Moduli Spaces of Coherent Sheaves and its Related Topics

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Contents

1 Introductions

First of course you need to know the basic theory of algebraic geometry like [7] and the basic theory of algebraic stacks like [10] or [14] or the first few chapters of [1].

We will first use the new language of good moduli theory due to Jarod Alper to write the theory of moduli space of semistable sheaves. For the basic theory of semistable sheaves and its openness and boundedness we following the first part of the book [8].

Need to add.

2 Basic Theory of Good Moduli Spaces

Here we will introduce some basic background about good moduli theory and the theory of Θ-complete and S-completedue to J. Alper in [3] and [4]. These will play an important role in our foundamental theory.

We will give the main properties, theorems and their motivations and some idea of proofs. For the detailed proof we refer reader to the original paper [3][4] or the book draft [1] of J. Alper.

2.1 Properties of Good Moduli Spaces

As we all know, in the modern construction of the moduli space of stable curves follows from the following way:

(a) Construct the stack $\overline{\mathscr{M}}_{g,n}$ and show that it is a Deligne-Mumford stack;

(b) show the stable-reduction of stable curves and find that $\overline{\mathscr{M}}_{g,n}$ is proper;

(c) use Keel-Mori theorem to construct the coarse moduli space $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ and show that it is projective.

But in our case, we can not use Keel-Mori theorem to the moduli stack of semistable sheaves because the inertia stack $\mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ is not finite. In order to this the similar modern way (instead of GIT-construction), J.Alper developed a nice similar (but much more complicated) theory to solve this problem – the theory of good moduli space ([3] and [4]) for linear reductive groups and the theory of adequate moduli spaces ([2]) for geometric reductive groups.

For now, the theory of good moduli space plays a central role in the construction of moduli spaces, such as moduli stack of semistable sheaves $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ and K-moduli stack $\mathscr{X}_{n,V}^{\mathrm{Kss}}$ which aim to construct a good moduli space of Fano varieties (see the book draft due to C. Xu).

Definition 2.1 (Good moduli space)**.** *For an algebraic stack $\mathscr{X}$, its* good moduli space *is an algebraic space X together with a qcqs morphism $\pi : \mathscr{X} \to X$ such that* *(i) the natural map $\mathscr{O}_X \to \pi_* \mathscr{O}_{\mathscr{X}}$ is an isomorphism;*

(ii) the functor $\pi_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact.

Note that the condition in (ii) is called ***cohomologically affine****.*

The definition of good moduli space is inspired from the GIT-quotient of linear reductive group G (that is, $V \mapsto V^G$ is exact. Hence G is linear reductive if and only if $\mathbf{B}G$ is cohomologically affine)

$$[X/G] \dashrightarrow [X^{\mathrm{ss}}/G] \to X /\!\!/ G = \operatorname{Proj} \bigoplus_{d \geq 0} \Gamma(X, \mathscr{O}_X(d))^G.$$

Or locally, the map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. Of coarse, a tame coarse moduli space is a good moduli space by the local structure of coarse moduli spaces.

Here we state several basic properties of cohomologically affine morphisms.

Lemma 2.2. *Consider a cartesian*

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\ \mathscr{Y}' \arrow[r, "g"] & \mathscr{Y} \end{tikzcd}$$

of algebraic stacks, then:

(i) If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.

(ii) If Y has quasi-affine diagonal and π is cohomologically affine, then π' is cohomologically affine.

If we consider a cartesian

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of algebraic stacks where X, X' are quasi-separated algebraic spaces, then:

(iii) If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.

(iv) If π is a good moduli space, so is π'.

(v) Let π is a good moduli space. For $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(X)$, the adjunction map $\pi_*\mathscr{F} \otimes \mathscr{G} \cong \pi_*(\mathscr{F} \otimes \pi^*\mathscr{G})$ is an isomorphism. In particular, the adjunction map $\mathscr{G} \cong \pi_*\pi^*\mathscr{G}$ is an isomorphism.

(vi) *For $\mathscr{F} \in \mathrm{QCoh}(X)$, then $g^*\pi_*\mathscr{F} \cong \pi'_*(g')^*\mathscr{F}$.*

(vii) *For a quasi-coherent sheaf of ideals $\mathscr{J} \subset \mathscr{O}_X$, the natural map $\mathscr{J} \cong \pi_*(\pi^{-1}\mathscr{J} \cdot \mathscr{O}_{\mathscr{X}})$ is an isomorphism.*

If $\pi : \mathscr{X} \to X$ be a good moduli space with X quasi-separated, then

(viii) *If $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras, then $\underline{\mathrm{Spec}}_{\mathscr{X}}\mathscr{A} \to \underline{\mathrm{Spec}}_X \pi_*\mathscr{A}$ is a good moduli space.*

(ix) *If $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space.*

Proof. See section 4 in fundamental paper [3]. □

Now some important properties of good moduli spaces and give some comments. Actually these are similar as the properties of GIT.

Theorem 2.3. *Let $\pi : \mathscr{X} \to X$ be a good moduli space where $\mathscr{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(i) *π is surjective and universally closed (and universally submersive);*

(ii) *for closed substacks $\mathscr{Z}_1, \mathscr{Z}_2 \subset \mathscr{X}$, we have $\mathrm{Im}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \mathrm{Im}(\mathscr{Z}_1) \cap \mathrm{Im}(\mathscr{Z}_2)$. For geometric points $x_1, x_2 \in \mathscr{X}(k)$, $\pi(x_1) = \pi(x_2) \in \mathscr{X}(k)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_1\}} \neq \emptyset$ in $|\mathscr{X} \times_S k|$. In particular, π induces a bijection between closed points in $\mathscr{X}$ and closed points in X;*

(iii) *if $\mathscr{X}$ is noetherian, so is X. If $\mathscr{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence;*

(iv) *If X is noetherian, then π is universal for maps to algebraic spaces.*

Proof. Here we give some idea. The proof we refer the Theorem 4.16 in [3].

For (i), by Lemma 2.2 (iv) we know that $\mathscr{X} \times_X \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ is good moduli space. Hence $\Gamma(\mathscr{X} \times_X \mathrm{Spec}\, k, \mathscr{O}_{\mathscr{X} \times_X \mathrm{Spec}\, k}) = k$ and $|\mathscr{X} \times_X \mathrm{Spec}\, k| \neq \emptyset$. Hence π is surjective. Again by Lemm 2.2 (ix) we know that if $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space. Hence it is surjective and hence π is closed. By Lemma 2.2 (iv) we know that it is universally closed.

For (ii), let ideal sheaves be $\mathscr{I}_1, \mathscr{I}_2$, then by the exactness of π_* we have

$$\begin{tikzcd}
 & & \pi_*\mathscr{I}_2 \arrow[d] \arrow[dr, two heads] & & \\
0 \arrow[r] & \pi_*\mathscr{I}_1 \arrow[r] & \pi_*(\mathscr{I}_1 + \mathscr{I}_2) \arrow[r, two heads] & \pi_*\mathscr{I}_2/\pi_*(\mathscr{I}_1 \cap \mathscr{I}_2) \arrow[r] & 0
\end{tikzcd}$$

Hence the inclution $\pi_*(\mathscr{I}_1 + \mathscr{I}_2) \to \pi_*(\mathscr{I}_1 + \mathscr{I}_2)$ is surjective.

For (iii), X is noetherian follows from Lemma 2.2 (vii). We omit others and (iv). $\square$

There is an interesting result which we will use it:

Proposition 2.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a cohomologically affine morphism of algebraic stacks where $\mathscr{Y}$ has quasi-affine diagonal. If f is representable (that is, $\mathscr{I}_{\mathscr{X}/\mathscr{Y}} \to \mathscr{X}$ is trivial, or equivalently, Δ_{Δ_f} is an isomorphism), then f is affine.*

Proof. Trivial by faithfully flat descent and Serre's Criterion. $\square$

2.2 Luna's Results and Étale Local Structure of Algebraic Stacks

2.2.1 Luna's Fundamental Lemma and Luna's Étale Slice Theorem

Luna's results are classical and you can find them even in [12].

Theorem 2.5 (Luna's Fundamental Lemma)**.** *Consider a commutative diagram:*

$$\begin{array}{ccc} \mathscr{X}' & \xrightarrow{f} & \mathscr{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{g} & X \end{array}$$

where f is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in \mathscr{X}'$ be a point such that

(a) f is étale at x';

(b) f induces an isomorphism of stabilizer groups at x', and

(c) $x' \in \mathscr{X}'$ and $x = f(x') \in \mathscr{X}$ are closed points.

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathscr{X} \cong (\pi')^{-1}(U')$.

Sketch. Using limit-argument, we may let $X = \operatorname{Spec} A$, where A is a strictly henselian local ring. After shrink $\mathscr{X}'$, we may let f is étale. Then by Zariski main theorem we get $\mathscr{X}' \to \widetilde{\mathscr{X}} = \underline{\operatorname{Spec}}_{\mathscr{X}} \mathscr{A} \to \mathscr{X}$. Hence $\widetilde{\mathscr{X}} \to \widetilde{X} = \underline{\operatorname{Spec}}_{\mathscr{X}} \pi_* \mathscr{A}$ is a good moduli space with $\widetilde{X} \to X$ finite. Hence we can let $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ of henselian local rings. Take $U' = \operatorname{Spec} A_i$ contains image of x'. Well done. $\square$

Hence we have an very important corollary we will use:

Corollary 2.6. *With the same hypotheses, suppose that f is étale and that for all closed points $x' \in \mathscr{X}'$ we have*

(a) $f(x')$ closed;

(b) f induces an isomorphism of stabilizer groups at x'.

Then $g : X' \to X$ étale and that commutative diagram is cartesian.

This is our main motivation to define the Θ-completeness and S-completeness. We will discuss this deeply later.

Next we introduce Luna's étale slice theorem which was motivated the étale local structure of algebraic stacks.

Lemma 2.7 (Luna Map)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, there exists a G_x-equivariant morphism (Luna map)*

$$f : X \to T_{X,x} := \underline{\mathrm{Spec}}\mathrm{Sym}\mathfrak{m}_x/\mathfrak{m}_x^2$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \operatorname{Spec} A$, then $\mathfrak{m}_x$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ are G_x-representations and we see that G_x acts naturally on the tangent space $T_{X,x} := \underline{\mathrm{Spec}}\mathrm{Sym}\mathfrak{m}_x/\mathfrak{m}_x^2$. Since G_x is linearly reductive, the surjection $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$ of G_x-representations has a section $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x$. This induces a G_x-equivariant ring map $\mathrm{Sym}\mathfrak{m}_x/\mathfrak{m}_x^2 \to A$ and thus a G_x-equivariant morphism $f : X \to T_{X,x}$ sending x to the origin. If x is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x. ☐

Theorem 2.8 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$[W/G_x] \to [X/G]$$

is affine étale. If in addition $Gx \subset X$ closed (then by Matsushima's theorem G_X is linearly reductive), then there is a cartesian

$$\begin{tikzcd} {[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {[X/G]} \arrow[d] \\ W /\!\!/ G_x \arrow[r] & X /\!\!/ G \end{tikzcd}$$

where $W /\!/ G_x \to X /\!/ G$ is also étale.

Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then it can be arranged that there is an G_x-invariant étale morphism $W \to N_x$ which is the pullback of an étale map $W /\!/ G_x \to N_x /\!/ G_x$ of GIT quotients.

Proof. Pick a finite G-representation V and a G-equivariant closed immersion $X \subset \mathbb{A}(V)$. Then using this we can reduce to the case where $x \in X$ is smooth.

Hence we have Luna map $f : X \to T_{X,x}$ is G_x-equivariant and étale at x. The subspace $T_{Gx,x} \subset T_{X,x}$ is G_x-invariant and again since G_x is linearly reductive, the surjection $T_{X,x} \to N_x$ has a section $N_x \to T_{X,x}$. We define W as

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N_x \arrow[d] \\ X \arrow[r, "f"] & T_{X,x} \end{tikzcd}$$

Then $[W/G_x] \to [X/G]$ and $[W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at x. Hence we have commutative diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l] \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ N_x /\!/ G_x & W /\!/ G_x \arrow[l] \arrow[r] & X /\!/ G \end{tikzcd}$$

Hence using Luna's fundamental lemma 2.5 twice and well done. □

2.2.2 Coherent Tannaka Duality and Coherent Completeness

Here we introduce some very importent results aiming to extend to morphisms.

Theorem 2.9 (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks $\mathscr{X}$ and $\mathscr{Y}$ with affine diagonal, the functor*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})), \quad f \mapsto f^*$$

is an equivalence of categories where the latter category denote the right exact additive tensor functors $\mathrm{Coh}(\mathscr{Y}) \to \mathrm{Coh}(\mathscr{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.

Proof. This follows from a nice observation of Lurie in [11]. For the proof we refer [1] Theorem 6.4.1. □

Definition 2.10. *A noetherian algebraic stack $\mathscr{X}$ is* coherently complete *along a closed substack $\mathscr{X}_0$ if the natural functor*

$$\mathrm{Coh}(\mathscr{X}) \to \varprojlim \mathrm{Coh}(\mathscr{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathscr{X}_n$ denotes the n-th nilpotent thickening of $\mathscr{X}_0$.

Remark 2.11. *(i) This motivated by the* Grothendieck's Existence Theorem *asserts that if X is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then X is coherently complete along X_0.*

Actually this is right even for proper algebraic stack over some I-adically complete noetherian ring. We refer [15].

(ii) Let k be an algebraically closed field and R be a complete noetherian local k-algebra with residue field k. Let G be a linearly reductive group over k acting on an affine scheme $\mathrm{Spec}\, A$ of finite type over R. Suppose that $A^G = R$ and that there is a G-fixed k-point $x \in \mathrm{Spec}\, A$. Then $[\mathrm{Spec}\, A/G]$ is coherently complete along the closed substack $\mathbf{B}G$ defined by x. See the Theorem 6.4.11 in [1] for the proof.

We will use the follows corollary many times:

Corollary 2.12. *Let $\mathscr{X}$ and $\mathscr{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathscr{X}$ is coherently complete along $\mathscr{X}_0$. Then there is an equivalence of categories*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}), \quad f \mapsto (f_n).$$

Proof. This is directly:

$$\begin{aligned} \mathrm{MOR}(\mathscr{X}, \mathscr{Y}) &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})) \\ &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \varprojlim \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}) \end{aligned}$$

and well done. □

2.2.3 Some Deformation Theory

Proposition 2.13. *Consider a commutative diagram*

$$\begin{tikzcd} \mathscr{W} \arrow[r, "f"] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \mathscr{W}' \arrow[r] \arrow[ur, dashed] & \mathscr{Y} \end{tikzcd}$$

of noetherian algebraic stacks with affine diagonal where $\mathscr{X} \to \mathscr{Y}$ is smooth and affine-and $\mathscr{W} \to \mathscr{W}'$ is a closed immersion defined by a square-zero sheaf of ideals $\mathscr{J}$. If $\mathscr{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. As the case of schemes, the set of liftings is a torsor under $\mathrm{Hom}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J})$. Hence let $\mathscr{F} := f^*\Omega^\vee_{\mathscr{X}/\mathscr{Y}} \otimes \mathscr{J}$. Consider

$$\begin{tikzcd}
(U/\mathscr{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathscr{W} \arrow[r] \arrow[d, hook, "f'_U"'] & \mathscr{X} \arrow[d] \\
(U'/\mathscr{W}')^2 \arrow[r, shift left] \arrow[r, shift right] & U' \arrow[r] \arrow[urr] & \mathscr{W}' \arrow[r] \arrow[ur, dashed, "f'"'] & \mathscr{Y}
\end{tikzcd}$$

where $(U/\mathscr{W})^2 = U \times_{\mathscr{W}} U$ is affine. Because $\mathscr{X} \to \mathscr{Y}$ is representable, to check that f'_U descends to a morphism f', we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. As $f'_U \circ p_1 - f'_U \circ p_2 \in \Gamma((U/\mathscr{W})^2, q_2^*\mathscr{F})$, this follows from the $\mathscr{W}$ is cohomologically affine and exact sequences. Omitted and see [1] Proposition 6.5.8.

There is another way, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathscr{O}_{\mathscr{W}}}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J}) \cong H^1(\mathscr{W}, \mathscr{F})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Proposition 2.14. *Let $\mathscr{W} \to \mathscr{W}'$ be a closed immersion of algebraic stacks of finite type over k with affine diagonal defined by a square-zero sheaf of ideals $\mathscr{J}$. Let G be an affine algebraic group over k. If $\mathscr{W}$ is cohomologically affine, then every principal G-bundle $\mathscr{P} \to \mathscr{W}$ extends to a principal G-bundle $\mathscr{P}' \to \mathscr{W}'$.*

Proof. Similar as the proof above and we need to take $\mathscr{F} = \mathfrak{g} \otimes \mathscr{J}$ from the deformation theory of principal G-bundles in [1] D.2.9.

There is also another way. Note that this is equivalent to the deformation of $f : \mathscr{W} \to \mathbf{B}G$ to $\mathscr{W}' \to \mathbf{B}G$ which is the same problem in Proposition 2.13 to $\mathbf{B}G \to \operatorname{Spec} k$ which is not affine. See the arguments in Remark 6.5.11 in [1], we can see the obstruction lies in $H^2(\mathscr{W}, \mathfrak{g} \otimes \mathscr{J})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Remark 2.15. *All these results are the special case in Theorem 1.5 in [13].*

2.2.4 Étale Local Structure of Algebraic Stacks

There is a fundamental theorem about algebraic stacks as follows:

Theorem 2.16 (Minimal Presentations)**.** *Let $\mathscr{X}$ be a noetherian algebraic stack and let $x \in |\mathscr{X}|$ be a finite type point with smooth stabilizer G_x. Then there exists a scheme U with a closed point $u \in U$ and a smooth morphism $(U, u) \to (\mathscr{X}, x)$ of relative dimension $\dim G_x$ such that the diagram*

$$\begin{tikzcd}
\operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathscr{X}
\end{tikzcd}$$

is cartesian.

Proof. This is easy in Theorem 3.6.1 in [1]. Let $(U, u) \to (\mathscr{X}, x)$ be a smooth morphism of relative dimension n, hence we have

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathscr{X} \end{tikzcd}$$

As $\dim \mathcal{G}_x = -\dim G_x$, then $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. By Nakayama's lemma, we pick a reguler sequence $f_1, ..., f_c \in \mathscr{O}_U$ and consider $W = V(f_1, ..., f_c)$ and then $W \cap O(u) = \operatorname{Spec} \kappa(u)$. By the local criterion for flatness and smooth descent to $U \times_{\mathscr{X}} U \rightrightarrows \mathscr{X}$, we know that $W \to \mathscr{X}$ is flat. Checking on the fibers we can conclude the result. □

Before giving the statement of the étale local structure of algebraic stacks, we will give a useful criteria for morphisms to be closed immersions or isomorphisms.

Lemma 2.17. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field k with affine diagonal. Assume that $|\mathscr{X}| = \{x\}$ and $|\mathscr{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism of residue gerbes $\mathscr{X}_0 := \mathcal{G}_x = \mathbf{B}G_x$ with $\mathscr{Y}_0 := \mathcal{G}_y = \mathbf{B}G_y$. Let $\mathfrak{m}_x, \mathfrak{m}_y$ be the ideal sheaves defining them, and let $f_1 : \mathscr{X}_1 \to \mathscr{Y}_1$ be the induced morphism between the first nilpotent thickenings.*

(i) If f_1 is a closed immersion, then so is f.

(ii) If f_1 is a closed immersion and there is an isomorphism

$$\bigoplus_{n \geq 0} \mathfrak{m}_y^n / \mathfrak{m}_y^{n+1} \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

of graded $\mathscr{O}_{\mathscr{X}_0}$-modules, then f is an isomorphism.

Proof. By Theorem 2.16, we may choose a minimal smooth presentations $V = \operatorname{Spec} B \to \mathscr{Y}$ such that $V \times_{\mathscr{Y}} \mathscr{Y}_0 \cong \operatorname{Spec} k$. Hence B is an artinian local ring, then so is $U = \operatorname{Spec} B \cong V \times_{\mathscr{Y}} \mathscr{X}$. Hence we can let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian rings.

For (i), this follows from [7] Lemma II.7.4. For (ii), this is trivial. □

Lemma 2.18. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f : \mathscr{W} := [\operatorname{Spec} A/G] \to \mathscr{X}$ be a finite type morphism with G linearly reductive. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Let $\pi : \mathscr{W} \to \operatorname{Spec} A^G$. We may let $F : \mathscr{W} \to \mathscr{X}$ is quasi-finite as it is quasi-finite over some open set.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathscr{X}$, then

$$\begin{tikzcd} \mathscr{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V = \operatorname{Spec} B \arrow[d] \\ \mathscr{W} \arrow[r, "f"] & \mathscr{X} \end{tikzcd}$$

As $\mathscr{X}$ with affine diagonal, the map $V \to \mathscr{X}$ is affine. Hence $\mathscr{W}_V$ is cohomologically affine. By Proposition 6.3.28 in [1] we have:

- Suppose $\mathscr{Z}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi : \mathscr{Z} \to Z$. If the diagonal Δ_π is quasi-finite, then it is finite.

Hence $\mathscr{W}_V \to V$ is separated. From descent $\mathscr{W} \to \mathscr{X}$ is also separated and that the relative inertia $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is finite. Since the fiber over w is trivial, there is an open neighborhood over which the relative inertia is trivial. Hence replace this we may let $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is trivial. Hence it is representable. By Serre's criteria we get the result. □

Theorem 2.19 (Étale Local Structure of Algebraic Stacks)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. For every point $x \in X(k)$ with linearly reductive stabilizer G_x there exists an affine étale morphism*

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

If $x \in \mathscr{X}$ is a smooth point, then there is also an étale morphism

$$f : ([\operatorname{Spec} A/G_x], w) \to ([T_{\mathscr{X},x}/G_x], 0).$$

Proof of the Smooth Case. Here we only give the proof of smooth case and tell you the difficulties of proof the general case in the remark.

Since x is locally closed, we may let it is closed. Hence $\mathscr{X}_0 := \mathbf{B}G_x \subset \mathscr{X}$ defined by $\mathscr{I}$. Let $\mathscr{X}_n$ to be the n-th nilpotent thickening of it. The Zariski tangent space $T_{\mathscr{X},x}$ can be identified with the normal space $(\mathscr{I}/\mathscr{I}^2)^\vee$, hence with a G_x-representation. Hence we can define $\mathscr{T} = [T_{\mathscr{X},x}/G_x]$ with $\mathscr{T}_0 := \mathbf{B}G_x$ and the n-th nilpotent thickening $\mathscr{T}_n$.

By Proposition 2.14 we get an affine $\mathscr{X}_n \to \mathbf{B}G_x$. By Proposition 2.13 inductively we have lifts:

$$\begin{tikzcd} \mathscr{X}_n \arrow[r] \arrow[d, hook] & \mathscr{T} \arrow[d] \\ \mathscr{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x \end{tikzcd}$$

By some easy commutative algebra via smooth descent, we have $\mathscr{X}_1 \cong \mathscr{T}_1$. Hence by Lemma 2.17(ii) we have $\mathscr{X}_n \cong \mathscr{T}_n$.

Consider $\pi : \mathscr{T} \to T := T_{\mathscr{X},x} /\!\!/ G_x$ and $\widehat{\mathscr{T}} := \operatorname{Spec} \widehat{\mathscr{O}}_{T,\pi(0)} \times_T \mathscr{T} = [\operatorname{Spec} B/G]$ where B is of finite type over the noetherian complete local k-algebra $B^G = \widehat{\mathscr{O}}_{T,\pi(0)}$. By Remark 2.11 (ii) we know that $\widehat{\mathscr{T}}$ is coherently complete along $\mathscr{T}_0$ and by coherent Tannaka duality we get

$$\mathrm{MOR}(\widehat{\mathscr{T}}, \mathscr{X}) \to \varprojlim \mathrm{MOR}(\mathscr{T}_n, \mathscr{X}).$$

Hence we have

$$\begin{tikzcd}
 & X & \\
\mathscr{X}_n \cong \mathscr{T}_n \arrow[ur] \arrow[r] & \widehat{\mathscr{T}} \arrow[u, dashed] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{T} \arrow[d] \\
 & \operatorname{Spec} \mathscr{O}_{T,\pi(0)} \arrow[r] & T
\end{tikzcd}$$

Now by Artin Approximation, there exists an étale morphism $(U, u) \to (T, 0)$ where U is an affine scheme with a k-point $u \in U$ and a morphism $(U \times_T \mathscr{T}, (u, 0)) \to (\mathscr{X}, x)$ agreeing with $(\widehat{\mathscr{T}}, 0) \to (\mathscr{X}, x)$ in the first order. As $U \times_T \mathscr{T}$ is smooth at $(u, 0)$ and $\mathscr{X}$ is smooth at x, and as $U \times_T \mathscr{T} \to \mathscr{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u, 0)$, hence the morphism $U \times_T \mathscr{T} \to \mathscr{X}$ is étale at $(u, 0)$. Finally, by Lemma 2.18 we get the result. $\square$

Remark 2.20. *We refer Section 6.5.5 in [1] for the proof of the general case. Now we point out that in the general case we also have $\mathscr{X}_1 \cong \mathscr{T}_1$. But we can only use the Lemma 2.17(i) to get a closed immersion $\mathscr{X}_n \to \mathscr{T}_n$. Also in the general case we can not deduce $U \times_T \mathscr{T} \to \mathscr{X}$ is étale from the isomorphism of tangent spaces! In order to solve this, we need a more general fact called equivariant Artin algebraization theorem. See Theorem 6.5.14 in [1] for the statement and the proof.*

2.3 One Parameter Subgroups and Filtrations

2.4 Existence of Good Moduli Space

2.4.1 Basic Properties of Θ-Complete and S-Complete

Definition 2.21 (Θ-completeness)**.**

Definition 2.22 (S-completeness)**.**

2.4.2 Θ-Surjectivity and Θ-Complete

Definition 2.23 (Θ-surjective)**.**

2.4.3 Unpunctured Inertia and S-Complete

Definition 2.24 (Unpunctured Inertia)**.**

2.4.4 The Finally Statement and the Proof

3 Moduli Stack of Coherent Sheaves

In the next sections we will just consider the stacks over $\mathbb{C}$.

3.1 Construction of the Moduli Stack of Coherent Sheaves

Now we consider the moduli space of coherent sheaves over some smooth projective complex variety X. Then we have the Chern character map

$$\gamma : K(X) \xrightarrow{\text{ch}} \mathrm{CH}^*(X)_{\mathbb{Q}} \xrightarrow{\text{cl}} H^{2*}(X, \mathbb{Q}).$$

(or we can use ℓ-adic cohomology) Let Γ be the image of this map.

By Grothendieck-Riemann-Roch theorem (see Chpter 15 in [5]),

$$P(\mathscr{F}, m) = \chi(\mathscr{F}(m)) = \int_X \mathrm{ch}(\mathscr{F}(m))\mathrm{td}(\mathcal{T}_X),$$

then we find that the information of $v \in \Gamma$ is equivalent to the information of the Hilbert polynomial χ. So we can use both of them when X is smooth. If X is just a projective scheme, then we will only to use the Hilbert polynomial.

Theorem 3.1. *Let X be a connected projective $\mathbb{C}$-scheme, we let $\underline{\mathrm{Coh}}_P(X)$ the category fibred in groupoid over $\mathrm{Sch}/\mathbb{C}$ sending a $\mathbb{C}$-scheme T to the groupoid of T-flat families $\mathscr{E} \in \mathrm{Coh}(X \times T)$ such that any restriction $\mathscr{E}_t \in \mathrm{Coh}(X)$ has the Hilbert polynomial P, the morphisms in the above groupoid are given by isomorphisms of $\mathscr{E}$.*

Then $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$ of affine diagonal. Also, we have the algebraic stack $\underline{\mathrm{Coh}}(X) = \coprod_P \underline{\mathrm{Coh}}_P(X)$.

Proof. Easy to see that $\underline{\mathrm{Coh}}_P(X)$ is actually a stack, we first claim that it is an algebraic stack in a natural way.

For each integer N, we claim there is an open substack $\mathscr{U}_N \subset \underline{\mathrm{Coh}}_P(X)$ parameterizing coherent sheaves $\mathscr{E}$ such that $\mathscr{E}(N)$ generated by global sections and $H^i(X, \mathscr{E}(N)) = 0$ for any $i > 0$. Actually this is trivial by some application of cohomology and base change. As $\underline{\mathrm{Coh}}_P(X) = \bigcup_N \mathscr{U}_N$, we just need to show $\mathscr{U}_N$ is an algebraic stack locally of finite type over $\mathbb{C}$.

For each N, we consider the quotient scheme

$$Q_N := \underline{\mathrm{Quot}}_X^P(\mathscr{O}_X(-N)^{P(N)}).$$

Again by some application of cohomology and base change, we find that there is an open subscheme $Q'_N \subset Q_N$ parameterizing quotients $q : \mathscr{O}_X(-N)^{P(N)} \twoheadrightarrow \mathscr{F}$ such that $H^0(q(N))$ is surjective and $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$.

We have a natural map $Q'_N \to \mathscr{U}_N$ maps $[\mathscr{O}_X(-N)^{P(N)}]$ to $\mathscr{F}$. We observe that Q'_N is also $\mathrm{GL}_{P(N)}$-invariant, then this map descends to

$$\Psi^{\mathrm{pre}} : [Q'_N/\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathscr{U}_N$$

which is fully faithful since every automorphism of a coherent sheaf $\mathscr{E}$ on $X \times S$ induces an automorphism of $p_{2,*}\mathscr{E}(N) = \mathscr{O}_S^{P(N)}$ i.e. an element of $\mathrm{GL}_{P(N)}(S)$, and this element acts on $\mathscr{O}_X(-N)^{P(N)}$ preserving the quotient $\mathscr{E}$.

After stackification, we have another fully faithful map $\Psi : [Q'_N/\mathrm{GL}_{P(N)}] \to \mathscr{U}_N$ which is also essentially surjective by the constructions. Hence we have

$$\mathscr{U}_N \cong [Q'_N/\mathrm{GL}_{P(N)}], \quad \underline{\mathrm{Coh}}_P(X) = \bigcup_N [Q'_N/\mathrm{GL}_{P(N)}].$$

Hence $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$. □

3.2 Basic Facts of the Moduli Stack of Coherent Sheaves

Proposition 3.2. *Let X be a projective scheme over an algebraically closed field k. For a noetherian k-algebra R, $\mathrm{MOR}_k(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(\mathscr{E}, \mathscr{E}_*)$ where $\mathscr{E}$ is a coherent sheaf on X_R flat over R and*

$$\mathscr{E}_* : 0 \subset \cdots \subset \mathscr{E}_{i-1} \subset \mathscr{E}_i \subset \cdots \subset \mathscr{E}$$

is a filtration such that $\mathscr{E}i = 0$ for $i \ll 0$, $\mathscr{E}_i = E$ for $i \gg 0$, and each factor $\mathscr{E}_i/\mathscr{E}_{i-1}$ is flat over R. A morphism is an isomorphism $\mathscr{E} \to \mathscr{E}'$ of coherent sheaves compatible with the filtration.

Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to E and 0 to the associated graded $\mathrm{gr}\,\mathscr{E}_* = \bigoplus_i \mathscr{E}_i/\mathscr{E}_{i-1}$.

Proof. □

Theorem 3.3. *For every projective scheme X over an algebraically closed field k, the algebraic stack $\underline{\mathrm{Coh}}(X)$ (and hence $\underline{\mathrm{Coh}}_P(X)$) is Θ-complete and S-complete.*

Proof. □

Theorem 3.4.

Proof. □

4 Basic Theory of Semistable Sheaves

Our aim is to find a moduli space of sheaves which is of finite type! Actually $\underline{\mathrm{Coh}}_P(X)$ is never of finite type. Consider $\{\mathscr{O}(n)\oplus\mathscr{O}(-n)\}$ on $\mathbb{P}^1$, then this can not parametrized by a scheme of finite type. Hence we need some more conditions.

4.1 Basic Properties

Fix X be a projective scheme over a field k with $H=\mathscr{O}(1)$. Now if $\mathscr{F}$ be a coherent sheaf of dimension $d=\dim X$ with Hilbert polynomial $P(\mathscr{F},m)=\sum_{i=0}^d \alpha_i(\mathscr{F})\frac{m^i}{i!}$, then we can define $\mathrm{rank}(\mathscr{F}):=\frac{\alpha_d(\mathscr{F})}{\alpha_d(\mathscr{O}_X)}$. If X is integral, this is the usual definition.

For polynomials $f_i\in\mathbb{Q}[m]$ for $i=1,2$, we define $f_1<(\le)f_2$ if $f_1(m)<(\le)f_2(m)$ for $m\gg 0$.

Definition 4.1. *Fix (X,H) as above and $\mathscr{F}$ be a coherent sheaf of dimension d.*

(i) We define the slope $\mu_H(\mathscr{F}):=\frac{c_1(\mathscr{F})\cdot H^{d-1}}{\mathrm{rank}(\mathscr{F})}$*;*

(ii) we call $\mathscr{F}$ is μ_H-(semi)stable *if for any $0\subset\mathscr{E}\subset\mathscr{F}$ with $0<\mathrm{rank}\,\mathscr{E}<\mathrm{rank}\,\mathscr{F}$ we have $T_{d-2}(\mathscr{F})=T_{d-1}(\mathscr{F})$ and $\mu_H(\mathscr{E})<(\le)\mu_H(\mathscr{F})$;*

(iii) we consider the Hilbert polynomial $P(\mathscr{F},m)=\sum_{i=0}^d\alpha_i(\mathscr{F})\frac{m^i}{i!}$, then we have $\alpha_d(\mathscr{F})=\mathrm{rank}(\mathscr{F})\cdot H^d$ and $\alpha_{d-1}(\mathscr{F})=\frac{1}{2}\mathrm{rank}(\mathscr{F})\deg T_X+\deg\mathscr{F}$. We define the reduced Hilbert polynomial *is*

$$p(\mathscr{F},m)=\frac{P(\mathscr{F},m)}{\alpha_d(\mathscr{F})}=\frac{m^d}{d!}+\frac{1}{H^d}\left(\frac{1}{2}\deg\mathscr{F}+\mu_H(\mathscr{F})\right)\frac{m^{d-1}}{(d-1)!}+\textit{lower terms.}$$

(iv) Define $\mathscr{F}$ is H-(semi)stable *if it is pure and for any $0\subsetneq\mathscr{E}\subsetneq\mathscr{F}$, we have $p(\mathscr{E},m)<(\le)p(\mathscr{F},m)$.*

(v) Define $\mathscr{F}$ is geometrically H-stable *if for any base field extension $X_K=X\times_k\mathrm{Spec}(K)$ the pull-back $\mathscr{F}_K$ is stable.*

Remark 4.2. *Here we have some remarks.*

- *As the Harder-Narasimhan filtration is unique (Theorem 4.9) and stable under field extension (Proposition 4.10), we don't need the* geometrically H-ss.
- *We can define $\mathscr{F}$ is μ_H-(semi)stable if for any $0\subsetneq\mathscr{E}\subsetneq\mathscr{F}$ with $0<\mathrm{rank}\,\mathscr{E}<\mathrm{rank}\,\mathscr{F}$, we have $\mathrm{rank}(\mathscr{F})\deg(\mathscr{E})<(\le)\mathrm{rank}(\mathscr{E})\deg(\mathscr{F})$. This is obviously the same definition except that it does not require explicitly that $T_{d-2}(\mathscr{F})=T_{d-1}(\mathscr{F})$. But this can be easy to be deduced.*

- *Similarly, we can define $\mathscr{F}$ is H-(semi)stable if for any $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$, we have $\alpha_d(\mathscr{F})P(\mathscr{E}, m) < (\leq)\alpha_d(\mathscr{E})p(\mathscr{F}, m)$. This is obviously the same definition except that it does not require explicitly that $\mathscr{F}$ is pure. But applying the inequality to $\mathscr{E} = T_{d-1}(\mathscr{F})$ (maximal subsheaf of dimension $\leq d-1$), this implies $T_{d-1}(\mathscr{F}) = 0$, i.e. it is pure.*

- *If $\mathscr{F}$ is pure of dimension d, then we also can use saturated subsheaves, proper quotient sheaves with $\alpha_d > 0$ and even proper purely d-dimensional quotient sheaves to define the H-(semi)stable!*

  The proof is trivial by using the trivial exact sequence. See Proposition 1.2.6 in [8] for the proof.

Remark 4.3.
- *Easy to see that when it is pure, then*

$$\mu_H\text{-stable} \Rightarrow H\text{-stable} \Rightarrow H\text{-ss} \Rightarrow \mu_H\text{-ss};$$

- *if $\dim X = 1$, then μ_H-(semi)stable iff H-(semi)stable.*

Lemma 4.4. *Let $\mathscr{F}, \mathscr{G}$ are H-ss of dimension d. Then*

(i) if $p(\mathscr{F}) > p(\mathscr{G})$, then $\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = 0$;

(ii) let $p(\mathscr{F}) = p(\mathscr{G})$. If $\mathscr{F}$ is moreover H-stable, then any $\phi : \mathscr{F} \to \mathscr{G}$ either zero or injection. Similarly if $\mathscr{G}$ is moreover H-stable, then any $\phi : \mathscr{F} \to \mathscr{G}$ either zero or surjection.

(iii) If $p(\mathscr{F}) = p(\mathscr{G})$ and $\alpha_d(\mathscr{F}) = \alpha_d(\mathscr{G})$, then any non-trivial homomorphism $f : \mathscr{F} \to \mathscr{G}$ is an isomorphism provided $\mathscr{F}$ or $\mathscr{G}$ is H-stable.

Proof. For (i), let nontrivial f with image $\mathscr{E}$, then $p(\mathscr{F}) \leq p(\mathscr{E}) \leq p(\mathscr{G})$ which is impossible. Hence $\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = 0$.

For (ii), this is the similar reason in the proof of (i).

For (iii), this is the similar reason in the proof of (i). □

Corollary 4.5. *If $\mathscr{E}$ is a H-stable sheaf, then $\operatorname{End}(\mathscr{E})$ is a finite dimensional division algebra over k. In particular, if k is algebraically closed, then $k \cong \operatorname{End}(\mathscr{E})$, i.e. $\mathscr{E}$ is a simple sheaf.*

Example 4.1. *(i) Any line bundles over smooth projective curves are H-stable. See Example 1.2.10 in [8].*

(ii) For an algebraically closed field k of zero characteristic, the bundle $\Omega_{\mathbb{P}^n}$ is H-stable. See Section 1.4 in [8].

4.2 The Harder-Narasimhan Filtration

We consider a classical result due to Grothendieck as a motivation of the Harder-Narasimhan filtration.

Theorem 4.6 (Grothendieck)**.** *Let $\mathscr{E}$ be a vector bundle of rank r on $\mathbb{P}^1$, then there is a uniquely determined decreasing sequence of integers $a_1 \geq \cdots \geq a_r$ such that $E \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$.*

Proof. For $r = 1$ this is trivial. Let the theorem holds for all vector bundles of rank $< r$ and that $\mathscr{E}$ is a vector bundle of rank r.

Take any saturation of any rank 1 subsheaf of $\mathscr{E}$. As $\mathbb{P}^1$ is a smooth curve, then it is a line bundle of form $\mathscr{O}(a)$. Let a_1 be the maximal number with this property. Hence $\mathscr{E}/\mathscr{O}(a_1) \cong \bigoplus_{i=2}^r \mathscr{O}(a_i)$ with $a_2 \geq \cdots \geq a_r$. We claim that $a_1 \geq a_2$. Indeed, consider

$$0 \to \mathscr{O}(-1) \to \mathscr{E}(-1-a_1) \to \bigoplus_{i=2}^{r} \mathscr{O}(a_i - a_1 - 1) \to 0.$$

Since $\Gamma(\mathscr{E}(-1-a_1)) = \mathrm{Hom}(\mathscr{O}(1+a_1), \mathscr{E})$ and a_1 be the maximal number with non-trivial $\mathrm{Hom}(\mathscr{O}(a), \mathscr{E})$, then $\Gamma(\mathscr{E}(-1-a_1)) = 0$. By the long exact sequence we get $H^0(\mathscr{O}(a_i - 1 - a_1)) = 0$ for all i. Hence $a_i < a_1 + 1$. Hence we get the claim.

Next we claim the sequence $0 \to \mathscr{O}(a_1) \to \mathscr{E} \to \bigoplus_{i=2}^r \mathscr{O}(a_i) \to 0$ split. This follows from the Serre duality

$$\mathrm{Ext}^1\left(\bigoplus_{i=2}^{r} \mathscr{O}(a_i), \mathscr{O}(a_1)\right)^\vee \cong \bigoplus_{i=2}^{r} \mathrm{Hom}(\mathscr{O}(a_1), \mathscr{O}(a_i - 2)) = 0.$$

Finally, the uniqueness is not hard to prove. We omit it. □

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 4.7. *Fix $\mathscr{E} \in \mathrm{Coh}(X)$ is pure of dimension d. A* Harder-Narasimhan filtration *(or HN-filtration) of $\mathscr{E}$ is*

$$0 = \mathrm{HN}_0(\mathscr{E}) \subset \mathrm{HN}_1(\mathscr{E}) \subset \cdots \subset \mathrm{HN}_l(\mathscr{E}) = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E}) := \mathrm{HN}_i(\mathscr{E})/\mathrm{HN}_{i-1}(\mathscr{E})$ which are H-ss of dimension d and $p(\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})) > p(\mathrm{gr}_{i+1}^{\mathrm{HN}}(\mathscr{E}))$ for all i. We define $p_{\max}(\mathscr{E}) := p(\mathrm{gr}_1^{\mathrm{HN}}(\mathscr{E}))$ and $p_{\min}(\mathscr{E}) := p(\mathrm{gr}_l^{\mathrm{HN}}(\mathscr{E}))$;

Lemma 4.8. *If $\mathscr{F}, \mathscr{G}$ is pure of dimension d with $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$, then $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$.*

Proof. If $f : \mathscr{F} \to \mathscr{G}$ is non-trivial. Let $i > 0$ be the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \neq 0$ and $j > 0$ the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \subset \mathrm{HN}_j(\mathscr{G})$. Hence we get a non-trivial $\bar{f} : \mathrm{gr}_i^{\mathrm{HN}}(\mathscr{F}) \to \mathrm{gr}_j^{\mathrm{HN}}(\mathscr{G})$. But this is impossible by $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$ and Lemma 4.4(i). $\square$

Theorem 4.9. *Let $\mathscr{E}$ be a pure coherent sheaf of dimension d. Then there always exists a unique Harder-Narasimhan filtration.*

Proof. Here we will use a result (see Lemma 1.3.5 in [8]):

- Let $\mathscr{E}$ be a purely d-dimensional sheaf. Then there is a subsheaf $\mathscr{F} \subset \mathscr{E}$ such that for all subsheaves $\mathscr{G} \subset \mathscr{E}$ one has $p(\mathscr{F}) \geq p(\mathscr{G})$, and in case of equality $\mathscr{F} \supset \mathscr{G}$. Moreover, $\mathscr{F}$ is uniquely determined and semistable. It is called the maximal destabilizing subsheaf of $\mathscr{E}$.

Let $\mathscr{E}_1$ be its maximal destabilizing subsheaf. By induction we may assume $\mathscr{E}/\mathscr{E}_1$ has a Harder-Narasimhan filtration

$$0 \subset \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_{l-1} = \mathscr{E}/\mathscr{E}_1.$$

Let $\mathscr{E}_{i+1} \subset \mathscr{E}$ be the preimage of $\mathscr{G}_i$. Just need to show that $p(\mathscr{E}_1) > p(\mathscr{E}_2/\mathscr{E}_1)$. If this were false, we would have $p(\mathscr{E}_2) \geq p(\mathscr{E}_1)$ contradicting the maximality of $\mathscr{E}_1$.

For the uniqueness, consider two Harder-Narasimhan filtrations $\mathscr{E}_*, \mathscr{E}'_*$. Let $p(\mathscr{E}'_1) \geq p(\mathscr{E}_1)$. Let j be minimal with $\mathscr{E}'_1 \subset \mathscr{E}_j$. Then we have

$$p(\mathscr{E}_j/\mathscr{E}_{j-1}) \geq p(\mathscr{E}'_1) \geq p(\mathscr{E}_1) \geq p(\mathscr{E}_j/\mathscr{E}_{j-1}).$$

Hence $p(\mathscr{E}'_1) = p(\mathscr{E}_1)$ and $j = 1$ and $\mathscr{E}'_1 \subset \mathscr{E}_1$. Similarly we get $\mathscr{E}'_1 \supset \mathscr{E}_1$, hence $\mathscr{E}'_1 = \mathscr{E}_1$. Using induction again we get the result. $\square$

Proposition 4.10. *Let $\mathscr{E}$ be a pure sheaf of dimension d and let K/k be a field extension. Then*

$$\mathrm{HN}_*(E \otimes_k K) = \mathrm{HN}_*(E) \otimes_k K.$$

In particular, the H-ss sheaves stable under base field extension.

Proof. We do not care about this. We refer the proof of Theorem 1.3.7 in [8]. $\square$

4.3 The Jordan-Hölder Filtration

As we all know, the Harder-Narasimhan filtration shows that the H-ss sheaves form the building blocks for all the coherent sheaves. But the Jordan-Hölder filtration shows that the H-stable sheaves form the building blocks for all H-ss sheaves.

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 4.11. *Fix $\mathscr{E} \in \mathrm{Coh}(X)$. Let $\mathscr{E}$ is H-ss, a* Jordan-Hölder filtration *(or JH-filtration) of $\mathscr{E}$ is*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ are H-stable and $p(\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})) = p(\mathscr{E})$ for all i. We define $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) := \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$.

Remark 4.12. *Unlike the Harder-Narasimhan filtration, the Jordan-Hölder filtration is NOT unique. For example we let the direct sum of two line bundles of the same degree one.*

Theorem 4.13. *Jordan-Hölder filtrations always exist. Up to isomorphism, the sheaf $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. Any filtration of $\mathscr{E}$ by semistable sheaves with reduced Hilbert polynomial $p(\mathscr{E})$ has a maximal refinement, whose factors are necessarily stable. The uniqueness of $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$ is not hard to show. We refer 1.5.2 in [8]. □

Definition 4.14. *Two H-ss sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same reduced Hilbert polynomial are called* S-equivalent *if $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_1) \cong \mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_2)$.*

Definition 4.15. *If $\mathscr{E}$ is H-ss, we call $\mathscr{E}$ is H-*polystable *if it is the direct sum of stable sheaves. In this case $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \mathscr{E}$.*

Remark 4.16. *We will show that the good moduli space of moduli stack of H-ss sheaves actually parametrizes only* S*-equivalence classes of H-ss sheaves! As we saw above, every* S*-equivalence class of H-ss sheaves contains exactly one polystable sheaf up to isomorphism. Thus, the good moduli space of H-ss sheaves in fact parametrizes polystable sheaves.*

Actually the S *stands for* Seshadri *as* S*-completeness is a geometric property reminiscent of how the* S*-equivalence relation on sheaves implies separatedness of the moduli space.*

Remark 4.17. *(i) By the similar arguments of Jordan-Hölder filtrations, on can show that every semistable sheaf $\mathscr{E}$ contains a unique non-trivial maximal H-polystable subsheaf of the same reduced Hilbert polynomial. This sheaf is called the* socle *of $\mathscr{E}$.*

(ii) One can use some basic properties of socles to find that if $\mathscr{E}$ is a simple sheaf, then it is H-stable if and only if it is geometrically H-stable. Hence in particular if k is algebraically closed and $\mathscr{E}$ is a H-stable sheaf, then $\mathscr{E}$ is also gemetrically H-stable. See 1.5.10 and 1.5.11 in [8].

(iii) There also have the Harder-Narasimhan filtrations and Jordan-Hölder filtrations for μ_H-(semi)stable sheaves. For the general arguments we refer Section 1.6 in [8].

5 Moduli Stack of Semistable Sheaves

5.1 The Mumford-Castelnuovo Regularity and Boundedness

In this section we will give some useful criterion about boundedness of families of sheaves.

Let X be a projective scheme over k with very ample $H = \mathscr{O}_X(1)$.

Definition 5.1. *Let $\mathscr{E}$ be a coherent sheaf of dimension d, we define $\hat{\mu}(\mathscr{E}) = \frac{\alpha_{d-1}(\mathscr{E})}{\alpha_d(\mathscr{E})}$.*

Definition 5.2. *Let m be an integer. A coherent sheaf $\mathscr{F}$ is said to be m-regular , if for all $i > 0$ we have $H^i(X, \mathscr{F}(m-i)) = 0$.*

The Mumford-Castelnuovo regularity of a coherent sheaf $\mathscr{F}$ is the number

$$\mathrm{reg}(\mathscr{F}) = \inf\{m \in \mathbb{Z} : \mathscr{F} \textit{ is } m\textit{-regular}\}.$$

Lemma 5.3. *There are universal polynomials $P_i \in \mathbb{Q}[T_0, ..., T_i]$ such that the following holds: Let $\mathscr{F}$ be a coherent sheaf of dimension $\leq d$ and let $H_1, ..., H_d$ be an $\mathscr{F}$-regular sequence of hyperplane sections. If $\chi(\mathscr{F}|_{\bigcap_{j\leq i} H_j}) = a_i$ and $h^0(\mathscr{F}|_{\bigcap_{j\leq i} H_j}) \leq b_i$, then*

$$\mathrm{reg}(\mathscr{F}) \leq P_d(a_0 - b_0, ..., a_d - b_d).$$

Proof. See [9] for the original proof. □

Lemma 5.4. *The following properties of a flat family of sheaves $\mathscr{F}$ on $X \to S$ are equivalent:*

(i) The family is bounded.

(ii) There is a uniform bound $\mathrm{reg}(\mathscr{F}_s) \leq \rho$ for all $s \in S$.

Proof. See [6] for the original proof. □

Then we have two nice criterion about boundedness of sheaves.

Theorem 5.5 (Kleiman Criterion)**.** *Let flat family of sheaves $\mathscr{F}$ on $X \to S$ with the same Hilbert polynomial P. Then this family is bounded if and only if there are constants $C_i, i = 0, ..., d = \deg P$ such that for every $\mathscr{F}_s$ there exists an $\mathscr{F}_s$-regular sequence of hyperplane sections $H_1, ..., H_d$, such that*

$$h^0(\mathscr{F}_s|_{\bigcap_{j\leq i} H_j}) \leq C_i.$$

Proof. Follows from Lemma 5.3 and Lemma 5.4. □

Theorem 5.6 (Grothendieck)**.** *Let P be a polynomial and ρ an integer. Then there is a constant C depending only on P and ρ such that the following holds:*

- *If X be a projective scheme on k with very ample divisor H and if $\mathscr{E} \in \mathrm{Coh}(X)$ is a d-dimensional sheaf with Hilbert polynomial P and Mumford-Castelnuovo regularity $\mathrm{reg}(\mathscr{E}) \le \rho$ and if $\mathscr{F} \in \mathrm{Coh}(X)$ is a purely d-dimensional quotient sheaf of $\mathscr{E}$ then $\hat{\mu}(\mathscr{F}) \ge C$.*

Moreover, the family of purely d-dimensional quotients $\mathscr{F}$ with $\hat{\mu}(\mathscr{F})$ bounded from above is bounded.

Proof. After embedding them into the projective space $\mathbb{P}^d$, we may consider $X = \mathbb{P}^d$. Hence we have $\mathscr{G} := V \otimes \mathscr{O}(-\rho) \twoheadrightarrow \mathscr{E}$ where $\operatorname{rank} V = P(\rho)$, so we just need to consider $\mathscr{G}$. Pick a quotient $q : \mathscr{G} \to \mathscr{F}$ of rank s, then

$$\bigwedge^s q : \bigwedge^s V \otimes \mathscr{O}(-s\rho) \to \det \mathscr{F} = \mathscr{O}(\deg \mathscr{F})$$

gives $\deg \mathscr{F} \ge -s\rho$. Hence

$$\hat{\mu}(\mathscr{F}) = \frac{\deg \mathscr{F} + \operatorname{rank} \mathscr{F} \alpha_{d-1}(\mathscr{O}_X)}{\alpha_d(\mathscr{F})} \ge -\rho + \alpha_{d-1}(\mathscr{O}_X).$$

For the final part, we let $\hat{\mu} \le C'$. It is enough to show that the family of pure quotient sheaves $\mathscr{F}$ of rank $0 < s \le \operatorname{rank}(\mathscr{G}) = P(\rho)$ and with $l = \deg \mathscr{F} = s(C' - \alpha_{d-1}(\mathscr{O}_X))$ is bounded. Consider $\psi : \mathscr{G} \otimes \bigwedge^{s-1} \mathscr{G} \xrightarrow{\wedge} \bigwedge^s \mathscr{G} \xrightarrow{\det q} \mathscr{O}(l)$ and $\psi^\vee : G \to \mathscr{O}(l) \otimes \bigwedge^{s-1} \mathscr{G}^\vee$. Let U denote the dense open subscheme where $\mathscr{F}$ is locally free. Then $\ker(\psi^\vee)|_U = \ker(q)|_U$. Since the quotients of $\mathscr{G}$ corresponding to these two subsheaves of $\mathscr{G}$ are torsion free and since they coincide on a dense open subscheme of $\mathbb{P}^d$, we must have $\ker(\psi^\vee) = \ker(q)$ everywhere, i.e. $\mathscr{F} \cong \mathrm{Im}\psi^\vee$. Now, the family of such image sheaves certainly is bounded. □

5.2 Basic Construction and Openness of Semistable Sheaves

Definition 5.7. *We define stack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ send a scheme T to a families of H-ss sheaves on $X \times T \to T$. Similarly we define $\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X)$ send a scheme T to a families of geometrically H-stable sheaves on $X \times T \to T$.*

Theorem 5.8. *The following properties of coherent sheaves are open in flat families: being simple, of pure dimension, H-ss, or geometrically H-stable.*

Proof. □

Corollary 5.9. *We have open substacks*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X) \subset \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$$

which parameterizing H-ss sheaves and geometrically H-stable sheaves, are all algebraic stacks locally of finite type.

Proof. Follows from the Theorem 5.8. □

5.3 Boundedness of Moduli Stack of Semistable Sheaves

5.3.1 The Grauert-Mülich Theorem

5.3.2 The Le Potier-Simpson Estimate

5.3.3 Boundedness of Semistable Sheaves

Theorem 5.10. *Let $f : X \to S$ be a projective morphism of schemes of finite type over k and let $\mathscr{O}_X(1)$ be an f-ample line bundle. Let P be a polynomial of degree d, and let μ_0 be a rational number. Then the family of purely d-dimensional sheaves on the fibres of f with Hilbert polynomial P and maximal slope $\hat{\mu}_{\max} \leq \mu_0$ is bounded. In particular, the family of H-ss sheaves with Hilbert polynomial P is bounded.*

Proof. □

Corollary 5.11. *The open moduli substack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$ of H-ss sheaves is an algebraic stack of finite type.*

5.4 Semistable reduction: Langton's Theorem

6 Good Moduli Space of Semistable Sheaves

6.1 Existence of Good Moduli Space of Semistable Sheaves

6.2 More Properties

6.3 Projectivity

6.4 Need to add

We know that the automorphism of stable sheaves is $\mathbb{C}^*$, hence the left morphism above is a $\mathbb{C}^*$-gerbe.

Definition 6.1. *We say $M_X^{H-st}(v)$*

(i) is a fine moduli space *if $\mathscr{M}_X^{H-st}(v)$ is a trivial $\mathbb{C}^*$-gerbe over $M_X^{H-st}(v)$;*

(ii) satisfies the ss = st condition if the equality $M_X^{H-ss}(v) = M_X^{H-st}(v)$ holds.

Remark 6.2. *(i) If $\gcd\{\chi(E \otimes F) : [E] \in M_X^{H-st}(v), F \in K(X)\} = 1$, then $M_X^{H-st}(v)$ s a fine moduli space by [8] Theorem 4.6.5;*

(ii) Actually $\mathscr{M}_X^{H-st}(v)$ is a trivial $\mathbb{C}^*$-gerbe over $M_X^{H-st}(v)$ is that there is a universal sheaf over $X \times M_X^{H-st}(v)$. Also, by the basic theory of algebraic gerbes, this is equivalent to $\mathscr{M}_X^{H-st}(v) \cong M_X^{H-st}(v) \times B\mathbb{C}^*$;

(iii) Let $v = (v_i) \in \Gamma$ with $v_0 = 1$, then any H-stable sheaf $\mathscr{E}$ of $\mathrm{ch}(\mathscr{E}) = v$ is of form $\mathscr{E} \cong \mathscr{L} \otimes \mathscr{I}_Z$ for a line bundle $\mathscr{L}$ on X and a closed subscheme Z with $\mathrm{codim}_X Z \geq 2$. Then

$$\mathrm{Pic}_X(v_1) \times \mathrm{Hilb}_X(e^{-v_1}v) \cong M_X^{H-st}(v)$$

and $M_X^{H-st}(v)$ is fine with ss = st condition. Here $\mathrm{Pic}_X(v_1)$ is the Picard scheme parameterizing line bundles with $c_1(\mathscr{L}) = v_1$ and $\mathrm{Hilb}_X(v)$ is the Hilbert scheme parameterizing closed subschemes $Z \subset X$ with $\mathrm{ch}(I\mathscr{I}_Z) = v$.

7 DT-invariants

7.1 Virtual Fundamental Classes

7.2 DT-invariants

8 Moduli Space of Complexes and Bridgeland Stability

9 Need to add

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