Algebraic Cycles and Hodge Theory

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Contents

1 Introduction

The reader of course need to be familiar with the book [3] including the basic theory and schemes, cohomology, curves and surfaces. We will also use the intersection theory frequently such as the main contents of [2] or [1] and the reader should familiar with these. Finally we will omit the most basic theory of complex Hodge theory, such as the first seven chapters in [5].

We will focus on the final part of the book [6]. There are three topics of Hodge theory in this book but we just discuss the final part of them. We will also use the Serre's GAGA-principle without explanation.

2 Some Background of Mixed Hodge Theory

2.1 Basic Definition and Properties

Definition 2.1. *A rational (real) mixed Hodge structure of weight n is given by a $\mathbb{Q}$-vector space ($\mathbb{R}$-vector space) H equipped with an increasing filtration W_iH called the weight filtration , and a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$, called the Hodge filtration $F^kH_{\mathbb{C}}$. Such that the induced Hodge filtration on each $\mathrm{Gr}_i^W H$ make $\mathrm{Gr}_i^W H$ to be a Hodge structure of weight $n+i$.*

These filtrations are required to be bounded. Recall that a morphism $\alpha : (U, F) \to (V, G)$ is said to be strict if $\mathrm{Im}\alpha \cap G^pV = \alpha(F^pU)$. It's easy to show that the morphism of rational pure Hodge structures are strict for Hodge filtration (even in type (r, r), see [5] Lemma 7.23).

This is an analogue theory of Hodge decomposition of pure Hodge structures:

Lemma 2.2. *Let (H, W, F) be a mixed Hodge structure. Then there exists a decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

with $H^{p,q} \subset F^pH_{\mathbb{C}} \cap W_{p+q-n}H_{\mathbb{C}}$, such that via the projection $W_{p+q-n}H_{\mathbb{C}} \to \mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}$, the space $H^{p,q}$ can be identified with

$$H^{p,q}(\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}) := F^p\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}} \cap \overline{F^q\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}}.$$

More generally, we have

$$W_iH_{\mathbb{C}} = \bigoplus_{p+q \le n+i} H^{p,q}, F^iH_{\mathbb{C}} = \bigoplus_{p \ge i} H^{p,q}.$$

This decomposition is preserved by the morphisms of mixed Hodge structures.

Proof. This is pure linear algebra, we omit it and refer [6] Lemma 4.21. □

Remark 2.3. *Unlike the pure case, the decomposition above may satisfies* $H^{p,q} \neq \overline{H^{p,q}}$, *although this does become true after projection to* $\mathrm{Gr}^W_{p+q} H_{\mathbb{C}}$.

Theorem 2.4 (P. Deligne, 1971)**.** *The morphisms*

$$\alpha : (H, W, F) \to (H', W', F')$$

of (rational or real) mixed Hodge structures are strict for the filtrations W *and* F.

Proof. We will only show the statement for W since the statement for H is similar.

Pick $l' \in \alpha(H_{\mathbb{C}}) \cap W_i H'$ and we write $l' = \alpha(l)$ with $l = \sum_{p,q} l^{p,q}$ by Lemma 2.2. As $l' \in W_i' H_{\mathbb{C}}'$, then $\alpha(l^{p,q}) = 0$ for $p + q > n + i$ by Lemma 2.2 again. Hence $l' \in \alpha(W_i H_{\mathbb{C}})$ and well done. □

2.2 A Classical Example of Mixed Hodge Structure

We consider a smooth complex variety U with a compactification X such that $X \backslash U = D$, a effective normal crossing divisor.

Definition 2.5. *Define a subsheaf* $\Omega^k_X(\log D) \subset \Omega^k_X(*D)$ *such that* $\alpha \in \Gamma(V, \Omega^k_X(\log D))$ *if* α *is a meromorphic differential form on* V, *holomorphic on* $V \backslash D$ *and admits a pole of order at most* 1 *along (each component of)* D, *and the same holds for* $d\alpha$. *Hence* $d = \partial$ *in it and we call the complex* $(\Omega^*_X(\log D), \partial)$ *the* logarithmic de Rham complex .

Lemma 2.6. *Let* $z_1, ..., z_n$ *be local coordinates on an open set* $V \subset X$, *in which* $D \cap V$ *is defined by the equation* $z_1 \cdots z_r = 0$. *Then* $\Omega^k_X(\log D)|_V$ *is a sheaf of free* $\mathscr{O}|_U$*-modules with basis*

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$$

where $i_s \leq r$, $j_s > r$ *and* $l + m = k$. *In particular,* $\Omega^k_X(\log D)$ *is locally free.*

Proof. Almost trivial, see [5] Lemma 8.16. □

Proposition 2.7. *Let inclusion* $j : U \hookrightarrow X$, *then we have a canonical inclusion* $\Omega^k_X(\log D) \subset j_* \Omega^k_U \subset j_* \mathscr{A}^k_U$ *which give us a morphism of complex*

$$\Omega^*_X(\log D) \to j_* \mathscr{A}^*_U.$$

Then this is a quasi-isomorphism. In particular we have

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_X(\log D)).$$

Proof. This is not hard to see and we refer [5] Proposition 8.18. From this we have $\mathbb{H}^k(X, \Omega_X^*(\log D)) \cong \mathbb{H}^k(X, j_*\mathscr{A}_U^*)$. As $\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_U^\infty$-modules which is a resolution of $\mathbb{C}_U$, then $j_*\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_X^\infty$-modules, so it is acyclic and

$$\mathbb{H}^k(X, j_*\mathscr{A}_U^*) \cong H^k\Gamma(X, j_*\mathscr{A}_U^*) = H^k\Gamma(U, \mathscr{A}_U^*) = H^k(U, \mathbb{C}).$$

Hence we get the result. □

For now we will give $H^k(U, \mathbb{Q})$ (or $H^k(U, \mathbb{R})$) a mixed Hodge structure. First we will give two filtrations over $\Omega_X^*(\log D)$.

We define the Hodge filtration over $\Omega_X^*(\log D)$ to be

$$F^p\Omega_X^*(\log D) = \Omega_X^{\geq p}(\log D).$$

For weight filtration, we define $W_l\Omega_X^*(\log D)$ to be

$$W_l\Omega_X^*(\log D) = \begin{cases} \bigwedge^l \Omega_X^1(\log D) \wedge \Omega_X^{*-l}, & 0 \leq l \leq r, \\ 0, & l > r. \end{cases}$$

(We often let $W^k := W_{-k}$)

Now for simplicity, we let the divisor D is simply normal crossing with $D = \bigcup_i D_i$ where each $D_i \subset X$ is a smooth hypersurface, and the intersection of any l hypersurfaces $D_{i_1}, ..., D_{i_l}$ is transverse. We equip I with a total order. We let

$$D^{(k)} := \coprod_{K \subset I, |K| = k} D_K = \coprod_{K \subset I, |K| = k} \bigcap_{i \in K} D_i$$

with inclusions $j_k : D^{(k)} \to X$ and $j_M : D_M \to X$.

Proposition 2.8. *There exists a natural isomorphism*

$$W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

Proof. This morphism defined by Poincaré residue map . Give a local coordinates in $V \subset X$ we define $\mathrm{Res}^V : \Gamma(V, W_k\Omega_X^*(\log D)) \to \Gamma(V, j_{k,*}\Omega_{D^{(k)}}^{*-k})$ as

$$\begin{aligned} \alpha &= \sum_{K \subset \{1,...,r\} \subset I, |K| \leq k} \alpha_{K,L} dz_L \wedge \frac{dz_K}{z_K} \\ &\mapsto (\mathrm{Res}^V \alpha)_M = \left((2\pi\sqrt{-1})^k \sum_L \alpha_{M,L} dz_L|_{D_M \cap V} \right)_M. \end{aligned}$$

Note that this annihilates the sections of $W_{k-1}\Omega_X^*(\log D)$ and change coordinates only change the elements in $W_{k-1}\Omega_X^*(\log D)$, Hence we get a well-defined residue map:

$$\alpha : W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

This is an isomorphism is easy to see. We refer [5] Proposition 8.32. □

Now these two filtrations induce two filtrations over $R\Gamma(X, \Omega_X^*(\log D))$, and hence over $H^k(U, \mathbb{C})$ by Proposition 2.7. So the arguments in [5] is far from complete and we need some derived-version filtration of these, such as mixed Hodge complex. We omitted this and we refer section 3.3 in [4].

Theorem 2.9 (P. Deligne, 1971)**.** *The discussion above equip $H^k(U, \mathbb{C})$ a mixed Hodge structure which is independent with X, D.*

Proof. This follows from some analysis of the weight spectral sequence (induced by $W^* = W_{-*}$), here we give a sketch.

By the general theory of spectral sequence, we have

$$_W E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)).$$

By Proposition 2.8 we have $\mathrm{Gr}_W^p \Omega_X^*(\log D) \cong j_{-p,*}\Omega_{D^{(-p)}}^{*+p}$, hence

$$\begin{aligned}\mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)) &= \mathbb{H}^{2p+q}(X, j_{-p,*}\Omega_{D^{(-p)}}^*) \\ &= \mathbb{H}^{2p+q}(D^{(-p)}, \Omega_{D^{(-p)}}^*) = H^{2p+q}(D^{(-p)}, \mathbb{C}).\end{aligned}$$

We can also get that the differential

$$d_1 : \quad \begin{array}{ccc} H^{2p+q}(D^{(-p)}, \mathbb{C}) & \longrightarrow & H^{2p+q+2}(D^{(-p-1)}, \mathbb{C}) \\ \downarrow\cong & & \downarrow\cong \\ \bigoplus_{|K|=-p} H^{2p+q}(D_K, \mathbb{C}) & \longrightarrow & \bigoplus_{|L|=-p-1} H^{2p+q+2}(D_L, \mathbb{C}) \end{array}$$

has component $d_{1,K}^L$ equal to zero for $L \nsubseteq K$, and equal to $(-1)^{q+s} j_{K,*}^L$ when $K = \{i_1 < \cdots < i_p\}$ and $L = K \backslash \{i_s\}$ where $j_K^L : D_K \to D_L$ (see Proposition 8.34 in [5]). Hence we can deduce any pages of weight spectral sequence! By some analysis we can get the result which omitted, we refer Theorem 3.4.7 and section 3.4.1.5 in [4]. □

3 Cycle Classes and Abel–Jacobi Map

3.1 Cycle Classes and Cycle Map

The case of general complex manifolds with closed analytic subsets

Let X be a $n + r$-dimensional complex manifold with a codimension r closed analytic subset Z, we will associated Z to be a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$.

Lemma 3.1. *If $Y \subset X$ be a closed complex submanifold of codimension k, then the natural map $H^l(X, \mathbb{Z}) \to H^l(X \backslash Y, \mathbb{Z})$ is an isomorphism for $l \leq 2k - 2$.*

Proof. Trivial, just need to look at the long exact sequence induced by the good pair $(X, X\backslash Y)$ and using Thom's isomorphism and the excision theorem. $\square$

Come back to our case, as in algebraic geometry, we can have a filtration

$$\emptyset = Z_{n+1} \subset \cdots \subset Z_0 = Z$$

where $\dim Z_i = n - i$ and $Z_k\backslash Z_{k-1}$ is a closed complex submanifold of dimension $n-k$ in $X\backslash Z_{k-1}$ (see [5] Theorem 11.11 for the proof).

We apply this Lemma to each $X\backslash Z_k \subset X\backslash Z_{k+1}$, we have

$$H^{2r}(X, \mathbb{Z}) \cong H^{2r}(X\backslash Z_1, \mathbb{Z}).$$

Here $Z\backslash Z_1$ is smooth. So we just need to consider the case when Z is a smooth complex submanifold in X!

If Z is a smooth complex submanifold of codimension r in X, then by Thom's isomorphism and the excision theorem, we have the following diagram

$$\begin{array}{ccc} H^{2r}(X, X\backslash Z; \mathbb{Z}) & \xrightarrow{j_Z} & H^{2r}(X, \mathbb{Z}) \\ \big\downarrow = & & \\ H^{2r}(X, X\backslash Z; \mathbb{Z}) & \xrightarrow{\cong, T} & H^0(Z, \mathbb{Z}) \end{array}$$

Then we define $[Z] = j_Z(T^{-1}(1)) \in H^{2r}(X, \mathbb{Z})$.

Remark 3.2. *We an also using the most natural way: if $Z = \sum_i n_i Z_i$, we can define $[Z] = \sum_i n_i [Z_i]$ where $[Z_i] = \mathrm{PD}(j_{i,*}([Z_i']_{\text{fund}}))$ and $j_i : Z_i' \to X$ is a resolution of singularity of Z_i, $[Z_i']_{\text{fund}}$ is the fundamental homology class and PD denotes Poincaré duality!*

Here we give some discription using the de Rham cohomology without proof:

Proposition 3.3. *Let $U \subset X$ be a neighbourhood of Z isomorphic to a neighbourhood V of the section in the normal bundle $N_{Z/X}$. Let ω be a closed form of degree k with support in V, satisfying*

$$\int_{\pi^{-1}(z)} \omega = 1$$

where $\pi : V \to Z$ is the projection. Then the form ω is a representative in de Rham cohomology of the class $[Z]$.

Proof. See Lemma 11.14 in [5]. $\square$

The case of compact Kähler manifolds

Let X be a $n+r$-dimensional compact Kähler manifold with a codimension r closed analytic subset Z. We have associated Z to be a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$. Now using Hodge decomposition, we will discuss the type of $[Z]$ in $H^{2r}(X, \mathbb{C}) = \bigoplus_{p+q=2r} H^{p,q}(X)$.

Theorem 3.4. *The image of* $[Z]$ *in* $H^{2r}(X, \mathbb{C})$ *lies in* $H^{r,r}(X)$.

Proof. Here we need to use the following two results (for the proof, see [5] Lemma 7.30 and Theorem 11.21, using the de Rham discription we discussed above this is easy to prove):

(i) If Y be a compact Kähler manifold of dimension m, then

$$H^{p,q}(Y) = \left(\bigoplus_{k+l=2m-p-q,(k,l)\neq(m-p,m-q)} H^{k,l}(Y) \right)^{\perp}$$

where the orthogonality is relative to the Poincaré duality on Y.

(ii) (Lelong, 1957) The current $\omega \mapsto \int_{Z_{\text{smooth}}} \omega$ maps to zero on the exact forms. Hence it is an element in $H^{2n}(X, \mathbb{C})^*$. Then this element is equal to the image of $[Z]$under the morphism

$$H^{2r}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C})^*.$$

By (i), we just need to show that $\int_X [Z] \wedge \alpha = 0$ for any α of type (p,q), $p+q = 2n$, $(p,q) \neq (n,n)$. Then this is trivial by (ii). $\square$

The case of complex smooth (quasi-)projective varieties

Let X be a complex smooth quasi-projective variety of dimension n and $Z \in \mathcal{Z}_k(X)$, then we give $[Z] \in H^{2n-2k}(X, \mathbb{Z})$ as above.

Proposition 3.5. *If* $Z \sim_{\text{rat}} 0$, *then* $[Z] = 0 \in H^{2n-2k}(X, \mathbb{Z})$, *hence we give the class map*

$$\text{cl} : \text{CH}_l(X) \to H^{2n-2l}(X, \mathbb{Z}), Z \mapsto [Z].$$

We denote its kernal $\text{CH}_l(X)_{\text{hom}}$.

Proof. WLOG we can assume X is projective. Let $W \subset X$ is of dimension $k+1$ and $\phi \in K(W)^*$, we just need to show $[\tau_* \text{div}(\phi)] = 0$ where $\tau : W' \to W \to X$ be a resolution of singularity of W.

We can easy to see $[\tau_* \text{div}(\phi)] = \tau_*[\text{div}(\phi)]$ where $\tau_* : H^2(W', \mathbb{Z}) \to H^{2n-2k}(X, \mathbb{Z})$ defined by Poincaré duality by Remark 3.2. Hence we just need to show $[\text{div}(\phi)] = 0 \in H^2(W', \mathbb{Z})$. This follows from Lelong's fundamental theorem that $[D] = c_1(\mathcal{O}(D)) \in H^2(W', \mathbb{Z})$. $\square$

Proposition 3.6. *Let $f : X \to Y$ be morphism of smooth quasi-projective varieties.*

(i) If $Z \in \mathrm{CH}^l(X), Z' \in \mathrm{CH}^k(X)$, then

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(Z) \cup \mathrm{cl}(Z') \in H^{2k+2l}(X, \mathbb{Z});$$

(ii) if $Z \in \mathrm{CH}^k(Y)$, then $f^* \mathrm{cl}(Z) = \mathrm{cl}(f^* Z) \in H^{2k}(X, \mathbb{Z})$;

(iii) if f proper and $Z \in \mathrm{CH}^k(X)$, then $f_* \mathrm{cl}(Z) = \mathrm{cl}(f_* Z) \in H^{2k-2\dim X+2\dim Y}(Y, \mathbb{Z})$.

Proof. We have showed (iii) in the proof of Proposition 3.5. We first show the case of closed immersion of (ii). Indeed, by moving lemma we may assume Z and X intersect generically transverse. We may assume Z is irreducible. By Lemma 3.1 we may assume Z and X intersect transversely. Then compare the normal bundle and well done.

Then we prove (i). We know by definition that

$$[Z \times Z'] = p_1^*[Z] \cup p_2^*[Z'].$$

Let diagonal $\delta : \Delta_X = X \subset X \times X$, then by the case of closed immersion of (ii) we have

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(\delta^*(Z \times Z')) = \delta^* \mathrm{cl}(Z \times Z') = \delta^*(p_1^*[Z] \cup p_2^*[Z']) = \mathrm{cl}(Z) \cup \mathrm{cl}(Z'),$$

well done.

For the general case of (ii), this follows from (i) and decomposition

$$f : X \xrightarrow{\text{graph}} X \times Y \to Y,$$

well done. □

3.2 Hodge Classes and Hodge Conjecture

Definition 3.7. *If $(V_{\mathbb{Z}}, F^* V_{\mathbb{C}})$ is a pure Hodge structure of weight $2p$, then we denote the set of* Hodge classes

$$\mathrm{Hdg}(V) := V_{\mathbb{Z}} \cap V^{p,p}.$$

Now consider a compact Kähler manifold X with its standard Hodge structure, then we have:

Theorem 3.8. *The cohomology class of analytic subsets of X and the Chern classes of the holomorphic vector bundles over X are all Hodge classes.*

Proof. The first way is Theorem 3.4. For the second one, we know that if X is algebraic, this is trivial since the Chern classes in the cohomology group can come from the Chern classes in the Chow group! Indeed, we can tensoring some higher times ample bundle and get a map from X to a Grassmannian such that the Chern class of this bundle comes from the Schubert classes in the Grassmannian (see [1] Proposition 10.2). In general, we have a classical result that

$$H^*(P(\mathscr{E})) \cong \frac{H^*(X)[\zeta]}{\zeta^r + c_1(\mathscr{E})\zeta^{r-1} + \cdots + c_r(\mathscr{E})}$$

and $\zeta = c_1(\mathscr{O}_{P\mathscr{E}}(1))$ is of type $(1,1)$. Hence well done. □

Corollary 3.9. *If X is algebraic, these two subgroups of $\mathrm{Hdg}(X)$ coincide.*

Proof. Omitted. □

Converse is true in codimension 1 which is classical Lefschetz $(1,1)$-Theorem:

Theorem 3.10. *In this case, the group $\mathrm{Hdg}^2(X,\mathbb{Z})$ is equal to the image of $c_1 : \mathrm{Pic}X \to H^2(X,\mathbb{Z})$.*

But this is false in general. Actually, J. Kollár in 1992 gave a counterexample. His counterexample is given by hypersurfaces X of degree d in the projective space $\mathbb{P}^4$. Such a hypersurface satisfies $H^2(X,\mathbb{Z}) = \mathbb{Z}$, and the class of a plane curve $\mathbb{P}^2 \cap X$ is equal to d times the generator of $H^2(X,\mathbb{Z})$. He shows that this generator is not, however, in general, the class of an algebraic cycle.

In $\mathbb{Q}$-coefficient, we have the following famous conjectures:

Conjecture 1 (Hodge Conjecture)**.** *Let X be a projective manifold, and $\alpha \in \mathrm{Hdg}^{2k}(X)$. Then a multiple $N\alpha$ with $N \neq 0$ is the class of an algebraic cycle.*

Conjecture 2 (Generalized Hodge Conjecture, Grothendieck)**.** *Let X be a smooth algebraic variety, and $L \subset H^{2k+l}(X,\mathbb{Q})$ a rational sub-Hodge structure contained in $F^kH^{2k+l}(X)$. Then there exist (not necessarily smooth) algebraic subvarieties $j_i : Y_i \to X$ of codimension k such that L is contained in $\sum_i j_{i,*}H_{2n-l}(Y_i,\mathbb{Q})$ where $\dim X = n+k$.*

Finally, we have an easy relation between Hodge class in product space and morphism of Hodge structures which is useful:

Lemma 3.11. *Let $k+l$ is even, then the class*

$$\alpha \in H^k(X,\mathbb{Z}) \otimes H^l(Y,\mathbb{Z}) \subset H^{k+l}(X \times Y,\mathbb{Z})$$

is a Hodge class if and only if the corresponding map

$$\alpha' \in \mathrm{Hom}_{\mathbb{Z}}(H^{2n-k}(X,\mathbb{Z}), H^l(Y,\mathbb{Z}))$$

via Poincaré duality is a morphism of Hodge structures (of degree $(k+l-n, k+l-n)$).

Proof. This is not hard, we refer [5] Lemma 11.41. □

3.3 The Abel–Jacobi Map

Let X be a compact Kähler manifold.

Definition 3.12. *We know $H^{2k-1}(X,\mathbb{R}) \cong H^{2k-1}(X,\mathbb{C})/F^kH^{2k-1}(X)$ and consider the lattice $H^{2k-1}(X,\mathbb{Z}) \subset H^{2k-1}(X,\mathbb{R})$, we get the k-th intermediate Jacobian $J^{2k-1}(X)$ as a complex torus*

$$J^{2k-1}(X) := H^{2k-1}(X,\mathbb{C})/(F^kH^{2k-1}(X) \oplus H^{2k-1}(X,\mathbb{Z})).$$

Remark 3.13. *Note that in general $J^{2k-1}(X)$ is not an abelian variety! But $J^1(X) = \mathrm{Pic}^0(X)$ is.*

Definition 3.14. *Pick $Z \in \mathcal{Z}^k(X)_{\mathrm{hom}}$, then we can find a differentiable chain $\Gamma \subset X$ of real dimension $2n-2k+1$ such that $\partial\Gamma = Z$. Consider $\int_\Gamma \in A^{2n-2k+1}(X)^*$, When we restrict it into $F^{n-k-1}H^{2n-2k+1}(X)^*$, we find that by the reason of type and Stokes's formula, $\int_\Gamma \in F^{n-k-1}H^{2n-2k+1}(X)^*$ is independent of the choice of the representative of cohomology class. One can also easy to see that if we descend $\int_\Gamma$ in*

$$F^{n-k-1}H^{2n-2k+1}(X)^*/H_{2n-2+1}(X,\mathbb{Z}) \cong J^{2k-1}(X),$$

it is independent of the choice of Γ such that $\partial\Gamma = Z$! Hence we give a morphism called the Abel-Jacobi map

$$\Phi_X^k : Z \in \mathcal{Z}^k(X)_{\mathrm{hom}} \to J^{2k-1}(X), \quad Z \mapsto \int_\Gamma.$$

Theorem 3.15 (Griffiths, 1968)**.** *Let Y be a connected complex manifold, $y_0 \in Y$ a reference point, and $Z \subset Y \times X$ a cycle of codimension k. We will assume that $Z = \sum_i n_i Z_i$ where each Zi is flat over Y, then the map*

$$\phi : Y \to J^{2k-1}(X), \quad y \mapsto \Phi_X^k(Z_y - Z_{y_0})$$

is holomorphic.

Proof. See Theorem 12.4 in [5]. □

Now we consider two special cases when $k = 1$ and $k = \dim X$.

When $k = 1$, by Proposition 12.7 in [5] we can show that $c_1(\mathscr{O}(Z)) = \Phi_X^1(Z) \in J^1(X) = \mathrm{Pic}^0(X)$. Then we get:

Corollary 3.16 (Abel's Theorem)**.** *If D be a divisor homologous to 0 in X, then $\Phi_X^1(D) = 0$ if and only if $\mathscr{O}_X(D)$ is trivial.*

When $k = n = \dim X$, we find that if X is connected, such a cycle is homologous to 0 if and only if it is of degree 0.

Definition 3.17. *We define the* Albanese variety $\mathrm{Alb}(X) := J^{2n-1}(X)$ *and the holomorphic map*

$$\mathrm{alb}_X : X \to \mathrm{Alb}(X), \quad x \mapsto \Phi_X^{2n-1}(x - x_0)$$

is called the Albanese map .

Remark 3.18. *(a) The Albanese map satisfies the following universal property: For every holomorphic map $f : X \to T$ with values in a complex torus and satisfying $f(x_0) = 0$, there exists a unique morphism of complex tori such that:*

$$\begin{tikzcd} X \arrow[r, "\mathrm{alb}"] \arrow[d, "f"'] & \mathrm{Alb}(X) \arrow[dl, dashed, "\exists! g"] \\ T & \end{tikzcd}$$

Hence this give us a purely algebraic construction. For details see Theorem 12.15 in [5];

(b) actually for sufficiently large k, the morphism

$$\mathrm{alb}_X^k : X^k \to \mathrm{Alb}(X), \quad (x_1, ..., x_k) \mapsto \sum_i \mathrm{alb}_X(x_i)$$

is surjective. Hence we can use this to show that $\mathrm{Alb}(X)$ is an abelian variety. For details see Lemma 12.11 and Corollary 12.12 in [5].

Theorem 3.19. *Let X, Y are two compact Kähler manifolds with Y connected and $Z \subset Y \times X$ be a cycle of codimension k which is flat over Y. Hence we have the holomorphic map $\phi : Y \to J^{2k-1}(X)$ given by $y \mapsto \Phi_X^k(Z_y - Z_{y_0})$. Hence by the universal property, we get a morphsim of complex tori $\psi : \mathrm{Alb}(Y) \to J^{2k-1}(X)$ factor through ϕ. Then ψ induced by $[Z]^{1,2k-1} : H^{2\dim Y - 1}(Y, \mathbb{Z}) \to H^{2k-1}(X, \mathbb{Z})$, the $(1, 2k-1)$-Künneth component. In particular, the image of ψ is a complex subtorus of $J^{2k-1}(X)$ having the property that its tangent space at 0 is contained in $H^{k-1,k}(X)$.*

Proof. The main result is not hard to see (Theorem 12.17 in [5]). Let $m = \dim Y$, then the morphism of Hodge structures $[Z]$ is of bidegree $(k - m, k - m)$, and as the Hodge structure on $H^{2m-1}(Y)$ is of type $(m, m-1) + (m-1, m)$, the image of $[Z]$ is thus contained in $H^{k,k-1}(X) \oplus H^{k-1,k}(X)$. Well done. $\square$

When X be a smooth (quasi-)projective variety, we can descend this map into Chow-group level:

Lemma 3.20. *If $Z \in \mathcal{Z}_k(X)$ with $Z \sim_{\mathrm{rat}} 0$, then $\Phi_X^{n-k}(Z) = 0$. Hence we get the map:*

$$\Phi_X^{n-k} : \mathrm{CH}_k(X)_{\mathrm{hom}} \to J^{2n-2k-1}(X).$$

Proof. Let Z be a subvariety. As $Z \sim_{\mathrm{rat}} 0$, we have a subvariety $W \subset Z \times \mathbb{P}^1$ dominates $\mathbb{P}^1$ such that $W_0 = Z$ and $W_\infty = 0$. By Theorem 3.15 we have a holomorphic map from $\mathbb{P}^1$ to a torus:

$$f : \mathbb{P}^1 \to J^{2n-2k-1}(X), \quad t \mapsto \Phi_X^{n-k}(W_t - W_0).$$

Since $\mathbb{P}^1$ has no non-zero holomorphic forms of degree 1, so that the pullbacks by f of the holomorphic 1-forms on the torus must vanish, which is equivalent to $df = 0$. Thus, we have

$$\Phi_X^{n-k}(Z) = f(\infty) = f(0) = \Phi_X^{n-k}(0) = 0,$$

well done. $\square$

Moreover, there is also a compactible relation with the intersection product.

Proposition 3.21. *Now for any $Z \in \mathrm{CH}^k(X)$, we have the map $\mathrm{cl}(Z)\cup : H^{2l-1}(X, \mathbb{Z}) \to H^{2l+2k-1}(X, \mathbb{Z})$ which can descend to $\mathrm{cl}(Z)\cup : J^{2l-1}(X) \to J^{2l+2k-1}(X)$. If let $Z' \in \mathrm{CH}^l(X)_{\mathrm{hom}}$, then by Proposition 3.6(i) we have $Z \cdot Z' \in \mathrm{CH}^{k+l}(X)_{\mathrm{hom}}$. Then we have*

$$\Phi_X^{k+l}(Z \cdot Z') = \mathrm{cl}(Z) \cup \Phi_X^l(Z').$$

Proof. This is not hard but we omit the proof and refer to [6] Proposition 9.23. Note that this is the special case of Theorem 3.33. $\square$

Finally in this case with $\dim X = n$, we introduce two more maps induced by the Abel-Jacobi map.

We know that if $Z \sim_{\mathrm{alg}} 0$, then $Z \sim_{\mathrm{hom}} 0$ for any Weil cohomology theory (see [2]). Then we define $\mathrm{Griff}^k(X) := \mathcal{Z}^k(X)_{\mathrm{hom}}/\mathcal{Z}^k(X)_{\mathrm{alg}}$.

Remark 3.22. *When $k = 1, n$ and X connected, we have $\mathrm{Griff}^k(X) = 0$. Here we need a classical result:*

- *Let X by any variety and $x_i \in X$. Then there is an irreducible curve C on X containing x_i. (Sketch of the proof, which is so classical: Given any two points on a projective variety, blow them up and re embed the blownup variety in $\mathbb{P}^N$. Then by Bertini, any general linear section of the right codimension will meet the variety in an irreducible curve which also meets both exceptional divisors. Then blowing back down gives an irreducible curve connecting the original points.)*

For $k = 1$, we know that any two homologous trivial divisor lies in the connected variety $\mathrm{Pic}^0(X)$, then by this results with normalizing that curve gives a map from just one smooth connected curve that connects your two points. Hence $\mathrm{Griff}^1(X) = 0$.

For $k = n$, as any 0-cycle lies over an irreducible curve in X by this result, normalizing that curve and we get $\mathrm{Griff}^n(X) = 0$.

But when $2 \leq k < n$ in general $\mathrm{Griff}^k(X) \neq 0$. So we define $\mathrm{Griff}^k(X)$ to find their difference and to deduce some new Abel-Jacobi maps as follows.

We define $J^{2k-1}(X)_{\mathrm{alg}} \subset J^{2k-1}(X)$ be the largest subtorus with tangent space contained in $H^{k-1,k}(X)$. Hence by Theorem 3.19 we know that $\Phi_X^k(\mathcal{Z}^k(X)_{\mathrm{alg}}) \subset J^{2k-1}(X)_{\mathrm{alg}}$. We can also define $J^{2k-1}(X)_{\mathrm{trans}} := J^{2k-1}(X)/J^{2k-1}(X)_{\mathrm{alg}}$ as the transcendental part of the intermediate Jacobian.

Remark 3.23. *This is because in general $J^{2k-1}(X)$ is not algebraic but $J^{2k-1}(X)_{\mathrm{alg}}$ is always algebraic, and hence $J^{2k-1}(X)_{\mathrm{alg}}$ is an abelian variety! Indeed, by the classical Hodge–Riemann bilinear relation: (see [5] Theorem 6.32)*

- *Consider a Hermitian form H_k on $H^k(X,\mathbb{C})$ defined by*

$$H_k(\alpha,\beta) = (\sqrt{-1})^k \int_X \omega^{n-k} \wedge \alpha \wedge \bar{\beta},$$

  then the form $(-1)^{\frac{k(k-1)}{2}}(\sqrt{-1})^{p-q-k} H_k$ is positive definite on the complex subspace $H^{p,q}(X) \cap H^k(X,\mathbb{C})_{\mathrm{prim}}$.

we can get the result directly.

Hence we can define the continuous (algebraic) part of the Abel-Jacobi map $\Phi^k_{X,\mathrm{alg}} : \mathcal{Z}^k(X)_{\mathrm{alg}} \to J^{2k-1}(X)_{\mathrm{alg}}$ and the transcendental part of the Abel-Jacobi map:

$$\begin{tikzcd}
\mathcal{Z}^k(X)_{\mathrm{hom}} \arrow[r, "\Phi_X^k"] \arrow[d, two heads] & J^{2k-1}(X) \arrow[d, two heads] \\
\mathrm{Griff}^k(X) \arrow[r, "\Phi^k_{X,\mathrm{trans}}"] & J^{2k-1}(X)_{\mathrm{trans}}
\end{tikzcd}$$

Conjecture 3. *The map $\Phi^k_{X,\mathrm{alg}} : \mathcal{Z}^k(X)_{\mathrm{alg}} \to J^{2k-1}(X)_{\mathrm{alg}}$ is surjective.*

Proof using Hodge conjecture. By Remark 3.23, we know that $A := J^{2k-1}(X)_{\mathrm{alg}}$ is an abelian variety. Let $N = \dim A$ and we get an isomorphism of Hodge structures

$$\alpha : H^{2N-1}(A,\mathbb{Z}) \cong H^{2k-1}(X,\mathbb{Z})_{\mathrm{alg}}.$$

By Lemma 3.11, this α corresponds to a Hodge class $[\alpha] \in \mathrm{Hdg}^{2k}(A \times X)$. Then by Hodge conjecture we can find a cycle Z of codimension k with $\mathrm{cl}(Z) = N[\alpha]$. Then by Theorem 3.19 we know that the map $A = \mathrm{Alb}(A) \to J^{2k}(X)$ induced by Z is equal to N times of $A \cong J^{2k-1}(X)_{\mathrm{alg}}$, hence surjective. $\square$

Remark 3.24. *This is right when X be a 3-dimensional projective manifold covered by rational curves. See Exercise 12.2 in [5].*

3.4 Deligne Cohomology and Deligne Classes

Definition 3.25. *Let X be a complex manifold and $p \geq 1$, we define the* Deligne complex *$\mathbb{Z}_D(p)$ is*

$$0 \to \mathbb{Z} \xrightarrow{(2\pi\sqrt{-1})^p} \mathscr{O}_X \xrightarrow{d} \Omega_X \to \cdots \to \Omega_X^{p-1} \to 0.$$

We define the Deligne cohomology *$H_D^k(X, \mathbb{Z}(p)) := \mathbb{H}^k(X, \mathbb{Z}(p))$.*

Remark 3.26. *We have $\mathbb{Z}_D(1) \simeq_{\text{qis}} \mathscr{O}_X^*[-1]$ and $H_D^2(X, \mathbb{Z}(1)) = H^1(X, \mathscr{O}_X^*)$.*

Proposition 3.27. *If X is a compact Kähler manifold, then there exists a long exact sequence*

$$\begin{aligned}\cdots \to H_D^k(X, \mathbb{Z}(p)) &\to H^k(X, \mathbb{Z}) \\ &\to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \to H^{k+1}D(X, \mathbb{Z}(p)) \to \cdots .\end{aligned}$$

Proof. First we consider

$$0 \to \Omega_X^{\leq p-1}[-1] \to \mathbb{Z}_D(p) \to \mathbb{Z} \to 0$$

which induce a long exact sequence and we see that we just need to show $\mathbb{H}^k(X, \Omega_X^{\leq p-1}) = H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C})$. From the basic fact of Hodge structure (e.g. Proposition 7.5 in [5]) $\mathbb{H}^k(X, \Omega_X^{\geq p}) = F^p H^k(X, \mathbb{C})$ and the exact sequence

$$0 \to \Omega_X^{\geq p} \to \Omega_X^* \to \Omega_X^{\leq p-1} \to 0$$

we can get the result directly. □

Corollary 3.28. *In this case, we have exact sequence*

$$0 \to J^{2p-1}(X) \to H_D^{2p}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^{2p}(X, \mathbb{Z}) \to 0.$$

Proof. This follows directly from the $k = 2p$ in theorem and the fact

$$\mathrm{Hdg}^{2p}(X, \mathbb{Z}) = \ker(H^{2p}(X, \mathbb{Z}) \to H^k(X, \mathbb{C})/F^p H^{2p}(X)).$$

Well done. □

Here we give another method to compute the Deligne cohomology $H_D^{2p}(X, \mathbb{Z}(p))$.

Definition 3.29. *Let X is a differentiable manifold. Let the* group of differential characters *$\Xi_{\text{diff}}^l(X)$ be a subgroup of $\mathrm{Hom}(Z_l^{\text{diff}}, \mathbb{R}/\mathbb{Z})$ consist of $\chi : Z_l^{\text{diff}} \to \mathbb{R}/\mathbb{Z}$ such that there exists a real differential form (obviously uniquely determined by χ) $\omega \in A^{l+1}(X)$ satisfying*

$$\chi(\partial\phi) = \int_{\Delta_{l+1}} \phi^*\omega \mod \mathbb{Z}, \quad \forall \phi \in C_{l+1}^{\text{diff}}(X).$$

Remark 3.30. *(i) We have $d\omega = 0$. Indeed, for any $\phi : \Delta_{l+2} \to X$, we have*

$$\int_{\Delta_{l+2}} \phi^*(d\omega) = \int_{\partial\Delta_{l+2}} \phi^*\omega \mod \mathbb{Z} = \chi(\partial\partial\phi) = 0 \in \mathbb{R}/\mathbb{Z}.$$

Hence we get $d\omega = 0$.

(ii) The de Rham class $[\omega] \in H^{l+1}(X, \mathbb{Z})$. Indeed, this follows from definition such that $\int_{\Delta_{l+1}} \phi^*\omega \subset \mathbb{Z}$ for $\partial\phi = 0$.

(iii) If X is a complex manifold and $\mu \in A^{l-1}(X)$ is real, then by Stokes' formula we have

$$\overline{\int_{\Delta_l} \phi^*(i\partial\mu)} = \int_{\Delta_l} \phi^*(i\bar\partial\mu) = \int_{\Delta_l} \phi^*(-id\mu + i\partial\mu) = \int_{\Delta_l} \phi^*(i\partial\mu).$$

Hence $\int_{\Delta_l} \phi^*(i\partial\mu) \in \mathbb{R}$ and we get a canonical differential character $\int i\partial\mu : \phi \mapsto \int_{\Delta_l} \phi^*(i\partial\mu)$.

Theorem 3.31. *Let X is a compact Kähler manifold. Let $\Xi^{2p-1}_{\text{diff}}(X)^{p,p}$ be a subgroup which consist of characters whose associated form ω of type (p,p). Then*

$$H^{2p}_D(X, \mathbb{Z}(p)) \cong K^{2p-1}_{\text{diff}}(X) := \Xi^{2p-1}_{\text{diff}}(X)^{p,p} / \left\{ \int i\partial\mu : \mu \in A^{p-1,p-1}_{\mathbb{R}}(X) \right\}.$$

Proof. Omitted, see Theorem 12.29 in [5]. □

Finally we give an analogue of cycle map correspond to Deligne cohomology using differential characters.

Theorem 3.32. *Let X be a smooth projective variety, then there exists the Deligne cycle class map*

$$\mathrm{cl}_D : \mathrm{CH}_k(X) \to H^{2n-2k}_D(X, \mathbb{Z}(n-k))$$

such that $c \circ \mathrm{cl}_D = \mathrm{cl}$ and $\mathrm{cl}_D|_{\mathrm{CH}_k(X)_{\mathrm{hom}}} = \Phi^{n-k}_X$ where $c : H^{2n-2k}_D(X, \mathbb{Z}(n-k)) \to \mathrm{Hdg}^{2n-2k}(X, \mathbb{Z})$ in Corollary 3.28.

Proof. Here we give a very concise sketch of the construction when $Z \in \mathcal{Z}_k(X)$ is smooth. First we need the following result:

- (Soulé, 1992) For $Z \in \mathcal{Z}^p(X)$ is smooth, there exists a real form ψ of type $(p-1, p-1)$ on $X \backslash Z$ such that it is integrable and with the equality of currents $i\partial\bar\partial\psi = Z - \omega$ such that the following construction well-defined (see Theorem 12.31 in [5]):

For a closed differentiable chain γ of dimension $2p-1$, there exists a closed differentiable chain γ' of dimension $2p-1$ which does not meet Z, and a differentiable chain Γ of dimension $2p$, such that $\gamma = \gamma' + \partial\Gamma$. We define a differential character

$$\chi_{Z,\psi} : \gamma \mapsto \int_{\gamma'} i\partial\psi + \int_\Gamma \omega \mod \mathbb{Z}.$$

Moreover, descend in $K_{\text{diff}}^{2p-1}(X)$, this is independent to the choice of ψ.

Then we use this to define $[Z]_D = \chi_{Z,\psi} \in K_{\text{diff}}^{2p-1}(X) = H_D^{2p}(X, \mathbb{Z}(p))$. □

We can also define the product in the Deligne cohomology

$$- \cdot_D - : H_D^{2p}(X, \mathbb{Z}(p)) \otimes H_D^{2q}(X, \mathbb{Z}(q)) \to H_D^{2p+2q}(X, \mathbb{Z}(p+q))$$

induced by

$$\Xi_{\text{diff}}^{2p-1}(X) \times \Xi_{\text{diff}}^{2q-1}(X) \to \Xi_{\text{diff}}^{2p+2q-1}(X), \quad (\phi, \psi) \mapsto \phi \cdot \psi$$

defined as follows:

- Pick $Z \in Z_l^{\text{diff}}(X)$ and consdier the diagonal $\delta : X \to X \times X$. Let $Z_i \in Z_{k_i}^{\text{diff}}(X)$ and $Z_i' \in Z_{l_i}^{\text{diff}}(X)$ with $k_i + l_i = 2p + 2q - 1$ such that by Künneth formula we have a differentiable chain Γ of dimension $2p + 2q$ on $X \times X$ such that

  $$\delta(Z) = \sum_i Z_i \times Z_i' + \partial\Gamma.$$

  We define

  $$\phi \cdot \psi(Z) := \sum_{i, k_i = k-1} \phi(Z_i) \int_{Z_i'} \omega_\psi + \sum_{i, k_i = k} \psi(Z_i') \int_{Z_i} \omega_\phi + \int_\Gamma p_1^* \omega_\phi \wedge p_2^* \omega_\psi.$$

  In this case $\phi \cdot \psi$ associated to the form $p_1^*\omega_\phi \wedge p_2^*\omega_\psi$.

By these construction we have:

Theorem 3.33. *For $Z \in \mathrm{CH}_k(X), Z' \in \mathrm{CH}_l(X)$ we have*

$$\mathrm{cl}_D(Z \cdot Z') = \mathrm{cl}_D(Z) \cdot_D \mathrm{cl}_D(Z').$$

Note that in this case Proposition 3.6(i) and Proposition 3.21 are the special cases of this result.

4 Mumford's Theorem and its Generalizations

4.1 Representability and Roitman's Theorem

Fix a complex connected projective variety X.

Definition 4.1. *We say* $\mathrm{CH}_0(X)$ *is representable if the map*

$$\sigma_d : X^{(d)} \times X^{(d)} \to \mathrm{CH}_0(X)_{\mathrm{hom}}, \quad (Z_1, Z_2) \mapsto Z_1 - Z_2$$

is surjective for $d \gg 0$ where $X^{(d)} = X^d/\mathfrak{S}_d$.

Lemma 4.2. *The fibers of σ_d are countable unions of closed algebraic subsets of $X^{(d)} \times X^{(d)}$.*

Proof. Pick $Z = Z^+ - Z^-$ of degree 0 in $\mathrm{CH}_0(X)_{\mathrm{hom}}$ and we need to learn the fiber $\sigma_d^{-1}(Z)$. It consist of the pairs (Z_1, Z_2) such that $Z_1 - Z_2 \sim_{\mathrm{rat}} Z$, which means that there exist curves $C_i \subset X$ with normalizations $\nu_i : C_i \to X$ and rational functions ϕ_i over C_i such that

$$\sum_i \nu_{i,*}\mathrm{div}(\phi_i) = Z_1 - Z_2 - Z.$$

That is, there exist effective 0-cycles A, B such that

$$Z_1 + Z^- + A = \sum_i \nu_{i,*}\phi_i^{-1}(0) + B, \tag{1}$$

$$Z_2 + Z^+ + A = \sum_i \nu_{i,*}\phi_i^{-1}(\infty) + B. \tag{2}$$

Now C_i parametrized by a countable union of Hilbert scheme $\coprod_i \mathrm{Hilb}_X^{P_i}$ where P_i are degree 1 integral polynomials. Moreover, if we fix these Hilbert schemes and the degrees d_i of the divisors $\phi_i^{-1}(0)$, each function ϕ_i can be viewed as a pencil of degree d_i on C_i. Finally, if we fix the degrees a and b of A and B, the 0-cycles A and B are parametrised by the symmetric products $X^{(a)}$ and $X^{(b)}$!

Thus, on the condition of fixing certain discrete data, the objects $(C_i, \phi_i, \nu_i, A, B)$ are parametrised by a projective algebraic variety K. Moreover, the equations (1) (2) define a closed algebraic subset $X^{(d)} \times X^{(d)} \times K$. Then the result is trivial now. $\square$

Remark 4.3. *By the proof, we know that there exists a countable union B of proper algebraic subsets of $X^{(d)} \times X^{(d)}$ such that for any $x \in X^{(d)} \times X^{(d)} \backslash B$, the dimension of the fibre $\sigma^{-1}(\sigma(x))$ is constant and equal to r.*

Definition 4.4. *(i) The dimension of* $\mathrm{Im}\sigma_d$ *is defined to be equal to $2nd - r$ which is with d, where r is defined as above and $n = \dim X$ (so $2nd = \dim(X^{(d)} \times X^{(d)})$).*

(ii) We say $\mathrm{CH}_0(X)$ *is infinite-dimensional if*

$$\lim_{d\to\infty} \dim \mathrm{Im}\sigma_d = \infty.$$

Proposition 4.5. *Group* $\mathrm{CH}_0(X)$ *is representable if and only if it is finite-dimensional.*

Proof. If $\mathrm{CH}_0(X)$ is representable, then there is some $D \gg 0$ such that σ_D is surjective. For any d we consider

$$R = \{(Z_1, Z_2, Z_1', Z_2') \in X^{(d)} \times X^{(d)} \times X^{(D)} \times X^{(D)} : \sigma_d(Z_1, Z_2) = \sigma_D(Z_1', Z_2')\}.$$

Then $p : R \to X^{(d)} \times X^{(d)}$ is surjective. By the proof of previous Lemma, we know that R is a countable union of closed algebraic subsets. By Baire's Theorem we have a irreducible component R_0 maps along p has a non-empty interior. Hence $p_0 : R_0 \to X^{(d)} \times X^{(d)}$ is surjective. If $(Z_1, Z_2, Z_1', Z_2') \in R_0$, we have

$$\dim_{(Z_1,Z_2)} R_0 \cap (X^{(d)} \times X^{(d)} \times (Z_1', Z_2')) \geq 2nd - 2nD.$$

As it is contained in $\sigma_d^{-1}\sigma_D(Z_1', Z_2')$ and p_0 is surjective, we know that any fibers of σ_d are of dimension$\geq 2nd - 2nD$. Hence $\dim \mathrm{Im}\sigma_d \leq 2nD$.

Conversely, if $\mathrm{CH}_0(X)$ is finite-dimensional, then we have $\dim \mathrm{Im}\sigma_D = \dim \mathrm{Im}\sigma_{D+1} = \cdots$ for some $D \gg 0$. For any point $x \in X$ we define

$$\begin{tikzcd} X^{(D)} \times X^{(D)} \arrow[r, "i_x"] \arrow[d, "\sigma_D"] & X^{(D+1)} \times X^{(D+1)} \arrow[dl, "\sigma_{D+1}"] \\ \mathrm{CH}_0(X)_{\mathrm{hom}} & \end{tikzcd}$$

where $i_x : (Z_1, Z_2) \mapsto (Z_1 + x, Z_2 + x)$. Let $F = \sigma_D^{(Z_1,Z_2)}$ for some general point (Z_1, Z_2) and F' be a general fiber of σ_{D+1}. Hence $\dim F' = \dim F + 2n$. Let $F'' = \sigma_D^{-1}(Z_1 + x, Z_2 + x)$. By semicontinuity of the dimensions of the fibres, we have $\dim F'' \geq \dim F'$. Define

$$R = \{(Z_1, Z_2, Z_1', Z_2') \in X^{(D+1)} \times X^{(D+1)} \times X^{(D)} \times X^{(D)} : \sigma_{D+1}(Z_1, Z_2) = \sigma_D(Z_1', Z_2')\}.$$

It is now clear that R has an algebraic component R_0 passing through $(Z_1 + x, Z_2 + x, Z_1, Z_2)$, which dominates $X^{(D)} \times X^{(D)}$, and is such that the fibre of the second projection $q : R_0 \to X^{(D)} \times X^{(D)}$ is of dimension equal to $\dim F''$, and so of dimension greater than or equal to $\dim F + 2nD$, whereas the fibres of the first projection $p : R_0 \to X^{(D+1)} \times X^{(D+1)}$ are of generic dimension at most equal to $\dim F$. As a conclusion, we have

$$\dim p(R_0) \geq \dim F' - \dim F + 2nD = 2n(D+1) = \dim X^{(D+1)} \times X^{(D+1)}.$$

Therefore p is surjective and $\mathrm{Im}\sigma_D = \mathrm{Im}\sigma_{D+1}$. Hence σ_D is surjective. $\square$

There is another equivalent statement of representability:

Proposition 4.6. *Group* $\mathrm{CH}_0(X)$ *is representable if and only if for every smooth curve* $C = Y_1 \cap \cdots \cap Y_{n-1}$ *which is a complete intersection of ample hypersurfaces* $Y_i \subset X$*, letting* $j : C \hookrightarrow X$ *be the inclusion, the map*

$$j_* : \mathrm{CH}_0(C)_{\mathrm{hom}} = J(C) \to \mathrm{CH}_0(X)_{\mathrm{hom}}$$

is surjective.

Proof. If we have such C, then by Remark 3.18(b) we have surjection

$$C^{(g)} \to \mathrm{CH}_0(C)_{\mathrm{hom}} = J(C), \quad z \mapsto z - g \cdot z_0.$$

By the surjectivity of j_* we know that

$$X^{(g)} \to \mathrm{CH}_0(X)_{\mathrm{hom}}, \quad z \mapsto z - g \cdot z_0$$

is surjective.

Conversely, we will only give a sketch. Fix $0 \in C$, let $n \geq 2$ and consider

$$\sigma_m^0 : X^{(m)} \to \mathrm{CH}_0(X)_{\mathrm{hom}}, \quad Z \mapsto \sigma_m(Z, m0).$$

Now let $\dim \mathrm{Im}\sigma_m^0 = K$ for any $m \gg 0$, the fibers of σ_m^0 has dimension $mn - K$. We claim that an irreducible component of maximal dimension Z of a general fibre of σ_m^0 cannot be contained in a set of the form $X^{(m-i)} + W, i \geq 1$, with $W \subset X^{(i)}$ and $\dim W < i$. We omitted the proof of the claim and we refer to Page 285 in [6].

- Let Y_1 be an ample hypersurface of X and let Z be an irreducible subset of $X^{(m)}$ not contained in any subset of the form $X^{(m-i)} + W, i \geq 1$, with $W \subset X^{(i)}$ and $\dim W < i$. Then if $\dim Z \geq m$, we have $Z \cap Y_1^{(m)} \neq \emptyset$. (Lemma 10.13 in [6])

Hence by the claim and this result, we know that a general fibre of σ_m^0 intersects $Y_1^{(m)}$ for $m \gg 0$. Hence σ_m^0 and $\sigma_m^0|_{Y_1^{(m)}}$ have the same image for $m \gg 0$. As the images of the maps $\sigma_m^0|_{Y_1^{(m)}}$ are also of bounded dimension, so we can iterate the reasoning to finally conclude that σ_m^0 and $\sigma_m^0|_{C^{(m)}}$ have the same image for $m \gg 0$. Hence for any 0-cycle of form $Z - m0$, we have $Z - m0 \sim_{\mathrm{rat}} Z' - m0$ for $Z' \in C^{(m)}$, so

$$j_* : \mathrm{CH}_0(C)_{\mathrm{hom}} = J(C) \to \mathrm{CH}_0(X)_{\mathrm{hom}}$$

is surjective. □

Now we will use the following theorem:

Theorem 4.7 (Roitman 1980, Bloch 1979)**.** *The Albanese map*

$$\mathrm{alb}_X : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$$

induces an isomorphism on torsion points for every smooth projective variety X.

Finally we can use these results to show that the representability can be justified by the following theorem of Roitman:

Theorem 4.8 (Roitman, 1972)**.** *If* $\mathrm{CH}_0(X)$ *is representable, then the Albanese map is an isomorphism:*

$$\mathrm{alb}\, X : \mathrm{CH}_0(X)_{\mathrm{hom}} \cong \mathrm{Alb}(X).$$

In particular, in this case $\mathrm{CH}_0(X)_{\mathrm{hom}}$ *is an algebraic group.*

Proof. By Proposition 4.6, if $\mathrm{CH}_0(X)$ is representable, then for every smooth curve $C = Y_1 \cap \cdots \cap Y_{n-1}$ which is a complete intersection of ample hypersurfaces $Y_i \subset X$, letting $j : C \hookrightarrow X$ be the inclusion, the map

$$j_* : \mathrm{CH}_0(C)_{\mathrm{hom}} = J(C) \to \mathrm{CH}_0(X)_{\mathrm{hom}}$$

is surjective. We claim that j_* induced by a correspondence Γ satisfies:

(i) Γ_* takes values in $\mathrm{CH}_0(C)_{\mathrm{hom}}$ and is a group homomorphism;

(ii) Γ_* is surjective.

Indeed, consider $\phi : C^{(g)} \to J(C)$ by $z_1 + \cdots + z_g \mapsto \mathrm{alb}_C(\sum_i z_i - gc_0)$ where $g = g(C)$ and $c_0 \in C$ fixed. We define

$$\Gamma_C^1 := \{(a, c) \in J(C) \times C : \exists z \in C^{(g)}, c \in z, \phi(z) = a\}$$

and $\Gamma_C^2 := g(J(C) \times c_0)$. We define $\Gamma_C = \Gamma_C^1 - \Gamma_C^2$. Then easy to see that Γ_C satisfies the condition (i). Define $\Gamma := (\mathrm{id}, j)_* \Gamma_C \subset J(C) \times X$, then one can see that Γ induce j_* and satisfies (i)(ii).

By the similar argument in the proof of Lemma 4.2, we know that $\ker \Gamma_*$ is a countable union of algebraic subsets of $A = J(C)$ which is also a subgroup (hence the countable union of translates of an abelian subvariety A_0). Then there exists an abelian subvariety $B \subset A$, which is supplementary to A_0 up to isogeny, i.e. such that the morphism $A_0 \times B \to A, (a, b) \mapsto a + b$ is surjective with finite kernel. Hence replace A by B we let assume $\ker \Gamma_*$ is countable.

Define $R := \{(x, a) : \Gamma_*(a) = x - x_0\} \subset X \times A$. Then it is a countable union of algebraic sets whose projection onto X is surjective, since Γ_* is surjective. Thus, R has an algebraic component R_0 which dominates X. Furthermore, as $\ker \Gamma_*$ is countable, the projection $R_0 \to X$ is finite of degree r. R_0 is thus a correspondence of dimension equal to $\dim X$ between X and A, and provides a morphism $\alpha : X \to A$, given by $x \mapsto \mathrm{alb}_A(R_{0,*}(x - x_0))$. By the definition of R_0 and property (i), it is clear that

$$\Gamma_* \circ \alpha(x) = r(x - x_0).$$

Moreover, by the universal property of the Albanese map, there exists a morphism of groups β such that we have the following comuutative diagram:

$$\begin{tikzcd}
\mathrm{Alb}(X) \arrow[r, "\beta"] & A \arrow[dr, "\Gamma_*"] & \\
\mathrm{CH}_0(X)_{\mathrm{hom}} \arrow[u, "\mathrm{alb}_X"] \arrow[ur, "\alpha"] \arrow[rr, "r\cdot\mathrm{id}"] & & \mathrm{CH}_0(X)_{\mathrm{hom}}
\end{tikzcd}$$

Hence $\ker \mathrm{alb}_X$ is torsion, so that

$$\mathrm{alb}\, X : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$$

is an isomorphism up to torsion, since alb_X is always surjective by Remark 3.18(b). Then by Theorem 4.7, well done. □

4.2 The Bloch-Srinivas Construction and Mumford's Theorem

4.2.1 Decomposition of the Diagonal

Here we introduce the heart result due to Bloch-Srinivas, in order to prove Mumford's Theorem 4.12. Here we give a more generalized result 4.9 and we only use the special case 4.11 to prove the Mumford's Theorem.

Let $f : X \to Y$ be a projective fibration with X, Y smooth and connected variety. Let $Z \in \mathrm{CH}^k(X)$.

(*) There exists a subvariety $X' \subset X$ such that for every $y \in Y$, the cycle $Z_y := Z|_{X_y} \in \mathrm{CH}^k(X_y)$ vanishes in $\mathrm{CH}^k(X_y - X'_y)$.

Theorem 4.9 (Bloch-Srinivas, 1980-1983)**.** *If Z satisfies the property (*), then there exist an integer $m > 0$, and a cycle Z' supported in X', such that we have the equality*

$$mZ = Z' + Z'' \in \mathrm{CH}^k(X),$$

where Z'' is a cycle supported in $f^{-1}(Y')$, for a proper closed algebraic subset $Y' \subset Y$.

Proof. One omitted, we refer [6] Theorem 10.19. □

Corollary 4.10. *Let $\Gamma \in \mathrm{CH}^k(X_1 \times Y)$ for some connected smooth variety X_1, and assume that for every $y \in Y$, the cycle $\Gamma^*(y) \in \mathrm{CH}^k(X_1)$ restricts to zero in $\mathrm{CH}^k(X_1 \backslash X_1')$ for a subvariety $X_1' \subset X_1$. Then we have adecomposition*

$$m\Gamma = Z' + Z'' \in \mathrm{CH}^k(X_1 \times Y),$$

where Z' is supported in $X_1' \times Y$ and Z'' is supported in $X_1 \times Y'$, for a proper closed algebraic subset $Y' \subset Y$.

Proof. Take $X = X_1 \times Y$ in Theorem 4.9. □

Corollary 4.11 (Decomposition of the Diagonal)**.** *Consider $\Delta_X \subset X \times X$ be the diagonal scheme. Assume there is a closed subvariety $j : X' \subset X$ such that*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective. Then there exists a proper closed algebraic subset $T \subset X$, and a decomposition

$$m\Delta_X = Z' + Z'',$$

where Z' is supported in $T \times X$ and Z'' is supported in $X \times X'$.

Proof. Take $X_1 = Y = X$ in Corollary 4.10 and since j_* is surjective, we know that $\mathrm{CH}_0(X \backslash X') = 0$. Well done. □

4.2.2 Mumford's Theorem

We will use these results to prove the following kind of generalizations of Mumford's Theorem 4.16 and Roitman's Theorem 4.14. Note that when $k = n = 2$ this is weaker than Theorem 4.16.

Theorem 4.12. *Let X be a smooth complex projective variety. If there exists a subvariety $j : X' \hookrightarrow X$ such that $\dim X' < k$ and the map*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective, then $H^0(X, \Omega_X^k) = 0$.

Proof. By Corollary 4.11, there exists a proper closed algebraic subset $T \subset X$, and a decomposition

$$m\Delta_X = Z' + Z'',$$

where Z' is supported in $T \times X$ and Z'' is supported in $X \times X'$. Hence $m\,\mathrm{cl}(\Delta_X) = \mathrm{cl}(Z') + \mathrm{cl}(Z'')$. By Lemma 3.11, their Künneth components induce

$$m[\Delta_X]^*, [Z']^*, [Z'']^* : H^r(X, \mathbb{Z}) \to H^r(X, \mathbb{Z})$$

for each r. As $[\Delta_X]^* = \mathrm{id}$, we have

$$m \cdot \mathrm{id} = [Z']^* + [Z'']^* \in \mathrm{End}(H^r(X, \mathbb{Z})).$$

Pick $l : \hat{T} \to X$, a resolution of singularity of T. As the cycle Z'' is supported in $T \times X$, it comes from a cycle $\hat{Z}''$ of $\hat{T} \times X$, so we have

$$\mathrm{cl}(Z'') = (l, \mathrm{id})_*(\mathrm{cl}(\hat{Z}'')).$$

By the definition we have $[Z'']^* = l_* \circ [\hat{Z}'']^*$. Similarly, let $\hat{j} : \hat{X}' \to X$ be a resolution of singularity of X', and let $\hat{Z}'$ be a cycle of $X \times \hat{X}'$ such that $(\mathrm{id}, \hat{j})_*(\hat{Z}') = Z'$. Hence $[Z']^* = [\hat{Z}']^* \circ \hat{j}^*$.

Combining these we have

$$m\eta = ([\hat{Z}']^* \circ \hat{j}^*)\eta + (l_* \circ [\hat{Z}'']^*)\eta$$

for any $\eta \in H^0(X, \Omega_X^r)$. As $\dim X' < k$, then $\hat{j}^*\eta = 0$ for all $r \geq k$. As $\dim T < \dim X$, then l_* is of bidegree (s, s) for $s > 0$. Combining these, we have $\eta = 0$ for all $r \geq k$. □

Corollary 4.13. *In this case, if* $\mathrm{CH}_0(X)_{\mathrm{hom}} = 0$*, then* $H^0(X, \Omega_X^k) = 0$ *for all* $k > 0$.

Proof. In the Theorem 4.12 we pick X' be a single point and well done. □

Theorem 4.14 (Roitman 1972)**.** *Let X be a smooth complex projective variety of dimension n. Let d_m be the dimension of the general fiber of σ_m. If $H^0(X, \Omega_X^k) \neq 0$, then for $m \gg 0$ we have*

$$d_m < 2(n - k + 1)m.$$

Proof. We give the main idea of the proof. Let $d_m \geq 2(n - k + 1)m$ and let X' be the complete intersection of $n - k + 1$ ample hypersurfaces, and use the same proof of Proposition 4.6 show that $X'^{(m)} \times X'^{(m)}$ must intersect the fibres of σ_m for $m \gg 0$, on the condition that we have $d_m \geq 2(n-k+1)m$. Then we can get j_* is surjective! Then by Theorem 4.12 and well done. □

The Roitman's Theorem 4.14 is also some kind of generalizations of Mumford's Theorem and note that when $k = n = 2$ this is weaker than Theorem 4.16.

Lemma 4.15. *Let S be a smooth complex projective surface such that $H^0(S, K_S) \neq 0$. Define an algebraic component*

$$R = \left\{ (x_1, ..., x_{2n}) \in S^{2n} : \sum_{1 \leq i \leq 2n} n_i x_i = 0 \in \mathrm{CH}_0(S) \right\}.$$

Assume that R dominates S^n via the first projection. Then the first projection $p_1 : R \to R^n$ is generically finite.

Proof. By the proof of Theorem 4.12 and Corollary 4.10, we have:

- Let W be a smooth projective variety, and let $Z \subset W \times X$ be a cycle of codimension $n = \dim X$. Assume that there exists a k_0-dimensional subvariety $X' \subset X$ such that for every $w \in W$, the 0-cycle Z_w is rationally equivalent in X to a cycle supported on X'. Then for every $k > k_0$ and every $\eta \in H^0(X, \Omega_X^k)$, we have $[Z]^*\eta = 0$ in $H^0(W, \Omega_W^k)$.

Then we can use this to prove the Lemma. The detail we refer [6] Lemma 10.25. □

Theorem 4.16 (Mumford 1968)**.** *Let S be a smooth complex projective surface such that $H^0(S, K_S) \neq 0$. Then for any integer m, the map σ_m has countable general fiber. In particular,* $\mathrm{CH}_0(S)$ *is not representable.*

Proof. If the general fiber of σ_m are not countable, then there would exist an algebraic subvariety $R \subset S^{(n)} \times S^{(n)} \times S^{(n)} \times S^{(n)}$ such that the first projection $R \to S^{(n)} \times S^{(n)}$ is dominant with positive- dimensional fibres, and such that

$$(Z_1, Z_2, Z_3, Z_4) \in R \Rightarrow \sigma_n(Z_1, Z_2) = \sigma_n(Z_3, Z_4).$$

But then, taking the inverse image of R, we find that there exists $R' \subset S^n \times S^n \times S^n \times S^n$ such that the first projection $R' \to S^n \times S^n$ is dominant with positive-dimensional fibres, and such that the similar relation holds as R. But this would contradict Lemma 4.15. Well done. □

4.2.3 Some Other Applications

Theorem 4.17 (Bloch-Srinivas 1983)**.** *Let X be a smooth complex projective variety such that there exists a subvariety $j : X' \subset X$, of dimension ≤ 3, such that the map*

$$j_* : \mathrm{CH}_0(X') \to \mathrm{CH}_0(X)$$

is surjective. Then the Hodge conjecture holds for classes of degree 4 on X.

Proof. By Corollary 4.11, there exists a proper closed algebraic subset $T \subset X$, which we may assume to be of codimension 1, and a decomposition

$$m\Delta_X = Z' + Z'',$$

where Z' is supported in $T \times X$ and Z'' is supported in $X \times X'$.

Take $k : \hat{T} \to X, \hat{j} : \hat{X}' \to X$ be the resolution of singularities of T, X'. Note that we have $m\,\mathrm{cl}(\Delta_X) = \mathrm{cl}(Z') + \mathrm{cl}(Z'')$ which induce the morphism of Hodge structures

$$m[\Delta_X]^* = [Z']^* + [Z'']^* \in \mathrm{End}(H^4(X, \mathbb{Z}))$$

via the Künneth components of type $(2n-4,4)$.

Now pick $\hat{Z}' \subset \hat{T} \times X$ be a cycle of codimension n such that $(k, \mathrm{id})_*\hat{Z}' = Z'$, and pick $\hat{Z}'' \subset X \times \hat{X}'$ be a cycle of codimension n such that $(\mathrm{id}, \hat{j})_*\hat{Z}'' = Z''$. We have $[Z']^* = k_*[\hat{Z}']^*$ and $[Z'']^* = [\hat{Z}'']^*\hat{j}^*$.

Now as $\dim X' \leq 3$, the rational Hodge conjecture holds for $\hat{X}'$ by Theorem 3.10 and Hard Lefschetz Theorem. If $\alpha \in \mathrm{Hdg}^2(X)_{\mathbb{Q}}$, then the classes $\hat{j}^*\alpha$ and $[Z']^*\alpha$ are classes of algebraic cycles. The relation

$$m[\Delta_X]^*\alpha = m\alpha = k_*[\hat{Z}']^*\alpha + [\hat{Z}'']^*\hat{j}^*\alpha$$

and the compatibility of the cycle class map with correspondences then show that α is also the class of an algebraic cycle with rational coefficients. Well done. $\square$

Theorem 4.18 (Bloch-Srinivas 1983)**.** *Let X be a smooth complex projective variety such that there exists a subvariety $j : S \subset X$, of dimension ≤ 2, such that*

$$j_* : \mathrm{CH}_0(S) \to \mathrm{CH}_0(X)$$

is surjective. Then the group $\mathrm{Griff}^2(X)$ *is a torsion group.*

Proof. By Corollary 4.11, there exists a proper closed algebraic subset $T \subset X$, which we may assume to be of codimension 1, and a decomposition

$$m\Delta_X = Z' + Z'',$$

where Z' is supported in $T \times X$ and Z'' is supported in $X \times S$.

Take $k : \hat{T} \to X, \hat{j} : \hat{S} \to X$ be the resolution of singularities of T, S. Now pick $\hat{Z}' \subset \hat{T} \times X$ be a cycle of codimension n such that $(k, \mathrm{id})_*\hat{Z}' = Z'$, and pick $\hat{Z}'' \subset X \times \hat{S}$ be a cycle of codimension n such that $(\mathrm{id}, \hat{j})_*\hat{Z}'' = Z''$. We have $[Z']^* = k_*[\hat{Z}']^*$ and $[Z'']^* = [\hat{Z}'']^*\hat{j}^*$.

Now pick $z \in \mathrm{CH}^2(X)_{\mathrm{hom}}$, then

$$mz = k_*[\hat{Z}']^*z + [\hat{Z}'']^*\hat{j}^*z.$$

As S is a surface, then $\hat{j}^*z \in \mathrm{CH}_0(\hat{S})_{\mathrm{hom}}$. Hence $\hat{j}^*z \sim_{\mathrm{alg}} 0$. Similarly $[\hat{Z}']^*z \in \mathrm{CH}^1(\hat{T})_{\mathrm{hom}}$ hence $[\hat{Z}']^*z \sim_{\mathrm{alg}} 0$ by Remark 3.22. As the algebraic cycles which equivalent to 0 are stable under the action of correspondences, we have $mz \sim_{\mathrm{alg}} 0$ and hence $\mathrm{Griff}^2(X)$ is a torsion group. $\square$

4.3 Generalizations

5 The Bloch Conjecture and its Generalizations

5.1 Surfaces

5.2 Filtrations on Chow groups

5.3 The case of Abelian varieties

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