PERVERSE SHEAVES

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Abstract. In this note we will introduce the basic theory of perverse sheaves, including constructible sheaves, perverse sheaves, nearby and vanishing cycles. Moreover we will also give a glimpse of $\mathscr{D}$-modules, the Riemann-Hilbert correspondence and mixed Hodge modules. Finally we will consider some applications of the theory, such as enumerative geometry and representation theory.

Contents

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1. Introduction

1.1. **Background/Motivation.** Perverse sheaves were discovered in the fall of 1980 by Beilinson-Bernstein- Deligne-Gabber in [BBDG18], sitting at the confluence of two major developments of the 1970s: the intersection homology theory of Goresky-MacPherson, and the Riemann-Hilbert correspondence, due to Kashiwara and Mebkhout.

We will first follows the book [Ach21] to learn the basic theory of perverse sheaves. We will focus the theory of algebraic varieties over $\mathbb{C}$ and using analytic topology. We will also give a quike discussion about pure-algebraic theory using étale topology and étale cohomology. See also original [BBDG18].

We will also discuss some applications of this theory. Such as representation theory and enumerative geometry, especially the relative DT conjecture.

The prerequisites are: familiarity with the language of derived and triangulated categories; familiarity with introductory algebraic topology and some topology of complex algebraic varieties; familiarity with basic algebraic geometry.

1.2. **Related works and some future direction.** Need to add.

Acknowledgments. Need to add.

2. Recollection of the basic theory of sheaves

2.1. **Six functors.** Here we recollect some definitions of sheaves. Including six functors.

Definition 2.1. *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

- *Let $\mathscr{F} \in \mathsf{Sh}(Y, R)$, then the* pullback *$f^{-1}(\mathscr{F})$ of $\mathscr{F}$ is the sheafification of*
$$f^{-1}_{\mathrm{pre}}(\mathscr{F}) : U \mapsto \varinjlim_{V \subset Y \text{ open}, V \supset f(U)} \mathscr{F}(V).$$
This is an exact functor.
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* pushforward *$f_*(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_*(\mathscr{F})(U) := \mathscr{F}(f^{-1}(U))$.*
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* proper pushforward *$f_!(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_!(\mathscr{F})(U) := \{s \in \mathscr{F}(f^{-1}(U)) : f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U$ is proper$\}$.*
- *Let $\mathscr{F} \in \mathsf{D}^-(X, R)$ and $\mathscr{G} \in \mathsf{D}^-(Y, R)$, we can define* external tensor product *as*
$$\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G} := p_1^{-1}\mathscr{F} \otimes^{\mathbf{L}} p_2^{-1}\mathscr{G}$$
where p_i are projections.
- *We can define*
$$\mathbb{R}\mathscr{H}om(-,-) : \mathsf{D}^-(X, R)^{\mathrm{op}} \times \mathsf{D}^+(X, R) \to \mathsf{D}^+(X, R),$$
$$- \otimes^{\mathbf{L}} - : \mathsf{D}^{\pm}(X, R) \times \mathsf{D}^{\pm}(X, R) \to \mathsf{D}^{\pm}(X, R).$$

Here we recollect some useful and basic results about these functors.

Proposition 2.2 ([Ach21])**.** *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

(1) f^{-1} is exact and $f_*, f_!$ are left exact functor. So we can define
$$\mathbf{R}f_*, \mathbf{R}f_! : \mathsf{D}^+(X, R) \to \mathsf{D}^+(Y, R), \quad f^{-1} : \mathsf{D}(Y, R) \to \mathsf{D}(X, R).$$
Moreover, consider $f : X \to Y$ and $g : Y \to Z$, then we have $(g \circ f)^{-1} = f^{-1}g^{-1}$ and $\mathbf{R}(g \circ f)_* = \mathbf{R}g_* \circ \mathbf{R}f_*$. If X, Y, Z are Hausdorff and locally compact, then $\mathbf{R}(g \circ f)_! = \mathbf{R}g_! \circ \mathbf{R}f_!$.

(2) If $h : Y \hookrightarrow X$ is a locally closed embedding, then for any $\mathscr{F} \in \mathsf{Sh}(Y, R)$ the sheaf $h_!(\mathscr{F})$ is the sheafification of $h_{!,\mathrm{pre}}\mathscr{F}$ which maps U to $\Gamma(U \cap Y, \mathscr{F})$ if $U \cap \overline{Y} \subset Y$ and 0 otherwise. Moreover in this case $h_!$ is exact. In this case of locally closed embeddings, without assuming X, Y, Z are Hausdorff and locally compact, we also have $\mathbf{R}(g \circ f)_! = \mathbf{R}g_! \circ \mathbf{R}f_!$. Note that $h_!(\mathscr{F})_x \cong \begin{cases} \mathscr{F}_x & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$

(3) *We have*
$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om(f^{-1}\mathscr{F},\mathscr{G}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F},\mathbf{R}f_*\mathscr{G})$$
for any $\mathscr{F} \in \mathsf{D}^-(Y,R)$ *and* $\mathscr{G} \in \mathsf{D}^+(X,R)$.

(4) *We have*
$$\mathbf{R}\mathscr{H}om(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G},\mathscr{H}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F},\mathbf{R}\mathscr{H}om(\mathscr{G},\mathscr{H}))$$
for any $\mathscr{F},\mathscr{G} \in \mathsf{D}^-(X,R)$ *and* $\mathscr{H} \in \mathsf{D}^+(X,R)$.

Proposition 2.3 ([Ach21], Prop. 1.4.21)**.** *Let* $f : X \to X'$ *and* $g : Y \to Y'$ *be continuous maps.*

(1) *For* $\mathscr{F} \in \mathsf{D}^-(X',R)$ *and* $\mathscr{G} \in \mathsf{D}^-(Y',R)$*, there is a natural isomorphism*
$$f^{-1}\mathscr{F} \boxtimes^{\mathbf{L}} g^{-1}\mathscr{G} \cong (f \times g)^{-1}(\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}).$$

(2) *For* $\mathscr{F},\mathscr{F}' \in \mathsf{D}^-(X,R)$ *and* $\mathscr{G},\mathscr{G}' \in \mathsf{D}^-(Y,R)$*, there is a natural isomorphism*
$$(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{F}') \boxtimes^{\mathbf{L}} (\mathscr{G} \otimes^{\mathbf{L}} \mathscr{G}') \cong (\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}) \otimes^{\mathbf{L}} (\mathscr{F}' \boxtimes^{\mathbf{L}} \mathscr{G}').$$

(3) *Assume that* $\mathrm{gl.dim}(R) < \infty$ *has finite global dimension and that our spaces are Hausdorff and locally compact. For* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *and* $G \in \mathsf{D}^+(Y,R)$*, there is a natural isomorphism*
$$\mathbf{R}f_!\mathscr{F} \boxtimes^{\mathbf{L}} \mathbf{R}f_!\mathscr{G} \cong \mathbf{R}(f \times g)_!(\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}).$$

Remark 2.4. *If* $\mathrm{gl.dim}(R) < \infty$*, (1)(2) also hold for* D^+.

Theorem 2.5 (Proper base change, [Ach21] Thm. 1.2.13)**.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

(1) *If all the spaces are Hausdorff and locally compact, then for any* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *we have isomorphism*
$$g^{-1}f_!\mathscr{F} \cong f'_!(g')^{-1}\mathscr{F}.$$

(2) *If* f *is proper, then for any* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *we have isomorphism*
$$g^{-1}f_*\mathscr{F} \cong f'_*(g')^{-1}\mathscr{F}.$$

Theorem 2.6 (Projection formula, [Ach21] Thm. 1.4.9)**.** *Let* $f : X \to Y$ *be a continuous map of Hausdorff and locally compact spaces, and assume that* $\mathrm{gl.dim}(R) < \infty$*. For* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *and* $\mathscr{G} \in \mathsf{D}^+(Y,R)$*, there is a natural isomorphism*
$$\mathbf{R}f_!\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G} \cong \mathbf{R}f_!(\mathscr{F} \otimes^{\mathbf{L}} f^{-1}\mathscr{G}).$$

Remark 2.7 (Change of scalars)**.** *We can also consider change of scalars. For ring map* $\phi : R \to R'$*, we can define* $\mathrm{Res}_{R,R'} : \mathsf{Sh}(X,R') \to \mathsf{Sh}(X,R)$ *and* $R' \otimes - : \mathsf{Sh}(X,R) \to \mathsf{Sh}(X,R')$ *as usual. Note that* $\mathrm{Res}_{R,R'}$ *is exact and* $R' \otimes -$ *is right exact. So we have* $\mathrm{Res}_{R,R'} : \mathsf{D}(X,R') \to \mathsf{D}(X,R)$ *and* $R' \otimes^{\mathbf{L}} - : \mathsf{D}^-(X,R) \to \mathsf{D}^-(X,R')$*. We have the following results about these. Let* $f : X \to Y$ *be a continuous map, and let* $\phi : R \to R'$ *be a ring homomorphism.*

- *For* $\mathscr{F} \in \mathsf{D}^-(X,R)$*, there is a natural isomorphism*
$$f^{-1}(R' \otimes^{\mathbf{L}} \mathscr{F}) \cong R' \otimes^{\mathbf{L}} f^{-1}(\mathscr{F}).$$
If $\mathrm{gl.dim}(R) < \infty$*, this holds for* D^+.
- *If* X, Y *are Hausdorff and locally compact and* R *is Noetherian with* $\mathrm{gl.dim}(R) < \infty$*, then*
$$R' \otimes^{\mathbf{L}} \mathbf{R}f_!\mathscr{F} \cong \mathbf{R}f_!(R' \otimes^{\mathbf{L}} \mathscr{F}).$$

Finally, we will consider the functor $f^!$.

Theorem 2.8 (Verdier)**.** *Let* $f : X \to Y$ *be a continuous map of Hausdorff and locally compact spaces with a Noetherian ring* R *such that* $\mathrm{gl.dim}(R) < \infty$*. Assume that* $f_!$ *has finite cohomological dimension. Then there exists a triangulated functor*
$$f^! : \mathsf{D}^+(Y,R) \to \mathsf{D}^+(X,R)$$
such that we have
$$\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{F},\mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{F},f^!\mathscr{G})$$
for any $\mathscr{F} \in \mathsf{D}^-(X,R)$ *and* $\mathscr{G} \in \mathsf{D}^+(Y,R)$.

Sketch. Here we give a sketch and the details we refer [KS94] Theorem 3.1.5 and Proposition 3.1.10. Now first using the assumption that R is Noetherian with $\mathrm{gl.dim}(R) < \infty$, we can show that $\underline{R}_X$ has a flat and soft resolution $\mathscr{K}$ of finite terms. For $\mathscr{F} \in \mathsf{Ch}^-(\mathsf{Sh}(X,R))$ and $\mathscr{G} \in \mathsf{Ch}^+(\mathsf{Sh}(Y,R))$, we can define a chain complex of presheaves

$$\mathsf{E}(\mathscr{F},\mathscr{G}) : U \mapsto \mathrm{Hom}_{\mathsf{Ch}}(f_!(\mathscr{F} \otimes j_{U!}(\mathscr{K}|_U)), \mathscr{G})$$

with inclusion $j_U : U \subset X$. It is actually a complex of sheaves and hence we have a functor

$$\mathsf{E} : \mathsf{K}^-(\mathsf{Sh}(X,R))^{\mathrm{op}} \times \mathsf{K}^+(\mathsf{Sh}(Y,R)) \to \mathsf{K}^+(\mathsf{Sh}(X,R)).$$

We can find that $\mathsf{E}(\mathscr{F},\mathscr{G}) \cong \mathscr{H}om_{\mathsf{Ch}}(\mathscr{F}, \mathsf{E}(\underline{R}_X,\mathscr{G}))$ with $\mathscr{H}om_{\mathsf{Ch}}(f_!(\mathscr{F} \otimes \mathscr{K}),\mathscr{G}) \cong f_*\mathsf{E}(\mathscr{F},\mathscr{G})$.

Moreover, we find that $\mathsf{E}(\mathscr{F},-)$ maps injective complex into injective complex. Hence we have

$$\mathbf{R}\mathsf{E} : \mathsf{D}^-(\mathsf{Sh}(X,R))^{\mathrm{op}} \times \mathsf{D}^+(\mathsf{Sh}(Y,R)) \to \mathsf{D}^+(\mathsf{Sh}(X,R))$$

with $\mathbf{R}\mathsf{E}(\mathscr{F},\mathscr{G}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}\mathsf{E}(\underline{R}_X,\mathscr{G}))$ and $\mathbf{R}\mathscr{H}om_{\mathsf{Ch}}(\mathbf{R}f_!(\mathscr{F} \otimes \mathscr{K}),\mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathsf{E}(\mathscr{F},\mathscr{G})$. Finally, we define $f^!$ as:

$$f^! : \mathsf{D}^+(Y,R) \to \mathsf{D}^+(X,R), \quad \mathscr{G} \mapsto \mathbf{R}\mathsf{E}(\underline{R}_X,\mathscr{G}).$$

Hence $\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{F},\mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G})$ for any $\mathscr{F} \in \mathsf{D}^-(X,R)$ and $\mathscr{G} \in \mathsf{D}^+(Y,R)$. □

Remark 2.9. *Why we define as this? Note that if we have proved this theorem and $f^!$ maps sheaves to sheaves, then we have:*

$$\mathscr{H}om(\mathscr{F}, f^!\mathscr{G})(U) = \mathrm{Hom}(j_{U!}j_U^{-1}\mathscr{F}, f^!\mathscr{G}) = \mathrm{Hom}(\mathbf{R}f_!(j_{U!}j_U^{-1}\mathscr{F}),\mathscr{G})$$

with $j_{U!}j_U^{-1}\mathscr{F} = \mathscr{F} \otimes j_{U!}\underline{R}_U$.

Proposition 2.10 ([Ach21], 1.3)**.** *If $h : Y \hookrightarrow X$ is a locally closed embedding, then for any $\mathscr{F} \in \mathsf{Sh}(Y,R)$ we define*

$$^\circ h^!\mathscr{F} : U \mapsto \varinjlim_{V \subset x \ \text{open}, V \cap \overline{Y} = U} \{s \in \mathscr{F}(V) : \mathrm{supp}(s) \subset U\}.$$

Then $^\circ h^!$ is left exact and we have $\mathbf{R}^\circ h^!$. When h is an open embedding, then $^\circ h^! = h^{-1}$.

If moreover $h : Y \hookrightarrow X$ is a locally closed embedding of Hausdorff and locally compact spaces and R is a Noetherian ring with $\mathrm{gl.dim}(R) < \infty$, then $h_!$ is exact which implies the existence of $h^!$ and we have $h^! = \mathbf{R}^\circ h^!$.

Note that without assuming the conditions of spaces and ring, we also have $\mathbf{R}^\circ(h \circ k)^! = \mathbf{R}^\circ k^! \circ \mathbf{R}^\circ h^!$.

Remark 2.11. *In the case of locally closed embeddings without assuming the conditions of spaces and ring, we also denote $\mathbf{R}^\circ h^!$ by $h^!$.*

Proposition 2.12 ([Ach21], Prop. 1.5.6-1.5.9)**.** *Assume R is a Noetherian ring with $\mathrm{gl.dim}(R) < \infty$.*

(1) Let $f : X \to Y$ and $g : Y \to Z$ are continuous maps of Hausdorff and locally compact spaces such that $f_!, g_!$ have finite cohomological dimension, then we have $f^!g^! \cong (g \circ f)^!$.

(2) Consider cartesian

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

of continuous maps of Hausdorff and locally compact spaces with $f_!$ has finite cohomological dimension, then $g'_*(f')^! \cong f^!g_*$.

(3) Let $f : X \to Y$ is a continuous map of Hausdorff and locally compact spaces such that $f_!$ have finite cohomological dimension, then

$$f^!\mathbf{R}\mathscr{H}om(\mathscr{F},\mathscr{G}) \cong \mathbf{R}\mathscr{H}om(f^{-1}\mathscr{F}, f^!\mathscr{G})$$

for any $\mathscr{F} \in \mathsf{D}^b(Y,R)$ and $\mathscr{G} \in \mathsf{D}^+(Y,R)$.

(4) Let $f : X \to Y$ is a continuous map of Hausdorff and locally compact spaces of finite soft dimension, then there is a natural map

$$f^!\mathscr{F} \otimes^{\mathbf{L}} f^{-1}\mathscr{G} \to f^!(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G})$$

and when $\mathscr{G}$ is a local system of finite type, this map is an isomorphism.

Definition 2.13. *Let X be a Hausdorff and locally compact space of finite soft dimension. The* dualizing complex *of X, denoted by $\omega_X \in \mathsf{D}^+(X,R)$ given by*

$$\omega_X := f^! \underline{R}_{pt}$$

for $f : X \to \{pt\}$. The Borel-Moore-Verdier duality functor *is the functor*

$$\mathbb{D} : \mathsf{D}^-(X,R)^{op} \to \mathsf{D}^+(X,R), quad \mathscr{F} \mapsto \mathbf{R}\mathscr{H}om(\mathscr{F}, \omega_X).$$

Corollary 2.14. *Assume R is Noetherian ring with $\mathrm{gl.dim}(R) < \infty$ and let $f : X \to Y$ be a continuous map of locally compact spaces of finite soft dimension.*

(1) *There is a canonical isomorphism $f^!\omega_Y \cong \omega_X$.*
(2) *For any $\mathscr{F} \in \mathsf{D}^-(X,R)$, there is a natural isomorphism $\mathbf{R}f_*\mathbb{D}(F) \cong \mathbb{D}(\mathbf{R}f_! F)$. For any $\mathscr{G} \in \mathsf{D}^-(Y,R)$, there is a natural isomorphism $f^!\mathbb{D}(\mathscr{F}) \cong \mathbb{D}(f^{-1}\mathscr{F})$.*

2.2. **Open and closed embeddings.**

Proposition 2.15 ([Ach21], Prop. 1.3.9)**.** *If $h : Y \hookrightarrow X$ is a locally closed embedding, then for any $\mathscr{F} \in \mathsf{D}^+(Y,R)$ we have isomorphisms*

$$\mathscr{F} \to h^! h_! \to h^{-1}h_!\mathscr{F}, \quad h^!\mathbf{R}h_*\mathscr{F} \to h^{-1}\mathbf{R}h_*\mathscr{F} \to \mathscr{F}.$$

Theorem 2.16 ([Ach21], Thm. 1.3.10)**.** *Let $i : Z \hookrightarrow X$ is a closed embedding with complement $j : U := X\backslash Z \hookrightarrow X$ is an open embedding.*

(1) *We have $i^{-1} \circ j_! = 0$, $i^! \circ \mathbf{R}j_* = 0$ and $j^{-1} \circ i_* = 0$.*
(2) *We have $j_!j^{-1}\mathscr{F} \to \mathscr{F} \to i_*i^{-1}\mathscr{F} \to$ a distinguished triangle for any $\mathscr{F} \in \mathsf{D}^+(X,R)$ by adjunctions. Conversely, if $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to$ is a distinguished triangle with $i^{-1}\mathscr{F}'' = 0$ and $j^{-1}\mathscr{F}' = 0$, then it is $j_!j^{-1}\mathscr{F} \to \mathscr{F} \to i_*i^{-1}\mathscr{F} \to$.*
(3) *We have $i_*i^!\mathscr{F} \to \mathscr{F} \to \mathbf{R}j_*j^{-1}\mathscr{F} \to$ a distinguished triangle for any $\mathscr{F} \in \mathsf{D}^+(X,R)$ by adjunctions. Conversely, if $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to$ is a distinguished triangle with $i^!\mathscr{F}'' = 0$ and $j^{-1}\mathscr{F}' = 0$, then it is $i_*i^!\mathscr{F} \to \mathscr{F} \to \mathbf{R}j_*j^{-1}\mathscr{F} \to$.*

Corollary 2.17. *Let $i : Z \hookrightarrow X$ is a closed embedding with complement $j : U := X\backslash Z \hookrightarrow X$ is an open embedding.*

(1) *Then i_* is fully faithful which induce an equivalence*
$$\mathsf{D}^+(Z,R) \cong \{\mathscr{F} \in \mathsf{D}^+(X,R) : \mathrm{supp}(\mathscr{F}) \subset Z\}.$$
(2) *For any $\mathscr{G} \in \mathsf{D}^+(U,R)$, then $i^{-1}\mathbf{R}j_*\mathscr{F} \cong i^!j_!\mathscr{F}[1]$.*

Corollary 2.18 ([Ach21], Prop. 1.3.13)**.** *Let Y_1 and Y_2 be subsets of X that are either both open or both closed, and such that $Y_1 \cup Y_2 = X$. Let $h_1 : Y_1 \hookrightarrow X$, $h_2 : Y_2 \hookrightarrow X$ and $h : Y_1 \cap Y_2 \hookrightarrow X$ be the inclusion maps. For any $\mathscr{F} \in \mathsf{D}^+(X,R)$, there are distinguished triangles*

$$\mathscr{F} \to \mathbf{R}h_{1*}h_1^{-1}\mathscr{F} \oplus \mathbf{R}h_{2*}h_2^{-1}\mathscr{F} \to \mathbf{R}h_*h^{-1}\mathscr{F} \to,$$
$$h_!h^!\mathscr{F} \to \mathbf{R}h_{1!}h_1^!\mathscr{F} \oplus \mathbf{R}h_{2!}h_2^!\mathscr{F} \to \mathscr{F} \to .$$

2.3. **Local Systems.**

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