Fano Varieties and The Geometry of the Kuznetsov components

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October 28, 2023

Contents

Preface

[4][5]. We only consider the schemes and categories over $\mathbb{C}$.

Chapter 1

Derived Category and Semi-Orthogonal Decomposition

Here we follows some definitions and results in [12] and [18]. Note that when I working in the derived category, we will omit the **R** or **L** of the derived functors.

1.1 Basic Definitions

Definition 1.1.1. *A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called* right (left) admissible *if the inclusion has a right (left) adjoint $\pi : \mathscr{D} \to \mathscr{D}'$. If it is both right and left admissible, we call it* admissible *.*

The orthogonal complement *of a(an admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory ${\mathscr{D}'}^{\perp}$ of all objects $C \in \mathscr{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{D}'$. (one can also assume ${}^{\perp}\mathscr{D}'$ similarly)*

Remark 1.1.2. *When we let the inclusion is $j : \mathscr{D}' \to \mathscr{D}$, then its right (left) adjoint functor will be denoted by $j^!$ (j^*). But we will not use them when it will be confused with the true maps of derived functors.*

Definition 1.1.3. *An object $E \in \mathscr{D}$ in a k-linear triangulated category $\mathscr{D}$ is called* exceptional *if*

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k, \text{if } \ell = 0, \\ 0, \text{if } \ell \neq 0. \end{cases}$$

An exceptional sequence *is a sequence $E_1, ..., E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.*

An exceptional sequence is full *if $\mathscr{D}$ is generated by $\{E_i\}$.*

An exceptional collection $E_1, ..., E_n$ is strong *if in addition $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \neq 0$.*

Definition 1.1.4. *A sequence of full triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ defines a semi-orthogonal decomposition if the following holds:*

(a) For all $i > j$ we have $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$.

(b) For any $F \in \mathscr{D}$ there is a sequence of distinguished triangles:

$$\begin{tikzcd} 0 = F_m \arrow[rr] & & F_{m-1} \arrow[dl] \to \cdots \arrow[rr] & & F_1 \arrow[rr] & & F_0 = F \arrow[dl] \\ A_m = \mathrm{cone}(F_m \to F_{m-1}) \arrow[u] & & & & A_1 = \mathrm{cone}(F_1 \to F_0) \arrow[u] & & \end{tikzcd}$$

where $A_i = \mathrm{cone}(F_i \to F_{i-1}) \in \mathscr{D}_i$ for any i.

In this case we denote it $\mathscr{D} = \langle \mathscr{D}_1, ..., \mathscr{D}_n \rangle$.

Remark 1.1.5. *Some remarks:*

(a) The condition (a) in the definition implies that the "filtration" in (b) and its "factors" are unique and functorial.

(b) When we consider a sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ such that $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$ for all $i > j$ and let them generates a subcategory $\mathscr{A}$, then this defines an S.O.D:

$$\mathscr{D} = \left\langle \mathscr{A}^{\perp}, \mathscr{D}_1, ..., \mathscr{D}_n \right\rangle.$$

Hence moreover if $\mathscr{D}_i$ generates $\mathscr{D}$, then these becomes an S.O.D. This is just the definition of S.O.D in [12].

(c) If X is a smooth projective variety and $\mathbf{D}^b(X) = \langle \mathscr{D}_1, ..., \mathscr{D}_n \rangle$ is an S.O.D, then each component $\mathscr{D}_i$ is admissible. See [3].

Remark 1.1.6. *Some other remarks:*

(a) If $E \in \mathscr{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle \subset \mathscr{D}$.

(b) Let $E_1, ..., E_n$ be an exceptional sequence in $\mathscr{D}$. Then the admissible triangulated subcategories $\langle E_1 \rangle, ..., \langle E_n \rangle$ form a semi-orthogonal sequence. In this case if E_i generates a subcategory $\mathscr{C}$, then one can easy to show that $\mathscr{C}$ is admissible (Proposition 2.6 in [20]). Hence we have S.O.Ds

$$\mathscr{D} = \left\langle \mathscr{C}^{\perp}, E_1, ..., E_n \right\rangle = \left\langle E_1, ..., E_n, {}^{\perp}\mathscr{C} \right\rangle.$$

If this sequence is a full exceptional sequence, then this forms an S.O.D. of $\mathscr{D}$ by trivial reason.

(c) *Any semi-orthogonal sequence of full admissible triangulated subcategories* $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ *defines an S.O.D. of* $\mathscr{D}$*, if and only if any object* $A \in \mathscr{D}$ *with* $A \in \mathscr{D}_i^{\perp}$ *for all* $i = 1, ..., n$ *is trivial. See Lemma 1.61 in [12].*

(d) *If* $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ *is an S.O.D., then* $\mathscr{D}_1 \subset \langle \mathscr{D}_2, ..., \mathscr{D}_n \rangle^{\perp}$ *is an equivalence. See Exercise 1.62 in [12].*

So the admissible triangulated subcategories will be useful. Here we give a nice property about the admissible triangulated subcategories of the derived category of smooth projective varieties. First we recall that a triangulated category $\mathscr{D}$ of finite type is called right (left) saturated if any contravariant (covariant) cohomological functor of finite type $\mathscr{D} \to \mathrm{Vect}$ is representable.

Theorem 1.1.7. *This separated as two important parts:*

(i) *Let the triangulated category* $\mathscr{A}$ *be right (left) saturated. Assume that* $\mathscr{A}$ *is embedded in a triangulated category* $\mathscr{D}$ *as a full triangulated subcategory. Then* $\mathscr{A}$ *is right (left) admissible.*

(ii) *Let* X *be a smooth projective variety. Then* $\mathbf{D}^b(X)$ *is right and left saturated.*

Proof. We refer [3] for the original proof. □

Definition 1.1.8. *Fix an algebraic variety* X *and a line bundle* $\mathscr{L}$ *over it.*

(a) *A right Lefschetz decomposition of* $\mathbf{D}^b(X)$ *with respect to* $\mathscr{L}$ *is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_0, \mathscr{D}_1 \otimes \mathscr{L}, ..., \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(m-1)} \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

(b) *A left Lefschetz decomposition of* $\mathbf{D}^b(X)$ *with respect to* $\mathscr{L}$ *is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(1-m)}, ..., \mathscr{D}_1 \otimes \mathscr{L}^{\otimes(-)}, \mathscr{D}_0 \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

The subcategories $\mathscr{D}_i$ *forming a Lefschetz decomposition will be called blocks, the largest will be called the first block. Usually we will consider right Lefschetz decompositions. So, we will call them simply Lefschetz decompositions. We call a Lefschetz decompositions is rectangular if* $\mathscr{D}_{m-1} = \cdots = \mathscr{D}_1 = \mathscr{D}_0$.

If we need to consider the moduli space, we need to consider the family version of S.O.D:

Definition 1.1.9. *A triangulated category $\mathscr{T}$ is S-linear if it is equipped with a module structure over the tensor triangulated category $\mathbf{D}^b(S)$. In particular, if X is a scheme over S and $f : X \to S$ is the structure morphism then an S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle$$

is S-linear if each of the subcategories $\mathscr{A}_k$ satisfies that for $A \in \mathscr{A}_k$ and $F \in \mathbf{D}^b(S)$ one has $A \otimes f^*F \in \mathscr{A}_k$.

Theorem 1.1.10 (Kuznetsov)**.** *If X is an algebraic variety over S with an S-linear S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle ,$$

then for a change of base morphism $T \to S$ there is, under a certain technical condition, a T-linear S.O.D

$$\mathbf{D}^b(X \times_S T) = \langle \mathscr{A}_{1T}, ..., \mathscr{A}_{mT} \rangle ,$$

such that $\pi^*A \in \mathscr{A}_{iT}$ for any $A \in \mathscr{A}_i$ and $\pi_*(A') \in \mathscr{A}_i$ for any $A' \in \mathscr{A}_{iT}$ which has proper support over X.

Proof. See [16]. □

1.2 Example I – Projective Bundles

Proposition 1.2.1. *For a smooth projective variety Y we consider the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to Y$ of locally free sheaf $\mathscr{E}$ of rank r on Y, in the sense of Grothendieck. Then for any $a \in \mathbb{Z}$ we claim that $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a + r - 1)$ is an S.O.D. of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.*

Remark 1.2.2. *Hence this is a rectangular Lefschetz decomposition where all $\mathscr{D}_i = \pi^*\mathbf{D}^b(Y)$ and $\mathscr{L} = \mathscr{O}(1)$.*

This combined by the following three things:

Step 1. The subcategories $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(i)$ are all admissible of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$. This follows from Theorem 1.1.7.

Step 2. For any $E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(m), F \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(n)$, we have $\mathrm{Hom}(E, F) = 0$ for any $r - 1 \geq m - n > 0$.

Indeed, we can let $m = 0$ and hence $-r + 1 \leq n < 0$. Let $E = \pi^*E'$ and $F = \pi^*F' \otimes \mathscr{O}(n)$, hence

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(E', \pi_*(\pi^*F' \otimes \mathscr{O}(n))) = \mathrm{Hom}(E', F' \otimes \pi_*\mathscr{O}(n)).$$

It's well-known that $\mathbf{R}^i\pi_*\mathscr{O}(n) = \begin{cases} \mathrm{Sym}^n\mathscr{E}, \text{for } i = 0, \\ 0, \text{for } 0 < i < r - 1, \\ \mathrm{Sym}^{-n-r}\mathscr{E}^\vee, \text{for } i = r - 1. \end{cases}$ Well done.

Step 3. Categories $\pi^*\mathbf{D}^b(Y)\otimes\mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y)\otimes\mathscr{O}(a+r-1)$ generates $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.

Here we generalize the proof for $\mathbb{P}^n$ in [12] Corollary 8.29. Consider

$$\begin{tikzcd}
& \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}) \arrow[dl, "p"] \arrow[dr, "q"'] \arrow[dd, phantom, "\wedge"] & \\
\mathbb{P}(\mathscr{E}) \arrow[dr, "\pi_1"'] & & \mathbb{P}(\mathscr{E}) \arrow[dl, "\pi_2"] \\
& Y &
\end{tikzcd}$$

then by the canonical identification

$$\begin{aligned}
&H^0(\mathbb{P}(\mathscr{E})\times_Y\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\boxtimes\mathscr{Q}^\vee)\\
&=H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\otimes p_*q^*\mathscr{Q}^\vee)\\
&=H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\otimes \pi_1^*\pi_{2,*}\mathscr{Q}^\vee)\\
&=H^0(Y,\pi_{1,*}\mathscr{O}(1)\otimes \pi_{2,*}\mathscr{Q}^\vee)\\
&=H^0(Y,\mathscr{E}\otimes\mathscr{E}^\vee)
\end{aligned}$$

where $0\to\mathscr{Q}\to\pi^*\mathscr{E}\to\mathscr{O}(1)\to 0$ is the universal exact sequence. Let s correspond to the $\mathrm{id}_{\mathscr{E}}$, then $Z(s)=\Delta\subset\mathbb{P}(\mathscr{E})\times_Y\mathbb{P}(\mathscr{E})$. By the Koszul resolution of $\mathscr{O}_\Delta$ respect to the s, we have an exact sequence:

$$\begin{aligned}
0\to\bigwedge^{r-1}(\mathscr{O}(-1)\boxtimes\mathscr{Q})\to\bigwedge^{r-2}(\mathscr{O}(-1)\boxtimes\mathscr{Q})\\
\to\cdots\to\mathscr{O}(-1)\boxtimes\mathscr{Q}\to\mathscr{O}\boxtimes\mathscr{O}\to\mathscr{O}_\Delta\to 0.
\end{aligned}$$

(you can also use the Euler exact sequence instead of the universal exact sequence, just as in [12] Corollary 8.29)

Now there is to way to slove this.

The First Way: for any coherent sheaf $\mathscr{F}\in\mathrm{Coh}(\mathbb{P}(\mathscr{E}))$, tensoring $q^*\mathscr{F}$ we have

$$\begin{aligned}
0\to\mathscr{O}(-r+1)\boxtimes\bigwedge^{r-1}\mathscr{Q}\otimes\mathscr{F}\to\mathscr{O}(-r+2)\boxtimes\bigwedge^{r-2}\mathscr{Q}\otimes\mathscr{F}\\
\to\cdots\to\mathscr{O}(-1)\boxtimes(\mathscr{Q}\otimes\mathscr{F})\to\mathscr{O}\boxtimes\mathscr{F}\to q^*\mathscr{F}|_\Delta\to 0.
\end{aligned}$$

Consider a spectral sequence

$$\begin{aligned}
E_1^{ij}=\mathbf{R}^ip_*(\mathscr{O}(j)\boxtimes\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F})=\mathscr{O}(j)\otimes\mathbf{R}^ip_*q^*\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F}\\
=\mathscr{O}(j)\otimes\pi_1^*\mathbf{R}^i\pi_{2,*}\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F}\Rightarrow\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta.
\end{aligned}$$

We know that $\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta = 0$ if $i+j \neq 0$ and $\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta = \mathscr{F}$ if $i+j=0$. Since any E_1^{ij} contained in

$$\left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(0) \right\rangle,$$

so is $\mathscr{F}$. Hence well done (if you use the Euler exact sequence instead of the universal exact sequence, the similar spectral sequence called the generalized Beilinson spectral sequence as Proposition 8.28 in [12]).

The Second Way: Consider again the Koszul resolution

$$\begin{aligned} 0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) &\to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \\ &\to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Split it into short exact sequences

$$\begin{aligned} &0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-2} \to 0, \\ &0 \to M_{r-2} \to \bigwedge^{r-3}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-3} \to 0, \\ &\cdots, \\ &0 \to M_1 \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Tensor product with q^*F and direct image under the first projection p yields distinguished triangles of Fourier-Mukai transforms:

$$\Phi_{M_{i+1}}(\mathscr{F}) \to \Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \to \Phi_{M_i}(\mathscr{F}) \to \Phi_{M_{i+1}}(\mathscr{F})[1].$$

Easy to see that

$$\Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-i) \right\rangle.$$

By induction we get $F = \Phi_{\mathscr{O}_\Delta}F \in \langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O} \rangle$. Well done.

Fully Exceptional Sequence. By the discussed above, we know that pick any fully exceptional sequence $E_1, ..., E_n$ of Y, the set

$$\{\pi^*E_1 \otimes \mathscr{O}(a), ..., \pi^*E_n \otimes \mathscr{O}(a), ..., \pi^*E_1 \otimes \mathscr{O}(a+r-1), ..., \pi^*E_n \otimes \mathscr{O}(a+r-1)\}$$

is a fully exceptional sequence of $\mathbb{P}(\mathscr{E})$ for any $a \in \mathbb{Z}$.

Example 1.2.1. *More general case, such as Grassmann-bundle and even the flag bundle has the similar things. We refer [27].*

We even have the similar about the general Brauer-Severi variety which need the twist derived category. See [2].

1.3 Example II – Blow-Ups

Here we follows section 11.1 in [12]. First we need some results about closed immersions.

Lemma 1.3.1. *Suppose $j : Y \hookrightarrow X$ of codimension C with normal bundle $\mathscr{N}$ is the zero locus of a regular section of a locally free sheaf $\mathscr{E}$ of rank c. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following canonical isomorphisms:*

$$
\begin{aligned}
&\text{(i)} j^*j_*\mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{N}^\vee[k],\\
&\text{(ii)} j_*j^*j_*F \simeq j_*\mathscr{O}_Y \otimes j_*F \simeq j_*\left(\bigoplus \bigwedge^k \mathscr{N}^\vee[k] \otimes F\right),\\
&\text{(iii)} \mathscr{H}om_X(j_*\mathscr{O}_Y, j_*F) \simeq j_*\left(\bigoplus \bigwedge^k \mathscr{N}[-k] \otimes F\right).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\mathcal{H}^\ell(j^*j_*F) &\simeq \bigoplus_{s-r=\ell} \bigwedge^r \mathscr{N}^\vee \otimes \mathcal{H}^s(F)\\
\mathscr{E}xt^\ell_X(j_*\mathscr{O}_Y, j_*F) &\simeq j_*\left(\bigoplus_{r+s=\ell} \bigwedge^r \mathscr{N} \otimes \mathcal{H}^s(F)\right).
\end{aligned}
$$

Proof. For (i), by Koszul resolution we get $j^*j_*\mathscr{O}_Y \simeq \bigwedge^* \mathscr{E}^\vee|_Y$. As the differentials in the Koszul complex $\bigwedge^* \mathscr{E}^\vee$ are given by contraction with the defining section, they become trivial on Y. Hence $j^*j_*\mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{E}^\vee[k]|_Y$. As $\mathscr{E}|_Y \cong \mathscr{N}$, well done.

For (ii), we split the Koszul resolution into the following short exact sequences:

$$
\begin{tikzcd}
 & & & M_i \arrow[dr] & & \\
\cdots \arrow[r] & \bigwedge^{i+1} \mathscr{E}^\vee \arrow[rr] \arrow[dr] & & \bigwedge^i \mathscr{E}^\vee \arrow[rr] \arrow[u] & & \bigwedge^{i-1} \mathscr{E}^\vee \arrow[r] & \cdots \\
 & & M_{i+1} \arrow[ur] & & &
\end{tikzcd}
$$

Again all these morphisms vanish on Y, we have

$$M_i \otimes j_*F \simeq \left(\bigwedge^i \mathscr{E}^\vee \otimes j_*F\right) \oplus (M_{i+1}[1] \otimes j_*F).$$

Putting these togetherand we get the result.

For (iii), as we have $\mathscr{H}om_X(j_*\mathscr{O}_Y, j_*F) \simeq \left(\bigwedge^i \mathscr{E}^\vee\right)^\vee \otimes j_*F$, then by the similar argument of (ii) we get the result.

The final part follows from (ii)(iii) and the fact that j_* is exact and tensor product with the locally free sheaf commutes with taking cohomology. □

Corollary 1.3.2. *Let $j : Y \hookrightarrow X$ be a smooth hypersurface. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following distinguished triangle*

$$F \otimes \mathscr{O}_Y(-Y)[1] \to j^*j_*F \to F \to F \otimes \mathscr{O}_Y(-Y)[2].$$

Proof. We omit it and refer [12] Corollary 11.4. □

Lemma 1.3.3. *Let $j : Y \hookrightarrow X$ be an arbitrary closed embedding of smooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^*j_*\mathscr{O}_Y) \simeq \bigwedge^{-i} \mathscr{N}_{Y/X}^\vee, \quad \mathscr{E}xt^i_X(j_*\mathscr{O}_Y, j_*\mathscr{O}_Y) \simeq \bigwedge^{i} \mathscr{N}_{Y/X}.$$

Proof. Here we just give an idea, the detail we refer Proposition 11.8 in [12]. Here we first pick a global resolution of locally free sheaves $\mathscr{G}^* \to \mathscr{O}_Y$ and get the free resolution $\mathscr{G}^*_y \to \mathscr{O}_{Y,y}$. Also we can let Y defined by a section of a vector bundle near y, hence we get a local Koszul resolution. Hence at the point y we can get the result from before. Easy to see that this is independent of any choice, we get the result. □

Proposition 1.3.4. *Let $q : \widetilde{X} \to X$ be the blow-up along a smooth subvariety $Y \subset X$. Then for the structure sheaf $\mathscr{O}_Z$ of a subvariety $Z \subset Y$ considered as an object in $\mathbf{D}^b(X)$ one has*

$$\mathcal{H}^k(q^*\mathscr{O}_Z) \simeq (\Omega_\pi^{\otimes -k} \otimes \mathscr{O}_\pi(-k))|_{\pi^{-1}(Z)}$$

where $\pi : \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor.

Proof. We will only show the case that $Y \subset X$ is given as the zero set of a regular section $s \in H^0(X, \mathscr{E})$ of a locally free sheaf $\mathscr{E}$ of rank c. The general case follows from this and the similar argument of Lemma 1.3.3, we refer [12] Proposition 11.12 for details.

Consider $g : \mathbb{P}(\mathscr{E}) \to X$ and consider the Euler sequence

$$0 \to \mathscr{O}_g(-1) \to g^*\mathscr{E} \xrightarrow{\phi} \mathscr{T}_g \otimes \mathscr{O}_g(-1) \to 0.$$

Let $t := \phi(g^*(s)) \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{T}_g \otimes \mathscr{O}_g(-1))$ and consider the zero scheme $Z(t) \subset \mathbb{P}(\mathscr{E})$.

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Hence g induced $Z(t) \to X$ can be identified with the blow-up $q : \widetilde{X} \to X$. Pick the Koszul resolution $\bigwedge^*(\mathscr{O}_g(1) \otimes \Omega_g) \to \mathscr{O}_{\widetilde{X}} \to 0$ of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules, hence

$$\begin{aligned}\iota_*(\mathcal{H}^k(q^*\mathscr{O}_Z)) &\simeq \iota_*(\mathcal{H}^k(\iota^*g^*\mathscr{O}_Z)) \simeq \mathcal{H}^k(\iota_*\iota^*g^*\mathscr{O}_Z) \\ &\simeq \mathcal{H}^k(g^*\mathscr{O}_Z \otimes \mathscr{O}_{\widetilde{X}}) \simeq \mathcal{H}^k(\bigwedge^{*}(\mathscr{O}_g(1) \otimes \Omega_g)|_{g^{-1}(Z)})\end{aligned}$$

where $\iota : \widetilde{X} = Z(t) \hookrightarrow \mathbb{P}(\mathscr{E})$. If Z is contained in Y , the differentials, which are given by contraction with the section t, vanish and, therefore

$$\mathcal{H}^k(q^*\mathscr{O}_Z) \simeq (\Omega_g^{\otimes -k} \otimes \mathscr{O}_g(-k))|_{g^{-1}(Z)}.$$

Well done. □

Lemma 1.3.5. *Suppose $f : S \to T$ is a projective morphism of smooth projective varieties such that $f_* : \mathbf{D}^b(S) \to \mathbf{D}^b(T)$ sends $\mathscr{O}_S$ to $\mathscr{O}_T$. Then $f^* : \mathbf{D}^b(T) \to \mathbf{D}^b(S)$ is fully faithful and thus describes an equivalence of $\mathbf{D}^b(T)$ with an admissible triangulated subcategory of $\mathbf{D}^b(S)$.*

Proof. Trivial by the projection formula and $f^* \dashv f_*$, which shows directly $\mathrm{id} \simeq f_* f^*$, hence fully faithful. □

Lemma 1.3.6. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Then the functor*

$$\Phi_k = i_*(\mathscr{O}_E(kE) \otimes \pi^*(-)) : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X})$$

is fully faithful for any k. Moreover, Φ_k admits a right adjoint functor.

Proof. The functor Φ_k is a Fourier-Mukai transform with kernel $\mathscr{O}_E(kE)$ considered as on object in $\mathbf{D}^b(Y \times \widetilde{X})$. As such, Φ_k admits in particular right and left adjoint. Now we will use a result due to Bondal-Orlov (Proposition 7.1 in [12]):

- Consider the Fourier-Mukai transform $\Phi_{\mathscr{P}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ between the derived categories of two smooth projective varieties X and Y given by an object $\mathscr{P} \in \mathbf{D}^b(X \times Y)$. Then the functor $\Phi_{\mathscr{P}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\mathrm{Hom}(\Phi_{\mathscr{P}}(\kappa(x)), \Phi_{\mathscr{P}}(\kappa(y))[i]) = \begin{cases} k, \text{if } x = y \text{ and } i = 0; \\ 0, \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

For any j and $x \neq y$, this follows from the fact that the result objects have disjoint supports.

Now we let $x = y \in Y$. We need to show that $\mathrm{Ext}^i_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})$ is trivial for $i \notin [0, d = \dim Y]$ and of dimension one for $i = 0$. By Lemma 1.3.3 we get the spectral sequence

$$\begin{aligned} E_2^{p,q} = H^p(\widetilde{X}, \mathscr{E}xt^q_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})) = H^p\left(E_x, \bigwedge^q \mathscr{N}_{E_x/\widetilde{X}}\right) \\ \Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x}). \end{aligned}$$

Hence we need to determine $\mathscr{N}_{E_x/\widetilde{X}}$. Consider the exact sequence

$$0 \to \mathscr{N}_{E_x/E} \to \mathscr{N}_{E_x/\widetilde{X}} \to \mathscr{N}_{E/\widetilde{X}}|_{E_x} \to 0,$$

as $\mathscr{N}_{E/\widetilde{X}} = \mathscr{O}_E(E)$ and $\mathscr{N}_{E_x/E} = \mathscr{O}_{E_x}^{\oplus d}$ and since $E_x \cong \mathbb{P}^{c-1}$ one get

$$\mathscr{N}_{E_x/\widetilde{X}} \cong \mathscr{O}_{E_x}(-1) \oplus \mathscr{O}_{E_x}^{\oplus d}$$

by computing the Ext^1. Hence we can directly get the result. $\square$

Proposition 1.3.7. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Define*

$$\mathscr{D}_k := \mathrm{Im}(\Phi_{-k} : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X}))$$

for $k = -c+1, ..., -1$ and $\mathscr{D}_0 := q^*\mathbf{D}^b(X)$.

Then $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ forms an S.O.D of $\mathbf{D}^b(\widetilde{X})$.

Proof. We divided this into three parts:
Step 1. For $-c+1 \leq \ell < k < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_k^\perp$.

For any $E, F \in \mathbf{D}^b(Y)$ we have

$$\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), i_*(\pi^*E \otimes \mathscr{O}_\pi(\ell))) = \mathrm{Hom}(i^*i_*\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

By Corollary 1.3.2, we get the distinguished triangle:

$$\pi^*F \otimes \mathscr{O}_\pi(1)[1] \to i^*i_*\pi^*F \to \pi_*F \to \pi^*F \otimes \mathscr{O}_\pi(1)[2].$$

Hence we just need to show that

$$\mathrm{Hom}(\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)) = 0 = \mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(1), \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

Both are easily deduced from adjunction $\pi^* \dashv \pi_*$, the projection formula, and $\pi_*\mathscr{O}_\pi(\ell - k) = 0$ for $-c+1 \leq \ell - k < 0$.
Step 2. For $-c+1 \leq \ell < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_0^\perp$.

Again use $\pi_*\mathscr{O}_\pi(\ell) = 0$ for $-c+1 \leq \ell < 0$ to conclude this.
Step 3. We have $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ generates $\mathbf{D}^b(\widetilde{X})$.

For this we let $E \in \mathscr{D}_k^\perp$ for all $-c+1 \leq k < 0$, then we claim that then exists an object $G \in \mathbf{D}^b(Y)$ with $i^*E \otimes \mathscr{O}_\pi(c-1) \simeq \pi^*G$.

By assumption, for any $-c+1 \leq k < 0$ one has $\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), E) = 0$ for all $F \in \mathbf{D}^b(Y)$. By Grothendieck duality we get for any $-c+2 \leq k < 1$ one has $\mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(k), i^*E) = 0$. By Proposition 1.2.1 we have $i^*E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}_\pi(-c+1)$.

Hence if we let $E' := E \otimes \mathscr{O}((-c+1)E)$, then $i^*E' \in \pi^*\mathbf{D}^b(Y)$. Pick such $G \in \mathbf{D}^b(Y)$ such that $i^*E' \simeq \pi^*G$.

If $i^*E' \simeq 0$, then $\mathrm{supp}(E') \subset E$ and $E' \in \mathscr{D}_0$.

If not, consider the spectral sequence

$$E_2^{r,s} = \mathrm{Hom}(E', \mathcal{H}^s(q^*\kappa(x))[r]) \Rightarrow \mathrm{Hom}(E', q^*\kappa(x)[r+s]).$$

By Proposition 1.3.4 we have $\mathcal{H}^s(q^*\kappa(x)) \simeq \Omega_{E_x}^{\otimes -s}(-s)$. Hence

$$\begin{aligned} E_2^{r,s} &\simeq \mathrm{Hom}(E', i_*\Omega_{E_x}^{\otimes -s}(-s)[r]) \\ &\simeq \mathrm{Hom}(\pi^*G, \Omega_{E_x}^{\otimes -s}(-s)[r]) \\ &\simeq \mathrm{Hom}(G, \pi_*\Omega_{E_x}^{\otimes -s}(-s)[r]) = 0 \end{aligned}$$

except for $s = 0$. Hence

$$E_2^{m,0} \simeq \mathrm{Hom}(G, \kappa(x)[m]) \simeq \mathrm{Hom}(q^*\kappa(x), E[\dim X - m])^\vee \neq 0$$

for some $m \in \mathbb{Z}$ and some $x \in Y$. Hence if $E \in \mathscr{D}_k^\perp$ for all $-c+1 \leq k < 0$, we cannot have $E \in \mathscr{D}_0^\perp$. Hence well done. □

1.4 Example III – Smooth Quadrics and Grassmannians

Here we follows the results in [15] and just give some results.

Proposition 1.4.1. *Let* $\mathsf{Gr}(k, V)$ *be the Grassmannian of* k*-dimensional subspaces in a vector space* V *of dimension* n*. Let* $\mathscr{U}$ *be thebtautological subbundle of rank* k*. If* char$k = 0$ *then there is a strong S.O.D*

$$\mathbf{D}^b(\mathsf{Gr}(k, V)) = \left\langle \Sigma^\alpha \mathscr{U}^\vee \right\rangle$$

where α *is a Young diagram in the* $k \times (n-k)$ *rectangle and* Σ^α *is the associated Schur functor.*

Proof. We will not prove this. We refer the original proof in [15]. Note that as in the proof of the projective bundles, if we let $\mathscr{U}^\perp = ((V \otimes \mathscr{O}_{\mathsf{Gr}(k,V)})/\mathscr{U})^\vee$, then we can let a canonical section

$$s \in H^0(\mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V), \mathscr{U}^\vee \boxtimes (\mathscr{U}^\perp)^\vee) = V^\vee \otimes V = \mathrm{End}(V, V)$$

correspond to the id_V. Then s vanishes exactly along the diagonal $\Delta \subset \mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V)$ which induce the Koszul resolution

$$\cdots \to \bigwedge^2(\mathscr{U} \boxtimes \mathscr{U}^\perp) \to \mathscr{U} \boxtimes \mathscr{U}^\perp \to \mathscr{O}_{\mathsf{Gr}(k,V)\times\mathsf{Gr}(k,V)} \to \mathscr{O}_\Delta \to 0$$

where the i-th term is just the sum $\bigoplus_\alpha \Sigma^\alpha \mathscr{U} \boxtimes \Sigma^{\alpha^*} \mathscr{U}^\perp$ where α runs through Young diagrams with i cells. Hence as before this deduce another generalised Beilinson spectral sequence

$$E_1^{p,q} = \bigoplus_{|\alpha|=-p} \mathbb{H}^q(F \otimes \Sigma^{\alpha^*} \mathscr{U}^\perp) \otimes \Sigma^\alpha \mathscr{U} \Rightarrow \mathcal{H}^{p+q}(F)$$

for any $F \in \mathbf{D}^b(\mathsf{Gr}(k, V))$. □

Remark 1.4.2. *Note that we even have the Lefschetz decomposition on Grassmannians. We refer [10].*

Proposition 1.4.3. *Let $Q \subset \mathbb{P}_k^{n+1}$ be a smooth quadric hypersurface where* $\operatorname{char} k \neq 2$, *then there is a full exceptional collection*

$$\mathbf{D}^b(Q) = \begin{cases} \langle S, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1) \rangle, n \text{ odd}; \\ \left\langle S^-, S^+, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1) \right\rangle, n \text{ even}; \end{cases}$$

where $S, S^\pm$ are the spinor bundles.

Remark 1.4.4. *This is also right for the family version, that is, consider a flat fibration in quadrics $f : X \to S$. In other words, assume that $X \subset \mathbb{P}_S(\mathscr{E})$ is a divisor of relative degree 2 where $\mathscr{E}$ is of rank $n+2$ on a scheme S corresponding to a line subbundle $\mathscr{L} \subset \mathrm{Sym}^2 \mathscr{E}^\vee$. For each i there is a fully faithful functor $\Phi_i : \mathbf{D}^b(S) \to \mathbf{D}^b(X)$ given by $F \mapsto f^*F \otimes \mathscr{O}_{X/S}(i)$. Then we have a S.O.D*

$$\mathbf{D}^b(X) = \left\langle \mathbf{D}^b(S, \mathscr{C}\ell_0), \Phi_0(\mathbf{D}^b(S)), ..., \Phi_{n-1}(\mathbf{D}^b(S)) \right\rangle$$

where $\mathscr{C}\ell_0$ is the sheaf of even parts of Clifford algebras on S associated with the quadric fibration $X \to S$.

1.5 Example IV – Curves

Here we will follows [25]. Let C be a smooth projective curve over $\mathbb{C}$.

Proposition 1.5.1. *When $g(C) = 0$, then $C \cong \mathbb{P}^1$ and we have S.O.D*

$$\mathbf{D}^b(C) = \langle \mathscr{O}_C, \mathscr{O}_C(1) \rangle.$$

Proof. Special case of Proposition 1.2.1. □

Now we consider $g(C) \geq 1$ and show a lemma.

Lemma 1.5.2. *Let $g(C) \geq 1$. Suppose $\mathscr{E} \in \mathrm{Coh}(C)$ is included in a triangle*

$$Y \to \mathscr{E} \to X \to Y[1]$$

with $\mathrm{Hom}^{\leq 0}(Y, X) = 0$*, then* $X, Y \in \mathrm{Coh}(C)$.

Proof. Almost the pure homological algebra, using the fact that $\deg K_C \geq 0$ here. See [11] Lemma 7.2. □

Corollary 1.5.3. *Let $g(C) \geq 1$ and $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. Then for any $\mathscr{E} \in \mathrm{Coh}(C)$, there exist coherent sheaves $B \in \mathscr{B} \cap \mathrm{Coh}(C)$ and $A \in \mathscr{A} \cap \mathrm{Coh}(C)$, and an exact sequence of sheaves*

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Proposition 1.5.4. *When $g(C) \geq 1$, then $\mathbf{D}^b(C)$ admits no non-trivial S.O.Ds.*

Proof. Let $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. By Corollary 1.5.3, for any closed point $x \in C$ there exist $B \in \mathscr{B} \cap \mathrm{Coh}(C)$, $A \in \mathscr{A} \cap \mathrm{Coh}(C)$ such that both of them are sheaves and there exists an exact sequence

$$0 \to B \to \mathscr{O}_x \to A \to 0.$$

Hence $\mathscr{O}_x$ is contained in only one of $\mathscr{A}$ or $\mathscr{B}$. Hence $C(\mathrm{Spec}\,\mathbb{C}) = C_{\mathscr{A}} \sqcup C_{\mathscr{B}}$ by this fact.

By Proposition 3.17 in [12] we know that the set of closed points forms a spanning class, hence if $C_{\mathscr{B}} = \emptyset$ or $C_{\mathscr{A}} = \emptyset$, then $\mathscr{B}$ or $\mathscr{A}$ is trivial. Hence we may let both $C_{\mathscr{B}}$ and $C_{\mathscr{A}}$ are not empty.

We now claim that any coherent sheaf in $\mathscr{B}$ must be torsion. Indeed, otherwise the support of the sheaf coincides with the whole variety C, hence there exists a non-trivial morphism from the sheaf to a closed point which belongs to $\mathscr{A}$. This is a contradiction.

Next we claim that any torsion free sheaf belongs to $\mathscr{A}$. Indeed, let $\mathscr{E}$ be a torsion free sheaf. As before, we have an exact sequence

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Since $\mathscr{E}$ is torsion free, so is B. Combined with the first claim, we see B must be zero, hence $A = \mathscr{E}$.

By Corollary 3.19 in [12] we know that the set of torsion free sheaves forms a spanning class of $\mathbf{D}^b(C)$. Hence $\mathscr{B}$ must be trivial. Well done. □

Remark 1.5.5. *Actually the only thing we use the $g(C) \geq 1$ is Corollary 1.5.3. So any smooth projective variety satisfies Corollary 1.5.3 admits no non-trivial S.O.Ds.*

1.6 Example V – Other Examples

Proposition 1.6.1. *Let X be a smooth projective variety with $\omega_X \cong \mathscr{O}_X$, then $\mathbf{D}^b(X)$ admits no non-trivial S.O.Ds.*

Proof. Let there exists an S.O.D $\mathbf{D}^b(X) = \langle \mathscr{A}, \mathscr{B} \rangle$. Hence for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and for any i we have $\mathrm{Hom}(B, A[i]) = 0$. Hence by Serre duality we have

$$\mathrm{Hom}(B, A[i]) = \mathrm{Hom}(A[i], B[n])^\vee = \mathrm{Hom}(A, b[n-i])^\vee = 0.$$

Hence $\mathbf{D}^b(X) = \langle \mathscr{B}, \mathscr{A} \rangle$ is also an S.O.D. Hence $\mathscr{A}, \mathscr{B}$ forms an orthogonal decomposition. Hence by Proposition 3.10 in [12] and the fact that X is connected, this S.O.D must be trivial. □

Lemma 1.6.2. *Let X be a smooth projective variety and $F \in \mathbf{D}^b(X)$ is non-trivial, and $\mathscr{L}$ be a globally generated line bundle. Then*

$$\mathrm{Hom}_X(F, F \otimes \mathscr{L}) \neq 0.$$

Proof. Here we follows [25]. Let $m = \min\{i : \mathscr{H}^i(F) \neq 0\}$ and consider the following standard distinguished triangle

$$\tau_{\leq m} F \to F \to \tau_{\geq m+1} F \to \tau_{\leq m} F[1].$$

Since $\tau_{\leq m} F$ is isomorphic to a shift of a sheaf, we can find $s \in H^0(X, \mathscr{L})$ which induce a non-trivial $\tau_{\leq m} F \to \mathscr{L} \otimes \tau_{\leq m} F$. Consider

$$\begin{tikzcd}
\tau_{\geq m+1} F[-1] \arrow[r] \arrow[d, "\sigma_{\geq m+1}[-1]"] & \tau_{\leq m} F \arrow[r] \arrow[d, "\sigma_{\leq m}"] & F \arrow[r] \arrow[d, "\sigma"] & \tau_{\geq m+1} F \arrow[d, "\sigma_{\geq m+1}"] \\
\tau_{\geq m+1} F \otimes \mathscr{L}[-1] \arrow[r] & \tau_{\leq m} F \otimes \mathscr{L} \arrow[r] & F \otimes \mathscr{L} \arrow[r] & \tau_{\geq m+1} F \otimes \mathscr{L}
\end{tikzcd}$$

where these four vertical arrows are defined by taking tensor products with the section s. Hence here $\sigma_{\leq m} \neq 0$. Suppose that $\sigma = 0$. Then $\sigma_{\leq m} \neq 0$ factors through a morphism from to $\tau_{\geq m+1} F \otimes \mathscr{L}[-1]$, which is zero since $\tau_{\geq m+1} F \otimes \mathscr{L}[-1]$ has trivial cohomologies up to degree $m+1$. Thus we obtain a contradiction, well done. □

Proposition 1.6.3. *Let X be a smooth projective variety whose canonical line bundle is globally generated. Then $\mathbf{D}^b(X)$ has no exceptional objects.*

Proof. This follows from Lemma 1.6.2 and the duality

$$\mathrm{Hom}(F, F[\dim X]) = \mathrm{Hom}(F, F \otimes \omega_X)^\vee \neq 0.$$

Well done. □

Chapter 2

Non-Commutative Smooth Projective Varieties

Here we will follows the fundamental paper [18] and a nice survey [20].

2.1 Basic Definition

Definition 2.1.1. *Let $\mathscr{D}$ be a triangulated category linear over $\mathbb{C}$. We say that $\mathscr{D}$ is a* (geometric) non-commutative smooth projective variety *if there exists a smooth projective variety X over $\mathbb{C}$ and a fully faithful $\mathbb{C}$-linear exact functor $\mathscr{D} \to \mathbf{D}^b(X)$ having left and right adjoints.*

Remark 2.1.2. *Actually there is another more general non-commutative smooth projective variety (without geometric), but we will just consider these case. Note that by identifying $\mathscr{D}$ with its essential image in $\mathbf{D}^b(X)$, then the definition is only asking that $\mathscr{D}$ is an admissible subcategory.*

2.2 Functors of Fourier-Mukai Type

Before introduce this, we will introduce the products of the non-commutative smooth projective varieties. Note that there is a generalization – gluing of categories, but we will omit it and we refer [26].

Proposition 2.2.1. *If $\mathscr{D}_1 \subset \mathbf{D}^b(X_1)$ and $\mathscr{D}_2 \subset \mathbf{D}^b(X_2)$ are non-commutative smooth projective varieties, we can define $\mathscr{D}_1 \boxtimes \mathscr{D}_2$ as the smallest triangulated subcategory of $\mathbf{D}^b(X_1 \times X_2)$ which is closed under taking direct summands and contains all objects of the form $F_1 \boxtimes F_2$, with $F_i \in \mathscr{D}_i$. Then $\mathscr{D}_1 \boxtimes \mathscr{D}_2 \subset \mathbf{D}^b(X_1 \times X_2)$ is admissible.*

Proof. By Remark 1.1.5(c), we just need to show that there is an S.O.D:

$$\mathbf{D}^b(X_1 \times X_2) = \left\langle \mathscr{D}_1 \boxtimes \mathscr{D}_2, {}^\perp\mathscr{D}_1 \boxtimes \mathscr{D}_2, \mathscr{D}_1 \boxtimes {}^\perp\mathscr{D}_2, {}^\perp\mathscr{D}_1 \boxtimes {}^\perp\mathscr{D}_2 \right\rangle.$$

The first condition is trivial by Künneth formula. The second condition follows from the finite locally free resolution and stupid truncations. See [20] Proposition 2.15 for details. □

Definition 2.2.2. *Let X_1, X_2 be algebraic varieties. Let $\mathscr{D}_1 \hookrightarrow \mathbf{D}^b(X_1)$ and $\mathscr{D}_2 \hookrightarrow \mathbf{D}^b(X_2)$ be admissible categories. A functor $F : \mathscr{D}_1 \to \mathscr{F}_2$ is called of* Fourier-Mukai type *if the composite functor*

$$\mathbf{D}^b(X_1) \xrightarrow{\delta_1} \mathscr{D}_1 \xrightarrow{F} \mathscr{D}_2 \hookrightarrow \mathbf{D}^b(X_2)$$

is equivalent to a Fourier-Mukai transform where δ_1 is the left adjoint of $\mathscr{D}_1 \hookrightarrow \mathbf{D}^b(X_1)$.

Proposition 2.2.3. *Let X be a smooth projective variety with an S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{D}_1, ..., \mathscr{D}_n \rangle .$$

Then the induced projection $\delta_i : \mathbf{D}^b(X) \to \mathscr{D}_i$ is of Fourier-Mukai type whose kernel is unique up to an isomorphism.

Proof. There is a more general case in [16] Theorem 7.1. But in our case this is very easy. By Proposition 2.2.1 the subcategories $\mathscr{D}_i \boxtimes \mathbf{D}^b(X)$ are admissible. Hence consider $K_i \in \mathbf{D}^b(X \times X)$ as the projection of the structure sheaf of the diagonal $\Delta_* \mathscr{O}_X \in \mathbf{D}^b(X \times X)$ onto the category $\mathscr{D}_i \boxtimes \mathbf{D}^b(X)$ Then easy to see that there are just the Fourier-Mukai kernel here. The uniqueness follows from Theorem 1.1.10. □

Remark 2.2.4. *In the case of the Proposition, if we let*

$$\mathscr{B}_i := {}^\perp \langle \mathscr{D}_1, ..., \mathscr{D}_{i-1}, \mathscr{D}_{i+1}, ..., \mathscr{D}_n \rangle ,$$

then the kernel P_i of δ_i will be contained in $\mathscr{D}_i \boxtimes \mathscr{B}_i^\vee$. This is easy but we will omit the proof and we refer [17] Proposition 3.8. From this we find that $\mathrm{Ext}^*(P_i, P_j \circ S_X) = 0$ *for any $i \neq j$. See Corollary 3.10 in [17].*

Note that the Serre functor play a vital role in the whole theory. Here we will state some facts about it.

Proposition 2.2.5. *Some basic facts about the Serre functor:*

(i) If a Serre functor exists, then it is unique up to unique isomorphism and it is an exact functor of triangulated categories.

(ii) *If $j : \mathscr{C} \hookrightarrow \mathscr{D}$ is an admissible subcategory and $\mathscr{D}$ has a Serre functor $S_{\mathscr{D}}$, then $\mathscr{C}$ also has a Serre functor as:*

$$S_{\mathscr{C}} \cong j^! \circ S_{\mathscr{D}} \circ j, \quad j^! \cong S_{\mathscr{C}} \circ j^* \circ S_{\mathscr{D}}^{-1}.$$

Furthermore, a Serre functor on $\mathscr{C}^{\perp}$ exists as well and satisfies

$$S_{\mathscr{C}^{\perp}} \cong S_{\mathscr{D}} \circ \mathbf{R}_{\mathscr{C}}, \quad S_{\mathscr{C}^{\perp}}^{-1} \cong \mathbf{L}_{\mathscr{C}} \circ S_{\mathscr{D}}^{-1}.$$

In particular any non-commutative smooth projective variety has a Serre functor.

(iii) *The Serre functor and its inverse of any non-commutative smooth projective variety are both of Fourier-Mukai type.*

(iv) *If $\mathscr{A} \to \mathscr{B}$ be a $\mathbb{C}$-linear equivalence with $\mathscr{A}, \mathscr{B}$ are all have a Serre functor such that the* Hom*s are all of finite dimension, then this equivalence commute with the Serre functors.*

(v) *If $F : \mathscr{A} \to \mathscr{B}$ be a $\mathbb{C}$-linear functor with $\mathscr{A}, \mathscr{B}$ are all have a Serre functor $S_{\mathscr{A}}, S_{\mathscr{B}}$ such that the* Hom*s are all of finite dimension, then if G is a left adjoint of F, then F has a right adjoint $S_{\mathscr{A}} \circ G \circ S_{\mathscr{B}}^{-1}$. Similar for another side.*

(vi) *Given two non-commutative smooth projective varieties $\mathscr{D}_1 \subset \mathbf{D}^b(X_1)$ and $\mathscr{D}_2 \subset \mathbf{D}^b(X_2)$, let us denote by $P_{S_{\mathscr{D}_1}} \in D^b(X_1 \times X_1)$, respectively $P_{S_{\mathscr{D}_2}} \in D^b(X_2 \times X_2)$, kernels representing the Serre functors. Then the Serre functor of the product $\mathscr{D}_1 \boxtimes \mathscr{D}_2 \subset \mathbf{D}^b(X_1 \times X_2)$ is representable by $P_{S_{\mathscr{D}_1}} \boxtimes P_{S_{\mathscr{D}_2}}$.*

Proof. See Tag 0FY6 for (i). See Lemma 1.30 in [12] for (iv). See Remark 1.31 in [12] for (v). Now we will prove (ii)(iii)(vi).

For (ii), this directly follows from the definitions and the Yoneda's lemma. When $\mathscr{D} = \mathbf{D}^b(X)$ where X is a smooth projective variety, then $S_{\mathscr{D}}$ exists by Serre duality.

For (iii), consider the admissible $j : \mathscr{D} \hookrightarrow \mathbf{D}^b(X)$ for some smooth projective variety. By using (ii) and (v) we know that

$$S_{\mathscr{D}}^{-1} \cong j^* \circ S_X^{-1} \circ j.$$

By Proposition 2.2.3 and the fact that S_X is of course a Fourier-Mukai transform, we find that $S_{\mathscr{D}}^{-1}$ is of Fourier-Mukai type. Hence $j \circ S_{\mathscr{D}}^{-1} \circ j^*$ is a Fourier-Mukai transform. As its inverse is just $j \circ S_{\mathscr{D}} \circ j^*$, then it is an equivalence and hence a Fourier-Mukai transform. Hence $S_{\mathscr{D}}$ is of Fourier-Mukai type.

For (vi), this can be showed directly. We omit it. □

2.3 Hochschild homology and cohomology

Definition 2.3.1. *Let $\mathscr{D} \subset \mathbf{D}^b(X)$ be a non-commutative smooth projective variety and $P \in \mathbf{D}^(X \times X)$ the Fourier-Mukai kernel of the projection functor onto $\mathscr{D}$ and let S_X*

is the Fourier-Mukai kernel of the Serre functor on X. Then we define the Hochschild cohomology *and* Hochschild homology *of $\mathscr{D}$ as:*

$$\mathsf{HH}^*(\mathscr{D}) := \mathrm{Ext}^*(P, P), \quad \mathsf{HH}_*(\mathscr{D}) := \mathrm{Ext}^*(P, P \circ S_X).$$

More generally, the Hochschild cohomology of X with support in T and coefficients in E *is defined as*

$$\mathsf{HH}^*_T(X, E) := \mathrm{Ext}^*(E, E \circ T)$$

for any kernels $E, T \in \mathbf{D}^b(X \times X)$. In case of $T = \Delta_*\mathscr{O}_X$ we call it the Hochschild cohomology of X with coefficients in E*. Similarly, in case of $T = S_X$ we call it the* Hochschild homology of X with coefficients in E*. Hence when $(E, T) = (P, \Delta_*\mathscr{O}_X)$ and (P, S_X), this is just the Hochschild cohomology and Hochschild homology of $\mathscr{D}$.*

Remark 2.3.2. *Some important remarks:*

(a) We can see that $\mathsf{HH}_*(\mathscr{D})$ is a graded right module over $\mathsf{HH}^*(\mathscr{D})$ by Yoneda's lemma. Moreover, by the definition of Serre functor, the graded structure is given by

$$\mathsf{HH}^i(\mathscr{D}) \times \mathsf{HH}_j(\mathscr{D}) \to \mathsf{HH}_{i+j}(\mathscr{D}).$$

(b) We can define a perfect pairing

$$\mathsf{HH}_*(\mathscr{D}) \times \mathsf{HH}_{-*}(\mathscr{D}) \to \mathbb{C}$$

which called a Mukai pairing*. This follows from*

$$\begin{aligned}\mathsf{HH}_i(\mathscr{D}) &= \mathrm{Ext}^i(P, P \circ S_X) = \mathrm{Ext}^i(P, P \circ S_{X\times X} S_X)\\ &= \mathrm{Ext}^{-i}(P \circ S_X^{-1}, P)^\vee = \mathsf{HH}_{-i}(\mathscr{D})^\vee\end{aligned}$$

by Serre duality.

Lemma 2.3.3. *Let $\mathscr{D} \subset \mathbf{D}^b(X)$ be a non-commutative smooth projective variety and let $P \in \mathbf{D}^(X \times X)$ the Fourier-Mukai kernel of the projection functor onto $\mathscr{D}$, then we have*

$$\mathsf{HH}^*(\mathscr{D}) = H^*(X, \Delta^! P), \quad \mathsf{HH}_*(\mathscr{D}) = H^*(X, \Delta^* P).$$

Proof. Let R be the kernel of the right projection onto ${}^\perp\mathscr{D}$. Then we have a distinguished triangle $R \to \Delta_*\mathscr{O}_X \to P$. On the other hand, $\mathrm{Ext}^*(R, P) = \mathrm{Ext}^*(R, P \circ S_X) = 0$ by Remark 2.2.4. Since

$$\Delta^!(P \circ S_X) = \Delta^!(P \circ S_X) = \Delta^!(P \otimes p_2^*\omega_X[\dim X]) = \Delta^* P,$$

we can get the result directly. $\square$

Lemma 2.3.4. *If $\mathscr{A} = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle$ is an S.O.D, then*

$$\mathsf{HH}_*(\mathscr{A}) = \mathsf{HH}_*(\mathscr{A}_1) \oplus \cdots \oplus \mathsf{HH}_*(\mathscr{A}_m)$$

which is orthogonal with respect to the Mukai pairing. Moreover, if $\mathscr{D} = \langle \mathscr{A}, \mathscr{B} \rangle$ is an orthogonal decomposition, then

$$\mathsf{HH}^*(\mathscr{D}) = \mathsf{HH}^*(\mathscr{A}) \oplus HH^*(\mathscr{B}).$$

Proof. See [17] or [20] Proposition 2.25 for the proof. □

Example 2.3.1. *Let E be an exceptional object, then $\mathsf{HH}_*(\langle E \rangle) = \mathsf{HH}^*(\langle E \rangle) = \mathbb{C}$.*

Let $E_1, ..., E_m$ be an exceptional collection and let $\mathscr{D} = \langle E_1, ..., E_m \rangle$. Then $\mathsf{HH}_*(\mathscr{D}) = \mathbb{C}^{\oplus m}$, concentrated in degree 0.

Theorem 2.3.5. *Let $\mathscr{C} \subset \mathbf{D}^b(X), \mathscr{D} \subset \mathbf{D}^b(Y), \mathscr{E} \subset \mathbf{D}^b(Z)$ be non-commutative smooth projective varieties.*

(i) Any Fourier-Mukai functor $\Phi : \mathscr{C} \to \mathscr{D}$ induces a morphism of graded k-vector spaces $\Phi_{\mathsf{HH}} : \mathsf{HH}_*(\mathscr{C}) \to \mathsf{HH}_*(\mathscr{D})$ such that $\mathrm{id}_{\mathsf{HH}} = \mathrm{id}$ *and, given another functor $\Psi : \mathscr{D} \to \mathscr{E}$, we have $(\Psi \circ \Phi)_{\mathsf{HH}} = \Psi_{\mathsf{HH}} \circ \Phi_{\mathsf{HH}}$.*

(ii) If (Ψ, Φ) is a pair of adjoint Fourier-Mukai functors, then

$$(-, \Phi_{\mathsf{HH}}(-)) = (\Psi_{\mathsf{HH}}(-), -)$$

according to the Mukai pairing.

(iii) There is a Chern character $\mathrm{ch} : \mathrm{K}_0(\mathscr{D}) \to \mathsf{HH}_0(\mathscr{D})$ *such that, for all $F, G \in \mathscr{D}$,*

$$(\mathrm{ch}\, E, \mathrm{ch}\, F) = \chi(E, F) := \sum_i (-1)^i \dim \mathrm{Ext}^i_Y(E, F).$$

(iv) The Hochschild structure is invariant under exact equivalences of Fourier-Mukai type.

Proof. For (i), we first consider the case of Fourier-Mukai functor $\Phi_E : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ of kernel E. Then its left and right adjoint are of kernels $E_L := E^\vee \otimes p_Y^* \omega_Y[\dim Y]$ and $E_R := E^\vee \otimes p_X^* \omega_X[\dim X]$, respectively. In our language, these are $E_L = E^\vee \circ S_Y$ and $E_R = S_X \circ E^\vee$. Hence this induce $\Phi_E \circ \Phi_{E_R} \to \mathrm{id}_Y$ and $\mathrm{id}_Y \to \Phi_E \circ \Phi_{E_L}$. In our language, these are $\Delta_* \mathscr{O}_Y \to E \circ E_L$ and $E \circ E_R \to \Delta_* \mathscr{O}_Y$. Hence for any $\mu \in \mathsf{HH}_i(X)$, we induce

$$\begin{aligned} \Delta_* \mathscr{O}_Y &\to E \circ E_L = E \circ E^\vee \circ S_Y \\ &\simeq E \circ S_X^{-1} \circ S_X \circ E^\vee \circ S_Y \\ &\xrightarrow{\mu} E \circ S_X \circ E^\vee \circ S_Y[i] \\ &= E \circ E_R \circ S_Y[i] \to S_Y[i]. \end{aligned}$$

Hence we get an element in $\mathsf{HH}_i(Y)$. This induce $\Phi_{E,\mathsf{HH}}:\mathsf{HH}_i(X)\to\mathsf{HH}_i(Y)$. In general case, this follows from this special case and Lemma 2.3.4.

For (ii) this follows from the direct calculation and we omit it and refer [6] Theorem 7.3. For (iii) this also follows from the direct calculation and we omit it and refer [6] Theorem 7.1 and Theorem 7.6. For (iv) we refer [17] Section 7. □

Now we consider a special case: $\mathscr{D}=\mathbf{D}^b(X)$ for a smooth projective variety X of dimension n. In this case by Lemma 2.3.3 we have

$$\mathsf{HH}^*(X):=\mathsf{HH}^*(\mathbf{D}^b(X))=H^*(X,\Delta^!\Delta_*\mathscr{O}_X),$$
$$\mathsf{HH}_*(X):=\mathsf{HH}_*(\mathbf{D}^b(X))=H^*(X,\Delta^*\Delta_*\mathscr{O}_X).$$

Consider the universal Atiyah class $\mathsf{At}\in\mathrm{Ext}^1(\Delta_*\mathscr{O}_X,\Delta_*\Omega_X)$ correspond to

$$0\to I_\Delta/I_\Delta^2\cong\Delta_*\Omega_X\to\mathscr{O}_{X\times X}/I_\Delta^2\to\Delta_*\mathscr{O}_X\to 0.$$

Repeat this we get

$$\mathsf{At}^p:\Delta_*\mathscr{O}_X\to\Delta_*\Omega_X^p[p],\quad \mathsf{At}^p:\Delta_*\Omega_X^{n-p}[n-p]\to\Delta_*\omega_X[n]$$

which by adjunction we get

$$\Delta^*\Delta_*\mathscr{O}_X\to\Omega_X^p[p],\quad \bigwedge^p T_X[-p]\to\Delta^!\Delta_*\omega_X[n].$$

Proposition 2.3.6 (Hochschild-Kostant-Rosenberg)**.** *For a smooth projective variety X of dimension n, then the previous maps induce isomorphisms*

$$\Delta^*\Delta_*\mathscr{O}_X\cong\bigoplus_{p=0}^n\Omega_X^p[p],\quad \bigoplus_{p=0}^n\bigwedge^p T_X[-p]\cong\Delta^!\Delta_*\omega_X[n].$$

In particular we have

$$\mathsf{HH}^i(X)\cong\bigoplus_{p+q=i}H^q(X,\bigwedge^p T_X),\quad \mathsf{HH}_i(X)\cong\bigoplus_{q-p=i}H^q(X,\Omega_X^p).$$

Proof. We refer [21] and [22]. □

Chapter 3

Examples of Fano Varieties

We always consider the schemes and vector spaces over $\operatorname{Spec}\mathbb{C}$.

3.1 Basic Results of Fano Varieties

Theorem 3.1.1 (Fujita 1980-1984)**.** *Let X be a smooth Fano n-fold of index $r \geq n-1$. Then the general element in the fundamental divisor is smooth.*

Proof. See [28] Theorem 2.3.2. $\square$

Theorem 3.1.2 (Mella 1996)**.** *Let X be a smooth Fano n-fold of index $n-2$. Then the general element in the fundamental divisor is smooth.*

Proof. See [23] Theorem 2.5. $\square$

Corollary 3.1.3. *Let X be a smooth Fano 3-fold of index 1 and $H^3 \geq 8$ and $\rho(X) = 1$. Then the linear system $|-K_X|$ is very ample and X is projectively normal which is an intersection of quadrics.*

Proof. See [28] Corollary 4.1.13. $\square$

3.2 Cubics

3.3 Gushel-Mukai Varieties

3.3.1 Basic Definitions and Properties

Let V_5 be a vector space of dimension 5 and consider the Plücker embedding $\mathsf{Gr}(2,V_5) \hookrightarrow \mathbb{P}\left(\bigwedge^2 V_5\right)$. For any vector space K, consider the cone $\mathsf{C}_K(\mathsf{Gr}(2,V_5)) \subset \mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$

of vertex $\mathbb{P}(K)$. Choose a vector subspace $W \subset \bigwedge^2 V_5 \oplus K$ and a subscheme $Q \subset \mathbb{P}(W)$ defined by defined by one quadratic equation (possibly zero).

Definition 3.3.1. *The scheme*

$$X = \mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W) \cap Q$$

is called a Gushel-Mukai intersection (GM intersection). A GM intersection X is called a Gushel-Mukai variety (GM variety) if X is a smooth variety of dimension $\dim W - 5 \geq 1$.

Remark 3.3.2. *Some remarks:*

(a) *In the original paper [7] they defined without the smoothness (but always Gorenstein).*

(b) *Note that all Q and $\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W)$ are Gorenstein, hence all Cohen-Macaulay. So the dimension condition means they are dimensionally transverse, that is, $\mathrm{Tor}_{>0}(\mathscr{O}_Q, \mathscr{O}_{\mathsf{C}_K(\mathsf{Gr}(2,V_5)) \cap \mathbb{P}(W)}) = 0$.*

(c) *A GM variety X has a canonical polarization, the restriction H of the hyperplane class on $\mathbb{P}(W)$; we will call (X, H) a polarized GM variety.*

The definition of a GM variety is not intrinsic. We actually have an intrinsic characterization. But before giving these, we will introduce a new definition:

Definition 3.3.3. *Let W be a vector space and let $Y \subset \mathbb{P}(W)$ be a closed subscheme which is an intersection of quadrics, i.e., the twisted ideal sheaf $\mathscr{I}_X(2)$ on $\mathbb{P}(W)$ is globally generated.*

Define $V_X := H^0(\mathbb{P}(W), \mathscr{I}_X(2))$, this yields a surjection $V_X \otimes \mathscr{O}_{\mathbb{P}(W)}(-2) \twoheadrightarrow \mathscr{I}_X$ which induce

$$V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{I}_X/\mathscr{I}_X^2 = \mathscr{N}^{\vee}_{X/\mathbb{P}(W)}.$$

We define the excess conormal sheaf $\mathscr{E}\mathscr{N}^{\vee}_{X/\mathbb{P}(W)}$ to be the kernel of this map.

Theorem 3.3.4. *A smooth polarized projective variety (X, H) of dimension $n \geq 1$ is a polarized GM variety if and only if all the following conditions hold:*

(a) *$H^n = 10$ and $K_X = -(n-2)H$.*

(b) *H is very ample and the vector space $W := H^0(X, \mathscr{O}_X(H))^{\vee}$ has dimension $n + 5$.*

(c) *X is an intersection of quadrics in $\mathbb{P}(W)$ and the vector space*

$$V_6 := H^0(\mathbb{P}(W), \mathscr{I}_X(2)) \subset \mathrm{Sym}^2 W^{\vee}$$

of quadrics through X has dimension 6.

(d) *The twisted excess conormal sheaf $\mathscr{U}_X := \mathscr{E}\mathscr{N}^{\vee}_{X/\mathbb{P}(W)}(2H)$ of X in $\mathbb{P}(W)$ is simple.*

Proof. We first need to show a smooth polarized GM variety (X, H) satisfies (a)-(d).

For (a), as $\deg(\mathsf{C}_K(\mathsf{Gr}(2, V_5))) = 5$ and they are dimensionally transverse, then $\deg(X) = 10$. Let $\dim K = k$ and hence $K_{\mathsf{C}_K(\mathsf{Gr}(2,V_5))} = -(5+k)H$ by Lemma 3.3.7. Finally we have

$$K_X = (-(5+k) + (10+k) - (n+5) + 2)H = -(n-2)H.$$

For (b), we just need to show $W = H^0(X, \mathscr{O}_X(H))^\vee$. Consider the resolution

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{C}_K\,\mathsf{Gr}(2,V_5)} \to 0.$$

Restrict it into $\mathbb{P}(W)$ and tensor the resolution of Q as $0 \to \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_Q$, then tensor $\mathscr{O}(1)$ again we get the resolution

$$\begin{aligned} 0 \to \mathscr{O}(-6) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-4) &\to (V_5 \otimes \mathscr{O}(-3)) \oplus (V_5^\vee \otimes \mathscr{O}(-2)) \\ &\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-1) \to \mathscr{O}(1) \to \mathscr{O}_X(H) \to 0 \end{aligned}$$

on $\mathbb{P}(W)$. Hence $H^0(X, \mathscr{O}_X(H)) = H^0(\mathbb{P}(W), \mathscr{O}_{\mathbb{P}(W)}(1)) = W^\vee$.

For (c), consider the resolution again:

$$\begin{aligned} 0 \to \mathscr{O}(-5) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-3) &\to (V_5 \otimes \mathscr{O}(-2)) \oplus (V_5^\vee \otimes \mathscr{O}(-1)) \\ &\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O} \to \mathscr{O}(2) \to \mathscr{O}_X(2H) \to 0 \end{aligned}$$

Hence one can show that $H^0(\mathbb{P}(W), \mathscr{I}_X(2)) = V_5 \oplus \mathbb{C}$, hence well done.

For (d), we will use the induction of the dimension. For $n = 1$, this follows from some basic fact of excess normal sheaf and the Mukai's construction about a stable vector bundle of rank 2 on X to show that $\mathscr{U}_X$ is stable, and hence simple. For the detail we refer [7] Theorem 2.3. Hence we now assume $n \geq 2$. Pick a smooth hyperplane section $X' \subset X$ which is also irreducible since $n \geq 2$ by Bertini's theorem. Hence X' is also a GM variety. One can easy to show that in this case $\mathscr{U}_X|_{X'} = \mathscr{U}_{X'}$ (see Lemma A.5 in [7]). Hence we have $0 \to \mathscr{U}_X(-H) \to \mathscr{U}_X \to \mathscr{U}_{X'} \to 0$. Hence

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}).$$

If $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X)) > 1$, then $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H))) > 0$. By the similar argument we get

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}(-H)) = 0.$$

Hence $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \neq 0$. By induction we get $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-kH)) \neq 0$ for any $k > 0$. Hence for any $k > 0$ we have $\Gamma(X, \mathscr{U}_X^\vee \otimes \mathscr{U}_X(-kH)) \neq 0$. But these are vector bundles and X is integral of dimension ≥ 2, hence this is impossible.

Now we let a smooth polarized projective variety (X, H) of dimension $n \geq 1$ which satisfies (a)-(d). We need to show that (X, H) is a polarized GM variety.

We know that

$$\det \mathscr{U}_X^\vee = \det(\mathscr{N}^\vee_{X/\mathbb{P}(W)}(2H)) = \mathscr{O}_X(H)$$

and the embedding $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$. Taking wedge product, duality and global sections we get

$$\bigwedge^2 V_6^\vee \to H^0(X, \mathscr{O}_X(H)) = W^\vee.$$

Hence we get $W \to \bigwedge^2 V_6$ which can be factored through an injection $W \to \bigwedge^2 V_6 \oplus K$ for some vector space K. Hence we have

$$\begin{tikzcd}
 & \mathbb{P}(W) \arrow[r, hook] & \mathbb{P}\left(\bigwedge^2 V_6 \oplus K\right) \arrow[d, dashed] \\
X \arrow[ur, hook] \arrow[r] & \mathsf{Gr}(2, V_6) \arrow[r, hook] & \mathbb{P}\left(\bigwedge^2 V_6\right)
\end{tikzcd}$$

where $X \to \mathsf{Gr}(2, V_6)$ induced by $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$ and is commutative since these are the same linear system. Hence we get $X \subset \mathsf{C}_K^\circ \mathsf{Gr}(2, V_6) = \mathsf{C}_K \mathsf{Gr}(2, V_6) \backslash \mathbb{P}(K)$.

Now by some facts of excess normal sheaves (see Proposition A.3 in [7]), then excess normal sequence induce a functorial diagram:

$$\begin{tikzcd}
0 \arrow[r] & (V_6 \otimes \mathscr{U}_X)/\mathrm{Sym}^2\mathscr{U}_X \arrow[r] \arrow[d] & \bigwedge^2 V_6 \otimes \mathscr{O}_X \arrow[r] \arrow[d] & \det V_6 \otimes \mu^* \mathscr{N}^\vee_{\mathsf{Gr}(2,V_6)/\mathbb{P}(\bigwedge^2 V_6)}(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \det V_6 \otimes \mathscr{U}_X \arrow[r] & \det V_6 \otimes V_6 \otimes \mathscr{O}_X \arrow[r] & \det V_6 \otimes \mathscr{N}_{X/\mathbb{P}(W)}(2) \arrow[r] & 0
\end{tikzcd}$$

which follows from the expression of the excess normal sheaf of $\mathsf{Gr}(2, V_6) \subset \mathbb{P}(\bigwedge^2 V_6)$. The left vertical arrow induces a morphism $\lambda' : V_6 \otimes \mathscr{U}_X \to \det V_6 \otimes \mathscr{U}_X$. As $\mathscr{U}_X$ is simple by (d) we get $\lambda : V_6 \to \det V_6$. Since λ' vanishes on $\mathrm{Sym}^2\mathscr{U}_X$, the image of $\mathscr{U}_X$ in $_6 \otimes \mathscr{O}_X$ is contained in $\ker \lambda \otimes \mathscr{O}_X$. Moreover, the middle vertical map in the diagram above is given by $v_1 \wedge v_2 \mapsto \lambda(v_1)v_2 - \lambda(v_2)v_1$.

We claim that $\lambda \neq 0$. If $\lambda = 0$, the middle vertical map in the diagram is zero, which means that all the quadrics cutting out $\mathsf{C}_K \mathsf{Gr}(2, V_6)$ contain $\mathbb{P}(W)$, i.e. $\mathbb{P}(W) \subset \mathsf{C}_K \mathsf{Gr}(2, V_6)$. In other words, $\mathbb{P}(W)$ is a cone over $\mathbb{P}(W') \subset \mathsf{Gr}(2, V_6)$ with vertex a subspace of K. Hence $X \to \mathsf{Gr}(2, V_6)$ factor through $\mathbb{P}(W')$. Hence the vector bundle $\mathscr{U}_X$ is a pullback from $\mathbb{P}(W')$ of the restriction of the tautological bundle of $\mathsf{Gr}(2, V_6)$ to $\mathbb{P}(W')$.

There are two types of linear spaces on $\mathsf{Gr}(2, V_6)$: the first type corresponds to 2-dimensional subspaces containing a given vector and the second type to those contained in a given 3-subspace $V_3 \subset V_6$. If W' is of the first type, the restriction of the tautological bundle to $\mathbb{P}(W')$ is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$, hence $\mathscr{U}_X \cong \mathscr{O} \oplus \mathscr{O}(-H)$ by Lemma 3.3.8. In particular, it is not simple, which is a contradiction. If W' is of the second type, the embedding $\mathscr{U}_X \to V_6 \otimes \mathscr{O}_X$ factors through a subbundle $V_3 \otimes \mathscr{O}_X \subset V_6 \otimes \mathscr{O}_X$. Recall that V_6 is the space of quadrics passing through X in $\mathbb{P}(W)$. Consider the scheme-theoretic intersection M of the quadrics corresponding to the vector subspace V_3. Since the embedding of the excess conormal sheaf factors through $V_3 \otimes \mathscr{O}_X$, the variety X is the complete intersection of M with the 3 quadrics corresponding to the quotient space V_6/V_3. But the degree of X is then divisible by 8, which contradicts the fact that it is 10 by (a). Hence we conclude that $\lambda \neq 0$.

Now let $V_5 := \ker(\lambda)$ is a hyperplane in v_6 which fits in the exact sequence $0 \to V_5 \to V_6 \xrightarrow{\lambda} \det V_6 \to 0$. The composition $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X \xrightarrow{\lambda} \det V_6 \otimes \mathscr{O}_X$ vanish, hence we get $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathscr{O}_X$.

We now replace V_6 with V_5 and repeat the above argument, then we get a linear map $W \to \bigwedge^2 W_5$ which factor through $\mu : W \hookrightarrow \bigwedge^2 W_5 \oplus K$ which induce again the embedding $X \subset \mathsf{C}_K^\circ \operatorname{\mathsf{Gr}}(2, V_5) = \mathsf{C}_K \operatorname{\mathsf{Gr}}(2, V_5) \backslash \mathbb{P}(K)$. By the functorial of the excess normal sequence (see Proposition A.3 in [7]) again we get that inside the space V_6 of quadrics cutting out X in $\mathbb{P}(W)$, the hyperplane V_5 is the space of quadratic equations of $\operatorname{\mathsf{Gr}}(2, V_5)$, i.e., of Plücker quadrics.

As the Plücker quadrics cut out the cone $\mathsf{C}_K \operatorname{\mathsf{Gr}}(2, V_5)$ in $\mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$, they cut out $\mathsf{C}_K \operatorname{\mathsf{Gr}}(2, V_5) \cap \mathbb{P}(W)$ in $\mathbb{P}(W)$. Since X is the intersection of 6 quadrics by condition (c), we finally obtain

$$X = \mathsf{C}_K \operatorname{\mathsf{Gr}}(2, V_5) \cap \mathbb{P}(W) \cap Q$$

where Q is some non-Plücker quadric corresponding to a point in $V_6 \backslash V_5$, so X is a GM variety. □

Remark 3.3.5. *This is right for all normal varieties with the similar proof.*

Remark 3.3.6. *The twisted excess conormal sheaf $\mathscr{U}_X$ that was crucial for the proof will be called its* **Gushel sheaf***. As we showed in the proof, the projection of X from the vertex $\mathbb{P}(K)$ of the cone $\mathsf{C}_K \operatorname{\mathsf{Gr}}(2, V_5)$ defines a morphism $X \to \operatorname{\mathsf{Gr}}(2, V_5)$ and the Gushel sheaf $\mathscr{U}_X$ is isomorphic to the pullback under this map of the tautological vector bundle on $\operatorname{\mathsf{Gr}}(2, V_5)$. The map $X \to \operatorname{\mathsf{Gr}}(2, V_5)$ is thus determined by $\mathscr{U}_X$ and is canonically associated with X. We call this map the* **Gushel map** *of X.*

When X have some mild singularity, then the Gushel map is just a rational map and $\mathscr{U}_X$ is isomorphic to the pullback under this map of the tautological vector bundle on $\operatorname{\mathsf{Gr}}(2, V_5)$ in the smooth locus.

Lemma 3.3.7. *Let $X \subset \mathbb{P}^n$ be a subvariety such that $K_X = rH$. Let $\mathsf{C}(X) \subset \mathbb{P}^{n+1}$ be a cone over X, then $K_{\mathsf{C}(X)} = (r-1)H$.*

Proof. We know that the blow-up of of the vertex of $\mathsf{C}(X)$ is

$$\begin{tikzcd} & X' = \mathbb{P}_X(\mathscr{O}_X \oplus \mathscr{O}_X(-H)) \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathsf{C}(X) & & X \end{tikzcd}$$

Let H' be the relative hyperplane class of p. Then

$$K_{X'} = p^*(K_X + H) - 2H' = (r+1)p^*H - 2H'.$$

On the other hand, the morphism π contracts the exceptional section $E \subset X'$ and H' is the pullback of $H_{\mathsf{C}(X)}$. Finally $E \sim_{\mathrm{lin}} H' - p^*H$, hence

$$K_{X'} = (r-1)H' - (r+1)E.$$

Hence $K_{\mathsf{C}(X)} = (r-1)H$. □

Lemma 3.3.8. *Let $Z_p \subset \mathsf{Gr}(k, V)$ be the subscheme parameterizing all k-planes containing the vector p. Then $Z_p \cong \mathsf{Gr}(k-1, n-1)$ and the restriction of the tautological subbundle $\mathscr{S}_{\mathsf{Gr}(k,V)}$ to Z_p splits as the sum of $\mathscr{O}$ and the tautological subbundle $\mathscr{S}_{Z_p}$ of $Z_p \cong \mathsf{Gr}(k-1, n-1)$.*

Proof. This is almost trivial. Indeed, let $V_1 \subset V$ be the 1-dimensional subspace generated by the vector p. Let $V = V_1 \oplus V'$ be a direct sum decomposition. Then for each $k-1$-dimensional subspace $U' \subset V'$ the sum $V_1 \oplus U'$ is a k-dimensional subspace of V. Hence the corresponding subbundle

$$V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{\mathsf{Gr}(k-1,V')} \subset V_1 \otimes \mathscr{O} \oplus V' \otimes \mathscr{O} = V \otimes \mathscr{O}$$

induces a morphism $\mathsf{Gr}(k-1, V') \to \mathsf{Gr}(k, V)$ which is an isomorphism onto Z_p and such that the pullback of the tautological bundle is $V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{Z_p}$. □

3.3.2 Some Classifications

Lemma 3.3.9. *Let (X, H) be a polarized variety. If it is projective normal, that is, the canonical map* $\mathrm{Sym}^m H^0(X, \mathscr{O}_X(H)) \to H^0(X, \mathscr{O}_X(mH))$ *is surjective for any $m \geq 0$, then H must be very ample.*

Proof. By the commutative diagram

$$\begin{tikzcd} & \mathbb{P}H^0(X, \mathscr{O}_X(nH)) \arrow[dr, hook] & \\ X \arrow[ur, hook, "|nH|"] \arrow[dr, "|H|"'] & & \mathbb{P}H^0(X, \mathrm{Sym}^n \mathscr{O}_X(H)) \\ & \mathbb{P}H^0(X, \mathscr{O}_X(H)) \arrow[ur, hook, "n\text{-uple}"'] & \end{tikzcd}$$

we know that $|H|$ also induce an immersion. Hence H is very ample. $\square$

Proposition 3.3.10. *Let (X,H) be a smooth polarized variety of dimension $n \geq 2$ such that $K_X = -(n-2)H$ and $H^1(X, \mathscr{O}_X) = 0$. If there is a hypersurface $X' \subset X$ in the linear system $|H|$ such that $(X', H|_{X'})$ is a smooth polarized GM variety, (X,H) is also a smooth polarized GM variety.*

Proof. First we note that for any smooth GM variety (Y,H) the resolution

$$0 \to \mathscr{O}(m-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(m-5) \to (V_5 \otimes \mathscr{O}(m-4)) \oplus (V_5^\vee \otimes \mathscr{O}(m-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(m-2) \to \mathscr{O} \to \mathscr{O}_Y(mH) \to 0$$

can imply Y is projective normal, that is, the canonical map $\mathrm{Sym}^m H^0(Y, \mathscr{O}_Y(H)) \to H^0(Y, \mathscr{O}_Y(mH))$ is surjective for any $m \geq 0$.

Back to the result, we need to check the conditions in Theorem 3.3.4. For (a), this follows from $H^n = H \cdot H^{n-1} = H|_{X'}^{n-1} = 10$. Now we know X' is projective normal, so is X by [14] Lemma (2.9). By Lemma 3.3.9 we know H is very ample. By $H^1(X, \mathscr{O}_X) = 0$ we know that $h^0(X, \mathscr{O}_X(H)) = n+5$ by the case of X'. This proves (b), and [14] Lemma (2.10) proves (c). For (d), since $\mathscr{U}_{X'}$ is simple, by the similar proof of (d) in Theorem 3.3.4 we can also show that $\mathscr{U}_X$ is simple. $\square$

Theorem 3.3.11. *Let X be a complex smooth projective variety of dimension $n \geq 1$, together with an ample Cartier divisor H such that $K_X \sim_{\mathrm{lin}} -(n-2)H$ and $H^n = 10$. If we assume that*

- *when $n \geq 3$, we have $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$;*
- *when $n = 2$, the surface X is a Brill-Noether general K3 surface (a K3 surface is called Brill-Noether general if $h^0(S,D)h^0(S,H-D) < h^0(S,H)$ for all divisors D on S not linearly equivalent to 0 or H. When $H^2 = 10$, this is equivalent to the fact that $|H|$ contains a Clifford general smooth curve);*
- *when $n = 1$, the genus-6 curve X is Clifford general (that is, it is neither hyperelliptic, nor trigonal, nor a plane quintic).*

then X is a GM variety.

Before proving this, we need some Lemmas:

Lemma 3.3.12. *Let X be a complex smooth projective variety of dimension $n \geq 3$ with an ample divisor H such that $H^n = 10$ and $K_X \sim_{\mathrm{lin}} -(n-2)H$.*

Then the linear system $|H|$ is very ample and a smooth general $X' \in |H|$ satisfies the same conditions: if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and $K_{X'} \sim_{\mathrm{lin}} -(n-3)H'$.

Proof. First we need to show that $h^0(H) > 0$. This follows from the follows result:

- **Lemma 3.3.12.A.** Let X be a smooth Fano variety of dimension $n \geq 3$ such that $-K_X \sim_{\text{lin}} rH$ where H is ample. Then when $r \geq n-2$, then $h^0(H) > 0$.

Proof of Lemma 3.3.12.A. Now we separate it as two cases.

When $r \geq n-1$, use Kodaira vanishing theorem to $(x+r)H+K_X$ we have $h^i(xH) = 0$ for all $i > 0$ and all $x \geq -(n-2)$. Now we let $h^0(H) = 0$ and in these cases we have $\chi(xH) = h^0(xH)$. Hence $\chi(xH)$, as a polynomial, has roots $-1, -2, ..., -(n-2), 1$. As $\chi(0) = 1$ and $\chi(xH)$ as the top coefficient $\frac{H^n}{n!}$, we know that

$$\begin{aligned}\chi(xH) &= \frac{1}{n!}(x+1)(x+2)\cdots(x+n-2)(x-1)(H^n x - n(n-1)) \\ &= \frac{1}{n!}\left(H^n x^n + \left(n(n-3)\frac{H^n}{2} - n(n-1)\right)x^{n-1} + \text{lower terms}\right).\end{aligned}$$

On the other hand, by HRR we get

$$\chi(xH) = \frac{1}{n!}\left(H^n x^n + \frac{1}{2}nrH^n x^{n-1} + \text{lower terms}\right).$$

Hence $\frac{1}{2}nrH^n = n(n-3)\frac{H^n}{2} - n(n-1)$, that is, $r = n-3-\frac{2n-2}{H^n}$. But $r \geq n-1$, this is impossible. Hence $h^0(H) > 0$.

When $r = n-2$, we will go through this directly. By Kodaira vanishing theorem again we have $h^i(xH) = 0$ for all $i > 0$ and all $x \geq -(n-3)$. For $x = -(n-2)$, we only have

$$h^i(-(n-2)H) = h^i(K_X) = \begin{cases} 1, & i = n; \\ 0, & 0 \leq i < n. \end{cases}$$

Hence again we have

$$\chi(xH) = \frac{1}{n!}(x+1)(x+2)\cdots(x+n-3)(H^n x^3 + bx^2 + cx + n(n-1)(n-2)).$$

Now as $\chi(-(n-2)H) = (-1)^n$, we can find that $b = \frac{3}{2}H^n(n-2)$ and $c = 2n(n-1) + \frac{1}{2}H^n(n-2)^2$. Hence $h^0(H) > 0$ by taking $x = 1$. $\square$

Hence now $|H|$ is non-empty. Note that in this case H is already the fundamental divisor since $H^n = 10$. Hence by Theorem 3.1.1 and Theorem 3.1.2 as in this case the index of X is $\geq n-2$, then the general elements are smooth. Pick such X'. Then if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and by adjunction formula we have $K_{X'} \sim_{\text{lin}} -(n-3)H'$. By Kodaira vanishing theorem we have $H^1(X, \mathscr{O}_X) = 0$. Hence the linear series $|H'|$ is just the restriction of $|H|$ to X' and the base loci of $|H|$ and $|H'|$ are the same. Taking successive linear sections, we arrive at a linear section Y of dimension 3 which is smooth and $K_Y \sim_{\text{lin}} -H_Y$ and $H_Y^3 = 10$.

If $\text{Pic}(Y) = \mathbb{Z} \cdot H_Y$, then by Corollary 3.1.3 the pair (Y, H_Y) is projectively normal.

If not, then $\rho(X) \geq 2$. By the classification theory (NEED TO ADD) of the Fano threefold, Y must be a divisor of bidegree $(3,1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and the pair (Y, H_Y) is again projectively normal.

Hence in both case, we can use the [14] Lemma (2.9) repeatly which imply that (X, H) is projectively normal. Hence by Lemma 3.3.9 we know H is very ample. □

Remark 3.3.13. *Note that in Lemma 3.3.12.A, by the similar arguments we can show that when X is a smooth Fano variety of dimension n and index r with fundamental divisor H we have $h^0(H) = \frac{1}{2}H^n(r-n+3)+n-1$ when $r > n-2$ and $h^0(H) = \frac{1}{2}H^n+n$ when $r = n-2$.*

Lemma 3.3.14. *Let (X, H) be a polarized complex variety of dimension $n \geq 2$ which satisfies the hypotheses of Theorem 3.3.11. A general element of $|H|$ then satisfies the same properties.*

Proof. Assume first $n \geq 4$. By Lemma 3.3.12 we need only to prove that a general smooth $X' \in |H|$ satisfies $\mathrm{Pic}(X') = \mathbb{Z} \cdot H'$ where $H' := H|_{X'}$. By Grothendieck-Lefschetz theorem we have $\mathrm{Cl}(X) \cong \mathrm{Cl}(X')$. Hence $\mathrm{Pic}(X') = \mathbb{Z} \cdot H'$ as $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$.

When $n = 2$, this follows from definitions.

When $n = 3$, X is a smooth Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Then by Corollary 3.1.3 X is an intersection of quadrics. Any smooth hyperplane section S of X is a degree-10 smooth K3 surface which is still an intersection of quadrics. A general hyperplane section of S is still an intersection of quadrics, hence is a Clifford general curve. This proves that S is Brill-Noether general. □

Proof of Theorem 3.3.11. Induction on n. The case $n = 1$ was proved in Proposition 3.3.15, so we assume $n \geq 2$. A general hyperplane section X' of X has the same properties by Lemma 3.3.14, hence is a GM variety by the induction hypothesis. On the other hand, we have $H^1(X, \mathscr{O}_X) = 0$. By Proposition 3.3.10, we conclude that X is a GM variety. Well done. □

Some inverse results:

Proposition 3.3.15. *A smooth projective curve is a GM curve if and only if it is a Clifford general curve of genus* 6.

Proof. Follows from the Theorem 3.3.4 and the Enriques-Babbage theorem in [1] Section III.3. □

Proposition 3.3.16. *A smooth projective surface X is a GM surface if and only if X is a Brill-Noether general polarized K3 surface of degree* 10.

Proof. By Theorem 3.3.11, we just need to show that if X is a GM surface, then X is a Brill-Noether general polarized K3 surface of degree 10. In this case, we have $K_X = 0$ by Theorem 3.3.4(a), and the resolution

$$\begin{aligned}0 \to \mathscr{O}(-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-5) \to (V_5 \otimes \mathscr{O}(-4)) \oplus (V_5^\vee \otimes \mathscr{O}(-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0\end{aligned}$$

implies $H^1(X, \mathscr{O}_X) = 0$, hence X is a K3 surface. Moreover, a general hyperplane section of X is a GM curve, hence a Clifford general curve of genus 6, hence X is Brill-Noether general. □

Proposition 3.3.17. *Let (X, H) be a polarized complex smooth GM variety of dimension $n \geq 3$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. In particular, the polarization H is the unique GM polarization on X.*

Proof. By Theorem 3.3.11, we just need to show that if (X, H) be a polarized complex smooth GM variety of dimension $n \geq 3$, then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. By Theorem 3.3.4, X is a Fano variety of degree 10 and is an intersection of quadrics. When $n = 3$, by the proof of Lemma 3.3.12 we know that $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Now consider $n \geq 4$, a general hyperplane section X' of X satisfies the same properties by Lemma 3.3.14 and by Grothendieck-Lefschetz theorem again (for general case we refer Theorem 1 in [29]) we have injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X')$. Hence by induction we get the result. □

3.3.3 Grassmannian Hulls

Fix $V_5, V_6, K, W \subset \bigwedge^2 V_5 \oplus K, Q \subset \mathbb{P}(W)$ which defines a smooth GM variety

$$X = \mathsf{C}_K \,\mathsf{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q.$$

Definition 3.3.18. *Define $M_X := \mathsf{C}_K \,\mathsf{Gr}(2, V_5) \cap \mathbb{P}(W)$ to be the Grassmannian hull of X. Hence $X = M_X \cap Q$ which is a quadric section of M_X.*

Define $M'_X := \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W')$ to be the projected Grassmannian hull of X where W' defined as the image of the projection $\mu : W \subset \bigwedge^2 V_5 \oplus K \to \bigwedge^2 V_5$.

Remark 3.3.19. *Note that these two schemes are canonically associated to X via GM datas. See [7] Section 2.*

Now consider the Gushel map $X \to \mathsf{Gr}(2, V_5)$.

Proposition 3.3.20. *Let X be a such smooth GM variety.*

(i) If $\mu : W \to \bigwedge^2 V_5$ is injective, that is, μ induce $W \cong W'$, then $M_X \cong M'_X$ and Gushel map $X \to \mathsf{Gr}(2, V_5)$ is an embedding which induce

$$X \cong M'_X \cap Q = \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q.$$

In this case we call X a **ordinary GM variety**. *Hence in this case*

$$\dim X = \dim W - 5 \leq \dim \bigwedge^2 V_5 - 5 = 5.$$

(ii) If $\ker \mu \neq 0$, then $\dim \ker \mu = 1$, $Q \cap \mathbb{P}(\ker \mu) = \emptyset$ and $M_X = \mathsf{C}_{\mathbb{P}(\ker \mu)} M'_X$ and the Gushel map $X \to \mathsf{Gr}(2, V_5)$ induce $X \to M'_X$ which is a double covering branched at a quadric (which is a ordinary GM variety if $\dim X \geq 2$). In this case we call X a **special GM variety**. *Hence in this case it comes with a canonical involution from the double covering and*

$$\dim X = \dim W - 5 \leq \dim \bigwedge^2 V_5 + 1 - 5 = 6.$$

Proof. For (i), this is trivial by the conditions.

For (ii), note that the blow up $\mathrm{Bl}_{\mathbb{P}(\ker \mu)} M_X$ at its vertex is a $\mathbb{P}^{\dim \ker \mu}$-bundle over M'_X. As X is smooth, then $X \cap \mathbb{P}(K) = Q \cap \mathbb{P}(\ker \mu) = \emptyset$. Hence $\dim \ker \mu = 1$ as $\dim Q = \dim \mathbb{P}(W) - 1$. Now as Q is a quadric, then the Gushel map induce $X \to M'_X$ which is a double covering. We have $X \to M'_X$ branched along $\mathsf{Gr}(2, V_5) \cap \mathbb{P}(W') \cap Q$ which is a ordinary GM variety if $\dim X \geq 2$, □

Remark 3.3.21. *In (ii), if X is not smooth, then there is two more cases which from the similar arguments:*

If $\mathbb{P}(\ker \mu) \subset Q$. In this case $\tilde{\mu} : \widetilde{X} := \mathrm{Bl}_{\mathbb{P}(\ker \mu) \cap Q} X \to M'_X$ is generically $\mathbb{P}^{\dim \ker \mu - 1}$-bundle. If $\mathbb{P}(\ker \mu) \not\subseteq Q$ but $\mathbb{P}(\ker \mu) \cap Q \neq \emptyset$, then $\tilde{\mu} : \widetilde{X} := \mathrm{Bl}_{\mathbb{P}(\ker \mu) \cap Q} X \to M'_X$ is generically $(\dim \ker \mu - 1)$-dim quadric bundle.

Hence in the world of singular varieties there are many bad situations. But fortunately, we are living in a smooth world.

Remark 3.3.22. *By (ii), we can turn the special GM variety into a ordinary GM variety (as its branched locus). This leads to an important birational operation on the set of all GM varieties which can be described by GM datas. This actually gives a correspondence between special GM n-folds and ordinary GM $(n-1)$-folds. For details we refer [7] Lemma 2.33.*

Remark 3.3.23. *Hence in this case we know that we only need to assume $\dim K = 1$ to construct the whole theory if we just consider the smooth GM varieties.*

Proposition 3.3.24. *Let X be a smooth GM variety of dimension $2 \leq n \leq 6$, then the Hodge diamonds as follows:*

Proof. Follows from [19], [13], [24] and [8]. □

$\dim(X)=2$	$\dim(X)=3$	$\dim(X)=4$	$\dim(X)=5$	$\dim(X)=6$
1 0 0 1 20 1 0 0 1	1 0 0 0 1 0 0 10 10 0 0 1 0 0 0 1	1 0 0 0 1 0 0 0 0 0 0 1 22 1 0 0 0 0 0 0 1 0 0 0 1	1 0 0 0 1 0 0 0 0 0 0 0 2 0 0 0 0 10 10 0 0 0 0 2 0 0 0 0 0 0 0 1 0 0 0 1	1 0 0 0 1 0 0 0 0 0 0 0 2 0 0 0 0 0 0 0 0 0 0 1 22 1 0 0 0 0 0 0 0 0 0 0 2 0 0 0 0 0 0 0 1 0 0 0 1

Proposition 3.3.25. *Let X be a smooth GM variety of dimension $2 \leq n \leq 6$, then*

$$\mathsf{HH}_*(X) \cong \begin{cases} \mathbb{C}[2] \oplus \mathbb{C}^{\oplus 2n+18} \oplus \mathbb{C}[-2], n \ \text{even}; \\ \mathbb{C}^{\oplus 10}[1] \oplus \mathbb{C}^{\oplus 2n-2} \oplus \mathbb{C}^{\oplus 10}[-1], n \ \text{odd}. \end{cases}$$

Proof. Follows directly from Proposition 2.3.6 and Proposition 3.3.24. □

3.4 Debarre-Voisin Varieties

Definition 3.4.1. *Let X be the smooth hyperplane section of $\mathsf{Gr}(3, V_{10})$, then we call X a Debarre-Voisin variety.*

Hence it is of dimension 20 with canonical bundle

$$\omega_X = \omega_{\mathsf{Gr}(3,V_{10})}|_X \otimes \mathscr{O}(X) = \mathscr{O}_X(-9).$$

Hence it is a Fano variety.

This was first introduced in [9] aim of constructing new examples of locally complete families of polarized hyperkähler fourfolds.

Theorem 3.4.2 (Debarre-Voisin)**.** *Let X be a Debarre-Voisin variety, then the only nonzero Hodge numbers of the vainshing cohomology $H^{20}(X, \mathbb{Q})_{\mathrm{van}}$ are*

$$h^{9,11}_{\mathrm{van}} = h^{11,9}_{\mathrm{van}} = 1, \quad h^{10,10}_{\mathrm{van}} = 20$$

where $H^{20}(X, \mathbb{Q})_{\mathrm{van}} = \ker(H^{20}(X, \mathbb{Q}) \to H^{22}(\mathsf{Gr}(3, V_{10}), \mathbb{Q}))$.

Proof. See Theorem 2.2(1) in [9]. □

Corollary 3.4.3. *Let X be a Debarre-Voisin variety, then the Hodge number of X satisfies $h^{p,q} = 0$ for all $p \neq q, p+q \neq 20$ and $p < 9$ or > 11 when $p+q = 20$ and $\sum_p h^{p,p} = 130$.*

Proof. As X is a hyperplane section, by the weak Lefschetz theorem we get

$$H^{40-j}(X,\mathbb{Q}) = H_j(X,\mathbb{Q}) \to H_j(\mathsf{Gr}(3,V_{10}),\mathbb{Q}) = H^{42-j}(\mathsf{Gr}(3,V_{10}),\mathbb{Q})$$

which is isomorphism when $j < 20$ and surjective when $j = 20$. Hence by the cohomology of $\mathsf{Gr}(3,V_{10})$, some duality theorem and Theorem 3.4.2 we get the result. $\square$

Proposition 3.4.4. *Let X be a Debarre-Voisin variety, then*

$$\mathsf{HH}_*(X) = \mathbb{C}[2] \oplus \mathbb{C}^{\oplus 130} \oplus \mathbb{C}[-2].$$

Proof. By Proposition 2.3.6 and Corollary 3.4.3 we get the result. $\square$

3.5 Iliev-Manivel Varieties

Chapter 4

Calabi-Yau Categories and its General Properties

4.1 Fractional Calabi-Yau Categories

The definition of Calabi-Yau categories is an analogue of the Calabi-Yau varieties:

Definition 4.1.1. *A triangulated category $\mathscr{D}$ is a* fractional Calabi-Yau category *if it has a Serre functor $S_{\mathscr{D}}$ and there are integers p and $q \neq 0$ such that $S_{\mathscr{D}}^q \cong [p]$.*

*In the case of $(p, q) = (n, 1)$, we call $\mathscr{D}$ is an n-*Calabi-Yau category*. Sometimes we call n its* dimension*.*

4.2 Indecomposability

4.3 Hochschild (co-)homology

4.4 The Dimension of Calabi-Yau Subcategories

Chapter 5

Kuznetsov Components and Examples

5.1 The Construction of Kuznetsov Components

5.1.1 Spherical Functors

First let $\mathscr{T}_1$ and $\mathscr{T}_2$ are triangulated categories and $\Phi : \mathscr{T}_1 \to \mathscr{T}_2$ be a functor with right and left adjoints $\Phi^* \dashv \Phi \dashv \Phi^!$, then we define units and counits:

$$\eta_{\Phi,\Phi^*} : \mathrm{id}_{\mathscr{T}_2} \to \Phi \circ \Phi^*, \quad \varepsilon_{\Phi^*,\Phi} : \Phi^* \circ \Phi \to \mathrm{id}_{\mathscr{T}_1};$$
$$\eta_{\Phi^!,\Phi} : \mathrm{id}_{\mathscr{T}_1} \to \Phi^! \circ \Phi, \quad \varepsilon_{\Phi,\Phi^!} : \Phi \circ \Phi^! \to \mathrm{id}_{\mathscr{T}_2}.$$

Definition 5.1.1. *A Fourier-Mukai functor $\Phi : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ is spherical if*

(a) the map $\eta_{\Phi^!,\Phi} \circ \Phi^* + \Phi^! \circ \eta_{\Phi,\Phi^*} : \Phi^* \oplus \Phi^! \to \Phi^! \circ \Phi \circ \Phi^*$ is an isomorphism, and

(b) the map $(\Phi^* \circ \varepsilon_{\Phi,\Phi^!}, \varepsilon_{\Phi^*,\Phi} \circ \Phi^!) : \Phi^* \circ \Phi \circ \Phi^! \to \Phi^* \oplus \Phi^!$ is an isomorphism.

Proposition 5.1.2. *Consider a Fourier-Mukai functor $\Phi : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ which is spherical, we define T_X, T_Y, T'_X, T'_Y by the following distinguished triangles:*

$$\Phi^* \circ \Phi \stackrel{\varepsilon_{\Phi^*,\Phi}}{\to} \mathrm{id}_X \to T_X, \quad T_Y \to \mathrm{id}_Y \stackrel{\eta_{\Phi,\Phi^*}}{\to} \Phi \circ \Phi^*,$$
$$T'_X \to \mathrm{id}_X \stackrel{\eta_{\Phi^!,\Phi}}{\to} \Phi^! \circ \Phi, \quad \Phi \circ \Phi^! \stackrel{\varepsilon_{\Phi,\Phi^!}}{\to} \mathrm{id}_Y \to T'_Y.$$

Then these are mutually inverse autoequivalences of $\mathbf{D}^b(X)$ and $\mathbf{D}^b(Y)$.

In this case we call T_X, T_Y are spherical twist functors. Moreover, we have

$$\Phi \circ T_X \cong T_Y \circ \Phi \circ [2], \quad T_X \circ \Phi^* \cong \Phi^* \circ T_Y \circ [2].$$

Proof. The first statment is direct and we refer Proposition 2.13 in [18]. For the second, we can show directly that $\Phi^*[1] \cong T_X \circ \Phi^!$ and $\Phi^*[-1] \cong \Phi^! \circ T_Y$. Then by these the second statments are trivial. □

5.1.2 The Main Construction

Here is our fundamental theorem in this chapter due to Kuznetsov which follows [18]:

Theorem 5.1.3 (Kuznetsov, 2015)**.** *Let M and X are smooth projective varieties with a spherical functor $\Phi : \mathbf{D}^b(X) \to \mathbf{D}^b(M)$ between their derived categories. Let T_M and T_X be the spherical twists. Assume we have:*

(a) *$\mathbf{D}^b(M)$ has a rectangular Lefschetz decomposition*

$$\mathbf{D}^b(M) = \left\langle \mathscr{B}, \mathscr{B} \otimes \mathscr{L}_M, ..., \mathscr{B} \otimes \mathscr{L}_M^{\otimes(m-1)} \right\rangle.$$

(b) *There is some $1 \leq d < m$ such that for all $i \in \mathbb{Z}$ we have $T_M(\mathscr{B} \otimes \mathscr{L}_M^i) = \mathscr{B} \otimes \mathscr{L}_M^{i-d}$.*

(c) *There is a line bundle $\mathscr{L}_X$ on X such that: $\mathscr{L}_M \circ \Phi \cong \Phi \circ \mathscr{L}_X$.*

(d) *Finally, assume that $T_X \circ \mathscr{L}_X \cong \mathscr{L}_X \circ T_X$.*

Then the functor $\Phi^*|_{\mathscr{B}} : \mathscr{B} \to \mathbf{D}^b(X)$ is fully faithful. If we let $\mathscr{B}_X := \Phi^*\mathscr{B} \subset \mathbf{D}^b(X)$, then they induce an S.O.D

$$\mathbf{D}^b(X) = \left\langle \mathsf{Ku}(X), \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X, ..., \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes(m-d-1)} \right\rangle$$

where $\mathsf{Ku}(X) := \left\langle \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X, ..., \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes(m-d-1)} \right\rangle^{\perp}$ is called the Kuznetsov component *of X associated to our data: M, Φ, and the rectangular Lefschetz decomposition of $\mathbf{D}^b(M)$. Finally if we consider the following autoequivalences of $\mathbf{D}^b(X)$:*

$$\rho := T_X \circ \mathscr{L}_X^d, \quad \sigma := S_X \circ T_X \circ \mathscr{L}_X^m,$$

if $c = \gcd(d, m)$ then the Serre functor of the Kuznetsov component $\mathsf{Ku}(X)$ can be expressed as

$$S_{\mathsf{Ku}(X)}^{\frac{d}{c}} \cong \rho^{-\frac{m}{c}} \circ \sigma^{\frac{d}{c}}.$$

In particular, if some powers of ρ and σ are shifts then $\mathsf{Ku}(X)$ is a fractional Calabi-Yau category.

Proof. Note that the facts that functor $\Phi^*|_{\mathscr{B}} : \mathscr{B} \to \mathbf{D}^b(X)$ is fully faithful and the existence of Kuznetsov component $\mathsf{Ku}(X)$ are directly by arguments of category theory. We omit it and we refer Lemma 3.10 in [18] for details.

For the final argument, consider the degree shift functor:

$$\mathsf{O}_{\mathsf{Ku}(X)} := \delta_{\mathsf{Ku}(X)} \circ \mathscr{L}_X : \mathsf{Ku}(X) \to \mathsf{Ku}(X).$$

Then we have the following properties:

- (A) $\sigma \circ \rho = \rho \circ \sigma$.
- (B) $S_X^{-1} = \mathscr{L}_X^{\otimes m} \circ T_X \circ \sigma^{-1}$.
- (C) All components $\mathsf{Ku}(X), \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X^i$ are preserved by ρ and σ.
- (i) $\mathsf{O}_{\mathsf{Ku}(X)}$ is an autoequivalence.
- (ii) $\mathsf{O}_{\mathsf{Ku}(X)}$ commute with σ, ρ.
- (iii) $\mathsf{O}^i_{\mathsf{Ku}(X)} = \delta_{\mathsf{Ku}(X)} \circ \mathscr{L}_X^{\otimes i}$ for all $0 \leq i \leq m-d$.
- (iv) $S^{-1}_{\mathsf{Ku}(X)} = \mathsf{O}^{m-d}_{\mathsf{Ku}(X)} \circ \rho \circ \sigma^{-1}$.
- (v) $\mathsf{O}^d_{\mathsf{Ku}(X)} = \rho$.

Proof of Properties. (A)-(C) are direct and we refer Lemma 3.11 in [18]. Property (i) follows by either (iv) or (v). Property (ii) follows by a direct check. Property (iv) follows from (iii), Proposition 2.2.5(ii) and (B). Hence the key results are (iii) and (v).

For (iii), observe that the formula is true for $i = 0, 1$. Let us assume the formula is true for $0 \leq i < m-d$; we want to show it is true for $i+1$ as well. Let $F \in \mathsf{Ku}(X)$, consider

$$\begin{aligned} F \otimes \mathscr{L}_X^{\otimes i+1} &\to \delta_{\mathsf{Ku}(X)}(F \otimes \mathscr{L}_X^{\otimes i}) \otimes \mathscr{L}_X = \mathsf{O}^i_{\mathsf{Ku}(X)}(F) \otimes \mathscr{L}_X \\ &\to \delta_{\mathsf{Ku}(X)}(\mathsf{O}^i_{\mathsf{Ku}(X)}(F) \otimes \mathscr{L}_X) = \mathsf{O}^{i+1}_{\mathsf{Ku}(X)}(F). \end{aligned}$$

Then we need to show the cone of this map is in $\left\langle \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X, ..., \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes(m-d-1)} \right\rangle$. By the octahedral axiom, this cone is an extension of two objects

$$G_1 \otimes \mathscr{L}_X \to \text{cone} \to G_2 \to$$

where $G_1 \in \left\langle \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X, ..., \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes(i-1)} \right\rangle$ and $G_2 \in \mathscr{B}_X$. Hence we get the result.

For (v), the general case need many categorial calculations and we refer Corollary 3.18 in [18]. Here we show the case $d \leq m-d$. Let $F \in \mathsf{Ku}(X)$, we have a distinguished triangle

$$\Phi^*\Phi(F \otimes \mathscr{L}_X^{\otimes d}) \to F \otimes \mathscr{L}_X^{\otimes d} \to \rho(F) \to .$$

As $\rho(F) \in \mathsf{Ku}(X)$, we need to show $\Phi^*\Phi(F \otimes \mathscr{L}_X^{\otimes d}) \in {}^\perp\mathsf{Ku}(X)$. By adjointness we have $\Phi(F \otimes \mathscr{L}_X^{\otimes d}) \in \left\langle \mathscr{B}, \mathscr{B} \otimes \mathscr{L}_M, ..., \mathscr{B} \otimes \mathscr{L}_M^{\otimes(d-1)} \right\rangle$. Easy to see that

$$\Phi^*(\mathscr{B} \otimes \mathscr{L}_M^{\otimes i}) = \Phi^*(\mathscr{B}) \otimes \mathscr{L}_X^{\otimes i} = \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes i}.$$

Hence

$$\Phi^*\Phi(F \otimes \mathscr{L}_X^{\otimes d}) \in \left\langle \mathscr{B}_X, \mathscr{B}_X \otimes \mathscr{L}_X, ..., \mathscr{B}_X \otimes \mathscr{L}_X^{\otimes(d-1)} \right\rangle \subset {}^\perp\mathsf{Ku}(X).$$

Well done! ∎

Now come back to the proof. By property (iv), we can express the (inverse of the) Serre functor $S_{\mathsf{Ku}(X)}$ in terms of the functors $\mathsf{O}_{\mathsf{Ku}(X)}, \rho$ and σ. By properties (A), (i) and (ii), all these functors commute. By raising everything to the power d/c, and by using property (v), the statement follows. □

Remark 5.1.4. *Note that in this case we have*

$$\mathsf{Ku}(X) = \left\{ F \in \mathbf{D}^b(X) : \Phi(F) \in \left\langle \mathscr{B} \otimes \mathscr{L}_M^{-d}, ..., \mathscr{B} \otimes \mathscr{L}_X^{-1} \right\rangle \subset \mathbf{D}^b(M) \right\}$$

by trivial reasons. See Lemma 3.12 in [18].

Remark 5.1.5. *The same result and proof with holds if consider the more general conditions:*

(a) We could replace $\mathscr{L}_M$ and $\mathscr{L}_X$ by arbitrary autoequivalences.

(b) Second, we could replace $\mathbf{D}^b(X)$ and $\mathbf{D}^b(M)$ by non-commutative smooth projective varieties.

We refer Remark 3.21 in [18] to consider the case of not rectangular case.

5.2 Example I – Hypersurfaces

Proposition 5.2.1. *Let M be a smooth projective variety with an rectangular Lefschetz decomposition*

$$\mathbf{D}^b(M) = \left\langle \mathscr{B}, \mathscr{B} \otimes \mathscr{L}_M, ..., \mathscr{B} \otimes \mathscr{L}_M^{\otimes(m-1)} \right\rangle.$$

Consider the map $f : X \to M$ is a divisorial embedding with the image $f(X)$ being a smooth divisor in the linear system $\mathscr{L}_M^{\otimes d}$ for some $1 \leq d \leq m$.

Then the $\Phi := f_* : \mathbf{D}^b(X) \to \mathbf{D}^b(M)$ is spherical. Let $\mathscr{L}_X := f^*\mathscr{L}_M$, then T_X commute with $\mathscr{L}_X$. Moreover we have an appropriate power of the functor $\rho = T_X \circ \mathscr{L}_X^d$ is a shift. Finally, if $\omega_M = \mathscr{L}_M^{-m}$ then an appropriate power of the functor $\sigma = S_X \circ T_X \circ \mathscr{L}_X^m$ is also a shift.

Proof. In this case $f^!F = f^*F \otimes \mathscr{L}_X^d[-1]$. Hence for any $F \in \mathbf{D}^b(M)$ we have

$$\begin{aligned} f^! f_* f^* F &\cong f^!(F \otimes f_*\mathscr{O}_X) \cong f^!(F \otimes (\mathscr{L}_M^{-d} \xrightarrow{\phi} \mathscr{O}_M)) \\ &\cong f^*(F \otimes (\mathscr{L}_M^{-d} \xrightarrow{\phi} \mathscr{O}_M)) \otimes \mathscr{L}_X^d[-1] \\ &\cong f^*F \otimes (\mathscr{L}_X^{-d}[1] \oplus \mathscr{O}_X) \otimes \mathscr{L}_X^d[-1] \\ &\cong f^*F \oplus f^!F. \end{aligned}$$

Similar for $f^* \circ f_* \circ f^!$. Hence $\Phi = f_*$ is spherical. □

Corollary 5.2.2. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n+1$ and $c = \gcd(d, n+1)$. Then we have S.O.D*

$$\mathbf{D}^b(X) = \langle \mathsf{Ku}(X), \mathscr{O}_X, \mathscr{O}_X(1), ..., \mathscr{O}_X(n-d) \rangle$$

and the Serre functor of $\mathsf{Ku}(X)$ has the property $S_{\mathsf{Ku}(X)}^{d/c} = \left[\frac{(n+1)(d-2)}{c}\right]$. In particular, if $d|n+1$ then $\mathsf{Ku}(X)$ is a Calabi-Yau category of dimension $\frac{(n+1)(d-2)}{d}$.

5.3 Example II – Double coverings

5.4 Example III – K3 categories

5.5 Example IV – Some Other Cases

Chapter 6

Examples of Derived Equivalences of Kuznetsov Components

Chapter 7

Stability Conditions on K3 Categories

Chapter 8

Applications: Mukai's program

Chapter 9

Application to Cubic Fourfolds and Gushel-Mukai Manifolds

Index

Bibliography

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I.* Springer-Verlag, New York, 1985.

[2] M. Bernardara. A semiorthogonal decomposition for brauer-severi schemes. *Math. Nachr.*, 282(10):1406–1413, 2009.

[3] Alexei Bondal and Mikhail Kapranov. Representable functors, serre functors, and mutations. *Mathematics of the USSR-Izvestiya*, 35:519, 1990.

[4] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. Math.*, 166(2):317–345, 2007.

[5] Tom Bridgeland. Stability conditions on k3 surfaces. *Duke Math. J.*, 166(2):241–291, 2008.

[6] Andrei Căldăraru. The mukai pairing. i. the hochschild structure. *arXiv: 0308079v2*, 2003.

[7] O. Debarre and A. Kuznetsov. Gushel-mukai varieties: classification and birationalities. *Algebr. Geom.*, 5:15–76, 2018.

[8] Olivier Debarre and Alexander Kuznetsov. Gushel-mukai varieties: linear spaces and periods. *Kyoto J. Math.*, 59:897–953, 2019.

[9] Olivier Debarre and Claire Voisin. Hyper-kähler fourfolds and grassmann geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal).*, 649:63–87, 2010.

[10] A. Fonarev. On minimal lefschetz decompositions for grassmannians. *Izv. RAN. Ser. Mat.*, 77:203–224, 2013.

[11] A.L. Gorodentsev, S.A. Kuleshov, and A.N. Rudakov. t-stabilities and tstructures on triangulated categories. *Translation in Izv. Math.*, 68, 2004.

[12] D. Huybrechts. *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford University Press, 2006.

[13] Atanas Iliev and Laurent Manivel. Fano manifolds of degree ten and epw sextics. *Ann. Sci. Ec. Norm. Super.*, 44:393–426, 2011.

[14] V A Iskovskih. Fano 3-folds. i. *Mathematics of the USSR-Izvestiya*, 11(3):485, 1977.

[15] M. Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. *Invent. Math.*, 92(3):479–508, 1988.

[16] A. Kuznetsov. Base change for semiorthogonal decompositions. *Compos. Math.*, 147:852–876, 2011.

[17] Alexander Kuznetsov. Hochschild homology and semiorthogonal decompositions. *arXiv: 0904.4330*, 2009.

[18] Alexander Kuznetsov. Calabi-yau and fractional calabi-yau categories. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015.

[19] Dmitry Logachev. Fano threefolds of genus 6. *Asian J. Math.*, 16:515–559, 2012.

[20] Emanuele Macrì and Paolo Stellari. *Lectures on Non-commutative K3 Surfaces, Bridgeland Stability, and Moduli Spaces*, pages 199–265. Springer International Publishing, 2019.

[21] N. Markarian. Poincaré-birkoff-witt isomorphism, hochshild homology and riemann-roch theorem. *preprint MPIM2001-52*, 2001.

[22] N. Markarian. The atiyah class, hochschild cohomology and the riemann-roch theorem. *J. London Math. Soc.*, 379:129–143, 2009.

[23] Massimiliano Mella. Existence of good divisors on mukai varieties. *J. Algebraic Geom.*, 8:197–206, 1999.

[24] J. Nagel. The generalized hodge conjecture for the quadratic complex of lines in projective four-space. *Math. Ann.*, 312:387–401, 1998.

[25] S. Okawa. Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math.*, 228:2869–2873, 2011.

[26] D. Orlov. Smooth and proper noncommutative schemes and gluing of dg categories. *Adv. Math.*, 302:59–105, 2016.

[27] Dmitri Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izvestiya Mathematics*, 41(1):133–141, 1993.

[28] A.N. Parshin and I.R. Shafarevich. *Algebraic Geometry V: Fano Varieties*. Springer Berlin, Heidelberg, 1999.

[29] G.V. Ravindra and V. Srinivas. The grothendieck-lefschetz theorem for normal projective varieties. *J. Algebraic Geo.*, 15(3):563–590, 2006.