Algebraic Cycles and Hodge Theory

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Contents

1 Introduction

The reader of course need to be familiar with the book [3] including the basic theory and schemes, cohomology, curves and surfaces. We will also use the intersection theory frequently such as the main contents of [2] or [1] and the reader should familiar with these. Finally we will omit the most basic theory of complex Hodge theory, such as the first seven chapters in [5].

We will focus on the final part of the book [6]. There are three topics of Hodge theory in this book but we just discuss the final part of them. We will also use the Serre's GAGA-principle without explanation.

2 Some Background of Mixed Hodge Theory

2.1 Basic Definition and Properties

Definition 2.1. *A rational (real) mixed Hodge structure of weight n is given by a $\mathbb{Q}$-vector space ($\mathbb{R}$-vector space) H equipped with an increasing filtration W_iH called the weight filtration , and a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$, called the Hodge filtration $F^kH_{\mathbb{C}}$. Such that the induced Hodge filtration on each $\mathrm{Gr}_i^W H$ make $\mathrm{Gr}_i^W H$ to be a Hodge structure of weight $n+i$.*

These filtrations are required to be bounded. Recall that a morphism $\alpha : (U, F) \to (V, G)$ is said to be strict if $\mathrm{Im}\alpha \cap G^pV = \alpha(F^pU)$. It's easy to show that the morphism of rational pure Hodge structures are strict for Hodge filtration (even in type (r, r), see [5] Lemma 7.23).

This is an analogue theory of Hodge decomposition of pure Hodge structures:

Lemma 2.2. *Let (H, W, F) be a mixed Hodge structure. Then there exists a decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

with $H^{p,q} \subset F^pH_{\mathbb{C}} \cap W_{p+q-n}H_{\mathbb{C}}$, such that via the projection $W_{p+q-n}H_{\mathbb{C}} \to \mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}$, the space $H^{p,q}$ can be identified with

$$H^{p,q}(\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}) := F^p\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}} \cap \overline{F^q\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}}.$$

More generally, we have

$$W_iH_{\mathbb{C}} = \bigoplus_{p+q \le n+i} H^{p,q}, F^iH_{\mathbb{C}} = \bigoplus_{p \ge i} H^{p,q}.$$

This decomposition is preserved by the morphisms of mixed Hodge structures.

Proof. This is pure linear algebra, we omit it and refer [6] Lemma 4.21. □

Remark 2.3. *Unlike the pure case, the decomposition above may satisfies $H^{p,q} \neq \overline{H^{p,q}}$, although this does become true after projection to $\mathrm{Gr}^W_{p+q} H_{\mathbb{C}}$.*

Theorem 2.4 (P. Deligne, 1971)**.** *The morphisms*

$$\alpha : (H, W, F) \to (H', W', F')$$

of (rational or real) mixed Hodge structures are strict for the filtrations W and F.

Proof. We will only show the statement for W since the statement for H is similar.

Pick $l' \in \alpha(H_{\mathbb{C}}) \cap W_i H'$ and we write $l' = \alpha(l)$ with $l = \sum_{p,q} l^{p,q}$ by Lemma 2.2. As $l' \in W_i' H_{\mathbb{C}}'$, then $\alpha(l^{p,q}) = 0$ for $p+q > n+i$ by Lemma 2.2 again. Hence $l' \in \alpha(W_i H_{\mathbb{C}})$ and well done. □

2.2 A Classical Example of Mixed Hodge Structure

We consider a smooth complex variety U with a compactification X such that $X \backslash U = D$, a effective normal crossing divisor.

Definition 2.5. *Define a subsheaf $\Omega^k_X(\log D) \subset \Omega^k_X(*D)$ such that $\alpha \in \Gamma(V, \Omega^k_X(\log D))$ if α is a meromorphic differential form on V, holomorphic on $V \backslash D$ and admits a pole of order at most 1 along (each component of) D, and the same holds for $d\alpha$. Hence $d = \partial$ in it and we call the complex $(\Omega^*_X(\log D), \partial)$ the* logarithmic de Rham complex .

Lemma 2.6. *Let $z_1, ..., z_n$ be local coordinates on an open set $V \subset X$, in which $D \cap V$ is defined by the equation $z_1 \cdots z_r = 0$. Then $\Omega^k_X(\log D)|_V$ is a sheaf of free $\mathscr{O}|_U$-modules with basis*

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$$

where $i_s \leq r$, $j_s > r$ and $l + m = k$. In particular, $\Omega^k_X(\log D)$ is locally free.

Proof. Almost trivial, see [5] Lemma 8.16. □

Proposition 2.7. *Let inclusion $j : U \hookrightarrow X$, then we have a canonical inclusion $\Omega^k_X(\log D) \subset j_* \Omega^k_U \subset j_* \mathscr{A}^k_U$ which give us a morphism of complex*

$$\Omega^*_X(\log D) \to j_* \mathscr{A}^*_U.$$

Then this is a quasi-isomorphism. In particular we have

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_X(\log D)).$$

Proof. This is not hard to see and we refer [5] Proposition 8.18. From this we have $\mathbb{H}^k(X, \Omega_X^*(\log D)) \cong \mathbb{H}^k(X, j_*\mathscr{A}_U^*)$. As $\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_U^\infty$-modules which is a resolution of $\mathbb{C}_U$, then $j_*\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_X^\infty$-modules, so it is acyclic and

$$\mathbb{H}^k(X, j_*\mathscr{A}_U^*) \cong H^k\Gamma(X, j_*\mathscr{A}_U^*) = H^k\Gamma(U, \mathscr{A}_U^*) = H^k(U, \mathbb{C}).$$

Hence we get the result. □

For now we will give $H^k(U, \mathbb{Q})$ (or $H^k(U, \mathbb{R})$) a mixed Hodge structure. First we will give two filtrations over $\Omega_X^*(\log D)$.

We define the Hodge filtration over $\Omega_X^*(\log D)$ to be

$$F^p\Omega_X^*(\log D) = \Omega_X^{\geq p}(\log D).$$

For weight filtration, we define $W_l\Omega_X^*(\log D)$ to be

$$W_l\Omega_X^*(\log D) = \begin{cases} \bigwedge^l \Omega_X^1(\log D) \wedge \Omega_X^{*-l}, & 0 \leq l \leq r, \\ 0, & l > r. \end{cases}$$

(We often let $W^k := W_{-k}$)

Now for simplicity, we let the divisor D is simply normal crossing with $D = \bigcup_i D_i$ where each $D_i \subset X$ is a smooth hypersurface, and the intersection of any l hypersurfaces $D_{i_1}, ..., D_{i_l}$ is transverse. We equip I with a total order. We let

$$D^{(k)} := \coprod_{K \subset I, |K|=k} D_K = \coprod_{K \subset I, |K|=k} \bigcap_{i \in K} D_i$$

with inclusions $j_k : D^{(k)} \to X$ and $j_M : D_M \to X$.

Proposition 2.8. *There exists a natural isomorphism*

$$W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

Proof. This morphism defined by Poincaré residue map . Give a local coordinates in $V \subset X$ we define $\mathrm{Res}^V : \Gamma(V, W_k\Omega_X^*(\log D)) \to \Gamma(V, j_{k,*}\Omega_{D^{(k)}}^{*-k})$ as

$$\begin{aligned} \alpha &= \sum_{K \subset \{1,\dots,r\} \subset I, |K| \leq k} \alpha_{K,L} dz_L \wedge \frac{dz_K}{z_K} \\ &\mapsto (\mathrm{Res}^V \alpha)_M = \left((2\pi\sqrt{-1})^k \sum_L \alpha_{M,L} dz_L|_{D_M \cap V} \right)_M . \end{aligned}$$

Note that this annihilates the sections of $W_{k-1}\Omega_X^*(\log D)$ and change coordinates only change the elements in $W_{k-1}\Omega_X^*(\log D)$, Hence we get a well-defined residue map:

$$\alpha : W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

This is an isomorphism is easy to see. We refer [5] Proposition 8.32. □

Now these two filtrations induce two filtrations over $R\Gamma(X, \Omega_X^*(\log D))$, and hence over $H^k(U, \mathbb{C})$ by Proposition 2.7. So the arguments in [5] is far from complete and we need some derived-version filtration of these, such as mixed Hodge complex. We omitted this and we refer section 3.3 in [4].

Theorem 2.9 (P. Deligne, 1971)**.** *The discussion above equip $H^k(U, \mathbb{C})$ a mixed Hodge structure which is independent with X, D.*

Proof. This follows from some analysis of the weight spectral sequence (induced by $W^* = W_{-*}$), here we give a sketch.

By the general theory of spectral sequence, we have

$$_W E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)).$$

By Proposition 2.8 we have $\mathrm{Gr}_W^p \Omega_X^*(\log D) \cong j_{-p,*}\Omega_{D^{(-p)}}^{*+p}$, hence

$$\begin{aligned}\mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)) &= \mathbb{H}^{2p+q}(X, j_{-p,*}\Omega_{D^{(-p)}}^*) \\ &= \mathbb{H}^{2p+q}(D^{(-p)}, \Omega_{D^{(-p)}}^*) = H^{2p+q}(D^{(-p)}, \mathbb{C}).\end{aligned}$$

We can also get that the differential

$$d_1 : \quad \begin{array}{ccc} H^{2p+q}(D^{(-p)}, \mathbb{C}) & \longrightarrow & H^{2p+q+2}(D^{(-p-1)}, \mathbb{C}) \\ \big\downarrow \cong & & \big\downarrow \cong \\ \bigoplus_{|K|=-p} H^{2p+q}(D_K, \mathbb{C}) & \longrightarrow & \bigoplus_{|L|=-p-1} H^{2p+q+2}(D_L, \mathbb{C}) \end{array}$$

has component $d_{1,K}^L$ equal to zero for $L \not\subseteq K$, and equal to $(-1)^{q+s} j_{K,*}^L$ when $K = \{i_1 < \cdots < i_p\}$ and $L = K \backslash \{i_s\}$ where $j_K^L : D_K \to D_L$ (see Proposition 8.34 in [5]). Hence we can deduce any pages of weight spectral sequence! By some analysis we can get the result which omitted, we refer Theorem 3.4.7 and section 3.4.1.5 in [4]. □

3 Cycle Classes and Abel–Jacobi Map

3.1 Cycle Classes and Cycle Map

The case of general complex manifolds with closed analytic subsets

Let X be a $n + r$-dimensional complex manifold with a codimension r closed analytic subset Z, we will associated Z to be a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$.

Lemma 3.1. *If $Y \subset X$ be a closed complex submanifold of codimension k, then the natural map $H^l(X, \mathbb{Z}) \to H^l(X \backslash Y, \mathbb{Z})$ is an isomorphism for $l \leq 2k - 2$.*

Proof. Trivial, just need to look at the long exact sequence induced by the good pair $(X, X\backslash Y)$ and using Thom's isomorphism and the excision theorem. $\square$

Come back to our case, as in algebraic geometry, we can have a filtration

$$\emptyset = Z_{n+1} \subset \cdots \subset Z_0 = Z$$

where $\dim Z_i = n - i$ and $Z_k \backslash Z_{k-1}$ is a closed complex submanifold of dimension $n-k$ in $X\backslash Z_{k-1}$ (see [5] Theorem 11.11 for the proof).

We apply this Lemma to each $X\backslash Z_k \subset X\backslash Z_{k+1}$, we have

$$H^{2r}(X, \mathbb{Z}) \cong H^{2r}(X\backslash Z_1, \mathbb{Z}).$$

Here $Z\backslash Z_1$ is smooth. So we just need to consider the case when Z is a smooth complex submanifold in X!

If Z is a smooth complex submanifold of codimension r in X, then by Thom's isomorphism and the excision theorem, we have the following diagram

$$\begin{array}{ccc} H^{2r}(X, X\backslash Z; \mathbb{Z}) & \xrightarrow{j_Z} & H^{2r}(X, \mathbb{Z}) \\ \big\downarrow{=} & & \\ H^{2r}(X, X\backslash Z; \mathbb{Z}) & \xrightarrow{\cong, T} & H^0(Z, \mathbb{Z}) \end{array}$$

Then we define $[Z] = j_Z(T^{-1}(1)) \in H^{2r}(X, \mathbb{Z})$.

Remark 3.2. *We an also using the most natural way: if $Z = \sum_i n_i Z_i$, we can define $[Z] = \sum_i n_i [Z_i]$ where $[Z_i] = \mathrm{PD}(j_{i,*}([Z_i']_{\mathrm{fund}}))$ and $j_i : Z_i' \to X$ is a resolution of singularity of Z_i, $[Z_i']_{\mathrm{fund}}$ is the fundamental homology class and PD denotes Poincaré duality!*

Here we give some discription using the de Rham cohomology without proof:

Proposition 3.3. *Let $U \subset X$ be a neighbourhood of Z isomorphic to a neighbourhood V of the section in the normal bundle $N_{Z/X}$. Let ω be a closed form of degree k with support in V, satisfying*

$$\int_{\pi^{-1}(z)} \omega = 1$$

where $\pi : V \to Z$ is the projection. Then the form ω is a representative in de Rham cohomology of the class $[Z]$.

Proof. See Lemma 11.14 in [5]. $\square$

The case of compact Kähler manifolds

Let X be a $n+r$-dimensional compact Kähler manifold with a codimension r closed analytic subset Z. We have associated Z to be a cohomology class $[Z] \in H^{2r}(X,\mathbb{Z})$. Now using Hodge decomposition, we will discuss the type of $[Z]$ in $H^{2r}(X,\mathbb{C}) = \bigoplus_{p+q=2r} H^{p,q}(X)$.

Theorem 3.4. *The image of* $[Z]$ *in* $H^{2r}(X,\mathbb{C})$ *lies in* $H^{r,r}(X)$.

Proof. Here we need to use the following two results (for the proof, see [5] Lemma 7.30 and Theorem 11.21, using the de Rham discription we discussed above this is easy to prove):

(i) If Y be a compact Kähler manifold of dimension m, then

$$H^{p,q}(Y) = \left(\bigoplus_{k+l=2m-p-q,(k,l)\neq(m-p,m-q)} H^{k,l}(Y)\right)^{\perp}$$

where the orthogonality is relative to the Poincaré duality on Y.

(ii) (Lelong, 1957) The current $\omega \mapsto \int_{Z_{\text{smooth}}} \omega$ maps to zero on the exact forms. Hence it is an element in $H^{2n}(X,\mathbb{C})^*$. Then this element is equal to the image of $[Z]$under the morphism

$$H^{2r}(X,\mathbb{Z}) \to H^{2r}(X,\mathbb{C}) \to H^{2n}(X,\mathbb{C})^*.$$

By (i), we just need to show that $\int_X [Z] \wedge \alpha = 0$ for any α of type (p,q), $p+q=2n$, $(p,q) \neq (n,n)$. Then this is trivial by (ii). □

The case of complex smooth (quasi-)projective varieties

Let X be a complex smooth quasi-projective variety of dimension n and $Z \in \mathcal{Z}_k(X)$, then we give $[Z] \in H^{2n-2k}(X,\mathbb{Z})$ as above.

Proposition 3.5. *If* $Z \sim_{\text{rat}} 0$, *then* $[Z] = 0 \in H^{2n-2k}(X,\mathbb{Z})$, *hence we give the class map*

$$\text{cl} : \text{CH}_l(X) \to H^{2n-2l}(X,\mathbb{Z}), Z \mapsto [Z].$$

We denote its kernal $\text{CH}_l(X)_{\text{hom}}$.

Proof. WLOG we can assume X is projective. Let $W \subset X$ is of dimension $k+1$ and $\phi \in K(W)^*$, we just need to show $[\tau_*\text{div}(\phi)] = 0$ where $\tau : W' \to W \to X$ be a resolution of singularity of W.

We can easy to see $[\tau_*\text{div}(\phi)] = \tau_*[\text{div}(\phi)]$ where $\tau_* : H^2(W',\mathbb{Z}) \to H^{2n-2k}(X,\mathbb{Z})$ defined by Poincaré duality by Remark 3.2. Hence we just need to show $[\text{div}(\phi)] = 0 \in H^2(W',\mathbb{Z})$. This follows from Lelong's fundamental theorem that $[D] = c_1(\mathscr{O}(D)) \in H^2(W',\mathbb{Z})$. □

Proposition 3.6. *Let $f : X \to Y$ be morphism of smooth quasi-projective varieties.*

(i) If $Z \in \mathrm{CH}^l(X), Z' \in \mathrm{CH}^k(X)$, then

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(Z) \cup \mathrm{cl}(Z') \in H^{2k+2l}(X, \mathbb{Z});$$

(ii) if $Z \in \mathrm{CH}^k(Y)$, then $f^* \mathrm{cl}(Z) = \mathrm{cl}(f^* Z) \in H^{2k}(X, \mathbb{Z})$;

(iii) if f proper and $Z \in \mathrm{CH}^k(X)$, then $f_* \mathrm{cl}(Z) = \mathrm{cl}(f_* Z) \in H^{2k-2\dim X+2\dim Y}(Y, \mathbb{Z})$.

Proof. □

3.2 Hodge Classes and Hodge Conjecture

3.3 The Abel–Jacobi Map

3.4 Deligne Cohomology

Definition 3.7. *Let X be a complex manifold and $p \geq 1$, we define the* Deligne complex *$\mathbb{Z}_D(p)$ is*

$$0 \to \mathbb{Z} \xrightarrow{(2\pi\sqrt{-1})^p} \mathscr{O}_X \xrightarrow{d} \Omega_X \to \cdots \to \Omega_X^{p-1} \to 0.$$

We define the Deligne cohomology *$H_D^k(X, \mathbb{Z}(p)) := \mathbb{H}^k(X, \mathbb{Z}(p))$.*

Remark 3.8. *We have $\mathbb{Z}_D(1) \simeq_{\mathrm{qis}} \mathscr{O}_X^*[-1]$ and $H_D^2(X, \mathbb{Z}(1)) = H^1(X, \mathscr{O}_X^*)$.*

Proposition 3.9. *If X is a compact Kähler manifold, then there exists a long exact sequence*

$$\begin{aligned} \cdots \to H_D^k(X, \mathbb{Z}(p)) &\to H^k(X, \mathbb{Z}) \\ &\to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) \to H^{k+1} D(X, \mathbb{Z}(p)) \to \cdots. \end{aligned}$$

Proof. First we consider

$$0 \to \Omega_X^{\leq p-1}[-1] \to \mathbb{Z}_D(p) \to \mathbb{Z} \to 0$$

which induce a long exact sequence and we see that we just need to show $\mathbb{H}^k(X, \Omega_X^{\leq p-1}) = H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C})$. From the basic fact of Hodge structure (e.g. Proposition 7.5 in [5]) $\mathbb{H}^k(X, \Omega_X^{\geq p}) = F^p H^k(X, \mathbb{C})$ and the exact sequence

$$0 \to \Omega_X^{\geq p} \to \Omega_X^* \to \Omega_X^{\leq p-1} \to 0$$

we can get the result directly. □

Corollary 3.10. *In this case, we have exact sequence*

$$0 \to J^{2p-1}(X) \to H_D^{2p}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^{2p}(X, \mathbb{Z}) \to 0.$$

Proof. This follows directly from the $k = 2p$ in theorem and the fact

$$\mathrm{Hdg}^{2p}(X, \mathbb{Z}) = \ker(H^{2p}(X, \mathbb{Z}) \to H^k(X, \mathbb{C})/F^p H^{2p}(X)).$$

Well done. $\square$

Here we give another method to compute the Deligne cohomology $H_D^{2p}(X, \mathbb{Z}(p))$.

Definition 3.11. *Let* $\Xi^l_{\mathrm{diff}}(X)$ *be a subgroup of* $\mathrm{Hom}(Z_l^{\mathrm{diff}}, \mathbb{R}/\mathbb{Z})$ *consist of*

4 Mumford's Theorem and its Generalizations

5 The Bloch Conjecture and its Generalizations

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