A Quick Tour of Derived Algebraic Geometry

Xiaolong Liu

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Contents

1 Introduction

We will follows [EP23] and Joyce's slides as in `https://people.maths.ox.ac.uk/~joyce/DAG2022/index.html`.

2 Basic Concepts of Infinity Categories

In this lecture '∞-category' always means '$(\infty, 1)$-category', that is, all n-morphisms are invertible for $n \geqslant 2$. (Although 'n-morphism' may not make sense, depending on your model for ∞-categories.)

There are a bunch of different but related structures which are more-or-less kinds of ∞-category:

- Model categories.
- Categories enriched in topological spaces.
- Simplicial categories; simplicial model categories.
- Quasicategories.

Of these, model categories are the oldest (Quillen 1967), and look least like an ∞-category (they have no visible higher morphisms). But most of the other kinds of ∞-category use model categories under the hood. Toën-Vezzosi's DAG is written in terms of model categories and simplicial categories. Lurie works with quasicategories, which may be the best and coolest version.

- If you start with an ordinary category $\mathcal{C}$ and invert some class of morphisms $\mathcal{W}$ in $\mathcal{C}$ ('weak equivalences'), the result $\mathcal{C}[\mathcal{W}^{-1}]$ should really be an ∞-category with homotopy category $\mathsf{Ho}(\mathcal{C}[\mathcal{W}^{-1}])$ an ordinary category.

This idea is similar as derived category $\mathbf{D}(\mathcal{A})$ construct from $\mathsf{Ho}(\mathrm{Com}(\mathcal{A}))$ by inverting the class $\mathcal{W}$ of quasi-isomorphisms. Note that $\mathbf{D}(\mathcal{A}) = \mathsf{Ho}(\mathbb{D}(\mathcal{A}))$ for a stable ∞-category $\mathbb{D}(\mathcal{A})$.

Here we give some idea how to consider the ∞-category. We know that a $(2,1)$-category $\mathfrak{C}$ is acategory enriched in groupoids. A $(3,1)$-category $\mathfrak{C}$ is acategory enriched in 2-groupoids. So similarly, an $(\infty,1)$-category is really a 'category enriched in ∞-groupoids'. But what is an ∞-groupoid?

Two models for the (model/∞-)category of ∞-groupoids are topological spaces **Top** (up to homotopy), and simplicial sets **sSets**. Note that **Top** and **sSets** are Quillen equivalent as model categories, theories of ∞-categories based on **Top** and **sSets** are essentially equivalent. But it seems no one uses categories enriched in **Top** except as motivation.

2.1 Categories Enriched in Topological Spaces

Our first model for an $(\infty,1)$-category is the categories enriched in topological spaces:

Definition 2.1. *A category enriched in topological spaces is a category $\mathcal{C}$ such that for all objects X, Y in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms $f : X \to Y$ is given the structure of a topological space (generally a nice topological space, e.g. Hausdorff,..., and homotopy equivalent to a CW complex), and for objects X, Y, Z the composition $\mu_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is a continuous map. Moreover, there is a homotopy between $\mu_{W,X,Y} \circ (\mu_{W,X,Y} \times \mathrm{id}) \to \mu_{W,X,Y} \circ (\mathrm{id} \times \mu_{X,Y,Z})$.*

The higher-morphism of $\mathcal{C}$ defined as follows:

- *A 1-morphism $f : X \to Y$ is a point of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.*
- *If $f, g : X \to Y$ are 1-morphisms, a 2-morphism $\eta : f \Rightarrow g$ is a continuous path $\eta : [0,1] \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ with $\eta(0) = f$ and $\eta(1) = g$. Note that η is invertible with $\eta^{-1}(s) = \eta(1-s)$.*
- *If $\eta, \zeta : f \Rightarrow g$ are 2-morphisms, a 3-morphism $\aleph : \eta \Rrightarrow \zeta$ is a continuous map $\aleph : [0,1]^2 \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that for $s, t \in [0,1]$ we have*

$$\aleph(0,t) = f, \aleph(1,t) = g, \aleph(s,0) = \eta, \aleph(s,1) = \zeta.$$

- *n-morphisms are continuous maps $[0,1]^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ with prescribed boundary conditions on $\partial([0,1]^{n-1})$.*
- *Moreover, if $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms, the vertical composition $\zeta \odot \eta : f \Rightarrow h$ is $(\zeta \odot \eta)(s) = \eta(2s)$ if $s \in [0,1/2]$ and $(\zeta \odot \eta)(s) = \zeta(2s-1)$ if $s \in [1/2,1]$. This is not associative, but is associative up to homotopy, i.e. up to 3-isomorphism. Other kinds of composition can be defined in a similar way.*

Definition 2.2. *For any higher category $\mathcal{C}$ (as we will used), the homotopy category $\mathsf{Ho}(\mathcal{C})$ which is an ordinary category, where objects X of $\mathsf{Ho}(\mathcal{C})$ are objects of $\mathcal{C}$, and morphisms $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathsf{Ho}(\mathcal{C})$ are 2-isomorphism classes of 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{C}$.*

Now for a category enriched in topological spaces $\mathcal{C}$. we have $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{C})}(X,Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y))$.

2.2 Model Categories

Now we introduce some model categories invented by Quillen to abstract methods of homotopy theory into category theory.

Definition 2.3. *A model category is a complete and cocomplete category $\mathcal{M}$ equipped with three distinguished classes of morphisms: the weak equivalences $\mathcal{W}$, the fibrations $\mathcal{F}$, and the cofibrations $\mathcal{C}$. These must satisfy:*

(a) $\mathcal{W}, \mathcal{F}, \mathcal{C}$ *are closed under composition and include identities.*

(b) $\mathcal{W}, \mathcal{F}, \mathcal{C}$ *are closed under retracts. Here* f *is a retract of* g *if there exist* i, j, r, s *such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "f"] & Y \arrow[r, "r"'] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}_{X'}"'] & Y' \arrow[r, "s"] & X' \end{tikzcd}$$

(c) For $f : X \to Y, g : Y \to Z$ *in* $\mathcal{M}$*, if two of* $f, g, g \circ f$ *are in* $\mathcal{W}$ *then so is the third.*

(d) A (co)fibration which is also a weak equivalence is called acyclic*. Acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with respect to acyclic fibrations. Explicitly, if the square below commutes, where* i *is a cofibration,* p *is a fibration, and* i *or* p *is acyclic, then there exists* h *as shown:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

(e) Every morphism f *in* $\mathcal{M}$ *can be written as* $f = p \circ i$ *for a fibration* p *and an acyclic cofibration* i*.*

(f) Every morphism f *in* $\mathcal{M}$ *can be written as* $f = p \circ i$ *for an acyclic fibration* p *and a cofibration* i*.*

Some basic elements in a model category:

Definition 2.4. *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ *be a model category with initial object* $\emptyset$ *and final object* $*$*.*

(a) An object $X \in \mathcal{M}$ *is called* fibrant *if* $[X \to *] \in \mathcal{F}$*, and* cofibrant *if* $[\emptyset \to X] \in \mathcal{C}$*.*

(b) If $X \in \mathcal{M}$ *and there is a weak equivalence* $w : C \to X$ *with* C *cofibrant then* C *is a* cofibrant replacement *for* X*. If there is a weak equivalence* $w : X \to F$ *with* F *fibrant then* F *is a* fibrant replacement *for* X*. Such replacements always exist.*

(c) If $X \in \mathcal{M}$*, a* cylinder object *is an object* $X \times [0, 1]$ *in* $\mathcal{M}$ *with a factorization* $X \sqcup X \xrightarrow{c} X \times [0, 1] \xrightarrow{w} X$ *of the codiagonal* $X \sqcup X \to X$*, with* c *a cofibration and* w *a weak equivalence. Cylinder objects exist by Definition 2.3(d).*

(d) If $X \in \mathcal{M}$*, a* path object *is an object* $\mathrm{Map}([0, 1], X)$ *in* $\mathcal{M}$ *with a factorization* $X \xrightarrow{w} \mathrm{Map}([0, 1], X) \xrightarrow{f} X \times X$ *of the diagonal* $X \to X \times X$*, with* w *a weak equivalence and* f *a fibration. Path objects exist by Definition 2.3(e).*

(e) Morphisms $f, g : X \to Y$ *are called* (left) homotopy equivalent *if there exists* $h : X \times [0, 1] \to Y$ *with* $h \circ c = f \sqcup g$ *where* c *as in (c).*

(f) The homotopy category *is* $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$*, the category obtained by formally inverting all weak equivalences. Note that this is independent of* $\mathcal{C}, \mathcal{F}$*.*

Note that two of $\mathcal{W}, \mathcal{F}, \mathcal{C}$ determine the third. Now we introduce an important theorem:

Theorem 2.5 (Fundamental Theorem of Model Categories)**.** *Let* $\mathcal{M}$ *be a model category. Then* $\mathrm{Ho}(\mathcal{M})$ *is equivalent to the category whose objects are fibrant-cofibrant objects in* $\mathcal{M}$*, and whose morphisms are homotopy classes of morphisms in* $\mathcal{M}$*.*

Example 2.6. *(a) The category* **Top** *of topological spaces has a model structure with* $\mathcal{W}$ *the weak homotopy equivalences, and* $\mathcal{F}$ *the Serre fibrations (maps with the homotopy lifting property for CW complexes). In this case by Theorem 2.5,* $\mathrm{Ho}(\mathbf{Top})$*, which is the homotopy category of homotopy types, can be described as the category whose objects are CW complexes and morphisms are homotopy classes of continuous maps.*

(b) *If R is a commutative ring then* $\mathrm{Com}(\mathrm{Mod}_\mathsf{R})$ *has two canonical model structures with weak equivalences quasi-isomorphisms and*

- *cofibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *injective for all* k;
- *fibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *surjective for all* k.

The first one is the injective model category and the second one is the projective model category. In this case by Theorem 2.5, $\mathsf{Ho}(\mathrm{Com}(\mathrm{Mod}_\mathsf{R})) = \mathbf{D}(\mathsf{R})$, *can be described as the category whose objects are either K-injective, or K-projective, complexes, and morphisms are homotopy classes of maps between these complexes.*

(c) *Let $\mathcal{A}$ be a Grothendieck abelian category (such as* $\mathrm{Qcoh}(\mathsf{X})$ *for any scheme* X*). Then* $\mathrm{Com}(\mathcal{A})$ *has a model structure with weak equivalences quasi-isomorphisms and cofibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *injective for all* k. *In this case by Theorem 2.5,* $\mathsf{Ho}(\mathrm{Com}(\mathcal{A})) = \mathbf{D}(\mathcal{A})$, *can be described as the category whose objects are either K-injective complexes, and morphisms.*

Next we consider some another contructions about model categories.

Definition 2.7 (Derived Functors)**.** *Let* $(\mathcal{C}, \mathcal{W})$ *and* $(\mathcal{D}, \mathcal{V})$ *be a relative category (that is, a category with a subcategory of weak equivalences). Let* $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ *be a functor and we say* $\mathbf{R}\mathsf{F} : \mathsf{Ho}(\mathcal{C}) \to \mathsf{Ho}(\mathcal{D})$ *be a right-derived functor of* F *we have*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathsf{F}"] \arrow[d, "\lambda_\mathcal{C}"'] & \mathcal{D} \arrow[d, "\lambda_\mathcal{D}"] \arrow[dl, phantom, "\overset{\eta}{\Longleftarrow}"] \\ \mathsf{Ho}(\mathcal{C}) \arrow[r, "\mathbf{R}\mathsf{F}"'] & \mathsf{Ho}(\mathcal{D}) \end{tikzcd}$$

then any natural transformation $\lambda_\mathcal{D} \circ \mathsf{F} \to \mathbf{R}\mathsf{F} \circ \lambda_\mathcal{C}$ *factors through* η, *and this factorisation is unique up to natural isomorphism in* $\mathsf{Ho}(\mathcal{D})$ *— this condition ensures that* $\mathbf{R}\mathsf{F}$ *is unique up to weak equivalence. Similarly for left-derived functor.*

Definition 2.8. *A functor* $\mathsf{G} : \mathcal{C} \to \mathcal{D}$ *of model categories is right Quillen if it has a left adjoint* F *and preserves fibrations and trivial fibrations. Dually,* F *is left Quillen if it has a right-adjoint and* F *preserves cofibrations and trivial cofibrations.* $\mathsf{F} \dashv \mathsf{G}$ *is in that case called a Quillen adjunction. A Quillen adjunction* $\mathsf{F} \dashv \mathsf{G}$ *is said to be a Quillen equivalence if* $\mathbf{R}\mathsf{G} : \mathsf{Ho}(\mathcal{C}) \to \mathsf{Ho}(\mathcal{D})$ *is an equivalence of categories, in which case it has quasi-inverse* $\mathbf{L}\mathsf{F}$.

Lemma 2.9. *Let* $\mathsf{F} \dashv \mathsf{G}$ *be a adjunction of functors of model categories, then* F *is left Quillen is and only if* G *is right Quillen.*

Proof. Easy from lifting properties. □

Theorem 2.10 (Quillen)**.** *If* G *is right Quillen, then the right-derived functor* $\mathbf{R}\mathsf{G}$ *exists and is given on objects by* $\mathsf{A} \mapsto \mathsf{G}\widehat{\mathsf{A}}$, *for* $\mathsf{A} \to \widehat{\mathsf{A}}$ *a fibrant replacement. Dually, left Quillen functors give left-derived functors by cofibrant replacement.*

Remark 2.11. *To get a functor, we can take fibrant replacements functorially, but on objects the choice of fibrant replacement doesn't matter (and in particular need not be functorial), because it turns out that right Quillen functors preserve weak equivalences between fibrant objects. The proof is an exercise with path objects.*

Next we consider another construction.

Definition 2.12. *Let* $\mathcal{C}$ *be a model category with a small category* I. *Then we have constant functor* $\Delta : \mathcal{C} \to \mathcal{C}^\mathrm{I}$. *This desecend to* $\Delta : \mathsf{Ho}(\mathcal{C}) \to \mathsf{Ho}(\mathcal{C}^\mathrm{I})$ *where now* $\mathcal{C}^\mathrm{I}$ *has two model structures, projective one and injective one (see Section A.2.8 in [Lur09]).*

Now $\Delta : \mathcal{C} \to \mathcal{C}^\mathrm{I}$ *is left Quillen for the injective model structure on* $\mathcal{C}^\mathrm{I}$, *and right Quillen for the projective model structure. So* $\Delta : \mathsf{Ho}(\mathcal{C}) \to \mathsf{Ho}(\mathcal{C}^\mathrm{I})$ *possesses both a right and a left adjoint, called the homotopy limit and homotopy colimit* $\mathrm{Holim}_\mathrm{I}, \mathrm{Hocolim}_\mathrm{I} : \mathsf{Ho}(\mathcal{C}^\mathrm{I}) \to \mathsf{Ho}(\mathcal{C})$.

Here are some motivation:

Example 2.13. *In* **Top**, *fibre products* $X \times_{f,Z,g} Y$ *and pushouts* $X \sqcup_{f,Z,g} Y$ *exist, but if we change* f, g *by homotopies, then* $X \times_{f,Z,g} Y$, $X \sqcup_{f,Z,g} Y$ *need not stay homotopy equivalent. So fibre products and pushouts in* **Top** *are the wrong idea for homotopy theory. Instead one defines the homotopy fibre product and homotopy pushout as before. In this case, we have*

$$X \times^h_{f,Z,g} Y := X \times_{f,Z,\mathrm{ev}_0} \mathrm{Map}([0,1], Z) \times_{\mathrm{ev}_1,Z,g} Y$$

and

$$X \sqcup^h_{f,Z,g} Y := X \sqcup_{f,Z,\iota_0} ([0,1] \times Z) \sqcup_{\iota_1,Z,g} Y.$$

For example $\{y\} \times^h_Y \{y\} = \Omega(Y; y)$ *be the loop space.*

What do these mean? Let $\mathcal{C}$ be a model category, in general homotopy limits and colimits are not limits and colimits (do not satisfy universal properties) in either the ordinary categories $\mathcal{C}$, or the ordinary categories $\mathrm{Ho}(\mathcal{C})$. Instead, the correct interpretation is that there are secretly $(\infty, 1)$-categories $\mathcal{C}^\infty$ with homotopy categories $\mathrm{Ho}(\mathcal{C})$, and homotopy limits and colimits are actually ∞-category limits and colimits in $\mathcal{C}^\infty$. Thus model category techniques effectively give ways to do constructions in an ∞-category, without defining ∞-categories.

- Model categories are strict forms of $(\infty, 1)$-categories.

That is, composition of morphisms in a model category is strictly associative (other notions of $(\infty, 1)$-category have composition non-associative, or even not uniquely defined). There are 'strictification theorems' which allow you to pass from looser forms of $(\infty, 1)$-categories (e.g. Segal categories) to model categories. Typically, general constructions are done in the looser kinds of $(\infty, 1)$-category, and explicit computations are done in model categories.

2.3 Simplicial Sets and Simplicial Categories

Definition 2.14. *(a) The simplex category* Δ *has objects set* $[n] = \{0, 1, ..., n\}$ *for* $n \geqslant 0$ *and morphisms* $f : [n] \to [m]$ *the order preserving functions, that is, if* $0 \leqslant i \leqslant j \leqslant n$ *then* $f(i) \leqslant f(j)$.

A simplicial set is a functor $F : \Delta^{\mathrm{op}} \to \mathbf{Sets}$. *A morphism of simplicial sets* $\eta : F \to G$ *is a natural transformation of functors* $\eta : F \Rightarrow G$. *This makes simplicial sets into a category* **sSets**.

(b) The topological n*-simplex* Δ^n_{top} *is*

$$\Delta^n_{\mathrm{top}} := \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} : x_i \geqslant 0, x_0 + \cdots + x_n = 1\}.$$

If $f : [n] \to [m]$ *is order-preserving we define* $f_{\mathrm{top}} : \Delta^m_{\mathrm{top}} \to \Delta^n_{\mathrm{top}}$ *by*

$$f_{\mathrm{top}}(x_0, ..., x_m) = (y_0, ..., y_n), \quad y_i = \sum_{j \in \{0,...,m\}: f(j)=i} x_j.$$

This defines a functor $G : \Delta^{\mathrm{op}} \to \mathbf{Top}$ *mapping* $[n] \mapsto \Delta^n_{\mathrm{top}}$ *and* $f \mapsto f_{\mathrm{top}}$.

(c) Let $F : \Delta^{\mathrm{op}} \to \mathbf{Sets}$ *be a simplicial set. We define a topological space* X_F *with a triangulation, the topological realization of* F, *by*

$$X_F := \left(\coprod_{n \geqslant 0} F([n]) \times \Delta^n_{\mathrm{top}} \right) / \sim$$

where $\sim$ *generated by* $(s, f_{\mathrm{top}}(x_0, ..., x_m)) \sim (F(f)s, (x_0, ..., x_m))$. *In this way we can define a topological realization functor* $\mathbf{TR} : \mathbf{sSets} \to \mathbf{CGHaus} \subset \mathbf{Top}$, *for* **CGHaus** *the subcategory of compactly generated Hausdorff spaces. It is a right Kan extension*

$$\begin{tikzcd} \Delta^{\mathrm{op}} \arrow[r, hook, "Yoneda"] \arrow[dr, "G"'] & \mathbf{sSets} \arrow[d, dashed, "\mathbf{TR}"] & \\ & \mathbf{CGHaus} \arrow[r, hook] & \mathbf{Top} \end{tikzcd}$$

(d) *There is also a functor* **Sing** : **Top** $\to$ **sSets** *which maps a space* X *to the functor* $F : \Delta^{\mathrm{op}} \to$ **Sets** *with* $F([n]) := \mathrm{Map}(\Delta^n_{\mathrm{top}}, X)$ *for objects* $[n]$ *and* $F(f) := - \circ f_{\mathrm{top}} : \mathrm{Map}(\Delta^n_{\mathrm{top}}, X) \to \mathrm{Map}(\Delta^m_{\mathrm{top}}, X)$ *for morphisms* $f : [n] \to [m]$*, so that* $f_{\mathrm{top}} : \Delta^m_{\mathrm{top}} \to \Delta^n_{\mathrm{top}}$.

Note that there are model category structures on **Top** and **sSets** such that **TR** : **sSets** $\to$ **Top** and **Sing** : **Top** $\to$ **sSets** are homotopy inverses (**Top** and **sSets** are Quillen equivalent model categories), and the homotopy categories $\mathsf{Ho}(\mathbf{Top})$ and $\mathsf{Ho}(\mathbf{sSets})$ are equivalent categories.

The weak equivalences in the model category **sSets** are morphisms η for which $\mathbf{TR}(\eta)$ is a weak homotopy equivalence of topological spaces. The fibrations are 'Kan fibrations', and the cofibrations morphisms $\eta : F \Rightarrow G$ such that $\eta([n]) : F([n]) \Rightarrow G([n])$ is injective for all n. All simplicial sets are cofibrant. The fibrant objects are called 'Kan complexes'. The category **sSets** is used as a model for ∞-groupoids.

Definition 2.15. *A simplicial category* $\mathscr{S}$ *is a category enriched in simplicial sets. That is,* $\mathscr{S}$ *is a 'category' in which for all objects* X, Y *in* $\mathscr{S}$*, the morphisms* $\mathrm{Hom}(X, Y)$ *is a simplicial set, and composition* $\mu_{X,Y,Z} : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$ *is a morphism of simplicial sets. Composition is strictly associative, that is,*

$$\mu_{W,Y,Z} \circ (\mathrm{id}_{\mathrm{Hom}(Y,Z)} \times \mu_{W,X,Y}) = \mu_{W,Y,Z} \circ (\mu_{X,Y,Z} \times \mathrm{id}_{\mathrm{Hom}(W,X)}),$$

rather than 'associative up to homotopy'.

Note that unlike additive categories, in which $\mathrm{Hom}(X, Y)$ is an abelian group, is still an ordinary category since it is a set. For simplicial categories, it is probably best not to think of $\mathrm{Hom}(X, Y)$ as a set with extra structure (though you could think of the underlying set as $\mathbf{TR}(\mathrm{Hom}(X, Y))$), so a simplicial category is not an ordinary category with extra structure.

Remark 2.16. *Any ordinary category* $\mathcal{C}$ *can be made into a simplicial category* $\mathscr{S}$ *by defining* $\mathrm{Hom}_{\mathscr{S}}(X, Y)$ *to map* $[n] \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *and* $f \mapsto \mathrm{id}_{\mathcal{C}}(X, Y)$ *and for all* $[n]$ *and* $f : [m] \to [n]$.

Here is a related notion:

Definition 2.17. *Given a category* $\mathcal{C}$*, a simplicial object in* $\mathcal{C}$ *is a functor* $F : \Delta^{\mathrm{op}} \to \mathcal{C}$*. In particular, a simplicial object in* $\mathfrak{Cat}$ *is a functor* $F : \Delta^{\mathrm{op}} \to \mathfrak{Cat}$.

Given a simplicial category $\mathscr{S}$, we can define $F_{\mathscr{S}} : \Delta^{\mathrm{op}} \to \mathfrak{Cat}$, a simplicial object in $\mathfrak{Cat}$, by taking $F_{\mathscr{S}}([n])$ to be the category with the same objects as $\mathscr{S}$ and with morphisms $X \to Y$ to be the set $\mathrm{Hom}_{\mathscr{S}}(X, Y)([n])$. Then simplicial categories correspond to special simplicial objects in $\mathfrak{Cat}$.

For example, a simplicial commutative $\mathbb{K}$-algebra is a functor $A : \Delta^{\mathrm{op}} \to \mathbf{Alg}_{\mathbb{K}}$. This will be important for Derived Algebraic Geometry as one model for 'derived commutative K-algebras' are simplicial commutative $\mathbb{K}$-algebras. (If $\mathrm{char}\mathbb{K} = 0$, another model is cdgas as before.) So roughly speaking, a derived scheme should be a topological space with a homotopy sheaf (∞-sheaf) of simplicial commutative $\mathbb{K}$-algebras. A simplicial commutative $\mathbb{K}$-algebra is a cosimplicial affine $\mathbb{K}$-scheme.

Return to the simplex category Δ.

Definition 2.18. *Define morphisms in* Δ*, the face maps* $\delta^{n,i} : [n-1] \to [n]$ *and degeneracy maps* $\sigma^{n,i} : [n+1] \to [n]$ *for* $i = 0, ..., n$ *by*

$$\delta^{n,i}(j) = \begin{cases} j, & j < i \\ j+1, & j \geqslant i \end{cases}, \quad \sigma^{n,i}(j) = \begin{cases} j, & j \leqslant i \\ j-1, & j > i \end{cases}$$

That is, $\delta^{n,i}$ *misses* i*, and* $\sigma^{n,i}$ *repeats* i.

Now let $S : \Delta^{\mathrm{op}} \to$ **Sets** *be a simplicial set. Write* $S_n = S([n])$ *and define face maps* $d_{n,i} = S(\delta^{n,i}) : S_n \to S_{n-1}$ *and degeneracy maps* $s_{n,i} = S(\sigma^{n,i}) : S_n \to S_{n+1}$.

These satisfy the identities

$$\begin{aligned}
&\delta^{n,j} \circ \delta^{n-1,i} = \delta^{n,i} \circ \delta^{n-1,j-1}, && 0 \leqslant i < j \leqslant n,\\
&\sigma^{n,j} \circ \sigma^{n+1,i} = \sigma^{n,i} \circ \sigma^{n+1,j+1}, && 0 \leqslant i \leqslant j \leqslant n,\\
&\sigma^{n,j} \circ \delta^{n+1,i} = \delta^{n,i} \circ \sigma^{n-1,j-1}, && 0 \leqslant i < j \leqslant n,\\
&\sigma^{n-1,j} \circ \delta^{n,i} = \delta^{n-1,i-1} \circ \sigma^{n-2,j}, && 0 \leqslant j < j+1 < i \leqslant n\\
&\sigma^{n,j} \circ \delta^{n+1,i} = \mathrm{id}, && i = j \text{ or } j+1.
\end{aligned}$$

The category Δ is generated by the $\delta^{n,i}$, $\sigma^{n,i}$ subject only to these relations.

For $S : \Delta^{\mathrm{op}} \to \mathbf{Sets}$, eversing directions of morphisms, these satisfy

$$
\begin{aligned}
&d_{n-1,i} \circ d_{n,j} = d_{n-1,j-1} \circ d_{n,i}, && 0 \leqslant i < j \leqslant n,\\
&s_{n+1,i} \circ s_{n,j} = s_{n+1,j+1} \circ s_{n,i}, && 0 \leqslant i \leqslant j \leqslant n,\\
&d_{n+1,i} \circ s_{n,j} = s_{n-1,j-1} \circ d_{n,i}, && 0 \leqslant i < j \leqslant n,\\
&d_{n,i} \circ s_{n-1,j} = s_{n-2,j} \circ d_{n-1,i-1}, && 0 \leqslant j < j+1 < i \leqslant n,\\
&d_{n+1,i} \circ s_{n,j} = \mathrm{id}, && i = j \text{ or } j+1.
\end{aligned}
$$

As Δ is generated by the $\delta^{n,i}$, $\sigma^{n,i}$ subject with those relations, to define a simplicial set S it is enough to give sets S_n for $n \geqslant 0$ and maps $d_{n,i} = S(\delta^{n,i}) : S_n \to S_{n-1}$ and $s_{n,i} = S(\sigma^{n,i}) : S_n \to S_{n+1}$ for $0 \leqslant i \leqslant n$ satisfying these relations, and all the other morphisms $S(f)$ in S for $f : [k] \to [l]$ can be written as compositions of the $d_{n,i}$ and $s_{n,i}$. This gives us a way to draw a picture of a simplicial set:

$$
\begin{tikzcd}[column sep=large]
S_0 \arrow[r, "s_{0,0}" description] & S_1 \arrow[l, shift right=3, "d_{1,0}"'] \arrow[l, shift left=3, "d_{1,1}"] \arrow[r, shift left=2, "s_{1,1}" description] \arrow[r, shift right=2, "s_{1,0}" description] & S_2 \arrow[l, shift right=4, "d_{2,0}"'] \arrow[l, "d_{2,1}" description] \arrow[l, shift left=4, "d_{2,2}"] \arrow[r, shift left=4, "s_{2,0}" description] \arrow[r, "s_{2,1}" description] \arrow[r, shift right=4, "s_{2,2}" description] & S_3 \arrow[l, shift right=6, "d_{3,0}"'] \arrow[l, shift right=2, "d_{3,1}" description] \arrow[l, shift left=2, "d_{3,2}" description] \arrow[l, shift left=6, "d_{3,3}"] & \cdots
\end{tikzcd}
$$

2.4 Kan Complexes and Weak Kan Complexes

Definition 2.19. *(a) For $n \geqslant 0$, the standard n-simplex Δ^n is the simplicial set* $\mathrm{Hom}(-, [n]) : \Delta^{\mathrm{op}} \to$ **Sets**.

(b) Define morphisms of simplicial sets $\delta^{n,i} : \Delta^{n-1} \to \Delta^n$ for $i = 0, 1, ..., n$ by $\delta^{n,i} = - \circ \delta^{n,i}$, where $\delta^{n,i} : [n-1] \to [n]$ is the face map. Then $\delta^{n,i}$ is an injective morphism in **sSets**, *and $\delta^{n,i}(\Delta^{n-1})$ is a simplicial subset of Δ^n, that is, $\delta^{n,i}(\Delta^{n-1})([k]) \subset \Delta^n([k])$ in* **Sets** *for each $k \geqslant 0$.*

(c) Define the $n-1$-sphere $\partial\Delta^n$, as a simplicial subset of Δ^n, by

$$\partial\Delta^n = \bigcup_{i=0,...,n} \delta^{n,i}(\Delta^{n-1}),$$

where for each $k \geqslant 0$ we take the union in subsets of $\Delta^n([k])$. It is a simplicial set with an inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

(d) For $k = 0, ..., n$, define the k-horn Λ^n_k, as a simplicial subset of Δ^n, by

$$\Lambda^n_k = \bigcup_{i=0,...,n;\, i \neq k} \delta^{n,i}(\Delta^{n-1}),$$

It is a simplicial set with an inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. It is an inner horn if $0 < k < n$.

Definition 2.20. *A simplicial set S is a Kan complex if for all $0 \leqslant k \leqslant n$ and all $f : \Lambda^n_k \to S$ in* **sSets**, *there exists a (not necessarily unique) morphism $g : \Delta^n \to S$ making the following diagram commute:*

$$
\begin{tikzcd}
\Lambda^n_k \arrow[rr, "f"] \arrow[dr, hook] & & S \\
& \Delta^n \arrow[ur, dashed, "g"'] &
\end{tikzcd}
$$

Then we say that all horns in S have fillers.

We call S a weak Kan complex if the above holds for all $0 < k < n$. Then we say that all inner horns in S have fillers.

Example 2.21. *If* X *is a topological space then* $\mathbf{Sing}(X)$ *is a Kan complex.*

Indeed, we must fill in the diagram in **Top***:*

$$\begin{tikzcd}
\mathbf{TR}(\Lambda^n_k) \arrow[rr, "f"] \arrow[dr, hook] & & X \\
& \mathbf{TR}(\Delta^n) \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

This is possible as $\mathbf{TR}(\Delta^n)$ *retracts onto* $\mathbf{TR}(\Lambda^n_k)$*.*

2.5 Quasicategories

uasicategories are a model (arguably the best) for $(\infty, 1)$-categories, developed by Joyal and Lurie. Lurie went on to use them as the foundation for his theory of Derived Algebraic Geometry.

Definition 2.22. *A quasicategory is a weak Kan complex.*

Next we first define the nerve $\mathcal{N}(\mathcal{C})$ of a category $\mathcal{C}$, then we will use them to explain how to treat a quasicategory like an $(\infty, 1)$-category.

Definition 2.23. *Let* $\mathcal{C}$ *be a small category. Define a simplicial set* $\mathcal{N}(\mathcal{C})$ *called the nerve of* $\mathcal{C}$ *as follows:*

- 0*-simplices (elements of* $\mathcal{N}(\mathcal{C})([0])$*) are objects* $X \in \mathcal{C}$*;*
- 1*-simplices (in* $\mathcal{N}(\mathcal{C})([1])$*) are morphisms* $X_0 \xrightarrow{f_1} X_1$ *in* $\mathcal{C}$*.*
- n*-simplices are sequences* $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ *in* $\mathcal{C}$*.*
- *Face maps* $d_{n,i} : \mathcal{N}(\mathcal{C})([n]) \to \mathcal{N}(\mathcal{C})([n-1])$ *omit* X_0, f_1 *for* $i = 0$*, omit* X_n, f_n *for* $i = n$*, and compose* f_i, f_{i+1} *for* $0 < i < n$*.*
- *Degeneracy maps* $s_{n,i} : \mathcal{N}(\mathcal{C})([n]) \to \mathcal{N}(\mathcal{C})([n+1])$ *insert* $\mathrm{id}_{X_i} : X_i \to X_i$ *into the sequence.*

Functors $F : \mathcal{C} \to \mathcal{D}$ *induce morphisms* $\mathcal{N}(\mathcal{C}) \to \mathcal{N}(\mathcal{D})$*.*

We can characterize when a simplicial set is a nerve via horn-filling.

Proposition 2.24. *(a) A simplicial set* S *is isomorphic to the nerve* $\mathcal{N}(\mathcal{C})$ *of a small category* $\mathcal{C}$ *iff all inner horns* $f : \Lambda^n_k \to S$ *have unique fillers.*

This implies that $\mathcal{N}(\mathcal{C})$ *is a weak Kan complex for all* $\mathcal{C}$*.*

(b) A simplicial set S *is isomorphic to the nerve* $\mathcal{N}(\mathcal{C})$ *of a small groupoid* C *iff all horns* $f : \Lambda^n_k \to S$ *for* $n > 1$ *have unique fillers.*

(c) The nerve $\mathcal{N}(\mathcal{C})$ *of a small category* $\mathcal{C}$ *is a Kan complex iff* $\mathcal{C}$ *is a groupoid.*

Note that Proposition suggests that we can see weak Kan complexes as generalizations of categories, and Kan complexes as generalizations of groupoids. We already know that Kan complexes are the fibrant-cofibrant objects in **sSets**, so by the fundamental theorem of model theory, the category with objects Kan complexes and homotopy classes of morphisms between them is equivalent to $\mathsf{Ho}(\mathbf{sSets})$, so a model for ∞-groupoids.

Return to the quasicategories. Based on our definition of the nerve $\mathcal{N}(\mathcal{C})$ of a category $\mathcal{C}$, we will explain how to treat a quasicategory like an ∞-category.

Definition 2.25. *Let* $\mathcal{Q}$ *be a quasicategory.*

- *An object* X *of* $\mathcal{Q}$ *is a* 0*-simplex (element* $X \in \mathcal{Q}([0])$*).*
- *A* 1*-morphism* $f : X \to Y$ *of* $\mathcal{Q}$ *is a* 1*-simplex (element* $f \in \mathscr{Q}([1])$*) with face maps* $d_{1,1}(f) = X$ *and* $d_{1,0}(f) = Y$*. The identity* 1*-morphism is* $\mathrm{id}_X = s_{0,0}(X)$*, from the degeneracy map* $s_{0,0} : \mathcal{Q}([0]) \to \mathcal{Q}([1])$*.*

- *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be 1-morphisms in* $\mathcal{Q}$. *We say that* $h : X \to Z$ *is a choice of composition in* $\mathcal{Q}$ *if there exists a 2-simplex* $\eta \in \mathcal{Q}([2])$ *with* $d_{2,2}(\eta) = f, d_{2,0}(\eta) = g$ *and* $d_{2,1}(\eta) = h$. *We think of* $\eta : g \circ f \Rightarrow h$ *as a 2-morphism in* $\mathcal{Q}$*, and draw it as a picture of a 2-simplex, with* X, Y, Z *as the vertices,* f, g, h *as the edges, and* η *as the 2-simplex:*

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z \end{tikzcd}$$

  Note that compositions are nonunique. But as $\mathcal{Q}$ *is a weak Kan complex, compositions always exist, as* X, Y, Z, f, g *define a morphism* $\Lambda^2_1 \to \mathcal{Q}$*, and* h, η *fill the horn to a morphism* $\Delta^2 \to \mathcal{Q}$.

- *Let* $f, f' : X \to Y$ *be 1-morphisms. We say that* f, f' *are 2-isomorphic, or homotopic, written* $f \sim f'$*, if there exists*

$$\begin{tikzcd} & X \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f'"'] & {} & Y \end{tikzcd}$$

  This is an equivalence relation on f, f'.

  Using the horn-filling condition for 3-horns we can show that if h_1, h_2 *are possible compositions* $g \circ f$ *then* $h_1 \sim h_2$.

- *The homotopy category* $\mathsf{Ho}(\mathcal{Q})$ *is the category with objects* X *the objects* X *of* $\mathcal{Q}$*, and morphisms* $[f] : X \to Y$ *the* $\sim$*-equivalence classes of 1-morphisms* $f : X \to Y$. *Identity morphisms are* $[\mathrm{id}_X] : X \to X$. *The composition of* $[f] : X \to Y$ *and* $[g] : Y \to Z$ *is* $[g] \circ [f] = [h] : X \to Z$*, where* $h : X \to Z$ *is a choice of composition of 1-morphisms in* $\mathcal{Q}$*,and* $[h]$ *is independent of choices.*
- *If* $\mathcal{Q} = \mathcal{N}(\mathcal{C})$ *then* $\mathsf{Ho}(\mathcal{Q}) \simeq \mathcal{C}$.

Many definitions in category theory have well-behaved analogues for quasicategories. Here are some examples:

Definition 2.26.

- *Let* $\mathcal{Q}, \mathcal{R}$ *be quasicategories. A functor* $F : \mathcal{Q} \to \mathcal{R}$ *is a morphism of simplicial sets.*
- *If* $F, G : \mathcal{Q} \to \mathcal{R}$ *are functors, a natural transformation* $\eta : F \Rightarrow G$ *is a morphism of simplicial sets* $\eta : \Delta^1 \times \mathcal{Q} \to \mathcal{R}$ *which restricts to* F *on* $0 \times \mathcal{Q}$ *and to* G *on* $1 \times \mathcal{Q}$.
- *Let* $\mathcal{Q}$ *be a quasi-category and* X, Y *be objects in* $\mathcal{Q}$. *Define the right Hom object* $\mathrm{Hom}^R_{\mathcal{Q}}(X, Y)$ *to be the simplicial set whose* n*-simplices are morphisms* $\Delta^{n+1} \to \mathcal{Q}$ *which restrict to the constant map to* X *on* $\delta^{n+1,n+1}(\Delta^n) \subset \Delta^{n+1}$*, and restrict to* Y *on vertex* $n+1$ *of* Δ^{n+1}.
- *An object* Y *in* $\mathcal{Q}$ *is a terminal object in* $\mathcal{Q}$ *if* $\mathrm{Hom}^R_{\mathcal{Q}}(X, Y)$ *is contractible for all objects* X.
- *Left Hom objects* $\mathrm{Hom}^L_{\mathcal{Q}}(X, Y)$ *and initial objects have the dual definition.*
- *Let* K *be a simplicial set and* $k : K \to \mathcal{Q}$ *a morphism. We can define a quasicategory* $\mathcal{Q}_{/k}$ *with objects* (X, η) *an object* X *in* $\mathcal{Q}$ *and a natural transformation* $\eta : \mathbb{1}_X \Rightarrow k$*, where* $\mathbb{1}_X : K \to \mathcal{Q}$ *is the constant functor with value* X. *A* limit *of* $k : K \to \mathcal{Q}$ *is a terminal object in* $\mathcal{Q}_{/k}$. *So, for example, a fibre product* $X \times_{g,Z,h} Y$ *in* $\mathcal{Q}$ *is a limit of the following morphism:*

$$\begin{tikzcd} \bullet(\Delta^0) \arrow[dr, "\Delta^1"] & & \bullet(\Delta^0) \arrow[dl, "\Delta^1"'] \arrow[r, Rightarrow, "k"] & X \arrow[dr, "g"] & & Y \arrow[dl, "h"'] \\ & \bullet(\Delta^0) & & & Z & \end{tikzcd}$$

A general principle:

- When working in quasicategories, everything is a simplicial set.

- Something is 'unique' if it lies in a contractible simplicial set.

Definition 2.27. *Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 1)$-categories. Then*

$$\mathbf{Fun}_\infty(\mathcal{C}, \mathcal{D}) := \mathbf{sSets}(\mathcal{C}, \mathcal{D})$$

is the simplicial set of morphisms of simplicial sets as $[n] \mapsto \mathrm{Hom}_{\mathbf{sSets}}(\Delta^n \times \mathcal{C}, \mathcal{D})$. It is a quasicategory by directly check.

Definition 2.28. *A ∞-groupoid is a $(\infty, 0)$-category, is an ∞-category in which all k-morphisms for all k are equivalences.*

Moreover we consider $\mathbf{Grpd}^\infty$ is the $(\infty, 1)$-category of ∞-groupoids. Note that $\mathbf{Grpd}^\infty$ is just the Dwyer-Kan localization (see below) of $\mathbf{sSets}$ of weak equivalences.

Definition 2.29. *The full $\mathbf{sSets}$-enriched-subcategory of $\mathbf{sSets}$ on those simplicial sets which are quasi-categories is a quasi-category-enriched category. This is the $(\infty, 2)$-category of $(\infty, 1)$-categories. So here we need some modification. The $\mathbf{sSets}$-subcategory of that obtained by picking of each hom-object the core, i.e. the maximal ∞-groupoid yields an ∞-groupoid-enriched category. This is the $(\infty, 1)$-category of $(\infty, 1)$-categories and we denote it as $\mathbf{Cat}^\infty$.*

2.6 Dwyer-Kan Localization

Let $\mathcal{C}$ be a model category with weak equivalence $\mathcal{W}$. We will define an ∞-category associated to it.

Definition 2.30. *For a $\mathcal{C}$ be a model category with weak equivalence $\mathcal{W}$, we define its Dwyer-Kan localization $L^{\mathcal{W}}\mathcal{C}$ is a simplicial category (category enriched in simplicial sets, hence is an $(\infty, 1)$-category) with objects are objects of $\mathcal{C}$, and in the simplicial set $\mathrm{Hom}_{L^{\mathcal{W}}\mathcal{C}}(X, Y)$, n-simplices are equivalence classes of commutative diagrams for $m \geqslant 0$:*

$$\begin{tikzcd}
 & K_{10} \arrow[d, "\sim"'] & L_{10} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"'] & K_{20} \arrow[d, "\sim"'] & L_{20} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"'] & K_{30} \arrow[r] \arrow[d, "\sim"'] & \cdots & L_{m0} \arrow[l, "\sim"'] \arrow[d, "\sim"'] \arrow[dr] & \\
X \arrow[ur] \arrow[r] \arrow[ddr] & K_{11} \arrow[d, "\sim"] & L_{11} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"'] & K_{21} \arrow[d, "\sim"'] & L_{21} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"'] & K_{31} \arrow[r] \arrow[d, "\sim"'] & \cdots & L_{m1} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"'] & Y \\
 & \vdots \arrow[d, "\sim"] & \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] & \\
 & K_{1n} & L_{1n} \arrow[l, "\sim"'] \arrow[r] & K_{2n} & L_{2n} \arrow[l, "\sim"'] \arrow[r] & K_{3n} \arrow[r] & \cdots & L_{mn} \arrow[l, "\sim"'] \arrow[uur] &
\end{tikzcd}$$

with $n+1$ rows, where morphisms $\sim$ are weak equivalences, and the equivalence relation omits identities and composes composable morphisms, changing m, and for $m = 0$ we take morphisms $X \to Y$.

This give us a right one to invert weak equivalences.

2.7 Stable Infinity Categories

We will give a reason why we need stable ∞-categories instead of triangulated categories:

Example 2.31. *I would argue that triangulated categories are not quite the 'right' theory. However, they are a very good approximation - you can work with them for years and not notice the problems.*

As a signal that there should be something more, recall that if $\mathcal{T}$ is a triangulated category, and $u : X \to Y$ a morphism in $\mathcal{T}$, there is $\mathrm{cone}(u) \in \mathcal{T}$, in a distinguished triangle $X \to Y \to \mathrm{cone}(u) \to X[1]$ in $\mathcal{T}$. This is begging to be turned into a cone functor: we would like a category $\mathrm{Mor}(\mathcal{T})$ of morphisms in $\mathcal{T}$, and a functor $\mathrm{cone} : \mathrm{Mor}(\mathcal{T}) \to \mathcal{T}$ mapping $u \mapsto \mathrm{cone}(u)$ on objects. To try to define cone on morphisms in $\mathrm{Mor}(\mathcal{T})$, consider the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & \mathrm{cone}(u) \arrow[r] \arrow[d, dashed, "\mathrm{cone}(f,g)"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & X'[1]
\end{tikzcd}$$

and extension via the definition of triangulated categories. But it is not unique, so we cannot define cone.

So the explanation is $\mathcal{T}$ should be a higher category (an ∞-category)! In this case n-morphisms in $\mathrm{Mor}(\mathcal{T})$ *correspond to $(n+1)$-morphisms in $\mathcal{T}$. So to define* cone *on 1-morphisms in* $\mathrm{Mor}(\mathcal{T})$*, we should be using 2-morphisms in $\mathcal{T}$. We consider the diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & \mathrm{cone}(u) \arrow[r] \arrow[d, dashed, "\mathrm{cone}(f,g;\eta)"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"'] \arrow[ur, phantom, "\Nearrow\eta"] & Y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & X'[1] \end{tikzcd}$$

where $\eta : u' \circ f \Rightarrow g \circ u$ is a 2-morphism. Then $\mathrm{cone}(f, g; \eta)$ *should exist and be unique up to 2-isomorphism. When we pass to the homotopy category $\mathsf{Ho}(\mathcal{T})$, this choice of η is forgotten, which is why we lose uniqueness of* $\mathrm{cone}(f, g)$*. Note moreover that if we want $\mathcal{T}$ and* $\mathrm{Mor}(\mathcal{T})$ *to be objects of the same type we cannot truncate to n-categories for any finite n — we need $n = \infty$.*

To define the stable ∞-category, we first consider some basuc things.

Definition 2.32. *Let $\mathcal{Q}$ be a quasicategory.*

- *A* zero object *is an object $0 \in \mathcal{Q}$ which is both an initial object and a terminal object.*
- *Suppose $\mathcal{Q}$ has a zero object, and consider commuting squares*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"'] & Y \arrow[d, "g"] \\ 0 \arrow[r, "0"'] \arrow[ur, phantom, "\Nearrow\eta"] & Z \end{tikzcd}$$

  comes from the following map of simplicial sets

$$\begin{tikzcd} \bullet(\Delta^0) \arrow[r, "\Delta^1"] \arrow[d, "\Delta^1"'] \arrow[dr, "\Delta^1"] \arrow[dr, phantom, "\Delta^2", shift right=3ex] \arrow[dr, phantom, "\Delta^2", shift left=3ex] & \bullet(\Delta^0) \arrow[d, "\Delta^1"] \\ \bullet(\Delta^0) \arrow[r, "\Delta^1"'] & \bullet(\Delta^0) \end{tikzcd} \overset{k}{\Longrightarrow} \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "0"'] \arrow[dr] & Y \arrow[d, "g"] \\ 0 \arrow[r, "0"'] & Z \end{tikzcd}$$

  In this case we call $X \xrightarrow{f} Y \xrightarrow{g} Z$ a triangle *in $\mathcal{Q}$.*

- *We say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an* exact triangle*, and write $X = \ker(g)$, if the diagram before is an ∞-Cartesian square.*

  Dually, we say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is an coexact triangle*, and write $Z = \mathrm{coker}(g)$, if the diagram before is an ∞-coCartesian square.*

Definition 2.33. *A quasicategory $\mathcal{Q}$ is a* stable ∞-category *if*

- *$\mathcal{Q}$ has a zero object 0.*
- *Every morphism in $\mathcal{Q}$ has a kernel and a cokernel.*
- *Every exact triangle is coexact, and vice versa.*

This is a simple definition with remarkable consequences. There are lots of stable ∞-categories in nature, and stable ∞-categories have very good properties, you can do lots of beautiful mathematics in them. For example:

Theorem 2.34. *Let $\mathcal{Q}$ be a stable ∞-category, then $\mathsf{Ho}(\mathcal{Q})$ is a triangulated category.*

Moreover, we have many other models such as dg-categories and Segal Categories. From an ∞-category point of view, dg-categories are more-or-less equivalent to $\mathbb{K}$-linear stable ∞-categories. Segal categories are yet another model for ∞-categories, a weak form of simplicial categories. We will omit them.

3 Derived Schemes and Derived Stacks

Here we fix some field $\Bbbk$ (of characteristic zero if we want).

3.1 Higher Stacks

Here we fix the model of $(\infty,1)$-theory is quasicategories. For simplicial, we consider all categories have pullbacks (if not, we need to use $(\infty,1)$-Yoneda embedding into the category of presheaves).

Definition 3.1. *For a $(\infty,1)$-category $\mathcal{C}$, a $(\infty,1)$-presheaf $\mathcal{F}$ of $\mathcal{C}$ is a $(\infty,1)$-functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to$ $\mathbf{Grpd}^{\infty}$. The $(\infty,1)$-category of $(\infty,1)$-presheaves is the*

$$\mathbf{PSh}_{\infty}(\mathcal{C}) := \mathbf{Fun}_{\infty}(\mathcal{C}^{\mathrm{op}}, \mathbf{Grpd}^{\infty}).$$

Definition 3.2. *For a $(\infty,1)$-category $\mathcal{C}$, a Grothendieck topology on $\mathcal{C}$ is similar as ordinary one, see [Lur09] Definition 6.2.2.1. Then $(\infty,1)$-category $\mathcal{C}$ equipped with a Grothendieck topology is called an $(\infty,1)$-site.*

Definition 3.3. *Fix a $(\infty,1)$-site $\mathcal{C}$ and a $(\infty,1)$-presheaf $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}^{\infty}$. We call $\mathcal{F}$ is a $(\infty,1)$-sheaf or ∞-stack if for every covering $f : U \to X$ in $\mathcal{C}$, let $U_{\bullet}$ denote the the simplicial object (a hypercovering)*

$$U \leftleftarrows\!\!\dashrightarrow U \times_X U \;\substack{\dashrightarrow\\ \leftarrow\\ \dashrightarrow\\ \leftarrow\\ \dashrightarrow\\ \leftarrow\\ \dashrightarrow}\; U \times_X U \times_X U \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow}\; \cdots$$

whose n-th term is the $(n+1)$-fold fibre power $U \times_X \cdots \times_X U$. Then $\mathcal{F}(X)$ is a limit of $\mathcal{F}(U_{\bullet})$, that is, limit of

$$\mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_X U) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \mathcal{F}(U \times_X U \times_X U) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots$$

in $\mathbf{Grpd}^{\infty}$. We call the $(\infty,1)$-category of $(\infty,1)$-sheaves is $\mathbf{Sh}_{\infty}(\mathcal{C})$.

Remark 3.4. *This construction is also right if we replace $\mathbf{Grpd}^{\infty}$ by some other infinity categories.*

Definition 3.5. *A higher stack is an $(\infty,1)$-sheaf on the category of schemes with the étale (or fppf) topology. For a field $\Bbbk$, we define the higher $\Bbbk$-stack is an $(\infty,1)$-sheaf on the category of $\Bbbk$-schemes with the étale (or fppf) topology. The $(\infty,1)$-category $\mathbf{HSta}_{\Bbbk}$ of higher $\Bbbk$-stacks is the full $(\infty,1)$-subcategory of $\mathbf{Fun}_{\infty}(\mathbf{Sch}_{\Bbbk}, \mathbf{Grpd}^{\infty})$ (or $\mathbf{Fun}_{\infty}(\mathbf{Aff}_{\Bbbk}, \mathbf{Grpd}^{\infty})$ if we want)*

Definition 3.6. *We define a notion of n-Artin stack for $n \geqslant -1$,by induction on n:*

- *A -1-Artin stack is an affine scheme.*
- *A morphism $h : Y \to Z$ in $\mathbf{HSta}_{\Bbbk}$ is -1-representable if $X \times_{g,Z,h} Y$ is an affine scheme for all $g : X \to Z$ with $X \in \mathbf{Aff}_{\Bbbk}$.*
- *Suppose by induction that $(n-1)$-Artin stacks and $(n-1)$-representable morphisms are defined. Then $X \in \mathbf{HSta}_{\Bbbk}$ is an n-Artin stack if there exists an $(n-1)$-representable, smooth, surjective morphism $\pi : U \to X$ with U a disjoint union of affine schemes. A morphism $h : Y \to Z$ in $\mathbf{HSta}_{\Bbbk}$ is n-representable if $X \times_{g,Z,h} Y$ is an n-Artin stack for all $g : X \to Z$ with $X \in \mathbf{Aff}_{\Bbbk}$.*

A higher stack X which is locally an n-Artin stack for some n is called a higher Artin stack. We define the full ∞-subcategory of higher Artin stacks is $\mathbf{HArt}_{\Bbbk} \subset \mathbf{HSta}_{\Bbbk}$.

Here is a similar but more useful notion:

Definition 3.7. *Let X be a higher Artin stack. We say that X is a (higher) Artin n-stack if for all $A \in \mathbf{Alg}_{\Bbbk}$, the simplicial set $X(A)$ is n-truncated, that is, $\pi_i(X(A)) = 0$ for all $i > n$, or equivalently morphism of simplicial sets $\partial\Delta^m \to X(A)$ can be extended to an m-simplex of X for any $m > n+1$. Then n-Artin stacks are Artin $n+1$-stacks. Write $\mathbf{HArt}^n_{\Bbbk} \subset \mathbf{HArt}_{\Bbbk}$ for the full ∞-subcategory of higher Artin n-stacks.*

Artin 1-stacks are equivalent to ordinary Artin stacks. Artin 0-stacks are more-or-less the same thing as smooth version of algebraic spaces.

Example 3.8. *For a smooth projective $\Bbbk$-scheme X, the ∞-moduli stack of complexes E in $\mathbf{D}^b\mathrm{Coh}(X)$ with $\mathrm{Ext}^{\leqslant -n}(E,E)=0$ is an Artin n-stack. Moreover, Toën-Vaquié prove that the ∞-moduli stack $\mathbf{M}(\mathbf{D}^b\mathrm{Coh}(X))$ of objects exists as a higher Artin $\Bbbk$-stack.*

Over $\Bbbk=\mathbb{C}$ a higher $\mathbb{C}$-stack X has a 'topological realization' X^{top}, a topological space natural up to homotopy equivalence, so we can define the (co)homology $H_*(X) := H_*(X^{\mathrm{top}})$ and $H^*(X) := H^*(X^{\mathrm{top}})$. It turns out that the (co)homology of the higher stack $\mathbf{M}(\mathbf{D}^b\mathrm{Coh}(X))$ is often computable, and is much nicer than the (co)homology of the Artin stack $\mathbf{M}(\mathrm{Coh}(X))$, which is usually not computable. Basically this is because $\mathbf{M}(\mathrm{Coh}(X))$ is like an 'abelian monoid in stacks', with addition $\oplus$ in $\mathrm{Coh}(X)$. But $\mathbf{M}(\mathbf{D}^b\mathrm{Coh}(X))$ is like an 'abelian group in stacks', as $[1] : \mathbf{D}^b\mathrm{Coh}(X) \to \mathbf{D}^b\mathrm{Coh}(X)$ acts like an (up to homotopy) inverse for addition $\oplus$ in $\mathbf{D}^b\mathrm{Coh}(X)$; and abelian groups are much simpler than monoids.

3.2 Commutative Differential Graded Algebras

Now we will consider some models about derived commutative rings. There are several models such as commutative differential graded algebras and simplicial commutative rings. In $\mathrm{char}(\Bbbk)=0$, they give the same theory. Lurie in his SAG using $\mathbb{E}_\infty$-ring spectra but we will use them. Here we will assume the base ring $\mathbb{K}$ containing $\mathbb{Q}$.

Definition 3.9. *(a) A differential graded $\mathbb{K}$-algebra (dga or dg-algebra for short) $A^\bullet=(A^*,d)$ consists of a chain complex with a unital associative multiplication. Concretely, that is a family of $\mathbb{K}$-modules $\{A^i\}_{i\in\mathbb{Z}}$, an associative $\mathbb{K}$-linear multiplication $(-\cdot-) : A^i \times A^j \to A^{i+j}$ (for all i,j), a unit $1\in A^0$ and a differential $d : A^i \to A^{i+1}$ (for all i) which is $\mathbb{K}$-linear, satisfies $d^2=0$ and is a derivation with respect to the multiplication, which means $d(a\cdot b)=d(a)\cdot b+(-1)^{\deg(a)}a\cdot d(b)$.*

(b) A graded $\mathbb{K}$-algebra A is graded-commutative if $a\cdot b=(-1)^{\deg(a)\cdot\deg(b)}b\cdot a$.

(c) A morphism of dg-algebras is a map $f : A^\bullet \to B^\bullet$ that respects the differentials (i.e. $fd_A=d_Bf$), and the multiplication (i.e. $f(a\cdot_A b)=f(a)\cdot_B f(b)$).

In this note we mainly focus on the following:

Definition 3.10. *We define the category $\mathbf{cdg^-Alg}_{\mathbb{K}}$ of the graded-commutative differential graded $\mathbb{K}$-algebras which are concentrated in non-positive degree. Hence $A^\bullet\in\mathbf{cdg^-Alg}_{\mathbb{K}}$ we have $A^\bullet=\bigoplus_{k=0}^{-\infty}A^k$. Moreover, for $R^\bullet\in\mathbf{cdg^-Alg}_{\mathbb{K}}$, we can define $\mathbf{cdg^-Alg}_{R^\bullet} := R^\bullet\downarrow\mathbf{cdg^-Alg}_{\mathbb{K}}$ of cdga $R^\bullet$-algebras.*

Definition 3.11. *(a) Let $A^\bullet,B^\bullet\in\mathbf{cdg^-Alg}_{R^\bullet}$. A morphism $f : A^\bullet\to B^\bullet$ of $R^\bullet$-cdga is a quasi-isomorphism (or weak equivalence) if it induces an isomorphism on cohomology $H^*(A^\bullet)\cong H^*(B^\bullet)$.*

(b) We say that $R^\bullet$-cdga $A^\bullet$ and $B^\bullet$ are quasi-isomorphic if there exists a diagram $A^\bullet\leftarrow C^\bullet\to B^\bullet$ of quasi-isomorphisms in $\mathbf{cdg^-Alg}_{R^\bullet}$.

Remark 3.12. *Note that there is a global version of these.*

A very important example of cdgas and derived schemes:

Example 3.13. *Let $M^\bullet$ be a graded $\mathbb{K}$-module. The free graded-commutative $\mathbb{K}$-algebra generated by $M^\bullet$ is*

$$\mathbb{K}[M^\bullet] := \left(\bigoplus \mathrm{Sym}^n M^{\mathrm{even}}\right)\otimes_{\mathbb{K}}\left(\bigoplus\bigwedge^n M^{\mathrm{odd}}\right)$$

with the degree of a product of elements being the sum of the degrees of those elements.

In particular, we consider the free graded-commutative $\mathbb{K}$-algebra $A^\bullet$ generated by $x_1,...,x_m;y_1,...,y_n$ where $\deg x_i=0$ and $\deg y_j=-1$. Hence

$$A^k=\mathbb{K}[x_1,...,x_m]\otimes_{\mathbb{K}}\bigwedge^{-k}\mathbb{K}^n$$

for $k=0,-1,...,-n$ and $A^k=0$ for otherwise. Pick $p_1,...,p_n\in\mathbb{K}[x_1,...,x_m]$, as $A^\bullet$ is free there are unique maps $d : A^k\to A^{k+1}$ satisfying the Leibniz rule, such that $d(y_i)=p_i(x_1,...,x_m)$ for $i=1,...,n$. Also $d^2=0$ and $A^\bullet\in\mathbf{cdg^-Alg}_{\mathbb{K}}$.

Now $\mathsf{H}^0(\mathsf{A}^\bullet) = \mathbb{K}[x_1, ..., x_m]/(p_1, ..., p_n)$ *and hence* $\operatorname{Spec} \mathsf{H}^0(\mathsf{A}^\bullet)$ *is a subscheme defined by these polynomials. Now the derived scheme* $\mathbf{Spec}\, \mathsf{A}^\bullet$ *remembers information about the dependencies between* $p_1, ..., p_n$ *which have more information than the truncated classical scheme* $\operatorname{Spec} \mathsf{H}^0(\mathsf{A}^\bullet)$*.*

3.3 Simplicial Commutative Rings

Definition 3.14. *A simplicial commutative ring is a simplicial object in* $\mathbf{Aff}$*, that is, a functor* $\mathsf{A}_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Aff}$*. In particular, a simplicial commutative* R*-algebra is a functor* $\mathsf{A}_\bullet : \Delta^{\mathrm{op}} \to \mathbf{Aff}_\mathsf{R}$*.*

We define its ∞*-category* $\mathbf{sComm}$ *and* $\mathbf{sComm}_\mathsf{R}$ *is the Dwyer-Kan localization of* sComm *and* $\mathrm{sComm}_\mathsf{R}$ *of weak equivalences (the ones induce weak homotopy equivalence on the underlying simplicial sets).*

Here we give some comparation with simplicial commutative ring and cdgas.

Theorem 3.15 (Dold-Kan)**.** *For an abelian category* $\mathcal{A}$*, there is an equivalence of categories*

$$\mathsf{N} : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathcal{A}) \to \mathrm{Ch}^+(\mathcal{A})$$

Theorem 3.16 (Quillen)**.** *For* $\mathbb{Q} \subset \mathsf{R}$*, Dold-Kan denormalisation gives a right Quillen equivalence*

$$\mathsf{N} : \mathrm{sComm}_\mathsf{R} \to \mathbf{cdg}^-\mathbf{Alg}_\mathsf{R}.$$

This give use the equivalence of ∞*-categories* $\mathbf{sComm}_\mathsf{R}$ *and* $\mathbf{cdg}^-\mathbf{Alg}_\mathsf{R}$*.*

3.4 Derived Ringed Spaces and Derived Schemes

Definition 3.17. *A derived ringed space* $\mathbf{X} = (\mathsf{X}, \mathscr{O}_\mathsf{X})$ *is a topological space* X *with a* ∞*-sheaf* $\mathscr{O}_\mathsf{X} \in \mathbf{Sh}_\infty(\mathbf{Open}(\mathsf{X}), \mathbf{sComm})$*. Then* $\pi_0(\mathscr{O}_\mathsf{X})$ *is an ordinary sheaf of ordinary commutative rings, and* $\pi_i(\mathscr{O}_\mathsf{X})$ *is an ordinary sheaf of modules over* $\pi_0(\mathscr{O}_\mathsf{X})$ *for all* $i > 0$*.*

The morphism is defined by

$$\mathrm{Mor}((\mathsf{X}, \mathscr{O}_\mathsf{X}), (\mathsf{Y}, \mathscr{O}_\mathsf{Y})) := \coprod_{u : \mathsf{X} \to \mathsf{Y}} \mathrm{Mor}_{\mathbf{Sh}_\infty(\mathbf{Open}(\mathsf{Y}), \mathbf{sComm})}(\mathscr{O}_\mathsf{Y}, u_* \mathscr{O}_\mathsf{X}).$$

Hence we have an ∞*-category* $\mathbf{DRgSp}$ *of derived ringed spaces.*

Remark 3.18. *We haven't define the homotopy groups of objects in some* $(\infty, 1)$*-topos, here I give some comments. Consider* $\mathbb{S}^n := \partial \Delta^{n+1}$ *and consider the constant diagram* $\mathrm{const}_\mathscr{F} : \mathbb{S}^n \to \mathbf{Grpd}^\infty$*. Then we define* $\mathscr{F}^{\mathbb{S}^n} := \varprojlim \mathrm{const}_\mathscr{F}$ *which play a role as the mapping from sphere to the space. Then we define* $\pi_n(\mathscr{F}) := \tau_{\leqslant 0} \mathscr{F}^{\mathbb{S}^n}$*. This can be seen as the* n*-truncated sheaf module the* $n+1$ *ones (like* $\mathscr{I}^n / \mathscr{I}^{n+1}$ *in the* $\mathsf{X}_{\mathrm{red}} \subset \mathsf{X}$ *for some scheme).*

Definition 3.19. *We call a derived ring space* $\mathbf{X} = (\mathsf{X}, \mathscr{O}_\mathsf{X})$ *is a derived scheme if* $(\mathsf{X}, \pi_0(\mathscr{O}_\mathsf{X}))$ *is a scheme and* $\pi_i(\mathscr{O}_\mathsf{X})$ *are quasi-coherent* $\pi_0(\mathscr{O}_\mathsf{X})$*-modules. We call* $\mathbf{X}$ *is a derived affine scheme if* $(\mathsf{X}, \pi_0(\mathscr{O}_\mathsf{X}))$ *is an affine scheme. Their* ∞*-categories are* $\mathbf{DAff} \subset \mathbf{DSch}$*. The relative case is similar.*

Hence we have an adjoint pair t_0, i of $\mathbf{Sch}$ and $\mathbf{DSch}$.

Proposition 3.20. *We have following useful results.*

(a) We have an ∞*-functor* $\Gamma : \mathbf{DAff}^{\mathrm{op}} \to \mathbf{sComm}$ *defined as* $(\mathsf{X}, \mathscr{O}_\mathsf{X}) \mapsto \pi_* \mathscr{O}_\mathsf{X}$ *where* $\pi : \mathsf{X} \to * = \operatorname{Spec} \mathbb{Z}$*. This induce an* ∞ *equivalence with* ∞*-inverse* $\mathbf{Spec} : \mathbf{sComm} \to \mathbf{DAff}^{\mathrm{op}}$*.*

(b) The ∞*-category* $\mathbf{DSch}$ *of derived schemes has all finite limits (see [TV08]). Again, when* X, Y *and* S *are merely (underived) schemes,* $\mathsf{Z} := \mathsf{X} \times_\mathsf{S} \mathsf{Y}$ *is a derived scheme whose truncation is the usual fibered product of schemes. The* ∞*-sheaf of the derived structure sheaf* $\mathscr{O}_\mathsf{Z}$ *are* $\pi_n(\mathscr{O}_\mathsf{Z}) = \mathscr{T}\mathrm{or}_n^{\mathscr{O}_\mathsf{S}}(\mathscr{O}_\mathsf{X}, \mathscr{O}_\mathsf{Y})$*.*

Example 3.21 (Self-Intersections). *For simplicity we assume that* Y *is a local complete intersection in* X *of ordinary schemes, and we let* $\mathscr{I} \subset \mathscr{O}_{\mathsf{X}}$ *be its ideal sheaf. The conormal bundle of* Y *inside* X *is then* $\mathscr{N}^{\vee} \cong \mathscr{I}/\mathscr{I}^2$*, which is a vector bundle on* Y*. When* $\mathsf{X} = \mathbf{Spec}\mathsf{A}$ *is affne, and* $\mathsf{Y} = \mathbf{Spec}\mathsf{A}/\mathrm{I}$*, the derived scheme* $\mathbf{Z} := \mathsf{Y} \times_{\mathsf{X}} \mathsf{Y}$ *can be understood in a very explicit manner. Let* $(\mathsf{f}_1, \dots, \mathsf{f}_r)$ *en regular sequence generating* $\mathscr{I}$*. We consider the derived ring* $\mathsf{K}(\mathsf{A}, \mathsf{f})$*, which is obtained by freely adding a 1-simplices* h_i *to* A *such that* $\mathsf{d}_0(\mathsf{h}_i) = 0$ *and* $\mathsf{d}_1(\mathsf{h}_1) = \mathsf{f}_i$ *(see [TV08], proof of proposition 4.9, for details). The derived ring* $\mathsf{K}(\mathsf{A}, \mathsf{f})$ *has a natural augmentation* $\mathsf{K}(\mathsf{A}, \mathsf{f}) \to \mathsf{A}/\mathrm{I}$ *which is an equivalence because the sequence is regular. It is moreover a cellular* A*-algebra, by construction, and thus the derived ring* $\mathsf{A}/\mathrm{I} \otimes^{\mathbb{L}}_{\mathsf{A}} \mathsf{A}/\mathrm{I}$ *can be identified with* $\mathsf{B} = \mathsf{K}(\mathsf{A}, \mathsf{f}) \otimes_{\mathsf{A}} \mathsf{A}/\mathrm{I}$*. This derived ring is an* A/I*-algebra such that* $\pi_1(\mathsf{B}) \simeq \mathrm{I}/\mathrm{I}^2$*. As* I/I^2 *is a projective* A/I*-module we can represent the isomorphism* $\pi_1(\mathsf{B}) \cong \mathrm{I}/\mathrm{I}^2$ *by a morphism of simplicial* A/I*-modules* $\Sigma(\mathrm{I}/\mathrm{I}^2) \longrightarrow \mathsf{B}$*, where* Σ *denotes the suspension in the* ∞*-category of simplicial modules. This produces a morphism of derived rings* $\mathrm{Sym}_{\mathsf{A}/\mathrm{I}}(\Sigma(\mathrm{I}/\mathrm{I}^2)) \to \mathsf{B}$ *where* $\mathsf{Sym}_{\mathsf{A}/\mathrm{I}}$ *denotes here the* ∞*-functor sending an* A/I*-module* M *to the derived* A/I*-algebras it generates. This morphism is an equivalence in characteristio zero, and thus we have in this case*

$$\mathsf{Z} \cong \mathbf{Spec}\left(\mathrm{Sym}_{\mathsf{A}/\mathrm{I}}(\Sigma(\mathrm{I}/\mathrm{I}^2))\right).$$

3.5 Derived Stacks

Definition 3.22. *A derived stack is an* $(\infty, 1)$*-sheaf on the category of derived affine schemes (or simplicial commutative rings) with some Grothendieck topology. The* $(\infty, 1)$*-category* $\mathbf{DSta}$ *of derived stacks is the full* $(\infty, 1)$*-subcategory of* $\mathbf{Fun}_\infty(\mathbf{DAff}, \mathbf{Grpd}^\infty)$ *(or* $\mathbf{Fun}_\infty(\mathbf{sComm}, \mathbf{Grpd}^\infty)$ *if we want).*

We then define derived n*-Artin stacks for* $\mathsf{n} \geqslant -1$ *by induction, exactly as for higher stacks, but starting with derived* -1*-Artin stacks being derived affine schemes. A derived stack* $\mathbf{X}$ *which is locally a derived n-Artin stack for some* n *is a derived Artin stack.*

If $\mathbf{X}$ *is a derived Artin stack then* $\mathsf{t}_0(\mathbf{X})$ *is a higher Artin stack. We call* $\mathbf{X}$ *a derived Artin* n*-stack if* $\mathsf{t}_0(\mathbf{X})$ *is a higher Artin* n*-stack. Write* $\mathbf{DArt} \subset \mathbf{DSta}$ *for the full* ∞*-subcategory of derived Artin stacks, and* $\mathbf{DArt}^{\mathsf{n}} \subset \mathbf{DArt}$ *for the full* ∞*-subcategory of derived Artin* n*-stacks.*

Definition 3.23. *There is a notion of when a simplicial commutative* $\Bbbk$*-algebra is finitely presented. Roughly, it means there are only finitely many generators and (higher) relations. When* $\mathrm{char}(\Bbbk) = 0$*, the parallel notion for cdgas is a free graded polynomial* $\Bbbk$*-algebra with finitely many generators.*

A derived $\Bbbk$*-stack* $\mathbf{X}$ *is locally finitely presented if it is locally modelled on* $\mathbf{Spec}\mathsf{A}^\bullet$ *for finitely presented* $\mathsf{A}^\bullet$*. Locally finitely presented* $\mathbf{X}$ *are particularly nice. They have perfect cotangent complexes* $\mathbb{L}_{\mathbf{X}}$ *(see later).*

Theorem 3.24 (Toën-Vaquié). *If* X *is a smooth projective* $\Bbbk$*-scheme then the derived moduli stack* $\mathbf{M}(\mathrm{D^b Coh}(\mathrm{X}))$ *is a locally finitely presented derived Artin stack.*

Remark 3.25. *Note that if* X *is a singular scheme or stack then* $\mathsf{i}(\mathsf{X})$ *is generally not locally finitely presented, so* $\mathsf{i}(\mathsf{X})$ *is not 'nice' as a derived stack where* $\mathsf{i} : \mathbf{Sta} \hookrightarrow \mathbf{DSta}$*. Often it is better to consider a derived version* $\mathbf{X}$ *of* X *with* $\mathsf{X} \neq \mathsf{i}(\mathsf{X})$ *but* $\mathsf{t}_0(\mathbf{X}) = \mathsf{X}$*.*

4 Geometry of Derived Stacks

4.1 Several Properties of Derived Schemes

Lemma 4.1. *For a derived stack* $\mathbf{X}$*, then* $\mathbf{X}$ *and* $\mathsf{t}_0(\mathbf{X})$ *have the same topology. Moreover, the truncation* $\mathsf{t}_0(-)$ *gives a bijective between Zariski open substacks of* $\mathsf{t}_0(\mathbf{X})$ *and Zariski open derived substacks of* $\mathbf{X}$*.*

Proof. See Proposition 2.1 in [STV13]. □

Definition 4.2 (Coherent Sheaves). *Fix a derived scheme* $\mathbf{X}$*, we define the quasi-coherent derived* ∞*-category* $\mathrm{L}_{\mathrm{Qcoh}}(\mathbf{X})$*.*

We consider the ∞*-category* $\mathbf{Zaff}(\mathbf{X})$ *of affine open derived subschemes* $\mathbf{U} \subset \mathbf{X}$*. For each object* $\mathbf{U} \in \mathbf{Zaff}(\mathbf{X})$ *we have its derived ring of functions* $\mathsf{A}_{\mathbf{U}} := \Gamma(\mathbf{U}, \mathscr{O}_{\mathbf{U}})$*. The simplicial ring* $\mathsf{A}_{\mathbf{U}}$ *can* *be normalized to a cdga $\mathcal{N}(\mathsf{A}_{\mathbf{U}})$, for which we can consider the category of unbounded $\mathcal{N}(\mathsf{A}_{\mathbf{U}})$-dg-modules (see [Sch03] for more about the monoidal properties of the normalization functor). Consider Dwyer-Kan localization of this category along quasi-isomorphisms defines an ∞-category $\mathsf{L}_{\mathsf{Qcoh}}(\mathbf{U}) := \mathsf{L}^{\textit{quasi-iso}}(\mathcal{N}(\mathsf{A}_{\mathbf{U}}))$.*

For each inclusion of open derived affine subschemes $\mathbf{V} \subset \mathbf{U} \subset \mathbf{X}$, we have a morphism of cdgas $\mathcal{N}(\mathsf{A}_{\mathbf{U}}) \to \mathcal{N}(\mathsf{A}_{\mathbf{A}})$ and thus an induced base change ∞-functor

$$(-) \otimes^{\mathbb{L}}_{\mathcal{N}(\mathsf{A}_{\mathbf{U}})} \mathcal{N}(\mathsf{A}_{\mathbf{V}}) : \mathsf{L}_{\mathsf{Qcoh}}(\mathbf{U}) \to \mathsf{L}_{\mathsf{Qcoh}}(\mathbf{V}).$$

This defines an ∞-functor $\mathsf{L}_{\mathsf{Qcoh}}(-) : \mathbf{Zaff}(\mathbf{X})^{\mathrm{op}} \to \mathbf{Cat}^{\infty}$ which moreover is a higher stack actually. Then we define

$$\mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X}) := \varprojlim_{\mathbf{U} \in \mathbf{Zaff}(\mathbf{X})^{\mathrm{op}}} \mathsf{L}_{\mathsf{Qcoh}}(\mathbf{U}) \in \mathbf{Cat}^{\infty}.$$

For $\mathsf{E} \in \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X})$, we have cohomology sheaves $\mathsf{H}^{\mathsf{i}}(\mathsf{E})$ that are quasi-coherent on $\mathsf{t}_0(\mathbf{X})$, and which we will also denote by $\pi_{\mathsf{i}}(\mathsf{E})$.

For a morphism between derived schemes $\mathsf{f} : \mathbf{X} \to \mathbf{Y}$ there is an natural pull-back ∞-functor $\mathsf{f}^* : \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{Y}) \to \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X})$, as well as its right adjoint the push-forward $\mathsf{f}_* : \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X}) \to \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{Y})$. These are first defined locally on the level of affne derived schemes: the ∞-functor f^* is induced by the base change of derived rings whereas the ∞-functor f_* is a forgetful ∞-functor. The general case is done by gluing the local constructions (see Section 1.1 in [Toë12] for details).

Proposition 4.3. *For any commutative square of derived schemes*

$$\begin{array}{ccc} \mathbf{X}' & \xrightarrow{\mathsf{g}} & \mathbf{X} \\ \downarrow{\scriptstyle \mathsf{q}} & & \downarrow{\scriptstyle \mathsf{p}} \\ \mathbf{Y}' & \xrightarrow{\mathsf{f}} & \mathbf{Y} \end{array}$$

then we have a natural morphism between ∞-functors $\mathsf{f}^*\mathsf{p}_* \Rightarrow \mathsf{q}_*\mathsf{g}^* : \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X}) \to \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{Y}')$. Moreover, if those schemes are all qcqs and the square is cartesian, then this is an equivalence of ∞-functors.

Proof. See Proposition 1.4 in [Toë12]. □

When $\mathsf{X}, \mathsf{Y}, \mathsf{Y}'$ are ordinary schemes with f and p are not Tor-independent, then we have the ordinary base-change theorem since in this case the derived fiber product is again an ordinary scheme. When it is not, the derived scheme $\mathbf{X}'$ is not a scheme and the difference between $\mathbf{X}'$ and its truncation $\mathsf{t}_0(\mathbf{X}')$ measures the excess of intersection. All the classical excess intersection formulae can be recovered from the base change formula for derived schemes.

Definition 4.4. *For $\mathbf{X}$ a derived scheme and $\mathsf{E} \in \mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X})$ whose cohomology is concentrated in nonpositive degrees, we can form a derived scheme $\mathbf{X}[\mathsf{E}] := \mathbf{Spec}\,(\mathscr{O}_{\mathsf{X}} \oplus \mathsf{E})$. The derived scheme $\mathbf{X}[\mathsf{E}]$ sits under the derived scheme $\mathbf{X}$ itself and is considered in the comma ∞-category $\mathbf{X}/\mathbf{dSch}$ of derived schemes under $\mathbf{X}$. The mapping space $\mathrm{Map}_{\mathbf{X}/\mathbf{dSch}}(\mathbf{X}[\mathsf{E}], \mathbf{X})$ is called the space of derivations on $\mathbf{X}$ with coefficients in E. It is possible to show the existence of an object $\mathbb{L}_{\mathbf{X}} \in \mathsf{L}^{\leqslant 0}_{\mathrm{Qcoh}}(\mathbf{X})$ together with a universal derivation $\mathbf{X}[\mathbb{L}_{\mathbf{X}}] \to \mathbf{X}$. The object $\mathbb{L}_{\mathbf{X}}$ together with the universal derivation are characterized by the following universal property*

$$\mathrm{Map}_{\mathbf{X}/\mathbf{dSch}}(\mathbf{X}[\mathsf{E}], \mathbf{X}) \simeq \mathrm{Map}_{\mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X})}(\mathbb{L}_{\mathbf{X}}, \mathsf{E}).$$

The object $\mathbb{L}_{\mathsf{X}}$ is called the absolute cotangent complex *of $\mathbf{X}$.*

For any morphism of derived schemes $\mathsf{f} : \mathbf{X} \to \mathbf{Y}$, there is a natural morphism $\mathsf{f}^*(\mathbb{L}_{\mathbf{Y}}) \to \mathbb{L}_{\mathbf{X}}$ in $\mathsf{L}_{\mathrm{Qcoh}}(\mathbf{X})$, and the relative cotangent complex *of f defined as one*

$$\mathbb{L}_{\mathbf{X}/\mathbf{Y}} = \mathbb{L}_{\mathsf{f}} := \mathrm{cofiber}\,(\mathsf{f}^*(\mathbb{L}_{\mathbf{Y}}) \to \mathbb{L}_{\mathbf{X}})\,.$$

Or we can seen them as $\mathbb{L}_{\mathsf{f}} : \mathsf{f}^*(\mathbb{L}_{\mathbf{Y}}) \to \mathbb{L}_{\mathbf{X}}$. It is an object in $\mathsf{L}^{\leqslant 0}_{\mathrm{Qcoh}}(\mathbf{X})$, and equiped with a universal derivation $\mathsf{X}[\mathbb{L}_{\mathsf{f}}] \to \mathsf{X}$ which is now a morphism in the double comma ∞-category $\mathbf{X}/\boldsymbol{dSch}/\mathbf{Y}$.

Proposition 4.5. *Over derived schemes, we have the following facts of cotangent complexes:*

(a) For any cartesian square of derived schemes

$$\begin{tikzcd} \mathbf{X}' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbf{X} \arrow[d, "p"] \\ \mathbf{Y}' \arrow[r, "f"'] & \mathbf{Y} \end{tikzcd}$$

Then the natural morphism $g^*(\mathbb{L}_p) \to \mathbb{L}_q$ is an equivalence in $L_{\mathrm{Qcoh}}(\mathbf{X}')$. Moreover there is a distinguished triangle of cotangent complexes

$$(f \circ q)^* \mathbb{L}_{\mathbf{Y}} \xrightarrow{q^*\mathbb{L}_f \oplus g^*\mathbb{L}_p} q^*\mathbb{L}_{\mathbf{Y}'} \oplus g^*\mathbb{L}_{\mathbf{X}} \xrightarrow{\mathbb{L}_q \oplus -\mathbb{L}_g} \mathbb{L}_{\mathbf{X}'} \longrightarrow \cdots$$

(b) If $\mathbf{X}$ is locally finitely presented then $\mathbb{L}_{\mathbf{X}}$ is a perfect complex. We define the virtual dimension *$d^{\mathrm{vir}}\mathbf{X} = \mathrm{rank}(\mathbb{L}_{\mathbf{X}})$. In this case we can define the* tangent complex *$\mathbb{T}_{\mathbf{X}} := \mathbb{L}_{\mathbf{X}}^{\vee}$.*

Definition 4.6.
- *A derived scheme $\mathbf{X}$ with $\mathbb{L}_{\mathbf{X}}$ perfect in $[-1, 0]$ is called* quasi-smooth.
- *A morphism of derived schemes $f : \mathbf{X} \to \mathbf{Y}$ will be called* étale (resp. smooth) *if it is locally of finite presentation and if $\mathbb{L}_f$ vanishes (resp. $\mathbb{L}_f$ is a vector bundle on $\mathbf{X}$).*

An étale (resp. smooth) morphism $f : \mathbf{X} \to \mathbf{Y}$ of derived schemes induces an étale (resp. smooth) morphism on the truncations $t_0(f) : t_0(\mathbf{X}) \to t_0(\mathbf{Y})$, which is moreover flat: for all i the natural morphism $t_0(f)^*(\pi_i(\mathscr{O}_Y)) \to \pi_i(\mathscr{O}_X)$ is an isomorphism of quasi-coherent sheaves.

4.2 Several Properties of Derived Artin Stacks

Similar as derived schemes, we have the following definitions.

Definition 4.7. *Let $\mathbf{X}$ be a derived Artin stack.*

- *We define its* quasi-coherent derived ∞-category *$L_{\mathrm{Qcoh}}(\mathbf{X})$ as:*

$$L_{\mathrm{Qcoh}}(\mathbf{X}) := \varprojlim_{\mathbf{S} \in \mathbf{dSch}/\mathbf{X}} L_{\mathrm{Qcoh}}(\mathbf{S}),$$

  where the limit is taken along the ∞-category of all derived schemes over $\mathbf{X}$. By using descent, we could also restrict to affne derived scheme over X and get an equivalent definition.

- *For $M \in L_{\mathrm{Qcoh}}(\mathbf{X})$, with cohomology sheaves concentrated in nonpositive degrees, we set $\mathbf{X}[M] := \mathbf{Spec}\,(\mathscr{O}_X \oplus M)$, the trivial square zero infnitesimal extension of X by M. The object $\mathbf{X}[M]$ sits naturally under X. The* cotangent complex *of X is the object $\mathbb{L}_{\mathbf{X}} \in L_{\mathrm{Qcoh}}(\mathbf{X})$ such that for all $M \in L_{\mathrm{Qcoh}}(\mathbf{X})$ as above, we have functorial equivalences*

$$\mathrm{Map}_{\mathbf{X}/\mathbf{dSt}}(\mathbf{X}[M], \mathbf{X}) \simeq \mathrm{Map}_{L_{\mathrm{Qcoh}}(\mathbf{X})}(\mathbb{L}_{\mathbf{X}}, M).$$

The existence of such the object $\mathbb{L}_X$ is a theorem, whose proof can be found in [TV08] Corollary 2.2.3.3.

We define the relative cotangent complex as before. Cotangent complexes of derived Artin stacks behave similarly to the case of derived schemes: functoriality and stability by base-change. The smooth and étale morphisms between derived Artin stacks have similar characterizations using cotangent complexes (see [TV08] section 2.2.5).

A finitely presented morphism $f : \mathbf{X} \to \mathbf{Y}$ between derived Artin stacks is étale if and only if the relative cotangent complex $\mathbb{L}_f$ vanishes. The same morphism is smooth if and only if the relative cotangent complex $\mathbb{L}_f$ has positive Tor amplitude.

Remark 4.8. *We note here that cotangent complexes of derived Artin stacks might not be themselves cohomologically concentrated in nonpositive degrees. It is a general fact that if $\mathbf{X}$ is a derived n-Artin stack, then $\mathbb{L}_{\mathbf{X}} \in L_{\mathrm{Qcoh}}^{\leqslant n}(\mathbf{X})$.*

Definition 4.9. *A morphism $f : \mathbf{X} \to \mathbf{Y}$ of derived Artin stacks is* quasi-smooth *if it is locally of finite presentation and the relative cotangent complex $\mathbb{L}_{\mathbf{X}/\mathbf{Y}}$ is perfect in the interval $[-1, 1]$.*

The relative virtual dimension *of a quasi-smooth morphism $f : \mathbf{X} \to \mathbf{Y}$ is $\operatorname{rank}(\mathbb{L}_{\mathbf{X}/\mathbf{Y}})$.*

Here we give some examples.

Example 4.10 (Derived Moduli of Perfect Complexes)**.** *For this we fix two integers $a \leqslant b$ and we define a derived stack $\mathbb{R}\mathbf{Perf}^{[a,b]} \in \mathbf{DSta}$, classifying perfect complexes of amplitude contained in $[a, b]$. As an ∞-functor it sends a derived scheme $\mathbf{S}$ to the ∞-groupoid of perfect objects in $L_{\mathrm{Qcoh}}(\mathbf{S})$ with amplitude contained in $[a, b]$. We remind here that the amplitude of a perfect complex E on S is contained in $[a, b]$ if its cohomology sheaves are universally concentrated in degree $[a, b]$: for every derived scheme S' and every morphism $u : S' \to S$, we have $H^i(u^*(E)) = 0$ for $i \notin [a, b]$.*

Theorem 4.11. *The derived stack $\mathbb{R}\mathbf{Perf}^{[a,b]}$ is a derived Artin stack locally of fnite presentation over* $\operatorname{Spec}\mathbb{Z}$.

There is also a derived stack $\mathbb{R}\mathbf{Perf}$, classifying all perfect complexes, without any restriction on the amplitude. The derived stack $\mathbb{R}\mathbf{Perf}$ is covered by open derived substacks $\mathbb{R}\mathbf{Perf}^{[a,b]}$ and is itself an increasing union of substacks. Such derived stacks are called locally geometric *but we will allow ourselves to keep using the expression derived Artin stack. To be more precise, $\mathbb{R}\mathbf{Perf}^{[a,b]}$ is a derived $(b - a + 1)$-Artin stack. We refer [TV07] for more details.*

Example 4.12 (Derived Moduli of Stable Maps)**.** *Let X be a smooth and projective scheme over $\mathbb{C}$. We fix $\beta \in H_2(X(\mathbb{C}), \mathbb{Z})$ a curve class. We consider $\mathcal{M}^{\mathrm{pre}}_{g,n}$ the smooth Artin stack of pre-stable curves of genus g an n marked points. It can be considered as a derived Artin stack $\boldsymbol{\mathcal{M}}^{\mathrm{pre}}_{g,n}$ and thus as an object in $\mathbf{DSta}_{\mathbb{C}}$. We let $\boldsymbol{\mathcal{C}}_{g,n} \to \boldsymbol{\mathcal{M}}^{\mathrm{pre}}_{g,n}$ the universal pre-stable curve. We let*

$$\boldsymbol{\mathcal{M}}^{\mathrm{pre}}_{g,n}(X, \beta) = \mathbb{R}\mathrm{Map}_{\mathbf{DSta}/\boldsymbol{\mathcal{M}}^{\mathrm{pre}}_{g,n}}(\boldsymbol{\mathcal{C}}_{g,n}, X)$$

be the relative derived mapping stack of $\boldsymbol{\mathcal{C}}_{g,n}$ to X with fixed class β. The derived stack $\boldsymbol{\mathcal{M}}^{\mathrm{pre}}_{g,n}(X, \beta)$ is a derived Artin stack, as this can be deduced from the representability of the derived mapping scheme. It contains an open derived Deligne-Mumford substack $\overline{\boldsymbol{\mathcal{M}}}_{g,n}(X, \beta)$ which consists of stable maps. The derived stack $\overline{\boldsymbol{\mathcal{M}}}_{g,n}(X, \beta)$ is proper and locally of fnite presentation over $\operatorname{Spec}\mathbb{C}$*, and can be used in order to recover Gromov-Witten invariants of X.*

Finally we state the representability theorem of Lurie, an extremely powerful tool for proving that a given derived stack is a derived Artin stack, thus extending to the derived setting the famous Artin's representability theorem.

Theorem 4.13 (Artin-Lurie Representability Theorem)**.** *Let R be a noetherian commutative ring. A derived stack $\mathbf{X} \in \mathbf{DSta}_R$ is a derived Artin stack locally of finite presentation over* $\operatorname{Spec} R$ *if and only if the following conditions are satisfied.*

- *There is an integer $n \geqslant 0$ such that for any underived affine scheme S over R the simplicial set $\mathbf{X}(S)$ is n-truncated.*
- *For any filtered system of derived R-algebras $A = \varinjlim A_\alpha$ the natural morphism $\varinjlim \mathbf{X}(A_\alpha) \to \mathbf{X}(A)$ is an equivalence.*
- *Take any derived R-algebra A with Postnikov tower*

  $$A \to \cdots \to A_{\leqslant k} \to A_{\leqslant k-1} \to \cdots \to \pi_0(A),$$

  that is, as a tower of morphisms in the ∞-category of stacks of derived rings with $\pi_i(A_{\leqslant n}) = 0$ for any $i > n$ and the morphism $A \to A_{\leqslant n}$ induces isomorphisms $\pi_i(A) \cong \pi_i(A_{\leqslant n})$ for $i \leqslant n$. Then

  $$\mathbf{X}(A) \to \varprojlim \mathbf{X}(A_{\leqslant n})$$

  is an equivalence.
- *The derived stack $\mathbf{X}$ has an obstruction theory (see Section 1.4.2 in [TV08] for details).*

- *For any local noetherian R-algebra A with maximal ideal $\mathfrak{m} \subset \mathsf{A}$, the natural morphism*

$$\mathbf{X}(\widehat{\mathsf{A}}) \to \varprojlim \mathbf{X}(\mathsf{A}/\mathfrak{m}^{\mathsf{k}})$$

is an equivalence.

Proof. See the Lurie's DAG series which can be find in `http://www.math.harvard.edu/~lurie/`. □

Corollary 4.14. *Let X be a flat and proper scheme over some base scheme S and $\mathbf{M}$ be a derived Artin stack which is locally of finite presentation over S. Then the derived mapping stack $\mathbb{R}\mathrm{Map}_{\mathbf{DSta}/\mathsf{S}}(\mathsf{X}, \mathbf{M})$ is again a derived Artin stack locally of finite presentation over S.*

4.3 Obstruction Theories and Virtual Classes

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