Kuznetsov components, Stability, and Moduli Spaces

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Preface

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Chapter 1

Derived Category and Kuznetsov Components

Here we follows some definitions and results in [7]. Note that when I working in the derived category, we will omit the **R** or **L** of the derived functors. Also, we only consider the schemes over $\mathbb{C}$.

1.1 Basic Definitions

Definition 1.1.1. *A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called **admissible** if the inclusion has a right adjoint $\pi : \mathscr{D} \to \mathscr{D}'$*

*The **orthogonal complement** of a(an admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory $\mathscr{D}'^{\perp}$ of all objects $C \in \mathscr{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{D}'$.*

Definition 1.1.2. *An object $E \in \mathscr{D}$ in a k-linear triangulated category $\mathscr{D}$ is called **exceptional** if*

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k, & \text{if } \ell = 0, \\ 0, & \text{if } \ell \neq 0. \end{cases}$$

*An **exceptional sequence** is a sequence $E_1, ..., E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.*

*An exceptional sequence is **full** if $\mathscr{D}$ is generated by $\{E_i\}$.*

*An exceptional collection $E_1, ..., E_n$ is **strong** if in addition $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \neq 0$.*

Definition 1.1.3. *A sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is **semi-orthogonal** if for all $i > j$ we have $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$. Such a sequence defines a **semi-orthogonal decomposition** (S.O.D.) of $\mathscr{D}$ if $\mathscr{D}$ is generated by the $\mathscr{D}_i$.*

Remark 1.1.4. *Some remarks:*

(a) *If $E \in \mathscr{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle \subset \mathscr{D}$.*

(b) *Let $E_1, ..., E_n$ be an exceptional sequence in $\mathscr{D}$. Then the admissible triangulated subcategories $\langle E_1 \rangle, ..., \langle E_n \rangle$ form a semi-orthogonal sequence. If this sequence is a full exceptional sequence, then this forms an S.O.D. of $\mathscr{D}$.*

(c) *Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ defines an S.O.D. of $\mathscr{D}$, if and only if any object $A \in \mathscr{D}$ with $A \in \mathscr{D}_i^{\perp}$ for all $i = 1, ..., n$ is trivial. See Lemma 1.61 in [7].*

(d) *If $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is an S.O.D., then $D_1 \subset \langle \mathscr{D}_2, ..., \mathscr{D}_n \rangle^{\perp}$ is an equivalence. See Exercise 1.62 in [7].*

Definition 1.1.5. *Fix an algebraic variety X and a line bundle $\mathscr{L}$ over it.*

(a) *A right Lefschetz decomposition of $\mathbf{D}^b(X)$ with respect to $\mathscr{L}$ is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_0, \mathscr{D}_1 \otimes \mathscr{L}, ..., \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(m-1)} \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

(b) *A left Lefschetz decomposition of $\mathbf{D}^b(X)$ with respect to $\mathscr{L}$ is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(1-m)}, ..., \mathscr{D}_1 \otimes \mathscr{L}^{\otimes(-)}, \mathscr{D}_0 \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

The subcategories $\mathscr{D}_i$ forming a Lefschetz decomposition will be called blocks, the largest will be called the first block. Usually we will consider right Lefschetz decompositions. So, we will call them simply Lefschetz decompositions. We call a Lefschetz decompositions is rectangular if $\mathscr{D}_{m-1} = \cdots = \mathscr{D}_1 = \mathscr{D}_0$.

If we need to consider the moduli space, we need to consider the family version of S.O.D:

Definition 1.1.6. *A triangulated category $\mathscr{T}$ is S-linear if it is equipped with a module structure over the tensor triangulated category $\mathbf{D}^b(S)$. In particular, if X is a scheme over S and $f : X \to S$ is the structure morphism then an S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle$$

is S-linear if each of the subcategories $\mathscr{A}_k$ satisfies that for $A \in \mathscr{A}_k$ and $F \in \mathbf{D}^b(S)$ one has $A \otimes f^*F \in \mathscr{A}_k$.

Theorem 1.1.7 (Kuznetsov)**.** *If X is an algebraic variety over S with an S-linear S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle,$$

then for a change of base morphism $T \to S$ there is, under a certain technical condition, a T-linear S.O.D

$$\mathbf{D}^b(X \times_S T) = \langle \mathscr{A}_{1T}, ..., \mathscr{A}_{mT} \rangle,$$

such that $\pi^* A \in \mathscr{A}_{iT}$ for any $A \in \mathscr{A}_i$ and $\pi_*(A') \in \mathscr{A}_i$ for any $A' \in \mathscr{A}_{iT}$ which has proper support over X.

Proof. See [9]. □

1.2 Example I – Projective Bundles

Proposition 1.2.1. *For a smooth projective variety Y we consider the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to Y$ of locally free sheaf $\mathscr{E}$ of rank r on Y, in the sense of Grothendieck. Then for any $a \in \mathbb{Z}$ we claim that $\pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(a),..., \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(a + r - 1)$ is an S.O.D. of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.*

Remark 1.2.2. *Hence this is a rectangular Lefschetz decomposition where all $\mathscr{D}_i = \pi^* \mathbf{D}^b(Y)$ and $\mathscr{L} = \mathscr{O}(1)$.*

This combined by the following two things:

Step 1. For any $E \in \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(m)$, $F \in \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(n)$, we have $\mathrm{Hom}(E, F) = 0$ for any $r - 1 \geq m - n > 0$.

Indeed, we can let $m = 0$ and hence $-r + 1 \leq n < 0$. Let $E = \pi^* E'$ and $F = \pi^* F' \otimes \mathscr{O}(n)$, hence

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(E', \pi_*(\pi^* F' \otimes \mathscr{O}(n))) = \mathrm{Hom}(E', F' \otimes \pi_* \mathscr{O}(n)).$$

It's well-known that $\mathbf{R}^i \pi_* \mathscr{O}(n) = \begin{cases} \mathrm{Sym}^n \mathscr{E}, \text{for } i = 0, \\ 0, \text{for } 0 < i < r - 1, \\ \mathrm{Sym}^{-n-r} \mathscr{E}^\vee, \text{for } i = r - 1. \end{cases}$ Well done.

Step 2. Categories $\pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(a),..., \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(a + r - 1)$ generates $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.

Here we generalize the proof for $\mathbb{P}^n$ in [7] Corollary 8.29. Consider

$$
\begin{array}{ccc}
& \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}) & \\
& {}^{p} \swarrow \quad {}^{\wedge} \quad \searrow {}^{q} & \\
\mathbb{P}(\mathscr{E}) & & \mathbb{P}(\mathscr{E}) \\
& {}_{\pi_1} \searrow \quad \swarrow {}_{\pi_2} & \\
& Y &
\end{array}
$$

then by the canonical identification

$$
\begin{aligned}
&H^0(\mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}), \mathscr{O}(1) \boxtimes \mathscr{Q}^\vee) \\
&= H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes p_* q^* \mathscr{Q}^\vee) \\
&= H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes \pi_1^* \pi_{2,*} \mathscr{Q}^\vee) \\
&= H^0(Y, \pi_{1,*} \mathscr{O}(1) \otimes \pi_{2,*} \mathscr{Q}^\vee) \\
&= H^0(Y, \mathscr{E} \otimes \mathscr{E}^\vee)
\end{aligned}
$$

where $0 \to \mathscr{Q} \to \pi^* \mathscr{E} \to \mathscr{O}(1) \to 0$ is the universal exact sequence. Let s correspond to the $\mathrm{id}_\mathscr{E}$, then $Z(s) = \Delta \subset \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E})$. By the Koszul resolution of $\mathscr{O}_\Delta$ respect to the s, we have an exact sequence:

$$
0 \to \overset{r-1}{\bigwedge}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \overset{r-2}{\bigwedge}(\mathscr{O}(-1) \boxtimes \mathscr{Q})
$$
$$
\to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.
$$

(you can also use the Euler exact sequence instead of the universal exact sequence, just as in [7] Corollary 8.29)

Now there is to way to slove this.

The First Way: for any coherent sheaf $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}(\mathscr{E}))$, tensoring $q^* \mathscr{F}$ we have

$$
0 \to \mathscr{O}(-r+1) \boxtimes \overset{r-1}{\bigwedge} \mathscr{Q} \otimes \mathscr{F} \to \mathscr{O}(-r+2) \boxtimes \overset{r-2}{\bigwedge} \mathscr{Q} \otimes \mathscr{F}
$$
$$
\to \cdots \to \mathscr{O}(-1) \boxtimes (\mathscr{Q} \otimes \mathscr{F}) \to \mathscr{O} \boxtimes \mathscr{F} \to q^* \mathscr{F}|_\Delta \to 0.
$$

Consider a spectral sequence

$$
\begin{aligned}
E_1^{ij} &= \mathbf{R}^i p_*(\mathscr{O}(j) \boxtimes \overset{-j}{\bigwedge} \mathscr{Q} \otimes \mathscr{F}) = \mathscr{O}(j) \otimes \mathbf{R}^i p_* q^* \overset{-j}{\bigwedge} \mathscr{Q} \otimes \mathscr{F} \\
&= \mathscr{O}(j) \otimes \pi_1^* \mathbf{R}^i \pi_{2,*} \overset{-j}{\bigwedge} \mathscr{Q} \otimes \mathscr{F} \Rightarrow \mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta.
\end{aligned}
$$

We know that $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = 0$ if $i+j \neq 0$ and $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = \mathscr{F}$ if $i+j = 0$. Since any E_1^{ij} contained in

$$
\left\langle \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(0) \right\rangle,
$$

so is $\mathscr{F}$. Hence well done (if you use the Euler exact sequence instead of the universal exact sequence, the similar spectral sequence called the generalized Beilinson spectral sequence as Proposition 8.28 in [7]).

The Second Way: Consider again the Koszul resolution

$$0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q})$$
$$\to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.$$

Split it into short exact sequences

$$0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-2} \to 0,$$
$$0 \to M_{r-2} \to \bigwedge^{r-3}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-3} \to 0,$$
$$\cdots ,$$
$$0 \to M_1 \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.$$

Tensor product with q^*F and direct image under the first projection p yields distinguished triangles of Fourier-Mukai transforms:

$$\Phi_{M_{i+1}}(\mathscr{F}) \to \Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \to \Phi_{M_i}(\mathscr{F}) \to \Phi_{M_{i+1}}(\mathscr{F})[1].$$

Easy to see that

$$\Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-i) \right\rangle.$$

By induction we get $F = \Phi_{\mathscr{O}_\Delta} F \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O} \right\rangle$. Well done.

Fully Exceptional Sequence. By the discussed above, we know that pick any fully exceptional sequence $E_1, ..., E_n$ of Y, the set

$$\{\pi^*E_1 \otimes \mathscr{O}(a), ..., \pi^*E_n \otimes \mathscr{O}(a), ..., \pi^*E_1 \otimes \mathscr{O}(a+r-1), ..., \pi^*E_n \otimes \mathscr{O}(a+r-1)\}$$

is a fully exceptional sequence of $\mathbb{P}(\mathscr{E})$ for any $a \in \mathbb{Z}$.

Example 1.2.1. *More general case, such as Grassmann-bundle and even the flag bundle has the similar things. We refer [11].*

We even have the similar about the general Brauer-Severi variety which need the twist derived category. See [1].

1.3   Example II − Blow-Ups

Here we follows section 11.1 in [7]. First we need some results about closed immersions.

Lemma 1.3.1. *Suppose $j : Y \hookrightarrow X$ of codimension C with normal bundle $\mathcal{N}$ is the zero locus of a regular section of a locally free sheaf $\mathcal{E}$ of rank c. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following canonical isomorphisms:*

$$(i) j^* j_* \mathcal{O}_Y \simeq \bigoplus \bigwedge^k \mathcal{N}^\vee[k],$$

$$(ii) j_* j^* j_* F \simeq j_* \mathcal{O}_Y \otimes j_* F \simeq j_* \left(\bigoplus \bigwedge^k \mathcal{N}^\vee[k] \otimes F \right),$$

$$(iii) \mathcal{H}om_X(j_* \mathcal{O}_Y, j_* F) \simeq j_* \left(\bigoplus \bigwedge^k \mathcal{N}[-k] \otimes F \right).$$

In particular, we have

$$\mathcal{H}^\ell(j^* j_* F) \simeq \bigoplus_{s-r=\ell} \bigwedge^r \mathcal{N}^\vee \otimes \mathcal{H}^s(F)$$

$$\mathcal{E}xt^\ell_X(j_* \mathcal{O}_Y, j_* F) \simeq j_* \left(\bigoplus_{r+s=\ell} \bigwedge^r \mathcal{N} \otimes \mathcal{H}^s(F) \right).$$

Proof. For (i), by Koszul resolution we get $j^* j_* \mathcal{O}_Y \simeq \bigwedge^* \mathcal{E}^\vee|_Y$. As the differentials in the Koszul complex $\bigwedge^* \mathcal{E}^\vee$ are given by contraction with the defining section, they become trivial on Y. Hence $j^* j_* \mathcal{O}_Y \simeq \bigoplus \bigwedge^k \mathcal{E}^\vee[k]|_Y$. As $\mathcal{E}|_Y \cong \mathcal{N}$, well done.

For (ii), we split the Koszul resolution into the following short exact sequences:

Again all these morphisms vanish on Y, we have

$$M_i \otimes j_* F \simeq \left(\bigwedge^i \mathcal{E}^\vee \otimes j_* F \right) \oplus (M_{i+1}[1] \otimes j_* F).$$

Putting these togetherand we get the result.

For (iii), as we have $\mathcal{H}om_X(j_* \mathcal{O}_Y, j_* F) \simeq \left(\bigwedge^i \mathcal{E}^\vee \right)^\vee \otimes j_* F$, then by the similar argument of (ii) we get the result.

The final part follows from (ii)(iii) and the fact that j_* is exact and tensor product with the locally free sheaf commutes with taking cohomology. $\qquad\square$

Corollary 1.3.2. *Let $j : Y \hookrightarrow X$ be a smooth hypersurface. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following distinguished triangle*

$$F \otimes \mathscr{O}_Y(-Y)[1] \to j^* j_* F \to F \to F \otimes \mathscr{O}_Y(-Y)[2].$$

Proof. We omit it and refer [7] Corollary 11.4.   □

Lemma 1.3.3. *Let $j : Y \hookrightarrow X$ be an arbitrary closed embedding of smooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^* j_* \mathscr{O}_Y) \simeq \overset{-i}{\bigwedge} \mathscr{N}^\vee_{Y/X}, \quad \mathscr{E}xt^i_X(j_* \mathscr{O}_Y, j_* \mathscr{O}_Y) \simeq \overset{i}{\bigwedge} \mathscr{N}_{Y/X}.$$

Proof. Here we just give an idea, the detail we refer Proposition 11.8 in [7]. Here we first pick a global resolution of locally free sheaves $\mathscr{G}^* \to \mathscr{O}_Y$ and get the free resolution $\mathscr{G}^*_y \to \mathscr{O}_{Y,y}$. Also we can let Y defined by a section of a vector bundle near y, hence we get a local Koszul resolution. Hence at the point y we can get the result from before. Easy to see that this is independent of any choice, we get the result.   □

Proposition 1.3.4. *Let $q : \widetilde{X} \to X$ be the blow-up along a smooth subvariety $Y \subset X$. Then for the structure sheaf $\mathscr{O}_Z$ of a subvariety $Z \subset Y$ considered as an object in $\mathbf{D}^b(X)$ one has*

$$\mathcal{H}^k(q^* \mathscr{O}_Z) \simeq (\Omega_\pi^{\otimes -k} \otimes \mathscr{O}_\pi(-k))|_{\pi^{-1}(Z)}$$

where $\pi : \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor.

Proof. We will only show the case that $Y \subset X$ is given as the zero set of a regular section $s \in H^0(X, \mathscr{E})$ of a locally free sheaf $\mathscr{E}$ of rank c. The general case follows from this and the similar argument of Lemma 1.3.3, we refer [7] Proposition 11.12 for details.

Consider $g : \mathbb{P}(\mathscr{E}) \to X$ and consider the Euler sequence

$$0 \to \mathscr{O}_g(-1) \to g^* \mathscr{E} \overset{\phi}{\to} \mathscr{T}_g \otimes \mathscr{O}_g(-1) \to 0.$$

Let $t := \phi(g^*(s)) \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{T}_g \otimes \mathscr{O}_g(-1))$ and consider the zero scheme $Z(t) \subset \mathbb{P}(\mathscr{E})$.

BLABLABLA

Hence g induced $Z(t) \to X$ can be identified with the blow-up $q : \widetilde{X} \to X$. Pick the Koszul resolution $\bigwedge^*(\mathscr{O}_g(1) \otimes \Omega_g) \to \mathscr{O}_{\widetilde{X}} \to 0$ of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules, hence

$$\iota_*(\mathcal{H}^k(q^* \mathscr{O}_Z)) \simeq \iota_*(\mathcal{H}^k(\iota^* g^* \mathscr{O}_Z)) \simeq \mathcal{H}^k(\iota_* \iota^* g^* \mathscr{O}_Z)$$

$$\simeq \mathcal{H}^k(g^* \mathscr{O}_Z \otimes \mathscr{O}_{\widetilde{X}}) \simeq \mathcal{H}^k(\overset{*}{\bigwedge}(\mathscr{O}_g(1) \otimes \Omega_g)|_{g^{-1}(Z)})$$

where $\iota : \widetilde{X} = Z(t) \hookrightarrow \mathbb{P}(\mathscr{E})$. If Z is contained in Y, the differentials, which are given by contraction with the section t, vanish and, therefore

$$\mathcal{H}^k(q^* \mathscr{O}_Z) \simeq (\Omega_g^{\otimes -k} \otimes \mathscr{O}_g(-k))|_{g^{-1}(Z)}.$$

Well done.   □

Lemma 1.3.5. *Suppose $f : S \to T$ is a projective morphism of smooth projective varieties such that $f_* : \mathbf{D}^b(S) \to \mathbf{D}^b(T)$ sends $\mathscr{O}_S$ to $\mathscr{O}_T$. Then $f^* : \mathbf{D}^b(T) \to \mathbf{D}^b(S)$ is fully faithful and thus describes an equivalence of $\mathbf{D}^b(T)$ with an admissible triangulated subcategory of $\mathbf{D}^b(S)$.*

Proof. Trivial by the projection formula and $f^* \dashv f_*$, which shows directly $\mathrm{id} \simeq f_* f^*$, hence fully faithful. $\qquad\square$

Lemma 1.3.6. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Then the functor*

$$\Phi_k = i_*(\mathscr{O}_E(kE) \otimes \pi^*(-)) : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X})$$

is fully faithful for any k. Moreover, Φ_k admits a right adjoint functor.

Proof. The functor Φ_k is a Fourier-Mukai transform with kernel $\mathscr{O}_E(kE)$ considered as on object in $\mathbf{D}^b(Y \times \widetilde{X})$. As such, Φ_k admits in particular right and left adjoint. Now we will use a result due to Bondal-Orlov (Proposition 7.1 in [7]):

- Consider the Fourier-Mukai transform $\Phi_{\mathscr{P}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ between the derived categories of two smooth projective varieties X and Y given by an object $\mathscr{P} \in \mathbf{D}^b(X \times Y)$. Then the functor $\Phi_{\mathscr{P}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\mathrm{Hom}(\Phi_{\mathscr{P}}(\kappa(x)), \Phi_{\mathscr{P}}(\kappa(y))[i]) = \begin{cases} k, & \text{if } x = y \text{ and } i = 0; \\ 0, & \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

For any j and $x \neq y$, this follows from the fact that the result objects have disjoint supports.

Now we let $x = y \in Y$. We need to show that $\mathrm{Ext}^i_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})$ is trivial for $i \notin [0, d = \dim Y]$ and of dimension one for $i = 0$. By Lemma 1.3.3 we get the spectral sequence

$$E_2^{p,q} = H^p(\widetilde{X}, \mathscr{E}xt^q_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})) = H^p\left(E_x, \bigwedge^q \mathscr{N}_{E_x/\widetilde{X}}\right)$$

$$\Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x}).$$

Hence we need to determine $\mathscr{N}_{E_x/\widetilde{X}}$. Consider the exact sequence

$$0 \to \mathscr{N}_{E_x/E} \to \mathscr{N}_{E_x/\widetilde{X}} \to \mathscr{N}_{E/\widetilde{X}}|_{E_x} \to 0,$$

as $\mathscr{N}_{E/\widetilde{X}} = \mathscr{O}_E(E)$ and $\mathscr{N}_{E_x/E} = \mathscr{O}_{E_x}^{\oplus d}$ and since $E_x \cong \mathbb{P}^{c-1}$ one get

$$\mathscr{N}_{E_x/\widetilde{X}} \cong \mathscr{O}_{E_x}(-1) \oplus \mathscr{O}_{E_x}^{\oplus d}$$

by computing the Ext^1. Hence we can directly get the result. $\qquad\square$

Proposition 1.3.7. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Define*

$$\mathscr{D}_k := \mathrm{Im}(\Phi_{-k} : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X}))$$

for $k = -c+1, ..., -1$ and $\mathscr{D}_0 := q^*\mathbf{D}^b(X)$.
   Then $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ forms an S.O.D of $\mathbf{D}^b(\widetilde{X})$.

Proof. We divided this into three parts:
Step 1. For $-c+1 \leq \ell < k < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_k^\perp$.
   For any $E, F \in \mathbf{D}^b(Y)$ we have

$$\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), i_*(\pi^*E \otimes \mathscr{O}_\pi(\ell))) = \mathrm{Hom}(i^*i_*\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

By Corollary 1.3.2, we get the distinguished triangle:

$$\pi^*F \otimes \mathscr{O}_\pi(1)[1] \to i^*i_*\pi^*F \to \pi_*F \to \pi^*F \otimes \mathscr{O}_\pi(1)[2].$$

Hence we just need to show that

$$\mathrm{Hom}(\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)) = 0 = \mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(1), \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

Both are easily deduced from adjunction $\pi^* \dashv \pi_*$, the projection formula, and $\pi_*\mathscr{O}_\pi(\ell - k) = 0$ for $-c+1 \leq \ell - k < 0$.
Step 2. For $-c+1 \leq \ell < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_0^\perp$.
   Again use $\pi_*\mathscr{O}_\pi(\ell) = 0$ for $-c+1 \leq \ell < 0$ to conclude this.
Step 3. We have $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ generates $\mathbf{D}^b(\widetilde{X})$.
   For this we let $E \in \mathscr{D}_k^\perp$ for all $-c+1 \leq k < 0$, then we claim that then exists an object $G \in \mathbf{D}^b(Y)$ with $i^*E \otimes \mathscr{O}_\pi(c-1) \simeq \pi^*G$.
   By assumption, for any $-c+1 \leq k < 0$ one has $\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), E) = 0$ for all $F \in \mathbf{D}^b(Y)$. By Grothendieck duality we get for any $-c+2 \leq k < 1$ one has $\mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(k), i^*E) = 0$. By Proposition 1.2.1 we have $i^*E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}_\pi(-c+1)$. Hence if we let $E' := E \otimes \mathscr{O}((-c+1)E)$, then $i^*E' \in \pi^*\mathbf{D}^b(Y)$. Pick such $G \in \mathbf{D}^b(Y)$ such that $i^*E' \simeq \pi^*G$.
   If $i^*E' \simeq 0$, then $\mathrm{supp}(E') \subset E$ and $E' \in \mathscr{D}_0$.

If not, consider the spectral sequence

$$E_2^{r,s} = \operatorname{Hom}(E', \mathcal{H}^s(q^*\kappa(x))[r]) \Rightarrow \operatorname{Hom}(E', q^*\kappa(x)[r+s]).$$

By Proposition 1.3.4 we have $\mathcal{H}^s(q^*\kappa(x)) \simeq \Omega_{E_x}^{\otimes -s}(-s)$. Hence

$$\begin{aligned}
E_2^{r,s} &\simeq \operatorname{Hom}(E', i_*\Omega_{E_x}^{\otimes -s}(-s)[r]) \\
&\simeq \operatorname{Hom}(\pi^*G, \Omega_{E_x}^{\otimes -s}(-s)[r]) \\
&\simeq \operatorname{Hom}(G, \pi_*\Omega_{E_x}^{\otimes -s}(-s)[r]) = 0
\end{aligned}$$

except for $s = 0$. Hence

$$E_2^{m,0} \simeq \operatorname{Hom}(G, \kappa(x)[m]) \simeq \operatorname{Hom}(q^*\kappa(x), E[\dim X - m])^\vee \neq 0$$

for some $m \in \mathbb{Z}$ and some $x \in Y$. Hence if $E \in \mathscr{D}_k^\perp$ for all $-c + 1 \leq k < 0$, we cannot have $E \in \mathscr{D}_0^\perp$. Hence well done. $\qquad\square$

1.4   Example III – Smooth Quadrics and Grassmannians

Here we follows the results in [8] and just give some results.

Proposition 1.4.1. *Let* $\mathsf{Gr}(k, V)$ *be the Grassmannian of k-dimensional subspaces in a vector space V of dimension n. Let $\mathscr{U}$ be thebtautological subbundle of rank k. If $\operatorname{char} k = 0$ then there is a strong S.O.D*

$$\mathbf{D}^b(\mathsf{Gr}(k, V)) = \left\langle \Sigma^\alpha \mathscr{U}^\vee \right\rangle$$

where α is a Young diagram in the $k \times (n - k)$ rectangle and Σ^α is the associated Schur functor.

Proof. We will not prove this. We refer the original proof in [8]. Note that as in the proof of the projective bundles, if we let $\mathscr{U}^\perp = ((V \otimes \mathcal{O}_{\mathsf{Gr}(k,V)})/\mathscr{U})^\vee$, then we can let a canonical section

$$s \in H^0(\mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V), \mathscr{U}^\vee \boxtimes (\mathscr{U}^\perp)^\vee) = V^\vee \otimes V = \operatorname{End}(V, V)$$

correspond to the id_V. Then s vanishes exactly along the diagonal $\Delta \subset \mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V)$ which induce the Koszul resolution

$$\cdots \to \bigwedge^2 (\mathscr{U} \boxtimes \mathscr{U}^\perp) \to \mathscr{U} \boxtimes \mathscr{U}^\perp \to \mathcal{O}_{\mathsf{Gr}(k,V) \times \mathsf{Gr}(k,V)} \to \mathcal{O}_\Delta \to 0$$

where the i-th term is just the sum $\bigoplus_\alpha \Sigma^\alpha \mathscr{U} \boxtimes \Sigma^{\alpha^*} \mathscr{U}^\perp$ where α runs through Young diagrams with i cells. Hence as before this deduce another generalised Beilinson spectral sequence

$$E_1^{p,q} = \bigoplus_{|\alpha|=-p} \mathbb{H}^q(F \otimes \Sigma^{\alpha^*} \mathscr{U}^\perp) \otimes \Sigma^\alpha \mathscr{U} \Rightarrow \mathcal{H}^{p+q}(F)$$

for any $F \in \mathbf{D}^b(\mathsf{Gr}(k, V))$. $\square$

Remark 1.4.2. *Note that we even have the Lefschetz decomposition on Grassmannians. We refer [5].*

Proposition 1.4.3. *Let $Q \subset \mathbb{P}_k^{n+1}$ be a smooth quadric hypersurface where* char$k \neq 2$, *then there is a full exceptional collection*

$$\mathbf{D}^b(Q) = \begin{cases} \langle S, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1) \rangle, n \ odd; \\ \langle S^-, S^+, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1) \rangle, n \ even; \end{cases}$$

where $S, S^\pm$ are the spinor bundles.

Remark 1.4.4. *This is also right for the family version, that is, consider a flat fibration in quadrics $f : X \to S$. In other words, assume that $X \subset \mathbb{P}_S(\mathscr{E})$ is a divisor of relative degree 2 where $\mathscr{E}$ is of rank $n + 2$ on a scheme S corresponding to a line subbundle $\mathscr{L} \subset \mathrm{Sym}^2 \mathscr{E}^\vee$. For each i there is a fully faithful functor $\Phi_i : \mathbf{D}^b(S) \to \mathbf{D}^b(X)$ given by $F \mapsto f^*F \otimes \mathscr{O}_{X/S}(i)$. Then we have a S.O.D*

$$\mathbf{D}^b(X) = \left\langle \mathbf{D}^b(S, \mathscr{C}\ell_0), \Phi_0(\mathbf{D}^b(S)), ..., \Phi_{n-1}(\mathbf{D}^b(S)) \right\rangle$$

where $\mathscr{C}\ell_0$ is the sheaf of even parts of Clifford algebras on S associated with the quadric fibration $X \to S$.

1.5   Example IV – Curves

Here we will follows [10]. Let C be a smooth projective curve over $\mathbb{C}$.

Proposition 1.5.1. *When $g(C) = 0$, then $C \cong \mathbb{P}^1$ and we have S.O.D*

$$\mathbf{D}^b(C) = \langle \mathscr{O}_C, \mathscr{O}_C(1) \rangle.$$

Proof. Special case of Proposition 1.2.1. $\square$

Now we consider $g(C) \geq 1$ and show a lemma.

Lemma 1.5.2. *Let $g(C) \geq 1$. Suppose $\mathscr{E} \in \mathrm{Coh}(C)$ is included in a triangle*

$$Y \to \mathscr{E} \to X \to Y[1]$$

with $\mathrm{Hom}^{\leq 0}(Y, X) = 0$, *then* $X, Y \in \mathrm{Coh}(C)$.

Proof. Almost the pure homological algebra, using the fact that $\deg K_C \geq 0$ here. See [6] Lemma 7.2. $\qquad\square$

Corollary 1.5.3. *Let $g(C) \geq 1$ and $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. Then for any $\mathscr{E} \in \mathrm{Coh}(C)$, there exist coherent sheaves $B \in \mathscr{B} \cap \mathrm{Coh}(C)$ and $A \in \mathscr{A} \cap \mathrm{Coh}(C)$, and an exact sequence of sheaves*

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Proposition 1.5.4. *When $g(C) \geq 1$, then $\mathbf{D}^b(C)$ admits no non-trivial S.O.Ds.*

Proof. Let $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. By Corollary 1.5.3, for any closed point $x \in C$ there exist $B \in \mathscr{B} \cap \mathrm{Coh}(C)$, $A \in \mathscr{A} \cap \mathrm{Coh}(C)$ such that both of them are sheaves and there exists an exact sequence

$$0 \to B \to \mathscr{O}_x \to A \to 0.$$

Hence $\mathscr{O}_x$ is contained in only one of $\mathscr{A}$ or $\mathscr{B}$. Hence $C(\mathrm{Spec}\,\mathbb{C}) = C_{\mathscr{A}} \sqcup C_{\mathscr{B}}$ by this fact.

By Proposition 3.17 in [7] we know that the set of closed points forms a spanning class, hence if $C_{\mathscr{B}} = \emptyset$ or $C_{\mathscr{A}} = \emptyset$, then $\mathscr{B}$ or $\mathscr{A}$ is trivial. Hence we may let both $C_{\mathscr{B}}$ and $C_{\mathscr{A}}$ are not empty.

We now claim that any coherent sheaf in $\mathscr{B}$ must be torsion. Indeed, otherwise the support of the sheaf coincides with the whole variety C, hence there exists a non-trivial morphism from the sheaf to a closed point which belongs to $\mathscr{A}$. This is a contradiction.

Next we claim that any torsion free sheaf belongs to $\mathscr{A}$. Indeed, let $\mathscr{E}$ be a torsion free sheaf. As before, we have an exact sequence

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Since $\mathscr{E}$ is torsion free, so is B. Combined with the first claim, we see B must be zero, hence $A = \mathscr{E}$.

By Corollary 3.19 in [7] we know that the set of torsion free sheaves forms a spanning class of $\mathbf{D}^b(C)$. Hence $\mathscr{B}$ must be trivial. Well done. $\qquad\square$

Remark 1.5.5. *Actually the only thing we use the $g(C) \geq 1$ is Corollary 1.5.3. So any smooth projective variety satisfies Corollary 1.5.3 admits no non-trivial S.O.Ds.*

1.6 Example V – Other Examples

Proposition 1.6.1. *Let X be a smooth projective variety with $\omega_X \cong \mathscr{O}_X$, then $\mathbf{D}^b(X)$ admits no non-trivial S.O.Ds.*

Proof. Let there exists an S.O.D $\mathbf{D}^b(X) = \langle \mathscr{A}, \mathscr{B} \rangle$. Hence for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and for any i we have $\mathrm{Hom}(B, A[i]) = 0$. Hence by Serre duality we have

$$\mathrm{Hom}(B, A[i]) = \mathrm{Hom}(A[i], B[n])^\vee = \mathrm{Hom}(A, b[n-i])^\vee = 0.$$

Hence $\mathbf{D}^b(X) = \langle \mathscr{B}, \mathscr{A} \rangle$ is also an S.O.D. Hence $\mathscr{A}, \mathscr{B}$ forms an orthogonal decomposition. Hence by Proposition 3.10 in [7] and the fact that X is connected, this S.O.D must be trivial. $\qquad\square$

Lemma 1.6.2. *Let X be a smooth projective variety and $F \in \mathbf{D}^b(X)$ is non-trivial, and $\mathscr{L}$ be a globally generated line bundle. Then*

$$\mathrm{Hom}_X(F, F \otimes \mathscr{L}) \neq 0.$$

Proof. Here we follows [10]. Let $m = \min\{i : \mathcal{H}^i(F) \neq 0\}$ and consider the following standard distinguished triangle

$$\tau_{\leq m}F \to F \to \tau_{\geq m+1}F \to \tau_{\leq m}F[1].$$

Since $\tau_{\leq m}F$ is isomorphic to a shift of a sheaf, we can find $s \in H^0(X, \mathscr{L})$ which induce a non-trivial $\tau_{\leq m}F \to \mathscr{L} \otimes \tau_{\leq m}F$. Consider

$$
\begin{array}{ccccccc}
\tau_{\geq m+1}F[-1] & \longrightarrow & \tau_{\leq m}F & \longrightarrow & F & \longrightarrow & \tau_{\geq m+1}F \\
\downarrow{\scriptstyle\sigma_{\geq m+1}[-1]} & & \downarrow{\scriptstyle\sigma_{\leq m}} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma_{\geq m+1}} \\
\tau_{\geq m+1}F \otimes \mathscr{L}[-1] & \longrightarrow & \tau_{\leq m}F \otimes \mathscr{L} & \longrightarrow & F \otimes \mathscr{L} & \longrightarrow & \tau_{\geq m+1}F \otimes \mathscr{L}
\end{array}
$$

where these four vertical arrows are defined by taking tensor products with the section s. Hence here $\sigma_{\leq m} \neq 0$. Suppose that $\sigma = 0$. Then $\sigma_{\leq m} \neq 0$ factors through a morphism from to $\tau_{\geq m+1}F \otimes \mathscr{L}[-1]$, which is zero since $\tau_{\geq m+1}F \otimes \mathscr{L}[-1]$ has trivial cohomologies up to degree $m+1$. Thus we obtain a contradiction, well done. $\qquad\square$

Proposition 1.6.3. *Let X be a smooth projective variety whose canonical line bundle is globally generated. Then $\mathbf{D}^b(X)$ has no exceptional objects.*

Proof. This follows from Lemma 1.6.2 and the duality

$$\mathrm{Hom}(F, F[\dim X]) = \mathrm{Hom}(F, F \otimes \omega_X)^\vee \neq 0.$$

Well done. $\qquad\square$

Chapter 2

Examples of Fano Manifolds of Calabi-Yau Type

We just consider the schemes and vector spaces over $\operatorname{Spec} \mathbb{C}$.

2.1 Cubics

2.2 Gushel-Mukai Varieties

2.2.1 Basic Definitions and Properties

Let V_5 be a vector space of dimension 5 and consider the Plücker embedding $\mathsf{Gr}(2, V_5) \hookrightarrow \mathbb{P}\left(\bigwedge^2 V_5\right)$. For any vector space K, consider the cone $\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \subset \mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$ of vertex $\mathbb{P}(K)$. Choose a vector subspace $W \subset \bigwedge^2 V_5 \oplus K$ and a subscheme $Q \subset \mathbb{P}(W)$ defined by defined by one quadratic equation (possibly zero).

Definition 2.2.1. *The scheme*

$$X = \mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W) \cap Q$$

*is called a **Gushel-Mukai intersection** (GM intersection). A GM intersection X is called a **Gushel-Mukai variety** (GM variety) if X is a smooth variety of dimension* $\dim W - 5 \geq 1$.

Remark 2.2.2. *Some remarks:*

(a) *In the original paper [4] they defined without the smoothness (but always Gorenstein).*

(b) *Note that all Q and $\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W)$ are Gorenstein, hence all Cohen-Macaulay. So the dimension condition means they are dimensionally transverse, that is, $\operatorname{Tor}_{>0}(\mathscr{O}_Q, \mathscr{O}_{\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W)}) = 0$.*

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(c) *A GM variety X has a canonical polarization, the restriction H of the hyperplane class on $\mathbb{P}(W)$; we will call (X, H) a **polarized GM variety.***

The definition of a GM variety is not intrinsic. We actually have an intrinsic characterization. But before giving these, we will introduce a new definition:

Definition 2.2.3. *Let W be a vector space and let $Y \subset \mathbb{P}(W)$ be a closed subscheme which is an intersection of quadrics, i.e., the twisted ideal sheaf $\mathscr{I}_X(2)$ on $\mathbb{P}(W)$ is globally generated.*

Define $V_X := H^0(\mathbb{P}(W), \mathscr{I}_X(2))$, this yields a surjection $V_X \otimes \mathscr{O}_{\mathbb{P}(W)}(-2) \twoheadrightarrow \mathscr{I}_X$ which induce

$$V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{I}_X / \mathscr{I}_X^2 = \mathscr{N}_{X/\mathbb{P}(W)}^\vee.$$

*We define the **excess conormal sheaf** $\mathscr{E}\mathscr{N}_{X/\mathbb{P}(W)}^\vee$ to be the kernel of this map.*

Theorem 2.2.4. *A smooth polarized projective variety (X, H) of dimension $n \geq 1$ is a polarized GM variety if and only if all the following conditions hold:*

(a) *$H^n = 10$ and $K_X = -(n-2)H$.*

(b) *H is very ample and the vector space $W := H^0(X, \mathscr{O}_X(H))^\vee$ has dimension $n+5$.*

(c) *X is an intersection of quadrics in $\mathbb{P}(W)$ and the vector space*

$$V_6 := H^0(\mathbb{P}(W), \mathscr{I}_X(2)) \subset \mathrm{Sym}^2 W^\vee$$

   of quadrics through X has dimension 6.

(d) *The twisted excess conormal sheaf $\mathscr{U}_X := \mathscr{E}\mathscr{N}_{X/\mathbb{P}(W)}^\vee(2H)$ of X in $\mathbb{P}(W)$ is simple.*

Proof. We first need to show a smooth polarized GM variety (X, H) satisfies (a)-(d).

For (a), as $\deg(\mathsf{C}_K(\mathsf{Gr}(2, V_5))) = 5$ and they are dimensionally transverse, then $\deg(X) = 10$. Let $\dim K = k$ and hence $K_{\mathsf{C}_K(\mathsf{Gr}(2, V_5))} = -(5+k)H$ by Lemma 2.2.7. Finally we have

$$K_X = (-(5+k) + (10+k) - (n+5) + 2)H = -(n-2)H.$$

For (b), we just need to show $W = H^0(X, \mathscr{O}_X(H))^\vee$. Consider the resolution

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{C}_K \mathsf{Gr}(2, V_5)} \to 0.$$

Restrict it into $\mathbb{P}(W)$ and tensor the resolution of Q as $0 \to \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_Q$, then tensor $\mathscr{O}(1)$ again we get the resolution

$$0 \to \mathscr{O}(-6) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-4) \to (V_5 \otimes \mathscr{O}(-3)) \oplus (V_5^\vee \otimes \mathscr{O}(-2))$$
$$\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-1) \to \mathscr{O}(1) \to \mathscr{O}_X(H) \to 0$$

on $\mathbb{P}(W)$. Hence $H^0(X, \mathscr{O}_X(H)) = H^0(\mathbb{P}(W), \mathscr{O}_{\mathbb{P}(W)}(1)) = W^\vee$.

For (c), consider the resolution again:

$$0 \to \mathscr{O}(-5) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-3) \to (V_5 \otimes \mathscr{O}(-2)) \oplus (V_5^\vee \otimes \mathscr{O}(-1))$$
$$\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O} \to \mathscr{O}(2) \to \mathscr{O}_X(2H) \to 0$$

Hence one can show that $H^0(\mathbb{P}(W), \mathscr{I}_X(2)) = V_5 \oplus \mathbb{C}$, hence well done.

For (d), we will use the induction of the dimension. For $n = 1$, this follows from some basic fact of excess normal sheaf and the Mukai's construction about a stable vector bundle of rank 2 on X to show that $\mathscr{U}_X$ is stable, and hence simple. For the detail we refer [4] Theorem 2.3. Hence we now assume $n \geq 2$. Pick a smooth hyperplane section $X' \subset X$ which is also irreducible since $n \geq 2$ by Bertini's theorem. Hence X' is also a GM variety. One can easy to show that in this case $\mathscr{U}_X|_{X'} = \mathscr{U}_{X'}$ (see Lemma A.5 in [4]). Hence we have $0 \to \mathscr{U}_X(-H) \to \mathscr{U}_X \to \mathscr{U}_{X'} \to 0$. Hence

$$0 \to \text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \text{Hom}(\mathscr{U}_X, \mathscr{U}_X) \to \text{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}).$$

If $\dim(\text{Hom}(\mathscr{U}_X, \mathscr{U}_X)) > 1$, then $\dim(\text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H))) > 0$. By the similar argument we get

$$0 \to \text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \to \text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \text{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}(-H)) = 0.$$

Hence $\text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \neq 0$. By induction we get $\text{Hom}(\mathscr{U}_X, \mathscr{U}_X(-kH)) \neq 0$ for any $k > 0$. Hence for any $k > 0$ we have $\Gamma(X, \mathscr{U}_X^\vee \otimes \mathscr{U}_X(-kH)) \neq 0$. But these are vector bundles and X is integral of dimension ≥ 2, hence this is impossible.

Now we let a smooth polarized projective variety (X, H) of dimension $n \geq 1$ which satisfies (a)-(d). We need to show that (X, H) is a polarized GM variety.

We know that
$$\det \mathscr{U}_X^\vee = \det(\mathscr{N}_{X/\mathbb{P}(W)}^\vee(2H)) = \mathscr{O}_X(H)$$

and the embedding $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$. Taking wedge product, duality and global sections we get

$$\bigwedge^2 V_6^\vee \to H^0(X, \mathscr{O}_X(H)) = W^\vee.$$

Hence we get $W \to \bigwedge^2 V_6$ which can be factored through an injection $W \to \bigwedge^2 V_6 \oplus K$ for some vector space K. Hence we have

$$
\begin{array}{ccc}
& \mathbb{P}(W) \longhookrightarrow & \mathbb{P}\left(\bigwedge^2 V_6 \oplus K\right) \\
\nearrow & & \big\downarrow \\
X \longrightarrow \mathsf{Gr}(2, V_6) & \longhookrightarrow & \mathbb{P}\left(\bigwedge^2 V_6\right)
\end{array}
$$

where $X \to \mathsf{Gr}(2, V_6)$ induced by $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$ and is commutative since these are the same linear system. Hence we get $X \subset \mathsf{C}_K^\circ \, \mathsf{Gr}(2, V_6) = \mathsf{C}_K \, \mathsf{Gr}(2, V_6) \backslash \mathbb{P}(K)$.

Now by some facts of excess normal sheaves (see Proposition A.3 in [4]), then excess normal sequence induce a functorial diagram:

$$0 \to (V_6 \otimes \mathscr{U}_X)/\mathrm{Sym}^2 \mathscr{U}_X \longrightarrow \textstyle\bigwedge^2 V_6 \otimes \mathscr{O}_X \longrightarrow \det V_6 \otimes \mu^* \mathscr{N}^\vee_{\mathsf{Gr}(2,V_6)/\mathbb{P}(\bigwedge^2 V_6)}(2) \to 0$$

$$0 \longrightarrow \det V_6 \otimes \mathscr{U}_X \longrightarrow \det V_6 \otimes V_6 \otimes \mathscr{O}_X \longrightarrow \det V_6 \otimes \mathscr{N}_{X/\mathbb{P}(W)}(2) \longrightarrow 0$$

which follows from the expression of the excess normal sheaf of $\mathsf{Gr}(2, V_6) \subset \mathbb{P}(\bigwedge^2 V_6)$. The left vertical arrow induces a morphism $\lambda' : V_6 \otimes \mathscr{U}_X \to \det V_6 \otimes \mathscr{U}_X$. As $\mathscr{U}_X$ is simple by (d) we get $\lambda : V_6 \to \det V_6$. Since λ' vanishes on $\mathrm{Sym}^2 \mathscr{U}_X$, the image of $\mathscr{U}_X$ in $_6 \otimes \mathscr{O}_X$ is contained in $\ker \lambda \otimes \mathscr{O}_X$. Moreover, the middle vertical map in the diagram above is given by $v_1 \wedge v_2 \mapsto \lambda(v_1)v_2 - \lambda(v_2)v_1$.

We claim that $\lambda \neq 0$. If $\lambda = 0$, the middle vertical map in the diagram is zero, which means that all the quadrics cutting out $\mathsf{C}_K \, \mathsf{Gr}(2, V_6)$ contain $\mathbb{P}(W)$, i.e. $\mathbb{P}(W) \subset \mathsf{C}_K \, \mathsf{Gr}(2, V_6)$. In other words, $\mathbb{P}(W)$ is a cone over $\mathbb{P}(W') \subset \mathsf{Gr}(2, V_6)$ with vertex a subspace of K. Hence $X \to \mathsf{Gr}(2, V_6)$ factor through $\mathbb{P}(W')$. Hence the vector bundle $\mathscr{U}_X$ is a pullback from $\mathbb{P}(W')$ of the restriction of the tautological bundle of $\mathsf{Gr}(2, V_6)$ to $\mathbb{P}(W')$.

There are two types of linear spaces on $\mathsf{Gr}(2, V_6)$: the first type corresponds to 2-dimensional subspaces containing a given vector and the second type to those contained in a given 3-subspace $V_3 \subset V_6$. If W' is of the first type, the restriction of the tautological bundle to $\mathbb{P}(W')$ is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$, hence $\mathscr{U}_X \cong \mathscr{O} \oplus \mathscr{O}(-H)$ by Lemma 2.2.8. In particular, it is not simple, which is a contradiction. If W' is of the second type, the embedding $\mathscr{U}_X \to V_6 \otimes \mathscr{O}_X$ factors through a subbundle $V_3 \otimes \mathscr{O}_X \subset V_6 \otimes \mathscr{O}_X$. Recall that V_6 is the space of quadrics passing through X in $\mathbb{P}(W)$. Consider the scheme-theoretic intersection M of the quadrics corresponding to the vector subspace V_3. Since the embedding of the excess conormal sheaf factors through $V_3 \otimes \mathscr{O}_X$, the variety X is the complete intersection of M with the 3 quadrics corresponding to the quotient space V_6/V_3. But the degree of X is then divisible by 8, which contradicts the fact that it is 10 by (a). Hence we conclude that $\lambda \neq 0$.

Now let $V_5 := \ker(\lambda)$ is a hyperplane in v_6 which fits in the exact sequence $0 \to V_5 \to V_6 \xrightarrow{\lambda} \det V_6 \to 0$. The composition $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X \xrightarrow{\lambda} \det V_6 \otimes \mathscr{O}_X$ vanish, hence we get $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathscr{O}_X$.

We now replace V_6 with V_5 and repeat the above argument, then we get a linear map $W \to \bigwedge^2 W_5$ which factor through $\mu : W \hookrightarrow \bigwedge^2 W_5 \oplus K$ which induce again the embedding $X \subset \mathsf{C}_K^\circ \, \mathsf{Gr}(2, V_5) = \mathsf{C}_K \, \mathsf{Gr}(2, V_5) \backslash \mathbb{P}(K)$. By the functorial of the excess normal sequence (see Proposition A.3 in [4]) again we get that inside the space V_6 of

quadrics cutting out X in $\mathbb{P}(W)$, the hyperplane V_5 is the space of quadratic equations of $\mathsf{Gr}(2, V_5)$, i.e., of Plücker quadrics.

As the Plücker quadrics cut out the cone $\mathsf{C}_K \mathsf{Gr}(2, V_5)$ in $\mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$, they cut out $\mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W)$ in $\mathbb{P}(W)$. Since X is the intersection of 6 quadrics by condition (c), we finally obtain

$$X = \mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q$$

where Q is some non-Plücker quadric corresponding to a point in $V_6 \backslash V_5$, so X is a GM variety. $\qquad \square$

Remark 2.2.5. *This is right for all normal varieties with the similar proof.*

Remark 2.2.6. *The twisted excess conormal sheaf $\mathscr{U}_X$ that was crucial for the proof will be called its **Gushel sheaf**. As we showed in the proof, the projection of X from the vertex $\mathbb{P}(K)$ of the cone $\mathsf{C}_K \mathsf{Gr}(2, V_5)$ defines a rational map $X \dashrightarrow \mathsf{Gr}(2, V_5)$ and the Gushel sheaf $\mathscr{U}_X$ is isomorphic to the pullback under this map of the tautological vector bundle on $\mathsf{Gr}(2, V_5)$. The map $X \dashrightarrow \mathsf{Gr}(2, V_5)$ is thus determined by $\mathscr{U}_X$ and is canonically associated with X. We call this map the **Gushel map** of X.*

Lemma 2.2.7. *Let $X \subset \mathbb{P}^n$ be a subvariety such that $K_X = rH$. Let $\mathsf{C}(X) \subset \mathbb{P}^{n+1}$ be a cone over X, then $K_{\mathsf{C}(X)} = (r-1)H$.*

Proof. We know that the blow-up of of the vertex of $\mathsf{C}(X)$ is

$$X' = \mathbb{P}_X(\mathscr{O}_X \oplus \mathscr{O}_X(-H))$$

$$\mathsf{C}(X) \xleftarrow{\quad \pi \quad} X' \xrightarrow{\quad p \quad} X$$

Let H' be the relative hyperplane class of p. Then

$$K_{X'} = p^*(K_X + H) - 2H' = (r+1)p^*H - 2H'.$$

On the other hand, the morphism π contracts the exceptional section $E \subset X'$ and H' is the pullback of $H_{\mathsf{C}(X)}$. Finally $E \sim_{\text{lin}} H' - p^*H$, hence

$$K_{X'} = (r-1)H' - (r+1)E.$$

Hence $K_{\mathsf{C}(X)} = (r-1)H$. $\qquad \square$

Lemma 2.2.8. *Let $Z_p \subset \mathsf{Gr}(k, V)$ be the subscheme parameterizing all k-planes containing the vector p. Then $Z_p \cong \mathsf{Gr}(k-1, n-1)$ and the restriction of the tautological subbundle $\mathscr{S}_{\mathsf{Gr}(k,V)}$ to Z_p splits as the sum of $\mathscr{O}$ and the tautological subbundle $\mathscr{S}_{Z_p}$ of $Z_p \cong \mathsf{Gr}(k-1, n-1)$.*

Proof. This is almost trivial. Indeed, let $V_1 \subset V$ be the 1-dimensional subspace generated by the vector p. Let $V = V_1 \oplus V'$ be a direct sum decomposition. Then for each $k-1$-dimensional subspace $U' \subset V'$ the sum $V_1 \oplus U'$ is a k-dimensional subspace of V. Hence the corresponding subbundle

$$V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{\mathsf{Gr}(k-1,V')} \subset V_1 \otimes \mathscr{O} \oplus V' \otimes \mathscr{O} = V \otimes \mathscr{O}$$

induces a morphism $\mathsf{Gr}(k-1,V') \to \mathsf{Gr}(k,V)$ which is an isomorphism onto Z_p and such that the pullback of the tautological bundle is $V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{Z_p}$.   □

2.2.2   Some Classifications

Theorem 2.2.9. *Let X be a complex smooth projective variety of dimension $n \geq 1$, together with an ample Cartier divisor H such that $K_X \sim_{\mathrm{lin}} -(n-2)H$ and $H^n = 10$. If we assume that*

- *when $n \geq 3$, we have $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$;*

- *when $n = 2$, the surface X is a **Brill-Noether general K3 surface** (a K3 surface is called **Brill-Noether general** if $h^0(S,D)h^0(S,H-D) < h^0(S,H)$ for all divisors D on S not linearly equivalent to 0 or H. When $H^2 = 10$, this is equivalent to the fact that $|H|$ contains a Clifford general smooth curve);*

- *when $n = 1$, the genus-6 curve X is **Clifford general** (that is, it is neither hyperelliptic, nor trigonal, nor a plane quintic).*

then X is a GM variety.

Proof.   □

Some inverse results:

Proposition 2.2.10. *A smooth projective curve is a GM curve if and only if it is a Clifford general curve of genus 6.*

Proof.   □

Proposition 2.2.11. *A smooth projective surface X is a GM surface if and only if X is a Brill-Noether general polarized K3 surface of degree 10.*

Proof.   □

Proposition 2.2.12. *Let (X,H) be a polarized complex smooth GM variety of dimension $n \geq 3$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. In particular, the polarization H is the unique GM polarization on X.*

Proof.   □

2.3 Debarre-Voisin Varieties

2.4 Iliev-Manivel Varieties

Chapter 3

Kuznetsov Components

Chapter 4

Examples of Derived Equivalences of Kuznetsov Components with K3s

Chapter 5

Stability Conditions on K3 Categories

Chapter 6

Applications: Mukai's program

Chapter 7

Application to Cubic Fourfolds and Gushel-Mukai Manifolds

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Bibliography

[1] M. Bernardara. A semiorthogonal decomposition for brauer-severi schemes. *Math. Nachr.*, 282(10):1406–1413, 2009.

[2] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. Math.*, 166(2):317–345, 2007.

[3] Tom Bridgeland. Stability conditions on k3 surfaces. *Duke Math. J.*, 166(2):241–291, 2008.

[4] O. Debarre and A. Kuznetsov. Gushel-mukai varieties: classification and birationalities. *Algebr. Geom.*, 5:15–76, 2018.

[5] A. Fonarev. On minimal lefschetz decompositions for grassmannians. *Izv. RAN. Ser. Mat.*, 77:203–224, 2013.

[6] A.L. Gorodentsev, S.A. Kuleshov, and A.N. Rudakov. t-stabilities and tstructures on triangulated categories. *Translation in Izv. Math.*, 68, 2004.

[7] D. Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry.* Oxford University Press, 2006.

[8] M. Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. *Invent. Math.*, 92(3):479–508, 1988.

[9] A. Kuznetsov. Base change for semiorthogonal decompositions. *Compos. Math.*, 147:852–876, 2011.

[10] S. Okawa. Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math.*, 228:2869–2873, 2011.

[11] Dmitri Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izvestiya Mathematics*, 41(1):133–141, 1993.