Kuznetsov components, Stability, and Moduli Spaces

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Preface

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Chapter 1

Derived Category and Kuznetsov Components

1.1 Basic Definitions

Here we follows some definitions and results in [3]. Note that when I working in the derived category, we will omit the **R** or **L** of the derived functors.

1.1.1 Exceptional Sequences and S.O.Ds

Definition 1.1.1. *A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called admissible if the inclusion has a right adjoint $\pi : \mathscr{D} \to \mathscr{D}'$*

The orthogonal complement of a(an admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory $\mathscr{D}'^{\perp}$ of all objects $C \in \mathscr{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{D}'$.

Definition 1.1.2. *An object $E \in \mathscr{D}$ in a k-linear triangulated category $\mathscr{D}$ is called exceptional if*

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k, \text{if } \ell = 0, \\ 0, \text{if } \ell \neq 0. \end{cases}$$

An exceptional sequence is a sequence $E_1, ..., E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.

An exceptional sequence is full if $\mathscr{D}$ is generated by $\{E_i\}$.

An exceptional collection $E_1, ..., E_n$ is strong if in addition $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \neq 0$.

Definition 1.1.3. *A sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is semi-orthogonal if for all $i > j$ we have $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$. Such a sequence defines a semi-orthogonal decomposition (S.O.D.) of $\mathscr{D}$ if $\mathscr{D}$ is generated by the $\mathscr{D}_i$.*

Remark 1.1.4. *Some remarks:*

(a) If $E \in \mathscr{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle \subset \mathscr{D}$.

(b) Let $E_1, ..., E_n$ be an exceptional sequence in $\mathscr{D}$. Then the admissible triangulated subcategories $\langle E_1 \rangle, ..., \langle E_n \rangle$ form a semi-orthogonal sequence. If this sequence is a full exceptional sequence, then this forms an S.O.D. of $\mathscr{D}$.

(c) Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ defines an S.O.D. of $\mathscr{D}$, if and only if any object $A \in \mathscr{D}$ with $A \in \mathscr{D}_i^{\perp}$ for all $i = 1, ..., n$ is trivial. See Lemma 1.61 in [3].

(d) If $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is an S.O.D., then $D_1 \subset \langle \mathscr{D}_2, ..., \mathscr{D}_n \rangle^{\perp}$ is an equivalence. See Exercise 1.62 in [3].

Example 1.1.1 (Projective Bundle)**.** *For a smooth projective variety Y we consider the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to Y$ of locally free sheaf $\mathscr{E}$ of rank r on Y, in the sense of Grothendieck. Then for any $a \in \mathbb{Z}$ we claim that $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a + r - 1)$ is an S.O.D. of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$. This combined by the following two things:*
Step 1. *For any $E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(m), F \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(n)$, we have $\mathrm{Hom}(E, F) = 0$ for any $r - 1 \geq m - n > 0$.*

Indeed, we can let $m = 0$ and hence $-r + 1 \leq n < 0$. Let $E = \pi^*E'$ and $F = \pi^*F' \otimes \mathscr{O}(n)$, hence

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(E', \pi_*(\pi^*F' \otimes \mathscr{O}(n))) = \mathrm{Hom}(E', F' \otimes \pi_*\mathscr{O}(n)).$$

It's well-known that $\mathbf{R}^i\pi_*\mathscr{O}(n) = \begin{cases} \mathrm{Sym}^n\mathscr{E}, \text{for } i = 0, \\ 0, \text{for } 0 < i < r - 1, \\ \mathrm{Sym}^{-n-r}\mathscr{E}^\vee, \text{for } i = r - 1. \end{cases}$ *Well done.*

Step 2. *Categories $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a + r - 1)$ generates $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.*

Here we generalize the proof for $\mathbb{P}^n$ in [3] Corollary 8.29. Consider

$$\begin{tikzcd}
 & \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}) \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\wedge"] & \\
\mathbb{P}(\mathscr{E}) \arrow[dr, "\pi_1"'] & & \mathbb{P}(\mathscr{E}) \arrow[dl, "\pi_2"] \\
 & Y &
\end{tikzcd}$$

then by the canonical identification

$$
\begin{aligned}
& H^0(\mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}), \mathscr{O}(1) \boxtimes \mathscr{Q}^\vee) \\
& = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes p_* q^* \mathscr{Q}^\vee) \\
& = H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes \pi_1^* \pi_{2,*} \mathscr{Q}^\vee) \\
& = H^0(Y, \pi_{1,*} \mathscr{O}(1) \otimes \pi_{2,*} \mathscr{Q}^\vee) \\
& = H^0(Y, \mathscr{E} \otimes \mathscr{E}^\vee)
\end{aligned}
$$

where $0 \to \mathscr{Q} \to \pi^* \mathscr{E} \to \mathscr{O}(1) \to 0$ *is the universal exact sequence. Let* s *correspond to the* $\mathrm{id}_{\mathscr{E}}$*, then* $Z(s) = \Delta \subset \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E})$*. By the Koszul resolution of* $\mathscr{O}_\Delta$ *respect to the* s*, we have an exact sequence:*

$$
\begin{aligned}
& 0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \\
& \to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.
\end{aligned}
$$

(you can also use the Euler exact sequence instead of the universal exact sequence, just as in [3] Corollary 8.29)

Now there is to way to slove this.

The First Way: *for any coherent sheaf* $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}(\mathscr{E}))$*, tensoring* $q^* \mathscr{F}$ *we have*

$$
\begin{aligned}
& 0 \to \mathscr{O}(-r+1) \boxtimes \bigwedge^{r-1} \mathscr{Q} \otimes \mathscr{F} \to \mathscr{O}(-r+2) \boxtimes \bigwedge^{r-2} \mathscr{Q} \otimes \mathscr{F} \\
& \to \cdots \to \mathscr{O}(-1) \boxtimes (\mathscr{Q} \otimes \mathscr{F}) \to \mathscr{O} \boxtimes \mathscr{F} \to q^* \mathscr{F}|_\Delta \to 0.
\end{aligned}
$$

Consider a spectral sequence

$$
\begin{aligned}
E_1^{ij} & = \mathbf{R}^i p_*(\mathscr{O}(j) \boxtimes \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F}) = \mathscr{O}(j) \otimes \mathbf{R}^i p_* q^* \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F} \\
& = \mathscr{O}(j) \otimes \pi_1^* \mathbf{R}^i \pi_{2,*} \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F} \Rightarrow \mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta.
\end{aligned}
$$

We know that $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = 0$ *if* $i+j \neq 0$ *and* $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = \mathscr{F}$ *if* $i+j=0$*. Since any* E_1^{ij} *contained in*

$$
\left\langle \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(0) \right\rangle,
$$

so is $\mathscr{F}$*. Hence well done (if you use the Euler exact sequence instead of the universal exact sequence, the similar spectral sequence called the generalized Beilinson spectral sequence as Proposition 8.28 in [3]).*

The Second Way: *Consider again the Koszul resolution*

$$\begin{aligned} 0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \\ \to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Split it into short exact sequences

$$\begin{aligned} & 0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-2} \to 0, \\ & 0 \to M_{r-2} \to \bigwedge^{r-3}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-3} \to 0, \\ & \cdots, \\ & 0 \to M_1 \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Tensor product with q^*F and direct image under the first projection p yields distinguished triangles of Fourier-Mukai transforms:

$$\Phi_{M_{i+1}}(\mathscr{F}) \to \Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \to \Phi_{M_i}(\mathscr{F}) \to \Phi_{M_{i+1}}(\mathscr{F})[1].$$

Easy to see that

$$\Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-i) \right\rangle.$$

By induction we get $F = \Phi_{\mathscr{O}_\Delta} F \in \langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O} \rangle$. Well done.

Fully Exceptional Sequence. *By the discussed above, we know that pick any fully exceptional sequence $E_1, ..., E_n$ of Y, the set*

$$\{\pi^*E_1 \otimes \mathscr{O}(a), ..., \pi^*E_n \otimes \mathscr{O}(a), ..., \pi^*E_1 \otimes \mathscr{O}(a+r-1), ..., \pi^*E_n \otimes \mathscr{O}(a+r-1)\}$$

is a fully exceptional sequence of $\mathbb{P}(\mathscr{E})$ for any $a \in \mathbb{Z}$.

Example 1.1.2. *More general case, such as Grassmann bundle and even the flag bundle has the similar things. We refer [4].*

Example 1.1.3 (Blow-Up)**.**

1.2 Kuznetsov Components

Chapter 2

Examples of Fano Manifolds of Calabi-Yau Type

Chapter 3

Examples of Derived Equivalences of Kuznetsov Components with K3s

Chapter 4

Stability Conditions on K3 Categories

Chapter 5

Applications: Mukai's program

Chapter 6

Application to Cubic Fourfolds and Gushel-Mukai Manifolds

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Bibliography

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