Note for the Virtual Fundamental Class

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Contents

1 Introduction

We will follows [BF97][AB84][GP99] and we will also use [Ric22].

We need [Har77][Ful98][EH16].

Here we will consider $\mathbb{P}(-) = \mathbf{Proj}\,\mathrm{Sym}(-)^\vee$ for bundles and the vector bundle is both space and sheaf via $\mathbf{Spec}\,\mathrm{Sym}(-)^\vee$. For a cone $C = \mathbf{Spec}_X \mathscr{S}^*$, we define $\mathbb{P}(C) := \mathbf{Proj}_X \mathscr{S}^*$ and $\mathbb{P}(C \oplus \mathscr{O}) := \mathbf{Proj}_X \mathscr{S}^*[z]$ which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

2 Review of Basic Intersection Theory

We will follows [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field k.

2.1 Basic Facts of Refined Gysin Pullback

Here we will follows Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

Definition 2.1 (Intersection Product). *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $\mathscr{I}$ be the ideal of i and $\mathscr{J}$ be the ideal of j, then we have surjection

$$\bigoplus_n f^*(\mathscr{I}^n/\mathscr{I}^{n+1}) \to \bigoplus_n \mathscr{J}^n/\mathscr{J}^{n+1} \to 0$$

which induce embedding $C_{W/V} \hookrightarrow g^*N_{X/Y}$. Note that $C_{W/V}$ is also a scheme of pure dimension k since $\mathbb{P}(C_{W/V} \oplus \mathscr{O})$ is the exceptional divisor of $\mathrm{Bl}_W(Y \times \mathbb{A}^1)$. Let $0 : W \to g^*N_{X/Y}$ be the zero-section of $\pi : g^*N_{X/Y} \to W$, then we define

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \mathsf{CH}_{k-d}(W)$$

as the intersection class.

Proposition 2.2. *Consider the situation of Definition 2.1.*

(a) We have $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$.

(b) *Let $\mathscr{Q}$ be the universal quotient bundle of $q : \mathbb{P}(g^* N_{X/Y} \oplus \mathscr{O}) \to W$, then*

$$X \cdot V = q_*(c_d(\mathscr{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathscr{O})]).$$

(c) *If $j : W \hookrightarrow V$ is a regular embedding of codimension d', then $X \cdot V = c_{d-d'}(g^* N_{X/Y}/N_{W/V}) \cap [W]$.*

Proof. Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98]. □

Definition 2.3 (Refined Gysin Pullback)**.** *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick $f : Y' \to Y$ be a morphism. Consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Then we define $i^! : \mathsf{Z}_k\, Y' \to \mathsf{CH}_{k-d}\, X'$ as $\sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i$. Now $i^!$ can be decomposed as:

$$i^! : \mathsf{Z}_k\, Y' \xrightarrow{\sigma} \mathsf{Z}_k\, C_{X'/Y'} \to \mathsf{CH}_k(g^* N_{X/Y}) \xrightarrow{0^*} \mathsf{CH}_{k-d}\, X'$$

where $\sigma : \mathsf{Z}_k\, Y' \to \mathsf{Z}_k\, C_{X'/Y'}$ given by $[V] \mapsto [C_{V \cap X'/V}]$. By the technique of deformation to the normal cone, this can be descend to the Chow-group level as $\sigma : \mathsf{CH}_k\, Y' \to \mathsf{CH}_k\, C_{X'/Y'}$ (see Proposition 5.2 in [Ful98]) which is called the specialization to the normal cone*. Hence this induce the* refined Gysin pullback

$$i^! : \mathsf{CH}_k\, Y' \to \mathsf{CH}_{k-d}\, X', \quad \sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i.$$

Proposition 2.4. *Consider the situation of Definition 2.3. Consider*

$$\begin{tikzcd} X'' \arrow[r, hook, "i''"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y'' \arrow[d, "p"] \\ X' \arrow[r, hook, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

(a) *If p proper and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathsf{CH}_{k-d}(X')$.*

(b) *If p is flat of relative dimension n and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathsf{CH}_{k+n-d}(X'')$.*

(c) *If i' is also a regular embedding of codimension d and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! \alpha = (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'')$.*

(d) *If i' is a regular embedding of codimension d', then for $\alpha \in \mathsf{CH}_k(Y'')$ we have*

$$i^!(\alpha) = c_{d-d'}(q^*(g^* N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'').$$

We call $g^* N_{X/Y}/N_{X'/Y'}$ the excess normal bundle*.*

(e) *Let F be any vector bundle on Y', then for $\alpha \in \mathsf{CH}_k(Y'')$ we have*

$$i^!(c_m(F) \cap \alpha) = c_m((i')^* F) \cap i^!(\alpha) \in \mathsf{CH}_{k-d-m}(X').$$

Proof. See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98]. □

Corollary 2.5. *Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d, then*

$$i^* i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathsf{CH}_*(X).$$

Proof. By Proposition 2.4(d) directly. □

Proposition 2.6. *The refined Gysin pullback have the following properties.*

(a) *Let $i : X \hookrightarrow Y$ and $j : S \hookrightarrow T$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Y'' \arrow[r] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d, hook, "j"] \\
X' \arrow[r, hook, "i'"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[r, "g"'] \arrow[d, "f"'] & T \\
X \arrow[r, hook, "i"] & Y &
\end{tikzcd}$$

Then for any $\alpha \in \mathsf{CH}_k(Y'')$, we have

$$j^! i^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X'').$$

(b) *Let $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[r, hook, "j'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Z' \arrow[d, "f"'] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & Z
\end{tikzcd}$$

Then ji is a regular embedding of codimension $d+e$ and for all $\alpha \in \mathsf{CH}_k(Z')$ we have

$$(ji)^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X').$$

Proof. See Theorem 6.4 and Theorem 6.5 in [Ful98]. $\square$

Proposition 2.7. *Consider cartesians:*

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner"] & Y' \arrow[r, "p'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & Z' \arrow[d, "f"'] \\ X \arrow[r, "i"] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

(a) *If i is a regular embedding of codimension d and p and pi are flat of relative dimension $n, n-d$, respectively. Then i' is a regular embedding of codimension d and $p', p'i'$ are flat, and for $\alpha \in \mathsf{CH}_k(Z')$ we have*

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i^!((p')^*\alpha).$$

(b) *If i is a regular embedding of codimension d and p is smooth of relative dimension n, and pi is a regular embedding of codimension $d-n$ Then for $\alpha \in \mathsf{CH}_k(Z')$ we have*

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

Proof. See Proposition 6.5 in [Ful98]. $\square$

Remark 2.8. *Some remarks.*

(a) *For local complete intersection morphism $f : X \to Y$, we can decompose it into $f : X \xrightarrow{i} P \xrightarrow{p} Y$ where i is a closed regular embedding of constant codimension and p is smooth of constant relative dimension. Then we can define $f^! := i^!(p')^*$. See Section 6.6 in [Ful98] for more properties.*

(b) *If Y is nonsingular of dimension n, then we can define the following intersection product: Let $f : X \to Y$ and $p : X' \to X$ and $q : Y' \to Y$. Let $x \in \mathsf{CH}_k(X')$ and $y \in \mathsf{CH}_l(Y')$, consider the cartesian*

$$\begin{tikzcd} X' \times_Y Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \times Y' \arrow[d, "p\times q"] \\ X \arrow[r, "\gamma_f"] & X \times Y \end{tikzcd}$$

and define $x \cdot_f y := \gamma_f^!(x \times y) \in \mathsf{CH}_{k+l-n}(X' \times_Y Y')$.

So when $x, y \in \mathsf{CH}_*(Y)$*, then let* $X = Y$ *and* $X' = |x|, Y' = |y|$*, then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case* $CH_*(Y)$ *is a ring which is called* Chow ring*.*

Finally we will discuss something about equivalence and supportness.

Definition 2.9. *Let* $i : X \hookrightarrow Y$ *be a closed regular embedding of codimension* d *with normal bundle* $N_{X/Y}$*. Pick* V *be a scheme of pure dimension* k*. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $C_1, ..., C_r$ *be the irreducible components of* $C_{W/V}$*, then* $[C_{W/V}] = \sum_{i=1}^r m_i[C_i]$*. Let* $Z_i = \pi(C_i)$ *where* $\pi : g^*N_{X/Y} \to W$ *and we call them the* distinguished varieties *of the intersection of* V *by* X*. Let* $N_i := (g^*N_{X/Y})|_{Z_i}$ *and let* $0_i : Z_i \to N_i$ *be the zero-sections. Let* $\alpha_i := 0_i^*[C_i] \in \mathsf{CH}_{k-d}(Z_i)$ *and hence we have* $X \cdot V = \sum_{i=1}^r m_i\alpha_i \in \mathsf{CH}_{k-d}(W)$*.*

Pick any closed set $S \subset W$*, we define*

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i\alpha_i \in \mathsf{CH}_{k-d}(S)$$

as the part of $X \cdot V$ supported on S*.*

Definition 2.10. *Let* $X_i \hookrightarrow Y$ *be closed regular embeddings of codimension* d_i*. Let* $V \subset Y$ *be a* k*-dimensional subvariety. Consider*

$$\begin{tikzcd} \bigcap_i X_i \cap V \arrow[rr, hook] \arrow[d, hook] \arrow[drr, phantom, "\ulcorner"] & & V \arrow[d, hook, "\delta"] \\ X_1 \times \cdots \times X_r \arrow[rr, hook] & & Y \times \cdots \times Y \end{tikzcd}$$

Then we can get $X_1 \cdot \ldots \cdot x_r \cdot V \in \mathsf{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$*.*

Let Z *be a connected component of* $\bigcap_i X_i \cap V$*, we will consider*

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z \in \mathsf{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

Proposition 2.11. *As in the previous situation, we have*

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^{r} c(N_{X_i/Y}|_Z) \cap s(Z, V) \right\}_{\dim V - \sum_i d_i}.$$

If $Z \hookrightarrow V$ is a regular embedding, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^{r} c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If V, Z are both non-singular, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^{r} c(N_{X_i/Y}|_Z) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

Proof. See Proposition 9.1.1 in [Ful98]. □

2.2 Localized Chern Class

Here we will follows Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

Definition 2.12. *Let $E \to X$ be a vector bundle of rank e over a purely n-dimensional scheme X. Let $s : X \to E$ be a section, consider the cartesian*

$$\begin{tikzcd} Z(s) \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, hook, "s"'] \\ X \arrow[r, hook, "0"] & E \end{tikzcd}$$

with zero-section $0 : X \to E$ which is a regular section by trivial reason. We define

$$c_{\text{loc}}(E, s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \mathsf{CH}_{n-e}(Z(s))$$

be the localized (top) Chern class of E with respect to s.

Proposition 2.13. *Consider the situation of Definition 2.12.*

(a) We have $i_*(c_{\text{loc}}(E, s)) = c_e(E) \cap [X]$.

(b) Each irreducible component of $Z(s)$ has codimension at most e in X. If $\operatorname{codim}_{Z(s)} X = e$, then $c_{\text{loc}}(E, s)$ is a positive cycle whose support is $Z(s)$.

(c) If s is a regular section, then $c_{\text{loc}}(E, s) = [Z(s)]$.

(d) Let $f : X' \to X$ be a morphism, $s' = f^*s$ be a induced section of f^*E. Let $g : Z(s') \to Z(s)$ be the induced morphism.

(d1) If f flat, then $g^*c_{\mathrm{loc}}(E, s) = c_{\mathrm{loc}}(f^*E, s')$.

(d2) If f is proper of varieties, then $g_*c_{\mathrm{loc}}(f^*E, s') = \deg(X'/X)c_{\mathrm{loc}}(E, s)$.

Proof. For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_*0^![X] = 0^*s_*[X] = s^*s_*[X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{tikzcd} Z(s') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d, "s'"'] \\ X' \arrow[r, hook, "0_{f^*E}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f^*E \arrow[d] \\ X \arrow[r, hook, "0_E"] & E \end{tikzcd}$$

and Proposition 2.4. □

3 A Brief of Cotangent Complexes

Here we will give a quike introduction of cotangent complexes. We will consider Deligne-Mumford stacks locally of finite type over k. Morphisms are quasicompact and quasiseparated. We work over étale site.

Theorem 3.1. *For every morphism $f : X \to Y$ of DM-stacks (resp. finite type morphism of noetherian DM-stacks), there exists a complex*

$$\mathbb{L}_{X/Y} : \cdots \to \mathbb{L}^{-1}_{X/Y} \to \mathbb{L}^{0}_{X/Y} \to 0$$

of flat $\mathscr{O}_X$-modules with quasi-coherent (resp., coherent) cohomology, whose image $\mathbf{D}^-_{\mathrm{Qcoh}}(X_{ét})$ (resp. $\mathbf{D}^-_{\mathrm{Coh}}(X_{ét})$) is also denoted by $\mathbb{L}_{X/Y}$. This is called the cotangent complex of f. *It satisfies the following properties.*

(a) $H^0(X, \mathbb{L}_{X/Y}) = \Omega^1_{X/Y}$.

(b) The morphism f is smooth if and only if f is locally of finite presentation and $\mathbb{L}_{X/Y}$ is a perfect complex supported in degree 0. In this case, there is a quasi-isomorphism $\mathbb{L}_{X/Y} \cong \Omega^1_{X/Y}[0]$.

(c) *If f factors as $X \hookrightarrow Z$ defined by a sheaf of ideals $\mathscr{I}$ and a smooth morphism $Z \to Y$, then*

$$\mathbb{L}_{X/Y} \cong [0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{Z/Y}|_X \to 0]$$

in $\mathbf{D}^-_{\mathrm{Qcoh}}(X_{ét})$ with $\Omega^1_{X/Y}$ in degree 0. If in addition f is generically smooth, then $\mathbb{L}_{X/Y} \cong \Omega^1_{X/Y}[0]$. Moreover, if f is lci, then $\mathbb{L}_{X/Y}$ is perfect of perfect amplitude contained in $[-1, 0]$.

(d) *If we have a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

then there is a morphism $(\mathbf{L}g')^*\mathbb{L}_{X/Y} \to \mathbb{L}_{X'/Y'}$. When f or g is flat, then it is a quasi-isomorphism.

(e) *If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of DM-stacks, then there is an exact triangle*

$$\mathbf{L}f^*\mathbb{L}_{Y/Z} \to \mathbb{L}_{X/Z} \to \mathbb{L}_{X/Y} \to \mathbf{L}f^*\mathbb{L}_{Y/Z}[1]$$

in $\mathbf{D}^-_{\mathrm{Qcoh}}(X_{ét})$. This induces a long exact sequence on cohomology

$$\cdots \to H^{-1}(\mathbb{L}_{X/Z}) \to H^{-1}(\mathbb{L}_{X/Y}) \to f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$

Proof. In the level of ring maps $A \to B$, this constructed by standard simplicial free A-resolution $B \to P(B)_*$ where $P(B)_n = A[\cdots[A[B]]\cdots]$ as

$$\mathbb{L}_{B/A} := \Omega_{P(B)_*/A} \otimes_{P(B)_*} B.$$

See Tag 08UV Tag 0D0N Tag 0FK3 Tag 08QQ Tag 08T4. $\square$

Remark 3.2. *For the general algebraic stacks, any quasicompact and quasiseparated 1-morphism $f : \mathscr{X} \to \mathscr{Y}$ there exists a relative cotangent complex*

$$\mathbb{L}_f \in \mathbf{D}^{\leq 1}_{\mathrm{Coh}}(\mathscr{X}_{\textit{lis-ét}})$$

over lisse-étale site of $\mathscr{X}$. Existence is good, but the fact that the cotangent complex trespasses to positive degree forces one to pay more attention when performing the cutoff. If the diagonal of f is unramified (as we consider now), then this problem goes away, in the sense that $\mathbb{L}_f \in \mathbf{D}^{\leq 0}_{\mathrm{Coh}}(\mathscr{X}_{\textit{lis-ét}})$. We refer section C.3 in [Ric22] for more comments about this and the generalization of the properties as above.

4 Fundations of Virtual Fundamental Class

We will follows [BF97]. Here an algebraic stack (or Artin stack) over a field k is assumed to be quasi-separated and locally of finite type over k.

4.1 About Cones

We will let X be a Deligne-Mumford stack now.

Definition 4.1. *Let X be a DM-stack.*

(a) We call an affine X-scheme $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ is a cone over X *if the quasi-coherent algebra $\mathscr{S}$ is graded as $\mathscr{S} = \bigoplus_{i \geq 0} \mathscr{S}^i$ with $\mathscr{S}^0 = \mathscr{O}_X$ and $\mathscr{S}^1$ is coherent and $\mathscr{S}$ is generated by $\mathscr{S}^1$.*

(b) A morphism of cones over X *is an X-morphism induced by a graded morphism of graded sheaves of $\mathscr{O}_X$-algebras. A* closed subcone *is the image of a closed immersion of cones.*

Remark 4.2. *(a) The fiber product of cones over X is still a cone over X.*

(b) For every cone $C \to X$, it has a zero section $0 : X \to C$ induced by $\mathscr{S} \to \mathscr{S}^0$.

(c) For every cone $C \to X$, the grade induce a $\mathbb{G}_m$-action $\mathbb{G}_m \times C = \underline{\mathrm{Spec}}_X \mathscr{S}[t, t^{-1}] \to C$ induced by $\mathscr{S} \to \mathscr{S}[t, t^{-1}]$ via $s_0 + \cdots s_d \mapsto \sum_i a_i t^i$ where $s_i \in \mathscr{S}^i$. Since no negative power of t occurs, we can in fact replace $\mathbb{G}_m$ by $\mathbb{A}^1$. So we have the $\mathbb{A}^1$-action $\gamma : \mathbb{A}^1 \times C \to C$ induced by $\mathscr{S} \to \mathscr{S}[x]$ via $\mathscr{S}^i \ni s \mapsto sx^i$. Note that here $\mathbb{A}^1$ is not a group scheme and the action *here, as expected, to be the commutativity of the following diagrams:*

$$\begin{tikzcd} C \arrow[r, "{(1,\mathrm{id})/(0,\mathrm{id})}"] \arrow[dr, "\mathrm{id}/0"'] & \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"'] & \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ \mathbb{A}^1 \times C \arrow[r, "\gamma"'] & C \end{tikzcd}$$

where $m(x, y) = xy$.

(d) So a morphism of cones $f : C \to D$ over X is just the $\mathbb{A}^1$-equivariant X-morphism respecting the zero section, that is, the following commutativity of the diagram:

$$\begin{tikzcd} \mathbb{A}^1 \times C \arrow[r] \arrow[d, "\mathrm{id}\times f"'] & C \arrow[d, "f"'] \arrow[r, shift right] & X \arrow[l, shift right, "0_C"'] \arrow[dl, shift left, "0_D"] \\ \mathbb{A}^1 \times D \arrow[r] & D \arrow[ur, shift left] & \end{tikzcd}$$

Definition 4.3. Let $\mathscr{F}$ be a coherent sheaf of X, then we can define $C(\mathscr{F}) := \underline{\operatorname{Spec}}_X \operatorname{Sym}(\mathscr{F})$ which is a group scheme over X since it can be represented as $C(\mathscr{F})(T) = \operatorname{Hom}(\mathscr{F}_T, \mathscr{O}_T)$. We call a cone of this form is an *abelian cone over X*.

Remark 4.4. (a) A fibered product of abelian cones is an abelian cone.

(b) A vector bundle $E = \underline{\operatorname{Spec}}_X \operatorname{Sym}(\mathscr{E}^\vee)$ is a special case.

(c) Any cone $C = \underline{\operatorname{Spec}}_X \bigoplus_{i \geq 0} \mathscr{S}^i$ is canonically a closed subcone of an abelian cone $A(C) = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{S}^1$, called the *abelian hull of C*. The abelian hull is a vector bundle if and only if $\mathscr{S}^1$ is locally free.

(d) The *abelianization* $C \mapsto A(C)$ is a functor has the forgetful functor as a right adjoint. So we have

$$\operatorname{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \operatorname{Hom}_{\mathbf{Cone}_X}(C, A).$$

(e) Let $\mathbf{Alg}_X^o$ as the category of quasicoherent graded $\mathscr{O}_X$-algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:

$$\begin{tikzcd}
\mathbf{Alg}_X^o \arrow[r, "\underline{\operatorname{Spec}}_X"] & \mathbf{Cone}_X^{\mathrm{op}} \\
\mathbf{LocFree}_X \arrow[u, "\operatorname{Sym}"] \arrow[r, "\underline{\operatorname{Spec}}_X \operatorname{Sym}(-)^\vee"] \arrow[d, hook] & \mathbf{Vect}_X^{\mathrm{op}} \arrow[u, hook] \arrow[d, hook] \\
\mathbf{Coh}_X \arrow[r, "\underline{\operatorname{Spec}}_X \operatorname{Sym}"] & \mathbf{AbCone}_X^{\mathrm{op}}
\end{tikzcd}$$

Example 4.5. Tow important examples. Let $X \hookrightarrow Y$ be a closed immersion of ideal $\mathscr{I}$. Then $C_{X/Y} := \underline{\operatorname{Spec}}_X \bigoplus_{n \geq 0} \mathscr{I}^n / \mathscr{I}^{n+1}$ is called the *normal cone of X in Y*. The associated abelian cone $N_{X/Y} = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{I}/\mathscr{I}^2$ is called the *normal sheaf of X in Y*.

Lemma 4.6. About smoothness:

(a) Let $C = \underline{\operatorname{Spec}}_X \mathscr{S}$ be a cone over X. Then $C_{X/C} \cong \mathscr{S}^1 \cong 0^* \Omega_{C/X}$.

(b) A cone C over X is a vector bundle if and only if it is smooth over X.

(c) Let $C \to D$ be a smooth morphism of cones of relative dimension n over X. Then the induced morphism $A(C) \to A(D)$ is also smooth of relative dimension n.

Proof. For (a), note that $C_{X/C} \cong \mathscr{S}^1$ is trivial by definition. Morever, $0 : X \to C$ is the zero section and we have $0 \to C_{X/C} \to 0^*\Omega_{C/X} \to \Omega_{X/X} = 0$ exact (see Tag 0474). Well done.

For (b), let $C = \underline{\mathrm{Spec}}_X \bigoplus_{i\geq 0} \mathscr{S}^i$ and assume that $C \to X$ has constant relative dimension r. Then $\mathscr{S}^1 = 0^*\Omega_{C/X}$ is locally free of rank r. As $C \hookrightarrow A(C)$ where $A(C)$ is a vector bundle and $\dim C = \dim A(C)$, we know that C is a vector bundle.

For (c), apply the exact triangle of cotangent complex to $X \to C \to D$ and (a), we have an exact sequence

$$0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to 0_C^*\Omega_{C/D} \to 0$$

where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. So locally we have $A(C) = A(D) \times_X \underline{\mathrm{Spec}}_X \mathrm{Sym}(0_C^*\Omega_{C/D})$. Well done. $\square$

Definition 4.7. *A sequence of cone morphisms*

$$0 \to E \xrightarrow{i} C \to D \to 0$$

is called exact *if E is a vector bundle and locally over X there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \cong E \times_X D$.*

Remark 4.8. *As $E \to X$ is smooth and surjective by Lemma 4.6, if $0 \to E \xrightarrow{i} C \to D \to 0$ then locally we have $C \cong E \times_X D$ which force that $C \to D$ is smooth and surjective! Similarly $i : E \to C$ is a closed embedding.*

Lemma 4.9. *We have the following useful results.*

(a) Given a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0$ of coherent sheaves on X, with $\mathscr{E}$ locally free, then $0 \to C(\mathscr{E}) \to C(\mathscr{F}) \to C(\mathscr{F}') \to 0$ is exact, and conversely is also true.

(b) Let $0 \to E \to F \xrightarrow{f} G \to 0$ be an exact sequence of abelian cones over X with E a vector bundle. Assume that $D \subset G$ is a closed subcone, then the induced sequence $0 \to E \to f^{-1}(D) =: C \to D \to 0$ is exact.

(c) Let $f : C \to D$ be a morphisms of cones over X which is smooth surjective, then the induced diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, hook] & D \arrow[d, hook] \\ A(C) \arrow[r, "A(f)"] & A(D) \end{tikzcd}$$

is cartesian. Moreover, we have $D = [C/E]$ (see Lemma 4.12(a)) and $A(D) = [A(C)/E]$, where $E := C \times_{D,0} X = A(C) \times_{A(D),0} X$.

(d) Let E be a vector bundle over X and then the sequence $0 \to E \to C \to D \to 0$ is exact if and only if the abelian hulls $0 \to E \to A(C) \to A(D) \to 0$ is exact and $C \to D$ is smooth and surjective.

Proof. For (a), we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. As exactness is local, we may assume $\mathscr{E}$ is free. Then the first sequence is exact if and only if $\mathscr{F}' \oplus \mathscr{E} = \mathscr{F}$ if and only if the second sequence is exact as cones, since $\mathrm{Sym}(\mathscr{F}' \oplus \mathscr{E}) = \mathrm{Sym}(\mathscr{F}') \otimes \mathrm{Sym}(\mathscr{E}) = \mathrm{Sym}(\mathscr{F})$.

For (b), note that this can be checked locally, so we can let we can assume that $\mathscr{F} = \mathscr{G} \oplus \mathscr{E}^\vee$ where $E = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{E}^\vee$ and $F = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{F}$ and $G = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{G}$. Let $D = \underline{\mathrm{Spec}}_X \mathscr{T}$, then we have surjection $\mathrm{Sym}(\mathscr{G}) \to \mathscr{T}$. By definition, we have

$$\begin{aligned} C = F \times_G D &= \underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathscr{F}) \otimes_{\mathrm{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\mathrm{Spec}}_X((\mathrm{Sym}(\mathscr{G}) \otimes \mathrm{Sym}\, \mathscr{E}^\vee) \otimes_{\mathrm{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\mathrm{Spec}}_X(\mathrm{Sym}\, \mathscr{E}^\vee \otimes \mathscr{T}). \end{aligned}$$

This means locally $C = E \oplus D$ and the splitting $C \to E$ is induced by $F \to E$. Well done.

For (c), let $E := C \times_{D,0} X$ and $E' := A(C) \times_{A(D)} D$ with embedding $E \hookrightarrow E'$, then both of them are vector bundles by Lemma 4.6(b)(c) and hence $E = E'$. We have cartesians

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook] \\ A(C) \arrow[r] & A(D) \end{tikzcd}$$

By the properties of commutative affine group schemes, we have $A(D) = [A(C)/E]$. But how about $[C/E]$? Now we have

$$\begin{tikzcd} & & D \arrow[dl, hook] \\ C \arrow[r] \arrow[urr] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & {[C/E]} \arrow[d, hook] & \\ A(C) \arrow[r] & A(D) & \end{tikzcd}$$

Since $C \to [C/E]$ and $C \to D$ are both smooth and surjective, we know that $[C/E] \to D$ is flat and surjective. But by closed embeddings $[C/E] \to A(D)$ and $D \to A(D)$, we know that $[C/E] \to D$ is also a closed embedding. Thus $D = [C/E]$, well done.

For (d), note that all the question is locally on X. First we assume $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ is exact. Then by (a), to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, we only need to show that $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact where $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. First since f is faithfully flat and quasi-compact, we know that $\mathscr{T}^1 \to \mathscr{S}^1$ is injective. And since i is a closed embedding, $\mathscr{S}^1 \to \mathscr{E}^\vee$ is surjective. Now by local splitting, we know that locally we have $\operatorname{Sym}(E^\vee) \otimes \mathscr{T} = \mathscr{S}$. In particular, we have $\mathscr{T}^1 \oplus \mathscr{E}^\vee = \mathscr{S}^1$. Thus the exactness of $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is obtained. Conversely we assume that after taking abelian hull, the sequence is exact. Now the result follows from (a) and (c). □

Proposition 4.10. *Let $C \to D$ be a smooth, surjective morphism of cones. If we let $E = C \times_{D,0} X$, then the sequence*

$$0 \to E \to C \to D \to 0$$

is exact. Conversely if $0 \to E \to C \to D \to 0$ is exact, then $E \cong C \times_{D,0} X$.

Proof. Let $C = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathscr{S}^i$ and $D = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathscr{T}^i$.

Let $E = C \times_{D,0} X = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$, by Lemma 4.9(d) we just need to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, that is, $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact by Lemma 4.9(a). Note that $\operatorname{Sym} \mathscr{E}^\vee = \mathscr{S} \otimes_{\mathscr{T}} (\mathscr{T}/\mathscr{T}^{\geq 1})$ which force $\mathscr{E}^\vee \cong \mathscr{S}^1/\mathscr{T}^1$. Well done.

Conversely, assume that the sequence $0 \to E \to C \to D \to 0$ is exact and $F = C \times_{D,0} X$. Then by the universal property of fibre product, we get a morphism $E \to F$. From the construction, it is easy to see that $\mathscr{F}^\vee \to \mathscr{E}^\vee$ surjective. Since they are both bundles of the same rank over X, we know that $E = F$. □

Definition 4.11. *(a) If E is a vector bundle and $f : E \to C(\mathscr{F})$ a morphism of abelian cones. The there is an E-action as $E \times_X C(\mathscr{F}) \to C(\mathscr{F})$ as $(\nu, \gamma) \mapsto f\nu + \gamma$.*

(b) If E is a vector bundle and $d : E \to C$ a morphism of cones, we say that C is an E-cone, if C is invariant under the action of E on $A(C)$.

(c) A morphism ϕ from an E-cone C to an F-cone D is a commutative diagram of cones

$$\begin{tikzcd} E \arrow[r, "d_E"] \arrow[d, "\phi"] & C \arrow[d, "\phi"] \\ F \arrow[r, "d_F"] & D \end{tikzcd}$$

(d) If $\phi : (E, d_E, C) \to (F, d_F, D)$ and $\psi : (E, d_E, C) \to (F, d_F, D)$ are morphisms, we call them homotopic, if there exists a morphism of cones $k : C \to F$, such that $kd_E = \psi - \phi = d_F k$.

Lemma 4.12. *Some useful lemmas:*

(a) *Let $f : C \to D$ be a smooth surjective cone morphism with $E = C \times_{D,0} X$, then C is an E-cone.*

(b) *Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D = [C/E] \to 0$ be a sequence of algebraic X-spaces with E a bundle, C is a E-cone, i a closed embedding and $f : C \to D = [C/E]$ is the universal family. Then locally on X, there is a $j : C \to E$ split i and induces an isomorphism $(f, j) : C \to D \times_X E$.*

(c) *Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ be a sequence of algebraic X-spaces with sections and $\mathbb{A}^1$-actions such that E a bundle, C is a E-cone, i is a closed embedding and f is $\mathbb{A}^1$-equivariant. Then D is a cone with the sequence exact if and only if $D \cong [C/E]$.*

Proof. For (a), this follows from directly check. We omit it.

For (b), since the question is local we can assume that E is a trivial bundle and X is a scheme. Let $i' : E \to A(C)$ and $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Then the surjection $\mathscr{S}^1 \twoheadrightarrow \mathscr{E}^\vee$ has a splitting $\mathscr{E}^\vee \hookrightarrow \mathscr{S}^1$, which gives $j' : A(C) \to E$ such that $j' \circ i' = \mathrm{id}_E$. Then we just define $j : C \to E$ as composition with $C \to A(C)$ and j'. Hence $j \circ i = \mathrm{id}_E$.

Now since $C \to D$ is also a principal E-bundle, and we have a E-equivariant D-morphism $(f, j) : C \to D \oplus E$ from C to the trivial principal bundle. Since they are both E-principal bundle, we know that (f, j) is an isomorphism.

For (c), let $D = [C/E]$. We know that $D \to X$ is affine since locally on X we have $C \cong D \times_X E \to E$ is affine and (b) and faithfully flat descent. By construction we have $E = C \times_{D,0} X$, hence by Proposition 4.10 we just need to show D is a cone. Now as $D \to X$ affine we have $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. If $C = \underline{\mathrm{Spec}}_X \mathscr{S}$, then $\mathscr{T} \subset \mathscr{S}$ as $C \to D$ is faithfully flat. Hence it has graded structure $\mathscr{T} = \bigoplus_{i \geq 0} \mathscr{T} \cap \mathscr{S}^i$ as f is $\mathbb{A}^1$-equivariant. As it have zero section, we have $\mathscr{T}^0 = \mathscr{O}_X$. Finally we have $\mathbb{A}^1$-equivariant embedding $D \hookrightarrow [A(C)/E]$ and $[A(C)/E]$ is a cone by Lemma 4.9(c). Hence $\mathscr{T}$ generated by the coherent sheaf $\mathscr{T}^1$.

Conversely, we assume D is a cone and that sequence is exact. Let $D' = [C/E]$. By the universal property of quotient, we have a natural map $g : D' \to D$. Since $0 \to E \to C \to D' \to 0$ is also exact by the first case, by exactness we have locally $C \cong E \times_X D \cong E \times_X D'$. Note that these isomorphisms compatible with $g : D' \to D$, hence by faithfully flat descent we have g is an isomorphism. $\square$

Proposition 4.13. *Let X be a DM-stack.*

(a) *Let E be a vector bundle. Consider the sequence of cone morphisms $0 \to E \xrightarrow{i} C \xrightarrow{\phi} D \to 0$ with i a closed embedding. Then it is exact if and only if C is a E-cone, $\phi : C \to D$ is faithfully flat and the diagram*

$$\begin{tikzcd} E \times C \arrow[r, "\sigma"] \arrow[d, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "\phi"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

is cartesian with projection p and action σ.

(b) *Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions and let $\phi : C \to D$ be an $\mathbb{A}^1$-equivariant X-morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Assume that E is a vector bundle. Then C is an E-cone (resp. abelian cone, vector bundle) over X if and only if D is a cone (resp. abelian cone, vector bundle) over X and C is affine over X.*

Proof. For (a), if it is exact, locally we have $C \cong E \times_X D$. So E act on C locally as $E \times E \times_X D \to E \times_X D$ given by $(f, (e, d)) \mapsto (i(f) + e, d)$. So C is a E-cone. Now $\phi : C \to D$ is trivially faithfully flat. The cartesian diagram follows from Lemma 4.12(c).

Conversely, since ϕ is fppf, this diagram is also cocartesian by Proposition V.1.3.1 in [Li18] which force $D = [C/E]$. Hence the results follows from Lemma 4.12(c).

For (b), let C is an E-cone over X. Then we have $g : [C/E] \to D$. We claim that g is an isomorphism. Indeed, by the diagram in (a), we know that g induces an isomorphism $g' : E \times_X C = C \times_{[C/E]} C \to C \times_D C$. Note that we have a cartesian diagram:

$$\begin{tikzcd} C \times_{[C/E]} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \times_D C \arrow[d] \\ {[C/E]} \arrow[r, hook] & {[C/E] \times_D [C/E]} \end{tikzcd}$$

where $C \times_D C \to [C/E] \times_D [C/E]$ is faithfully flat, hence $[C/E] \hookrightarrow [C/E] \times_D [C/E]$ is an isomorphism. So g is a monomorphism. But since $C \to [C/E]$ and $C \to D$ are faithfully flat, hence epimorphism. Thus g is also an epimorphism, hence an isomorphism. Thus $D \cong [C/E]$ and the result follows from Lemma 4.12(c).

Now assume that $C = A(C)$ is an abelian cone, then taking hull to $0 \to E \to C \to D = [C/E] \to 0$. By Lemma 4.9(c)(d) we have $A(D) = [A(C)/E] = [C/E] = D$. Hence D is also an abelian cone.

Finally assume that C is a bundle. Then by the previous case we know that D is an abelian cone. The $\mathscr{T}^1 = \ker(\mathscr{S}^1 \twoheadrightarrow \mathscr{E}^\vee)$ is clearly locally free since $\mathscr{C}^1$ and $\mathscr{E}$ are where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$, $D = \underline{\mathrm{Spec}}_X \mathscr{T}$ and $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$.

Conversely we let D is a cone and C is affine over X. Hence we have $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ where $\mathscr{S} = \bigoplus_{i\geq 0} \mathscr{S}^i$ and $\mathscr{S}^1 = \mathscr{O}_X$. By the same reason E is affine over X. Hence we have $C = \underline{\mathrm{Spec}}_X \mathscr{F}$ where $\mathscr{F} = \bigoplus_{i\geq 0} \mathscr{F}^i$ and $\mathscr{F}^1 = \mathscr{O}_X$. If we let $D = \underline{\mathrm{Spec}}_X \mathscr{T}$, then $\mathscr{F} = \mathscr{S}/(\mathscr{T}^{\geq 1}\mathscr{S})$.

Apply the exact triangle of cotangent complex to $X \xrightarrow{0_C} C \to D$, we have an exact sequence

$$0 \to \mathscr{T}^1 \to \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 = C_{X/C} \to \mathscr{E}^\vee := 0_C^* \Omega_{C/D} \to 0.$$

As $\mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 = \mathscr{S}^1 \oplus \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2$, we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{T}^1 \arrow[r] \arrow[d, "="] & \mathscr{S}^1 \arrow[r] \arrow[d] & \mathscr{F}^1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{T}^1 \arrow[r] & \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 \arrow[r] \arrow[d] & \mathscr{E}^\vee \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2 \arrow[r, "="] \arrow[d] & \mathscr{E}^\vee/\mathscr{F} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Locally on X we can assume that $\mathscr{E}$ is free and $\mathscr{T}^1 \oplus \mathscr{E}^\vee = \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2$. Then as $\mathscr{F}^1 \subset \mathscr{E}^\vee$, we know that $\mathscr{F}^1$. Since $\mathscr{T}^1$ is also coherent, we know that so is $\mathscr{S}^1$. Finally we just need to show $\mathscr{S}$ generated by $\mathscr{S}^1$ as by Lemma 4.12(a) here C will be an E-cone.

Then locally on X we can choose generators of $\mathscr{T}^1, \mathscr{F}^1, \mathscr{E}^\vee/\mathscr{F}^1 = \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2$ such that gives a surjective $\mathscr{O}_X$-algebra morphism $\phi : \mathscr{T} \oplus \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{S}$ which induce $\mathscr{T} \oplus \operatorname{Sym} \mathscr{F}^1 \to \mathscr{T} \oplus \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{S}$ is graded. Tensoring $(-) \otimes_{\mathscr{T}} \mathscr{O}_X$ with ϕ we get surjection $\phi' : \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{F}$. This induce the closed immersion $E \hookrightarrow \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Since they are both smooth of a same relative dimension over X and $\underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ is a vector bundle, hence $E \cong \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and ϕ' is an isomorphism. Hence $\mathscr{F} = \operatorname{Sym}(\mathscr{F}^1)$ and $\mathscr{F}^1$ is locally free. As $\operatorname{Sym}(\mathscr{F}^1) \subset \operatorname{Sym} \mathscr{E}^\vee \xrightarrow{\phi'} \mathscr{F} = \operatorname{Sym}(\mathscr{F}^1)$ is identity, this force $\mathscr{E}^\vee = \mathscr{F}^1$. As this can be check locally, we have $\mathscr{E}^\vee = \mathscr{F}^1$ in whole X. By the diagram above, we have $\mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2 = \mathscr{E}^\vee/\mathscr{F}^1 = 0$. This means $\mathscr{S}$ generated by $\mathscr{S}^1$. Well done. $\square$

Remark 4.14. *In the original paper [BF97] they claim (a) is enough for the surjectivity of f.*

4.2 Cone Stack

Let X be a Deligne-Mumford stack.

Definition 4.15. *Let $\mathfrak{C}$ be an algebraic stack over X, together with a section $0 : X \to \mathfrak{C}$. An $\mathbb{A}^1$-action on $(\mathfrak{C}, 0)$ is given by a morphism of X-stacks $\gamma : \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$ and three 2-isomorphisms θ_1, θ_0 and θ_γ between the 1-morphisms in the following diagrams.*

$$\begin{tikzcd}
\mathfrak{C} \arrow[rr, "{(1,\mathrm{id})/(0,\mathrm{id})}"] \arrow[dr, "\mathrm{id}/0"'] & \arrow[d, phantom, "\Nearrow \theta_1/\theta_0"] & \mathbb{A}^1 \times \mathfrak{C} \arrow[dl, "\gamma"] \\
& \mathfrak{C} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"] \arrow[dr, phantom, "\Nearrow \theta_\gamma"] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\
\mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\gamma"'] & \mathfrak{C}
\end{tikzcd}$$

The 2-isomorphisms θ_1, θ_0 and θ_γ are required to satisfy certain compatibilities.

Definition 4.16. *Let $(\mathfrak{C}, 0, \gamma)$ and $(\mathfrak{D}, 0, \gamma)$ be X-stacks with sections and $\mathbb{A}^1$-actions. Then an $\mathbb{A}^1$-*equivariant morphism* $\phi : \mathfrak{C} \to \mathfrak{D}$ is a triple $(\phi, \eta_0, \eta_\gamma)$, where $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of algebraic X-stacks and η_0 and η_γ are 2-isomorphisms between the morphisms in the following diagrams.*

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[dr, "0"'] & \mathfrak{C} \arrow[d, "\phi"] \arrow[dl, phantom, "\Nearrow \eta_0"] \\
& \mathfrak{D}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\phi"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\Nearrow \eta_\gamma"] & \mathbb{A}^1 \times \mathfrak{D} \arrow[d, "\gamma"] \\
\mathfrak{C} \arrow[r, "\phi"] & \mathfrak{D}
\end{tikzcd}$$

Again, the 2-isomorphisms have to satisfy certain compatibilities.

Definition 4.17. *Let $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \to \mathfrak{D}$ and $(\psi, \eta_0', \eta_\gamma') : \mathfrak{C} \to \mathfrak{D}$ be two $\mathbb{A}^1$-equivariant morphisms. An $\mathbb{A}^1$-*equivariant isomorphism* $\zeta : \phi \to \psi$ is a*

2-isomorphism $\zeta : \phi \to \psi$ such that the diagrams

$$\begin{tikzcd} 0 \arrow[r, "\eta_0"] \arrow[dr, "\eta_0'"'] & \phi \circ 0 \arrow[d, "\zeta\circ 0"] \\ & \psi \circ 0 \end{tikzcd} \qquad \begin{tikzcd} \phi \circ \gamma \arrow[r] \arrow[d, "\zeta\circ\gamma"] & \gamma \circ (\mathrm{id} \times \phi) \arrow[d, "\gamma\circ(\mathrm{id}\times\zeta)"] \\ \psi \circ \gamma \arrow[r] & \gamma \circ (\mathrm{id} \times \psi) \end{tikzcd}$$

commute.

Example 4.18. *Let C be a E-cone, then consider the quotient stack $[C/E]$. We claim that $[C/E]$ a zero section and an $\mathbb{A}^1$-action.*

Indeed, the zero section $0 : X \to [C/E]$ given by $X \leftarrow E \to C$. The $\mathbb{A}^1$-action of $\alpha \in \mathbb{A}^1(T)$ on $(P, f) \in [C/E](T)$ defined by $(\alpha P, \alpha f)$ where $\alpha P = P \times^{E,\alpha} E$ and $\alpha f : P \times^{E,\alpha} E \to C$ given by $[p, v] \mapsto \alpha f(p) + d(v)$ where $d : E \to C$.

Moreover, if $\phi : (E, C) \to (F, D)$ is a morphism of vector bundle cones we get an induced $\mathbb{A}^1$-equivariant morphism $\tilde{\phi} : [C/E] \to [D/F]$.

Lemma 4.19. *Some useful results.*

- *(a) A homotopy $k : \phi \to \psi$ of two morphisms of vector bundle cones $\phi, \psi : (E, C) \to (F, D)$ gives rise to an $\mathbb{A}^1$-equivariant 2-isomorphism $\tilde{k} : \tilde{\phi} \to \tilde{\psi}$ of $\mathbb{A}^1$-equivariant morphisms of stacks with $\mathbb{A}^1$-action.*
- *(b) Conversely, let two morphisms of vector bundle cones $\phi, \psi : (E, C) \to (F, D)$ with an $\mathbb{A}^1$-equivariant 2-isomorphism $\zeta : \tilde{\phi} \to \tilde{\psi}$ of $\mathbb{A}^1$-equivariant morphisms of stacks with $\mathbb{A}^1$-action. Then $\zeta = \tilde{k}$ for unique homotopy $k : \phi \to \psi$.*

Proof. For (a), samilar to Proposition 4.29. For (b) TBC... □

Proposition 4.20. *Let C be an E-cone and D an F-cone and let $\phi : (E, C) \to (F, D)$ be a morphism. If the diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "\phi"] \\ F \arrow[r, "d"'] & D \end{tikzcd}$$

is cartesian and $F \times_X C \to D$ by $(\mu, \gamma) \mapsto d\mu + \phi(\gamma)$ is surjective, then $[C/E] \to [D/F]$ is an isomorphism of algebraic X-stacks with $\mathbb{A}^1$-action.

Proof. For the same proof of Proposition 4.30. □

Definition 4.21. *(a) We call an algebraic stack $(\mathfrak{C}, 0, \gamma)$ over X with section and $\mathbb{A}^1$-action a* cone stack*, if, étale locally on X, there exists a cone C over X and an $\mathbb{A}^1$-equivariant morphism $C \to \mathfrak{C}$ that is smooth and surjective and such that $E = C \times_{\mathfrak{C},0} X$ is a vector bundle over X.*

(b) *The morphism $C \to \mathfrak{C}$ is called a* local presentation of $\mathfrak{C}$. *The section $0 : X \to \mathfrak{C}$ is called the vertex of $\mathfrak{C}$.*

(c) *Let $\mathfrak{C}$ and $\mathfrak{D}$ be cone stacks over X. A* morphism of cone stacks $\phi : \mathfrak{C} \to \mathfrak{D}$ *is an $\mathbb{A}^1$-equivariant morphism of algebraic X-stacks. A 2-*isomorphism of cone stacks *is just an $\mathbb{A}^1$-equivariant 2-isomorphism.*

(d) *A cone stack $\mathfrak{C}$ over X is called* abelian cone stack *(resp.* vector bundle stack*), if, locally in X, one can find presentations $C \to \mathfrak{C}$, where C is an abelian cone (resp. vector bundle).*

Remark 4.22. *Some basic properties of cone stacks.*

(a) *If $C \to \mathfrak{C}$ is a global presentation with $E = C \times_{\mathfrak{C},0} X$, then C is an E-cone with $\mathfrak{C} \cong [C/E]$ as stacks with $\mathbb{A}^1$-action. This follows from Proposition 4.10 and 4.13 and Lemma 4.12.*

(b) *If $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of cone stacks, then, étale locally on X, ϕ is $\mathbb{A}^1$-equivariantly isomorphic to $[C/E] \to [D/F]$, where $E \to F$ is a morphism of vector bundles over X and $C \to D$ is a morphism from the E-cone C to the F-cone D.*

(c) *A 2-isomorphism of cone stacks $\zeta : \phi \to \psi$, where $\phi, \psi : \mathfrak{C} \to \mathfrak{D}$, is étale locally over X given by a homotopy of morphisms of vector bundle cones. This follows from Lemma 4.19(b).*

(d) *Let $C \to \mathfrak{C}$ and $D \to \mathfrak{D}$ be two local presentation of a cone stack $\mathfrak{C}$ over X, then so is $C \times_{\mathfrak{C}} D \to \mathfrak{C}$.*

Indeed, we only need to show that $C \times_{\mathfrak{C}} D$ is a cone. Since $C \to \mathfrak{C}$ and $D \to X$ are affine, we know that $C \times_{\mathfrak{C}} D \to D \to X$ is also affine. Then $C \times_{\mathfrak{C}} D$ is a cone a by Proposition 4.13(b) and the result follows.

(e) *Every fibered product of cone stacks is a cone stack.*

(f) *If $\mathfrak{C}$ is a representable cone stack over X, then it is a cone.*

Indeed, locally on X, $\mathfrak{C} \to X$ is $\mathbb{A}^1$-isomorphic to a cone. In particular, as $\mathfrak{C} \to X$ is representable, it is affine. Then we assume that $C = \underline{\mathrm{Spec}}_X \mathscr{S}$. Since there is a non-trivial $\mathbb{A}^1$-action on C and has a section, we know that $\mathscr{S}$ is a graded algebra with $\mathscr{S}^0 = \mathscr{O}_X$. To show C is a cone, we only need to show that $\mathscr{S}^1$ is coherent and $\mathscr{S}$ is locally generated by $\mathscr{S}^1$. These are both local property, then they hold since locally $\mathfrak{C} \to X$ is $\mathbb{A}^1$-isomorphic to a cone.

(g) *If $\mathfrak{C}$ is abelian (a vector bundle stack), then for every local presentation $C \to \mathfrak{C}$ the cone C will be abelian (a vector bundle).*

Example 4.23. *Note that all cones are cone stacks and all morphisms of cones are morphisms of cone stacks. For a vector bundle E on X, the* *classifying stack* $\mathbf{B}_X E$ *is a cone stack. Every homomorphism of vector bundles* $\phi : E \to F$ *gives rise to a morphism of cone stacks.*

Proposition 4.24. *Every cone stack is a closed subcone stack of an abelian cone stack. There exists a universal such abelian cone stack. It is called the* abelian hull.

Proof. Just glue the stacks obtained from the abelian hulls of local presentations. □

Definition 4.25. (a) *Let* $\mathfrak{E}$ *be a vector bundle stack and* $\mathfrak{E} \to \mathfrak{C}$ *a morphism of cone stacks. We say that* $\mathfrak{C}$ *is an* $\mathfrak{E}$-cone stack, *if* $\mathfrak{E} \to \mathfrak{C}$ *is locally isomorphic (as a morphism of cone stacks) to the morphism* $[E_1/E_0] \to [C/F]$ *coming from a commutative diagram*

$$\begin{array}{ccc} E_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & C \end{array}$$

where C *is both* E_1*- and* F*-cone. The natural action* $\mathfrak{E} \times_X \mathfrak{C} \to \mathfrak{C}$ *induced by* $E_1 \times C \to C$.

(b) *Let* $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ *be a sequence of morphisms of cone stacks where* $\mathfrak{C}$ *is an* $\mathfrak{E}$*-cone stack. If*

(b1) $\mathfrak{C} \to \mathfrak{D}$ *is a smooth epimorphism.*

(b2) *The diagram*

$$\begin{tikzcd} \mathfrak{E} \times_X \mathfrak{C} \arrow[r, "\sigma"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{C} \arrow[d] \\ \mathfrak{C} \arrow[r] & \mathfrak{D} \end{tikzcd}$$

is cartesian where σ *is action and* p *is projection.*

Then we call $0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ *is a* short exact sequence of cone stacks. *As before, this is equivalent to* $\mathfrak{C}$ *being locally isomorphic to* $\mathfrak{E} \times_X \mathfrak{D}$.

Proposition 4.26. *The sequence* $0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ *of morphisms of cone stacks is exact if and only if locally in* X *there exist commutative diagrams*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

where the top row is a short exact sequence of vector bundles and the bottom row is a short exact sequence of cones, such that $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ is isomorphic to $[E_1/E_0] \to [C/F] \to [D/G]$.

Proof. The statement is local on X. To prove the only if part we can assume $\mathfrak{C} = \mathfrak{E} \times_X \mathfrak{D}$, and then it is trivial. To prove the if part, note that both short exact sequences are locally split. $\square$

4.3 A Picard Stack of Special Type

General Theory

First we will consider the case of complex of two terms.

Definition 4.27. *Let X be a topos.*

(a) Let $d : E^0 \to E^1$ a homomorphism of abelian sheaves on X, which we shall consider as a complex of abelian sheaves on X. Via d, the abelian sheaf E^0 acts on E^1 and we may consider the quotient stack of this action, denoted

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := [E^1/E^0]$$

which is a Picard stack over X.

(b) Now if $d : F^0 \to F^1$ is another homomorphism of abelian sheaves on X and $\phi : E^\bullet \to F^\bullet$ is a homomorphism of complexes, then we get an induced morphism of Picard stacks

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

given by $(P, f) \mapsto (P \times^{E^0,\phi^0} F^0, \phi^1(f))$ where $\phi^1(f)$ is the map

$$\phi^1(f) : P \times^{E^0,\phi^0} F^0 \to F^1, \quad [p, \nu] \mapsto \phi^1(f(p) + d(\nu)).$$

(c) Now, if $\psi : E^\bullet \to F^\bullet$ is another homomorphism of complexes, then the homotopy *$k : \phi \to \psi$ is a homomorphism of abelian sheaves $k : E^1 \to F^0$, such that $kd = \psi^0 - \phi^0$ and $dk = \psi^1 - \phi^1$.*

Remark 4.28. *Note that roughly speaking, a Picard stack is a stack together with an 'addition' operation, that is both associative and commutative. For the precise definition of Picard stack see Sect. 1.4 of Exposé XVIII in [AGV73].*

Here the quotient stack is similar as before: the groupoid $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$ is the category of pairs (P, f), where P is an E^0-torsor over U and $f : P \to E^1|_U$ is an E^0-equivariant morphism of sheaves on U.

Proposition 4.29. *As in the considtion of definition, if we have a homotopy $k : \phi \to \psi$, then this can induce isomorphism $\theta : \mathcal{H}^1/\mathcal{H}^0(\phi) \to \mathcal{H}^1/\mathcal{H}^0(\psi)$ of morphisms of Picard stacks from $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ to $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)$.*

Proof. Pick object $U \in \mathrm{ob}(X)$ and $(P, f) \in \mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$, then $\theta(U)(P, f)$: $\mathcal{H}^1/\mathcal{H}^0(\phi)(U)(P, f) \to \mathcal{H}^1/\mathcal{H}^0(\psi)(U)(P, f)$ in $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)(U)$ is the isomorphism of $F^0|_U$-torsors

$$\theta(U)(P, f) : P \times^{E^0,\phi^0} F^0 \to P \times^{E^0,\psi^0} F^0$$

given by $[p, \nu] \mapsto [p, kf(p) + \nu]$ such that the diagram of $F^0|_U$-sheaves

$$\begin{tikzcd}
P \times^{E^0,\phi^0} F^0 \arrow[d, "\theta(U)(P,f)"'] \arrow[dr, "\phi^1(f)"] & \\
P \times^{E^0,\psi^0} F^0 \arrow[r, "\psi^1(f)"'] & F^1
\end{tikzcd}$$

commutes. □

Proposition 4.30. *Let $\phi : E^\bullet \to F^\bullet$ is a homomorphism of complexes of abelian sheaves in the topos X. If ϕ induces isomorphisms on kernels and cokernels (i.e. if ϕ is a quasi-isomorphism), then*

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

is an isomorphism of Picard stacks over X.

Proof. First let us treat the case that ϕ is a homotopy equivalence, that is, there is a homotopy inverse of ϕ such that compositions are homotopic to $\mathrm{id}_{E^\bullet}$ and $\mathrm{id}_{F^\bullet}$, respectively. By Proposition 4.29 well done.

Next we assume ϕ is an epimorphism. In this case $E^1 \to [F^1/F^0]$ is an epimorphism, so we just need to prove the diagram

$$\begin{tikzcd}
E^0 \times E^1 \arrow[r, "d+\mathrm{id}"] \arrow[d, "p"] & E^1 \arrow[d] \\
E^1 \arrow[r] & {[F^1/F^0]}
\end{tikzcd}$$

is cartesian as in this case this will be a cocartesian diagram! This quickly reduces to proving that

$$\begin{tikzcd}
E^1 \times E^0 \arrow[r] \arrow[d] & E^1 \arrow[d] \\
E^1 \times F^0 \arrow[r] & F^1
\end{tikzcd}$$

is cartesian, which, in turn, is equivalent to

$$\begin{array}{ccc} E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ F^0 & \longrightarrow & F^1 \end{array}$$

being cartesian, which is a consequence of the assumptions.

Finally in general case, let us note that a general ϕ factors as a homotopy equivalence followed by an epimorphism, then well done. Indeed, consider $E^\bullet \oplus F^0$, which is homotopy equivalent to $E^\bullet$. Define a homomorphism $\psi : E^\bullet \oplus F^0 \to F^\bullet$ by $\psi^0(\nu, \mu) = \phi^0(\nu) + \mu$ and $\psi^1(\xi, \mu) = \phi^1(\xi) + \mu$. Then ψ is surjective and $\phi = \psi \circ i$ where $i : E^\bullet \hookrightarrow E^\bullet \oplus F^0$ is the canonical embedding. $\square$

Now we consider the general case.

Definition 4.31. *Let X be a topos and $E^\bullet$ be a complex of abelian sheaves on X, then we define*

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := \mathcal{H}^1/\mathcal{H}^0(\tau^{[0,1]}E^\bullet).$$

Lemma 4.32. *Let X be a ringed topos with structure sheaf of rings $\mathscr{O}_X$.*

(a) We can define $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ and homomorphisms can defined over $\mathbf{D}(\mathscr{O}_X)$.

(b) Let $\phi, \psi : E^\bullet \to F^\bullet$ be two morphisms in $\mathbf{D}(\mathscr{O}_X)$. Then, if for some choice of $\mathcal{H}^1/\mathcal{H}^0(\phi)$ and $\mathcal{H}^1/\mathcal{H}^0(\psi)$ we have $\mathcal{H}^1/\mathcal{H}^0(\phi) \cong \mathcal{H}^1/\mathcal{H}^0(\psi)$ as morphisms of Picard stacks, then $\phi = \psi$.

(c) Consider the zero morphism $0(E, F) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$. Then $\mathrm{Aut}(0(E, F)) = \mathrm{Hom}^{-1}_{\mathbf{D}(\mathscr{O}_X)}(E^\bullet, F^\bullet)$.

Proof. For (b)(c), see Sect. 1.4 of Exposé XVIII in [AGV73]. For (a), the quasi-isomorphism induce an isomorphism of Picard stacks, see Proposition 4.30. $\square$

Example 4.33. *Consider $E^\bullet$ an we focus on $d^0 : E^0 \to E^1$.*

(1) If d^0 is a monomorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \mathrm{coker}(d^0)$ is a sheaf.

(2) If d^0 is a epimorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \mathbf{B}_X \ker(d^0)$ is a gerbe.

Application

Come back to our case, let X be a DM-stack over a field k, then consider the big fppf topos X_{fppf} and small étale topos $X_{\text{ét}}$. Then we have the morphism of topoi

$$v : X_{\text{fppf}} \to X_{\text{ét}}.$$

(a) Then we we can get $\mathbf{L}v^* : \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}}) \to \mathbf{D}^-(\mathscr{O}_{X_{\text{fppf}}})$. We may let $M^\bullet_{\text{fppf}} := \mathbf{L}v^* M^\bullet$ for any $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}})$.

(b) We also have $\mathbf{R}\mathscr{H}om(-, \mathscr{O}_{X_{\text{fppf}}}) : \mathbf{D}^-(\mathscr{O}_{X_{\text{fppf}}}) \to \mathbf{D}^+(\mathscr{O}_{X_{\text{fppf}}})$. We may let $M^{\bullet,\vee} := \mathbf{R}\mathscr{H}om(M^\bullet, \mathscr{O}_{X_{\text{fppf}}})$ for any $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\text{fppf}}})$.

Remark 4.34. *We will consider the stack $\mathcal{H}^1/\mathcal{H}^0(M^{\bullet,\vee}_{\text{fppf}})$ for $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}})$. Note that in this case*

$$\mathcal{H}^1/\mathcal{H}^0(M^{\bullet,\vee}_{\text{fppf}}) = \mathcal{H}^1/\mathcal{H}^0((\tau^{\geq -1} M^\bullet_{\text{fppf}})^\vee).$$

Remark 4.35. *For a complex $E^\bullet$, we define $Z^i(E^\bullet) = \ker(E^i \to E^{i+1})$ and $C^i(E^\bullet) = \operatorname{coker}(E^{i-1} \to E^i)$.*

Definition 4.36. *We call an object $L^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies Condition (*) if*

(1) $H^i(L^\bullet) = 0$ for all $i > 0$.

(2) $H^i(L^\bullet)$ is coherent for all $i = 0, -1$.

Here are some fundamental results:

Proposition 4.37. *Let X be a DM-stack.*

(a) Let $L^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfy Condition (*). Then the X-stack $\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\text{fppf}})$ is an abelian cone stack over X. Moreover, if $L^\bullet$ is of perfect amplitude contained in $[-1, 0]$, then $\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\text{fppf}})$ is a vector bundle stack.

(b) If $\phi : E^\bullet \to L^\bullet$ is a homomorphism in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $L^\bullet$ satisfy (*), then we get an induced morphism of algebraic stacks

$$\phi^\vee : \mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\text{fppf}}) \to \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\text{fppf}}).$$

Then $\phi^\vee$ is a morphism of abelian cone stacks. Moreover, $H^0(\phi)$ is surjective if and only if $\phi^\vee$ is representable.

(c) The morphism $\phi^\vee$ is a closed immersion if and only if $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. Moreover, $\phi^\vee$ is an isomorphism if and only if $H^0(\phi)$ and $H^{-1}(\phi)$ are isomorphisms.

(d) Let $E^\bullet \to F^\bullet \to G^\bullet \to E^\bullet[1]$ be a distinguished triangle in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $F^\bullet$ satisfy (*) and $G^\bullet$ is of perfect amplitude contained in $[-1, 0]$. Then the induced sequence

$$\mathcal{H}^1/\mathcal{H}^0(G^{\bullet,\vee}_{\text{fppf}}) \to \mathcal{H}^1/\mathcal{H}^0(F^{\bullet,\vee}_{\text{fppf}}) \to \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\text{fppf}})$$

is a short exact sequence of abelian cone stacks over X.

Proof. For (a), as the claim is étale local, we may assume $L^\bullet$ consists of free $\mathscr{O}_X$-modules with $L^i = 0$ for $i > 0$ and L^0, L^{-1} have finite rank. Then $L^\bullet_{\mathrm{fppf}} = v^* L^\bullet$ and $L^{\bullet,\vee}_{\mathrm{fppf}}$ is taking dual of $L^\bullet_{\mathrm{fppf}}$ component-wise. Hence we have

$$\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\mathrm{fppf}}) = [Z^1(L^{\vee,\bullet})/L^{\vee,0}]$$

which is an abelian cone stack given by $L^{\vee,0} \to Z^1(L^{\vee,\bullet}) = C(C^{-1}L^\bullet)$.

When $L^\bullet$ is of perfect amplitude contained in $[-1,0]$, then $\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\mathrm{fppf}})$ is a vector bundle stack since étale locally as above we have $Z^1(L^{\vee,\bullet}) = L^{\vee,1}$.

For (b), the fact that $\phi^\vee$ is a morphism of abelian cone stacks is immediate from the definition. The second question is étale local in X, so we may assume that $E^\bullet$ and $L^\bullet$ are complexes of free $\mathscr{O}_X$-modules and that $E^i = L^i = 0$, for $i > 0$, and that L^0, L^{-1}, E^0 and E^{-1} are of finite rank. Consider the commutative diagram

$$\begin{tikzcd}
C^{-1}(E^\bullet) \arrow[drr, bend left] \arrow[dr] & & \\
& F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E^0 \arrow[d] \\
& B^{-1}(L^\bullet) \arrow[r] & L^0
\end{tikzcd}$$

of coherent sheaves with fiber product F. This force $0 \to F \to E^0 \oplus C^{-1}(L^\bullet) \to L^0$ exact. Then its easy to see that $H^0(\phi)$ is surjective if and only if $0 \to F \to E^0 \oplus C^{-1}(L^\bullet) \to L^0 \to 0$ exact. Hence taking duality we get $0 \to L^{\vee,0} \to E^{\vee,0} \times_X Z^1(L^{\vee,\bullet}) \to C(F) \to 0$ exact. Then by Proposition 4.20 we get

$$[Z^1(L^{\vee,\bullet})/L^{\vee,0}] \cong [C(F)/E^{\vee,0}].$$

This force the following cartesians

$$\begin{tikzcd}
C(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z^1(E^{\vee,\bullet}) \arrow[d] \\
\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\mathrm{fppf}}) \arrow[r, "\phi^\vee"] & \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\mathrm{fppf}})
\end{tikzcd}$$

hence $\phi^\vee$ is representable.

For the converse, note that $\phi^\vee : [Z^1(L^{\vee,\bullet})/L^{\vee,0}] \to [Z^1(E^{\vee,\bullet})/E^{\vee,0}]$ representable implies that $[Z^1(L^{\vee,\bullet})/L^{\vee,0}] = [W/E^{\vee,0}]$. Then we have the commutative diagram:

$$\begin{tikzcd}
Z^1(L^{\vee,\bullet}) \times_X L^{\vee,0} \arrow[r] \arrow[d] & Z^1(L^{\vee,\bullet}) \arrow[d] \\
Z^1(L^{\vee,\bullet}) \times_X E^{\vee,0} \arrow[r] \arrow[d] & W \arrow[d] \\
Z^1(L^{\vee,\bullet}) \arrow[r] & {[W/E^{\vee,0}]}
\end{tikzcd}$$

such that the the whole diagram and the lower diagram are cartesians, then this force the upper square is cartesian. So we get cartesians

$$\begin{tikzcd}
L^{\vee,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z^1(L^{\vee,\bullet}) \times_X L^{\vee,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z^1(L^{\vee,\bullet}) \arrow[d] \\
E^{\vee,0} \arrow[r] & Z^1(L^{\vee,\bullet}) \times_X E^{\vee,0} \arrow[r] & W
\end{tikzcd}$$

Hence $L^{\vee,0} \cong E^{\vee,0} \times_W Z^1(L^{\vee,\bullet}) \to E^{\vee,0} \times_X Z^1(L^{\vee,\bullet})$ is a closed immersion. This implies that $E^0 \oplus C^{-1}(L^\bullet) \to L^0$ is an epimorphism.

For (c), following the previous argument in (b), $\phi^\vee$ is a closed immersion if and only if $C(F) \to Z^1(E^{\vee,\bullet})$ is. This is equivalent to $C^{-1}(E^\bullet) \to F$ being surjective. A simple diagram chase shows that this is equivalent to $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. The 'moreover' follows similarly.

For (d), the question is étale local, so assume that E^i and F^i are 0 for $i > 0$ and vector bundles for $i = 0, -1$, and that $G^i = E^{i+1} \oplus F^i$, that is, $G^\bullet = \mathrm{cone}(E^\bullet \to F^\bullet)$. If we consider the small enough étale locally, we may let $G^i = 0$ for $i \leq -2$ as $G^\bullet$ is of perfect amplitude contained in $[-1, 0]$. Now we have to prove that

$$0 \to [Z^1(G^{\vee,\bullet})/G^{\vee,0}] \to [Z^1(F^{\vee,\bullet})/F^{\vee,0}] \to [Z^1(E^{\vee,\bullet})/E^{\vee,0}] \to 0$$

is a short exact sequence of cone stacks. Now by directly check, we have the exact sequence of sheaves

$$0 \to C^{-1}(E^\bullet) \to C^{-1}(F^\bullet) \oplus E^0 \to C^{-1}(G^\bullet) \to 0.$$

Hence consider

$$\begin{tikzcd}
0 \arrow[r] & C^{-1}(E^\bullet) \arrow[r] \arrow[d] & C^{-1}(F^\bullet) \oplus E^0 \arrow[r] \arrow[d] & C^{-1}(G^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^0 \arrow[r] & F^0 \oplus E^0 \arrow[r] & G^0 = F^0 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Finally by Proposition 4.26 we get the result. □

4.4 About Normal Cones

Here we will consider some useful results about normal cones of DM-stacks.

Consider the commutative diagram of algebraic stacks

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

with where i and j are local immersions. Then there is a natural morphism of cones over X'

$$\alpha : C_{X'/Y'} \to C_{X/Y}.$$

If the diagram is cartesian, then α is a closed immersion. If, moreover, v is flat, then α is an isomorphism.

Proposition 4.38. *Consider a commutative diagram of DM-stacks*

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[dr, "i"'] & Y' \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where i and i' are local immersions and f is smooth. Then the sequence of morphisms of cones over X

$$(i')^* T_{Y'/Y} \to C_{X/Y'} \to C_{X/Y}$$

is exact.

Proof. The question is local, so we can assume them are schemes and that i' and i are immersions. This is then Example 4.2.6 in [Ful98]. □

Lemma 4.39. *Let $f : U \to M$ be a local immersion of affine k-schemes of finite type, where M is smooth over k. Then the normal cone $C_{U/M} \hookrightarrow N_{U/M}$ is invariant under the action of f^*T_M on $N_{U/M}$. In other words, $C_{U/M}$ is an f^*T_M-cone.*

Proof. Consider projections $p_i : M \times M \to M$, we consider two diagrams:

$$\begin{tikzcd} U \arrow[r, "\Delta f"] \arrow[dr, "f"'] & M \times M \arrow[d, "p_i"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\Delta f"'] & M \arrow[d, "\Delta"] \\ & M \times M \end{tikzcd}$$

The first one give us exact sequence of abelian cones on U:

$$0 \to f^*T_M \overset{j_i}{N_{U/M\times M}} \overset{p_{i,*}}{\to} N_{U/M} \to 0$$

and the second one give us a homomorphism of abelian cones $s : N_{U/M} \to N_{U/M\times M}$ which is a section of both $p_{i,*}$.

Using $(j_1, p_{1,*})$ we make the identification $N_{U/M\times M} = f^*T_M \times N_{U/M}$ and $p_{2,*}$ is identified with the action of f^*T_M on $N_{U/M}$. Since the same functorialities of normal sheaves used so far are enjoyed by normal cones, we get that under the identification above the subcone $C_{U/M\times M} \subset N_{U/M\times M}$ corresponds to $f^*T_M \times C_{U/M}$ and the action $p_{2,*} : f^*T_M \times N_{U/M} \to N_{U/M}$ restricts to $p_{2,*} : f^*T_M \times C_{U/M} \to C_{U/M}$. □

4.5 Intrinsic Normal Cone

Now let X be a Deligne-Mumford stack, locally of finite type over k. Now we will construct the intrinsic normal cone and intrinsic normal sheaf of X and their basic properties.

Definition 4.40. *We denote the abelian cone stack*

$$\mathfrak{N}_X := \mathcal{H}^1/\mathcal{H}^0((\mathbb{L}^\bullet_{X,\mathrm{fppf}})^\vee)$$

and call it the intrinsic normal sheaf of X *where* $\mathbb{L}^\bullet_X \in \mathbf{D}^{\leq 0}(\mathscr{O}_{X_{\text{ét}}})$ *is the cotangent complex which satisfies the condition (*).*

Definition 4.41. *(a) A* local embedding *of X is a pair (U, M) with a diagram $X \overset{i}{\leftarrow} U \overset{f}{\to} M$ where*

(a1) U is an affine k-scheme of finite type;

(a2) $i : U \to X$ is an étale morphism;

(a3) M is a smooth affine k-scheme of finite type;

(a4) $f : U \to M$ is a local immersion.

(b) A morphism of local embeddings *$\phi : (U', M') \to (U, M)$ is a pair of morphisms $\phi_U : U' \to U$ and $\phi_M : M' \to M$ such that*

(b1) ϕ_U is an étale X-morphism;

(b2) ϕ_M is smooth morphism such that

$$\begin{array}{ccc} U' & \xrightarrow{f'} & M' \\ \downarrow{\scriptstyle \phi_U} & & \downarrow{\scriptstyle \phi_M} \\ U & \xrightarrow{f} & M \end{array}$$

commutes.

Remark 4.42. *If (U', M') and (U, M) are local embeddings of X, then $(U' \times_X U, M' \times M)$ is naturally a local embedding of X which we call the product of local embeddings, even though it may not be the direct product in the category of local embeddings of X.*

Now we consider the local presentation of intrinsic normal sheaf $\mathfrak{N}_X$. Indeed, consider a local embedding $X \overset{i}{\leftarrow} U \overset{f}{\rightarrow} M$ of X, then we have a natural homomorphism

$$\phi : \mathbb{L}_X^\bullet|_U \to [\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_M^1]$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U_{\text{ét}}})$ where $\mathscr{I}$ be the ideal correspond to f and $[\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_M^1] \in \mathbf{D}^{[-1,0]}(\mathscr{O}_{U_{\text{ét}}})$. Moreover, by Theorem 3.1(c) we know that ϕ induces an isomorphism on H^{-1} and H^0. By Proposition 4.30 we get an induced isomorphism of cone stacks

$$\phi^\vee : [N_{U/M}/f^*T_M] \cong i^*\mathfrak{N}_X.$$

In other words, $N_{U/M}$ is a local presentation of the abelian cone stack $\mathfrak{N}_X$.

Theorem 4.43. *There exists a unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ such that for every local embedding (U, M) of X we have $\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]$, that is, the diagram*

$$\begin{tikzcd} C_{U/M} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & N_{U/M} \arrow[d] \\ \mathfrak{C}_X|_U \arrow[r, hook] & \mathfrak{N}_X|_U \end{tikzcd}$$

Proof. If $\chi : (U', M') \to (U, M)$ is a morphism of local embeddings, we have a commutative diagram

$$\begin{tikzcd} \mathbb{L}_X^\bullet|_{U'} \arrow[d, "\phi|_{U'}"] \arrow[dr, "\phi'"] & \\ {[\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_M^1]|_{U'}} \arrow[r, "\tilde{\chi}"] & {[\mathscr{I}'/(\mathscr{I}')^2 \to (f')^*\Omega_{M'}^1]} \end{tikzcd}$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U'_{\text{ét}}})$ because of the naturality of ϕ and thus induce the commutative diagram

$$\begin{tikzcd} {[N_{U'/M'}/(f')^*T_{M'}]} \arrow[r, "\tilde{\chi}^\vee"] \arrow[d, "{(\phi')^\vee, \cong}"'] & {[N_{U/M}/f^*T_M]|_{U'}} \arrow[dl, "{\phi^\vee|_{U'}, \cong}"] \\ \mathfrak{N}_X|_{U'} & \end{tikzcd}$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U'_{\text{ét}}})$. In particular, $\tilde{\chi}^\vee$ is an isomorphism of cone stacks over U'.

Now by Lemma 4.39 χ induce a morphism from the $(f')^*T_{M'}$-cone $C_{U'/M'}$ to the $f^*T_M|_{U'}$-cone $C_{U/M}|_{U'}$. By Proposition 4.26 the pair $(C_{U/M} \hookrightarrow N_{U/M})|_{U'}$ is the quotient of $(C_{U'/M'} \hookrightarrow N_{U'/M'})$ by the action of $(f')^*T_{M'/M}$ since the kernel of $(f')^*T_{M'} \to f^*T_M|_{U'}$ is $(f')^*T_{M'/M}$. This implies that the isomorphism above

$$\tilde{\chi}^\vee : [N_{U'/M'}/(f')^*T_{M'}] \cong [N_{U/M}/f^*T_M]|_{U'}$$

identifies the closed subcone stack $[C_{U'/M'}/(f')^*T_{M'}]$ with the closed subcone stack $[C_{U/M}/f^*T_M]|_{U'}$. This give us the unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ with the properties above. $\square$

Definition 4.44. *This unique closed subcone stack $\mathfrak{C}_X$ is called the intrinsic normal cone of X.*

Theorem 4.45. *The intrinsic normal cone $\mathfrak{C}_X$ is of pure dimension zero which abelian hull is the intrinsic normal sheaf $\mathfrak{N}_X$.*

Proof. The second claim follows because the normal sheaf is the abelian hull of the normal cone, for any local embedding.

To prove the claim about the dimension of $\mathfrak{C}_X$, consider a local embedding (U, M) of X, giving rise to the local presentation $C_{U/M}$ of $\mathfrak{C}_X$. Assume that M is of pure dimension. We then have a cartesian diagram of U-stacks

$$\begin{tikzcd} C_{U/M} \times f^*T_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C_{U/M} \arrow[d] \\ C_{U/M} \arrow[r] & {[C_{U/M}/f^*T_M]} \end{tikzcd}$$

Thus $C_{U/M} \to [C_{U/M}/f^*T_M]$ is a smooth epimorphism of relative dimension $\dim M$. So since $C_{U/M}$ is of pure dimension $\dim M$ (see the comments on the Definition 2.1), the stack $[C_{U/M}/f^*T_M]$ has pure dimension $\dim M - \dim M = 0$.Well done. $\square$

Finally, we discuss some basic properties of them.

Proposition 4.46. *Let X be a DM-stack.*

(a) The following are equivalent.

- *(a1) X is a local complete intersection.*
- *(a2) $\mathfrak{C}_X$ is a vector bundle stack.*
- *(a3) $\mathfrak{C}_X = \mathfrak{N}_X$.*

If X is smooth, we have $\mathfrak{C}_X = \mathfrak{N}_X = \mathbf{B}_X(T_X)$.

(b) We have $\mathfrak{N}_{X\times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y$ and $\mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$.

(c) Let $f : X \to Y$ be a local complete intersection morphism. Then we have a natural short exact sequence of cone stacks

$$\mathfrak{N}_{X/Y} := \mathcal{H}^1/\mathcal{H}^0(\mathbb{T}^\bullet_{X/Y}) \to \mathfrak{C}_X \to f^*\mathfrak{C}_Y.$$

Proof. (a) is trivial. (b) follows from the fact that if C is an E-cone and D is an F-cone, then $C\times D$ is an $E\times F$-cone and there is a canonical isomorphism of cone stacks $[C/E] \times [D/F] \to [C \times D/E \times F]$.

For (c), by Theorem 3.1(c)(e) we have an exact triangle

$$\mathbf{L}f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y} \to \mathbf{L}f^*\mathbb{L}_Y[1]$$

in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ and $\mathbb{L}_{X/Y}$ is of perfect amplitude contained in $[-1, 0]$. By Proposition 4.37(d) we have a short exact sequence of abelian cone stacks

$$\mathfrak{N}_{X/Y} \to \mathfrak{N}_X \to f^*\mathfrak{N}_Y.$$

So the claim is local in X and we may assume that we have a diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & M'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M' \arrow[d] \\ & Y \arrow[r] & M \end{tikzcd}$$

where the square is cartesian, the vertical maps are smooth, the horizontal maps are local immersions, i is regular and M is smooth. Then we have a morphism of short exact sequences of cones on X

$$\begin{tikzcd} i^*T_{M''/Y} \arrow[r] \arrow[d] & T_{M'}|_X \arrow[r] \arrow[d] & T_M|_X \arrow[d] \\ N_{X/M''} \arrow[r] & C_{X/M'} \arrow[r] & C_{Y/M}|_X \end{tikzcd}$$

Hence by Proposition 4.26 we get the result. □

4.6 About Obstruction Theories

Intrinsic Normal Sheaf as Obstruction

Let X be a DM-stack with intrinsic normal sheaf $\mathfrak{N}_X$. Let $T \hookrightarrow \overline{T}$ be a closed immersion with ideal $\mathscr{J}$ such that $\mathscr{J}^2 = 0$. If we have $g : T \to X$, may we have the extension $\overline{g} : \overline{T} \to X$ of g? What is the obstruction of this deformation?

First, by Theorem 3.1(e) we have a composition of canonical morphisms

$$\mathbf{L}g^*\mathbb{L}_X^\bullet \to \mathbb{L}_T^\bullet \to \mathbb{L}_{T/\overline{T}}^\bullet.$$

Since $\tau^{\geq -1}\mathbb{L}_{T/\overline{T}}^\bullet = \mathscr{J}[1]$, this homomorphism may be considered as an element $\omega(g) \in \mathrm{Ext}^1(g^*\mathbb{L}_X^\bullet, \mathscr{J})$. Then the basic deformation theory find that an extension $\overline{g} : \overline{T} \to X$ of g exists if and only if $\omega(g) = 0$ and if $\omega(g) = 0$ the extensions form a torsor under $\mathrm{Ext}^0(g^*\mathbb{L}_X^\bullet, \mathscr{J}) = \mathrm{Hom}(g^*\Omega_X, \mathscr{J})$.

Here we will use the intrinsic normal sheaf $\mathfrak{N}_X$ to interpret this. Recall the morphism as above

$$\mathbf{L}g^*\mathbb{L}_X^\bullet \to \mathbb{L}_T^\bullet \to \mathbb{L}_{T/\overline{T}}^\bullet.$$

This induce a morphism

$$\mathbf{ob}(g) : C(\mathscr{J}) = \mathcal{H}^1/\mathcal{H}^0(\mathbb{L}_{T/\overline{T},\mathrm{fppf}}^{\bullet,\vee}) \to \mathcal{H}^1/\mathcal{H}^0(\mathbf{L}g^*\mathbb{L}_{X,\mathrm{fppf}}^{\bullet,\vee}) = g^*\mathfrak{N}_X$$

since $\tau^{\geq -1}\mathbb{L}_{T/\overline{T}}^\bullet = \mathscr{J}[1]$ by Theorem 3.1(c). Consider another morphism $\mathbf{0}(g) : C(\mathscr{J}) \to X \xrightarrow{0} g^*\mathfrak{N}_X$.

- Consider a sheaf $\mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$ of 2-isomorphisms of cone stacks from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$, restricted to $T_{\text{ét}}$.
- Denote the sheaf of extensions $\overline{T} \to X$ of g by $\mathscr{E}xt(g, T)$ on $T_{\text{ét}}$.

Proposition 4.47. *There is a canonical isomorphism*

$$\mathscr{E}xt(g, T) \xrightarrow{\cong} \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$$

of sheaves on $T_{\text{ét}}$. Hence in particular, extensions of g to $\overline{T}$ exist if and only if $\mathbf{ob}(g)$ is $\mathbb{A}^1$-equivariantly isomorphic to $\mathbf{0}(g)$.

Proof. Locally we can have an embedding $i : X \hookrightarrow M$ where M is smooth of ideal $\mathscr{I}$. Then by the formally-smoothness of M we have the lifting:

$$\begin{tikzcd}
X \arrow[r, hook, "i"] & M \arrow[dr] & \\
T \arrow[u, "g"] \arrow[r, hook] & \overline{T} \arrow[u, dashed, "h"'] \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

and such extensions is a $\mathrm{Hom}(g^*i^*\Omega_M, \mathscr{J})$-torsor. Now, any such h induce $h^\sharp : g^*\mathscr{I}/\mathscr{I}^2 \to \mathscr{J}$. By the local description before the Theorem 4.43, $\mathbf{ob}(g)$ induced by

$$h^\sharp : \mathbf{L}g^*[\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_M] \to [\mathscr{J} \to 0].$$

Now the torsor structure induce the following homotopy

$$\begin{tikzcd} 0 \arrow[r] & g^*\mathscr{I}/\mathscr{I}^2 \arrow[r] \arrow[d, "h^\sharp,(\tilde{h})^\sharp"'] & g^*i^*\Omega_M \arrow[r] \arrow[d, "h^\sharp,(\tilde{h})^\sharp"] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & \mathscr{J} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

of extensions $h^\sharp, (\tilde{h})^\sharp$.

Now let $\overline{g} : \overline{T} \to X$ be an extension of g. Then easy to see that $(i \circ g)^\sharp = 0$, so that we get a homotopy from any local $h^\sharp$ as above to 0, or in other words a local $\mathbb{A}^1$-equivariant isomorphism from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$ by Proposition 4.29. Since these local isomorphisms glue, we get the required map

$$\mathscr{E}xt(g, T) \to \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g)).$$

Now we consider the inverse. Let $\theta : \mathbf{ob}(g) \to \mathbf{0}(g)$ be a 2-isomorphism of cone stacks. By Lemma 4.19(a), θ defines for every local h as above an extension of $h^\sharp$ to $\overline{h}^\sharp : g^*i^*\Omega_M \to \mathscr{J}$. So we can get $h' : \overline{T} \to M$ such that $(h')^\sharp = 0$ by the changing via homotopy $\overline{h}^\sharp$. So h' factor through X and we get $h' : \overline{T} \to X$. Gluing them we get the inverse. ◻

Proposition 4.48. *There is a canonical isomorphism*

$$\mathscr{A}ut(\mathbf{0}(g)) \xrightarrow{\cong} \mathscr{H}om(g^*\Omega_X, \mathscr{J})$$

of sheaves on $T_{\text{ét}}$.

Proof. Again similar as above, Lemma 4.19(a) shows that the automorphisms of $\mathbf{0}(g)$ are (locally) the homomorphisms from $g^*i^*\Omega_M$ to $\mathscr{J}$ vanishing on $g^*\mathscr{I}/\mathscr{I}^2$. The exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_M \to \Omega_X \to 0$$

give the result. ◻

Remark 4.49. *This shows that the sheaf $\mathscr{E}xt(g, T) \cong \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$ is a formal $\mathscr{H}om(g^*\Omega_X, \mathscr{J})$-torsor. So if $\mathbf{ob}(g) \cong \mathbf{0}(g)$, the set $\mathrm{Hom}(\mathbf{ob}(g), \mathbf{0}(g))$ is a torsor under the group $\mathrm{Hom}(g^*\Omega_X, \mathscr{J})$.*

Obstruction Theories

Here we consider more general setting.

Definition 4.50. *Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies condition (*). Then a homomorphism $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ is called an obstruction theory for X if $H^0(\phi)$ s an isomorphism and $H^{-1}(\phi)$ is surjective.*

4.7 Vistoli's Rational Equivalence

Before starting the theory of virtual class, we need some results of Vistoli. We will follows something in [Vis89].

Definition 4.51. *Let X be a stack.*

(a) The group $Z_k(X)$ of cycles of dimension k *is generated by all integral closed substacks of dimension k. And $Z_*(X) := \bigoplus_k Z_k(X)$.*

(b) The group of rational equivalences *on X is*

$$W_k(X) := \bigoplus_G K(G)^*, \quad W_*(X) := \bigoplus_k W_k(X)$$

where the direct sum is taken over all integral substacks G of X of dimension $k+1$.

(c) If X is a scheme, there is a canonical homorphism

$$\partial_X : W_*(X) \to Z_*(X).$$

This is commute with proper pushforward and flat pullback.

Remark 4.52. *Note that when X be a DM-stack, we can restricting Z_* and W_* to the étale site of X, we get two sheaves $\mathscr{Z}_*$ and $\mathscr{W}_*$ on X. As Z_* and W_* commute with proper pushforward and flat pullback, $\partial : \mathscr{W}_* \to \mathscr{Z}_*$ is a morphism of sheaves, so we get a homomorphism $\partial_X : W_*(X) \to Z_*(X)$.*

Recall that we consider again the cartesian diagram of algebraic stacks

$$\begin{tikzcd} X' \arrow[r, hook, "i"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, hook, "v"] \\ X \arrow[r, hook, "j"] & Y \end{tikzcd}$$

with i and j are local immersions and v is a regular local immersion and Y is smooth of constant dimension. Then this induce the cartesians

$$\begin{tikzcd} N_{Y'/Y} \times_Y C_{X/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & u^*C_{X/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C_{X/Y} \arrow[d] \\ j^*N_{Y'/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[r, "u"] \arrow[d, hook, "j"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook, "i"] \\ N_{Y'/Y} \arrow[r, "\rho"] & Y' \arrow[r, hook, "v"] & Y \end{tikzcd}$$

Theorem 4.53 (Vistoli)**.** *Consider the above situation, if Y is a scheme, then there is a canonical rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ such that*

$$\partial\beta(Y', X) = [C_{u^*C_{X/Y}/C_{X/Y}}] - [\rho^* C_{X'/Y'}].$$

Proof. See Lemma 4.6 in [Vis89]. □

Corollary 4.54. *In this case we have $v^![C_{X/Y}] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$.*

Proof. Let $0 : u^*C \to N \times_Y C$ be the zero section, then by definition of refined Gysin pullback

$$0^*[C_{u^*C_{X/Y}/C_{X/Y}}] = v^![C] \in \mathsf{CH}_*(u^*C_{X/Y}).$$

Moreover

$$0^*[\rho^* C_{X'/Y'}] = 0^*\rho^![C_{X'/Y'}] = C_{X'/Y'}.$$

By Theorem 4.53 we get $v^![C] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$. □

But now we need to consider the Vistoli rational equivalence at the level of stacks. So we need some base-change result about this:

Proposition 4.55. *Vistoli's rational equivalence commutes with any smooth base change $\phi : Y_1 \to Y$.*

Proof. If ϕ is étale, this is Lemma 4.6(ii) in [Vis89]. Vistoli's proof is based on the fact that the following commute with étale base change: blowing up a scheme along a closed subscheme; normalization; order of a Cartier divisor along an irreducible Weil divisor on a reduced, equidimensional scheme. But all these operations do in fact commute with smooth base change. Hence well done. □

Corollary 4.56. *We have Vistoli's rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ for any algebraic stacks. Moreover, if Y is a DM-stack, then $v^![C_{X/Y}] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$ holds.*

Proof. Follows directly from the previous Proposition. □

Now we again consider the general case. We assume $i : X \to Y$ can factor as $X \xrightarrow{\tilde{i}} \tilde{Y} \xrightarrow{\pi} Y$ where $\tilde{i}$ is another local immersion and π is of relative Deligne-Mumford type (i.e. has unramified diagonal) and is smooth of constant fiber dimension.

Then the previous diagram can be fused into a large diagram of cartesians:

$$\begin{tikzcd}
N_{Y'/Y} \times_Y C_{X/\tilde{Y}} \arrow[r] \arrow[d] & u^* C_{X/\tilde{Y}} \arrow[r] \arrow[d] & C_{X/\tilde{Y}} \arrow[d, "\alpha"] \\
N_{Y'/Y} \times_Y C_{X/Y} \arrow[r] \arrow[d] & u^* C_{X/Y} \arrow[r] \arrow[d] & C_{X/Y} \arrow[d] \\
j^* N_{Y'/Y} \arrow[r] \arrow[d] & X' \arrow[r, "u"] \arrow[d, "\tilde{j}"] & X \arrow[d, hook, "\tilde{i}"] \\
\pi^* N_{Y'/Y} \arrow[r, "\tilde{\rho}"] \arrow[d] & \tilde{Y}' \arrow[r, "\tilde{v}"] \arrow[d] & \tilde{Y} \arrow[d, "\pi"] \\
N_{Y'/Y} \arrow[r, "\rho"] & Y' \arrow[r, "v"] & Y
\end{tikzcd}$$

Hence by Proposition 4.38 $\alpha : C_{X/\tilde{Y}} \to C_{X/Y}$ is a $T_{\tilde{Y}/Y} \times_{\tilde{Y}} C_{X/Y}$-bundle.

Proposition 4.57. *We have* $\alpha^*(\beta(Y', X)) = \beta(\tilde{Y}', X) \in W_*(N_{Y'/Y} \times_Y C_{X/\tilde{Y}})$.

Proof. In the compatibilities of β proved in [Vis89] we reduce to the case that $\tilde{Y} = \mathbb{A}^n_Y$. Then one checks that Vistoli's construction in the case directly. $\square$

Proposition 4.58. *Back to the original diagram, assume that* Y *is of Deligne-Mumford type. Vistoli's rational equivalence* $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ *is invariant under the natural action of* $j^* N_{Y'/Y} \times_Y T_Y$ *on* $N_{Y'/Y} \times_Y C_{X/Y}$.

Proof. The vector bundle $i^* T_Y$ acts on the X-cone $C_{X/Y}$ by Lemma 4.39. Pulling back from X to $j^* N_{Y'/Y}$ gives the natural action of $j^* N_{Y'/Y} \times_Y T_Y$ on $N_{Y'/Y} \times_Y C_{X/Y}$. Using the construction of the proof of Lemma 4.39 the claim follows from Proposition 4.57 applied to $\tilde{Y} = Y \times Y$ and $\tilde{i} = \Delta \circ i : X \to Y \times Y$. $\square$

4.8 Virtual Fundamental Classes

4.9 Examples

5 Atiyah-Bott Localization

We will follows [AB84].

6 Localization of Virtual Fundamental Class

We will follows [GP99].

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