NOTES ON THE GEOMETRY OF HYPERTORIC VARIETIES

XIAOLONG LIU

Abstract. In this note we will introduce the basic theory of hypertoric varieties.

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We work over $\mathbb{C}$.

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We will follows [Ful93], [CLS11] and [Tel22] to recollect something we need.

3. Basic theories of hypertoric varieties

3.1. About symplectic varieties and symplectic resolutions.
Here we give an introduction of these and we refer [Bea00] and [Fu06] for more details.

Definition 3.1. *Let Y_0 be a normal variety.*

- *A pair (Y_0, ω_0) of the normal algebraic variety Y_0 and a 2-form ω_0 on the smooth locus $(Y_0)_{\mathsf{sm}}$ is called a symplectic variety if ω_0 is symplectic and there exists (or equivalently, for any) a resolution $\pi : Y \to Y_0$ such that the pull-back of ω_0 by π extends to a holomorphic 2-form ω on Y.*
- *The resolution $\pi : Y \to Y_0$ is called symplectic if ω is also symplectic.*

Some basic properties:

Proposition 3.2 (Prop.1.6 in [Fu06])**.** *Let W be a symplectic variety with a resolution $\pi : Z \to W$, then the following statements are equivalent:*

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(1) π is crepant;
(2) π is symplectic;
(3) K_Z is trivial.

Next, we now care about the following special case:

Definition 3.3. *An affine symplectic variety $(Y_0 = \operatorname{Spec} R, \omega_0)$ with $\mathbb{C}^*$-action (called conical $\mathbb{C}^*$-action) is called a conical symplectic variety if it satisfies:*

- *The grading induced from the $\mathbb{C}^*$-action to the coordinate ring R is positive, i.e., $R = \bigoplus_{i\geqslant 0} R_i$ and $R_0 = \mathbb{C}$.*
- *ω_0 is homogeneous with respect to the $\mathbb{C}^*$-action, i.e., there exists $\ell \in \mathbb{Z}$ (the weight of ω_0) such that $t^*\omega_0 = t^\ell \omega_0$ $(t \in \mathbb{C}^*)$.*

Remark 3.4. *We can show that the weight ℓ is always positive.*

3.2. **Algebraic symplectic quotients and hypertoric varieties.** Note that hypertoric varieties are examples of symplectic varieties.

Consider the exact sequence

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$

where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$ (the Gale duality of $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_n\}$). Acting $\operatorname{Hom}(-, \mathbb{C}^*)$ we get

$$1 \to \mathbb{T}^d \xrightarrow{A^T} \mathbb{T}^n \xrightarrow{B^T} \mathbb{T}^{n-d} \to 1$$

an exact sequence of algebraic tori.

Via the natural action of $\mathbb{T}^n$ on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$, we have the action of $\mathbb{T}^d$ on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ as

$$\boldsymbol{t} \cdot (z_1, ..., z_n, w_1, ..., w_n) = (\boldsymbol{t}^{\boldsymbol{a}_1} z_1, ..., \boldsymbol{t}^{\boldsymbol{a}_n} z_n, \boldsymbol{t}^{-\boldsymbol{a}_1} w_1, ..., \boldsymbol{t}^{-\boldsymbol{a}_n} w_n)$$

where $\boldsymbol{t}^{\boldsymbol{a}_i} := t_1^{a_{1,i}} \cdots t_d^{a_{d,i}}$. The moment map of this given by

$$\mu : T^*C^n \to \mathfrak{t}_d^* = \mathbb{C}^d, \quad (z_1, ..., z_n, w_1, ..., w_n) \mapsto \sum_{i=1}^n \boldsymbol{a}_i z_i w_i.$$

Definition 3.5. *Fix a character $\alpha \in \mathbb{Z}^d = \operatorname{Hom}(\mathbb{T}^d, \mathbb{C}^*)$ and a point $\xi \in \mathbb{C}^d$.*

- *We define the Lawrence toric variety as*
$$X(A, \alpha) := (\mathbb{C}^{2n})^{\alpha\text{-}ss} /\!/ \mathbb{T}^d = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha}\right)$$
where $(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \{u \in \mathbb{C}^{2n}$: there exists $f \in \mathbb{C}[z_i, w_j]$ such that $f(u) \neq 0$ and $\sigma(f) = \alpha^*(t)^k \otimes f$ for $k > 0\}$ where $\mathbb{C}^* = \operatorname{Spec} \mathbb{C}[t, 1/t]$ and coaction morphism $\sigma : \mathbb{C}[z_i, w_j] \to \Gamma(\mathscr{O}_{\mathbb{T}^d}) \otimes \mathbb{C}[z_i, w_j]$. Note that $\mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha} = \{f \in \mathbb{C}[z_i, w_j] : \sigma(f) = \alpha^*(t)^k \otimes f\}$.
- *We define the hypertoric variety (or toric hyperkähler variety) as*
$$Y(A, \alpha, \xi) := \mu^{-1}(\xi)^{\alpha\text{-}ss} /\!/ \mathbb{T}^d = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}\right)$$
similar as above.

Remark 3.6. *We can write the semistable locus as follows:*

$$(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \left\{(z_i, w_j) \in \mathbb{C}^{2n} : \alpha \in \sum_{i:z_i\neq 0} \mathbb{Q}_{\geqslant 0}\boldsymbol{a}_i + \sum_{j:w_j\neq 0} \mathbb{Q}_{\geqslant 0}(-\boldsymbol{a}_j)\right\}$$

and $\mu^{-1}(\xi)^{\alpha\text{-}ss} = \mu^{-1}(\xi) \cap (\mathbb{C}^{2n})^{\alpha\text{-}ss}$.

Remark 3.7. *Note that we have a natural morphism $\Pi : X(A, \alpha) \to X(A, 0)$ and $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$ with the same reason. Indeed, we consider the case of hypertoric varieties. Note that*

$$Y(A, 0, \xi) = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\cdot 0}\right) = \operatorname{Spec} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}.$$

Then inclusion $\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d} \subset \bigoplus_{k\geqslant 0}\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d,k\alpha}$ *induce* $\operatorname{Spec}\bigoplus_{k\geqslant 0}\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d,k\alpha} \to \operatorname{Spec}\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}$. *Since the grade induced by* $\mathbb{C}^*$*-action and this morphism is* $\mathbb{C}^*$*-invariant, then we get* $\pi : Y(A,\alpha,\xi) \to Y(A,0,\xi)$. *Note moreover that* $\mu^{-1}(\xi)^{\alpha\text{-}ss} \subset \mu^{-1}(\xi) = \mu^{-1}(\xi)^{0\text{-}ss}$.

Remark 3.8. *The hypertoric varieties are the special case of the following general contruction.*

Consider a reductive group G *and a representation* V. *Then we form* $T^*V = V \oplus V^*$ *which comes with a moment map* $\Phi : T^*V \to \mathfrak{g}^*$ *given by cup of* $T_xV^* \to \mathfrak{g}^*$ *as* $T_eG \to T_x(Gx) \subset T_xV$. *We fix a character* $\chi : G \to \mathbb{C}^\times$ *and form the GIT quotient*

$$\Phi^{-1}(\xi) /\!\!/_{\chi} G := \Phi^{-1}(\xi)^{\chi\text{-}ss} /\!\!/ G = \operatorname{Proj}\left(\bigoplus_{n\geqslant 0}\mathbb{C}[\Phi^{-1}(\xi)]^{G,n\chi}\right).$$

We have a natural projective morphism as before

$$\pi : Y := \Phi^{-1}(\xi) /\!\!/_{\chi} G \to X := \Phi^{-1}(\xi) /\!\!/_{0} G = \operatorname{Spec}\mathbb{C}[\Phi^{-1}(0)]^G$$

carry Poisson structures coming from the usual symplectic structure on T^*V. *This construction will not usually give a symplectic resolution; for example,* Y *may not be smooth and* $Y \to X$ *might not be birational. Here in the physics literature,* Y *is called the* Higgs branch of the 3d supersymmetric gauge theory defined by G, V. G *is called the* gauge group *and* N *is called the* matter.

There is a conical $\mathbb{C}^\times$ *action on* Y *coming from its scaling action of* T^*V. *In order to define a Hamiltonian torus action, we need one piece of data. We choose an extension* $1 \to G \to \widetilde{G} \to T \to 1$, *where* T *is the flavor torus, and an action of* $\widetilde{G}$ *on* V, *extending the action of* G. *Then we obtain a residual Hamiltonian action of* T *on* Y *and* X. *In general, this action does not have finitely many fixed points.*

Example 3.9. *Another special case, we introduce the Nakajima quiver varieties, first introduced by Nakajima. We fix a finite directed graph* $Q = (I, E)$, *with head and tail maps* $h, t : E \to I$. *Also, we fix two dimension vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$. *For* $i \in I$,*let* $V_i = \mathbb{C}^{u_i}, W_i = \mathbb{C}^{w_i}$ *and consider the space of representations of the quiver* Q *on the vector space* $\oplus V_i$ *framed by* $\oplus W_i$.

$$N = \bigoplus_{e\in E}\operatorname{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i\in I}\operatorname{Hom}(V_i, W_i).$$

This big vector space N *has a natural action of* $G = \prod_i \mathrm{GL}(V_i)$. *We form the cotangent bundle* T^*N *and take the Hamiltonian reduction by the action of* G. *The resulting space* $Y = \Phi^{-1}(0)/\!/_{\chi}G$ *is called a* Nakajima quiver variety. *Here we choose* $\chi : G \to \mathbb{C}^\times$ *to be given by the product of the determinants. On* Y, *we have a Hamiltonian action of* $T = \prod_i (\mathbb{C}^\times)^{w_i}$ *inherited from its action on* $\oplus W_i$. *(In other words, we take* $\widetilde{G} = G \times T$.*)*

Note that the space Y *is always smooth but* $\pi : Y \to X$ *is not always birational. Also, the Hamiltonian torus action does not always have finitely many fixed points.*

Here we give two examples of Nakajima quiver varieties.

- *Consider a linearly oriented type* A_{n-1}*-quiver with* $\mathbf{v} = (1, ..., n-1), \mathbf{w} = (0, ..., 0, n)$*:*

$$\bullet(V_1) \longrightarrow \bullet(V_2) \longrightarrow \cdots \longrightarrow \bullet(V_{n-1}) \longrightarrow \blacksquare(\mathbb{C}^n)$$

  Then $N = \bigoplus_{i=1}^{n-1}\operatorname{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$ *with* $G = \prod_{i=1}^{n-1}\mathrm{GL}_i$. *Then* $Y \cong T^*\mathsf{Fl}_n$ *with* $X = \mathcal{N}_{\mathfrak{sl}_n}$.
- *Another important example is a quiver with one vertex and one self-loop with* $V = \mathbb{C}^n$ *and* $W = \mathbb{C}^r$.

$$\begin{tikzcd} \bullet(\mathbb{C}^n) \arrow[loop above] \arrow[r] & \blacksquare(\mathbb{C}^r) \end{tikzcd}$$

  In this case, Y *is the moduli space of rank* r, *torsion-free sheaves on* $\mathbb{P}^2$, *framed at* ∞ *with second Chern class* n.

3.3. **Symplectic resolutions of hypertoric varieties.** We will consider when $\pi : Y(A,\alpha,\xi) \to Y(A,0,\xi)$ will be a symplectic resolution. So we need to consider the condition that $\mu^{-1}(\xi)^{\alpha\text{-ss}} = \mu^{-1}(\xi)^{\alpha\text{-st}}$. First we will compute their stabilizer group.

Let $(\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{C}^{2n}$ and set $J_{\boldsymbol{z},\boldsymbol{w}} := \{j \in \{1, ..., n\} : z_j \neq 0 \text{ or } w_j \neq 0\}$, then we have

$$\operatorname{Stab}_{\boldsymbol{z},\boldsymbol{w}}\mathbb{T}^d = \ker(\mathbb{T}^d \overset{A^T_{J_{\boldsymbol{z},\boldsymbol{w}}}}{\to} \mathbb{T}^{|J_{\boldsymbol{z},\boldsymbol{w}}|}).$$

Hence by some linear algebra we have

Corollary 3.10 (Coro.2.7 in [Nag21]). *We have:*
(1) $\mathrm{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d$ *is finite if and only if* $\sum_{j\in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Q}\boldsymbol{a}_j = \mathbb{Q}^d$*;*
(2) $\mathrm{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d = 1$ *if and only if* $\sum_{j\in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Z}\boldsymbol{a}_j = \mathbb{Z}^d$*.*

Definition 3.11. *In this setting, we call A is unimodular if all $d\times d$-minors of A are 0 or ± 1.*

Remark 3.12. *Note that A is unimodular if and only if B is.*

Hence for a unimodular A, we have $\sum_{j\in J} \mathbb{Q}\boldsymbol{a}_j = \mathbb{Q}^d$ iff $\sum_{j\in J} \mathbb{Z}\boldsymbol{a}_j = \mathbb{Z}^d$ for $J \subset \{1, ..., n\}$.

Let A is a unimodular matrix and we define $\mathcal{H}_A := \{H \subset \mathbb{R}^d : H$ is generated by some of the $\boldsymbol{a}_j$ and of codimension $=$ 1. We say α generic if $\alpha \notin \bigcup_{H\in\mathcal{H}_A} H$.

Lemma 3.13 (Lem.2.10 and Coro.2.11 in [Nag21]). *In the case, for any $\alpha \in \mathbb{Z}^d$ and $\xi \in \mathbb{C}^d$, we have $(\mu^{-1}(\xi))^{\alpha\text{-}ss} \neq \varnothing$. If α generic, then $(\mu^{-1}(\xi))^{\alpha\text{-}ss} = (\mu^{-1}(\xi))^{\alpha\text{-}st}$ with free action by $\mathbb{T}^d$. In particular, if α generic then $X(A,\alpha)$ is $2n-d$-dimensional smooth Poisson variety and for any ξ, $Y(A,\alpha,\xi)$ is a $2n-2d$-dimensional smooth symplectic variety.*

Theorem 3.14 (Thm.2.16 in [Nag21]). *For a unimodular A and generic α and any $\xi \in \mathbb{C}^d$, the morphism*
$$\pi_\xi : Y(A,\alpha,\xi) \to Y(A,0,\xi)$$
is a projective symplectic resolution and if $\xi = 0$, then it is conical.

Sketch. First, by $\mu : \mathbb{C}^{2n} \xrightarrow{\Psi} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^d$ with $\Psi : (\boldsymbol{z},\boldsymbol{w}) \mapsto \sum_j z_j w_j \boldsymbol{e}_j$ is flat. Then from dimension counting we get $\mu^{-1}(\xi)$ is of equidimension $2n-d$. As it define by d polynomials, we know that $\mu^{-1}(\xi) \in \mathbb{C}^{2d}$ is a complete intersection and hence Cohen-Macaulay. After showing that the codimension of singular locus $\geqslant 2$, then $\mu^{-1}(\xi)$ is normal by Serre's condition. Finally we can construct an open subset and show that π_ξ is identity over it which force it is birational. Moreover, the result follows from Lemma 3.13 and the following easy fact (see Proposition 2.15 in [Nag21]):

- If $\pi : Y \to Y_0$ is projective birational morphism with Y is a nonsingular sympecltic variety, then π is a symplectic resolution.

Well done. □

Remark 3.15. *Note that we have the more general results. In [Bel23] Lemma 2.4 and Proposition 2.5, without assuming A is unimodular, shows that if we choose α, α' such that $\mu^{-1}(\xi)^{\alpha'\text{-}ss} \subset \mu^{-1}(\xi)^{\alpha\text{-}ss}$, then there exists a projective birational Poisson morphism $Y(A,\alpha',\xi) \to Y(A,\alpha,\xi)$. Moreover, any hypertoric variety $Y(A,\alpha,\xi)$ has symplectic singularities.*

3.4. **Hypertoric varieties with hyperplane arrangements.** Here we consider the case $\xi = 0$. Then we define $Y(A,\alpha) := Y(A,\alpha,0)$. It defined by
$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$
where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$.

Then we can define $H_i := \{x \in \mathbb{R}^{n-d} : x\cdot \boldsymbol{b}_i + r_i = 0\}$ for $i = 1, ..., n$ where $\boldsymbol{r} = (r_1, ..., r_n) \in \mathbb{Z}^n$ be a lifting of α along A. This defines a hyperplane arrangement $\mathcal{A} := \{H_1, ..., H_n\}$. Here we can denote $Y(\mathcal{A}) := Y(A,\alpha)$.

Definition 3.16. *In this setting, for such hyperplane arrangement $\mathcal{A}$:*

- *we call $\mathcal{A}$ is simple if for any subset of m hyperplanes with nonempty intersections, they intersect of codimension m.*
- *we call $\mathcal{A}$ is unimodular if for any $n-d$ linear independent $\{\boldsymbol{b}_{i_1}, ..., \boldsymbol{b}_{i_{n-d}}\}$ spans $\mathbb{C}^{n-d}$ over $\mathbb{Z}$.*
- *we call $\mathcal{A}$ is smooth if it is simple and unimodular.*

Remark 3.17. *Note that $\mathcal{A}$ is unimodular if and only if B is unimodular if and only if A is unimodular.*

Proposition 3.18 (3.2/3.3 in [BD00]). *The hypertoric variety $Y(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth.*

Note that $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement, meaning that $r_i = 0$ for all i, so that all of the hyperplanes pass through the origin. Then we have the following result:

Corollary 3.19. *For any central arrangement $\mathcal{A}$, there exists a simplification $\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$ of $\mathcal{A}$ by which we mean an arrangement defined by the same vectors $\{\boldsymbol{b}_i\}$, but with a different choice of $\alpha, \boldsymbol{r}$ such that $\widetilde{\mathcal{A}}$ is simple. This will give us an equivariant orbifold resolution $Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$. When A is unimodular, this will give us a resolution of singularities which recover the special case of Theorem 3.14.*

3.5. **The cores and homotopy models.** Consider again $\xi = 0$. Then we have an equivariant orbifold resolution

$$\pi : Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$$

where $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement with simplification $\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$.

Definition 3.20. *In this case, we call $\mathfrak{c}(\widetilde{\mathcal{A}}) := \pi^{-1}(0)$ the core of $Y(\widetilde{\mathcal{A}})$.*

Now we will give a toric interpretation of the core $\mathfrak{c}(\widetilde{\mathcal{A}})$. For any $J \subset \{1, ..., n\}$, define the polyhedron

$$P_J := \{x \in \mathbb{R}^{n-d} : x \cdot \boldsymbol{b}_i + r_i \geqslant 0 \text{ if } i \in J \text{ and } x \cdot \boldsymbol{b}_i + r_i \leqslant 0 \text{ if } i \notin J\}.$$

Define

$$\mathfrak{E}_J := \{(\boldsymbol{z}, \boldsymbol{w}) \in T^*\mathbb{C}^n : w_i = 0 \text{ if } i \in J \text{ and } z_i = 0 \text{ if } i \notin J\}$$

and define $\mathfrak{X}_J := \mathfrak{E}_J /\!\!/_\alpha \mathbb{T}^d$, which induce the inclusion

$$\mathfrak{X}_J \hookrightarrow \mu^{-1}(0) /\!\!/_\alpha \mathbb{T}^d = Y(\widetilde{\mathcal{A}}).$$

Theorem 3.21 (Section 6 in [BD00]/ section 3.2 in [Pro04])**.** *In this setting, we have:*

1. *the scheme $\mathfrak{X}_J$ is isomorphic to the toric variety correspond to the weighted polytope P_J;*
2. *we have $\mathfrak{c}(\widetilde{\mathcal{A}}) = \bigcup_{J:P_J \text{ bounded}} \mathfrak{X}_J$, hence $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a union of compact toric varieties glued together along toric subvarieties as prescribed by the combinatorics of the polytopes P_J and their intersections in $\mathbb{R}^{n-d}$.*

Sketch. Note that (1) follows from the surjectivity real moment maps and some classification theorems, see Lemma 3.8 in [Pro04]. For (2), see Proposition 3.11 in [Pro04]. □

Remark 3.22. *This is right even for $\widetilde{\mathcal{A}}$ is not simple.*

Finally we consider some homotopy results.

Theorem 3.23 (6.5 in [BD00] and section 6 in [HS02])**.** *In this setting, we have:*

1. *the core $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a deformation retract of $Y(\widetilde{\mathcal{A}})$;*
2. *the inclusion*
$$Y(\widetilde{\mathcal{A}}) = \mu^{-1}(0) /\!\!/_\alpha \mathbb{T}^d \hookrightarrow T^*\mathbb{C}^n /\!\!/_\alpha \mathbb{T}^d = X(\widetilde{\mathcal{A}})$$
is a homotopy equivalence where $X(\widetilde{\mathcal{A}})$ is the corresponding Lawrence toric variety.

4. Universal Poisson structure of hypertoric varieties

5. Wall-crossing structures and Mukai flops

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Institute of Mathematics, AMSS, Chinese Academy of Sciences, 55 Zhongguancun East Road, Beijing, 100190, China

Email address: liuxiaolong@amss.ac.cn