

Varieties of Minimal Rational Tangents on the Fano Varieties

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Preface

Chapter 1

Basic Theory of Rational Curves

The main results here we follow the famous book [6].

1.1 Hilbert Schemes and Chow Schemes

1.1.1 Hilbert Schemes, a Basic Introduction

Definition 1.1.1. Let X be an S -scheme, we define the Hilbert functor $\mathcal{H}ilb_{X/S}$ sends an S -scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z .

Fix a Polynomial P and a relative ample line bundle $\mathcal{O}(1)$, we can define $\mathcal{H}ilb_{X/S}^P$ sends an S -scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z with Hilbert Polynomial P .

Theorem 1.1.2 (Grothendieck). Let S be a noetherian scheme, let $X \rightarrow S$ be a projective morphism, and \mathcal{L} a relatively very ample line bundle on X . Then for any polynomial P , the Hilbert functor $\mathcal{H}ilb_{X/S}^P$ is representable by a projective S -scheme $\text{Hilb}_{X/S}^P$. We also have $\text{Hilb}_{X/S} = \coprod_P \text{Hilb}_{X/S}^P$.

Proof. Note that this notion of projectivity is much general than [5], but is the same when $S = \text{Spec } k$. The proof is to embed it into Grassmannian. The original proof in [4] and we also refer [8], [6] and [3]. \square

Remark 1.1.3. In [2] we can remove the noetherian hypothesis, by instead assuming strong (quasi-)projectivity of $X \rightarrow S$. So also [1].

Example 1.1.1. Some examples and interesting results:

(a) We have $\text{Hilb}_{X/S}^1 = X/S$.

(b) Let C be a curve over a field k , then

$$\mathrm{Hilb}_{C/k}^m \cong S^m C := \underbrace{C \times \cdots \times C}_m / \mathfrak{S}_m.$$

Hence if C smooth, so is $\mathrm{Hilb}_{C/k}^m$. See also [3] Theorem 7.2.3(1) and Proposition 7.3.3.

(c) Let S be a smooth surface over a field k , then $\mathrm{Hilb}_{S/k}^m$ is also smooth of dimension $2m$ and hence $\mathrm{Hilb}_{S/k}^m \rightarrow S^m X$ (we will see this later for general settings) is a resolution of singularities. Note that $S^m X$ is smooth if and only if X is smooth and $\dim X = 1$ or $m < 2$. See [3] Theorem 7.2.3(2) and Theorem 7.3.4.

(d) Let X be a nonsingular variety. Then $\mathrm{Hilb}_{X/k}^m$ is nonsingular for $m \leq 3$. Moreover, for any nonsingular 3-fold the scheme $\mathrm{Hilb}_{X/k}^4$ is singular. See [3] Remark 7.2.5 and 7.2.6.

(e) Let \mathcal{E} be a vector bundle of rank $m+1$ over S and let $P_d(n) = \binom{m+n}{m} - \binom{m+n-d}{m}$, then

$$\mathrm{Hilb}_{\mathbb{P}(\mathcal{E})/S}^{P_d} \cong \mathbb{P}((\mathrm{Sym}^d \mathcal{E})^\vee).$$

(f) Let $Z \rightarrow S$, we have $\mathrm{Hilb}_{X \times_S Z/Z} \cong \mathrm{Hilb}_{X/S} \times_S Z$.

(g) **Hartshorne's Connectedness Theorem:** for every connected noetherian scheme S , $\mathrm{Hilb}_{\mathbb{P}_S^n}^P$ is connected.

(h) Let X be a connected variety over k , then $\mathrm{Hilb}_{X/k}^n$ is connected for all $n > 0$.

(i) **Murphy's Law:** It has many singularities, that is, for every scheme X finite type over \mathbb{Z} and point $x \in X$, there exists a point $q \in \mathrm{Hilb}_{\mathbb{P}^n/k}^P$ of some Hilbert scheme and an isomorphism

$$\widehat{\mathcal{O}}_{X,p}[[x_1, \dots, x_s]] \cong \widehat{\mathcal{O}}_{\mathrm{Hilb}_{\mathbb{P}^n/k,q}^P}[[y_1, \dots, y_t]].$$

See [11]. In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in \mathbb{P}^n for some n . It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

(j) In [10] they gave a full classification of the situation where $\mathrm{Hilb}_{\mathbb{P}^n/k}^P$ smooth.

Definition 1.1.4. Let $X/S, Y/S$ are S -schemes, then we have a functor $\mathcal{H}om_S(X, Y)$ send S -scheme T into a set of T -morphisms $X \times_S T \rightarrow Y \times_S T$.

For a subscheme $B \subset X$ proper over S and $g : B \rightarrow Y$, we have a functor $\mathcal{H}om_S(X, Y; g)$ send S -scheme T into a set of T -morphisms $X \times_S T \rightarrow Y \times_S T$ such that $f|_{B \times_S T} = g \times_S \mathrm{id}_T$.

Proposition 1.1.5. *If X/S and Y/S are both projective over S and X is flat over S , then $\mathcal{H}om_S(X, Y)$ represented by an open subscheme $\text{Hom}_S(X, Y) \subset \text{Hilb}_{X \times_S Y/S}$.*

Proof. Any $X \times_S T \rightarrow Y \times_S T$ correspond to its graph which is a closed immersion $\Gamma : X \times_S T \rightarrow X \times_S Y \times_S T$. As X is flat over S , then $X \times_S T$ is flat over T . Hence we get a morphism $\text{Hom}_S(X, Y) \rightarrow \text{Hilb}_{X \times_S Y/S}$. We omit the more details and refer Theorem I.1.10 in [6]. \square

Proposition 1.1.6. *If X/S and Y/S are both projective over S and X, B are both flat over S , then $\mathcal{H}om_S(X, Y; g)$ represented by a subscheme $\text{Hom}_S(X, Y; g) \subset \text{Hom}_S(X, Y)$.*

Proof. Consider the restriction map $R : \text{Hom}_S(X, Y) \rightarrow \text{Hom}_S(B, Y)$, then $g : B \rightarrow Y$ gives a section $G : S \rightarrow \text{Hom}_S(B, Y)$. Hence $\text{Hom}_S(X, Y; g) := R^{-1}(G(S)) \subset \text{Hom}_S(X, Y)$ represents $\mathcal{H}om_S(X, Y; g)$. \square

Now we state the deformation theory of Hilbert schemes. We only consider the simpler case that all schemes over a field k . For general case we refer Section 1.2 in [6].

Theorem 1.1.7. *Let Y be a projective scheme over a field k and $Z \subset Y$ is a subscheme. Then*

(a) *We have*

$$T_{[Z]} \text{Hilb}_Y \cong \text{Hom}_Z(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z).$$

(b) *The dimension of every irreducible components of Hilb_Y at $[Z]$ is at least*

$$\dim \text{Hom}_Z(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) - \dim \text{Ext}_Z^1(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z).$$

Proof. See Theorem I.2.8 in [6]. For family case we refer Theorem I.2.15 in [6]. \square

Corollary 1.1.8. *Let X, Y are projective varieties over a field k with a morphism $f : X \rightarrow Y$. Let Y is smooth over k . Then*

(a) *We have*

$$T_{[f]} \text{Hom}_k(X, Y) \cong \text{Hom}_X(f^* \Omega_Y^1, \mathcal{O}_X).$$

(b) *The dimension of every irreducible components of $\text{Hom}_k(X, Y)$ at $[f]$ is at least*

$$\dim \text{Hom}_X(f^* \Omega_Y^1, \mathcal{O}_X) - \dim \text{Ext}_X^1(f^* \Omega_Y^1, \mathcal{O}_X).$$

Proof. Let $Z \subset X \times_k Y$ be the graph of f , we claim that $\mathcal{I}_Z/\mathcal{I}_Z^2 \cong f^* \Omega_Y^1$. Indeed we have an exact sequence $\mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow \Omega_{X \times_k Y}^1|_Z \rightarrow \Omega_Z^1 \rightarrow 0$. This is split by $\mathcal{O}_Z \cong \mathcal{O}_X \xrightarrow{(\text{id}_X, 1)} \mathcal{O}_{X \times_k Y}$. Then we can show the claim. Hence the results follows from Theorem 1.1.7. The family version we refer Theorem I.2.17 in [6]. \square

1.1.2 Chow Schemes, a Basic Introduction

Here we only consider the schemes over a field k such that $\text{char}(k) = 0$. The positive characteristic case is very complicated and we refer Section I.4 in [6].

Definition 1.1.9. Let $g_i : U_i \rightarrow W$ be a proper morphism of schemes over W . Assume that W is reduced and U_i is irreducible. By generic flatness there is an open subset $W_i \subset g_i(U_i) \subset W$ such that g_i is flat of relative dimension d over W_i . Let $T = \text{Spec } \Delta$ be the spectrum of a DVR Δ and $h : T \rightarrow W$ a morphism such that $h(T_g) \in W_i$ and $h(T_0) = w \in W$. Let $h^*U_i = U_i \times_h T$ and $\mathcal{J} \subset \mathcal{O}_{h^*U_i}$ the ideal of those sections whose support is contained in the special fiber of $h^*U_i \rightarrow T$. Let $(U_i)'_T := \text{Spec}_T \mathcal{O}_{h^*U_i} / \mathcal{J}$ which is flat over T . Then we let $[Z_0]$ be the fundamental cycle of the central fiber of $(U_i)'_T \rightarrow T$, and define

$$\lim_{h \rightarrow w} (U_i/U) := [Z_0] \in Z_d(g_i^{-1}(w) \times_{\kappa(w)} T_0)$$

which is called the cycle theoretic fiber of g_i at w along h .

Definition 1.1.10. A well defined family of d -dimensional proper algebraic cycles over W is a pair $(g : U \rightarrow W)$ satisfying the following properties:

- (a) There is a reduced scheme $\text{supp } U$ with irreducible components U_i such that $U = \sum_i m_i [U_i]$ is an algebraic cycle.
- (b) W is a reduced scheme and $g : \text{supp } U \rightarrow W$ is a proper morphism.
- (c) Let $g_i := g|_{U_i}$. Then every g_i maps onto an irreducible component of W and every fiber of g_i is either empty or has dimension d . In particular there is a dense open subset $W_0 \subset W$ such that every g_i is flat over W_0 .
- (d) For every $w \in W$ there is a cycle $g^{[-1]}(w) \in Z_d(g^{-1}(w))$ such that for any $h : T \rightarrow W$ of spectrum of DVR such that $h(T_0) = w$ and $h(T_g) \in W_0$ we have

$$g^{[-1]}(w) =_{\text{ess}} \sum_i m_i \lim_{h \rightarrow w} (U_i/W).$$

That is, both two cycles from a single cycle of $Z_d(g^{-1}(w))$.

Definition 1.1.11. Let X be a scheme over S . A well defined family of proper algebraic cycles of X/S over W/S is a pair $(g : U/S \rightarrow W/S)$ satisfying the following properties:

- (a) $\text{supp } U$ is a closed subscheme of $X \times_S W$ and g is the natural projection morphism.
- (b) $(g : U \rightarrow W)$ is a well defined family of d -dimensional proper algebraic cycles over W for some d .

Definition 1.1.12 (Chow Schemes of Characteristic Zero). *Let X/S and we define a functor $\mathcal{C}how_{X/S}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$.*

Let a relative ample line bundle $\mathcal{O}(1)$, we can define $\mathcal{C}how_{X/S}^{d,d'}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$ which is of dimension d and degree d' .

Theorem 1.1.13. *Let X/S be a scheme, projective over S and $\mathcal{O}(1)$ relatively ample. Then the functor $\mathcal{C}how_{X/S}^{d,d'}$ is representable by a semi-normal and projective S -scheme $\text{Chow}_{X/S}^{d,d'}$. We also have $\text{Chow}_{X/S} = \coprod_{d,d'} \text{Chow}_{X/S}^{d,d'}$.*

Proof. Very complicated, we refer Theorem I.3.21 in [6]. □

Example 1.1.2. *Let X be a semi-normal variety, then $\text{Chow}_{X/k}^{0,m} \cong S^m X$.*

Remark 1.1.14. *Note that we have the Hilbert-Chow morphism in such zero-dimensional case. See Section I.6 in [6].*

1.1.3 Small Applications to Curves

For more applications we refer Section II.1 in [6]. Here we only need some easy case. We assume over a field k .

Theorem 1.1.15. *Let C be a proper curve and $f : C \rightarrow Y$ a morphism to a smooth variety Y of dimension n . Then*

$$\dim_{[f]} \text{Hom}(C, Y) \geq -C \cdot K_Y + n\chi(\mathcal{O}_C).$$

*And equality holds if $H^1(C, f^*T_Y) = 0$, in this case it is smooth at $[f]$.*

Proof. By Corollary 1.1.8(b) we have

$$\begin{aligned} \dim_{[f]} \text{Hom}(C, Y) &\geq \dim \text{Hom}_X(f^*\Omega_Y^1, \mathcal{O}_X) - \dim \text{Ext}_X^1(f^*\Omega_Y^1, \mathcal{O}_X) \\ &= h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y) \\ &= \deg f^*T_Y + n\chi(\mathcal{O}_C) \end{aligned}$$

by Riemann-Roch theorem. The final statement follows from Corollary 1.1.8(a). □

Proposition 1.1.16. *Assume that X/S is flat, B/S is flat and finite of degree m and Y/S is smooth of relative dimension n . Then $\dim \text{Hom}(X, Y; g) \geq \dim \text{Hom}(X, Y) - kn$.*

Proof. Let $p : B \rightarrow S$ be the projection. By Corollary 1.1.8 we find that $\text{Hom}(B, Y)$ is smooth over S of relative dimension rank kn . Thus $g(S) \subset \text{Hom}(B, Y)$ is locally defined by kn equations. Pulling back these equations by R we obtain local defining equations. □

Theorem 1.1.17. *Let C be a projective curve over k and Y a smooth variety over k . Let $B \subset C$ be a closed subscheme which is finite over k . Assume that C is smooth along B . Let $g : B \rightarrow Y$ be a morphism. Then*

(a) *We have*

$$T_{[f]} \operatorname{Hom}(C, Y; g) \cong H^0(C, f^*T_Y \otimes \mathcal{I}_B).$$

(b) *The dimension of every irreducible component of $\operatorname{Hom}(C, Y; g)$ at $[f]$ is at least*

$$h^0(C, f^*T_Y \otimes \mathcal{I}_B) - h^1(C, f^*T_Y \otimes \mathcal{I}_B).$$

Proof. The original proof we refer [7]. A simple case of family version we refer Theorem II.1.7 in [6]. Here we assume k is algebraically closed. Here $\mathcal{I}_B = \mathcal{O}_C(-s_1 - \dots - s_m)$.

Let $X_0 := C \times_k Y$ and let $\gamma_0 : C \cong \Gamma_0 \subset X_0$ be the graph of f . Let $\pi_1 : X_1 := \operatorname{Bl}_{\{s_1\}} X_0 \rightarrow X_0$ and Γ_1 be the strict transform of Γ_0 . Let $\gamma_1 : C \cong \Gamma_1 \subset X_1$ as C is smooth at s_1 . Repeat the process and finally we get $\pi_m : X_m := \operatorname{Bl}_{\{s_m\}} X_{m-1} \rightarrow X_{m-1}$ and Γ_m be the strict transform of Γ_{m-1} . Let $\gamma_m : C \cong \Gamma_m \subset X_m$. Then we have $\gamma_0^*(\mathcal{I}_{\Gamma_0}/\mathcal{I}_{\Gamma_0}^2) \cong f^*\Omega_Y^1$ and $\gamma_{i+1}^*(\mathcal{I}_{\Gamma_{i+1}}/\mathcal{I}_{\Gamma_{i+1}}^2) \cong \gamma_i^*(\mathcal{I}_{\Gamma_i}/\mathcal{I}_{\Gamma_i}^2) \otimes \mathcal{O}_C(-s_{i+1})$. Hence we get $\gamma_m^*(\mathcal{I}_{\Gamma_m}/\mathcal{I}_{\Gamma_m}^2) \cong f^*\Omega_Y^1 \otimes \mathcal{I}_B$.

Now we claim that there is an open neighborhood $[\Gamma_m] \in U \subset \operatorname{Hilb}_{X_m}$ such that $\operatorname{Hom}(C, Y; g) \cong U$. Indeed, let $U \subset \operatorname{Hilb}_{X_m}$ be the open set parametrizing those 1-cycles D for which the projection $D \rightarrow C$ is an isomorphism. This is open by Lemma I.1.10.1 in [6].

First, the universal family of U is contained in $\operatorname{Hom}(C, Y; g)(U)$. Conversely consider $[p_0 : C \times R \rightarrow Y \times R] \in \operatorname{Hom}(C, Y; g)(R)$. Let its graph is $G_0 \subset X_0 \times R$. As $\{s_1\} \times R \subset G_0$ and $G_0 \rightarrow R$ smooth along $\{s_1\} \times R$, we let $G_1 \subset X_1 \times R$ be the strict transform of G_0 . Then $G_1 \cong G_0 \cong C \times R$. Repeat the process and finally we get $X_m \times R \supset C \times R \cong G_m \in \operatorname{Hilb}_{X_m}(R)$. Hence this give the isomorphism $\operatorname{Hom}(C, Y; g) \cong U$. Hence by Theorem 1.1.7 and we get the result. \square

1.2 Families of Rational Curves

1.3 Free and Minimal Rational Curves

1.4 Bend and Break

1.5 Application I: Basic Theory of Fano Manifolds

[9]

1.6 Application II: Boundedness of Fano Manifolds

1.7 Application III: Hartshorne's Conjecture

Chapter 2

Varieties of Minimal Rational Tangents

We will assume the base field is \mathbb{C} .

2.1 Basic Properties

2.2 Examples of VMRT

2.3 Distributions and Its Properties

2.4 Cartan-Fubini Type Extension Theorem

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