A Quick Tour of Derived Algebraic Geometry

Xiaolong Liu

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Contents

1 Introduction

We will assume the base ring $\mathbb{K}$ containing $\mathbb{Q}$. We will follows [EP23] and Joyce's slides as in `https://people.maths.ox.ac.uk/~joyce/DAG2022/index.html`.

2 Commutative Differential Graded Algebras

Definition 2.1. *(a) A* differential graded $\mathbb{K}$-algebra *(dga or dg-algebra for short)* $A^\bullet = (A^*, d)$ *consists of a chain complex with a unital associative multiplication. Concretely, that is a family of* $\mathbb{K}$*-modules* $\{A^i\}_{i\in\mathbb{Z}}$*, an associative* $\mathbb{K}$*-linear multiplication* $(-\cdot-) : A^i \times A^j \to A^{i+j}$ *(for all* i, j*), a unit* $1 \in A^0$ *and a differential* $d : A^i \to A^{i+1}$ *(for all* i*) which is* $\mathbb{K}$*-linear, satisfies* $d^2 = 0$ *and is a derivation with respect to the multiplication, which means* $d(a\cdot b) = d(a)\cdot b + (-1)^{\deg(a)} a \cdot d(b)$.

(b) A graded $\mathbb{K}$*-algebra A is* graded-commutative *if* $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} b \cdot a$.

(c) A morphism of dg-algebras *is a map* $f : A^\bullet \to B^\bullet$ *that respects the differentials (i.e.* $fd_A = d_B f$*), and the multiplication (i.e.* $f(a \cdot_A b) = f(a) \cdot_B f(b)$*).*

In this note we mainly focus on the following:

Definition 2.2. *We define the* category $\mathbf{cdg^-Alg}_{\mathbb{K}}$ of the graded-commutative differential graded $\mathbb{K}$-algebras which are concentrated in non-positive degree. *Hence* $A^\bullet \in \mathbf{cdg^-Alg}_{\mathbb{K}}$ *we have* $A^\bullet = \bigoplus_{k=0}^{-\infty} A^k$.

Moreover, for $R^\bullet \in \mathbf{cdg^-Alg}_{\mathbb{K}}$, *we can define* $\mathbf{cdg^-Alg}_{R^\bullet} := R^\bullet \downarrow \mathbf{cdg^-Alg}_{\mathbb{K}}$ *of cdga* $R^\bullet$*-algebras.*

Definition 2.3. *(a) Let* $A^\bullet, B^\bullet \in \mathbf{cdg^-Alg}_{R^\bullet}$. *A morphism* $f : A^\bullet \to B^\bullet$ *of* $R^\bullet$*-cdga is a* quasi-isomorphism *(or* weak equivalence*) if it induces an isomorphism on cohomology* $H^*(A^\bullet) \cong H^*(B^\bullet)$.

(b) We say that $R^\bullet$*-cdga* $A^\bullet$ *and* $B^\bullet$ *are* quasi-isomorphic *if there exists a diagram* $A^\bullet \leftarrow C^\bullet \to B^\bullet$ *of quasi-isomorphisms in* $\mathbf{cdg^-Alg}_{R^\bullet}$.

Remark 2.4. *Note that there is a global version of these.*

A very important example of cdgas and derived schemes:

Example 2.5. *Let* $M^\bullet$ *be a graded* $\mathbb{K}$*-module. The free graded-commutative* $\mathbb{K}$*-algebra generated by* $M^\bullet$ *is*

$$\mathbb{K}[M^\bullet] := \left(\bigoplus \mathrm{Sym}^n M^{\mathrm{even}}\right) \otimes_{\mathbb{K}} \left(\bigoplus \bigwedge^n M^{\mathrm{odd}}\right)$$

with the degree of a product of elements being the sum of the degrees of those elements.

In particular, we consider the free graded-commutative $\mathbb{K}$*-algebra* $A^\bullet$ *generated by* $x_1, ..., x_m; y_1, ..., y_n$ *where* $\deg x_i = 0$ *and* $\deg y_j = -1$. *Hence*

$$A^k = \mathbb{K}[x_1, ..., x_m] \otimes_{\mathbb{K}} \bigwedge^{-k} \mathbb{K}^n$$

for $k = 0, -1, ..., -n$ *and* $A^k = 0$ *for otherwise. Pick* $p_1, ..., p_n \in \mathbb{K}[x_1, ..., x_m]$, *as* $A^\bullet$ *is free there are unique maps* $d : A^k \to A^{k+1}$ *satisfying the Leibnitz rule, such that* $d(y_i) = p_i(x_1, ..., x_m)$ *for* $i = 1, ..., n$. *Also* $d^2 = 0$ *and* $A^\bullet \in \mathbf{cdg^-Alg}_{\mathbb{K}}$.

Now $H^0(A^\bullet) = \mathbb{K}[x_1, ..., x_m]/(p_1, ..., p_n)$ *and hence* $\mathrm{Spec}\, H^0(A^\bullet)$ *is a subscheme defined by these polynomials. Now the derived scheme* $\mathbf{Spec}\, A^\bullet$ *remembers information about the dependencies between* $p_1, ..., p_n$ *which have more information than the truncated classical scheme* $\mathrm{Spec}\, H^0(A^\bullet)$.

3 Basic Concepts of Infinity Categories

In this lecture '∞-category' always means '$(\infty, 1)$-category', that is, all n-morphisms are invertible for $n \geqslant 2$. (Although 'n-morphism' may not make sense, depending on your model for ∞-categories.)

There are a bunch of different but related structures which are more-or-less kinds of ∞-category:

- Model categories.
- Categories enriched in topological spaces.
- Simplicial categories; simplicial model categories.
- Quasicategories.

Of these, model categories are the oldest (Quillen 1967), and look least like an ∞-category (they have no visible higher morphisms). But most of the other kinds of ∞-category use model categories under the hood. Toën-Vezzosi's DAG is written in terms of model categories and simplicial categories. Lurie works with quasicategories, which may be the best and coolest version.

- If you start with an ordinary category $\mathcal{C}$ and invert some class of morphisms $\mathcal{W}$ in $\mathcal{C}$ ('weak equivalences'), the result $\mathcal{C}[\mathcal{W}^{-1}]$ should really be an ∞-category with homotopy category $\mathsf{Ho}(\mathcal{C}[\mathcal{W}^{-1}])$ an ordinary category.

This idea is similar as derived category $\mathbf{D}(\mathcal{A})$ construct from $\mathsf{Ho}(\mathrm{Com}(\mathcal{A}))$ by inverting the class $\mathcal{W}$ of quasi-isomorphisms. Note that $\mathbf{D}(\mathcal{A}) = \mathsf{Ho}(\mathbb{D}(\mathcal{A}))$ for a stable ∞-category $\mathbb{D}(\mathcal{A})$.

Here we give some idea how to consider the ∞-category. We know that a $(2,1)$-category $\mathfrak{C}$ is acategory enriched in groupoids. A $(3,1)$-category $\mathfrak{C}$ is acategory enriched in 2-groupoids. So similarly, an $(\infty, 1)$-category is really a 'category enriched in ∞-groupoids'. But what is an ∞-groupoid?

Two models for the (model/∞-)category of ∞-groupoids are topological spaces $\mathbf{Top}$ (up to homotopy), and simplicial sets $\mathbf{sSets}$. Note that $\mathbf{Top}$ and $\mathbf{sSets}$ are Quillen equivalent as model categories, theories of ∞-categories based on $\mathbf{Top}$ and $\mathbf{sSets}$ are essentially equivalent. But it seems no one uses categories enriched in $\mathbf{Top}$ except as motivation.

3.1 Categories Enriched in Topological Spaces

Our first model for an $(\infty, 1)$-category is the categories enriched in topological spaces:

Definition 3.1. *A category enriched in topological spaces is a category $\mathcal{C}$ such that for all objects X, Y in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$ of morphisms $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is given the structure of a topological space (generally a nice topological space, e.g. Hausdorff,..., and homotopy equivalent to a CW complex), and for objects $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ the composition $\mu_{\mathsf{X},\mathsf{Y},\mathsf{Z}} : \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y}) \times \mathrm{Hom}_{\mathcal{C}}(\mathsf{Y}, \mathsf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Z})$ is a continuous map. Moreover, there is a homotopy between $\mu_{\mathsf{W},\mathsf{X},\mathsf{Y}} \circ (\mu_{\mathsf{W},\mathsf{X},\mathsf{Y}} \times \mathrm{id}) \to \mu_{\mathsf{W},\mathsf{X},\mathsf{Y}} \circ (\mathrm{id} \times \mu_{\mathsf{X},\mathsf{Y},\mathsf{Z}})$.*

The higher-morphism of $\mathcal{C}$ defined as follows:

- *A 1-morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is a point of $\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$.*
- *If $\mathsf{f}, \mathsf{g} : \mathsf{X} \to \mathsf{Y}$ are 1-morphisms, a 2-morphism $\eta : \mathsf{f} \Rightarrow \mathsf{g}$ is a continuous path $\eta : [0, 1] \to \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$ with $\eta(0) = \mathsf{f}$ and $\eta(1) = \mathsf{g}$. Note that η is invertible with $\eta^{-1}(\mathsf{s}) = \eta(1 - \mathsf{s})$.*
- *If $\eta, \zeta : \mathsf{f} \Rightarrow \mathsf{g}$ are 2-morphisms, a 3-morphism $\aleph : \eta \Rrightarrow \zeta$ is a continuous map $\aleph : [0, 1]^2 \to \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$ such that for $\mathsf{s}, \mathsf{t} \in [0, 1]$ we have*

$$\aleph(0, \mathsf{t}) = \mathsf{f}, \aleph(1, \mathsf{t}) = \mathsf{g}, \aleph(\mathsf{s}, 0) = \eta, \aleph(\mathsf{s}, 1) = \zeta.$$

- *n-morphisms are continuous maps $[0, 1]^{\mathsf{n}-1} \to \mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y})$ with prescribed boundary conditions on $\partial([0, 1]^{\mathsf{n}-1})$.*
- *Moreover, if $\eta : \mathsf{f} \Rightarrow \mathsf{g}$, $\zeta : \mathsf{g} \Rightarrow \mathsf{h}$ are 2-morphisms, the vertical composition $\zeta \odot \eta : \mathsf{f} \Rightarrow \mathsf{h}$ is $(\zeta \odot \eta)(\mathsf{s}) = \eta(2\mathsf{s})$ if $\mathsf{s} \in [0, 1/2]$ and $(\zeta \odot \eta)(\mathsf{s}) = \zeta(2\mathsf{s} - 1)$ if $\mathsf{s} \in [1/2, 1]$. This is not associative, but is associative up to homotopy, i.e. up to 3-isomorphism. Other kinds of composition can be defined in a similar way.*

Definition 3.2. *For any higher category $\mathcal{C}$ (as we will used), the homotopy category $\mathsf{Ho}(\mathcal{C})$ which is an ordinary category, where objects X of $\mathsf{Ho}(\mathcal{C})$ are objects of $\mathcal{C}$, and morphisms $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathsf{Ho}(\mathcal{C})$ are 2-isomorphism classes of 1-morphisms $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{C}$.*

Now for a category enriched in topological spaces $\mathcal{C}$. we have $\mathrm{Hom}_{\mathsf{Ho}(\mathcal{C})}(\mathsf{X}, \mathsf{Y}) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y}))$.

3.2 Model Categories

Now we introduce some model categories invented by Quillen to abstract methods of homotopy theory into category theory.

Definition 3.3. *A model category is a complete and cocomplete category $\mathcal{M}$ equipped with three distinguished classes of morphisms: the weak equivalences $\mathcal{W}$, the fibrations $\mathcal{F}$, and the cofibrations $\mathcal{C}$. These must satisfy:*

(a) $\mathcal{W}, \mathcal{F}, \mathcal{C}$ are closed under composition and include identities.

(b) $\mathcal{W}, \mathcal{F}, \mathcal{C}$ are closed under retracts. Here f is a retract of g if there exist $\mathsf{i}, \mathsf{j}, \mathsf{r}, \mathsf{s}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{X} \arrow[r, "\mathsf{i}"'] \arrow[rr, bend left, "\mathrm{id}_{\mathsf{X}}"] \arrow[d, "\mathsf{f}"] & \mathsf{Y} \arrow[r, "\mathsf{r}"'] \arrow[d, "\mathsf{g}"] & \mathsf{X} \arrow[d, "\mathsf{f}"] \\
\mathsf{X}' \arrow[r, "\mathsf{j}"] \arrow[rr, bend right, "\mathrm{id}_{\mathsf{X}'}"'] & \mathsf{Y}' \arrow[r, "\mathsf{s}"] & \mathsf{X}'
\end{tikzcd}$$

(c) For $\mathsf{f} : \mathsf{X} \to \mathsf{Y}, \mathsf{g} : \mathsf{Y} \to \mathsf{Z}$ in $\mathcal{M}$, if two of $\mathsf{f}, \mathsf{g}, \mathsf{g} \circ \mathsf{f}$ are in $\mathcal{W}$ then so is the third.

(d) A (co)fibration which is also a weak equivalence is called acyclic. Acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with

respect to acyclic fibrations. Explicitly, if the square below commutes, where i *is a cofibration,* p *is a fibration, and* i *or* p *is acyclic, then there exists* h *as shown:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

(e) *Every morphism* f *in* $\mathcal{M}$ *can be written as* $\mathsf{f} = \mathsf{p} \circ \mathsf{i}$ *for a fibration* p *and an acyclic cofibration* i.

(f) *Every morphism* f *in* $\mathcal{M}$ *can be written as* $\mathsf{f} = \mathsf{p} \circ \mathsf{i}$ *for an acyclic fibration* p *and a cofibration* i.

Some basic elements in a model category:

Definition 3.4. *Let* $(\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ *be a model category with initial object* $\emptyset$ *and final object* $*$.

(a) *An object* $\mathsf{X} \in \mathcal{M}$ *is called* fibrant *if* $[\mathsf{X} \to *] \in \mathcal{F}$, *and* cofibrant *if* $[\emptyset \to \mathsf{X}] \in \mathcal{C}$.

(b) *If* $\mathsf{X} \in \mathcal{M}$ *and there is a weak equivalence* $w : \mathsf{C} \to \mathsf{X}$ *with* C *cofibrant then* C *is a* cofibrant replacement *for* X. *If there is a weak equivalence* $w : \mathsf{X} \to \mathsf{F}$ *with* F *fibrant then* F *is a* fibrant replacement *for* X. *Such replacements always exist.*

(c) *If* $\mathsf{X} \in \mathcal{M}$, *a* cylinder object *is an object* $\mathsf{X} \times [0,1]$ *in* $\mathcal{M}$ *with a factorization* $\mathsf{X} \sqcup \mathsf{X} \xrightarrow{c} \mathsf{X} \times [0,1] \xrightarrow{w} \mathsf{X}$ *of the codiagonal* $\mathsf{X} \sqcup \mathsf{X} \to \mathsf{X}$, *with* c *a cofibration and* w *a weak equivalence. Cylinder objects exist by Definition 3.3(d).*

(d) *If* $\mathsf{X} \in \mathcal{M}$, *a* path object *is an object* $\mathrm{Map}([0,1], \mathsf{X})$ *in* $\mathcal{M}$ *with a factorization* $\mathsf{X} \xrightarrow{w} \mathrm{Map}([0,1], \mathsf{X}) \xrightarrow{f} \mathsf{X} \times \mathsf{X}$ *of the diagonal* $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$, *with* w *a weak equivalence and* f *a fibration. Path objects exist by Definition 3.3(e).*

(e) *Morphisms* $\mathsf{f}, \mathsf{g} : \mathsf{X} \to \mathsf{Y}$ *are called* (left) homotopy equivalent *if there exists* $\mathsf{h} : \mathsf{X} \times [0,1] \to \mathsf{Y}$ *with* $\mathsf{h} \circ \mathsf{c} = \mathsf{f} \sqcup \mathsf{g}$ *where* c *as in (c).*

(f) *The* homotopy category *is* $\mathsf{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$, *the category obtained by formally inverting all weak equivalences. Note that this is independent of* $\mathcal{C}, \mathcal{F}$.

Note that two of $\mathcal{W}, \mathcal{F}, \mathcal{C}$ determine the third. Now we introduce an important theorem:

Theorem 3.5 (Fundamental Theorem of Model Categories)**.** *Let* $\mathcal{M}$ *be a model category. Then* $\mathsf{Ho}(\mathcal{M})$ *is equivalent to the category whose objects are fibrant-cofibrant objects in* $\mathcal{M}$, *and whose morphisms are homotopy classes of morphisms in* $\mathcal{M}$.

Example 3.6. (a) *The category* **Top** *of topological spaces has a model structure with* $\mathcal{W}$ *the weak homotopy equivalences, and* $\mathcal{F}$ *the Serre fibrations (maps with the homotopy lifting property for CW complexes). In this case by Theorem 3.5,* $\mathsf{Ho}(\mathbf{Top})$, *which is the homotopy category of homotopy types, can be described as the category whose objects are CW complexes and morphisms are homotopy classes of continuous maps.*

(b) *If* R *is a commutative ring then* $\mathrm{Com}(\mathrm{Mod}_{\mathsf{R}})$ *has two canonical model structures with weak equivalences quasi-isomorphisms and*

- *cofibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *injective for all* k;
- *fibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *surjective for all* k.

The first one is the injective model category and the second one is the projective model category.

In this case by Theorem 3.5, $\mathsf{Ho}(\mathrm{Com}(\mathrm{Mod}_{\mathsf{R}})) = \mathbf{D}(\mathsf{R})$, *can be described as the category whose objects are either K-injective, or K-projective, complexes, and morphisms are homotopy classes of maps between these complexes.*

(c) *Let* $\mathcal{A}$ *be a Grothendieck abelian category (such as* $\mathrm{Qcoh}(\mathsf{X})$ *for any scheme* X*). Then* $\mathrm{Com}(\mathcal{A})$ *has a model structure with weak equivalences quasi-isomorphisms and cofibrations morphisms* $\phi : \mathsf{E}^\bullet \to \mathsf{F}^\bullet$ *with* $\phi^k : \mathsf{E}^k \to \mathsf{F}^k$ *injective for all* k. *In this case by Theorem 3.5,* $\mathsf{Ho}(\mathrm{Com}(\mathcal{A})) = \mathbf{D}(\mathcal{A})$, *can be described as the category whose objects are either K-injective complexes, and morphisms.*

3.3 Simplicial Sets and Simplicial Categories

3.4 Kan Complexes and Weak Kan Complexes

3.5 Quasicategories

3.6 Stable Infinity Categories

We will give a reason why we need stable ∞-categories:

Example 3.7. *I would argue that triangulated categories are not quite the 'right' theory. However, they are a very good approximation - you can work with them for years and not notice the problems.*

As a signal that there should be something more, recall that if $\mathcal{T}$ is a triangulated category, and $u : X \to Y$ a morphism in $\mathcal{T}$, there is $\mathrm{cone}(u) \in \mathcal{T}$, in a distinguished triangle $X \to Y \to \mathrm{cone}(u) \to X[1]$ in $\mathcal{T}$. This is begging to be turned into a **cone functor:** *we would like a category $\mathrm{Mor}(\mathcal{T})$ of morphisms in $\mathcal{T}$, and a functor $\mathrm{cone} : \mathrm{Mor}(\mathcal{T}) \to \mathcal{T}$ mapping $u \mapsto \mathrm{cone}(u)$ on objects. To try to define* cone *on morphisms in $\mathrm{Mor}(\mathcal{T})$, consider the diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & \mathrm{cone}(u) \arrow[r] \arrow[d, dashed, "\mathrm{cone}(f,g)"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & X'[1] \end{tikzcd}$$

and extension via the definition of triangulated categories. But it is not unique, so we cannot define cone.

So the explanation is $\mathcal{T}$ should be a higher category (an ∞-category)! In this case n-morphisms in $\mathrm{Mor}(\mathcal{T})$ correspond to $(n+1)$-morphisms in $\mathcal{T}$. So to define cone *on 1-morphisms in $\mathrm{Mor}(\mathcal{T})$, we should be using 2-morphisms in $\mathcal{T}$. We consider the diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & \mathrm{cone}(u) \arrow[r] \arrow[d, dashed, "\mathrm{cone}(f,g;\eta)"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] \arrow[ur, phantom, "\Nearrow\eta"] & Y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & X'[1] \end{tikzcd}$$

where $\eta : u' \circ f \Rightarrow g \circ u$ is a 2-morphism. Then $\mathrm{cone}(f, g; \eta)$ should exist and be unique up to 2-isomorphism. When we pass to the homotopy category $\mathrm{Ho}(\mathcal{T})$, this choice of η is forgotten, which is why we lose uniqueness of $\mathrm{cone}(f, g)$. Note moreover that if we want $\mathcal{T}$ and $\mathrm{Mor}(\mathcal{T})$ to be objects of the same type we cannot truncate to n-categories for any finite n — we need $n = \infty$.

3.7 Dg-categories and Segal Categories

3.8 Stable Infinity Categories

4 Derived Schemes and Derived Stacks

4.1 Higher Stacks

4.2 Derived Schemes and Derived Stacks

5 Geometry of Derived Stacks

6 Need to add

References

[EP23] Jon Eugster and Jon P Pridham. An introduction to derived (algebraic) geometry. *Rendiconti di Matematica*, September 2023.