Moduli Spaces of Coherent Sheaves and its Related Topics

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Contents

1 Introductions

[2][5][6][7][8]

2 Some Preliminaries

Here we will introduce some basic background about good moduli theory due to J. Alper in [2] and its relation to the GIT-theory. We will also use the theory of Θ-complete and S-complete in [3]. This will play an important role in our foundamental theory.

2.1 Theory of Good Moduli Spaces

As we all know, in the modern construction of the moduli space of stable curves follows from the following way:

(a) Construct the stack $\overline{\mathscr{M}}_{g,n}$ and show that it is a Deligne-Mumford stack;

(b) show the stable-reduction of stable curves and find that $\overline{\mathscr{M}}_{g,n}$ is proper;

(c) use Keel-Mori theorem to construct the coarse moduli space $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ and show that it is projective.

But in our case, we can not use Keel-Mori theorem to the moduli stack of semistable sheaves because the inertia stack $\mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ is not finite. In order to this the similar modern way (instead of GIT-construction), J.Alper developed a nice similar (but much more complicated) theory to solve this problem – the theory of good moduli space ([2] and [3]) for linear reductive groups and the theory of adequate moduli spaces ([1]) for geometric reductive groups.

For now, the theory of good moduli space plays a central role in the construction of moduli spaces, such as moduli stack of semistable sheaves $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ and K-moduli stack $\mathscr{X}_{n,V}^{\mathrm{Kss}}$ which aim to construct a good moduli space of Fano varieties (see the book draft due to C. Xu).

Definition 2.1 (Good moduli space)**.** *For an algebraic stack $\mathscr{X}$, its good moduli space is an algebraic space X together with a qcqs morphism $\pi : \mathscr{X} \to X$ such that*

(i) the natural map $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism;

(ii) the functor $\pi_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact.

Note that the condition in (ii) is called cohomologically affine.

The definition of good moduli space is inspired from the GIT-quotient of linear reductive group G (that is, $V \mapsto V^G$ is exact)

$$[X/G] \dashrightarrow [X^{\mathrm{ss}}/G] \to X//G = \mathrm{Proj} \bigoplus_{d\geq 0} \Gamma(X, \mathscr{O}_X(d))^G.$$

Or locally, the map $[\mathrm{Spec}\, A/G] \to \mathrm{Spec}\, A^G$. Of coarse, a tame coarse moduli space is a good moduli space by the local structure of coarse moduli spaces.

Here we state several basic properties of cohomologically affine morphisms.

Lemma 2.2.

Proof. □

Now some important properties of good moduli spaces and give some comments. Actually these are similar as the properties of GIT.

Theorem 2.3. *Let $\pi : \mathscr{X} \to X$ be a good moduli space where $\mathscr{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(i) π is surjective and universally closed (and universally submersive);

(ii) for closed substacks $\mathscr{Z}_1, \mathscr{Z}_2 \subset \mathscr{X}$, we have $\mathrm{Im}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \mathrm{Im}(\mathscr{Z}_1) \cap \mathrm{Im}(\mathscr{Z}_2)$. For geometric points $x_1, x_2 \in \mathscr{X}(k)$, $\pi(x_1) = \pi(x_2) \in \mathscr{X}(k)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_1\}} \neq \emptyset$ in $|\mathscr{X} \times_S k|$. In particular, π induces a bijection between closed points in $\mathscr{X}$ and closed points in X;

(iii) if $\mathscr{X}$ is noetherian, so is X. If $\mathscr{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence;

(iv) If X is noetherian, then π is universal for maps to algebraic spaces.

Proof. □

2.2 GIT Theory, Good Moduli Space and Some Applications

3 Moduli Stack of Coherent Sheaves

In the next sections we will just consider the stacks over $\mathbb{C}$. Now we consider the moduli space of coherent sheaves over some smooth projective complex variety X. Then we have the Chern character map

$$\gamma : K(X) \xrightarrow{\mathrm{ch}} \mathrm{CH}^*(X)_{\mathbb{Q}} \xrightarrow{\mathrm{cl}} H^{2*}(X, \mathbb{Q}).$$

(or we can use ℓ-adic cohomology) Let Γ be the image of this map.

By Grothendieck-Riemann-Roch theorem (see Chpter 15 in [4]),

$$P = \chi(\mathscr{F}(m)) = \int_X \mathrm{ch}(\mathscr{F}(m))\mathrm{td}(\mathcal{T}_X),$$

then we find that the information of $v \in \Gamma$ is equivalent to the information of the Hilbert polynomial χ. So we can use both of them when X is smooth. If X is just a projective scheme, then we will only to use the Hilbert polynomial.

Theorem 3.1. *Let X be a connected projective $\mathbb{C}$-scheme, we let $\underline{\mathrm{Coh}}_P(X)$ the category fibred in groupoid over $\mathrm{Sch}/\mathbb{C}$ sending a $\mathbb{C}$-scheme T to the groupoid of T-flat families $\mathscr{E} \in \mathrm{Coh}(X \times T)$ such that any restriction $\mathscr{E}_t \in \mathrm{Coh}(X)$ has the Hilbert polynomial P, the morphisms in the above groupoid are given by isomorphisms of $\mathscr{E}$.*

Then $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$ of affine diagonal. Also, we have the algebraic stack $\underline{\mathrm{Coh}}(X) = \coprod_P \underline{\mathrm{Coh}}_P(X)$.

Proof. Easy to see that $\underline{\mathrm{Coh}}_P(X)$ is actually a stack, we first claim that it is an algebraic stack in a natural way.

For each integer N, we claim there is an open substack $\mathscr{U}_N \subset \underline{\mathrm{Coh}}_P(X)$ parameterizing coherent sheaves $\mathscr{E}$ such that $\mathscr{E}(N)$ generated by global sections and $H^i(X, \mathscr{E}(N)) = 0$ for any $i > 0$. Actually this is trivial by some application of cohomology and base change. As $\underline{\mathrm{Coh}}_P(X) = \bigcup_N \mathscr{U}_N$, we just need to show $\mathscr{U}_N$ is an algebraic stack locally of finite type over $\mathbb{C}$.

For each N, we consider the quotient scheme

$$Q_N := \underline{\mathrm{Quot}}_X^P(\mathscr{O}_X(-N)^{P(N)}).$$

Again by some application of cohomology and base change, we find that there is an open subscheme $Q'_N \subset Q_N$ parameterizing quotients $q : \mathscr{O}_X(-N)^{P(N)} \twoheadrightarrow \mathscr{F}$ such that $H^0(q(N))$ is surjective and $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$.

We have a natural map $Q'_N \to \mathscr{U}_N$ maps $[\mathscr{O}_X(-N)^{P(N)}]$ to $\mathscr{F}$. We observe that Q'_N is also $\mathrm{GL}_{P(N)}$-invariant, then this map descends to

$$\Psi^{\mathrm{pre}} : [Q'_N/\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathscr{U}_N$$

which is fully faithful since every automorphism of a coherent sheaf $\mathscr{E}$ on $X \times S$ induces an automorphism of $p_{2,*}\mathscr{E}(N) = \mathscr{O}_S^{P(N)}$ i.e. an element of $\mathrm{GL}_{P(N)}(S)$, and this element acts on $\mathscr{O}_X(-N)^{P(N)}$ preserving the quotient $\mathscr{E}$.

After stackification, we have another fully faithful map $\Psi : [Q'_N/\mathrm{GL}_{P(N)}] \to \mathscr{U}_N$ which is also essentially surjective by the constructions. Hence we have

$$\mathscr{U}_N \cong [Q'_N/\mathrm{GL}_{P(N)}], \quad \underline{\mathrm{Coh}}_P(X) = \bigcup_N [Q'_N/\mathrm{GL}_{P(N)}].$$

Hence $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$. $\square$

4 Basic Theory of Semistable Sheaves

For polynomials $f_i \in \mathbb{Q}[m]$ for $i = 1, 2$, we write $f_1 \prec (\preceq) f_2$ if either $\deg f_1 > \deg f_2$ or $\deg f_1 = \deg f_2$ with $f_1(m) < (\leq) f_2(m)$ for $m \gg 0$.

Definition 4.1. *We call $\mathscr{E}$, a coherent sheaf over a complex projective variety X, is H-(semi)stable if for any non-zero subsheaf $\mathscr{E}' \subsetneq \mathscr{E}$ we have $\overline{\chi}(\mathscr{E}'(m)) \prec (\preceq) \overline{\chi}(\mathscr{E}(m))$ where $\overline{\chi}(\mathscr{E}(m)) = \chi(\mathscr{E}(m))/a_d$ be the reduced Hilbert polynomial where a_d be the top-degree coefficient of $\chi(\mathscr{E}(m))$ with $H = c_1(\mathscr{O}_X(1))$.*

We call $\mathscr{E}$ is strictly H-semistable if it is H-semistable but not H-stable. We call it is H-polystable if it is H-semistable and a direct sum of H-stable sheaves.

As we all know, these (semi)stable sheaves form the building blocks for all the coherent sheaves on X as the following two results.

Theorem 4.2 (Harder-Narasimhan filtration). *Let $\mathscr{E}$ be a coherent sheaf over a complex projective variety X. then there is a filtration (unique up to isomorphism)*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E}$$

with quotient $\mathscr{F}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ such that each $\mathscr{F}_i$ is H-semistable with $\overline{\chi}(\mathscr{F}_i(m)) \succ \overline{\chi}(\mathscr{F}_{i+1}(m))$ for all i.

Proof. See Theorem 1.3.4 in [6]. $\square$

Theorem 4.3 (Jordan-Hölder filtration). *Let $\mathscr{E}$ be a coherent sheaf over a complex projective variety X. then there is a filtration (the direct sum of $\mathscr{F}_i$ unique up to isomorphism)*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E}$$

with quotient $\mathscr{F}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ such that each $\mathscr{F}_i$ is H-stable with $\overline{\chi}(\mathscr{F}_i(m)) = \overline{\chi}(\mathscr{E}(m))$ for all i.

Proof. See Theorem 1.5.2 in [6]. $\square$

But this is not enough as $\mathscr{M}_X(v)$ is neither of finite type nor separated! Now we consider the open substacks of H-(semi)stable sheaves:

$$\mathscr{M}_X^{H-st}(v) \subset \mathscr{M}_X^{H-ss}(v) \subset \mathscr{M}_X(v).$$

By Proposition 2.3.1, Theorem 3.3.7 in [6], we get these two stacks are all of finite type. Hence now we can use the theory of good moduli space as we introduced.

Due to Jarod Alper in Example 8.7 at [2]: there exists the Cartesian diagram

$$\begin{tikzcd} \mathscr{M}_X^{H-st}(v) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{M}_X^{H-ss}(v) \arrow[d] \\ M_X^{H-st}(v) \arrow[r, hook] & M_X^{H-ss}(v) \end{tikzcd}$$

where $M_X^{H-ss}(v)$ be a projective scheme parametrizing H-polystable sheaves and $M_X^{H-st}(v)$ be an open subscheme corresponding to H-stable sheaves. The vertical arrows are good moduli space morphisms, sending an H-semistable sheaf to the associated graded sheaf of the Jordan-Hölder filtration. We know that the automorphism of stable sheaves is $\mathbb{C}^*$, hence the left morphism above is a $\mathbb{C}^*$-gerbe.

Definition 4.4. *We say $M_X^{H-st}(v)$*

(i) is a fine moduli space *if $\mathscr{M}_X^{H-st}(v)$ is a trivial $\mathbb{C}^*$-gerbe over $M_X^{H-st}(v)$;*

(ii) satisfies the ss = st condition if the equality $M_X^{H-ss}(v) = M_X^{H-st}(v)$ holds.

Remark 4.5. *(i) If $\gcd\{\chi(E\otimes F) : [E] \in M_X^{H-st}(v), F \in K(X)\} = 1$, then $M_X^{H-st}(v)$ s a fine moduli space by [6] Theorem 4.6.5;*

(ii) Actually $\mathscr{M}_X^{H-st}(v)$ is a trivial $\mathbb{C}^*$-gerbe over $M_X^{H-st}(v)$ is that there is a universal sheaf over $X \times M_X^{H-st}(v)$. Also, by the basic theory of algebraic gerbes, this is equivalent to $\mathscr{M}_X^{H-st}(v) \cong M_X^{H-st}(v) \times B\mathbb{C}^*$;

(iii) Let $v = (v_i) \in \Gamma$ with $v_0 = 1$, then any H-stable sheaf $\mathscr{E}$ of $\mathrm{ch}(\mathscr{E}) = v$ is of form $\mathscr{E} \cong \mathscr{L} \otimes \mathscr{I}_Z$ for a line bundle $\mathscr{L}$ on X and a closed subscheme Z with $\mathrm{codim}_X Z \geq 2$. Then

$$\mathrm{Pic}_X(v_1) \times \mathrm{Hilb}_X(e^{-v_1}v) \cong M_X^{H-st}(v)$$

and $M_X^{H-st}(v)$ is fine with ss = st condition. Here $\mathrm{Pic}_X(v_1)$ is the Picard scheme parameterizing line bundles with $\mathrm{c}_1(\mathscr{L}) = v_1$ and $\mathrm{Hilb}_X(v)$ is the Hilbert scheme parameterizing closed subschemes $Z \subset X$ with $\mathrm{ch}(I\mathscr{I}_Z) = v$.

5 Moduli Stacks and Spaces of Semistable Sheaves

6 Bridgeland Stability

7 Need to add

Index

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