

NOTES ON THE GEOMETRY OF HYPERTORIC VARIETIES

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ABSTRACT. In this note we will introduce the basic theory of hypertoric varieties.

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1. INTRODUCTION

1.1. Background/Motivation.

1.2. Related works and some future direction. Need to add.

1.3. Notations and remarks. We work over \mathbb{C} .

2. RECOLLECTION OF THE BASIC THEORY OF TORIC VARIETIES

We will follow [Ful93], [CLS11] and [Tel22] to recollect something we need.

3. BASIC DEFINITIONS AND RESOLUTIONS OF HYPERTORIC VARIETIES

3.1. About symplectic varieties and symplectic resolutions. Here we give an introduction of these and we refer [Bea00] and [Fu06] for more details.

Definition 3.1. *We consider complex algebraic schemes.*

- *We say a scheme X carries a Poisson structure if there is a \mathbb{C} -bilinear operation*

$$\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$$

which is a Lie bracket.

Date: October 22, 2024.

2020 Mathematics Subject Classification. 14M25, 14J42 .

Key words and phrases. hypertoric varieties, GIT quotient.

- Let $f : X \rightarrow Y$ be a morphism of Poisson schemes, we say it is a *Poisson morphism* if it induce a homomorphism of Lie algebras.

Remark 3.2. Any Poisson structure can be induced by the \mathcal{O}_X -linear homomorphism $H : \Omega_X^1 \rightarrow T_X = \text{Der}(\mathcal{O}_X, \mathcal{O}_X)$ such that $\{f, g\} = H(df)(g)$. In particular, any symplectic variety has a canonical Poisson structure.

We also have the relative version of Poisson schemes and we omit them here.

Definition 3.3. Let Y_0 be a normal variety.

- A pair (Y_0, ω_0) of the normal algebraic variety Y_0 and a 2-form ω_0 on the smooth locus $(Y_0)_{\text{sm}}$ is called a **symplectic variety** if ω_0 is symplectic and there exists (or equivalently, for any) a resolution $\pi : Y \rightarrow Y_0$ such that the pull-back of ω_0 by π extends to a holomorphic 2-form ω on Y .
- The resolution $\pi : Y \rightarrow Y_0$ is called **symplectic** if ω is also symplectic.

Some basic properties:

Proposition 3.4 (Prop.1.6 in [Fu06]). Let W be a symplectic variety with a resolution $\pi : Z \rightarrow W$, then the following statements are equivalent:

- (1) π is crepant;
- (2) π is symplectic;
- (3) K_Z is trivial.

Next, we now care about the following special case:

Definition 3.5. An affine symplectic variety $(Y_0 = \text{Spec } R, \omega_0)$ with \mathbb{C}^* -action (called **conical \mathbb{C}^* -action**) is called a **conical symplectic variety** if it satisfies:

- The grading induced from the \mathbb{C}^* -action to the coordinate ring R is positive, i.e., $R = \bigoplus_{i \geq 0} R_i$ and $R_0 = \mathbb{C}$.
- ω_0 is homogeneous with respect to the \mathbb{C}^* -action, i.e., there exists $\ell \in \mathbb{Z}$ (the weight of ω_0) such that $t^* \omega_0 = t^\ell \omega_0$ ($t \in \mathbb{C}^*$).

Remark 3.6. We can show that the weight ℓ is always positive.

3.2. Algebraic symplectic quotients and hypertoric varieties. Note that hypertoric varieties are examples of symplectic varieties.

Consider the exact sequence

$$0 \rightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \rightarrow 0$$

where $A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in M_{d \times n}(\mathbb{Z})$ and $B^T = [\mathbf{b}_1, \dots, \mathbf{b}_n] \in M_{(n-d) \times n}(\mathbb{Z})$ (the Gale duality of $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$). Acting $\text{Hom}(-, \mathbb{C}^*)$ we get

$$1 \rightarrow \mathbb{T}^d \xrightarrow{A^T} \mathbb{T}^n \xrightarrow{B^T} \mathbb{T}^{n-d} \rightarrow 1$$

an exact sequence of algebraic tori.

Via the natural action of \mathbb{T}^n on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$, we have the action of \mathbb{T}^d on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ as

$$\mathbf{t} \cdot (z_1, \dots, z_n, w_1, \dots, w_n) = (\mathbf{t}^{\mathbf{a}_1} z_1, \dots, \mathbf{t}^{\mathbf{a}_n} z_n, \mathbf{t}^{-\mathbf{a}_1} w_1, \dots, \mathbf{t}^{-\mathbf{a}_n} w_n)$$

where $\mathbf{t}^{\mathbf{a}_i} := t_1^{a_{1,i}} \dots t_d^{a_{d,i}}$. The moment map of this given by

$$\mu : T^*\mathbb{C}^n \rightarrow \mathfrak{t}_d^* = \mathbb{C}^d, \quad (z_1, \dots, z_n, w_1, \dots, w_n) \mapsto \sum_{i=1}^n \mathbf{a}_i z_i w_i.$$

Definition 3.7. Fix a character $\alpha \in \mathbb{Z}^d = \text{Hom}(\mathbb{T}^d, \mathbb{C}^*)$ and a point $\xi \in \mathbb{C}^d$.

- We define the **Lawrence toric variety** as

$$X(A, \alpha) := (\mathbb{C}^{2n})^{\alpha\text{-ss}} // \mathbb{T}^d = \text{Proj} \left(\bigoplus_{k \geq 0} \mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha} \right)$$

where $(\mathbb{C}^{2n})^{\alpha\text{-ss}} = \{u \in \mathbb{C}^{2n} : \text{there exists } f \in \mathbb{C}[z_i, w_j] \text{ such that } f(u) \neq 0 \text{ and } \sigma(f) = \alpha^*(t)^k \otimes f \text{ for } k > 0\}$ where $\mathbb{C}^* = \text{Spec } \mathbb{C}[t, 1/t]$ and coaction morphism $\sigma : \mathbb{C}[z_i, w_j] \rightarrow \Gamma(\mathcal{O}_{\mathbb{T}^d}) \otimes \mathbb{C}[z_i, w_j]$. Note that $\mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha} = \{f \in \mathbb{C}[z_i, w_j] : \sigma(f) = \alpha^*(t)^k \otimes f\}$.

- We define the hypertoric variety (or toric hyperkähler variety) as

$$Y(A, \alpha, \xi) := \mu^{-1}(\xi)^{\alpha-ss} // \mathbb{T}^d = \text{Proj} \left(\bigoplus_{k \geq 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha} \right)$$

similar as above.

Remark 3.8. We can write the semistable locus as follows:

$$(\mathbb{C}^{2n})^{\alpha-ss} = \left\{ (z_i, w_j) \in \mathbb{C}^{2n} : \alpha \in \sum_{i: z_i \neq 0} \mathbb{Q}_{\geq 0} \mathbf{a}_i + \sum_{j: w_j \neq 0} \mathbb{Q}_{\geq 0} (-\mathbf{a}_j) \right\}$$

and $\mu^{-1}(\xi)^{\alpha-ss} = \mu^{-1}(\xi) \cap (\mathbb{C}^{2n})^{\alpha-ss}$.

Remark 3.9. Note that we have a natural morphism $\Pi : X(A, \alpha) \rightarrow X(A, 0)$ and $\pi : Y(A, \alpha, \xi) \rightarrow Y(A, 0, \xi)$ with the same reason. Indeed, we consider the case of hypertoric varieties. Note that

$$Y(A, 0, \xi) = \text{Proj} \left(\bigoplus_{k \geq 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k \cdot 0} \right) = \text{Spec} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}.$$

Then inclusion $\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d} \subset \bigoplus_{k \geq 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}$ induce $\text{Spec} \bigoplus_{k \geq 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha} \rightarrow \text{Spec} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}$. Since the grade induced by \mathbb{C}^* -action and this morphism is \mathbb{C}^* -invariant, then we get $\pi : Y(A, \alpha, \xi) \rightarrow Y(A, 0, \xi)$. Note moreover that $\mu^{-1}(\xi)^{\alpha-ss} \subset \mu^{-1}(\xi) = \mu^{-1}(\xi)^{0-ss}$.

Remark 3.10. The hypertoric varieties are the special case of the following general construction.

Consider a reductive group G and a representation V . Then we form $T^*V = V \oplus V^*$ which comes with a moment map $\Phi : T^*V \rightarrow \mathfrak{g}^*$ given by cup of $T_x V^* \rightarrow \mathfrak{g}^*$ as $T_e G \rightarrow T_x(Gx) \subset T_x V$. We fix a character $\chi : G \rightarrow \mathbb{C}^\times$ and form the GIT quotient

$$\Phi^{-1}(\xi) //_{\chi} G := \Phi^{-1}(\xi)^{\chi-ss} // G = \text{Proj} \left(\bigoplus_{n \geq 0} \mathbb{C}[\Phi^{-1}(\xi)]^{G, n\chi} \right).$$

We have a natural projective morphism as before

$$\pi : Y := \Phi^{-1}(\xi) //_{\chi} G \rightarrow X := \Phi^{-1}(\xi) //_0 G = \text{Spec} \mathbb{C}[\Phi^{-1}(0)]^G$$

carry Poisson structures coming from the usual symplectic structure on T^*V . This construction will not usually give a symplectic resolution; for example, Y may not be smooth and $Y \rightarrow X$ might not be birational. Here in the physics literature, Y is called the **Higgs branch of the 3d supersymmetric gauge theory defined by G, V** . G is called the **gauge group** and N is called the **matter**.

There is a conical \mathbb{C}^\times action on Y coming from its scaling action of T^*V . In order to define a Hamiltonian torus action, we need one piece of data. We choose an extension $1 \rightarrow G \rightarrow \tilde{G} \rightarrow T \rightarrow 1$, where T is the flavor torus, and an action of \tilde{G} on V , extending the action of G . Then we obtain a residual Hamiltonian action of T on Y and X . In general, this action does not have finitely many fixed points.

Example 3.11. Another special case, we introduce the Nakajima quiver varieties, first introduced by Nakajima. We fix a finite directed graph $Q = (I, E)$, with head and tail maps $h, t : E \rightarrow I$. Also, we fix two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$. For $i \in I$, let $V_i = \mathbb{C}^{v_i}$, $W_i = \mathbb{C}^{w_i}$ and consider the space of representations of the quiver Q on the vector space $\bigoplus V_i$ framed by $\bigoplus W_i$.

$$N = \bigoplus_{e \in E} \text{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i \in I} \text{Hom}(V_i, W_i).$$

This big vector space N has a natural action of $G = \prod_i \text{GL}(V_i)$. We form the cotangent bundle T^*N and take the Hamiltonian reduction by the action of G . The resulting space $Y = \Phi^{-1}(0) //_{\chi} G$ is called a **Nakajima quiver variety**. Here we choose $\chi : G \rightarrow \mathbb{C}^\times$ to be given by the product of the determinants. On Y , we have a Hamiltonian action of $T = \prod_i (\mathbb{C}^\times)^{w_i}$ inherited from its action on $\bigoplus W_i$. (In other words, we take $\tilde{G} = G \times T$.)

Note that the space Y is always smooth but $\pi : Y \rightarrow X$ is not always birational. Also, the Hamiltonian torus action does not always have finitely many fixed points.

Here we give two examples of Nakajima quiver varieties.

- Consider a linearly oriented type A_{n-1} -quiver with $\mathbf{v} = (1, \dots, n-1)$, $\mathbf{w} = (0, \dots, 0, n)$:

$$\bullet(V_1) \longrightarrow \bullet(V_2) \longrightarrow \cdots \longrightarrow \bullet(V_{n-1}) \longrightarrow \blacksquare(\mathbb{C}^n)$$

Then $N = \bigoplus_{i=1}^{n-1} \text{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$ with $G = \prod_{i=1}^{n-1} \text{GL}_i$. Then $Y \cong T^*\text{Fl}_n$ with $X = \mathcal{N}_{\text{st}_n}$.

- Another important example is a quiver with one vertex and one self-loop with $V = \mathbb{C}^n$ and $W = \mathbb{C}^r$.

$$\begin{array}{c} \curvearrowright \\ \bullet(\mathbb{C}^n) \longrightarrow \blacksquare(\mathbb{C}^r) \end{array}$$

In this case, Y is the moduli space of rank r , torsion-free sheaves on \mathbb{P}^2 , framed at ∞ with second Chern class n .

3.3. Symplectic resolutions of hypertoric varieties. We will consider when $\pi : Y(A, \alpha, \xi) \rightarrow Y(A, 0, \xi)$ will be a symplectic resolution. So we need to consider the condition that $\mu^{-1}(\xi)^{\alpha\text{-ss}} = \mu^{-1}(\xi)^{\alpha\text{-st}}$. First we will compute their stabilizer group.

Let $(\mathbf{z}, \mathbf{w}) \in \mathbb{C}^{2n}$ and set $J_{\mathbf{z}, \mathbf{w}} := \{j \in \{1, \dots, n\} : z_j \neq 0 \text{ or } w_j \neq 0\}$, then we have

$$\text{Stab}_{\mathbf{z}, \mathbf{w}} \mathbb{T}^d = \ker(\mathbb{T}^d \xrightarrow{A_{J_{\mathbf{z}, \mathbf{w}}}^T} \mathbb{T}^{|J_{\mathbf{z}, \mathbf{w}}|}).$$

Hence by some linear algebra we have

Corollary 3.12 (Coro.2.7 in [Nag21]). *We have:*

- (1) $\text{Stab}_{\mathbf{z}, \mathbf{w}} \mathbb{T}^d$ is finite if and only if $\sum_{j \in J_{\mathbf{z}, \mathbf{w}}} \mathbb{Q} \mathbf{a}_j = \mathbb{Q}^d$;
- (2) $\text{Stab}_{\mathbf{z}, \mathbf{w}} \mathbb{T}^d = 1$ if and only if $\sum_{j \in J_{\mathbf{z}, \mathbf{w}}} \mathbb{Z} \mathbf{a}_j = \mathbb{Z}^d$.

Definition 3.13. In this setting, we call A is unimodular if all $d \times d$ -minors of A are 0 or ± 1 .

Remark 3.14. Note that A is unimodular if and only if B is.

Hence for a unimodular A , we have $\sum_{j \in J} \mathbb{Q} \mathbf{a}_j = \mathbb{Q}^d$ iff $\sum_{j \in J} \mathbb{Z} \mathbf{a}_j = \mathbb{Z}^d$ for $J \subset \{1, \dots, n\}$.

Let A is a unimodular matrix and we define $\mathcal{H}_A := \{H \subset \mathbb{R}^d : H \text{ is generated by some of the } \mathbf{a}_j \text{ and of codimension} = 1\}$. We say α generic if $\alpha \notin \bigcup_{H \in \mathcal{H}_A} H$.

Lemma 3.15 (Lem.2.10 and Coro.2.11 in [Nag21]). *In the case, for any $\alpha \in \mathbb{Z}^d$ and $\xi \in \mathbb{C}^d$, we have $(\mu^{-1}(\xi))^{\alpha\text{-ss}} \neq \emptyset$. If α generic, then $(\mu^{-1}(\xi))^{\alpha\text{-ss}} = (\mu^{-1}(\xi))^{\alpha\text{-st}}$ with free action by \mathbb{T}^d . In particular, if α generic then $X(A, \alpha)$ is $2n - d$ -dimensional smooth Poisson variety and for any ξ , $Y(A, \alpha, \xi)$ is a $2n - 2d$ -dimensional smooth symplectic variety.*

Theorem 3.16 (Thm.2.16 in [Nag21]). *For a unimodular A and generic α and any $\xi \in \mathbb{C}^d$, the morphism*

$$\pi_\xi : Y(A, \alpha, \xi) \rightarrow Y(A, 0, \xi)$$

is a projective symplectic resolution and if $\xi = 0$, then it is conical.

Sketch. First, by $\mu : \mathbb{C}^{2n} \xrightarrow{\Psi} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^d$ with $\Psi : (\mathbf{z}, \mathbf{w}) \mapsto \sum_j z_j w_j e_j$ is flat. Then from dimension counting we get $\mu^{-1}(\xi)$ is of equidimension $2n - d$. As it define by d polynomials, we know that $\mu^{-1}(\xi) \in \mathbb{C}^{2d}$ is a complete intersection and hence Cohen-Macaulay. After showing that the codimension of singular locus ≥ 2 , then $\mu^{-1}(\xi)$ is normal by Serre's condition. Finally we can construct an open subset and show that π_ξ is identity over it which force it is birational. Moreover, the result follows from Lemma 3.15 and the following easy fact (see Proposition 2.15 in [Nag21]):

- If $\pi : Y \rightarrow Y_0$ is projective birational morphism with Y is a nonsingular symplectic variety, then π is a symplectic resolution.

Well done. □

Remark 3.17. Note that we have the more general results. In [Bel23] Lemma 2.4 and Proposition 2.5, without assuming A is unimodular, shows that if we choose α, α' such that $\mu^{-1}(\xi)^{\alpha'\text{-ss}} \subset \mu^{-1}(\xi)^{\alpha\text{-ss}}$, then there exists a projective birational Poisson morphism $Y(A, \alpha', \xi) \rightarrow Y(A, \alpha, \xi)$. Moreover, any hypertoric variety $Y(A, \alpha, \xi)$ has symplectic singularities.

4. BASIC GEOMETRY OF HYPERTORIC VARIETIES

4.1. Hypertoric varieties with hyperplane arrangements. Here we consider the case $\xi = 0$. Then we define $Y(A, \alpha) := Y(A, \alpha, 0)$. It is defined by

$$0 \rightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \rightarrow 0$$

where $A = [\mathbf{a}_1, \dots, \mathbf{a}_n] \in M_{d \times n}(\mathbb{Z})$ and $B^T = [\mathbf{b}_1, \dots, \mathbf{b}_n] \in M_{(n-d) \times n}(\mathbb{Z})$.

Then we can define $H_i := \{x \in \mathbb{R}^{n-d} : x \cdot \mathbf{b}_i + r_i = 0\}$ for $i = 1, \dots, n$ where $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{Z}^n$ be a lifting of α along A . This defines a hyperplane arrangement $\mathcal{A} := \{H_1, \dots, H_n\}$. Here we can denote $Y(\mathcal{A}) := Y(A, \alpha)$.

Definition 4.1. *In this setting, for such hyperplane arrangement \mathcal{A} :*

- we call \mathcal{A} is *simple* if for any subset of m hyperplanes with nonempty intersections, they intersect of codimension m .
- we call \mathcal{A} is *unimodular* if for any $n - d$ linear independent $\{\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_{n-d}}\}$ spans \mathbb{C}^{n-d} over \mathbb{Z} .
- we call \mathcal{A} is *smooth* if it is simple and unimodular.

Remark 4.2. *Note that \mathcal{A} is unimodular if and only if B is unimodular if and only if A is unimodular.*

Proposition 4.3 (3.2/3.3 in [BD00]). *The hypertoric variety $Y(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if \mathcal{A} is simple, and is smooth if and only if \mathcal{A} is smooth.*

Note that $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement, meaning that $r_i = 0$ for all i , so that all of the hyperplanes pass through the origin. Then we have the following result:

Corollary 4.4. *For any central arrangement \mathcal{A} , there exists a simplification $\tilde{\mathcal{A}} = \{\tilde{H}_1, \dots, \tilde{H}_n\}$ of \mathcal{A} by which we mean an arrangement defined by the same vectors $\{\mathbf{b}_i\}$, but with a different choice of α, \mathbf{r} such that $\tilde{\mathcal{A}}$ is simple. This will give us an equivariant orbifold resolution $Y(\tilde{\mathcal{A}}) \rightarrow Y(\mathcal{A})$. When A is unimodular, this will give us a resolution of singularities which recover the special case of Theorem 3.16.*

4.2. The cores and homotopy models. Consider again $\xi = 0$. Then we have an equivariant orbifold resolution

$$\pi : Y(\tilde{\mathcal{A}}) \rightarrow Y(\mathcal{A})$$

where $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement with simplification $\tilde{\mathcal{A}} = \{\tilde{H}_1, \dots, \tilde{H}_n\}$.

Definition 4.5. *In this case, we call $\mathfrak{c}(\tilde{\mathcal{A}}) := \pi^{-1}(0)$ the core of $Y(\tilde{\mathcal{A}})$.*

Now we will give a toric interpretation of the core $\mathfrak{c}(\tilde{\mathcal{A}})$. For any $J \subset \{1, \dots, n\}$, define the polyhedron

$$P_J := \{x \in \mathbb{R}^{n-d} : x \cdot \mathbf{b}_i + r_i \geq 0 \text{ if } i \in J \text{ and } x \cdot \mathbf{b}_i + r_i \leq 0 \text{ if } i \notin J\}.$$

Define

$$\mathfrak{E}_J := \{(\mathbf{z}, \mathbf{w}) \in T^*\mathbb{C}^n : w_i = 0 \text{ if } i \in J \text{ and } z_i = 0 \text{ if } i \notin J\}$$

and define $\mathfrak{X}_J := \mathfrak{E}_J //_{\alpha} \mathbb{T}^d$, which induce the inclusion

$$\mathfrak{X}_J \hookrightarrow \mu^{-1}(0) //_{\alpha} \mathbb{T}^d = Y(\tilde{\mathcal{A}}).$$

Theorem 4.6 (Section 6 in [BD00]/ section 3.2 in [Pro04]). *In this setting, we have:*

- (1) *the scheme \mathfrak{X}_J is isomorphic to the toric variety correspond to the weighted polytope P_J ;*
- (2) *we have $\mathfrak{c}(\tilde{\mathcal{A}}) = \bigcup_{J: P_J \text{ bounded}} \mathfrak{X}_J$, hence $\mathfrak{c}(\tilde{\mathcal{A}})$ is a union of compact toric varieties glued together along toric subvarieties as prescribed by the combinatorics of the polytopes P_J and their intersections in \mathbb{R}^{n-d} .*

Sketch. Note that (1) follows from the surjectivity real moment maps and some classification theorems, see Lemma 3.8 in [Pro04]. For (2), see Proposition 3.11 in [Pro04]. \square

Remark 4.7. *This is right even for $\tilde{\mathcal{A}}$ is not simple.*

Finally we consider some homotopy results.

Theorem 4.8 (6.5 in [BD00] and section 6 in [HS02]). *In this setting, we have:*

- (1) *the core $\mathfrak{c}(\tilde{\mathcal{A}})$ is a deformation retract of $Y(\tilde{\mathcal{A}})$;*
- (2) *the inclusion*

$$Y(\tilde{\mathcal{A}}) = \mu^{-1}(0) //_{\alpha} \mathbb{T}^d \hookrightarrow T^*\mathbb{C}^n //_{\alpha} \mathbb{T}^d = X(\tilde{\mathcal{A}})$$

is a homotopy equivalence where $X(\tilde{\mathcal{A}})$ is the corresponding Lawrence toric variety.

4.3. Universal Poisson structure of hypertoric varieties. In this section we will give a concrete description of universal Poisson structure of hypertoric varieties. At the begining, we consider some general results. Here we will follows [Nag21].

Definition 4.9. *For a Poisson variety $(Y, \{-, -\}_0)$ and an affine scheme $(B, 0)$ with fixed point 0, we call a Poisson B -scheme $(\mathcal{Y}, \{-, -\})$ a **Poisson deformation** of Y if $\mathcal{Y} \rightarrow B$ is flat, each fiber is a Poisson scheme, and the central fiber is isomorphic to $(Y, \{-, -\}_0)$ as a Poisson variety.*

*A Poisson deformation $(\mathcal{Y}, \{-, -\}) \rightarrow B$ is called **infinitesimal** if $B = \text{Spec } A$ where A is an Artinian algebra with residue field \mathbb{C} .*

Definition 4.10. *A Poisson deformation $(\mathcal{Y}, \{-, -\}) \rightarrow B$ of a Poisson variety $(Y, \{-, -\}_0)$ is called **universal** at 0 if for each infinitesimal Poisson deformation $(\mathcal{X}, \{-, -\}') \rightarrow (\text{Spec } A, \mathfrak{m}_A)$ there exists a unique morphism $f : \text{Spec } A \rightarrow B$ such that $f(\mathfrak{m}_A) = 0$ and the diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\quad r \quad} & \mathcal{Y} \\ \downarrow & & \downarrow \\ \text{Spec } A & \xrightarrow{\quad f \quad} & B \end{array}$$

which is cartesian.

In general we have the following:

Theorem 4.11 ([Nam15]). *Let Y_0 be a conical symplectic variety with a projective symplectic resolution $\pi : Y \rightarrow Y_0$. Then there exists the universal Poisson deformation spaces $\mathcal{Y} \rightarrow H^2(Y, \mathbb{C})$ and $\mathcal{Y}_0 \rightarrow H^2(Y, \mathbb{C})/W$ of Y and Y_0 , respectively, and they satisfy the following \mathbb{C}^* -commutative diagram:*

$$\begin{array}{ccccc} & & Y & \xrightarrow{\quad \pi \quad} & Y_0 \\ & \swarrow & \downarrow & \searrow & \downarrow \\ \mathcal{Y} & \xleftarrow{\quad \Pi \quad} & \mathcal{Y}_0 & \xleftarrow{\quad} & \bar{0} \\ \downarrow \bar{\mu} & & \downarrow \bar{\mu}_W & & \downarrow \\ H^2(Y, \mathbb{C}) & \xrightarrow{\quad \psi \quad} & H^2(Y, \mathbb{C})/W & & \bar{0} \end{array}$$

*where ψ is a Galois cover with finite Galois group W acts linearly on $H^2(Y, \mathbb{C})$ which is called the **Namikawa–Weyl group** of Y_0 .*

Some comments. First, the singular locus $(Y_0)_{\text{sing}}$ is stratified by smooth symplectic varieties. Let $\Sigma_{\text{codim} \geq 4}$ denote the union of strata of codimension 4 or higher, and define $\Sigma_{\text{codim} 2} := (Y_0)_{\text{Sing}} \setminus \Sigma_{\text{codim} \geq 4}$. Then, for each component Z_k of the connected component decomposition $\Sigma_{\text{codim} 2} = \bigsqcup_{k=1}^s Z_k$, one can consider a transversal slice S_{ℓ_k} through a point $x \in Z_k$. Since $S_{\ell_k} = S_{\Delta_{\ell_k}}$ is a symplectic surface, i.e., the *ADE* type surface singularity with the corresponding Dynkin diagram Δ_{ℓ_k} , so $\pi : Y \rightarrow Y_0$ is locally (at x) isomorphic to $p \times \text{id} : \tilde{S}_{\ell_k} \times \mathbb{C}^{2m-2} \rightarrow S_{\ell_k} \times \mathbb{C}^{2m-2}$, where $2m = \dim Y_0$ and p is the minimal resolution of S_{ℓ_k} . We consider all (-2) -curves C_i ($1 \leq i \leq \ell_k$) in \tilde{S}_{ℓ_k} and set

$$\Phi_{\ell_k} := \left\{ \sum_{i=1}^{\ell_k} d_i [C_i] \mid d_i \in \mathbb{Z} \text{ s.t. } \left(\sum_{i=1}^{\ell_k} d_i [C_i] \right)^2 = -2 \right\} \subset H^2(\tilde{S}_{\ell_k}, \mathbb{R}).$$

Then, Φ_{ℓ_k} defines the corresponding *ADE* type root system in $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$, and the associated usual Weyl group $W_{S_{\ell_k}}$ acts on $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$. However this description is local at each point on Z_k , and the

number of irreducible components of $\pi^{-1}(Z_k)$ may be less than ℓ_k globally. In fact, the following homomorphism is defined by the monodromy:

$$\rho_k : \pi_1(Z_k) \rightarrow \text{Aut}(\Delta_{\ell_k}),$$

where Δ_{ℓ_k} is the associated Dynkin diagram and $\text{Aut}(\Delta_{\ell_k})$ is its graph automorphism group. Then, we can define the subgroup of $W_{S_{\ell_k}}$ as

$$W_{Z_k} := W_{S_{\ell_k}}^{\text{Im } \rho_k} := \{\sigma \in W_{S_{\ell_k}} \mid \sigma \iota = \iota \sigma^2 (\iota \in \text{Im } \rho_k)\}.$$

Finally, taking the direct product of them, we get the Namikawa-Weyl group:

$$W := \prod_k W_{Z_k}.$$

Well done. □

In our case of hypertoric varieties,

Theorem 4.12 (Thm 3.11 in [Nag21]). *Let A be a unimodular matrix and $\alpha \in \mathbb{Z}^d$ be a generic element. If for B , $\mathbf{b}_i \neq 0 (1 \leq i \leq n)$ and we take B as*

$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(s)} \end{pmatrix}, \quad B^{(k)} = \begin{pmatrix} \mathbf{b}^{(k)} \\ \mathbf{b}^{(k)} \\ \vdots \\ \mathbf{b}^{(k)} \end{pmatrix} \Bigg\} \ell_k$$

where if $k_1 \neq k_2$, then $\mathbf{b}^{(k_1)} \neq \pm \mathbf{b}^{(k_2)}$. Then the diagram of Theorem 4.11 for the affine hypertoric variety $Y(A, 0)$ is obtained as

$$\begin{array}{ccccc} Y(A, \alpha) & \xrightarrow{\pi} & Y(A, 0) & & \\ \swarrow & & \swarrow & & \downarrow \\ X(A, \alpha) & \xrightarrow{\Pi_{W_B}} & X(A, 0)/W_B & & \downarrow \\ \downarrow \bar{\mu}_\alpha & & \downarrow & & \downarrow \\ & 0 & \xrightarrow{\quad} & \bar{0} & \\ \swarrow & & \swarrow & & \downarrow \\ \mathbb{C}^d & \xrightarrow{\psi} & \mathbb{C}^d/W_B & & \end{array}$$

where Π_{W_B} is the composition of $X(A, \alpha) \rightarrow X(A, 0)$ and the quotient map of $X(A, 0)$ by $W_B := \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_s}$.

Sketch. First we need to show that $\bar{\mu}_\alpha : X(A, \alpha) \rightarrow \mathbb{C}^d$ and $\bar{\mu}_0 : X(A, 0) \rightarrow \mathbb{C}^d$ are Poisson deformations of $Y(A, \alpha)$ and $Y(A, 0)$, respectively. Note that $X(A, \alpha)$ is smooth and $X(A, 0)$ is Cohen-Macaulay by a result due to Hochster, then by miracle-flatness $\bar{\mu}_\alpha$ and $\bar{\mu}_0$ are flat. Then these are right by definition.

Next we need to analyze the structure of $\Sigma_{\text{codim}2}$ in order to describe the Namikawa-Weyl group. Note that in this case we already have the following diagram:

$$\begin{array}{ccccc} Y(A, \alpha) & \xrightarrow{\pi} & Y(A, 0) & & \\ \swarrow & & \swarrow & & \downarrow \\ X(A, \alpha) & \xrightarrow{\Pi} & X(A, 0) & & \downarrow \\ \downarrow \bar{\mu}_\alpha & & \downarrow & & \downarrow \\ & 0 & \xrightarrow{\quad} & \bar{0} & \\ \swarrow & & \swarrow & & \downarrow \\ \mathbb{C}^d & \xrightarrow{=} & \mathbb{C}^d/W_B & & \end{array}$$

If one can construct a good W_B -action on $X(A, 0)$ and \mathbb{C}^d , then one can show $W = W_B$ and construct the universal Poisson deformation of $Y(A, 0)$.

Need to add. □

5. WALL-CROSSING STRUCTURES, MUKAI FLOPS AND COUNTING CREPENT RESOLUTIONS

5.1. Mukai flops and its family-version.

5.2. Wall-crossing structures of hypertoric varieties.

5.3. An application: counting their projective crepent resolutions.

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