an ample line bundle over  $X$ .*

*Proof.* This is the stack-version of the Corollary 1.2.15 in [31]. Actually we consider the following diagram which come from (a) and the universal property of coarse moduli space:

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow \pi & \searrow f & \\ X & \xrightarrow{g} & \mathbb{P}(H^0(\mathcal{X}, L^{\otimes N})) \end{array}$$

By (b),  $f$  doesn't contract curves, so is  $g$ . Hence  $g$  is quasi-finite and proper, hence finite by Zariski main theorem. Hence  $M := g^* \mathcal{O}(1)$  is ample; moreover,  $\pi^* M = L^{\otimes N}$ , we win.  $\square$

**Theorem 13.1.2** (Nakai-Moishezon Criterion). *If  $X$  is a proper algebraic space, a line bundle  $L$  is ample if and only if for all irreducible closed subvarieties  $Z \subset X$ ,*

$$L^{\dim Z} \cdot Z > 0.$$

*Proof.* This is the algebraic space-version of the Theorem 1.21 in [8]. By Le Lemme de Gabber (Theorem C.2.1), there exists a finite surjection  $f : X' \rightarrow X$  and by the algebraic space version St 0GFB we get  $L$  is ample if and only if  $f^* L$  is ample. Hence by the scheme-version of Nakai-Moishezon Criterion ([8] Theorem 1.21), we win.  $\square$

Let  $X$  be a proper algebraic space over  $k$ . Let  $W \rightarrow Q$  be a surjection of vector bundles of rank  $w$  and  $q$ . Suppose that  $W$  has structure group  $G \rightarrow \mathrm{GL}_w$ . There is a classifying map

$$X \rightarrow [\mathrm{Gr}(q, w)/G], x \mapsto [W \otimes \kappa(x) \rightarrow Q \otimes \kappa(x)]$$

which is well defined because these killed by  $G$ .

Here we state our main theorem in this section. For simplicity, we only state it in characteristic 0. The criteria first appears in [28] and more general case we refer [27].

**Theorem 13.1.3** (Kollár's Criterion). *Let  $X$  be a proper algebraic space over a field  $k$  of characteristic 0. Let  $W \rightarrow Q$  be a surjection of vector bundles of rank  $w$  and  $q$ , where  $W$  has structure group  $G \rightarrow \mathrm{GL}_w$ . Suppose that*

- (a) *The classifying map  $X(k) \rightarrow \mathrm{Gr}(q, w)(k)/G(k)$  has finite fibers;*
- (b)  *$W$  is nef.*

*Then  $\det Q$  is ample.*

*Proof.* By Nakai-Moishezon criterion, for any irreducible subvariety  $Z \subset X$  we need to verify  $\det(Q)|_Z$  is big. As (a),(b) can restrict to  $Z$ , we can let  $X$  is an integral scheme and show that  $\det Q$  is big.

By Le Lemme de Gabber (Theorem C.2.1), there exists a finite projective surjection  $f : Y \rightarrow X$  of schemes. Hence we have  $\det(f^* Q)^{\dim Y} = \deg(f) \det(Q)^{\dim X}$  and  $\det Q$  is big if and only if  $\det(f^* Q)$  is big. By taking the normalization, we can assume  $Y$  is normal and integral. So by Lemma 13.1.4 we win.  $\square$

**Lemma 13.1.4.** *Let  $Y$  be a normal projective integral scheme over a field  $k$  of characteristic 0. Let  $W \rightarrow Q$  be a surjection of vector bundles of rank  $w$  and  $q$ , where  $W$  has structure group  $G \rightarrow \mathrm{GL}_w$ . Suppose that*

- (a) *The classifying map  $Y(k) \rightarrow \mathrm{Gr}(q, w)(k)/G(k)$  generically has finite fibers;*
- (b)  *$W$  is nef.*

*Then  $\det Q$  is big.*

*Sketch.* To add. See Proposition 5.8.9 in [1].  $\square$

## 13.2 Nefness of pluri-canonical bundles

**Theorem 13.2.1.** *Let  $\pi : \mathcal{C} \rightarrow T$  be a family of stable curves over a smooth curve  $T$  over  $k$ , then  $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$  is nef for  $k \geq 2$ .*

*Proof.* We following several steps:

•**Step 1. Reduction to characteristic  $p$ .** Now we let  $k$  is of characteristic 0. Since  $\mathcal{C}$  and  $T$  are finite type over  $k$ , their defining equations only involve finitely many coefficients of  $k$ . Thus there exists a finitely generated  $\mathbb{Z}$ -subalgebra  $A \subset k$  and a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \tilde{\mathcal{C}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \tilde{T} \\ \downarrow & & \downarrow \\ \text{Spec } k & \longrightarrow & \text{Spec } A \end{array}$$

where  $\tilde{\mathcal{C}}, \tilde{T}$  are schemes of finite type over  $A$ . By possibly enlarging  $A$ , we can arrange that  $\tilde{T} \rightarrow \text{Spec}(A)$  is smooth and projective family and  $\tilde{\mathcal{C}} \rightarrow \tilde{T}$  is a family of stable curves.

(Need to re-think, omitted here.)

•**Step 2. Second reductions.** We reduce to the case that

- (a)  $\mathcal{C}$  is a smooth and minimal surface;
- (b)  $\mathcal{C} \rightarrow T$  is generically smooth;
- (c) The genus of  $T$  is at least 2.

(To add.) These implies  $\mathcal{C}$  is of general type.

•**Step 3. Positive characteristic case.** Let  $p = \text{char}(k)$ . If  $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$  is not nef, then there exists a quotient line bundle  $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$  where  $d = \deg(M) > 0$ . Consider the absolute Frobenius

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Frob}_{\mathcal{C}}} & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\text{Frob}_T} & T \end{array}$$

By the property of the dualizing sheaf, we get  $\text{Frob}_T^* \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \cong \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ . And  $\deg \text{Frob}_T^* M = pd$ , we can let  $d \gg 0$ . Hence we can let  $M = \omega_T^{\otimes k} \otimes L$  where  $L$  is very ample.

The surjection  $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow (\omega_T^{\otimes k} \otimes L)^\vee$  induce

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T.$$

As  $h^1(T, \mathcal{O}_T) \geq 2$ , we have  $h^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$ . Use the Leray spectral sequence

$$H^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \Rightarrow H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L),$$

hence  $h^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$  by some calculation. By Lemma 13.2.2, we win.  $\square$

**Lemma 13.2.2** (Bombieri-Ekedahl). *Let  $S$  be a smooth projective surface over an algebraically closed field  $k$  which is minimal and of general type. Let  $D$  be an effective divisor with  $D^2 = 0$ . If  $\text{char}(k) \neq 2$ , then  $H^1(S, \omega_S^{\otimes n}(D)) = 0$  for all  $n \geq 2$ . If  $\text{char}(k) = 2$ , then  $h^1(S, \omega_S^{\otimes n}(D)) \leq 1$  for all  $n \geq 2$ .*

### 13.3 Positivity via positivity theory

For a morphism  $S \rightarrow \overline{\mathcal{M}}_g$  correspond to  $\mathcal{C} \rightarrow S$ . Consider an integral  $d$ , we have

$$\mathrm{Sym}^d \pi_* (\omega_{\mathcal{C}/S}^{\otimes k}) \rightarrow \pi_* (\omega_{\mathcal{C}/S}^{\otimes dk}).$$

When  $S = \mathrm{Spec}(K)$ ,  $\mathcal{C} = C$ , we get

$$\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \rightarrow H^0(C, \omega_C^{\otimes dk})$$

with kernel consists of degree  $d$  equations cutting out the image of  $|\omega_C^{\otimes k}| : C \rightarrow \mathbb{P}^{r(k)-1}$ . If  $k \geq 3$ ,  $\omega_{\mathcal{C}/S}^{\otimes k}$  is very ample and thus  $\mathcal{C} \rightarrow S$  can be recovered from the kernel of the multiplication map.

**Proposition 13.3.1.** *For  $k \gg 0$  and  $N$  sufficiently divisible, then  $\lambda_k = \det \pi_* (\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$  descends to an ample line bundle on  $\overline{\mathcal{M}}_g$ .*

*Proof.* Consider  $\mathcal{C} = \mathcal{U}_g$ ,  $S = \overline{\mathcal{M}}_g$ . Choose  $k, d$  such that

- (a)  $\omega_{\mathcal{C}/S}^{\otimes k}$  is relatively very ample and  $R^1 \pi_* \omega_{\mathcal{C}/S}^{\otimes k} = 0$ ;
- (b) Every curve  $|\omega_C^{\otimes k}| : C \hookrightarrow \mathbb{P}^{r(k)-1}$  is cut out by equations degree  $d$ ;
- (c)  $\pi_* (\omega_{\mathcal{C}/S}^{\otimes k})$  is nef (by Theorem 13.2.1).

These implies surjection

$$W := \mathrm{Sym}^d \pi_* (\omega_{\mathcal{C}/S}^{\otimes k}) \twoheadrightarrow \pi_* (\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q.$$

Let  $w, q$  be the rank of  $W, Q$ , respectively. Let  $W$  has structure group  $G \rightarrow \mathrm{GL}_w$ . Consider the classifying map

$$\overline{\mathcal{M}}_g \rightarrow [\mathrm{Gr}(q, w)/G], x \mapsto \underbrace{[\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k})]}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathcal{O}(d))}$$

is injective as the conditions on  $d$  and  $k$  imply that the kernel of the multiplication map uniquely determines  $C$ .

By Le Lemme de Gabber we get a finite cover  $X \rightarrow \overline{\mathcal{M}}_g$ . By Kollár's Criterion (Theorem 13.1.3), we get the pullback of  $\lambda_k$  to  $X$  is ample for  $k \gg 0$ . By Proposition C.2.2, we get for  $N$  sufficiently divisible,  $\lambda_k^{\otimes N}$  descends to a line bundle  $L$  on  $\overline{\mathcal{M}}_g$ . Since the pullback of  $L$  under the finite morphism  $X \rightarrow \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g$ , by St 0GFB we get the conclusion that  $L$  is ample.  $\square$

**Theorem 13.3.2.** *If  $2g - 2 + n > 0$ , then  $\overline{\mathcal{M}}_{g,n}$  is projective.*

*Proof.* The universal family  $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$  is projective by Proposition 7.3.3. Hence we just consider  $n = 0$ . This is right directly by the previous proposition.  $\square$

**Remark 13.3.3.** *If we consider  $\omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}^{\otimes k} (\Sigma_1 + \dots + \Sigma_n)$  from begining, we can prove the projectivity of  $\overline{\mathcal{M}}_{g,n}$  directly.*

### 13.4 Projectivity via GIT, a sketch

By our old way, we have  $\overline{\mathcal{M}}_g \cong [H'/\mathrm{PGL}_{r(k)}]$  for some locally closed  $\mathrm{PGL}_{r(k)}$ -invariant subscheme of  $\mathrm{Hilb}_{\mathbb{P}^{r(k)-1}}^P$  where  $P(t) = \chi(C, \omega_C^{\otimes kt})$  and  $r(k) = (2k-1)(g-1)$ .

**Remark 13.4.1.** In fact we have  $\overline{\mathcal{M}}_{g,n} \cong [H_{\nu,g,n}/\mathrm{PGL}(N)]$  where  $N = (2\nu - 1)(g - 1) + \nu n$  and  $H_{\nu,g,n} \subset \mathrm{Hilb}_{\mathbb{P}^{N-1}}^{P_\nu}$  be the Hilbert scheme of  $\nu$ -log-canonically embedded  $n$ -pointed stable curves of genus  $g$  where  $P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$  for  $\nu \geq 3$ . See [2] Theorem XII.5.6 for the proof.

Let  $H$  be the closure of  $H'$  in  $\mathrm{Hilb}_{\mathbb{P}^{r(k)-1}}^P$ . By the proof of the representability of the quotient scheme, we get a closed immersion for  $d \gg 0$ :

$$\begin{aligned} H &\hookrightarrow \mathrm{Hilb}_{\mathbb{P}^{r(k)-1}}^P \hookrightarrow \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))) \\ [C \hookrightarrow \mathbb{P}^{r(k)-1}] &\longmapsto [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] \end{aligned}$$

Next consider the Plücker embedding

$$\begin{aligned} \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))) &\hookrightarrow \mathbb{P} \left( \bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \right) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] &\mapsto [\bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \bigwedge^{P(d)} \Gamma(C, \mathcal{O}(d))] \end{aligned}$$

we can get  $L_d := \mathcal{O}_{\mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)))}(1)|_H$  be the very ample line bundle over  $H$ . All these morphisms are  $\mathrm{PGL}_{r(k)}$ -equivariant, hence  $L_d$  inherits a  $\mathrm{PGL}_{r(k)}$ -linearization. Hence  $L_d$  can be defined on  $[H/\mathrm{PGL}_{r(k)}]$ .

Using the theory of Hilbert-Mumford criteria, we can prove the following difficult result.

**Theorem 13.4.2.** Let  $k \geq 5$  and  $d \gg 0$ . For  $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$ , the curve  $C$  is stable if and only if  $h \in H$  is GIT semistable with respect to  $L_d$ , that is, there exists an equivariant section  $s \in \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$  with  $N > 0$  such that  $s(h) \neq 0$ . Moreover, we have

$$\overline{M}_g \cong \mathrm{Proj} \left( \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}} \right),$$

hence projective.





## **Part III**

# **Some geometry properties of the moduli space of curves**



# Chapter 14

## Preliminaries

We now consider  $\mathcal{M}_{g,n}$  and  $\overline{\mathcal{M}}_{g,n}$  as the groupoid over the category  $(Sch/Spec\mathbb{C})$ . Then by the same arguments in the previous part, we can get  $\overline{\mathcal{M}}_{g,n}$  is also a proper smooth Deligne-Mumford stack of dimension  $3g - 3 + n$  over  $\mathbb{C}$  with a coarse moduli space  $\overline{M}_{g,n}$  which is a projective variety over  $\mathbb{C}$ . Similarly for  $\mathcal{M}_{g,n}$  and  $M_{g,n}$ . We will refer [2].

### 14.1 Boundary geometry I. Graphs and dual graphs

We can associate a graph to a nodal curve with marked points.

**Definition 14.1.1.** *A graph  $\Gamma$  is the datum of:*

- (a) *a finite nonempty set  $V = V(\Gamma)$  (the set of vertices);*
- (b) *a finite set  $L = L(\Gamma)$  (the set of half-edges);*
- (c) *an involution  $\iota$  of  $L$ ;*
- (d) *a partition of  $L$  indexed by  $V$ , that is, the assignment to each  $v \in V$  of a (possibly empty) subset  $L_v$  of  $L$  such that  $L = \bigcup_{v \in V} L_v$  with  $L_v \cap L_w = \emptyset$  when  $v \neq w$ .*

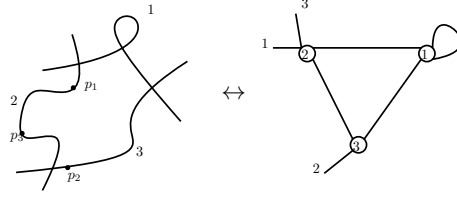
*A pair of distinct elements of  $L$  interchanged by the involution is called an edge of the graph. A fixed point of the involution is called a leg of the graph. The set of edges of  $\Gamma$  is denoted by  $E(\Gamma)$ . A dual graph is the datum of a graph together with the assignment of a nonnegative integer weight  $g_v$  to each vertex  $v$ . The genus of a dual graph  $\Gamma$  is defined to be*

$$g_\Gamma = \sum_{v \in V(\Gamma)} g_v + 1 - \chi(\Gamma).$$

*A graph (or a dual graph) endowed with a one-to-one correspondence between a finite set  $P$  and the set of its legs will be said to be  $P$ -marked, or numbered if  $P$  is of the form  $\{1, \dots, n\}$  for some nonnegative integer  $n$ .*

Let  $C$  be a nodal curve with a finite set  $D$  of smooth points of  $C$ . As in that case, there is a vertex for each component of the normalization of  $C$ , and its weight is the genus of the component. The half-edges issuing from a vertex are the points of the corresponding component which map either to a node of  $C$  or to a marked point. Easy to see that the edges of the graph are the pairs of half-edges mapping to the same node of  $C$ ; the legs are the half-edges coming from the marked points. This graph we denote it  $\text{Graph}(C; D)$ .

Easy to see by Theorem 6.2.1, we get the genus of the dual graph associated to  $(C; D)$  is equal to the genus of  $C$ ! For example:



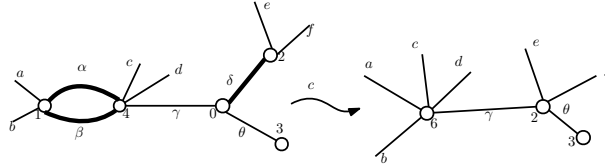
**Remark 14.1.2.** (i) Moreover, we also have some kind of localization. For  $(C; D)$  we fixed a set  $S$  of nodes of  $C$ . We let a graph  $\text{Graph}^S(C; D)$ : The vertices of this graph are the connected components of the partial normalization  $C^S$  of  $C$  at  $S$ , the weight  $g_v$  is the arithmetic genus of the corresponding component of  $C^S$ , the edges correspond to the nodes in  $S$ , and the half-edges are the marked points or the points of  $C^S$  mapping to nodes in  $S$ .

(ii) We can also define the stable dual graph similarly.

**Definition 14.1.3.** Let  $\Gamma$  be a  $P$ -marked dual graph and let  $I$  be a subgraph having no legs and containing all the vertices of  $\Gamma$ . Let  $\Gamma_I$  be the graph that contracting each connected component of  $I$  to a point (one can speak this seriously, see [2] page 313). Hence we have a continuous map  $c_I : \Gamma \rightarrow \Gamma_I$ .

Actually there is a bijection between vertices of  $\Gamma_I$  and the connected components of  $I$ , we can let  $g_w(\Gamma_I) = g(I_w)$  where  $w$  be a vertex of  $\Gamma_I$  and  $I_w$  be the corresponding connected component.

A  $P$ -marked dual graph  $\Gamma'$  is said to be a specialization of  $\Gamma$  if  $\Gamma$  is isomorphic to  $\Gamma'_I$  for some  $I \subset \Gamma'$ . We call  $c : \Gamma' \rightarrow \Gamma \cong \Gamma'_I$  an  $I$ -contraction or simply a contraction. For example:

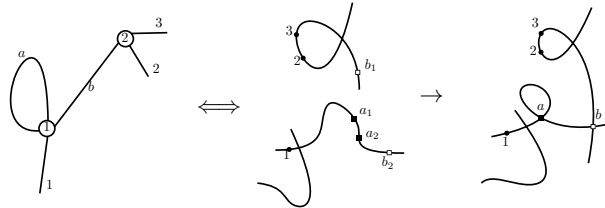


## 14.2 Boundary geometry II. More on gluing morphisms

### 14.2.1 Gluing via graphs

#### ► Gluing of curves.

Fix a  $P$ -marked dual graph  $\Gamma$  and for any  $v \in V$  we give a  $L_v$ -pointed nodal curve  $C_v$  of genus  $g_v$ . Let  $C' = \coprod_{v \in V} C_v$  and let  $C = C' / \sim$  where  $\sim$  means two points need to gluing together if and only if they are marked points labeled by the two halves of an edge of  $\Gamma$ . Hence  $C' \rightarrow C$  is actually a partial normalization. For example:



Here we need to note that this graph here is kind of partial diagram.

► **Gluing of families of curves.**

Fix a  $P$ -marked dual graph  $\Gamma$  and for any  $v \in V$  we give a family of stable  $L_v$ -pointed genus  $g_v$  curves  $F_v = (f_v : X_v \rightarrow S, \sigma_l \text{ where } l \in L_v)$ . Let  $X' = \coprod_v X_v$  and we get  $F' = (f' : X \rightarrow S, \sigma_i)$  a family of  $L$ -pointed nodal curves.

For any  $m \in L$ , by taking residue along  $\sigma_m$  we get a surjection

$$\omega_{f'}^k(k \sum \sigma_l) \rightarrow \mathcal{O}_{\sigma_m(S)}.$$

Hence we get

$$R_l^{(k)} : f_* \left( \omega_{f'}^k(k \sum \sigma_l) \right) \rightarrow \mathcal{O}_S$$

by some kind of positivity (see [2] Lemma X.6.1(i)) it is surjective for all  $k > 1$ . Consider

$$R^{(k)} : f_* \left( \omega_{f'}^k(k \sum \sigma_l) \right) \rightarrow \mathcal{O}_S^E$$

indexed by pairs of edges  $\{l, l'\}$  with components  $R_l^{(k)} + (-1)^{k-1} R_{l'}^{(k)}$ . The kernel of its fiber at  $s \in S$  is  $H^0(X_s, \omega_{f'}^k(k \sum_{p \in P} \sigma_{l_p}(s)))$  where  $X_s$  be the gluing of  $X'_s$  via  $\Gamma$ , hence its dimension is independent of  $s$ . Hence the kernel of  $R^{(k)}$ , which we denote by  $\mathcal{S}_k$ , is locally free. It is locally finitely generated (see [2] Corollary X.6.4). Let

$$X = \text{Proj}_S \bigoplus_{k \geq 0} \mathcal{S}_k$$

and hence the fibers of  $X \rightarrow S$  is gluing via  $\Gamma$ . Let  $\sigma'_p : S \rightarrow X$  is the composition of  $\sigma_{l_p}$  and  $X' \rightarrow X$ . Hence we get  $F = (X \rightarrow S, \sigma'_p)_{p \in P}$ .

### 14.2.2 Gluing functors

Fix a  $P$ -pointed dual genus  $g$  dual graph  $\Gamma$  and consider a Deligne-Mumford stack

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v}.$$

Fixed  $S$  and we let  $\eta = (\eta_v)_{v \in V} \in \overline{\mathcal{M}}_\Gamma(S)$  where  $\eta_v : X_v \rightarrow S$  be a family of stable  $L_v$ -pointed curves of genus  $g_v$ . The morphisms are isomorphisms between these families. Hence we get a gluing map via  $\Gamma$  to get  $\xi_\Gamma(\eta) : X \rightarrow S$ , a family of stable  $P$ -pointed genus  $g$  curves. Hence we get the gluing morphism of stacks

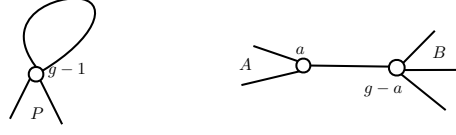
$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g, P}.$$

Let  $\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g, P}$  be a closed substack as

$$\mathcal{D}_\Gamma(S) = \left\{ \sigma : X \rightarrow S \text{ families of } P\text{-pointed stable curves of genus } g : \begin{array}{l} \text{fibers have dual graphs which are specializations of } \Gamma \end{array} \right\}$$

(as the image of  $\xi_\Gamma$ ). It is also a Deligne-Mumford stack with a coarse moduli space  $\Delta_\Gamma \subset \overline{M}_{g, P}$  as a closed subvariety. We often refer to the  $\mathcal{D}_\Gamma$  (or the  $\Delta_\Gamma$ ) as the boundary strata of  $\overline{\mathcal{M}}_{g, P}$  (or of  $\overline{M}_{g, P}$ ).

The simplest boundary strata are those of codimension 1 (as a divisor as before), which correspond to the stable graphs with a single edge. If we consider the following graphs:



then for the first we let  $\Gamma_{irr}$  and the second  $\Gamma_{a,A}$  (or  $\Gamma_{\mathcal{P}}$  if  $\mathcal{P} = \{(a, A), (b = g - a, B)\}$  be a stable bipartition). Hence we can also define  $\mathcal{D}_{irr} := \mathcal{D}_{\Gamma_{irr}}$  and  $\mathcal{D}_{a,A} := \mathcal{D}_{\Gamma_{a,A}}$  (or  $\mathcal{D}_{\mathcal{P}}$ ). The coarse case are the same  $\Delta_{irr}, \Delta_{a,A}, \Delta_{\mathcal{P}}$ . Moreover, in the case we get the old gluing way:

$$\xi_{irr} : \overline{\mathcal{M}}_{g-1, P \cup \{x, y\}} \rightarrow \overline{\mathcal{M}}_{g, P}, \xi_{a, A} : \overline{\mathcal{M}}_{a, A \cup \{x\}} \times \overline{\mathcal{M}}_{g-a, A^c \cup \{y\}} \rightarrow \overline{\mathcal{M}}_{g, P}.$$

**Definition 14.2.1** (Weak  $\Gamma$ -marking). *Consider a family of stable  $P$ -pointed genus  $g$  curves  $(\pi : \mathcal{C} \rightarrow S, \tau_P)$ . Let subvariety  $\Sigma \subset \text{Sing}(\mathcal{C})$  proper and étale over  $S$ , then for any  $s \in S$ , the fiber  $\Sigma_s$  be a finite set of nodes. Hence we can consider  $\text{Graph}^{\Sigma_s}(\mathcal{C}_s)$ .*

*Fix a  $P$ -marked graph  $\Gamma$  of genus  $g$ , if  $\text{Graph}^{\Sigma_s}(\mathcal{C}_s) \cong \Gamma$  for any  $s$ , then we call  $\Sigma$  is a weak  $\Gamma$ -marking. Hence we can define a stack  $\mathcal{E}_{\Gamma}$  as*

$$\mathcal{E}_{\Gamma}(S) = \left\{ \begin{array}{l} \pi : \mathcal{C} \rightarrow S \text{ families of } P\text{-pointed stable curves} \\ \text{of genus } g : \text{ endowed with a weak } \Gamma\text{-marking} \end{array} \right\}$$

**Definition 14.2.2** ( $\Gamma$ -marking). *If  $\mathcal{C} \rightarrow S$  coming from  $(X \rightarrow S) \in \overline{\mathcal{M}}_{\Gamma}(S)$  by gluing via  $\Gamma$  with  $\Sigma$  the locus of nodes produced by gluing. As  $\Sigma$  be a union of sections on  $\mathcal{C}$ , so is the preimage over  $X$  (partial normalization). Hence we can get  $\text{Graph}^{\Sigma}(\mathcal{C})$  with a family. Moreover  $\Gamma \cong \text{Graph}^{\Sigma}(\mathcal{C})$ . If these data exist for  $\mathcal{C} \rightarrow S$ , we called it endowed a  $\Gamma$ -marking.*

*Hence we can see that in this case we can do it conversely, hence we have  $\overline{\mathcal{M}}'_{\Gamma}$  as*

$$\overline{\mathcal{M}}_{\Gamma}(S) \Leftrightarrow \left\{ \begin{array}{l} \pi : \mathcal{C} \rightarrow S \text{ families of } P\text{-pointed stable curves} \\ \text{of genus } g : \text{ endowed with a } \Gamma\text{-marking} \end{array} \right\} := \overline{\mathcal{M}}'_{\Gamma}(S)$$

Hence we can find that the gluing map can be composited as

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{\Gamma} & & \searrow \xi_{\Gamma} & & \\ \downarrow \cong & & & & \\ \overline{\mathcal{M}}'_{\Gamma} & \xrightarrow{F} & \mathcal{E}_{\Gamma} & \xrightarrow{F'} & \mathcal{D}_{\Gamma} \hookrightarrow \overline{\mathcal{M}}_{g, P} \end{array}$$

where  $F, F'$  are forgetful maps.

**Proposition 14.2.3.** (i)  $\mathcal{E}_{\Gamma}$  be the normalization of substack  $\mathcal{D}_{\Gamma} \subset \overline{\mathcal{M}}_{g, P}$ ;  
(ii) The morphism  $\overline{\mathcal{M}}_{\Gamma} \rightarrow \mathcal{E}_{\Gamma}$  can be identified with  $\overline{\mathcal{M}}_{\Gamma} \rightarrow [\overline{\mathcal{M}}_{\Gamma} / \text{Aut}(\Gamma)]$ .

*Proof.* See [2] Proposition XII.10.11. □

**Corollary 14.2.4.** We can seen  $\text{Im}(\xi_{\Gamma}) = \mathcal{D}_{\Gamma}$  as before.

*Proof.* Trivial by the Proposition. □

**Corollary 14.2.5.** Let  $\Gamma$  be a stable  $P$ -marked dual graph of genus  $g$ . Assume that  $\text{Aut}(\Gamma) = \{\text{id}_{\Gamma}\}$ . Furthermore, assume that, for every graph  $\Gamma'$  which is a specialization of  $\Gamma$ , all the elements in  $\text{Aut}(\Gamma')$  are specializations of  $\text{id}_{\Gamma}$ . Then  $\xi_{\Gamma} : \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_{g, P}$  is a closed immersion.

*Proof.* See [2] Corollary XII.10.22.  $\square$

**Theorem 14.2.6.** *The map  $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$  is representable.*

*Proof.* ► **Step 1. Construct a new graph  $\widehat{\Gamma}$  from  $\Gamma$ .**

Fix an edge  $\ell = \{l, l'\} \in E(\Gamma)$ , consider the following graph  $\Gamma_l, \Gamma_{l'}$  and splitting  $\ell$  into  $l, l'$  and joint  $\Gamma_l, \Gamma_{l'}$ :



Repeat this operation for all edge of  $\Gamma$ , we get  $\widehat{\Gamma}$ . Hence  $\widehat{\Gamma}$  is  $P \cup H$ -marked where  $H$  the set of half-edges of  $\Gamma$  which are not legs.

► **Step 2. Decomposite  $\xi_\Gamma$  into closed immersion and projection.**

Consider maps

$$\begin{aligned} \iota_\Gamma : \overline{\mathcal{M}}_\Gamma &= \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \rightarrow \\ \overline{\mathcal{M}}_{\widehat{\Gamma}} &= \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l, l'\} \in E} (\overline{\mathcal{M}}_{0, \{l_0, l_1, l_\infty\}} \times \overline{\mathcal{M}}_{0, \{l'_0, l'_1, l'_\infty\}}) \end{aligned}$$

and

$$\xi_{\widehat{\Gamma}} : \overline{\mathcal{M}}_{\widehat{\Gamma}} = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l, l'\} \in E} (\overline{\mathcal{M}}_{0, \{l_0, l_1, l_\infty\}} \times \overline{\mathcal{M}}_{0, \{l'_0, l'_1, l'_\infty\}}) \rightarrow \overline{\mathcal{M}}_{g, P \cup H}$$

and  $\pi_H : \overline{\mathcal{M}}_{g, P \cup H} \rightarrow \overline{\mathcal{M}}_{g, P}$  be the natural projection. Then

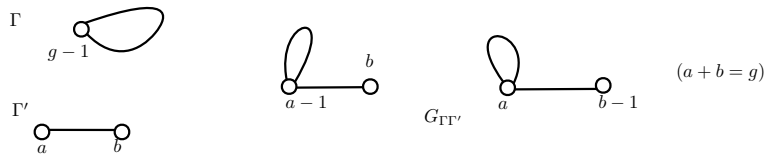
$$\xi_\Gamma = \pi_H \circ \xi_{\widehat{\Gamma}} \circ \iota_\Gamma.$$

As  $\iota_\Gamma$  is isomorphism (Why?) and  $\pi_H$  is representable (as a universal family) and  $\xi_{\widehat{\Gamma}}$  is a closed immersion by Corollary 14.2.5 and  $\text{Aut}(\widehat{\Gamma}) = \{\text{id}_{\widehat{\Gamma}}\}$  (Why?).  $\square$

It is important to describe how the various boundary strata intersect. Let  $\Gamma, \Gamma'$  are two  $P$ -marked dual graph of genus  $g$ . Consider

$$G_{\Gamma\Gamma'} = \left\{ \begin{array}{l} (\Lambda, c, c') / \cong : \Lambda \text{ be a } P\text{-marked dual graph of genus } g, \\ c : \Lambda \rightarrow \Gamma, c' : \Lambda \rightarrow \Gamma' \text{ are contractions with the} \\ \text{property that } E(\Lambda) = c^{-1}(E(\Gamma)) \cup c'^{-1}(E(\Gamma')) \end{array} \right\}.$$

For example:



**Proposition 14.2.7.** *If we let  $\overline{\mathcal{M}}_{\Gamma\Gamma'} := \overline{\mathcal{M}}_{\Gamma} \times_{\overline{\mathcal{M}}_{g,P}} \overline{\mathcal{M}}_{\Gamma'}$ , then*

$$\overline{\mathcal{M}}_{\Gamma\Gamma'} = \coprod_{\Lambda \in G_{\Gamma\Gamma'}} \overline{\mathcal{M}}_{\Lambda}.$$

*Proof.* Fix a scheme  $T$ .

First we let  $\xi : \mathcal{C} \rightarrow T$  in  $\overline{\mathcal{M}}_{\Lambda} = \overline{\mathcal{M}}'_{\Lambda}$ , then we are given a subvariety  $\Sigma \subset \text{Sing}(C)$ , proper and étale over  $T$ , whose inverse image in the partial normalization along  $\Sigma$  itself is a union of sections, plus an isomorphism  $\gamma : \text{Graph}^{\Sigma}(\mathcal{C}) \cong \Lambda$ . Let contractions  $c : \Lambda \rightarrow \Gamma$ ,  $c' : \Lambda \rightarrow \Gamma'$  and

$$\Sigma_1 = (c \circ \gamma)^{-1}(E(\Gamma)), \Sigma_2 = (c' \circ \gamma)^{-1}(E(\Gamma'))$$

such that  $\Sigma = \Sigma_1 \cup \Sigma_2$  with isomorphisms  $\gamma_1 : \text{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$  and  $\gamma_2 : \text{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$ . Hence  $\xi$  is both in  $\overline{\mathcal{M}}_{\Gamma}(T)$  and  $\overline{\mathcal{M}}_{\Gamma'}(T)$ . Hence in  $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$

Conversely, as we have

$$\overline{\mathcal{M}}_{\Gamma\Gamma'}(T) = \left\{ \begin{array}{l} (\xi, \xi', \phi) : \xi, \xi' \text{ are families of } \Gamma, \Gamma' \text{-marking stable} \\ P\text{-pointed genus } g \text{ curves over } T \text{ with } \phi : \xi \rightarrow \xi' \\ \text{a } T\text{-isomorphism} \end{array} \right\}.$$

Then let  $(\xi, \xi', \phi) \in \overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$ , hence we have  $\gamma : \text{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$  and  $\gamma' : \text{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$ . Hence we get contractions  $c, c' : \text{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}) \rightarrow \Gamma, \Gamma'$ . Hence we get  $(\text{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}), c, c') \in G_{\Gamma\Gamma'}$ , hence we win.  $\square$

### 14.3 Local structure of $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$

We also consider the case over  $\mathbb{C}$ . We will use the Kuranishi family and  $\nu$ -log canonical Hilbert scheme to describe the local structure of the moduli stack and (coarse) space of the stable curves.

Recall that we have the local structure of the Deligne-Mumford stack and its coarse moduli space, that is, the Theorem C.1.6 and Theorem C.1.5 as follows.

**Theorem A.** Let  $\mathcal{X}$  be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space  $S$ . Let  $\pi : \mathcal{X} \rightarrow X$  be its coarse moduli space. For any closed point  $x \in |X|$  with geometric stabilizer  $G_x$ , we have an étale neighborhood  $\text{Spec} A^{G_x} \rightarrow X$  of  $\pi(x) \in |X|$ .

**Theorem B.** Let  $\mathcal{X}$  be a separated Deligne-Mumford stack and  $x \in \mathcal{X}(k)$  be a geometric point with stabilizer  $G_x$ . Then exists an affine and étale map

$$f : ([\text{Spec} A / G_x], w) \rightarrow (\mathcal{X}, x)$$

where  $w \in (\text{Spec} A)(k)$  such that  $f$  induces an isomorphism of the stabilizer groups at  $w$ . Moreover, it can be arranged that  $f^{-1}(BG_x) \cong BG_w$ .

But now we will get a more coarse (but useful) local structure by using the Kuranishi family as follows. Actually as a set,  $\overline{M}_{g,n}$  is a set of isomorphism class of the  $n$ -pointed stable curves. Hence by Definition 8.5.3, for a  $n$ -pointed stable curve we have a standard Kuranishi family  $\xi : \mathcal{C} \rightarrow (X_0, x_0) \subset H_{\nu,g,n}$ . Hence we have a natural map

$$\psi : X_0 / G_{x_0} \rightarrow \overline{M}_{g,n}.$$

Recall some properties of  $X_0$  in Definition 8.5.3:

- For any  $y \in X_0$  we have  $G_y := \text{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \text{stab}_{G_{x_0}}(y)$ ;
- For any  $y \in X_0$ , there is a  $G_y$ -invariant analytic neighborhood  $U$  of  $y$  in  $X$  such that any isomorphism (of  $n$ -pointed curves) between fibers over  $U$  is induced by an element of  $G_y$ .



**Theorem 14.3.1.** *The map  $\psi : X_0/G_{x_0} \rightarrow \overline{M}_{g,n}$  is étale. Moreover there are finite many such  $X_i$  and  $G_i$  covers  $H_{\nu,g,n}$  such that the map*

$$\phi : Y := \coprod_i X_i/G_i \rightarrow \overline{M}_{g,n}$$

*is étale and surjective.*

*Proof.* See [2] Proposition XII.3.5. To add. □

**Theorem 14.3.2.** *The canonical map*

$$\alpha : X := \coprod_i X_i \rightarrow \overline{\mathcal{M}}_{g,n}$$

*is étale and surjective where  $X_i$  are Kuranishi families as before covers  $H_{\nu,g,n}$ .*

*Proof.* See [2] Theorem XII.8.3. To add. □



# Chapter 15

## Line bundles and Picard groups of the moduli of curves

We will refer [2] chapter XIII.

### 15.1 Line bundles on the moduli stack of stable curves

**Example 15.1.1** (Hodge bundle). *For any  $S \rightarrow \overline{\mathcal{M}}_{g,n}$  which correspond to  $\xi = (\pi : \mathcal{C} \rightarrow S)$ , we let  $\mathbb{E}_\xi := \pi_* \omega_\pi$ . Hence induce a sheaf  $\mathbb{E}$  over  $\overline{\mathcal{M}}_{g,n}$  called the Hodge bundle. As the relative dualizing sheaf is functorial with respect to morphisms of families, this is a quasi-coherent sheaf. By the cohomology and base change, it is actually a vector bundle of rank  $g$  as before. Let  $\det \mathbb{E} = \bigwedge^g \mathbb{E}$  and we call it the Hodge line bundle. Usually we denote  $\lambda := [\bigwedge^g \mathbb{E}] \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ .*

**Remark 15.1.2.** *For the canonical map  $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$  there are plenty of quasi-coherent sheaves on  $\overline{\mathcal{M}}_{g,n}$  which do not come by pullback from the quasi-coherent sheaves on  $\overline{M}_{g,n}$ . For example, Hodge bundle as follows.*

**Proposition 15.1.3.** *The Hodge bundle and its determinant do not descend to coherent sheaves on the moduli space  $\overline{M}_{g,n}$  except in genus zero.*

*Proof.* Consider a point  $\xi = (C; p_i)$ , then  $\mathbb{E}_\xi = H^0(C, \omega_C)$ . If  $\det \mathbb{E}$  comes from  $\overline{M}_{g,n}$ , then any automorphism of  $\xi$  will act trivially on  $\bigwedge^g H^0(C, \omega_C)$  by the basic theory of the coarse moduli space (Keel-Mori theory). Now we will find a curve and an automorphism of it which acts nontrivially.

For  $g$  odd, we let  $C$  be any hyperelliptic curve and consider hyperelliptic involution which acts as multiplication by  $-1$  on  $H^0(C, \omega_C)$ , hence nontrivial over  $\bigwedge^g H^0(C, \omega_C)$ ;

For  $g$  even, we let  $C$  be a ramified double covering of an elliptic curve and as  $\{p_i\}$  any set of  $n$  points which is invariant under the covering involution. The eigenvalues of the covering involution acting on  $H^0(C, \omega_C)$  are 1 with multiplicity 1 and  $-1$  with multiplicity  $g-1$  (**Why?**), hence the covering involution acts as  $-1$  on  $\bigwedge^g H^0(C, \omega_C)$ .  $\square$

**Example 15.1.4** (Generalization of the Hodge line bundle). *For any  $S \rightarrow \overline{\mathcal{M}}_{g,n}$  which corre-*

spond to  $\xi = (\pi : \mathcal{C} \rightarrow S)$  and any  $\nu \in \mathbb{Z}$ , we let

$$\Lambda(\nu)_\xi := \left( \bigwedge^{\max} R^1 \pi_* \omega_\pi^{\otimes \nu} \right)^{-1} \otimes \bigwedge^{\max} \pi_* \omega_\pi^{\otimes \nu}.$$

Hence induce a line bundle  $\Lambda(\nu)$  over  $\overline{\mathcal{M}}_{g,n}$ . Usually we denote  $\lambda(\nu) := [\Lambda(\nu)] \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ .

Actually when  $\nu = 1$ , by the same arguments in Lemma 2.1.1 we can show that  $R^1 \pi_* \omega_\pi \cong \mathcal{O}_S$ . So we have  $\Lambda(1)_\xi \cong \bigwedge^g \mathbb{E}_\xi$  canonically, hence  $\Lambda(1) \cong \bigwedge^g \mathbb{E}$ .

**Example 15.1.5** (Point bundles). For  $n > 0$  and for any  $S \rightarrow \overline{\mathcal{M}}_{g,n}$  which correspond to  $\xi = (\pi : \mathcal{C} \rightarrow S; \sigma_i)$ , we let  $(\mathcal{L}_i)_\xi := \sigma_i^* \omega_\pi$ . Hence we get  $\mathcal{L}_i$  be the line bundles over  $\overline{\mathcal{M}}_{g,n}$ . We usually Set

$$\psi_i = [\mathcal{L}_i] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}), \psi = \sum_i \psi_i.$$

**Remark 15.1.6.** As the Hodge bundle, in general,  $\mathcal{L}_i$  can't descend to a line bundle on  $\overline{M}_{g,n}$ .

**Example 15.1.7** (Boundary divisors and bundles). As before, we have

$$\partial \overline{\mathcal{M}}_{g,n} =: \mathcal{D} = \mathcal{D}_{irr} + \sum_{\mathcal{P}} \mathcal{D}_{\mathcal{P}},$$

where the sum runs through all stable bipartitions of  $(g, \{1, \dots, n\})$ . We denote

$$\delta_{irr} = [\mathcal{O}(\mathcal{D}_{irr})] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}), \delta_{\mathcal{P}} = [\mathcal{D}_{\mathcal{P}}] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}).$$

## 15.2 Tangent bundle, cotangent bundle and normal bundle

**Proposition 15.2.1.** Consider the moduli stack  $\overline{\mathcal{M}}_{g,P}$ , then tangent bundle  $\mathcal{T} = T_{\overline{\mathcal{M}}_{g,P}}$  can be described as: for any  $F = (f : X \rightarrow S, \{\sigma_p\}_{p \in P}) \in \overline{\mathcal{M}}_{g,P}(S)$ , we have

$$\mathcal{T}_F = f_*(\Omega_f^1 \otimes \omega_f(D))^\vee$$

where  $D = \sum \sigma_p(S)$ .

*Proof.* By Theorem 14.3.2, the Kuranishi families formed an étale covering. Hence consider a stable  $P$ -pointed curve  $\{C; x_p\}$  and its Kuranishi family (see Theorem 8.2.6)  $\mathcal{X} \rightarrow (U, u_0)$ , then we have

$$T_{u_0} U \cong \text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-\sum_p x_p)) \cong H^0(C, \Omega_C^1 \otimes \omega_C(\sum_p x_p))^\vee.$$

Hence we get the conclusion.  $\square$

**Example 15.2.2** (Canonical bundle). Hence the cotangent bundle  $\mathcal{T}^\vee$  given by  $\mathcal{T}_F^\vee = f_*(\Omega_f^1 \otimes \omega_f(D))$ . Hence we get the class of the canonical line bundle

$$K_{\overline{\mathcal{M}}_{g,P}} := \left[ \bigwedge^{\max} \mathcal{T}^\vee \right] \in \text{Pic}(\overline{\mathcal{M}}_{g,P}).$$

Now we consider the normal bundle of  $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$  where  $\Gamma$  be a stable graph (For a map of smooth schemes  $f : X \rightarrow Y$ , we let  $N_f = f^* T_Y / T_X$ ).

**Example 15.2.3** (Single curves). *Let  $N$  be a point of  $\overline{\mathcal{M}}_\Gamma$  with image  $C$  in  $\overline{\mathcal{M}}_{g,P}$ . One can consider  $N$  as a partial normalization of  $C$  at some  $y_e$ . By Claim 2 in Remark 8.2.5 we get*

$$0 \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - R)) \rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \bigoplus_{e \in E(\Gamma)} \text{Ext}^1(\Omega_{C, y_e}^1, \mathcal{O}_{C, y_e}) \rightarrow 0.$$

where  $D = \sum x_p$  with preimage  $\tilde{D}$  and  $R$  be the preimage of these  $y_e$ .

Easy to see that  $\text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - R))$  be the tangent space of  $\overline{\mathcal{M}}_\Gamma$  at  $N$  and  $\text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D))$  be the tangent space of  $\overline{\mathcal{M}}_{g,P}$  at  $C$ , hence the normal space to  $\xi_\Gamma$  at  $N$  is

$$\bigoplus_{e \in E(\Gamma)} \text{Ext}^1(\Omega_{C, y_e}^1, \mathcal{O}_{C, y_e}) = \bigoplus_{e \in E(\Gamma)} T_{N, y'_e} \otimes T_{N, y''_e}$$

by Claim 2 in Remark 8.2.4 (or Claim 3 in Remark 8.2.5).

**Proposition 15.2.4.** *The normal bundle of  $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$  can be expressed as*

$$N_{\xi_\Gamma} = \bigoplus_{\{l, l'\} \in E(\Gamma)} \eta_{v(l)}^* \mathcal{L}_l^\vee \otimes \eta_{v(l')}^* \mathcal{L}_{l'}^\vee$$

where  $\eta_v : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g_v, L_v}$  be the projections.

*Proof.* Let  $F$  in  $\overline{\mathcal{M}}_\Gamma$  is the datum of a family  $X_v \rightarrow S$  of stable  $L_v$ -pointed curves of genus  $g_v$  for each vertex  $v$  of  $\Gamma$ . We let  $X = \coprod_v X_v$ . For each  $l \in L(E)$ , we denote by  $\sigma_l$  the corresponding section of  $X \rightarrow S$ . The gluing construction yields a family  $X' \rightarrow S$  of stable  $P$ -pointed genus  $g$  curves. Then the normal

$$N_{\xi_\Gamma, F} = \bigoplus_{\{l, l'\} \in E(\Gamma)} \sigma_l^* T_{X/S} \otimes \sigma_{l'}^* T_{X/S} = \bigoplus_{\{l, l'\} \in E(\Gamma)} \mathcal{L}_{l, F}^\vee \otimes \mathcal{L}_{l', F}^\vee$$

where  $\mathcal{L}_i$  are point bundles. Hence we win.  $\square$

**Remark 15.2.5** (Excess intersection bundle). *By Proposition 14.2.7, we consider*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\Gamma\Gamma'} & = \coprod_{\Lambda \in G_{\Gamma\Gamma'}} \overline{\mathcal{M}}_\Lambda & \xrightarrow{\coprod \xi_{\Lambda\Gamma}} \overline{\mathcal{M}}_\Gamma \\ \downarrow & & \downarrow \xi_\Gamma \\ \overline{\mathcal{M}}_{\Gamma'} & \xrightarrow{\xi_{\Gamma'}} & \overline{\mathcal{M}}_{g,P} \end{array}$$

Then the excess intersection bundle is

$$F_{\Gamma\Gamma'} = \bigoplus_{\Lambda \in G_{\Gamma\Gamma'}} F_{\Lambda\Gamma\Gamma'} := \bigoplus_{\Lambda \in G_{\Gamma\Gamma'}} \xi_\Gamma^*(N_{\xi_{\Gamma'}})/N_{\xi_{\Lambda\Gamma}}.$$

We can show that (as [2] XIII.(3.8))

$$F_{\Gamma\Gamma'} = \bigoplus_{\Lambda \in G_{\Gamma\Gamma'}} \bigoplus_{\{l, l'\} \in c^{-1}(E(\Gamma)) \cap c'^{-1}(E(\Gamma'))} \eta_{v(l)}^* \mathcal{L}_l^\vee \otimes \eta_{v(l')}^* \mathcal{L}_{l'}^\vee.$$

**Corollary 15.2.6.** *We have*

$$\left[ \bigwedge^{\max} N_{\xi_\Gamma} \right] = - \sum_{l \in H(\Gamma)} \eta_{v(l)}^* \psi_l$$

where  $H(\Gamma)$  be the set of those half-edges of  $\Gamma$  which are not legs.

## 15.3 Determinant

### 15.3.1 Basic linear algebra

**Definition 15.3.1.** A  $\mathbb{Z}/2$ -graded line bundle is a pair  $(L, r)$  where  $L$  be a line bundle over a scheme  $X$  and  $r \in \{0, 1\}$ . We define the determinant of a finite vector bundle  $F$  over  $X$  is a  $\mathbb{Z}/2$ -graded line bundle

$$\det F := \left( \bigwedge^{\max} F, \text{rank } F \pmod{2} \right).$$

We say  $(L, r)$  is even/odd if  $r$  is even/odd. We define the tensor product of  $\mathbb{Z}/2$ -graded line bundles as  $(L, r) \otimes (T, s) := (L \otimes T, r + s)$ . Let  $A := (L, r), B := (T, s)$  and define the canonical isomorphism

$$\tau_{A,B} : A \otimes B \rightarrow B \otimes A, l \otimes m \mapsto (-1)^{rs} m \otimes l.$$

**Proposition 15.3.2.** (i) For  $\mathbb{Z}/2$ -graded line bundles  $A, B, C$  we have

$$\tau_{A \otimes B, C} = (\tau_{A, C} \otimes \text{id}) \circ (\text{id} \otimes \tau_{B, C});$$

(ii) For an exact sequence  $\mathcal{E} : 0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  of vector bundles, we have a canonical isomorphism  $\phi_{\mathcal{E}} : \det E \otimes \det G \rightarrow \det F$ ;

(iii) Define  $\mathbf{1}_X := (\mathcal{O}_X, 0)$  and  $A^{-1} = (L^{\vee}, a)$  for a  $\mathbb{Z}/2$ -graded line bundle  $A = (L, a)$ . Then  $A \otimes A^{-1} \cong \mathbf{1}_X, \alpha, \phi \mapsto \phi(\alpha)$  and

$$S_{A,B} : B^{-1} \otimes A^{-1} \cong (A \otimes B)^{-1}, \phi \otimes \psi \mapsto (\chi : \alpha \otimes \beta \mapsto \phi(\alpha)\psi(\beta));$$

(iv) We have  $\tau_{B,A}^{\vee} \circ S_{A,B} = S_{B,A} \circ \tau_{A^{-1}, B^{-1}}$ .

*Proof.* Trivial by some easy linear algebra and calculation.  $\square$

**Definition 15.3.3.** Let a finite complexes  $F^*$  of vector bundles on  $X$ , we define

$$\det F^* := \bigotimes_{q \in \mathbb{Z}} (\det F^q)^{(-1)^q}.$$

**Proposition 15.3.4.** (i) For a exact sequence of complexes  $\mathcal{E} : 0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow 0$ , we also have isomorphism

$$\phi_{\mathcal{E}} : \det E^* \otimes \det G^* \cong \det F^*;$$

(ii) The determinant and  $\phi_{\mathcal{E}}$  are functorial in the base space  $X$ ;

(iii) Consider

$$\begin{array}{ccccccc} & & \mathcal{E}_1 & & \mathcal{E}_2 & & \mathcal{E}_3 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_1 : & 0 \longrightarrow & A^* & \longrightarrow & B^* & \longrightarrow & C^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_2 : & 0 \longrightarrow & A'^* & \longrightarrow & B'^* & \longrightarrow & C'^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_3 : & 0 \longrightarrow & A''^* & \longrightarrow & B''^* & \longrightarrow & C''^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

then we have  $\phi_{\mathcal{E}_2} \circ (\phi_{\mathcal{R}_1} \otimes \phi_{\mathcal{R}_3}) = \phi_{\mathcal{R}_2} \circ (\phi_{\mathcal{E}_1} \otimes \phi_{\mathcal{E}_3}) \circ (\text{id} \otimes \tau_{C^*, A''^*} \otimes \text{id})$ ;

(iv) If  $A^*$  be a finite acyclic complex of vector bundles on  $X$ , there is a canonical isomorphism  $\det A^* \cong \mathbf{1}_X$ . More generally, if  $f : A^* \rightarrow B^*$  be a quasi-isomorphism of finite complexes of vector bundles, then there is an isomorphism  $\det f : \det A^* \rightarrow \det B^*$  which depends only on the homotopy class of  $f$ ;

(v) Consider

$$\begin{array}{ccccccc} \mathcal{E} : & 0 & \rightarrow & A_1^* & \rightarrow & A^* & \rightarrow A_2^* \rightarrow 0 \\ & & & \downarrow f_1 & & \downarrow f & \downarrow f_2 \\ \mathcal{E}' : & 0 & \rightarrow & B_1^* & \rightarrow & B^* & \rightarrow B_2^* \rightarrow 0 \end{array}$$

then  $\det f \circ \phi_{\mathcal{E}} = \phi_{\mathcal{E}'} \circ (\det f_1 \otimes \det f_2)$ ;

(vi) Consider the exact sequences  $\mathcal{E} : 0 \rightarrow A^* \xrightarrow{\alpha} B^* \rightarrow 0 \rightarrow 0$  and  $\mathcal{E}' : 0 \rightarrow 0 \rightarrow B^* \xrightarrow{\beta} C^* \rightarrow 0$ , then

$$\det \alpha = \phi_{\mathcal{E}} \circ (a \mapsto a \otimes 1), \det \beta = \phi_{\mathcal{E}'} \circ (b \mapsto 1 \otimes b).$$

*Proof.* These are more complicated linear algebra, we omit these here. We refer [2] XIII.4.  $\square$

### 15.3.2 Constructions and properties

**Proposition 15.3.5** (Determinant of the cohomology of coherent sheaves). *Aiming to construct the relative determinant of the cohomology here.*

• **Claim.** *Let  $f : X \rightarrow S$  be a flat morphism and let  $F$  be a coherent sheaf on  $X$  which is flat over  $S$ . Let  $Z$  be the subset of  $X$  where  $F$  is not locally free, then  $Z$  does not contain any component of any fiber of  $f$ .*

As  $F$  locally is  $\mathcal{O}_U^n \xrightarrow{\alpha} \mathcal{O}_U^n \rightarrow F|_U = \text{coker } \alpha \rightarrow 0$ , If an entire component of the fiber of  $f$  over a point  $s$  were contained in  $Z$ , the rank of  $\alpha$  would drop along the whole component. This could already be detected on a sufficiently thick infinitesimal neighborhood of  $s$ . Hence we may let  $S = \text{Spec } A$  where  $A$  be an artinian local ring. Let  $U = \text{Spec } B$  and  $F|_U = \widetilde{M}$  and we need to show  $U \not\subseteq Z$ . (Need to re-read. To add.)

• **Construction.** Consider a family of nodal curves  $\pi : X \rightarrow S$ . Let  $F$  is flat over  $S$ . Let  $S$  covered by  $U$  such that there is an effective Cartier divisor  $D$  in  $\pi^{-1}(U)$  which meets all the irreducible components of every fiber and does not contain any of them; in particular,  $D$  is relatively ample. We may replacing  $D$  with a multiple, then let  $R^1\pi_*F(D) = 0$ . By the Claim, we may also suppose that  $F$  is locally free at every point of  $D$ . We say such divisor admissible.

Hence  $F \subset F(D)$  and let  $F(D)|_D := F(D)/F$ . By some cohomology and base change, we get  $\pi_*F(D), \pi_*F(D)|_D$  are all locally free. We have

$$0 \rightarrow \pi_*F \rightarrow \pi_*F(D) \rightarrow \pi_*F(D)|_D \rightarrow R^1\pi_*F \rightarrow 0,$$

hence the complex  $E_D^* := (\pi_*F(D) \rightarrow \pi_*F(D)|_D)$  computes the higher direct image of  $F$ . Hence we let locally  $d_\pi F = \det E_D^*$ .

The independence on  $D$  and the gluing map are not hard to construct and we omitted them, see [2] page 356. Hence we get the determinant  $d_\pi F$  of the cohomology of  $F$  (relative to  $\pi$ ).

**Remark 15.3.6.** The flatness of  $F$  over  $S$  is unnecessary but simplifies the construction.

**Proposition 15.3.7** (Determinant of the (hyper)cohomology of complexes). *Consider a family of nodal curves  $\pi : X \rightarrow S$ . Similarly, Let  $F^*$  be a finite complex of coherent sheaves, flat over  $S$ . Let  $U$  be a sufficiently small open subset of  $S$ , and let  $D$  be a divisor in  $\pi^{-1}(U)$  which is*

admissible for each one of the  $F^i$  and that  $F^i \rightarrow F^{i+1}$  is a morphism of vector bundles at each point of  $D$  for each  $i$  (this is called admissible for  $F^*$ ). Hence we let  $E_D^{i,0} = \pi_*(F^i(D))$  and  $E_D^{i,1} = \pi_*(F^i(D)|_D)$ , then we get a double complex  $E_D^{*,*}$ . Regard it as a single complex graded by total degree, and we locally define  $d_\pi F^*$  to be its determinant.

**Proposition 15.3.8.** *Consider a family of nodal curves  $\pi : X \rightarrow S$ .*

(i) *For a coherent  $F$  with  $\pi_* F, R^1 \pi_* F$  are locally free, then we have*

$$d_\pi F \cong \det(R^1 \pi_* F)^{-1} \otimes \det(\pi_* F);$$

(ii) *For a finite complex  $F^*$  with  $\mathbb{R}^i \pi_* F^* := H^i R \pi_* F^*$  are locally free, then we have*

$$d_\pi F^* = \bigotimes_{i \in \mathbb{Z}} \det(\mathbb{R}^i \pi_* F^*)^{(-1)^i}.$$

*Proof.* I just prove (i) since (ii) is similar.

We split

$$0 \rightarrow \pi_* F \rightarrow \pi_* F(D) \rightarrow \pi_* F(D)|_D \rightarrow R^1 \pi_* F \rightarrow 0$$

into two sequences

$$0 \rightarrow \pi_* F \rightarrow E_D^0 \rightarrow Q \rightarrow 0, 0 \rightarrow Q \rightarrow E_D^1 \rightarrow R^1 \pi_* F \rightarrow 0.$$

Then we have  $\det(\pi_* F) \otimes \det Q \cong \det E_D^0$  and  $\det(R^1 \pi_* F) \otimes \det Q \cong \det E_D^1$ . Hence we have locally

$$d_\pi(F) = \det(E_D^1)^{-1} \otimes \det(E_D^0) = \det(R^1 \pi_* F)^{-1} \otimes \det(\pi_* F)$$

and well done.  $\square$

**Remark 15.3.9.** *These constructions are compatible with base change, hence for  $s \in S$  we have*

$$d_\pi F^* \otimes \kappa(s) \cong \bigotimes_{q \in \mathbb{Z}} (\det \mathbb{H}^q(X_s, F_s^*))^{(-1)^q}.$$

**Theorem 15.3.10.** *Let  $0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow 0$  be an exact sequence of finite complexes of coherent sheaves on  $X$ , all flat over  $S$ , then we have*

$$\phi : d_\pi(E^*) \otimes d_\pi(G^*) \cong d_\pi(F^*).$$

*Proof.* Not hard but it's hard to type and I omit it here. We refer [2] XIII.(4.17).  $\square$

### 15.3.3 Determinant, relative duality and applications

**Theorem 15.3.11.** *Consider a family of nodal curves  $\pi : X \rightarrow S$  and a coherent sheaf  $F$ , we have*

$$d_\pi(\omega_\pi \otimes F^\vee) \cong d_\pi(F).$$

*In particular, the Hodge bundle is  $d_\pi(\omega_\pi) = d_\pi(\mathcal{O}_X)$ .*

*Proof.* This is also not hard to prove by checking the construction of the determinant of cohomology. Using some canonical exact sequences and diagrams this is almost trivial. I omit these here and we refer [2] page 360.  $\square$



**Proposition 15.3.12** (Determinant and boundary of moduli). *We will describe  $\mathcal{O}(\mathcal{D})$  by using the determinant of cohomology.*

*Proof.* Let  $\pi : X \rightarrow S$  be a family of connected nodal curves of genus  $g$ .

•**Claim 1.**  $\Omega_\pi^1$  is  $S$ -flat.

WLOG we let  $S$  is smooth as these are pullbacked from a Kuranishi family. Shrinking  $S$ , we may assume that there exists an effective divisor  $D$  in  $X$  which cuts an ample divisor  $D_s$  on each fiber  $X_s$  and does not contain nodes of the fibers. We just need to show  $\chi(X_s, \Omega_\pi^1(nD) \otimes \kappa(s))$  is independent of  $s \in S$  for  $n \gg 0$ . By Corollary 6.4.5, we have

$$0 \rightarrow K \rightarrow \Omega_{X_s}^1 \xrightarrow{\rho_s} \omega_{X_s} \rightarrow Q \rightarrow 0$$

where  $\text{supp}(K), \text{supp}(Q) \subset \{\text{nodes}\}$ . As they both have one-dimensional stalks by Claim 1,2 in Corollary 6.4.5, hence we have  $\chi(X_s, \Omega_\pi^1(nD) \otimes \kappa(s)) = \chi(X_s, \omega_{X_s}(nD_s)) = 2g - 2 + n \deg(D)$ .

•**Claim 2.** Let  $L_\pi := d_\pi(\Omega_\pi^1 \xrightarrow{\rho_\pi} \omega_\pi)$ , then  $L_\pi = d_\pi(\omega_\pi) d_\pi(\Omega_\pi^1)^{-1}$  and induce  $\det(\rho_\pi) : d_\pi(\Omega_\pi^1) \rightarrow d_\pi(\omega_\pi)$  which is a canonical section of  $L_\pi$ .

As we have an exact sequence of complexes  $0 \rightarrow \omega_\pi[0] \rightarrow (\Omega_\pi^1 \xrightarrow{\rho_\pi} \omega_\pi) \rightarrow \Omega_\pi^1[1] \rightarrow 0$ , we get  $L_\pi = d_\pi(\omega_\pi) d_\pi(\Omega_\pi^1)^{-1}$ . The map  $\det(\rho_\pi) : d_\pi(\Omega_\pi^1) \rightarrow d_\pi(\omega_\pi)$  can be easily constructed step by step as the construction of  $d_\pi$ .

•**Claim 3.** Various  $L_\pi$  and  $\det(\rho_\pi)$  defines a line bundle  $L$  on  $\overline{\mathcal{M}}_{g,n}$  with a canonical section  $\det(\rho)$ . As  $\rho_\pi$  is an isomorphism on smooth fibers, we get  $L \cong \mathcal{O}(\sum_i n_i \mathcal{D}_i)$  where  $\mathcal{D}_i$  are components of  $\mathcal{D}$  with nonnegative integers  $n_i$ . We claim that all  $n_i = 1$  and hence  $L \cong \mathcal{O}(\mathcal{D})$ .

We consider the case when  $S$  is a disk (étale locally) centered at  $s$  and all the fibers of  $\pi$  are smooth except for  $X_s$ , which has a single node  $p$ . All we need is to calculate  $n_i$ , the order of vanishing of  $\det \rho_\pi$  at  $s$ . I omit it here and refer [2] page 363.  $\square$

**Proposition 15.3.13.** *Let  $\Gamma$  be a connected  $P$ -pointed genus  $g$  graph. Let  $H(\Gamma)$  for the set of the half-edges of  $\Gamma$  which are not legs. Suppose that for each  $v$ , we are given a family  $\pi_v : X_v \rightarrow S$  of connected nodal  $L_v$ -pointed genus  $g_v$  curves. Let  $\sigma_l$  the corresponding section of  $\pi_v$  and  $\mathcal{L}_l$  are point bundles on  $S$  where  $l \in L_v$ . Let  $D_l = \sigma_l(S)$  and let  $\pi : X \rightarrow S$  be the family gluing via  $\Gamma$  by  $X_v$ . Then*

$$\mathcal{O}(\mathcal{D})_\pi \cong \left( \bigotimes_{v \in V(\Gamma)} \mathcal{O}(\mathcal{D})_{\pi_v} \right) \otimes \left( \bigotimes_{h \in H(\Gamma)} \mathcal{L}_h^{-1} \right).$$

In particular, taking Chern classes we get

$$\xi_\Gamma^* \delta = \sum_{v \in V(\Gamma)} \eta_v^* \delta - \sum_{h \in H(\Gamma)} \eta_{v(h)}^* \psi_h$$

where  $\eta_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$  and  $\eta_v : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g_v, L_v}$ .

*Proof.* Let  $N = \coprod_v X_v \xrightarrow{\pi'} S$  with normalization  $\nu : N \rightarrow X$ . For  $e = \{h, h'\} \in E(\Gamma)$  and let  $\Sigma_e = \nu(D_h) = \nu(D_{h'})$ , we have

$$0 \rightarrow \omega_{\pi'} \rightarrow \omega_{\pi'} \left( \sum_{h \in H(\Gamma)} D_h \right) \xrightarrow{Res} \bigoplus_{\{h, h'\} \in E(\Gamma)} (\mathcal{O}_{D_h} \oplus \mathcal{O}_{D_{h'}}) \rightarrow 0$$

$$0 \rightarrow \omega_\pi \rightarrow \nu_* \left( \omega_{\pi'} \left( \sum_{h \in H(\Gamma)} D_h \right) \right) \rightarrow \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{\Sigma_e} \rightarrow 0$$

Taking cohomology we get

$$\begin{aligned} d_{\pi'}(\omega_{\pi'}) &\cong d_{\pi}(\nu_*\omega_{\pi'}) \cong d_{\pi}\left(\nu_*\omega_{\pi'}\left(\sum_{h \in H(\Gamma)} D_h\right)\right) \\ d_{\pi}(\omega_{\pi}) &\cong d_{\pi}\left(\nu_*\omega_{\pi'}\left(\sum_{h \in H(\Gamma)} D_h\right)\right). \end{aligned}$$

Hence

$$d_{\pi}(\omega_{\pi}) \cong d_{\pi'}(\omega_{\pi'}) \cong \bigotimes_{v \in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v}).$$

On the other hand, we have

$$0 \rightarrow K \rightarrow \Omega_{\pi}^1 \rightarrow \nu_*\Omega_{\pi'}^1 \rightarrow 0$$

which deduce

$$d_{\pi}(\Omega_{\pi}^1) \cong \left( \bigotimes_{e \in E(\Gamma)} \pi_* K_e \right) \otimes d_{\pi'}\Omega_{\pi'}^1.$$

•**Claim.** We have  $\pi_* K_{\{h, h'\}} \cong \mathcal{L}_h \otimes \mathcal{L}_{h'}$ .

Here we give a sketch of the claim. Consider  $e = \{h, h'\}$  with local coordinates  $x, y$ , then  $K_e$  locally generated by  $ydx (= -xdy)$ , then we define it mapping to section  $\sigma_h^*(dx) \otimes \sigma_{h'}^*(dy)$  of  $\mathcal{L}_h \otimes \mathcal{L}_{h'}$ . We omitted the verifying.

Finally, by the Claim 2,3 in Proposition 15.3.12 we get

$$\begin{aligned} \mathcal{O}(\mathcal{D})_{\pi} &= L_{\pi} = d_{\pi}(\omega_{\pi}) \otimes d_{\pi}(\Omega_{\pi}^1)^{-1} \\ &= \left( \bigotimes_{v \in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v}) \right) \otimes \left( \left( \bigotimes_{e \in E(\Gamma)} \pi_* K_e \right) \otimes d_{\pi'}\Omega_{\pi'}^1 \right)^{-1} \\ &= \left( \bigotimes_{v \in V(\Gamma)} \mathcal{O}(\mathcal{D})_{\pi_v} \right) \otimes \left( \bigotimes_{h \in H(\Gamma)} \mathcal{L}_h^{-1} \right) \otimes (d_{\pi'}\Omega_{\pi'}^1)^{-1}. \end{aligned}$$

(How to destroy  $(d_{\pi'}\Omega_{\pi'}^1)^{-1}$ ? Need to think this more.) □

**Remark 15.3.14.** We also get  $\xi_{\Gamma}^* \lambda = \sum_{v \in V(\Gamma)} \eta_v^* \lambda$ .

## 15.4 Deligne pairing, a quick tour

**Definition 15.4.1.** (a) Let  $C$  be a complete curve (need not be connected) and  $D = \sum_p n_p p$  a divisor on  $C$ . If  $f$  is a rational function on  $C$  whose divisor  $(f)$  is disjoint from  $D$ , we set  $f(D) := \prod_p f(p)^{n_p}$ ;

(b) Let  $\pi : X \rightarrow S$  be a family of nodal curves and  $D$  is an effective relative Cartier divisor not containing nodes of fibers,  $\pi_* \mathcal{O}(D)$  is locally free, and there is a norm map  $\text{Norm}_{D/S} : \pi_* \mathcal{O}(D) \rightarrow \mathcal{O}_S$  (as  $D \rightarrow S$  is proper and quasi-finite, hence finite). We also induce  $\text{Norm}_{D/S} : \pi_* \mathcal{O}(D)^{\times} \rightarrow \mathcal{O}_S^{\times}$ .

Hence for an divisor  $D = D_1 - D_2$  where  $D_i$  are effective, then we define

$$f(D) = \text{Norm}_{D_1/S}(f) \text{Norm}_{D_2/S}(f)^{-1}$$

which is well defined as if  $E_1, E_2$  are all effective, then  $f(E_1 + E_2) = f(E_1)f(E_2)$ .

**Proposition 15.4.2** (Weil reciprocity). (i) [Smooth case] Let  $C$  be a smooth proper curve (need not be connected) and  $f, g$  are rational functions which are nonzero on every component of  $C$  and with disjoint divisors. Then  $f((g)) = g((f))$ ;

(ii) [Nodal case] Let  $C$  be a possibly disconnected nodal curve, and let  $f$  and  $g$  be rational functions on  $C$  which do not vanish identically on any irreducible component of  $C$  and are regular and nonzero at all the nodes. Then, if the divisors of  $f$  and  $g$  are disjoint, we have  $f((g)) = g((f))$ ;

(iii) [Relative case] Let  $\pi : X \rightarrow S$  be a family of nodal curves and  $f$  and  $g$  are two meromorphic functions on  $X$  not vanish identically on any component of any fiber and be regular and nonzero at all the nodes, and their divisors be disjoint, then  $f((g)) = g((f))$ .

*Proof.* For (i) we refer [3] VI.B.2. For (ii), notice that what must be proved can reduce to the Weil reciprocity formula for the pullbacks of  $f$  and  $g$  to the normalization of  $C$ . For (iii), when  $S$  is reduced, we can do the same thing as single one. Otherwise, one can use the Kuranishi family and pullback.  $\square$

**Definition 15.4.3** (Deligne pairing for single case). Let  $L, M$  are two line bundles over a nodal curve  $C$ . Let  $V$  be a vector space generated by pairs  $(l, m)$  where  $l, m$  are rational sections of  $L, M$ , respectively, such that

- (a)  $l, m$  are nonzero on any component of  $C$ , and regular and nonzero at the nodes of  $C$ ;
- (b) the divisors  $(l)$  and  $(m)$  are disjoint.

Let  $\langle L, M \rangle$  be the quotient of  $V$  modulo the equivalence relation generated by

$$(fl, m) \sim f((m))(l, m), \quad (l, gm) \sim g((l))(l, m)$$

where  $f$  and  $g$  are rational functions on  $C$ . This space is the Deligne pairing of  $L$  and  $M$ . The class of  $(l, m)$  denoted by  $\langle l, m \rangle$ .

**Remark 15.4.4.** (i) Actually the meromorphic section  $l$  of  $L$  defined by a data  $(l_i, U_i)$  where  $l_i \in \mathcal{K}_C(U_i)$  of covering  $X = \bigcup_i U_i$  such that  $l_i = \psi_{ij} \cdot l_j$  where  $\psi_{ij} = \psi_i \circ \psi_j^{-1}$  are cocycles of trivializations  $\phi_i : L|_{U_i} \cong \mathcal{O}_{U_i}$ . In other words,  $l$  is a section of  $L \otimes_{\mathcal{O}_X} \mathcal{K}_X$ . Hence we have canonically divisor  $(l)$  associated to  $l$  and we have trivially  $\mathcal{O}_X((l)) \cong L$  (see [26] and [18]);

(ii) Two equivalence relations are called  $L$ -move and  $M$ -move, respectively.

**Proposition 15.4.5.** For any  $L, M$  on  $C$ , then  $\dim \langle L, M \rangle = 1$  be a line.

*sketch.* •**Claim 1.** We have  $\dim \langle L, M \rangle \leq 1$ .

For any  $(l, m), (l', m')$ , let  $\mu$  be a meromorphic divisor of  $M$  disjoint of  $l, l'$ . Hence let  $\mu = gm, m' = g'\mu, l' = fl$  where  $f, g, g'$  are rational functions, then

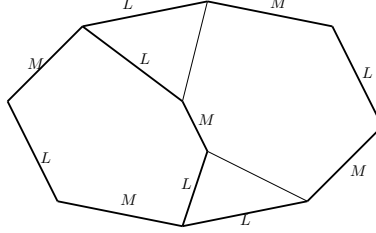
$$(l', m') \sim g'((l'))f((\mu))g((l))(l, m),$$

hence  $\dim \langle L, M \rangle \leq 1$ .

•**Claim 2.** A pair  $(l, m)$  cannot be equivalent to a strict multiple of itself (a cycle).

This is a very interesting proof by induction on the length of the cycle. After prove the case of 4 and 6 directly, we can let  $n \geq 8$  and using Weil reciprocity.

This method break a  $n$ -move cycle into two cycles of length  $n - 2$ , then one can use the induction. This proof is not so hard and much interesting, but I omit this and the detailed proof see [2] page 368. The main idea is the following diagram:



which tell us the cycle of 8 moves broken up in two cycles of 6.  $\square$

**Definition 15.4.6** (Deligne pairing for the families). A family  $\pi : X \rightarrow S$  of nodal curves and  $L$  and  $M$  are line bundles on  $X$ . For any  $s \in S$  we have a rank 1 free  $\mathcal{O}_{S,s}$ -module  $\langle L, M \rangle_s$  by Proposition 15.4.5. For any open  $U \subset S$ , we define a sheaf

$$\Gamma(U, \langle L, M \rangle_\pi) := \left\{ \left\{ u_s \in \langle L, M \rangle_s : s \in U \right\} \left| \begin{array}{l} \text{for every } s \in U, \text{ there are a neighborhood } U' \\ \text{and meromorphic sections } l, m \text{ of } L, M \text{ over} \\ \pi^{-1}(U') \text{ such that } u_t = \langle l, m \rangle \text{ for every } t \in U'. \end{array} \right. \right\}.$$

This is a line bundle on  $S$ , called the Deligne pairing of  $L$  and  $M$ , denoted by  $\langle L, M \rangle_\pi$ .

**Proposition 15.4.7.** Consider a family  $\pi : X \rightarrow S$  of nodal curves and  $L, L_1, L_2, L_3, M, M_1, M_2$  are line bundles on  $X$ .

(i) We have canonical isomorphisms

$$\begin{aligned} \langle L_1, M \rangle_\pi \otimes \langle L_2, M \rangle_\pi &\cong \langle L_1 \otimes L_2, M \rangle_\pi \\ \langle L, M_1 \rangle_\pi \otimes \langle L, M_2 \rangle_\pi &\cong \langle L, M_1 \otimes M_2 \rangle_\pi; \end{aligned}$$

(ii) We have canonical isomorphisms  $\langle L, \mathcal{O}_X \rangle_\pi \cong \mathcal{O}_S$  and  $\langle \mathcal{O}_X, M \rangle_\pi \cong \mathcal{O}_S$ ;

(iii) Of course, we have the canonical isomorphism  $\tau : \langle L, M \rangle_\pi \cong \langle M, L \rangle_\pi$  given by  $\langle l, m \rangle \mapsto \langle m, l \rangle$ . In particular when  $L = M$ , we have  $\tau(-) = (-1)^{\deg L} \cdot (-)$ .

*Proof.* See [2] XIII (5.4),(5.5) and Proposition 5.7.  $\square$

**Theorem 15.4.8.** Consider a family  $\pi : X \rightarrow S$  of nodal curves and  $L, M$  are line bundles on  $X$ . Then we have a canonical isomorphism

$$\langle M, L \rangle_\pi \cong d_\pi(L \otimes M) \otimes d_\pi(L)^{-1} \otimes d_\pi(M)^{-1} \otimes d_\pi(\mathcal{O}_X)$$

compatible with base change.

*Proof.* See [2] XIII Theorem 5.8.  $\square$

## 15.5 The Picard group of moduli space of curves

## 15.6 Formula of Mumford

## Chapter 16

# The kodaira dimension of moduli space of curves

We will refer [10] and [11] by using limit linear series instead of admissible covers in [23].



## Part IV

# Intersection Theory of moduli space of curves





We will refer [17].



## Part V

# Analytic construction via Teichmüller space



See [19] and [2] chapter XV.



## Part VI

# Cellular decomposition and cohomology





See [2] chapter XVIII and XIX.



## Part VII

# Alterations and the moduli space of stable curves



See He Tongmu's Answer for now. And [7] maybe used.



**Part VIII**

**Appendix**





# Appendix A

## Some basic result in scheme theory

### A.1 Some corollaries of semi-continuity theorem

**Review A.1.1** (Cohomology and Base Change, see [25] III.12.11). *Let  $f : X \rightarrow Y$  be a proper and finitely presented morphism of schemes with a finitely presented sheaf on  $X$  which is flat over  $Y$ . Let a point  $y \in Y$  and  $i \in \mathbb{Z}$ , the comparison map  $\phi_y^i : R^i f_* F \otimes \kappa(y) \rightarrow H^i(X_y, F_y)$  is surjective. Then*

(i) *There is an open neighborhood  $V \subset Y$  of  $y$  such that for any morphism  $g : Y' \rightarrow V$  of schemes, the comparison map  $\phi_{Y'}^i : g^* R^i f_* F \rightarrow R^i f'_*(g')^* F$  is an isomorphism. In particular  $\phi_y^i$  is an isomorphism;*

(ii)  *$\phi_y^{i-1}$  is surjective if and only if  $R^i f_* F$  is locally free in some neighborhood of  $y$ .*

**Review A.1.2** (Grauert's Corollary). (See [1] A.7.16) *Let  $f : X \rightarrow Y$  be a flat proper morphism of noetherian schemes such that  $h^0(X_y, \mathcal{O}_y) = 1$  for all  $y \in Y$  ( $\Leftrightarrow \mathcal{O}_Y = f_* \mathcal{O}_X$  and stable under base-change) (resp. the geometric fibers are integral).*

*For a line bundle  $L$  on  $X$ , consider the functor  $(Sch/Y) \rightarrow (Sets)$  by sending  $T \rightarrow Y$  to  $\{*\}$  if  $L_T$  is the pullback of a line bundle on  $T$  and to  $\emptyset$  otherwise. Then this functor is representable by a locally closed (resp. closed) subscheme of  $Y$ .*

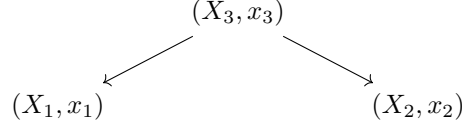
### A.2 Artin approximation and its corollaries

**Definition A.2.1.** *Let  $A \rightarrow B$  of noetherian rings is called geometrically regular if it is flat and for every prime ideal  $\mathfrak{p} \subset A$  and any finite field extension  $K/\kappa(\mathfrak{p})$ , the fiber  $B \otimes_A K$  is regular.*

*A noetherian local ring  $R$  is called a  $G$ -ring if  $A \rightarrow \hat{A}$  is geometrically regular.*

**Theorem A.2.2** (Artin approximation, see [1] A.10.9). *Let  $S$  be a scheme and  $s \in S$  be a point such that  $\mathcal{O}_{S,s}$  is  $G$ -ring. Let  $F : (Sch/S) \rightarrow (Sets)$  be a colimit preserving contravariant functor (commutes with systems of  $\mathcal{O}_S$ -algebras) and  $\hat{\xi} \in F(\text{Spec } \hat{\mathcal{O}}_{S,s})$ . For any integer  $N \geq 0$ , there exists an étale morphism  $(S', s') \rightarrow (S, s)$  and  $\xi' \in F(S')$  with  $\kappa(s) = \kappa(s)'$  such that the restrictions of  $\hat{\xi}$  and  $\xi'$  to  $\text{Spec}(\mathcal{O}_{S,s}/\mathfrak{m}_s^{N+1})$  are equal.*

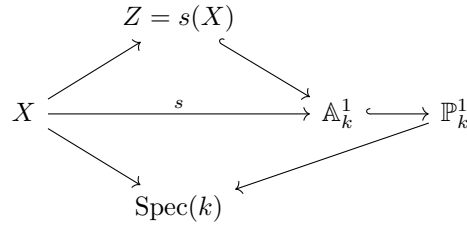
**Corollary A.2.3** (See [1] A.10.13). *Let  $X_1, X_2$  be schemes of finite type over  $S$  and let  $s \in S$  be a point such that  $\mathcal{O}_{S,s}$  is a  $G$ -ring. If  $x_1 \in X_1, x_2 \in X_2$  are points over  $s$  such that  $\widehat{\mathcal{O}}_{X_1, x_1}$  and  $\widehat{\mathcal{O}}_{X_2, x_2}$  are isomorphic as  $\mathcal{O}_{S,s}$ -algebras, then there exists a common residually-trivial étale neighborhood as*



### A.3 Miscellany

**Review A.3.1.** *Let  $k$  be a field and  $X$  be a proper geometrically connected and geometrically reduced  $k$ -scheme, then  $\Gamma(X, \mathcal{O}_X) = k$ .*

*Proof.* Just need to consider  $k$  is algebraically closed. Let  $s \in \Gamma(X, \mathcal{O}_X) = \text{Hom}(X, \mathbb{A}_k^1)$  and consider the following diagram:



Well done. □

**Review A.3.2** (Openness of ampleness). *Let  $X \rightarrow S$  be a proper morphism of schemes and  $L$  be a line bundle over  $X$ . Let  $S$  is noetherian. If for some  $s \in S$ , the fiber  $L_s$  over  $X_s$  is ample (resp. very ample), then exists an open neighborhood  $U$  of  $s$  such that  $L_U$  is ample (resp. very ample) over  $X_U$ .*

**Proposition A.3.3** (St 0C45). *Let  $X$  be a locally Noetherian scheme of dimension 1 with normalization  $f : \tilde{X} \rightarrow X$ . Then*

- (1)  $f$  is integral (finite if  $X$  is reduced locally finite type over a field), surjective, and induced a bijection on irreducible components;
- (2) there is a factorization  $\tilde{X} \rightarrow X_{\text{red}} \rightarrow X$  and the morphism  $\tilde{X} \rightarrow X_{\text{red}}$  is the normalization of  $X_{\text{red}}$  and birational;
- (3) for every closed point  $x \in X$ , stalk  $(f_* \mathcal{O}_{\tilde{X}})_x$  is the integral closure of  $\mathcal{O}_{X,x}$  in total ring of fractions of  $(\mathcal{O}_{X,x})_{\text{red}} = \mathcal{O}_{X_{\text{red}},x}$ ;
- (4)  $\tilde{X}$  is a disjoint union of integral normal Noetherian schemes.

**Proposition A.3.4** (0B5V). *Let  $R$  be a Noetherian ring. Let  $f : X \rightarrow Y$  be a morphism of schemes proper over  $R$ . Let  $L$  be an invertible  $\mathcal{O}_Y$ -module. Assume  $f$  is finite and surjective. Then  $L$  is ample if and only if  $f^*L$  is ample.*

## Appendix B

# Some results of resolution of singularities for surfaces

**Theorem B.0.1** (Minimal Resolutions). *Let  $X$  be a surface. There exists a unique projective birational morphism  $\pi : \tilde{X} \rightarrow X$  from a smooth surface such that every other resolution  $Y \rightarrow X$  factors as  $Y \rightarrow \tilde{X} \rightarrow X$  (or equivalently such that  $K_X \cdot E \geq 0$  for every  $\pi$ -exceptional curve  $E$ ).*

*Proof.* See [29] Theorem 2.16. □

**Theorem B.0.2** (Embedded Resolutions of Curves in Surfaces). *Let  $X$  be a surface and  $X_0 \subset X$  be a curve. There is a finite sequence of blow-ups at reduced points of  $X_0$  yielding a projective birational morphism  $\tilde{X} \rightarrow X$  such that  $\tilde{X}$  is smooth and such that the preimage  $\tilde{X}_0$  of  $X_0$  has set-theoretic normal crossings, i.e.  $(\tilde{X}_0)_{\text{red}}$  is nodal.*

*Proof.* See [29] Theorem 1.47. □

**Theorem B.0.3** (Castelnuovo's Contraction Theorem). *Let  $X$  be a smooth projective surface and  $E$  a smooth rational  $(-1)$ -curve. Then there is a projective morphism  $X \rightarrow Y$  to a smooth surface and a point  $y \in Y$  such that  $f^{-1}(y) = E$  and  $X \setminus E \rightarrow Y \setminus \{y\}$  is an isomorphism.*

*Proof.* See [29] Theorem 2.14. □

**Corollary B.0.4** (Existence of Relative Minimal Models). *A smooth surface  $X$  admits a projective birational morphism  $X \rightarrow X_{\min}$  to a smooth surface such that every projective birational morphism  $X_{\min} \rightarrow Y$  to a smooth surface is an isomorphism. In particular  $X_{\min}$  has no smooth rational  $(-1)$ -curves.*



# Appendix C

## Basic theory of algebraic spaces and stacks

### C.1 Some basic facts

**Theorem C.1.1.** (See [34] 8.3.3) Let  $\mathcal{X}/S$  be an algebraic stack, then the following statement

- (a)  $\mathcal{X}$  is a Deligne-Mumford stack;
- (b) the diagonal  $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$  is formally unramified;
- (c) for any algebraic closed field  $k$  and any point  $x \in \mathcal{X}(k)$ , the group scheme  $\underline{\text{Aut}}_x$  is reduced finite  $k$ -scheme.

Then (a) $\Leftrightarrow$ (b), and if  $\mathcal{X}$  noetherian, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

**Theorem C.1.2.** Let  $\mathcal{X}$  be a smooth noetherian algebraic stack over  $k$  and  $x \in \mathcal{X}(k)$  be a point with smooth stabilizer. Then

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

**Theorem C.1.3** (Valuative Criteria). Let  $f : \mathcal{X} \rightarrow \mathcal{Y}$  be a morphism of noetherian algebraic stacks. Assume  $f$  is of finite type and with separated diagonals. Then consider any DVR and its fraction field  $K$  with a 2-commutative diagram

$$\begin{array}{ccc} \text{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec} R & \longrightarrow & \mathcal{Y} \end{array}$$

Then

- (1)  $f$  is proper if and only if there exists an extension of DVRs  $R \rightarrow R'$  and  $K \rightarrow K'$  of fraction fields having finite transcendence degree and a lifting unique up to unique isomorphism

$$\begin{array}{ccccc} \text{Spec} K' & \longrightarrow & \text{Spec} K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec} R' & \longrightarrow & \text{Spec} R & \longrightarrow & \mathcal{Y} \end{array}$$

- (2)  $f$  is separated if and only if every two liftings of the first diagram are uniquely isomorphic.

(3)  $f$  is universally closed if for the first diagram, there exists an extension of DVRs  $R \rightarrow R'$  and  $K \rightarrow K'$  of fraction fields having finite transcendence degree and a lifting as in the second diagram.

*Proof.* See [1] Theorem 3.8.5 or St 0CLY.  $\square$

**Proposition C.1.4.** Let  $\mathcal{X}$  be a Deligne-Mumford stack with an étale cover  $X \rightarrow \mathcal{X}$ .

(A) Let  $\mathrm{Qcoh}(\mathcal{X})$  be the category of the quasi-coherent sheaves over  $\mathcal{X}$ , an object  $F$  of it defined as:

- (A1) A quasi-coherent sheaf  $F_f$  on  $S_{\mathrm{zar}}$  for any  $f : S \rightarrow \mathcal{X}$  where  $S$  be a scheme;
- (A2) An isomorphism  $\rho_H : h^*F_g \cong F_f$  for any 2-diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ f \downarrow & \searrow g & \\ \mathcal{X} & & \end{array}$$

of schemes;

(A3) For any pair of morphisms  $H_1 : f_1 \rightarrow f_2, H_2 : f_2 \rightarrow f_3$  where  $f_i : S_i \rightarrow \mathcal{X}$  are schemes, the diagram

$$\begin{array}{ccc} h_1^*(h_2^*(F_{f_3})) & \xrightarrow{\cong} & (h_2 \circ h_1)^*(F_{f_3}) \\ \downarrow h_1^*(\rho_{H_2}) & & \downarrow \rho_{H_2 \circ H_1} \\ h_1^*(F_{f_2}) & \xrightarrow{\rho_{H_1}} & F_{f_1} \end{array}$$

of isomorphisms of sheaves over  $S_1$  commutes.

(B) Let  $\mathrm{Eqcoh}(\mathcal{X})$  be the category of the extended quasi-coherent sheaves over  $\mathcal{X}$ , an object  $F$  of it defined as:

- (B1) A quasi-coherent sheaf  $F_f$  on  $S_{\mathrm{zar}}$  for any étale map  $f : S \rightarrow \mathcal{X}$  from a scheme;
- (B2) An isomorphism  $\rho_H : h^*F_g \cong F_f$  for any 2-diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ f \downarrow & \searrow g & \\ \mathcal{X} & & \end{array}$$

of étale maps of schemes;

(B3) For any pair of étale morphisms  $H_1 : f_1 \rightarrow f_2, H_2 : f_2 \rightarrow f_3$  where  $f_i : S_i \rightarrow \mathcal{X}$  are schemes, the diagram

$$\begin{array}{ccc} h_1^*(h_2^*(F_{f_3})) & \xrightarrow{\cong} & (h_2 \circ h_1)^*(F_{f_3}) \\ \downarrow h_1^*(\rho_{H_2}) & & \downarrow \rho_{H_2 \circ H_1} \\ h_1^*(F_{f_2}) & \xrightarrow{\rho_{H_1}} & F_{f_1} \end{array}$$

of isomorphisms of sheaves over  $S_1$  commutes.

(C) Let  $\mathrm{Qd}_X(\mathcal{X})$  be the category of the quasi-coherent sheaves over  $X$  with descent data related to  $X \rightarrow \mathcal{X}$ .

**Conclusion.** Then there are equivalence

$$\mathrm{Qcoh}(\mathcal{X}) \cong \mathrm{Eqcoh}(\mathcal{X}) \cong \mathrm{Qd}_X(\mathcal{X})$$

and their composition, in any one of the three possible orders, is isomorphic to the appropriate identity functor.

*Proof.* See [2] Proposition XIII.2.9.  $\square$

**Theorem C.1.5** (Local structure of DM-stacks). *Let  $\mathcal{X}$  be a separated Deligne-Mumford stack and  $x \in \mathcal{X}(k)$  be a geometric point with stabilizer  $G_x$ . Then exists an affine and étale map*

$$f : ([\mathrm{Spec} A / G_x], w) \rightarrow (\mathcal{X}, x)$$

where  $w \in (\mathrm{Spec} A)(k)$  such that  $f$  induces an isomorphism of the stabilizer groups at  $w$ . Moreover, it can be arranged that  $f^{-1}(BG_x) \cong BG_w$ .

*Proof.* See [1] Theorem 4.2.1.  $\square$

**Theorem C.1.6** (Local structure of coarse moduli space). *Let  $\mathcal{X}$  be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space  $S$ . Let  $\pi : \mathcal{X} \rightarrow X$  be its coarse moduli space. For any closed point  $x \in |\mathcal{X}|$  with geometric stabilizer  $G_x$ , there exists a cartesian*

$$\begin{array}{ccc} [\mathrm{Spec} A / G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec} A^{G_x} & \xrightarrow{s} & X \end{array}$$

such that  $s$  is an étale neighborhood of  $\pi(x) \in |X|$ .

*Proof.* Follows from the construction in the proof of the Keel-Mori theorem (see [1] Theorem 4.3.20). See [1] Corollary 4.3.23.  $\square$

## C.2 Miscellany

**Theorem C.2.1** (Le Lemme de Gabber). *Let  $\mathcal{X}$  be a Deligne-Mumford stack separated and of finite type over a noetherian scheme  $S$ . Then there exists a finite, generically étale and surjective morphism  $Z \rightarrow \mathcal{X}$  where  $Z$  be a scheme.*

**Proposition C.2.2.** *Let  $\mathcal{X}$  be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space  $S$ . Let  $\pi : \mathcal{X} \rightarrow X$  be the coarse moduli space. If  $\mathcal{L}$  is a line bundle on  $\mathcal{X}$ , then for  $N$  sufficiently divisible  $\mathcal{L}^{\otimes N}$  descends to  $X$ .*

*Proof.* See [1] Proposition 4.3.37.  $\square$

**Proposition C.2.3.** *Let  $G$  be an algebraic group acting on a scheme  $H$ , hence we get a quotient stack  $[H/G]$ . Then we have  $\mathrm{Qcoh}([H/G]) \cong \mathrm{Qcoh}(H, G)$  where the latter is the category of the  $G$ -equivariant quasi-coherent sheaf over  $H$ .*

*Proof.* See [2] Proposition XIII.2.19.  $\square$

**Corollary C.2.4.** *We have a group isomorphism  $\mathrm{Pic}([H/G]) \cong \mathrm{Pic}(H, G)$ .*





# Bibliography

- [1] Jarod Alper. *Notes on Stacks and Moduli*. <https://sites.math.washington.edu/~jarod/moduli.pdf>, 2022-9-28.
- [2] Enrico Arbarello, Maurizio Cornalba, and Phillip A. Griffiths. *Geometry of Algebraic Curves, Volume II*, volume 268. Springer, 2011.
- [3] Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths, and Joe Harris. *Geometry of Algebraic Curves, Volume I*, volume 267. Springer, 1985.
- [4] Michael Artin and Gayn Winters. Degenerate fibres and stable reduction of curves. *Topology*, 10(4):373–383, 1971.
- [5] The Stacks Project Authors. *The Stacks Project*. <https://stacks.math.columbia.edu/>, 2022.
- [6] G Codogni and Z Patakfalvi. Positivity of the cm line bundle for families of k-stable klt fano varieties. *Invent. math.*, 223:811–894, 2020.
- [7] Brian Conrad. Math 249b notes: Alterations. *Online Notes*, 2017.
- [8] Olivier Debarre. *Higher-Dimensional Algebraic Geometry*. Springer New York, NY, 2001.
- [9] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. *Publications Mathématiques de l’IHES*, 36:75–109, 1969.
- [10] David Eisenbud and Joe Harris. Limit linear series: Basic theory. *Invent. math.*, 85:337–371, 1986.
- [11] David Eisenbud and Joe Harris. The kodaira dimension of the moduli space of curves of genus  $\geq 23$ . *Invent. math.*, 90:359–387, 1987.
- [12] Finn F. Knudsen. The projectivity of the moduli space of stable curves, ii: The stacks  $m_{g,n}$ . *Mathematica Scandinavica*, 52(2):161–199, 1983.
- [13] Finn F. Knudsen. A closer look at the stacks of stable pointed curves. *Journal of Pure and Applied Algebra*, 216:2377–2385, 2012.
- [14] Lei Fu. *Algebraic Geometry*. Tsinghua University Press, 2006.
- [15] William Fulton. Hurwitz schemes and irreducibility of moduli of algebraic curves. *Ann. of Math.*, 90:542, 1969.

- [16] William Fulton. On the irreducibility of the moduli space of curves. *Invent. math.*, 67:87–88, 1982.
- [17] Letterio Gatto. *Intersection Theory on Moduli Space of Curves*. Instituto Nacional de Matemática Pura e Aplicada, 2000.
- [18] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 1994.
- [19] John H. Hubbard and Sarah Koch. An analytic construction of the Deligne-Mumford compactification of the moduli space of curves. *Journal of Differential Geometry*, 98:261 – 313, 2014.
- [20] Jack Hall. Moduli of singular curves. *Unknown*, 2010.
- [21] Jack Hall. The moduli stack of (all) curves. *Unknown*, 2013.
- [22] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187. Springer Science & Business Media, 2006.
- [23] Joe Harris and David Mumford. On the kodaira dimension of the moduli space of curves. *Invent. math.*, 67:23–86, 1982.
- [24] Robin Hartshorne. *Residues and Duality*, volume 20. Springer Berlin, Heidelberg, 1966.
- [25] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 1977.
- [26] Daniel Huybrechts. *Complex Geometry, An Introduction*. Springer Berlin, Heidelberg, 2005.
- [27] Sándor J. Kovács and Zsolt Patakfalvi. Projectivity of the moduli space of stable log-varieties and subadditivity of log-kodaira dimension. *J. Amer. Math. Soc.*, 30:959–1021, 2017.
- [28] János Kollár. Projectivity of complete moduli. *Journal of Differential Geometry*, 32:235–268, 1990.
- [29] János Kollár. *Lectures on Resolution of Singularities*, volume 166. PRINCETON UNIVERSITY PRESS, 2007.
- [30] Steven L. Kleiman. The picard scheme. *Arxiv: 0504020*, 2005.
- [31] Robert Lazarsfeld. *Positivity in Algebraic Geometry I*, volume 48. Springer Berlin, Heidelberg, 2004.
- [32] Qing Liu. *Algebraic Geometry and Arithmetic Curves*. Oxford University Press, USA, 2006.
- [33] Hideyuki Matsumura. *Commutative ring theory*. Cambridge university press, 1989.
- [34] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.
- [35] Edoardo Sernesi. *Deformations of Algebraic Schemes*. Springer-Verlag Berlin Heidelberg, 2006.

- [36] Ravi Vakil. *THE RISING SEA: Foundations of Algebraic Geometry*. Working Draft, August 29, 2022.
- [37] Eckart Viehweg. *Quasi-projective moduli for polarized manifolds*, volume 30. Springer, 1995.
- [38] Chenyang Xu and Ziquan Zhuang. On positivity of the cm line bundle on k-moduli spaces. *Ann. of Math*, 192:1005 – 1068, 2020.