Modern Theory of Moduli Spaces, Stability and Their Related Topics

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Contents

Preface

For the first part, we will mainly follows the paper [6] and the Chapter 6 in the book draft [3] which give us many results and applications.

For the second part, need to add.

Part I

Basic Theory and Constructions of Moduli Spaces

Chapter 1

General Theory of Good Moduli Space

Here we will introduce some basic background about good moduli theory and the theory of Θ-complete and S-completedue to J. Alper in [5] and [6]. These will play an important role in our foundamental theory.

We will give the main properties, theorems and their motivations and some idea of proofs. For the detailed proof we refer reader to the original paper [5][6] or the book draft [3] of J. Alper.

1.1 Properties of Good Moduli Spaces

As we all know, in the modern construction of the moduli space of stable curves follows from the following way:

(a) Construct the stack $\overline{\mathscr{M}}_{g,n}$ and show that it is a Deligne-Mumford stack;

(b) show the stable-reduction of stable curves and find that $\overline{\mathscr{M}}_{g,n}$ is proper;

(c) use Keel-Mori theorem to construct the coarse moduli space $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ and show that it is projective.

But in our case, we can not use Keel-Mori theorem to the moduli stack of semistable sheaves because the inertia stack $\mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ is not finite. In order to this the similar modern way (instead of GIT-construction), J.Alper developed a nice similar (but much more complicated) theory to solve this problem – the theory of good moduli space ([5] and [6]) for linear reductive groups and the theory of adequate moduli spaces ([4]) for geometric reductive groups.

For now, the theory of good moduli space plays a central role in the construction of moduli spaces, such as moduli stack of semistable sheaves $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ and K-moduli stack $\mathscr{X}^{\mathrm{Kss}}_{n,V}$ which aim to construct a good moduli space of Fano varieties (see the book draft due to C. Xu).

Of course we will just introduce some of them and there are many beautiful results we will not introduce, such as the Section 6.6 and 6.7 in [3] which gave us many applications and examples.

Definition 1.1.1 (Good moduli space). *For an algebraic stack $\mathscr{X}$, its good moduli space is an algebraic space X together with a qcqs morphism $\pi : \mathscr{X} \to X$ such that*

(i) the natural map $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism;

(ii) the functor $\pi_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact.

Note that the condition in (ii) is called cohomologically affine.

The definition of good moduli space is inspired from the GIT-quotient of linear reductive group G (that is, $V \mapsto V^G$ is exact. Hence G is linear reductive if and only if $\mathbf{B}G$ is cohomologically affine)

$$[X/G] \dashrightarrow [X^{\mathrm{ss}}/G] \to X /\!\!/ G = \operatorname{Proj} \bigoplus_{d\geq 0} \Gamma(X, \mathscr{O}_X(d))^G.$$

Or locally, the map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. Of coarse, a tame coarse moduli space is a good moduli space by the local structure of coarse moduli spaces.

Here we state several basic properties of cohomologically affine morphisms.

Lemma 1.1.2. *Consider a cartesian*

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\ \mathscr{Y}' \arrow[r, "g"] & \mathscr{Y} \end{tikzcd}$$

of algebraic stacks, then:

(i) If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.

(ii) If Y has quasi-affine diagonal and π is cohomologically affine, then π' is cohomologically affine.

If we consider a cartesian

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

of algebraic stacks where X, X' are quasi-separated algebraic spaces, then:

(iii) *If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.*

(iv) *If π is a good moduli space, so is π'.*

(v) *Let π is a good moduli space. For $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(X)$, the adjunction map $\pi_*\mathscr{F} \otimes \mathscr{G} \cong \pi_*(\mathscr{F} \otimes \pi^*\mathscr{G})$ is an isomorphism. In particular, the adjunction map $\mathscr{G} \cong \pi_*\pi^*\mathscr{G}$ is an isomorphism.*

(vi) *For $\mathscr{F} \in \mathrm{QCoh}(X)$, then $g^*\pi_*\mathscr{F} \cong \pi'_*(g')^*\mathscr{F}$.*

(vii) *For a quasi-coherent sheaf of ideals $\mathscr{J} \subset \mathscr{O}_X$, the natural map $\mathscr{J} \cong \pi_*(\pi^{-1}\mathscr{J} \cdot \mathscr{O}_{\mathscr{X}})$ is an isomorphism.*

If $\pi : \mathscr{X} \to X$ be a good moduli space with X quasi-separated, then

(viii) *If $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras, then $\underline{\mathrm{Spec}}_{\mathscr{X}}\mathscr{A} \to \underline{\mathrm{Spec}}_X \pi_*\mathscr{A}$ is a good moduli space.*

(ix) *If $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space.*

Proof. See section 4 in fundamental paper [5]. □

Now some important properties of good moduli spaces and give some comments. Actually these are similar as the properties of GIT.

Theorem 1.1.3. *Let $\pi : \mathscr{X} \to X$ be a good moduli space where $\mathscr{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(i) *π is surjective and universally closed (and universally submersive);*

(ii) *for closed substacks $\mathscr{Z}_1, \mathscr{Z}_2 \subset \mathscr{X}$, we have $\mathrm{Im}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \mathrm{Im}(\mathscr{Z}_1) \cap \mathrm{Im}(\mathscr{Z}_2)$. For geometric points $x_1, x_2 \in \mathscr{X}(k)$, $\pi(x_1) = \pi(x_2) \in \mathscr{X}(k)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_1\}} \neq \emptyset$ in $|\mathscr{X} \times_S k|$. In particular, π induces a bijection between closed points in $\mathscr{X}$ and closed points in X;*

(iii) *if $\mathscr{X}$ is noetherian, so is X. If $\mathscr{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence;*

(iv) *If X is noetherian, then π is universal for maps to algebraic spaces.*

Proof. Here we give some idea. The proof we refer the Theorem 4.16 in [5].

For (i), by Lemma 1.1.2 (iv) we know that $\mathscr{X} \times_X \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ is good moduli space. Hence $\Gamma(\mathscr{X} \times_X \mathrm{Spec}\, k, \mathscr{O}_{\mathscr{X} \times_X \mathrm{Spec}\, k}) = k$ and $|\mathscr{X} \times_X \mathrm{Spec}\, k| \neq \emptyset$. Hence π is surjective. Again by Lemm 1.1.2 (ix) we know that if $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space.

Hence it is surjective and hence π is closed. By Lemma 1.1.2 (iv) we know that it is universally closed.

For (ii), let ideal sheaves be $\mathscr{I}_1, \mathscr{I}_2$, then by the exactness of π_* we have

$$\begin{tikzcd} & & \pi_*\mathscr{I}_2 \arrow[d] \arrow[dr, two heads] & & \\ 0 \arrow[r] & \pi_*\mathscr{I}_1 \arrow[r] & \pi_*(\mathscr{I}_1 + \mathscr{I}_2) \arrow[r, two heads] & \pi_*\mathscr{I}_2/\pi_*(\mathscr{I}_1 \cap \mathscr{I}_2) \arrow[r] & 0 \end{tikzcd}$$

Hence the inclution $\pi_*(\mathscr{I}_1 + \mathscr{I}_2) \to \pi_*(\mathscr{I}_1 + \mathscr{I}_2)$ is surjective.

For (iii), X is noetherian follows from Lemma 1.1.2 (vii). We omit others and (iv). $\square$

There is an interesting result which we will use it:

Proposition 1.1.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a cohomologically affine morphism of algebraic stacks where $\mathscr{Y}$ has quasi-affine diagonal. If f is representable (that is, $\mathscr{I}_{\mathscr{X}/\mathscr{Y}} \to \mathscr{X}$ is trivial, or equivalently, Δ_{Δ_f} is an isomorphism), then f is affine.*

Proof. Trivial by faithfully flat descent and Serre's Criterion. $\square$

1.2 Luna's Results and Étale Local Structure of Algebraic Stacks

1.2.1 Luna's Fundamental Lemma and Luna's Étale Slice Theorem

Luna's results are classical and you can find them even in [16].

Theorem 1.2.1 (Luna's Fundamental Lemma)**.** *Consider a commutative diagram:*

$$\begin{tikzcd} \mathscr{X}' \arrow[r, "f"] \arrow[d, "\pi'"'] & \mathscr{X} \arrow[d, "\pi"'] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

where f is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in \mathscr{X}'$ be a point such that

(a) f is étale at x';

(b) f induces an isomorphism of stabilizer groups at x', and

(c) $x' \in \mathscr{X}'$ and $x = f(x') \in \mathscr{X}$ are closed points.

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathscr{X} \cong (\pi')^{-1}(U')$.

Sketch. Using limit-argument, we may let $X = \operatorname{Spec} A$, where A is a strictly henselian local ring. After shrink $\mathscr{X}'$, we may let f is étale. Then by Zariski main theorem we get $\mathscr{X}' \to \widetilde{\mathscr{X}} = \underline{\operatorname{Spec}}_{\mathscr{X}} \mathscr{A} \to \mathscr{X}$. Hence $\widetilde{\mathscr{X}} \to \widetilde{X} = \underline{\operatorname{Spec}}_{\mathscr{X}} \pi_* \mathscr{A}$ is a good moduli space with $\widetilde{X} \to X$ finite. Hence we can let $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ of henselian local rings. Take $U' = \operatorname{Spec} A_i$ contains image of x'. Well done. $\square$

Hence we have an very important corollary we will use:

Corollary 1.2.2. *With the same hypotheses, suppose that f is étale and that for all closed points $x' \in \mathscr{X}'$ we have*

(a) $f(x')$ closed;

(b) f induces an isomorphism of stabilizer groups at x'.

Then $g : X' \to X$ étale and that commutative diagram is cartesian.

This is our main motivation to define the Θ-completeness and S-completeness. We will discuss this deeply later.

Next we introduce Luna's étale slice theorem which was motivated the étale local structure of algebraic stacks.

Lemma 1.2.3 (Luna Map)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, there exists a G_x-equivariant morphism (Luna map)*

$$f : X \to T_{X,x} := \underline{\operatorname{Spec}} \operatorname{Sym} \mathfrak{m}_x/\mathfrak{m}_x^2$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \operatorname{Spec} A$, then $\mathfrak{m}_x$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ are G_x-representations and we see that G_x acts naturally on the tangent space $T_{X,x} := \underline{\operatorname{Spec}} \operatorname{Sym} \mathfrak{m}_x/\mathfrak{m}_x^2$. Since G_x is linearly reductive, the surjection $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$ of G_x-representations has a section $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x$. This induces a G_x-equivariant ring map $\operatorname{Sym} \mathfrak{m}_x/\mathfrak{m}_x^2 \to A$ and thus a G_x-equivariant morphism $f : X \to T_{X,x}$ sending x to the origin. If x is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x. $\square$

Theorem 1.2.4 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with* *an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$[W/G_x] \to [X/G]$$

is affine étale. If in addition $Gx \subset X$ closed (then by Matsushima's theorem G_X is linearly reductive), then there is a cartesian

$$\begin{tikzcd} {[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[X/G]} \arrow[d] \\ W /\!\!/ G_x \arrow[r] & X /\!\!/ G \end{tikzcd}$$

where $W /\!\!/ G_x \to X /\!\!/ G$ is also étale.

Moreover, if $x \in X$ is a smooth point and if we denote by $N_x = T_{X,x}/T_{Gx,x}$ the normal space to the orbit, then it can be arranged that there is an G_x-invariant étale morphism $W \to N_x$ which is the pullback of an étale map $W /\!\!/ G_x \to N_x /\!\!/ G_x$ of GIT quotients.

Proof. Pick a finite G-representation V and a G-equivariant closed immersion $X \subset \mathbb{A}(V)$. Then using this we can reduce to the case where $x \in X$ is smooth.

Hence we have Luna map $f : X \to T_{X,x}$ is G_x-equivariant and étale at x. The subspace $T_{Gx,x} \subset T_{X,x}$ is G_x-invariant and again since G_x is linearly reductive, the surjection $T_{X,x} \to N_x$ has a section $N_x \to T_{X,x}$. We define W as

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N_x \arrow[d] \\ X \arrow[r, "f"] & T_{X,x} \end{tikzcd}$$

Then $[W/G_x] \to [X/G]$ and $[W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at x. Hence we have commutative diagram

$$\begin{tikzcd} {[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l] \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ N_x /\!\!/ G_x & W /\!\!/ G_x \arrow[l] \arrow[r] & X /\!\!/ G \end{tikzcd}$$

Hence using Luna's fundamental lemma 1.2.1 twice and well done. □

1.2.2 Coherent Tannaka Duality and Coherent Completeness

Here we introduce some very importent results aiming to extend to morphisms.

Theorem 1.2.5 (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks $\mathscr{X}$ and $\mathscr{Y}$ with affine diagonal, the functor*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})), \quad f \mapsto f^*$$

is an equivalence of categories where the latter category denote the right exact additive tensor functors $\mathrm{Coh}(\mathscr{Y}) \to \mathrm{Coh}(\mathscr{X})$ of symmetric monoidal abelian categories where morphisms are tensor natural transformations.

Proof. This follows from a nice observation of Lurie in [14]. For the proof we refer [3] Theorem 6.4.1. □

Definition 1.2.6. *A noetherian algebraic stack $\mathscr{X}$ is coherently complete along a closed substack $\mathscr{X}_0$ if the natural functor*

$$\mathrm{Coh}(\mathscr{X}) \to \varprojlim \mathrm{Coh}(\mathscr{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathscr{X}_n$ denotes the n-th nilpotent thickening of $\mathscr{X}_0$.

Remark 1.2.7. *(i) This motivated by the Grothendieck's Existence Theorem asserts that if X is a proper scheme over a complete local ring $(R, \mathfrak{m})$ and $X_0 = X \times_R R/\mathfrak{m}$, then X is coherently complete along X_0.*

Actually this is right even for proper algebraic stack over some I-adically complete noetherian ring. We refer [18].

(ii) Let k be an algebraically closed field and R be a complete noetherian local k-algebra with residue field k. Let G be a linearly reductive group over k acting on an affine scheme $\mathrm{Spec}\, A$ of finite type over R. Suppose that $A^G = R$ and that there is a G-fixed k-point $x \in \mathrm{Spec}\, A$. Then $[\mathrm{Spec}\, A/G]$ is coherently complete along the closed substack $\mathbf{B}G$ defined by x. See the Theorem 6.4.11 in [3] for the proof.

We will use the follows corollary many times:

Corollary 1.2.8. *Let $\mathscr{X}$ and $\mathscr{Y}$ be noetherian algebraic stacks with affine diagonal. Suppose that $\mathscr{X}$ is coherently complete along $\mathscr{X}_0$. Then there is an equivalence of categories*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}), \quad f \mapsto (f_n).$$

Proof. This is directly:

$$\begin{aligned} \mathrm{MOR}(\mathscr{X}, \mathscr{Y}) &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})) \\ &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \varprojlim \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}) \end{aligned}$$

and well done. □

1.2.3 Some Deformation Theory

Proposition 1.2.9. *Consider a commutative diagram*

$$\begin{tikzcd} \mathscr{W} \arrow[r, "f"] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ \mathscr{W}' \arrow[r] \arrow[ur, dashed] & \mathscr{Y} \end{tikzcd}$$

of noetherian algebraic stacks with affine diagonal where $\mathscr{X} \to \mathscr{Y}$ is smooth and affine- and $\mathscr{W} \to \mathscr{W}'$ is a closed immersion defined by a square-zero sheaf of ideals $\mathscr{J}$. If $\mathscr{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. As the case of schemes, the set of liftings is a torsor under $\mathrm{Hom}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J})$. Hence let $\mathscr{F} := f^*\Omega^\vee_{\mathscr{X}/\mathscr{Y}} \otimes \mathscr{J}$. Consider

$$\begin{tikzcd} (U/\mathscr{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{W} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\ (U'/\mathscr{W}')^2 \arrow[r, shift left] \arrow[r, shift right] & U' \arrow[r] \arrow[urr, "f'_U"] & \mathscr{W}' \arrow[r] \arrow[ur, dashed, "f'"'] & \mathscr{Y} \end{tikzcd}$$

where $(U/\mathscr{W})^2 = U \times_{\mathscr{W}} U$ is affine. Because $\mathscr{X} \to \mathscr{Y}$ is representable, to check that f'_U descends to a morphism f', we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. As $f'_U \circ p_1 - f'_U \circ p_2 \in \Gamma((U/\mathscr{W})^2, q_2^*\mathscr{F})$, this follows from the $\mathscr{W}$ is cohomologically affine and exact sequences. Omitted and see [3] Proposition 6.5.8.

There is another way, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathscr{O}_{\mathscr{W}}}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J}) \cong H^1(\mathscr{W}, \mathscr{F})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Proposition 1.2.10. *Let $\mathscr{W} \to \mathscr{W}'$ be a closed immersion of algebraic stacks of finite type over k with affine diagonal defined by a square-zero sheaf of ideals $\mathscr{J}$. Let G be an affine algebraic group over k. If $\mathscr{W}$ is cohomologically affine, then every principal G-bundle $\mathscr{P} \to \mathscr{W}$ extends to a principal G-bundle $\mathscr{P}' \to \mathscr{W}'$.*

Proof. Similar as the proof above and we need to take $\mathscr{F} = \mathfrak{g} \otimes \mathscr{J}$ from the deformation theory of principal G-bundles in [3] D.2.9.

There is also another way. Note that this is equivalent to the deformation of $f : \mathscr{W} \to \mathbf{B}G$ to $\mathscr{W}' \to \mathbf{B}G$ which is the same problem in Proposition 1.2.9 to $\mathbf{B}G \to \operatorname{Spec} k$ which is not affine. See the arguments in Remark 6.5.11 in [3], we can see the obstruction lies in $H^2(\mathscr{W}, \mathfrak{g} \otimes \mathscr{J})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Remark 1.2.11. *All these results are the special case in Theorem 1.5 in [17].*

1.2.4 Étale Local Structure of Algebraic Stacks

There is a fundamental theorem about algebraic stacks as follows:

Theorem 1.2.12 (Minimal Presentations)**.** *Let $\mathscr{X}$ be a noetherian algebraic stack and let $x \in |\mathscr{X}|$ be a finite type point with smooth stabilizer G_x. Then there exists a scheme U with a closed point $u \in U$ and a smooth morphism $(U, u) \to (\mathscr{X}, x)$ of relative dimension $\dim G_x$ such that the diagram*

$$\begin{tikzcd} \operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathscr{X} \end{tikzcd}$$

is cartesian.

Proof. This is easy in Theorem 3.6.1 in [3]. Let $(U, u) \to (\mathscr{X}, x)$ be a smooth morphism of relative dimension n, hence we have

$$\begin{tikzcd} O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ \mathcal{G}_x \arrow[r, hook] & \mathscr{X} \end{tikzcd}$$

As $\dim \mathcal{G}_x = -\dim G_x$, then $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. By Nakayama's lemma, we pick a reguler sequence $f_1, ..., f_c \in \mathscr{O}_U$ and consider $W = V(f_1, ..., f_c)$ and then $W \cap O(u) = \operatorname{Spec} \kappa(u)$. By the local criterion for flatness and smooth descent to $U \times_{\mathscr{X}} U \rightrightarrows \mathscr{X}$, we know that $W \to \mathscr{X}$ is flat. Checking on the fibers we can conclude the result. $\square$

Before giving the statement of the étale local structure of algebraic stacks, we will give a useful criteria for morphisms to be closed immersions or isomorphisms.

Lemma 1.2.13. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field k with affine diagonal. Assume that $|\mathscr{X}| = \{x\}$ and $|\mathscr{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism of residue gerbes $\mathscr{X}_0 := \mathcal{G}_x = \mathbf{B}G_x$ with $\mathscr{Y}_0 := \mathcal{G}_y = \mathbf{B}G_y$. Let $\mathfrak{m}_x, \mathfrak{m}_y$ be the ideal sheaves defining them, and let $f_1 : \mathscr{X}_1 \to \mathscr{Y}_1$ be the induced morphism between the first nilpotent thickenings.*

(i) If f_1 is a closed immersion, then so is f.

(ii) If f_1 is a closed immersion and there is an isomorphism

$$\bigoplus_{n \geq 0} \mathfrak{m}_y^n / \mathfrak{m}_y^{n+1} \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

of graded $\mathscr{O}_{\mathscr{X}_0}$-modules, then f is an isomorphism.

Proof. By Theorem 1.2.12, we may choose a minimal smooth presentations $V = \operatorname{Spec} B \to \mathscr{Y}$ such that $V \times_{\mathscr{Y}} \mathscr{Y}_0 \cong \operatorname{Spec} k$. Hence B is an artinian local ring, then so is $U = \operatorname{Spec} B \cong V \times_{\mathscr{Y}} \mathscr{X}$. Hence we can let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian rings.

For (i), this follows from [9] Lemma II.7.4. For (ii), this is trivial. □

Lemma 1.2.14. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f : \mathscr{W} := [\operatorname{Spec} A/G] \to \mathscr{X}$ be a finite type morphism with G linearly reductive. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Let $\pi : \mathscr{W} \to \operatorname{Spec} A^G$. We may let $F : \mathscr{W} \to \mathscr{X}$ is quasi-finite as it is quasi-finite over some open set.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathscr{X}$, then

$$\begin{tikzcd} \mathscr{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V = \operatorname{Spec} B \arrow[d] \\ \mathscr{W} \arrow[r, "f"] & \mathscr{X} \end{tikzcd}$$

As $\mathscr{X}$ with affine diagonal, the map $V \to \mathscr{X}$ is affine. Hence $\mathscr{W}_V$ is cohomologically affine. By Proposition 6.3.28 in [3] we have:

- Suppose $\mathscr{Z}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi : \mathscr{Z} \to Z$. If the diagonal Δ_π is quasi-finite, then it is finite.

Hence $\mathscr{W}_V \to V$ is separated. From descent $\mathscr{W} \to \mathscr{X}$ is also separated and that the relative inertia $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is finite. Since the fiber over w is trivial, there is an open neighborhood over which the relative inertia is trivial. Hence replace this we may let $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is trivial. Hence it is representable. By Serre's criteria we get the result. □

Theorem 1.2.15 (Étale Local Structure of Algebraic Stacks)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. For every point $x \in X(k)$ with linearly reductive stabilizer G_x there exists an affine étale morphism*

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

If $x \in \mathscr{X}$ is a smooth point, then there is also an étale morphism

$$f : ([\operatorname{Spec} A/G_x], w) \to ([T_{\mathscr{X},x}/G_x], 0).$$

Proof of the Smooth Case. Here we only give the proof of smooth case and tell you the difficulties of proof the general case in the remark.

Since x is locally closed, we may let it is closed. Hence $\mathscr{X}_0 := \mathbf{B}G_x \subset \mathscr{X}$ defined by $\mathscr{I}$. Let $\mathscr{X}_n$ to be the n-th nilpotent thickening of it. The Zariski tangent space $T_{\mathscr{X},x}$ can be identified with the normal space $(\mathscr{I}/\mathscr{I}^2)^\vee$, hence with a G_x-representation. Hence we can define $\mathscr{T} = [T_{\mathscr{X},x}/G_x]$ with $\mathscr{T}_0 := \mathbf{B}G_x$ and the n-th nilpotent thickening $\mathscr{T}_n$.

By Proposition 1.2.10 we get an affine $\mathscr{X}_n \to \mathbf{B}G_x$. By Proposition 1.2.9 inductively we have lifts:

$$\begin{tikzcd}
\mathscr{X}_n \arrow[r] \arrow[d, hook] & \mathscr{T} \arrow[d] \\
\mathscr{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x
\end{tikzcd}$$

By some easy commutative algebra via smooth descent, we have $\mathscr{X}_1 \cong \mathscr{T}_1$. Hence by Lemma 1.2.13(ii) we have $\mathscr{X}_n \cong \mathscr{T}_n$.

Consider $\pi : \mathscr{T} \to T := T_{\mathscr{X},x} /\!\!/ G_x$ and $\widehat{\mathscr{T}} := \operatorname{Spec} \widehat{\mathscr{O}}_{T,\pi(0)} \times_T \mathscr{T} = [\operatorname{Spec} B/G]$ where B is of finite type over the noetherian complete local k-algebra $B^G = \widehat{\mathscr{O}}_{T,\pi(0)}$. By Remark 1.2.7 (ii) we know that $\widehat{\mathscr{T}}$ is coherently complete along $\mathscr{T}_0$ and by coherent Tannaka duality we get

$$\mathrm{MOR}(\widehat{\mathscr{T}}, \mathscr{X}) \to \varprojlim \mathrm{MOR}(\mathscr{T}_n, \mathscr{X}).$$

Hence we have

$$\begin{tikzcd}
 & X & \\
\mathscr{X}_n \cong \mathscr{T}_n \arrow[ur] \arrow[r] & \widehat{\mathscr{T}} \arrow[u, dashed] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{T} \arrow[d] \\
 & \operatorname{Spec} \mathscr{O}_{T,\pi(0)} \arrow[r] & T
\end{tikzcd}$$

Now by Artin Approximation, there exists an étale morphism $(U, u) \to (T, 0)$ where U is an affine scheme with a k-point $u \in U$ and a morphism $(U \times_T \mathscr{T}, (u, 0)) \to (\mathscr{X}, x)$ agreeing with $(\widehat{\mathscr{T}}, 0) \to (\mathscr{X}, x)$ in the first order. As $U \times_T \mathscr{T}$ is smooth at $(u, 0)$ and $\mathscr{X}$ is smooth at x, and as $U \times_T \mathscr{T} \to \mathscr{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u, 0)$, hence the morphism $U \times_T \mathscr{T} \to \mathscr{X}$ is étale at $(u, 0)$. Finally, by Lemma 1.2.14 we get the result. $\square$

Remark 1.2.16. *We refer Section 6.5.5 in [3] for the proof of the general case. Now we point out that in the general case we also have $\mathscr{X}_1 \cong \mathscr{T}_1$. But we can only use the Lemma 1.2.13(i) to get a closed immersion $\mathscr{X}_n \to \mathscr{T}_n$. Also in the general case we can not deduce $U \times_T \mathscr{T} \to \mathscr{X}$ is étale from the isomorphism of tangent spaces! In order* *to solve this, we need a more general fact called* equivariant Artin algebraization theorem*. See Theorem 6.5.14 in [3] for the statement and the proof.*

Remark 1.2.17. *Actually the property in some more general setting we only have the following (which we will not use):*

- *Let S be a quasi-separated algebraic space. Let $\mathscr{X}$ be an algebraic stack locally of finite presentation and quasi-separated over S, with affine stabilizers. If $x \in |\mathscr{X}|$ is a point with image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and the stabilizer of x is linearly reductive, then there exists $f : ([\operatorname{Spec} A/\mathrm{GL}_N], w) \to (\mathscr{X}, x)$ induces an isomorphism of stabilizer groups (such kind of maps called* quotient presentation*). If $\mathscr{X}$ has separated (resp. affine) diagonal, then there exists a such representable (resp. affine), étale quotient presentation.*

See [1] Theorem 1.1. In our case, this is proved in [2].

1.3 Existence of Good Moduli Space

Here we give a strategy for constructing good moduli spaces in 6.8.1 in [3].

Our main goal is to glue the étale local GIT quotient $[\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$ via the groupoid representations. Let $f : \mathscr{W} := [\operatorname{Spec} A/G_x] \to \mathscr{X}$ is affine étale with $W := \operatorname{Spec} A^{G_x}$. Let $\mathscr{R} := \mathscr{W} \times_{\mathscr{X}} \mathscr{W}$ which is of form $[\operatorname{Spec} B/G_x]$ as f is affine. Let $R = \operatorname{Spec} B^{G_x}$ and consider

$$\begin{tikzcd}
\mathscr{R} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{W} \arrow[r, "f"] \arrow[d] & \mathscr{X} \\
R \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & W &
\end{tikzcd}$$

Hence if q_1, q_2 defines an étale equivalence relation, the algebraic space quotient W/R is a good moduli space of $f(W)$. Then we have some chance to glue them.

By Luna's fundamental lemma 1.2.1 (its Corollary 1.2.2), in order to make q_1, q_2 as an étale equivalence relation, we need that for all closed points $r \in \mathscr{R}$ we have

(a) $p_1(r), p_2(r)$ are closed;

(b) p_1, p_2 induces isomorphisms of stabilizer groups at r.

As $f(w)$ is is closed and f induces an isomorphism of stabilizer groups. We just want to show that there is an open neighborhood $\mathscr{U}$ of w such that

(i) $f|_{\mathscr{U}}$ sends closed points map to closed points and stable under base change;

(ii) $f|_{\mathscr{U}}$ induces isomorphisms of stabilizer groups at closed points and stable under base change.

We will see that the Θ-completeness implies Θ-surjectivity which will implies (i); and S-completeness will implies (ii).

1.3.1 Basic Properties of Θ-Complete and S-Complete

Definition 1.3.1 (Θ-Completeness). *Define $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ over $\operatorname{Spec}\mathbb{Z}$ and $\Theta_R := \Theta \times \operatorname{Spec} R$ for any DVR R with fracion field K and residue field κ. We can describe it as following cartesians:*

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, "x\neq 0"] & & \mathbf{B}\mathbb{G}_{m,R} \arrow[dl, hook', "x=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr] \arrow[rr, phantom, "<" near start] & & \Theta_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul] \arrow[dl, hook'] \arrow[ll, phantom, ">" near start] \\
 & \Theta_K \arrow[ur, hook, "\pi\neq 0"] & & \Theta_\kappa \arrow[ul, hook', "\pi=0"'] &
\end{tikzcd}$$

Hence $\Theta_R\backslash 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \Theta_K$. Hence $\Theta_R\backslash 0 \to \mathscr{X}$ is the data of morphisms $\operatorname{Spec} R \to \mathscr{X}$ and $\Theta_K \to \mathscr{X}$ together with an isomorphism of their restrictions to $\operatorname{Spec} K$.

Then a locally noetherian algebraic stack $\mathscr{X}$ is called Θ-$\mathsf{complete}$ *if for any DVR R, every diagram*

$$\begin{tikzcd}
\Theta_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{X} \\
\Theta_R \arrow[ur, dashed] &
\end{tikzcd}$$

of solid arrows can be uniquely filled in.

Here is the figures of our stacks look like, see Remark 1.3.4:

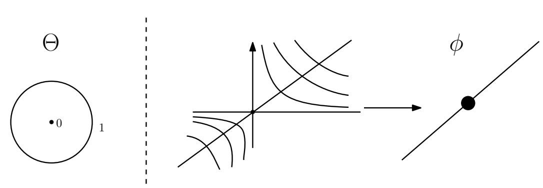

Figure 1.1: Θ and ϕ looks like

Definition 1.3.2 (S-Completeness). *For any DVR R with fracion field K and residue field κ, we define*

$$\phi_R := [\operatorname{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m]$$

where s and t have $\mathbb{G}_m$-weights 1 and -1, respectively. Now we have

$$\phi_R|_{s\neq 0} = [\operatorname{Spec}(R[s,t]_s/(t-\pi/s))/\mathbb{G}_m] = [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec} R$$

and similar for $t \neq 0$. Hence we can describe it as following cartesians:

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, "s\neq 0"] & & \Theta_\kappa \arrow[dl, hook, "s=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, phantom, "<" very near start] & & \phi_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul] \arrow[dl, hook] \arrow[ll, phantom, ">" very near start] \\
 & \operatorname{Spec} R \arrow[ur, hook, "t\neq 0"] & & \Theta_\kappa \arrow[ul, hook, "t=0"] &
\end{tikzcd}$$

Hence $\phi_R \backslash 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathscr{X}$ is the data of two morphisms $\xi, \xi' : \operatorname{Spec} R \to \mathscr{X}$ together with an isomorphism $\xi_K \cong \xi'_K$ over $\operatorname{Spec} K$.

Then a locally noetherian algebraic stack $\mathscr{X}$ is called S-*complete if for any DVR R, every diagram*

$$\begin{tikzcd}
\phi_R \backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{X} \\
\phi_R \arrow[ur, dashed] &
\end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 1.3.3. *In the original paper [6], they introduce the Θ-completeness and* S-*completenes for morphisms of algebraic stacks, but we won't use them.*

Remark 1.3.4. *There is an interesting fact that the symbols Θ and ϕ is used because they look like the stacks they represent! See figure 1.1.*

There are many properties of Θ-completeness and S-completeness, here we introduce some of them.

Proposition 1.3.5. *We have the following properties:*

(i) A locally noetherian algebraic stack with affine diagonal is Θ-complete (resp. S-*complete), if and only if these diagrams, there exists a lift after an extension of DVRs $R \subset R'$. In particular, Θ-completeness and* S-*completeness can be verified on complete DVRs with algebraically closed residue fields.*

(ii) *Let $f; \mathscr{X} \to \mathscr{Y}$ be an affine morphism of locally noetherian algebraic stacks. If $\mathscr{Y}$ is Θ-complete (resp. S-complete), so is $\mathscr{X}$.*

(iii) *If G is a reductive group over an algebraically closed field k, then every quotient stack $[\operatorname{Spec} A/G]$ is Θ-complete and S-complete.*

(iv) *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. If $\pi : \mathscr{X} \to X$ be a good moduli space, then $\mathscr{X}$ is Θ-complete. Moreover, $\mathscr{X}$ is S-complete if and only if X is separated.*

(v) *Let $\mathscr{X}$ be a noetherian algebraic stack with affine and quasi-finite diagonal. Then*

- *If R is a complete DVR, every map $\Theta_R \to \mathscr{X}$ (resp. $\phi_R \to \mathscr{X}$) factors through $\Theta_R \to \operatorname{Spec} R$ (resp. $\phi_R \to \operatorname{Spec} R$).*
- *$\mathscr{X}$ is Θ-complete. Moreover, $\mathscr{X}$ is S-complete if and only if it is separated.*

Some Comments of Proofs. We will not give the whole proofs, but we will give some comments on it. The proofs we refer [6] or [3].

For (i), this follows from the fpqc descent.

For (ii), since Θ_R is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \backslash 0 \to \Theta_R$ is the structure sheaf. Then by the definition we can get the result.

For (iii), first show the case of $\mathbf{B}\mathrm{GL}_n$. Indeed this follows from $0 \in \Theta_R$ is codimension 2 and Θ_R is regular, then vector bundles have unique extension. For general case, pick a faithful representation $G \subset \mathrm{GL}_n$ By the reductivity of G we get GL_n/G is affine by Matushima's result. As

$$\begin{tikzcd} \mathrm{GL}_n/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec} k \arrow[d] \\ \mathbf{B}G \arrow[r] & \mathbf{B}\mathrm{GL}_n \end{tikzcd}$$

and smooth descent we get $\mathbf{B}G \to \mathbf{B}\mathrm{GL}_n$ is affine. Hence the result follows from (ii).

For (iv), by the étale local structure of algebraic stacks and (i) we can show that the Θ-completeness follows from the local case (iii). For S-completeness, this from some arguments of valuative criterions.

For (v), the first one follows from deformation theory and coherent Tannaka duality. The second one follows from the first one and the valuative criterion. □

Actually the S-completeness also have some relation to the reductivity of groups:

Theorem 1.3.6 (Cartan Decomposition and S-Completeness)**.** *Let G be a smooth affine algebraic group over an algebraically closed field k. Then the following are equivalent:*

(a) *G is reducible.*

(b) $\mathbf{B}G$ *is* S*-complete.*

(c) For any complete DVR R *over* k *with residue field* κ *and fracion field* K *and for any* $g \in G(K)$*, there exists elements* $h_1, h_2 \in G(R)$ *and a 1-PS* $\lambda : \mathbb{G}_m \to G$ *such that* $g = h_1 \lambda|_K h_2$

In particular, if $\mathscr{X}$ *is an* S*-complete algebraic stack and* $x \in \mathscr{X}$ *is a closed point with smooth affine stabilizer* G_x*, then* G_x *is reductive.*

Proof. We omitted the proof. Actually the equivalence of (a) and (c) is classical in [16]. For the complete proof we refer Proposition 6.8.45 in [3]. □

Proposition 1.3.7 (Stacky Destabilization Theorem)**.** *Let* $\mathscr{X}$ *be an algebraic stack of finite type over an algebraically closed field* k *with affine diagonal. Let* $x \rightsquigarrow x_0$ *be a specialization of* k*-points such that the stabilizer* G_{x_0} *is linearly reductive. Then there exists a morphism* $[\mathbb{A}^1/\mathbb{G}_m] \to \mathscr{X}$ *representing the specialization* $x \rightsquigarrow x_0$*.*

Proof. Here we will use a classical destabilization theorem, see Page 53 in [16] or Theorem 6.6.28 in [3]:

- Let G be a reductive algebraic group over an algebraically closed field k acting on an affine scheme X of finite type over k. Given $x \in X(k)$, there exists a 1-PS $\lambda : \mathbb{G}_m \to G$ such that $x_0 := \lim_{t\to 0} \lambda(t) \cdot x$ exists and has closed G-orbit.

Back to our proof. By the Theorem 1.2.15, we have étale morphism $f : ([\operatorname{Spec} A/G_{x_0}], w_0) \to (\mathscr{X}, x_0)$ which induces an isomorphism of stabilizer groups at w_0. After possibly replacing $\operatorname{Spec} A$ with a G_{x_0}-invariant affine subscheme, we can assume that w_0 is a closed point. The specialization $x \rightsquigarrow x_0$ lifts a specialization $w \rightsquigarrow w_0$ in $\operatorname{Spec} A$, and we can choose a representative $\tilde{w} \in \operatorname{Spec} A$ of the orbit corresponding to w. The Destabilization Theorem gives a 1-PS $\lambda : \mathbb{G}_m \to G_{x_0}$ such that $\tilde{w}_0 := \lim_{t\to 0} \lambda(t)\cdot\tilde{w}$ exists and has closed orbit. By the affine version of Theorem 1.1.3 we get there is a unique closed orbit in $\overline{G\tilde{w}}$, and thus $\tilde{w}_0 \in \operatorname{Spec} A$ maps to w_0. Hence the extension of λ induce $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to \operatorname{Spec} A$. Hence we get $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}] \to \mathscr{X}$ representing the specialization $x \rightsquigarrow x_0$. □

1.3.2 Θ-Surjectivity and Θ-Complete

Definition 1.3.8 (Θ-surjective)**.** *Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a morphism of algebraic stacks and* $\mathscr{X}(k)$ *be a geometric point. We say* f *is* Θ-**surjective at** x *if every diagram:*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d, hook, "1"] & \mathscr{X} \arrow[d, "f"] \\ \Theta_k \arrow[r] \arrow[ur, dashed] & \mathscr{Y} \end{tikzcd}$$

has a lift. We say that f *is* Θ-**surjective** *if it is* Θ*-surjective at every geometric point.*

Remark 1.3.9. *This condition is table under base change as it is equivalent to the surjectivity of*

$$\mathrm{ev}(f)_1 : \underline{\mathrm{MOR}}(\Theta, \mathscr{X}) \to \mathscr{X} \times_{\mathscr{Y}, \mathrm{ev}(f)_1} \underline{\mathrm{MOR}}(\Theta, \mathscr{Y}).$$

If $f : \mathscr{X} \to \mathscr{Y}$ of noetherian algebraic stacks where $\mathscr{Y}$ with affine and quasi-finite diagonal, then by Propostion 1.3.5(v) we know that f is Θ-surjective.

Lemma 1.3.10. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a separated, representable, and finite type morphism of noetherian algebraic stacks, then the lift in the definition of Θ-surjectivity is unique and the Θ-surjectivity is not depend on the fields represent the same point.*

Proof. The first one follows from descent and valuative criterion. The second one follows from some limit result, we omit it. □

Proposition 1.3.11. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks, each of finite type over an algebraically closed field k with affine diagonal. Let the closed points of $\mathscr{Y}$ have linearly reductive stabilizers. If f is Θ-surjective, then f sends closed points to closed points.*

Proof. Let $x \in |\mathscr{X}|$ closed and $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By Proposition 1.3.7, we have $\Theta \to \mathscr{Y}$ sends $1 \mapsto f(x), 0 \mapsto y_0$. Hence by $Theta$-surjectivity, we get a lift $g : \Theta \to \mathscr{X}$ sends $1 \mapsto x$. As x closed, this map g is trivial. So is $\mathscr{Y}$. Well done. □

Proposition 1.3.12. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal such that the closed points of $\mathscr{X}$ have linearly reducetive stabilizers. Let $x \in \mathscr{X}$ be a closed point with affine étale morphism $f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$ inducing an isomorphism of stabilizers at w. Let $\pi : [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. Then if $\mathscr{X}$ is Θ-complete, then there exists an open affine $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathscr{X}$ is Θ-surjective.*

Proof. We omit the proof and refer Propisition 6.8.31 in [3] or Proposition 4.3(i) in [6] for more general case. □

Here we have a topology like GIT:

Proposition 1.3.13. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Assume that $\mathscr{X}$ is Θ-complete and that the closed points of $\mathscr{X}$ have linearly reductive stabilizer. Then the closure of every k-point contains a unique closed point.*

Proof. If we have two of them, we then have two $\Theta \to \mathscr{X}$. Then we can glue them into $[\mathbb{A}^2/\mathbb{G}_m]\backslash 0 \to \mathscr{X}$. Consider the diagonal action and Θ-completeness, we get extension $\Psi : [\mathbb{A}^2/\mathbb{G}_m] \to \mathscr{X}$. Hence $\Psi(0,0)$ is a common specialization of $x = \Psi(1,0)$ and $x' = \Psi(0,1)$. Since x and x' are closed points, we have that $x = \Psi(0,0) = x'$. □

1.3.3 Unpunctured Inertia and S-Complete

We will only give a sketch of these because the proof of this main theorem is very complicated.

Definition 1.3.14 (Unpunctured Inertia)**.** *We say that a noetherian algebraic stack $\mathscr{X}$ has* unpunctured inertia *if for every closed point $x \in |\mathscr{X}|$ and every formally smooth morphism $p : (T, t) \to (\mathscr{X}, x)$ where T is the spectrum of a local ring with closed point t, every connected component of the inertia group scheme $\operatorname{Aut}_{\mathscr{X}}(p) \to T$ has non-empty intersection with the fiber over t.*

Proposition 1.3.15. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Let $x \in |\mathscr{X}|$ be a closed point which have linearly reducetive stabilizers. Pick an affine étale morphism $f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$ inducing an isomorphism of stabilizers at w. and let $\pi : [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. Then if $\mathscr{X}$ has unpunctured inertia, then there exists an open affine $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathscr{X}$ induces isomorphisms of stabilizers at all points.*

Proof. Let $\mathscr{W} := [\operatorname{Spec} A/G_x]$. We just need to find an open $\mathscr{U} \subset \mathscr{W}$ of w such that $f|_{\mathscr{U}} : \mathscr{U} \to \mathscr{X}$ induce an isomorphism $\mathscr{I}_{\mathscr{U}} \cong \mathscr{U} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$. Consider

$$\begin{tikzcd} \mathscr{I}_{\mathscr{W}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \arrow[d] \\ \mathscr{W} \arrow[r] & \mathscr{W} \times_{\mathscr{X}} \mathscr{W} \end{tikzcd}$$

As f is affine étale, then $\mathscr{I}_{\mathscr{W}} \to \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$ is finite étale. Let $\mathscr{Z} \subset \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$ be the locus that is not an isomorphism. Then $\mathscr{Z}$ is closed and open substack. Let $p_1 : \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \to \mathscr{W}$ and then $w \notin p_1(\mathscr{Z})$.

Let a formally smooth morphism $p : (T, t) \to (\mathscr{X}, x)$ where T is the spectrum of a local ring with closed point t. Since $\mathscr{X}$ has unpunctured inertia, hence the preimage of $\mathscr{Z}$ in $\mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \times_{\mathscr{X}} T$ is empty. Then $w \notin \overline{p_1(\mathscr{Z})}$, hence pick $\mathscr{U} := \mathscr{W} \backslash \overline{p_1(\mathscr{Z})}$ and well done. ∎

Theorem 1.3.16. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathscr{X}$ is S-complete, then $\mathscr{X}$ has unpunctured inertia.*

Proof. Omitted since this is very complicated. For our case we refer the proof of Theorem 6.8.40 in [3] and the general case we refer Theorem 5.3 in [6]. ∎

1.3.4 The Finally Statement and the Proof

Theorem 1.3.17. *Let $\mathscr{X}$ be an algebraic stack of finite tyoe over an algebraically closed field k of characteristic 0 with affine diagonal. There exists a good moduli space $\pi : \mathscr{X} \to X$ with X a separated algebraic space if and only if $\mathscr{X}$ is Θ-complete and S-complete.*

Moreover, X is proper if and only if $\mathscr{X}$ satisfies the existence part of the valuative criterion for properness.

Remark 1.3.18. *Here we follows the proof in [3] which I talked before. In paper [6] Theorem 5.4, we have a more general form which is characteristic independent:*

- *Let $\mathscr{X}$ be an algebraic stack of finite presentation over a quasi-separated and locally noetherian algebraic space S, with affine stabilizers and separated diagonal. Then $\mathscr{X}$ admits a good moduli space X separated over S if and only if we have*

  (1) every closed point of $\mathscr{X}$ has linearly reductive stabilizer;

  (2) $\mathscr{X} \to S$ is Θ-complete;

  (3) $\mathscr{X} \to S$ is S-complete.

  If $\mathscr{X}$ is locally reductive and has affine diagonal, then $\mathscr{X}$ admits an adequate moduli space X separated over S if and only if (2) and (3) hold. In both cases, if S is locally excellent and $\mathscr{X} \to S$ has affine diagonal, it suffices to check the filling conditions of Θ-completeness and S-completeness only for DVRs that are essentially finite type over S.

  Furthermore, in both cases $X \to S$ is proper if and only if $\mathscr{X} \to S$ satisfies the existence part of the valuative criterion for properness.

But we will not use this.

Proof of Theorem 1.3.17. If $\mathscr{X}$ has a good moduli space X which is a separated algebraic space, then $\mathscr{X}$ is Θ-complete and S-complete by Proposition 1.3.5(iv). Hence we just need to consider the converse.

By Theorem 1.3.6, as $\mathscr{X}$ is S-complete and over characteristic zero, then stabilizers of every closed points are linearly reductive. For any closed $x \in |\mathscr{X}|$ there is an affine étale morphism $([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$ which is Θ-surjective and stabilizer preserving at all points since $\mathscr{X}$ is Θ-complete and S-complete by Proposition 1.3.12, Proposition 1.3.15 and Theorem 1.3.16. Since $\mathscr{X}$ is quasi-compact, we can choose finitely many closed points $x_i \in \mathscr{X}$ and morphisms $f_i : [\operatorname{Spec} A_i/G_{x_i}] \to \mathscr{X}$. Pick an embedding $G_{X_i} \hookrightarrow \mathrm{GL}_N$. Since $[\operatorname{Spec} A_i/G_{x_i}] \cong [\operatorname{Spec} A_i \times^{G_{x_i}} \mathrm{GL}_N/\mathrm{GL}_N]$, let $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_N)$ and we get a surjective, affine, and étale morphism

$$f : \mathscr{X}_1 := [\operatorname{Spec} A/\mathrm{GL}_N] \to \mathscr{X}$$

which is Θ-surjective and stabilizer preserving at all points. As the characteristic of k is zero, then GL_N is linear reductive. Hence we have a good moduli space $\mathscr{X}_1 \to X_1 := \operatorname{Spec} A^{\mathrm{GL}_N}$.

Let $\mathscr{X}_2 := \mathscr{X}_1 \times_{\mathscr{X}} \mathscr{X}_1$ with two affine, étale, Θ-surjective and stabilizer preserving projections $p_1, p_2 : \mathscr{X}_2 \to \mathscr{X}_1$. As f affine, then $\mathscr{X}_2 \cong [\operatorname{Spec} B/\mathrm{GL}_N]$ with good moduli space $\mathscr{X}_2 \to X_2 := \operatorname{Spec} B^{\mathrm{GL}_N}$. Hence we have two cartesian diagrams by Luna's fundamental lemma 1.2.1:

$$\begin{tikzcd}
\mathscr{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{X}_1 \arrow[d] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1
\end{tikzcd}$$

By the universal property of good moduli space, $q_1, q_2 : X_2 \rightrightarrows X_1$ is an étale groupoid.

We claim that $q_1, q_2 : X_2 \rightrightarrows X_1$ is an étale equivalence relation. Pick any $x_1 \in X_1(k)$ and let $x_2, x_2' \in X_2$ are two points in the preimage of (x_1, x_1) in $(q_1, q_2) : X_2 \to X_1 \times X_1$. Let $\hat{x}_2, \hat{x}_2'$ be the unique closed points in their preimages by Proposition 1.3.13. As f is Θ-surjective, then $p_1(\hat{x}_2), p_2(\hat{x}_2), p_1(\hat{x}_2')$ and $p_2(\hat{x}_2')$ are closed over $x_1 \in X_1$. Hence they are all identified with the unique closed point $\hat{x}_1$ over x_1. On the other hand, since f is stabilizer preserving, the stabilizer groups of $\hat{x}_2$ and $\hat{x}_2'$ are the same as the stabilizer groups of $\hat{x}_1$ and of its image in $\mathscr{X}$. Let this stabilizer group by G. It follows that the fiber product of $(p_1, p_2) : \mathscr{X}_2 \to \mathscr{X}_1 \times \mathscr{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\hat{x}_1, \hat{x}_1)} = \mathbf{B}G \times \mathbf{B}G \to \mathscr{X}_1 \times \mathscr{X}_1$ is isomorphic to $\mathbf{B}G$ and thus identified with the residual gerbe of a unique closed point. Therefore $x_2 = x_2'$. Hence we get the claim.

Now pick $X = X_1/X_2$ as an algebraic space. From étale descent, we have

$$\begin{tikzcd}
\mathscr{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{X}_1 \arrow[r, "f"] \arrow[d] & \mathscr{X} \arrow[d, dashed] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 \arrow[r, dashed] & X
\end{tikzcd}$$

By the descent of good moduli space we know $\mathscr{X} \to X$ is a good moduli space. As it is S-complete and Proposition 1.3.5(iv), we get X is separated. $\square$

1.4 Semistable Reduction and Θ-Stability

1.4.1 Preliminaries: Θ-Stratification

1.4.2 Semistable Reduction: Langton's Algorithm

1.4.3 Comparison Between a Stack and Its Semistable Locus

Chapter 2

Good Moduli Spaces for Objects in Abelian Categories

Chapter 3

Good Moduli Space of Semistable Sheaves

3.1 Moduli Stack of Coherent Sheaves

In the next sections we will just consider the stacks over $\mathbb{C}$.

3.1.1 Construction of the Moduli Stack of Coherent Sheaves

Now we consider the moduli space of coherent sheaves over some smooth projective complex variety X. Then we have the Chern character map

$$\gamma : K(X) \xrightarrow{\text{ch}} \text{CH}^*(X)_{\mathbb{Q}} \xrightarrow{\text{cl}} H^{2*}(X, \mathbb{Q}).$$

(or we can use ℓ-adic cohomology) Let Γ be the image of this map.

By Grothendieck-Riemann-Roch theorem (see Chpter 15 in [7]),

$$P(\mathscr{F}, m) = \chi(\mathscr{F}(m)) = \int_X \text{ch}(\mathscr{F}(m))\text{td}(\mathcal{T}_X),$$

then we find that the information of $v \in \Gamma$ is equivalent to the information of the Hilbert polynomial χ. So we can use both of them when X is smooth. If X is just a projective scheme, then we will only to use the Hilbert polynomial.

Theorem 3.1.1. *Let X be a connected projective $\mathbb{C}$-scheme, we let $\underline{\text{Coh}}_P(X)$ the category fibred in groupoid over* Sch/$\mathbb{C}$ *sending a $\mathbb{C}$-scheme T to the groupoid of T-flat families $\mathscr{E} \in \text{Coh}(X \times T)$ such that any restriction $\mathscr{E}_t \in \text{Coh}(X)$ has the Hilbert polynomial P, the morphisms in the above groupoid are given by isomorphisms of $\mathscr{E}$.*

Then $\underline{\text{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$ of affine diagonal. Also, we have the algebraic stack $\underline{\text{Coh}}(X) = \coprod_P \underline{\text{Coh}}_P(X)$.

Proof. Easy to see that $\underline{\mathrm{Coh}}_P(X)$ is actually a stack, we first claim that it is an algebraic stack in a natural way.

For each integer N, we claim there is an open substack $\mathscr{U}_N \subset \underline{\mathrm{Coh}}_P(X)$ parameterizing coherent sheaves $\mathscr{E}$ such that $\mathscr{E}(N)$ generated by global sections and $H^i(X, \mathscr{E}(N)) = 0$ for any $i > 0$. Actually this is trivial by some application of cohomology and base change. As $\underline{\mathrm{Coh}}_P(X) = \bigcup_N \mathscr{U}_N$, we just need to show $\mathscr{U}_N$ is an algebraic stack locally of finite type over $\mathbb{C}$.

For each N, we consider the quotient scheme

$$Q_N := \underline{\mathrm{Quot}}_X^P(\mathscr{O}_X(-N)^{P(N)}).$$

Again by some application of cohomology and base change, we find that there is an open subscheme $Q'_N \subset Q_N$ parameterizing quotients $q : \mathscr{O}_X(-N)^{P(N)} \twoheadrightarrow \mathscr{F}$ such that $H^0(q(N))$ is surjective and $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$.

We have a natural map $Q'_N \to \mathscr{U}_N$ maps $[\mathscr{O}_X(-N)^{P(N)}]$ to $\mathscr{F}$. We observe that Q'_N is also $\mathrm{GL}_{P(N)}$-invariant, then this map descends to

$$\Psi^{\mathrm{pre}} : [Q'_N/\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathscr{U}_N$$

which is fully faithful since every automorphism of a coherent sheaf $\mathscr{E}$ on $X \times S$ induces an automorphism of $p_{2,*}\mathscr{E}(N) = \mathscr{O}_S^{P(N)}$ i.e. an element of $\mathrm{GL}_{P(N)}(S)$, and this element acts on $\mathscr{O}_X(-N)^{P(N)}$ preserving the quotient $\mathscr{E}$.

After stackification, we have another fully faithful map $\Psi : [Q'_N/\mathrm{GL}_{P(N)}] \to \mathscr{U}_N$ which is also essentially surjective by the constructions. Hence we have

$$\mathscr{U}_N \cong [Q'_N/\mathrm{GL}_{P(N)}], \quad \underline{\mathrm{Coh}}_P(X) = \bigcup_N [Q'_N/\mathrm{GL}_{P(N)}].$$

Hence $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over $\mathbb{C}$. □

3.1.2 Basic Facts of the Moduli Stack of Coherent Sheaves

Proposition 3.1.2. *Let X be a projective scheme over an algebraically closed field k. For a noetherian k-algebra R, $\mathrm{MOR}_k(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(\mathscr{E}, \mathscr{E}_*)$ where $\mathscr{E}$ is a coherent sheaf on X_R flat over R and*

$$\mathscr{E}_* : 0 \subset \cdots \subset \mathscr{E}_{i-1} \subset \mathscr{E}_i \subset \cdots \subset \mathscr{E}$$

is a filtration such that $\mathscr{E}i = 0$ for $i \ll 0$, $\mathscr{E}_i = E$ for $i \gg 0$, and each factor $\mathscr{E}_i/\mathscr{E}_{i-1}$ is flat over R. A morphism is an isomorphism $\mathscr{E} \to \mathscr{E}'$ of coherent sheaves compatible with the filtration.

Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to E and 0 to the associated graded $\mathrm{gr}\,\mathscr{E}_* = \bigoplus_i \mathscr{E}_i/\mathscr{E}_{i-1}$.

Proof. A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ correspond to a coherent sheaf $\mathscr{F}$ on $X \times \Theta_R$ flat over Θ_R. By smooth descent, this corresponds to a coherent sheaf on $X \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$ together with a $\mathbb{G}_m$-action. Pushing forward $\mathscr{F}$ along the affine morphism $X \times \Theta_R \to X \times \mathbf{B}\mathbb{G}_{m,R}$, we see that $\mathscr{F}$ also corresponds to a graded $\mathscr{O}_{X_R}[x]$-module flat over $R[x]$. Then $\mathscr{F} = \bigoplus_i \mathscr{E}_i$ with each $\mathscr{E}_i$ a coherent sheaf on X_R, then multiplication by x induces maps $x : \mathscr{E}_i \to \mathscr{E}_{i+1}$ which are necessarily injective as $\mathscr{F}$ is flat over $R[x]$, hence torsion free. Since $\mathscr{F}$ is finitely generated as a graded $R[x]$-module, there exists finitely many homogeneous generators with bounded degree. Thus $\mathscr{E}_i = \mathscr{E}$ for $i \gg 0$. On the other hand, considering the $\mathscr{O}_{X_R}[x]$-module $\mathscr{E}_{\geq d} := \bigoplus_{i \geq d} \mathscr{E}_i \subset \mathscr{F}$, the ascending chain

$$\cdots \subset \mathscr{E}_{\geq d} \subset \mathscr{E}_{\geq d-1} \subset \cdots \subset \mathscr{F}$$

must terminate as $\mathscr{F}$ is noetherian. It follows that $\mathscr{E}_i = 0$ for $i \ll 0$. Since $\mathscr{F}$ is flat as an $R[x]$-module, the quotient $\mathscr{F}/x\mathscr{F} = \bigoplus_i \mathscr{E}_i/\mathscr{E}_{i-1}$ is flat as an R-module and thus each factor $\mathscr{E}_i/\mathscr{E}_{i-1}$ is flat over R. The converse is similar and we omit it. $\square$

Theorem 3.1.3. *For every projective scheme X over an algebraically closed field k, the algebraic stack $\underline{\mathrm{Coh}}(X)$ (and hence $\underline{\mathrm{Coh}}_P(X)$) is Θ-complete and S-complete.*

Remark 3.1.4. *We remark that a map $\phi_R \to \underline{\mathrm{Coh}}(X)$ is the same data as two opposite filtration $\mathscr{E}_*$ and $\mathscr{F}^*$ (that is, $\mathscr{E}_i/\mathscr{E}_{i-1} \cong \mathscr{F}^i/\mathscr{F}^{i+1}$) such that $\mathscr{E}_i = 0$ and $\mathscr{F}_i = \mathscr{F}$ for $i \ll 0$, and $\mathscr{E}_i = \mathscr{E}$ and $\mathscr{F}_i = 0$ for $i \gg 0$. In this case, under this map $(1,0) \mapsto \mathscr{E}$, $(0,1) \mapsto \mathscr{F}$ and $(0,0) \mapsto \mathrm{gr}\,\mathscr{E}_*$.*

Proof. Here we just give an idea. For the entire proof we refer Proposition 6.8.23 in [3].

For Θ-completeness, by Proposition 3.1.2 we know that a map $\Theta_R \backslash 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $\mathscr{E}$ on X_R flat over R and a $\mathbb{Z}$-graded filtration $F_* : \cdots F_{i-1} \subset F_i \subset \cdots \subset \mathscr{E}_K$ such that $F_i = \mathscr{E}_K$ for $i \gg 0$ and $F_i = 0$ for $i \ll 0$, and F_i/F_{i-1} is flat over R. Viewing $\mathscr{E}$ is a subsheaf of $\mathscr{E}_K$, we define $\mathscr{E}_i := F_i \cap \mathscr{E}$. Then $\mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free, hence flat over R. This defines $\Theta_R \to \underline{\mathrm{Coh}}(X)$.

For S-completeness, given a map $\phi_R \backslash 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves $\mathscr{E}$ and $\mathscr{F}$ flat over R and an isomorphism $\alpha : \mathscr{E}_K \cong \mathscr{F}_K$. Let $j : \phi_R \backslash 0 \subset \phi_R$, $j_s, j_t : \operatorname{Spec} R \to \phi_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st} : \operatorname{Spec} K \to \phi_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$0 \to (\mathrm{id} \times j)_* \mathscr{M} \to (\mathrm{id} \times j_s)_* \mathscr{E} \oplus (\mathrm{id} \times j_t)_* \mathscr{F} \to (\mathrm{id} \times j_{st})_* \mathscr{F}_K$$

where the last map is $(a,b) \mapsto a - \alpha(b)$. We can compute the last two sheaves and show that $j_* \mathscr{M}$ is cherent and flat over ϕ_R. Hence we get the result. $\square$

Theorem 3.1.5. *For every projective scheme X over an algebraically closed field k, let $\mathscr{U} \subset \underline{\mathrm{Coh}}(X)$ be an open substack.*

(i) The substack $\mathscr{U}$ is Θ-complete if and only if for every DVR R (with fraction field K and residue field κ), coherent sheaf $\mathscr{E}$ on X_R flat over R, and $\mathbb{Z}$-graded filtration $\mathscr{E}_*$ with $\mathscr{E}_i = 0$ for $i \ll 0$, $\mathscr{E}_i = \mathscr{E}$ for $i \gg 0$ and with each $\mathscr{E}_i/\mathscr{E}_{i-1}$ flat over R, then if $\mathscr{E}$ and $\mathrm{gr}(\mathscr{E}_*|_K)$ are in $\mathscr{U}$, so is $\mathrm{gr}(\mathscr{E}_*|_\kappa)$.

(ii) If for every pair of opposite filtrations $\mathscr{E}_*$ and $\mathscr{F}^*$ of $\mathscr{E}, \mathscr{F} \in \mathscr{U}(k)$, we have the associated graded $\mathrm{gr}\,\mathscr{E}_* \in \mathscr{U}(k)$, then the substack $\mathscr{U}$ is S-complete.

Proof. These are easy. As by Theorem 3.1.3, $\underline{\mathrm{Coh}}(X)$ is Θ-complete and S-complete, the valuative criteria for $\mathscr{U}$ are equivalent to the existence of lifts for all commutative diagrams:

$$\begin{tikzcd} \Theta_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d, hook] \\ \Theta_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X) \end{tikzcd} \qquad \begin{tikzcd} \phi_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d, hook] \\ \phi_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X) \end{tikzcd}$$

Hence we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(X)$ and $\phi_R \to \underline{\mathrm{Coh}}(X)$ are contained in $\mathscr{U}$. Hence theese two results are follows from this and the description as above. $\square$

3.2 Basic Theory of Semistable Sheaves

Our aim is to find a moduli space of sheaves which is of finite type! Actually $\underline{\mathrm{Coh}}_P(X)$ is never of finite type and one can show that even on the smooth projective curves, $\underline{\mathrm{Coh}}_P(X)$ has no good moduli space. Consider $\{\mathscr{O}(n) \oplus \mathscr{O}(-n)\}$ on $\mathbb{P}^1$, then this can not parametrized by a scheme of finite type. Hence we need some more conditions.

3.2.1 Basic Properties

Fix X be a projective scheme over a field k with $H = \mathscr{O}(1)$. Now if $\mathscr{F}$ be a coherent sheaf of dimension $d = \dim X$ with Hilbert polynomial $P(\mathscr{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathscr{F}) \frac{m^i}{i!}$, then we can define $\mathrm{rank}(\mathscr{F}) := \frac{\alpha_d(\mathscr{F})}{\alpha_d(\mathscr{O}_X)}$. If X is integral, this is the usual definition.

For polynomials $f_i \in \mathbb{Q}[m]$ for $i = 1, 2$, we define $f_1 < (\leq) f_2$ if $f_1(m) < (\leq) f_2(m)$ for $m \gg 0$.

Definition 3.2.1. *Fix (X, H) as above and $\mathscr{F}$ be a coherent sheaf of dimension d.*

*(i) We define the **slope** $\mu_H(\mathscr{F}) := \frac{c_1(\mathscr{F}) \cdot H^{d-1}}{\mathrm{rank}(\mathscr{F})}$;*

*(ii) we call $\mathscr{F}$ is **μ_H-(semi)stable** if for any $0 \subset \mathscr{E} \subset \mathscr{F}$ with $0 < \mathrm{rank}\,\mathscr{E} < \mathrm{rank}\,\mathscr{F}$ we have $T_{d-2}(\mathscr{F}) = T_{d-1}(\mathscr{F})$ and $\mu_H(\mathscr{E}) < (\leq) \mu_H(\mathscr{F})$;*

(iii) *we consider the Hilbert polynomial* $P(\mathscr{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathscr{F}) \frac{m^i}{i!}$*, then we have* $\alpha_d(\mathscr{F}) = \mathrm{rank}(\mathscr{F}) \cdot H^d$ *and* $\alpha_{d-1}(\mathscr{F}) = \frac{1}{2} \mathrm{rank}(\mathscr{F}) \deg T_X + \deg \mathscr{F}$*. We define the* **reduced Hilbert polynomial** *is*

$$p(\mathscr{F}, m) = \frac{P(\mathscr{F}, m)}{\alpha_d(\mathscr{F})} = \frac{m^d}{d!} + \frac{1}{H^d}\left(\frac{1}{2}\deg \mathscr{F} + \mu_H(\mathscr{F})\right)\frac{m^{d-1}}{(d-1)!} + \textit{lower terms.}$$

(iv) *Define* $\mathscr{F}$ *is* **H-(semi)stable** *if it is pure and for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$*, we have* $p(\mathscr{E}, m) < (\leq) p(\mathscr{F}, m)$*.*

(v) *Define* $\mathscr{F}$ *is* **geometrically H-stable** *if for any base field extension* $X_K = X \times_k \mathrm{Spec}(K)$ *the pull-back* $\mathscr{F}_K$ *is stable.*

Remark 3.2.2. *Here we have some remarks.*

- *As the Harder-Narasimhan filtration is unique (Theorem 3.2.9) and stable under field extension (Proposition 3.2.10), we don't need the* **geometrically H-ss.**
- *We can define* $\mathscr{F}$ *is* μ_H*-(semi)stable if for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$ *with* $0 < \mathrm{rank}\, \mathscr{E} < \mathrm{rank}\, \mathscr{F}$*, we have* $\mathrm{rank}(\mathscr{F}) \deg(\mathscr{E}) < (\leq) \mathrm{rank}(\mathscr{E}) \deg(\mathscr{F})$*. This is obviously the same definition except that it does not require explicitly that* $T_{d-2}(\mathscr{F}) = T_{d-1}(\mathscr{F})$*. But this can be easy to be deduced.*
- *Similarly, we can define* $\mathscr{F}$ *is H-(semi)stable if for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$*, we have* $\alpha_d(\mathscr{F}) P(\mathscr{E}, m) < (\leq) \alpha_d(\mathscr{E}) p(\mathscr{F}, m)$*. This is obviously the same definition except that it does not require explicitly that* $\mathscr{F}$ *is pure. But applying the inequality to* $\mathscr{E} = T_{d-1}(\mathscr{F})$ *(maximal subsheaf of dimension* $\leq d-1$*), this implies* $T_{d-1}(\mathscr{F}) = 0$*, i.e. it is pure.*
- *If* $\mathscr{F}$ *is pure of dimension d, then we also can use saturated subsheaves, proper quotient sheaves with* $\alpha_d > 0$ *and even proper purely d-dimensional quotient sheaves to define the H-(semi)stable!*

  The proof is trivial by using the trivial exact sequence. See Proposition 1.2.6 in [10] for the proof.

Remark 3.2.3.

- *Easy to see that when it is pure, then*

$$\mu_H\textit{-stable} \Rightarrow H\textit{-stable} \Rightarrow H\textit{-ss} \Rightarrow \mu_H\textit{-ss};$$

- *if* $\dim X = 1$*, then* μ_H*-(semi)stable iff H-(semi)stable.*

Lemma 3.2.4. *Let* $\mathscr{F}, \mathscr{G}$ *are H-ss of dimension d. Then*

(i) *if* $p(\mathscr{F}) > p(\mathscr{G})$*, then* $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$*;*

(ii) *let* $p(\mathscr{F}) = p(\mathscr{G})$. *If* $\mathscr{F}$ *is moreover* H*-stable, then any* $\phi : \mathscr{F} \to \mathscr{G}$ *either zero or injection. Similarly if* $\mathscr{G}$ *is moreover* H*-stable, then any* $\phi : \mathscr{F} \to \mathscr{G}$ *either zero or surjection.*

(iii) *If* $p(\mathscr{F}) = p(\mathscr{G})$ *and* $\alpha_d(\mathscr{F}) = \alpha_d(\mathscr{G})$, *then any non-trivial homomorphism* $f : \mathscr{F} \to \mathscr{G}$ *is an isomorphism provided* $\mathscr{F}$ *or* $\mathscr{G}$ *is* H*-stable.*

Proof. For (i), let nontrivial f with image $\mathscr{E}$, then $p(\mathscr{F}) \leq p(\mathscr{E}) \leq p(\mathscr{G})$ which is impossible. Hence $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$.

For (ii), this is the similar reason in the proof of (i).

For (iii), this is the similar reason in the proof of (i). □

Corollary 3.2.5. *If* $\mathscr{E}$ *is a* H*-stable sheaf, then* $\mathrm{End}(\mathscr{E})$ *is a finite dimensional division algebra over* k. *In particular, if* k *is algebraically closed, then* $k \cong \mathrm{End}(\mathscr{E})$, *i.e.* $\mathscr{E}$ *is a simple sheaf.*

Example 3.2.1. (i) *Any line bundles over smooth projective curves are* H*-stable. See Example 1.2.10 in [10].*

(ii) *For an algebraically closed field* k *of zero characteristic, the bundle* $\Omega_{\mathbb{P}^n}$ *is* H*-stable. See Section 1.4 in [10].*

3.2.2 The Harder-Narasimhan Filtration

We consider a classical result due to Grothendieck as a motivation of the Harder-Narasimhan filtration.

Theorem 3.2.6 (Grothendieck)**.** *Let* $\mathscr{E}$ *be a vector bundle of rank* r *on* $\mathbb{P}^1$, *then there is a uniquely determined decreasing sequence of integers* $a_1 \geq \cdots \geq a_r$ *such that* $E \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$.

Proof. For $r = 1$ this is trivial. Let the theorem holds for all vector bundles of rank $< r$ and that $\mathscr{E}$ is a vector bundle of rank r.

Take any saturation of any rank 1 subsheaf of $\mathscr{E}$. As $\mathbb{P}^1$ is a smooth curve, then it is a line bundle of form $\mathscr{O}(a)$. Let a_1 be the maximal number with this property. Hence $\mathscr{E}/\mathscr{O}(a_1) \cong \bigoplus_{i=2}^r \mathscr{O}(a_i)$ with $a_2 \geq \cdots \geq a_r$. We claim that $a_1 \geq a_2$. Indeed, consider

$$0 \to \mathscr{O}(-1) \to \mathscr{E}(-1 - a_1) \to \bigoplus_{i=2}^{r} \mathscr{O}(a_i - a_1 - 1) \to 0.$$

Since $\Gamma(\mathscr{E}(-1 - a_1)) = \mathrm{Hom}(\mathscr{O}(1 + a_1), \mathscr{E})$ and a_1 be the maximal number with non-trivial $\mathrm{Hom}(\mathscr{O}(a), \mathscr{E})$, then $\Gamma(\mathscr{E}(-1 - a_1)) = 0$. By the long exact sequence we get $H^0(\mathscr{O}(a_i - 1 - a_1)) = 0$ for all i. Hence $a_i < a_1 + 1$. Hence we get the claim.

Next we claim the sequence $0 \to \mathscr{O}(a_1) \to \mathscr{E} \to \bigoplus_{i=2}^{r} \mathscr{O}(a_i) \to 0$ split. This follows from the Serre duality

$$\mathrm{Ext}^1\left(\bigoplus_{i=2}^{r} \mathscr{O}(a_i), \mathscr{O}(a_1)\right)^{\vee} \cong \bigoplus_{i=2}^{r} \mathrm{Hom}(\mathscr{O}(a_1), \mathscr{O}(a_i - 2)) = 0.$$

Finally, the uniqueness is not hard to prove. We omit it. □

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 3.2.7. *Fix $\mathscr{E} \in \mathrm{Coh}(X)$ is pure of dimension d. A Harder-Narasimhan filtration (or HN-filtration) of $\mathscr{E}$ is*

$$0 = \mathrm{HN}_0(\mathscr{E}) \subset \mathrm{HN}_1(\mathscr{E}) \subset \cdots \subset \mathrm{HN}_l(\mathscr{E}) = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E}) := \mathrm{HN}_i(\mathscr{E})/\mathrm{HN}_{i-1}(\mathscr{E})$ which are H-ss of dimension d and $p(\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})) > p(\mathrm{gr}_{i+1}^{\mathrm{HN}}(\mathscr{E}))$ for all i. We define $p_{\max}(\mathscr{E}) := p(\mathrm{gr}_1^{\mathrm{HN}}(\mathscr{E}))$ and $p_{\min}(\mathscr{E}) := p(\mathrm{gr}_l^{\mathrm{HN}}(\mathscr{E}))$;

Lemma 3.2.8. *If $\mathscr{F}, \mathscr{G}$ is pure of dimension d with $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$, then $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$.*

Proof. If $f : \mathscr{F} \to \mathscr{G}$ is non-trivial. Let $i > 0$ be the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \neq 0$ and $j > 0$ the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \subset \mathrm{HN}_j(\mathscr{G})$. Hence we get a non-trivial $\bar{f} : \mathrm{gr}_i^{\mathrm{HN}}(\mathscr{F}) \to \mathrm{gr}_j^{\mathrm{HN}}(\mathscr{G})$. But this is impossible by $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$ and Lemma 3.2.4(i). □

Theorem 3.2.9. *Let $\mathscr{E}$ be a pure coherent sheaf of dimension d. Then there always exists a unique Harder-Narasimhan filtration.*

Proof. Here we will use a result (see Lemma 1.3.5 in [10]):

- Let $\mathscr{E}$ be a purely d-dimensional sheaf. Then there is a subsheaf $\mathscr{F} \subset \mathscr{E}$ such that for all subsheaves $\mathscr{G} \subset \mathscr{E}$ one has $p(\mathscr{F}) \geq p(\mathscr{G})$, and in case of equality $\mathscr{F} \supset \mathscr{G}$. Moreover, $\mathscr{F}$ is uniquely determined and semistable. It is called the maximal destabilizing subsheaf of $\mathscr{E}$.

Let $\mathscr{E}_1$ be its maximal destabilizing subsheaf. By induction we may assume $\mathscr{E}/\mathscr{E}_1$ has a Harder-Narasimhan filtration

$$0 \subset \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_{l-1} = \mathscr{E}/\mathscr{E}_1.$$

Let $\mathscr{E}_{i+1} \subset \mathscr{E}$ be the preimage of $\mathscr{G}_i$. Just need to show that $p(\mathscr{E}_1) > p(\mathscr{E}_2/\mathscr{E}_1)$. If this were false, we would have $p(\mathscr{E}_2) \geq p(\mathscr{E}_1)$ contradicting the maximality of $\mathscr{E}_1$.

For the uniqueness, consider two Harder-Narasimhan filtrations $\mathscr{E}_*, \mathscr{E}'_*$. Let $p(\mathscr{E}'_1) \geq p(\mathscr{E}_1)$. Let j be minimal with $\mathscr{E}'_1 \subset \mathscr{E}_j$. Then we have

$$p(\mathscr{E}_j/\mathscr{E}_{j-1}) \geq p(\mathscr{E}'_1) \geq p(\mathscr{E}_1) \geq p(\mathscr{E}_j/\mathscr{E}_{j-1}).$$

Hence $p(\mathscr{E}'_1) = p(\mathscr{E}_1)$ and $j = 1$ and $\mathscr{E}'_1 \subset \mathscr{E}_1$. Similarly we get $\mathscr{E}'_1 \supset \mathscr{E}_1$, hence $\mathscr{E}'_1 = \mathscr{E}_1$. Using induction again we get the result. □

Proposition 3.2.10. *Let $\mathscr{E}$ be a pure sheaf of dimension d and let K/k be a field extension. Then*

$$\mathrm{HN}_*(E \otimes_k K) = \mathrm{HN}_*(E) \otimes_k K.$$

In particular, the H-ss sheaves stable under base field extension.

Proof. We do not care about this. We refer the proof of Theorem 1.3.7 in [10]. □

3.2.3 The Jordan-Hölder Filtration

As we all know, the Harder-Narasimhan filtration shows that the H-ss sheaves form the building blocks for all the coherent sheaves. But the Jordan-Hölder filtration shows that the H-stable sheaves form the building blocks for all H-ss sheaves.

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 3.2.11. *Fix $\mathscr{E} \in \mathrm{Coh}(X)$. Let $\mathscr{E}$ is H-ss, a* Jordan-Hölder filtration *(or JH-filtration) of $\mathscr{E}$ is*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ are H-stable and $p(\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})) = p(\mathscr{E})$ for all i. We define $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) := \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$.

Remark 3.2.12. *Unlike the Harder-Narasimhan filtration, the Jordan-Hölder filtration is NOT unique. For example we let the direct sum of two line bundles of the same degree one.*

Theorem 3.2.13. *Jordan-Hölder filtrations always exist. Up to isomorphism, the sheaf $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. Any filtration of $\mathscr{E}$ by semistable sheaves with reduced Hilbert polynomial $p(\mathscr{E})$ has a maximal refinement, whose factors are necessarily stable. The uniqueness of $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$ is not hard to show. We refer 1.5.2 in [10]. □

Definition 3.2.14. *Two H-ss sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same reduced Hilbert polynomial are called* S-equivalent *if $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_1) \cong \mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_2)$.*

Definition 3.2.15. *If $\mathscr{E}$ is H-ss, we call $\mathscr{E}$ is H-polystable if it is the direct sum of stable sheaves. In this case $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \mathscr{E}$.*

Remark 3.2.16. *We will show that the good moduli space of moduli stack of H-ss sheaves actually parametrizes only* S*-equivalence classes of H-ss sheaves! As we saw above, every* S*-equivalence class of H-ss sheaves contains exactly one polystable sheaf up to isomorphism. Thus, the good moduli space of H-ss sheaves in fact parametrizes polystable sheaves.*

Actually the S *stands for* Seshadri *as* S*-completeness is a geometric property reminiscent of how the* S*-equivalence relation on sheaves implies separatedness of the moduli space.*

Remark 3.2.17. *(i) By the similar arguments of Jordan-Hölder filtrations, on can show that every semistable sheaf $\mathscr{E}$ contains a unique non-trivial maximal H-polystable subsheaf of the same reduced Hilbert polynomial. This sheaf is called the* socle *of $\mathscr{E}$.*

(ii) One can use some basic properties of socles to find that if $\mathscr{E}$ is a simple sheaf, then it is H-stable if and only if it is geometrically H-stable. Hence in particular if k is algebraically closed and $\mathscr{E}$ is a H-stable sheaf, then $\mathscr{E}$ is also gemetrically H-stable. See 1.5.10 and 1.5.11 in [10].

(iii) For μ_H-ss, they define $\mathrm{Coh}_{d,d'}(X) = \mathrm{Coh}_d(X)/\mathrm{Coh}_{d'-1}(X)$ and consider the μ-ss on it using $\hat{\mu}(\mathscr{E}) = \frac{\alpha_{d-1}(\mathscr{E})}{\alpha_d(\mathscr{E})}$. And when $d' = d-1$, this is just the definition before. In this space there also have the Harder-Narasimhan filtrations and Jordan-Hölder filtrations. For the general arguments we refer Section 1.6 in [10].

(iv) For μ, there are several properties for torsion-free sheaves $\mathscr{F}, \mathscr{G}$ on the normal variety ([10] Page 29):

- $\mu(\mathscr{E}(a)) = \mu(\mathscr{E}) + a \deg X$, *similar for* $\mu_{\min}, \mu_{\max}$;
- $\mu_{\min}(\mathscr{E} \oplus \mathscr{F}) = \min(\mu_{\min}(\mathscr{E}), \mu_{\min}(\mathscr{F}))$, *similar for* $\mu_{\max}$;
- $\mu_{\min}(\mathscr{F}) \geq \mu_{\min}(\mathscr{E})$ *for* $\mathscr{E} \twoheadrightarrow \mathscr{F}$;
- $\mu_{\max}(\mathscr{E}) \leq \mu_{\max}(\mathscr{F})$ *for* $\mathscr{E} \hookrightarrow \mathscr{F}$.

3.3 Moduli Stack of Semistable Sheaves

3.3.1 The Mumford-Castelnuovo Regularity and Boundedness

In this section we will give some useful criterion about boundedness of families of sheaves.

Let X be a projective scheme over k with very ample $H = \mathscr{O}_X(1)$.

Definition 3.3.1. *Let m be an integer. A coherent sheaf $\mathscr{F}$ is said to be m-regular , if for all $i > 0$ we have $H^i(X, \mathscr{F}(m-i)) = 0$.*

The Mumford-Castelnuovo regularity of a coherent sheaf $\mathscr{F}$ is the number

$$\operatorname{reg}(\mathscr{F}) = \inf\{m \in \mathbb{Z} : \mathscr{F} \text{ is } m\text{-regular}\}.$$

Lemma 3.3.2. *There are universal polynomials $P_i \in \mathbb{Q}[T_0, ..., T_i]$ such that the following holds: Let $\mathscr{F}$ be a coherent sheaf of dimension $\leq d$ and let $H_1, ..., H_d$ be an $\mathscr{F}$-regular sequence of hyperplane sections. If $\chi(\mathscr{F}|_{\bigcap_{j\leq i} H_j}) = a_i$ and $h^0(\mathscr{F}|_{\bigcap_{j\leq i} H_j}) \leq b_i$, then*

$$\operatorname{reg}(\mathscr{F}) \leq P_d(a_0 - b_0, ..., a_d - b_d).$$

Proof. See [11] for the original proof. □

Lemma 3.3.3. *The following properties of a flat family of sheaves $\mathscr{F}$ on $X \to S$ are equivalent:*

(i) The family is bounded.

(ii) There is a uniform bound $\operatorname{reg}(\mathscr{F}_s) \leq \rho$ for all $s \in S$.

Proof. See [8] for the original proof. □

Then we have two nice criterion about boundedness of sheaves.

Theorem 3.3.4 (Kleiman Criterion)**.** *Let flat family of sheaves $\mathscr{F}$ on $X \to S$ with the same Hilbert polynomial P. Then this family is bounded if and only if there are constants $C_i, i = 0, ..., d = \deg P$ such that for every $\mathscr{F}_s$ there exists an $\mathscr{F}_s$-regular sequence of hyperplane sections $H_1, ..., H_d$, such that*

$$h^0(\mathscr{F}_s|_{\bigcap_{j\leq i} H_j}) \leq C_i.$$

Proof. Follows from Lemma 3.3.2 and Lemma 3.3.3. □

Theorem 3.3.5 (Grothendieck)**.** *Let P be a polynomial and ρ an integer. Then there is a constant C depending only on P and ρ such that the following holds:*

- *If X be a projective scheme on k with very ample divisor H and if $\mathscr{E} \in \operatorname{Coh}(X)$ is a d-dimensional sheaf with Hilbert polynomial P and Mumford-Castelnuovo regularity $\operatorname{reg}(\mathscr{E}) \leq \rho$ and if $\mathscr{F} \in \operatorname{Coh}(X)$ is a purely d-dimensional quotient sheaf of $\mathscr{E}$ then $\hat{\mu}(\mathscr{F}) \geq C$.*

Moreover, the family of purely d-dimensional quotients $\mathscr{F}$ with $\hat{\mu}(\mathscr{F})$ bounded from above is bounded. In particular the set of Hilbert polynomials of pure quotients with fixed $\hat{\mu}(\mathscr{F})$ is finite.

Proof. After embedding them into the projective space $\mathbb{P}^d$, we may consider $X = \mathbb{P}^d$. Hence we have $\mathscr{G} := V \otimes \mathscr{O}(-\rho) \twoheadrightarrow \mathscr{E}$ where $\operatorname{rank} V = P(\rho)$, so we just need to consider $\mathscr{G}$. Pick a quotient $q : \mathscr{G} \to \mathscr{F}$ of rank s, then

$$\bigwedge^s q : \bigwedge^s V \otimes \mathscr{O}(-s\rho) \to \det \mathscr{F} = \mathscr{O}(\deg \mathscr{F})$$

gives $\deg \mathscr{F} \geq -s\rho$. Hence

$$\hat{\mu}(\mathscr{F}) = \frac{\deg \mathscr{F} + \operatorname{rank} \mathscr{F} \alpha_{d-1}(\mathscr{O}_X)}{\alpha_d(\mathscr{F})} \geq -\rho + \alpha_{d-1}(\mathscr{O}_X).$$

For the final part, we let $\hat{\mu} \leq C'$. It is enough to show that the family of pure quotient sheaves $\mathscr{F}$ of rank $0 < s \leq \operatorname{rank}(\mathscr{G}) = P(\rho)$ and with $l = \deg \mathscr{F} = s(C' - \alpha_{d-1}(\mathscr{O}_X))$ is bounded. Consider $\psi : \mathscr{G} \otimes \bigwedge^{s-1} \mathscr{G} \xrightarrow{\wedge} \bigwedge^s \mathscr{G} \xrightarrow{\det q} \mathscr{O}(l)$ and $\psi^\vee : G \to \mathscr{O}(l) \otimes \bigwedge^{s-1} \mathscr{G}^\vee$. Let U denote the dense open subscheme where $\mathscr{F}$ is locally free. Then $\ker(\psi^\vee)|_U = \ker(q)|_U$. Since the quotients of $\mathscr{G}$ corresponding to these two subsheaves of $\mathscr{G}$ are torsion free and since they coincide on a dense open subscheme of $\mathbb{P}^d$, we must have $\ker(\psi^\vee) = \ker(q)$ everywhere, i.e. $\mathscr{F} \cong \operatorname{Im}\psi^\vee$. Now, the family of such image sheaves certainly is bounded. □

3.3.2 Basic Construction and Openness of Semistable Sheaves

Definition 3.3.6. *We define the stack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ send a scheme T to a families of H-ss sheaves on $X \times T \to T$. Similarly we define $\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X)$ send a scheme T to a families of geometrically H-stable sheaves on $X \times T \to T$.*

Proposition 3.3.7. *The following properties of coherent sheaves are open in flat families: being simple, of pure dimension, H-ss, or geometrically H-stable.*

Proof. Let $f : X \to S$ be a projective morphism of Noetherian schemes (as the property is local) and let $\mathscr{O}_X(1)$ be an f-very ample invertible sheaf on X. Let $\mathscr{F}$ be a flat family of d-dimensional sheaves with Hilbert polynomial P on the fibres of f. For each $s \in S$, a sheaf $\mathscr{F}_s$ is simple iff $\hom_{\kappa(s)}(\mathscr{F}_s, \mathscr{F}_s) = 1$. Thus openness here is an immediate consequence of the semicontinuity properties for relative Ext-sheaves.

Next we consider pure dimension (P1), H-ss (P2), and geometrically H-stable (P3) which can be characteristics by the Hilbert polynomials of quotient sheaves. Consider the following several sets:

$$\begin{aligned} A = \{ & P'' : \deg(P'') = d, \hat{\mu}(P'') \leq \hat{\mu}(P) \text{ and there is a geometric point } s \in S \\ & \text{and a surjection } \mathscr{F}_s \to \mathscr{F}'' \text{ onto a pure sheaf with } P(\mathscr{F}'') = P'' \}; \\ A_1 = & \{P'' \in A : \deg(P - P'') \leq d - 1\}; \quad A_2 = \{P'' \in A : p'' < p\}; \\ A_3 = & \{P'' \in A : p'' \leq p \text{ and } P'' < P\}. \end{aligned}$$

By Theorem 3.3.5 we get the set A is finite. For each polynomial $P'' \in A$ we consider $\pi : Q(P'') = \underline{\mathrm{Quot}}_{X/S}(\mathscr{F}, P'') \to S$ be the projective morphism. Hence $\pi(Q(P''))$ is closed. As $\mathscr{F}_s$ has (Pi) if and only if $s \notin \bigcup_{P'' \in A_i} \pi(Q(P'')) \subset S$. Well done. □

Corollary 3.3.8. *We have open substacks*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X) \subset \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$$

which parameterizing H-ss sheaves and geometrically H-stable sheaves, are all algebraic stacks locally of finite type.

Proof. Follows from the Theorem 3.3.7. □

3.3.3 Boundedness I: The Grauert-Mülich Theorem

In these few sections we will assume the base field k is an algebraically closed field of characteristic zero!

In 2004, Langer in [13] and [12] proved the positive and mixed characteristic of the boundedness of semistable sheaves and also gives a generalized Le Potier-Simpson type bound for the number of global sections of a torsion free sheaf.

Since I don't care about the fields either not algebraically closed or not of characteristic zero, so we just introduce the characteristic zero case which is more easier.

We may use the Theorem 3.3.4 to show the boundedness. Hence we need to investigate the behavior of sheaves resticted to the intersections of hyperplanes. Actually the Grauert-Mülich theorem and the Le Potier-Simpson estimate are what we want.

Before we discuss the notations and main results, we will introduce a family-version of the Harder-Narasimhan filtration:

Theorem 3.3.9 (The Relative Harder-Narasimhan Filtration)**.** *Let S be an integral k-scheme of finite type, let $f : X \to S$ be a projective morphism and let H be an f-ample invertible sheaf on X. Let $\mathscr{F}$ be a flat family of d-dimensional coherent sheaves on the fibres of f. There is a projective birational morphism $g : T \to S$ of integral k-schemes and a filtration*

$$0 = \mathrm{HN}_0(\mathscr{F}) \subset \mathrm{HN}_1(\mathscr{F}) \subset \cdots \mathrm{HN}_l(\mathscr{F}) = \mathscr{F}_T$$

such that

- *(a)* $\mathrm{HN}_i(\mathscr{F})/\mathrm{HN}_{i-1}(\mathscr{F})$ *are T-flat;*
- *(b) there is a dense open subscheme $U \subset T$ such that* $\mathrm{HN}_*(\mathscr{F})_t = g_X^* \mathrm{HN}_*(\mathscr{F}_{g(t)})$ *for all $t \in U$.*

Moreover, $(g, \mathrm{HN}_*(\mathscr{F}))$ is universal in the sense that if $g' : T' \to S$ is any dominant morphism of integral schemes and if $\mathcal{F}'_*$ is a filtration of $\mathscr{F}_{T'}$ satisfying these two properties, then there is an S-morphism $h : T' \to T$ with $\mathcal{F}'_* = h_X^* \mathrm{HN}_*(\mathscr{F})$.

Proof. See [15] for the original proof. Also in Theorem 2.3.2 in [10]. □

Now let X be a normal projective variety over k of $\dim n \geq 2$ with very ample $H = \mathscr{O}_X(1)$. Let $V_a := H^0(X, \mathscr{O}_X(a))$ and $\Pi_a := \mathbb{P}(V_a^\vee) = |\mathscr{O}_X(a)|$. Let

$$\begin{array}{ccc} Z_a = \{(D,x) \in \Pi_a \times X : x \in D\} & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ \Pi_a & & \end{array}$$

The scheme-structure of Z_a is easy: consider $\mathscr{K}$ be the kernel of $V_a \otimes \mathscr{O}_X \to \mathscr{O}_X(a)$, then $Z_a = \mathbb{P}(\mathscr{K}^\vee)$.

Let $(a_1, ..., a_l)$ be a fixed finite sequence of positive integers, $0 < l < n$. Let $\Pi = \prod_i \Pi_{a_i}$ with $p_i : \Pi \to \Pi_{a_i}$ and $Z = Z_{a_1} \times_X \cdots \times_X Z_{a_l}$ and

$$\begin{array}{ccc} Z & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ \Pi & & \end{array}$$

with $q_i : Z \to Z_{a_i}$.

Lemma 3.3.10. *Let $\mathscr{E}$ be a torsion free coherent sheaf on X and $\mathscr{F} := q^*\mathscr{E}$.*

(i) There is a nonempty open subset $S' \subset \Pi$ such that the morphism $p_{S'} : Z_{S'} \to S'$ is flat and such that for all $s \in S'$ the fibre Z_s is a normal irreducible complete intersection of codimension l in X;

(ii) There is a nonempty open subset $S \subset S'$ such that the family $\mathscr{F}_S = q^*\mathscr{E}|_{Z_s}$ is flat over S and such that for all $s \in S$ the fibre $\mathscr{F}_s \cong \mathscr{E}|_{Z_s}$ is torsion free.

Proof. Lemma 3.3.1 in [10]. Just an easy Bertini-type lemma. □

By the relative Harder-Narasimhan filtration 3.3.9, we have

$$0 = \mathscr{F}_0 \subset \cdots \mathscr{F}_j = \mathscr{F}_S$$

such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ are S-flat and there is a dense open subscheme $S_0 \subset S$ such that for all $s \in S_0$ the fibres $(\mathscr{F}_*)_s$ form the Harder-Narasimhan filtration of $\mathscr{F}_s = \mathscr{E}|_{Z_s}$.

WLOG we let $S_0 = S$. Now S connected, we let $\mu_i = \mu((\mathscr{F}_i/\mathscr{F}_{i-1})_s)$ with $\mu_i > \mu_{i+1}$. Define the number

$$\delta\mu = \max\{\mu_i - \mu_{i+1} : i = 1, ..., j-1\}.$$

Remark 3.3.11. *Then $\delta\mu = \delta\mu(\mathscr{E}|_{Z_s})$ for a general point $s \in \Pi$, and $\delta\mu$ vanishes if and only if $\mathscr{E}|_{Z_s}$ is μ_H-ss for general s.*

Theorem 3.3.12 (Generalized Grauert-Mülich Theorem)**.** *Let $\mathscr{E}$ be a μ_H-ss torsion free sheaf. Then there is a nonempty open subset $S \subset \Pi$ such that for all $s \in S$ the following inequality holds:*

$$0 \leq \delta\mu(\mathscr{E}|_{Z_s}) \leq \max\{a_i\} \deg X \cdot \prod_i a_i.$$

Proof. WLOG we let $\delta\mu > 0$. Let i such that $\delta\mu = \mu_i - \mu_{i+1}$. Let $\mathscr{F}' = \mathscr{F}_i, \mathscr{F}'' = \mathscr{F}/\mathscr{F}'$ and for all $s \in S$ the sheaves $\mathscr{F}'_s, \mathscr{F}''_s$ are torsion-free. And $\mu_{\min}(\mathscr{F}'_s) = \mu_i$ and $\mu_{\max}(\mathscr{F}''_s) = \mu_{i+1}$. Pick Z_0 be a maximal open set of Z_S such that $\mathscr{F}|_{Z_0}$ and $\mathscr{F}''|_{Z_0}$ are locally free of rank r, r''. Then $\mathscr{F}|_{Z_0} \to \mathscr{F}''|_{Z_0}$ defines $\phi : Z_0 \to \underline{\mathrm{Grass}}_X(\mathscr{E}, r'')$.

Consider $d\phi : \mathcal{T}_{Z/X}|_{Z_0} \to \phi^*\mathcal{T}_{\underline{\mathrm{Grass}}_X(\mathscr{E},r'')/X}$. As

$$\phi^*\mathcal{T}_{\underline{\mathrm{Grass}}_X(\mathscr{E},r'')/X} = \mathscr{H}om(\mathscr{F}', \mathscr{F}'')|_{Z_0},$$

we get $d\phi$ correspond to $\Phi : (\mathscr{F}' \otimes \mathcal{T}_{Z/X})|_{Z_0} \to \mathscr{F}''|_{Z_0}$.

We claim that Φ_s were not zero for a general point $s \in S$. If it is, making S smaller if necessary, this supposition would imply that Φ is zero. As $q : Z \to X$ is a bundle, we have $X_0 := q(Z_0)$ is open and $\mathrm{codim}(X \backslash X_0, X) \geq 2$ and $\mathscr{E}|_{X_0}$ is locally free. Hence we have

$$\begin{tikzcd}
 & & \underline{\mathrm{Grass}}(\mathscr{E}|_{X_0}, r'') \arrow[d, shift right] \\
Z_0 \arrow[urr, "\phi"] \arrow[rr, "q_0"'] \arrow[d, hook] & & X_0 \arrow[u, dashed, shift right, "\rho"'] \arrow[d, hook] \\
Z_S \arrow[r, hook] \arrow[d, "p_S"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[r, "q"] \arrow[d, "p"] & X \\
S \arrow[r, hook] & \Pi &
\end{tikzcd}$$

Now q_0 is smooth of connected fibers and ϕ is constant on the fibres of q_0 and hence factors through a morphism ρ (here we need $\mathrm{char} k = 0$). But such ρ corresponds to a locally free quotient $\mathscr{E}|_{X_0} \to \mathscr{E}''$ of rank r'' with the property that $\mathscr{E}''|_{Z_s \cap X_0}$ is isomorphic to $\mathscr{F}''|_{Z_s \cap Z_0}$ for general s. Since by assumption $\mathscr{F}''_s$ is a destabilizing quotient of $\mathscr{F}_s$, any extension of $\mathscr{E}''$ as a quotient of $\mathscr{E}$ is destabilizing. This contradicts the assumption that $\mathscr{E}$ is μ_H-ss.

Hence Φ_s is nonzero for general $s \in S$, that is, Φ_s is a non-trivial element in $\mathrm{Hom}_{\mathcal{C}}(\mathscr{F}'_s \otimes \mathcal{T}_{Z/X}|_{Z_s}, \mathscr{F}''_s)$ where $\mathcal{C} := \mathrm{Coh}_{n-l,n-l-1}(Z_s)$. By the similar result of Lemma 3.2.8, we have

$$\mu_{\min}(\mathscr{F}'_s \otimes \mathcal{T}_{Z/X}|_{Z_s}) \leq \mu_{\max}(\mathscr{F}''_s).$$

The Koszul complex associated to the evaluation map $e : V_a \otimes \mathscr{O}_X \to \mathscr{O}_X(a)$ provides us a surjection $\bigwedge^2 V_a \otimes \mathscr{O}_X(-a) \to \ker e \cong \mathscr{K}$ and hence a surjection

$$\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i) \otimes p^*\mathscr{O}(1) \to \bigoplus_i q^*\mathscr{K}_{a_i} \otimes p^*\mathscr{O}(1) \to \mathcal{T}_{Z/X}.$$

Hence a surjection

$$\left(\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i)\right)\bigg|_{Z_s} \to \mathcal{T}_{Z/X}|_{Z_s}.$$

Hence we get

$$\begin{aligned}\mu_{\min}(\mathcal{T}_{Z/X}|_{Z_s} \otimes \mathscr{F}'_s) &\geq \mu_{\min}\left(\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i) \otimes \mathscr{F}'|_{Z_s}\right)\\ &= \min_i\{\mu_{\min}(\mathscr{O}_{Z_s}(-a_i) \otimes \mathscr{F}'_s)\}\\ &= \mu_{\min}(\mathscr{F}'_s) - \max\{a_i\} \cdot \deg Z_s.\end{aligned}$$

Hence combining these two inequality, we have

$$\begin{aligned}\delta\mu &= \mu_{\min}(\mathscr{F}'_s) - \mu_{\max}(\mathscr{F}''_s)\\ &\leq \max\{a_i\} \cdot \deg Z_s = \max\{a_i\} \deg X \cdot \prod_i a_i.\end{aligned}$$

Hence we get the result. □

Theorem 3.3.13. *Let X be a normal projective variety over an algebraically closed field of characteristic zero. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are μ_H-ss sheaves, then $\mathscr{F}_1 \otimes \mathscr{F}_2$ is μ_H-ss too.*

Proof. Omitted, see Section 3.2 in [10]. □

Remark 3.3.14. *As a corollary of this theorem, we have $\mu_{\min}(\mathscr{F}_1 \otimes \mathscr{F}_2) = \mu_{\min}(\mathscr{F}_1) + \mu_{\min}(\mathscr{F}_2)$ by tensoring their HN-filtrations, similar for $\mu_{\max}$ and μ.*

Corollary 3.3.15. *Let X be a normal projective variety of dimension n and let $H = \mathscr{O}_X(1)$ be a very ample line bundle. Let $\mathscr{F}$ be a μ_H-ss coherent $\mathscr{O}_X$-module of rank r. Let Y be the intersection of $s < n$ general hyperplanes in the linear system $|\mathscr{O}_X(1)|$. Then*

$$\mu_{\min}(\mathscr{F}|_Y) \geq \mu(\mathscr{F}) - \frac{r-1}{2}\deg(X)$$

and

$$\mu_{\max}(\mathscr{F}|_Y) \leq \mu(\mathscr{F}) + \frac{r-1}{2}\deg(X).$$

Proof. WLOG we let $\mathscr{F}$ is a torsion free sheaf. Pick $\mu_1, ..., \mu_j$ and $r_1, ..., r_j$ be the slopes and ranks of μ_H-HN filtration of $\mathscr{F}|_Y$. By Theorem 3.3.12 we have $0 \leq \mu_i - \mu_{i+1} \leq \deg X$. Hence $\mu_i \geq \mu_1 - (i-1)\deg X$. Hence we have

$$\begin{aligned}\mu(\mathscr{F}) = \sum_{i=1}^{j} \frac{r_i \mu_i}{r} &\geq \mu_1 - \sum_{i=1}^{j}(i-1)\frac{r_i}{r}\deg X \\ &\geq \mu_1 - \frac{\deg X}{r}\sum_{i=1}^{r}(i-1) = \mu_{\max}(\mathscr{F}|_Y) - \deg X \frac{r-1}{2}.\end{aligned}$$

Similar for $\mu_{\min}(\mathscr{F}|_Y)$. □

3.3.4 Boundedness II: The Le Potier-Simpson Estimate

Lemma 3.3.16. *Suppose that X is a normal projective variety of dimension d and that $\mathscr{F}$ is a torsion free sheaf of* $\operatorname{rank}(\mathscr{F})$. *Then for any $\mathscr{F}$-regular sequence of hyperplane sections $H_1, ..., H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ the following estimate holds for all $v = 1, ..., d$:*

$$\frac{h^0(X_v, \mathscr{F}|_{X_v})}{\deg X \cdot \operatorname{rank} \mathscr{F}} \leq \frac{1}{v!}\left[\frac{\mu_{\max}(\mathscr{F}|_{X_1})}{\deg X} + v\right]_+^v$$

where for any $x \in \mathbb{R}$ we define $[x]_+ := \max\{0, x\}$.

Proof. Let $\mathscr{F}_v := \mathscr{F}|_{X_v}$. Using induction on v.

Let $v = 1$. Since we have $h^0(X_1, \mathscr{F}_1) \leq \sum_i h^0(X_1, \operatorname{gr}_i^{\mathrm{HN}}(\mathscr{F}_1))$ and the right hand side of the estimate in the lemma is monotonously increasing with μ, we may assume WLOG that $\mu(\mathscr{F}_1) = \mu_{\max}(\mathscr{F}_1)$, i.e. that $\mathscr{F}_1$ is μ_H-semistable. Hence

$$h^0(x_1, \mathscr{F}_1) \leq h^0(x_1, \mathscr{F}_1(-l)) + \operatorname{rank}(\mathscr{F}) \cdot l \deg X.$$

By Lemma 3.2.4(i), we find that $h^0(x_1, \mathscr{F}_1(-l)) = \hom(\mathscr{O}_{X_1}(l), \mathscr{F}_1) = 0$ if $l > \mu(\mathscr{F}_1)/\deg X$. Now pick $l = \lfloor \mu(\mathscr{F}_1)/\deg X \rfloor + 1$ and well done.

Now let this is right for $v-1$. Consider

$$0 \to \mathscr{F}_v(-k-1) \to \mathscr{F}_v(-k) \to \mathscr{F}_{v-1}(-k) \to 0, \quad k = 0, 1, \cdots.$$

Hence inductively derives estimates

$$\begin{aligned}h^0(X_v, \mathscr{F}_v) &\leq h^0(X_v, \mathscr{F}_v(-l)) + \sum_{i=0}^{l-1} h^0(X_{v-1}, \mathscr{F}_{v-1}(-i)) \\ &\leq \sum_{i=0}^{\infty} h^0(X_{v-1}, \mathscr{F}_{v-1}(-i)).\end{aligned}$$

By induction hypothesis one has

$$\frac{h^0(X_v, \mathscr{F}_v)}{\operatorname{rank}(\mathscr{F}) \deg X} \leq \frac{1}{(v-1)!} \int_{-1}^{C} \left[\frac{\mu_{\max}(\mathscr{F}_1)}{\deg X} + (v-1) - t \right]_+^{v-1} dt$$

where C is the maximum of -1 and the smallest zero of the integrand. Evaluating the integral yields the bound of the lemma. □

Theorem 3.3.17 (The Le Potier-Simpson Estimate)**.** *Suppose that X is a projective variety over an algebraically closed k of characteristic zero. For any purely d-dimensional coherent sheaf $\mathscr{F}$ of multiplicity $\alpha_d(\mathscr{F}) = r(\mathscr{F})$ there is an $\mathscr{F}$-regular sequence of hyperplane sections $H_1, ..., H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ the following estimate holds for all $v = 1, ..., d$:*

$$\frac{h^0(X_v, \mathscr{F}|_{X_v})}{r(\mathscr{F})} \leq \frac{1}{v!} \left[\hat{\mu}_{\max}(\mathscr{F}) + r(\mathscr{F})^2 + \frac{1}{2}(r(\mathscr{F}) + d) - 1 \right]_+^v .$$

Proof. First we claim that when W is a normal projective variety of dimension d and that $\mathscr{K}$ is a torsion free sheaf of $\operatorname{rank}(\mathscr{K})$, there is an $\mathscr{K}$-regular sequence $H_1, ..., H_d$ such that the following estimate holds for all $v = 1, ..., d$:

$$\frac{h^0(W_v, \mathscr{K}|_{W_v})}{\deg W \cdot \operatorname{rank}(\mathscr{K})} \leq \frac{1}{v!} \left[\frac{\mu_{\max}(\mathscr{K})}{\deg W} + \frac{\operatorname{rank}(\mathscr{K}) - 1}{2} + v \right]_+^v .$$

Indeed,

Now we can use this claim to reduce to the general case. Let $i : X \hookrightarrow \mathbb{P}^N$ be the closed embedding induced by $H = \mathscr{O}_X(1)$. Let $\mathscr{F}$ as $i_*\mathscr{F}$ on $\mathbb{P}^N$, let $Z = \operatorname{supp}(\mathscr{F})$ and choose a linear subspace L of dimension $N - d - 1$ which does not intersect Z (right for infinite field). Consider projection $\pi : Z \hookrightarrow \mathbb{P}^N \backslash L \to Y \cong \mathbb{P}^d$ which is a finite map with $\mathscr{O}_Z(1) = \pi^*\mathscr{O}_Y(1)$. As $\mathscr{F}$ is pure, we know that $\pi_*\mathscr{F}$ is torsion-free and $\operatorname{rank}(\mathscr{F}) = \operatorname{rank}(\pi_*\mathscr{F})$. Hence

$$\hat{\mu}(\mathscr{F}) = \hat{\mu}(\pi_*\mathscr{F}) = \mu(\pi_*\mathscr{F}) + \frac{d+1}{2}.$$

A $\pi_*\mathscr{F}$-regular sequence of hyperplanes H_i' in Y induces an $\mathscr{F}$-regular sequence of hyperplane sections H_i on X. Let $Y_v = H_1' \cap \cdots \cap H_{d-v}'$, then $\pi_*\mathscr{F}|_{\mathscr{X}} = \pi_*(\mathscr{F})|_{X_v}$ and hence $h^0(F|_{X_v}) = h^0(\pi_*(\mathscr{F})|_{X_v})$.

- **Lemma 3.3.17.A.** The sheaf $\mathscr{A} := \pi_*\mathscr{O}_Z$ is a torsion free sheaf with

$$\mu_{\min}(\mathscr{A}) \geq -\operatorname{rank}(\mathscr{A}) \geq -\operatorname{rank}(\pi_*\mathscr{F})^2 = -r(\mathscr{F})^2.$$

Proof of Lemma 3.3.17.A. As $\pi_*\mathscr{F}$ is an $\mathscr{A}$-module,we have algebra homomorphism $\mathscr{A} \to \mathscr{E}nd(\pi_*\mathscr{F})$ which is injective since Z is the support of $\mathscr{F}$. Hence $\mathscr{A}$ is torsion free with rank less or equal to $\operatorname{rank}(\pi_*\mathscr{F})^2 = r(\mathscr{F})^2$.

Actually we have $\mathbb{P}^N \backslash L \cong \underline{\operatorname{Spec}}_Y \operatorname{Sym}\mathscr{O}_Y(-1)^{\oplus(N-d)}$, let $\mathscr{W} := \mathscr{O}_Y(-1)^{\oplus(N-d)}$. Then this induce a surjection $\phi : \operatorname{Sym}\mathscr{W} \to \mathscr{A}$. Consider the filtration $F_p\mathscr{A} := \phi\left(\bigoplus_{i=0}^p \operatorname{Sym}^i\mathscr{W}\right)$. As $\mathscr{A}$ is coherent the filtration is bounded. Moreover, since the multiplication $\mathscr{W} \otimes \operatorname{gr}_p^F \mathscr{A} \to \operatorname{gr}_{p+1}^F \mathscr{A}$ is surjective, hence if $\operatorname{gr}_p^F \mathscr{A}$ is torsion, the same is true for all $\operatorname{gr}_{p+i}^F \mathscr{A}$, $i \geq 0$. In particular, if $\operatorname{gr}_p^F \mathscr{A}$ is not torsion then $p \leq \operatorname{rank}(\mathscr{A})$. Hence the cokernel of $\phi : \bigoplus_{i=0}^{\operatorname{rank}(\mathscr{A})} \operatorname{Sym}^i\mathscr{W} \to \mathscr{A}$ is torsion. Hence $\mu_{\min}(\mathscr{A}) \geq \mu_{\min}(\operatorname{Sym}^{\operatorname{rank}(\mathscr{A})}\mathscr{W}) = -\operatorname{rank}(\mathscr{A})$. □

- **Lemma 3.3.17.B.** We have

$$\mu_{\max}(\pi_*\mathscr{F}) \leq \hat{\mu}_{\max}(\mathscr{F}) + r(\mathscr{F})^2 - \frac{d+1}{2}.$$

Proof of Lemma 3.3.17.B. Let $\mathscr{G}$ be the maximal destabilizing submodule of $\pi_*\mathscr{F}$, and let $\mathscr{G}'$ be the image of the multiplication map $\mathscr{A} \otimes \mathscr{G} \to \pi_*\mathscr{F}$. Then $\mathscr{G}' = \pi_*\mathscr{G}''$ for some $\mathscr{O}_Z$-submodule $\mathscr{G}'' \subset \mathscr{F}$. It follows that

$$\begin{aligned}
\hat{\mu}_{\max}(\mathscr{F}) &\geq \hat{\mu}(\mathscr{G}'') = \hat{\mu}(\mathscr{G}') = \mu(\mathscr{G}') + \hat{\mu}(\mathscr{O}_Y) \\
&\geq \mu_{\min}(\mathscr{A} \otimes \mathscr{G}) + \hat{\mu}(\mathscr{O}_Y) \\
&= \mu(\mathscr{G}) + \mu_{\min}(\mathscr{A}) + \hat{\mu}(\mathscr{O}_Y) \\
&\geq \mu_{\max}(\pi_*\mathscr{F}) - r(\mathscr{F})^2 + \frac{d+1}{2}
\end{aligned}$$

using Theorem 3.3.13 and Lemma 3.3.17.A. □

Now by Lemma 3.3.17.B, we have

$$\mu_{\max}(\pi_*\mathscr{F}) + v + \frac{\operatorname{rank}(\pi_*\mathscr{F}) - 1}{2} \leq \hat{\mu}_{\max}(\mathscr{F}) + r(\mathscr{F})^2 + \frac{r(\mathscr{F}) - 1}{2} + \frac{d-1}{2}.$$

By the claim for $\pi_*\mathscr{F}$ and this inequality, we get the result. □

3.3.5 Boundedness III: The Main Results

Theorem 3.3.18. *Let $f : X \to S$ be a projective morphism of schemes of finite type over k and let $\mathscr{O}_X(1)$ be an f-ample line bundle. Let P be a polynomial of degree d, and let μ_0 be a rational number. Then the family of purely d-dimensional sheaves on the fibres of f with Hilbert polynomial P and maximal slope $\hat{\mu}_{\max} \leq \mu_0$ is bounded. In particular, the family of H-ss sheaves with Hilbert polynomial P is bounded.*

Proof. Covering S by finitely many open subschemes and replacing H by an appropriate high tensor power, if necessary, we may assume that f factors through an embedding $X \hookrightarrow S \times \mathbb{P}^N$. Thus we may reduce to the case $S = \mathrm{Spec}(k), X = \mathbb{P}^N$. By Theorem 3.3.17, we can find for each purely d-dimensional coherent sheaf $\mathscr{F}$ a regular sequence of hyperplanes $H_1, ..., H_d$ such that $h^0(F|_{H_1 \cap \cdots \cap H_i}) \leq C$ for all $i = 0, ..., d$, where C is a constant depending only on the dimension and degree of X and the multiplicity and maximal slope of $\mathscr{F}$. Since these are given or bounded by P and μ_0, respectively, the bound is uniform for the family in question. Hence the result follows from this and the Kleiman Criterion 3.3.4. □

Corollary 3.3.19. *The open moduli substack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$ of H-ss sheaves is an algebraic stack of finite type.*

Remark 3.3.20. *Actually from Theorem 3.3.18, there is an integer m such that for any H-ss sheaf $\mathscr{F}$ with Hilbert polynomial P, $\mathscr{F}(m)$ is globally generated and $h^0(\mathscr{F}(m)) = P(m)$. Hence by the proofs of Proposition 3.3.7 and above, there is an open subscheme U of the Quot scheme $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-m)^{P(m)})$ parameterizing H-ss sheaves and inducing an isomorphism on H^0 which is invariant under the natural action of $\mathrm{GL}_{P(m)}$ on $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-m)^{P(m)})$. Hence*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \cong [U/\mathrm{GL}_{P(m)}].$$

3.3.6 Semistable Reduction: Langton's Theorem

3.3.7 More Properties of Moduli Stacks of Semistable Sheaves

3.4 Good Moduli Space of Semistable Sheaves

3.4.1 Existence of Good Moduli Space of Semistable Sheaves

Proposition 3.4.1. *The algebraic stack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ is Θ-complete and S-complete.*

Proof. By Theorem 3.1.5, □

Theorem 3.4.2. *The stack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ has a good moduli space*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \to \mathrm{Coh}_P^{\mathrm{H-ss}}(X).$$

Proof. □

3.4.2 Points In the $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$

3.4.3 Projectivity

3.4.4 Need to add

Chapter 4

Good Moduli Spaces of Complexes and Bridgeland Stability

4.1 Complexes and Stability Condition

4.2 Moduli Stack of Complexes and Bridgeland Stability

4.3 Need to add

Chapter 5

K-Stability and K-Moduli, a Glimpse

Part II

Geometry on These Moduli Spaces

Chapter 6

Donaldson-Thomas invariants

6.1 Virtual Fundamental Classes

6.2 Basic Facts of Donaldson-Thomas invariants

6.3 Need to add

Chapter 7

Need to add

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