

Note for the Virtual Fundamental Class

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1 Introduction

We will follow [BF97][AB84][GP99] and we will also use [Ric22].

We need [Har77][Ful98][EH16].

Here we will consider $\mathbb{P}(-) = \mathbf{Proj} \operatorname{Sym}(-)^\vee$ for bundles and the vector bundle is both space and sheaf via $\mathbf{Spec} \operatorname{Sym}(-)^\vee$. For a cone $C = \mathbf{Spec}_X \mathcal{S}^*$, we define $\mathbb{P}(C) := \mathbf{Proj}_X \mathcal{S}^*$ and $\mathbb{P}(C \oplus \mathcal{O}) := \mathbf{Proj}_X \mathcal{S}^*[z]$ which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

2 Review of Basic Intersection Theory

We will follow [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field k .

2.1 Basic Facts of Refined Gysin Pullback

Here we will follow Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

Definition 2.1 (Intersection Product). *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k . Consider the cartesian diagram*

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let \mathcal{I} be the ideal of i and \mathcal{J} be the ideal of j , then we have surjection

$$\bigoplus_n f^*(\mathcal{I}^n / \mathcal{I}^{n+1}) \rightarrow \bigoplus_n \mathcal{J}^n / \mathcal{J}^{n+1} \rightarrow 0$$

which induce embedding $C_{W/V} \hookrightarrow g^*N_{X/Y}$. Note that $C_{W/V}$ is also a scheme of pure dimension k since $\mathbb{P}(C_{W/V} \oplus \mathcal{O})$ is the exceptional divisor of $\text{Bl}_W(Y \times \mathbb{A}^1)$. Let $0 : W \rightarrow g^*N_{X/Y}$ be the zero-section of $\pi : g^*N_{X/Y} \rightarrow W$, then we define

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \text{CH}_{k-d}(W)$$

as the intersection class.

Proposition 2.2. *Consider the situation of Definition 2.1.*

- (a) *We have $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$.*
- (b) *Let \mathcal{Q} be the universal quotient bundle of $q : \mathbb{P}(g^*N_{X/Y} \oplus \mathcal{O}) \rightarrow W$, then*

$$X \cdot V = q_*(c_d(\mathcal{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathcal{O})]).$$

- (c) *If $j : W \hookrightarrow V$ is a regular embedding of codimension d' , then $X \cdot V = c_{d-d'}(g^*N_{X/Y}/N_{W/V}) \cap [W]$.*

Proof. Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98]. \square

Definition 2.3 (Refined Gysin Pullback). *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick $f : Y' \rightarrow Y$ be a morphism. Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xhookrightarrow{j} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Then we define $i^! : \mathbf{Z}_k Y' \rightarrow \mathbf{CH}_{k-d} X'$ as $\sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i$. Now $i^!$ can be decomposed as:

$$i^! : \mathbf{Z}_k Y' \xrightarrow{\sigma} \mathbf{Z}_k C_{X'/Y'} \rightarrow \mathbf{CH}_k(g^* N_{X/Y}) \xrightarrow{0^*} \mathbf{CH}_{k-d} X'$$

where $\sigma : \mathbf{Z}_k Y' \rightarrow \mathbf{Z}_k C_{X'/Y'}$ given by $[V] \mapsto [C_{V \cap X'/V}]$. By the technique of deformation to the normal cone, this can be descend to the Chow-group level as $\sigma : \mathbf{CH}_k Y' \rightarrow \mathbf{CH}_k C_{X'/Y'}$ (see Proposition 5.2 in [Ful98]) which is called the *specialization to the normal cone*. Hence this induce the refined Gysin pullback

$$i^! : \mathbf{CH}_k Y' \rightarrow \mathbf{CH}_{k-d} X', \quad \sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i.$$

Proposition 2.4. *Consider the situation of Definition 2.3. Consider*

$$\begin{array}{ccc} X'' & \xhookrightarrow{i''} & Y'' \\ q \downarrow & \lrcorner & p \downarrow \\ X' & \xhookrightarrow{i'} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

- (a) *If p proper and $\alpha \in \mathbf{CH}_k(Y'')$, then $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathbf{CH}_{k-d}(X')$.*
- (b) *If p is flat of relative dimension n and $\alpha \in \mathbf{CH}_k(Y'')$, then $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathbf{CH}_{k+n-d}(X')$.*
- (c) *If i' is also a regular embedding of codimension d and $\alpha \in \mathbf{CH}_k(Y'')$, then $i^! \alpha = (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X')$.*
- (d) *If i' is a regular embedding of codimension d' , then for $\alpha \in \mathbf{CH}_k(Y'')$ we have*

$$i^!(\alpha) = c_{d-d'}(q^*(g^* N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X').$$

We call $g^ N_{X/Y}/N_{X'/Y'}$ the excess normal bundle.*

(e) Let F be any vector bundle on Y' , then for $\alpha \in \mathbf{CH}_k(Y'')$ we have

$$i^!(c_m(F) \cap \alpha) = c_m((i')^*F) \cap i^!(\alpha) \in \mathbf{CH}_{k-d-m}(X').$$

Proof. See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98]. \square

Corollary 2.5. Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d , then

$$i^*i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathbf{CH}_*(X).$$

Proof. By Proposition 2.4(d) directly. \square

Proposition 2.6. The refined Gysin pullback have the following properties.

(a) Let $i : X \hookrightarrow Y$ and $j : S \hookrightarrow T$ are regular embeddings of codimension d, e , respectively. Consider cartesian:

$$\begin{array}{ccccc} X'' & \hookrightarrow & Y'' & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow j' & \lrcorner & \downarrow j \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{g} & T \\ \downarrow & \lrcorner & \downarrow f & & \\ X & \xrightarrow{i} & Y & & \end{array}$$

Then for any $\alpha \in \mathbf{CH}_k(Y'')$, we have

$$j^!i^!(\alpha) = i^!j^!(\alpha) \in \mathbf{CH}_{k-d-e}(X'').$$

(b) Let $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are regular embeddings of codimension d, e , respectively. Consider cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & Z \end{array}$$

Then ji is a regular embedding of codimension $d + e$ and for all $\alpha \in \mathbf{CH}_k(Z')$ we have

$$(ji)^!(\alpha) = i^!j^!(\alpha) \in \mathbf{CH}_{k-d-e}(X').$$

Proof. See Theorem 6.4 and Theorem 6.5 in [Ful98]. \square

Proposition 2.7. *Consider cartesian:*

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \end{array}$$

- (a) *If i is a regular embedding of codimension d and p and pi are flat of relative dimension $n, n-d$, respectively. Then i' is a regular embedding of codimension d and $p', p'i'$ are flat, and for $\alpha \in \text{CH}_k(Z')$ we have*

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i'^!((p')^*\alpha).$$

- (b) *If i is a regular embedding of codimension d and p is smooth of relative dimension n , and pi is a regular embedding of codimension $d-n$. Then for $\alpha \in \text{CH}_k(Z')$ we have*

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

Proof. See Proposition 6.5 in [Ful98]. □

Remark 2.8. *Some remarks.*

- (a) *For local complete intersection morphism $f : X \rightarrow Y$, we can decompose it into $f : X \xrightarrow{i} P \xrightarrow{p} Y$ where i is a closed regular embedding of constant codimension and p is smooth of constant relative dimension. Then we can define $f^! := i^!(p)^*$. See Section 6.6 in [Ful98] for more properties.*
- (b) *If Y is nonsingular of dimension n , then we can define the following intersection product: Let $f : X \rightarrow Y$ and $p : X' \rightarrow X$ and $q : Y' \rightarrow Y$. Let $x \in \text{CH}_k(X')$ and $y \in \text{CH}_l(Y')$, consider the cartesian*

$$\begin{array}{ccc} X' \times_Y Y' & \longrightarrow & X' \times Y' \\ \downarrow & \lrcorner & \downarrow p \times q \\ X & \xrightarrow{\gamma_f} & X \times Y \end{array}$$

and define $x \cdot_f y := \gamma_f^!(x \times y) \in \text{CH}_{k+l-n}(X' \times_Y Y')$.

So when $x, y \in \text{CH}_(Y)$, then let $X = Y$ and $X' = |x|, Y' = |y|$, then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case $\text{CH}_*(Y)$ is a ring which is called *Chow ring*.*

Finally we will discuss something about equivalence and supportness.

Definition 2.9. Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k . Consider the cartesian diagram

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let C_1, \dots, C_r be the irreducible components of $C_{W/V}$, then $[C_{W/V}] = \sum_{i=1}^r m_i [C_i]$. Let $Z_i = \pi(C_i)$ where $\pi : g^*N_{X/Y} \rightarrow W$ and we call them the **distinguished varieties** of the intersection of V by X . Let $N_i := (g^*N_{X/Y})|_{Z_i}$ and let $0_i : Z_i \rightarrow N_i$ be the zero-sections. Let $\alpha_i := 0_i^*[C_i] \in \mathbf{CH}_{k-d}(Z_i)$ and hence we have $X \cdot V = \sum_{i=1}^r m_i \alpha_i \in \mathbf{CH}_{k-d}(W)$.

Pick any closed set $S \subset W$, we define

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i \alpha_i \in \mathbf{CH}_{k-d}(S)$$

as the part of $X \cdot V$ supported on S .

Definition 2.10. Let $X_i \hookrightarrow Y$ be closed regular embeddings of codimension d_i . Let $V \subset Y$ be a k -dimensional subvariety. Consider

$$\begin{array}{ccc} \bigcap_i X_i \cap V & \xhookrightarrow{\quad} & V \\ \downarrow & \lrcorner & \downarrow \delta \\ X_1 \times \dots \times X_r & \xhookrightarrow{\quad} & Y \times \dots \times Y \end{array}$$

Then we can get $X_1 \cdot \dots \cdot X_r \cdot V \in \mathbf{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$.

Let Z be a connected component of $\bigcap_i X_i \cap V$, we will consider

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z \in \mathbf{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

Proposition 2.11. As in the previous situation, we have

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cap s(Z, V) \right\}_{\dim V - \sum_i d_i}.$$

If $Z \hookrightarrow V$ is a regular embedding, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If V, Z are both non-singular, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

Proof. See Proposition 9.1.1 in [Ful98]. \square

2.2 Localized Chern Class

Here we will follow Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

Definition 2.12. Let $E \rightarrow X$ be a vector bundle of rank e over a purely n -dimensional scheme X . Let $s : X \rightarrow E$ be a section, consider the cartesian

$$\begin{array}{ccc} Z(s) & \longrightarrow & X \\ i \downarrow & \lrcorner & s \downarrow \\ X & \xrightarrow{0} & E \end{array}$$

with zero-section $0 : X \rightarrow E$ which is a regular section by trivial reason. We define

$$c_{\text{loc}}(E, s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \text{CH}_{n-e}(Z(s))$$

be the localized (top) Chern class of E with respect to s .

Proposition 2.13. Consider the situation of Definition 2.12.

- (a) We have $i_*(c_{\text{loc}}(E, s)) = c_e(E) \cap [X]$.
- (b) Each irreducible component of $Z(s)$ has codimension at most e in X . If $\text{codim}_{Z(s)} X = e$, then $c_{\text{loc}}(E, s)$ is a positive cycle whose support is $Z(s)$.
- (c) If s is a regular section, then $c_{\text{loc}}(E, s) = [Z(s)]$.
- (d) Let $f : X' \rightarrow X$ be a morphism, $s' = f^*s$ be a induced section of f^*E . Let $g : Z(s') \rightarrow Z(s)$ be the induced morphism.
 - (d1) If f flat, then $g^*c_{\text{loc}}(E, s) = c_{\text{loc}}(f^*E, s')$.
 - (d2) If f is proper of varieties, then $g_*c_{\text{loc}}(f^*E, s') = \deg(X'/X)c_{\text{loc}}(E, s)$.

Proof. For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_*0^!([X]) = 0^*s_*[X] = s^*s_*[X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{array}{ccc}
Z(s') & \longrightarrow & X' \\
\downarrow & \lrcorner & \downarrow s' \\
X' & \xrightarrow{0_{f^*E}} & f^*E \\
\downarrow & \lrcorner & \downarrow \\
X & \xrightarrow{0_E} & E
\end{array}$$

and Proposition 2.4. □

3 Foundations of Virtual Fundamental Class

We will follows [BF97]. Here an algebraic stack (or Artin stack) over a field k is assumed to be quasi-separated and locally of finite type over k .

3.1 About Cones

We will let X be a Deligne-Mumford stack now.

Definition 3.1. *Let X be a DM-stack.*

- (a) *We call an affine X -scheme $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ is a **cone over X** if the quasi-coherent algebra \mathcal{S} is graded as $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$ with $\mathcal{S}^0 = \mathcal{O}_X$ and \mathcal{S}^1 is coherent and \mathcal{S} is generated by \mathcal{S}^1 .*
- (b) *A **morphism of cones over X** is an X -morphism induced by a graded morphism of graded sheaves of \mathcal{O}_X -algebras. A **closed subcone** is the image of a closed immersion of cones.*

Remark 3.2. (a) *The fiber product of cones over X is still a cone over X .*

- (b) *For every cone $C \rightarrow X$, it has a zero section $0 : X \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}^0$.*
- (c) *For every cone $C \rightarrow X$, the grade induce a \mathbb{G}_m -action $\mathbb{G}_m \times C = \underline{\mathrm{Spec}}_X \mathcal{S}[t, t^{-1}] \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}[t, t^{-1}]$ via $s_0 + \cdots s_d \mapsto \sum_i a_i t^i$ where $s_i \in \mathcal{S}^i$. Since no negative power of t occurs, we can in fact replace \mathbb{G}_m by \mathbb{A}^1 . So we have the \mathbb{A}^1 -action $\gamma : \mathbb{A}^1 \times C \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}[x]$ via $\mathcal{S}^i \ni s \mapsto sx^i$. Note that here \mathbb{A}^1 is not a*

group scheme and the **action** here, as expected, to be the commutativity of the following diagrams:

$$\begin{array}{ccc}
C & \xrightarrow{(1, \text{id})/(0, \text{id})} & \mathbb{A}^1 \times C \\
& \searrow \text{id}/0 & \downarrow \gamma \\
& & C
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{A}^1 \times \mathbb{A}^1 \times C & \xrightarrow{\text{id} \times \gamma} & \mathbb{A}^1 \times \mathbb{A}^1 \times C \\
m \times \text{id} \downarrow & & \downarrow \gamma \\
\mathbb{A}^1 \times C & \xrightarrow{\gamma} & C
\end{array}$$

where $m(x, y) = xy$.

- (d) So a morphism of cones $f : C \rightarrow D$ over X is just the \mathbb{A}^1 -equivariant X -morphism respecting the zero section, that is, the following commutativity of the diagram:

$$\begin{array}{ccccc}
\mathbb{A}^1 \times C & \longrightarrow & C & \xleftarrow{0_C} & X \\
\text{id} \times f \downarrow & & f \downarrow & \nearrow & \searrow 0_D \\
\mathbb{A}^1 \times D & \longrightarrow & D & &
\end{array}$$

Definition 3.3. Let \mathcal{F} be a coherent sheaf of X , then we can define $C(\mathcal{F}) := \underline{\text{Spec}}_X \text{Sym}(\mathcal{F})$ which is a group scheme over X since it can be represented as $C(\mathcal{F})(T) = \text{Hom}(\mathcal{F}_T, \mathcal{O}_T)$. We call a cone of this form is an **abelian cone** over X .

Remark 3.4. (a) A fibered product of abelian cones is an abelian cone.

(b) A vector bundle $E = \underline{\text{Spec}}_X \text{Sym}(\mathcal{E}^\vee)$ is a special case.

(c) Any cone $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$ is canonically a closed subcone of an abelian cone $A(C) = \underline{\text{Spec}}_X \text{Sym} \mathcal{S}^1$, called the **abelian hull** of C . The abelian hull is a vector bundle if and only if \mathcal{S}^1 is locally free.

(d) The **abelianization** $C \mapsto A(C)$ is a functor has the forgetful functor as a right adjoint. So we have

$$\text{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \text{Hom}_{\mathbf{Cone}_X}(C, A).$$

(e) Let \mathbf{Alg}_X^o as the category of quasicoherent graded \mathcal{O}_X -algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:

$$\begin{array}{ccc}
\mathbf{Alg}_X^o & \xrightarrow{\underline{\text{Spec}}_X} & \mathbf{Cone}_X^{\text{op}} \\
\text{Sym} \uparrow & & \uparrow \\
\mathbf{LocFree}_X & \xrightarrow{\underline{\text{Spec}}_X \text{Sym}(-)^\vee} & \mathbf{Vect}_X^{\text{op}} \\
\downarrow & & \downarrow \\
\mathbf{Coh}_X & \xrightarrow{\underline{\text{Spec}}_X \text{Sym}} & \mathbf{AbCone}_X^{\text{op}}
\end{array}$$

Lemma 3.5. *A cone C over X is a vector bundle if and only if it is smooth over X .*

Proof. Let $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{I}^i$ and assume that $C \rightarrow X$ has constant relative dimension r . Then $\mathcal{I}^1 = 0^* \Omega_{C/X}$ is locally free of rank r . As $C \hookrightarrow A(C)$ where $A(C)$ is a vector bundle and $\dim C = \dim A(C)$, we know that C is a vector bundle. \square

Example 3.6. *Two important examples. Let $X \hookrightarrow Y$ be a closed immersion of ideal \mathcal{I} . Then $C_{X/Y} := \underline{\text{Spec}}_X \bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$ is called the **normal cone** of X in Y . The associated abelian cone $N_{X/Y} = \underline{\text{Spec}}_X \text{Sym } \mathcal{I} / \mathcal{I}^2$ is called the **normal sheaf** of X in Y .*

Definition 3.7. *A sequence of cone morphisms*

$$0 \rightarrow E \xrightarrow{i} C \rightarrow D \rightarrow 0$$

*is called **exact** if E is a vector bundle and locally over X there is a morphism of cones $C \rightarrow E$ splitting i and inducing an isomorphism $C \cong E \times_X D$.*

Lemma 3.8. *We have the following useful results.*

- (a) *Given a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0$ of coherent sheaves on X , with \mathcal{E} locally free, then $0 \rightarrow C(\mathcal{E}) \rightarrow C(\mathcal{F}) \rightarrow C(\mathcal{F}') \rightarrow 0$ is exact, and conversely is also true.*
- (b) *Let $C \rightarrow D$ be a smooth, surjective morphism of cones, and let $E = C \times_{D,0} X$. Then the sequence*

$$0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$$

is exact.

Proof. For (a) we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. Now we consider (b).

For (b), let $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{I}^i$ and $D = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{J}^i$. We first claim that

$$0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$$

exact. Indeed, as now $E \rightarrow X$ smooth, hence $E = \underline{\text{Spec}}_X \text{Sym } \mathcal{E}$ is a vector bundle by Lemma 3.5. Moreover, by base change we may assume $\mathcal{I}^i = 0$ for $i \geq 2$ (why??). Hence $E = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} (\mathcal{I}^i / \mathcal{I}^1 \mathcal{I}^{i-1})$. So $0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I} \rightarrow \mathcal{E} \rightarrow 0$ exact. By (a) we get the claim.

For general case, as $C \rightarrow A(C) \times_{A(D)} D$ is a closed immersion by trivial reason and both these schemes are smooth of the same relative dimension over D . Well done. \square

Definition 3.9. (a) If E is a vector bundle and $f : E \rightarrow C(\mathcal{F})$ a morphism of abelian cones. Then there is an E -action as $E \times_X C(\mathcal{F}) \rightarrow C(\mathcal{F})$ as $(\nu, \gamma) \mapsto f\nu + \gamma$.

(b) If E is a vector bundle and $d : E \rightarrow C$ a morphism of cones, we say that C is an E -cone, if C is invariant under the action of E on $A(C)$.

(c) A morphism ϕ from an E -cone C to an F -cone D is a commutative diagram of cones

$$\begin{array}{ccc} E & \xrightarrow{d_E} & C \\ \downarrow \phi & & \downarrow \phi \\ F & \xrightarrow{d_F} & D \end{array}$$

(d) If $\phi : (E, d_E, C) \rightarrow (F, d_F, D)$ and $\psi : (E, d_E, C) \rightarrow (F, d_F, D)$ are morphisms, we call them *homotopic*, if there exists a morphism of cones $k : C \rightarrow F$, such that $kd_E = \psi - \phi = d_F k$.

Proposition 3.10. Let X be a DM-stack.

(a) Let E be a vector bundle. Then a sequence of cone morphisms $0 \rightarrow E \xrightarrow{i} C \xrightarrow{\phi} D \rightarrow 0$ is exact if and only if C is a E -cone, $\phi : C \rightarrow D$ is surjective and the diagram

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ \downarrow p \quad \uparrow & & \downarrow \phi \\ C & \xrightarrow{\phi} & D \end{array}$$

is cartesian with projection p and action σ .

(b) Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X -spaces with sections and \mathbb{A}^1 -actions and let $\phi : C \rightarrow D$ be an \mathbb{A}^1 -equivariant X -morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Assume that E is a vector bundle. Then C is an E -cone over X if and only if D is a cone over X . Moreover, C is abelian (resp. vector bundle) if and only if D is.

Proof. For (a), if it is exact, locally we have $C \cong E \times_X D$. So E act on C locally as $E \times E \times_X D \rightarrow E \times_X D$ given by $(f, (e, d)) \mapsto (i(f) + e, d)$. So C is a E -cone. Now $\phi : C \rightarrow D$ is surjective is trivial. The cartesian diagram follows from local case.

Conversely, we need to find that $C \cong E \times_X D$ in local.

For (b), let C is an E -cone over X . By faithfully flat descent, we know that D is a relative spectrum over X . As $(D, 0, \gamma)$ be algebraic X -spaces with sections and \mathbb{A}^1 -actions, we know that D is a relative spectrum of graded algebra with zero section and $\phi : C \rightarrow D$ induced by graded morphism and preserving zero sections which is smooth and surjective.

We first assume C is abelian, that is, $C = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{F}$. Let $E = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{E}^\vee$. Hence we have surjection $\mathcal{F} \rightarrow \mathcal{E}^\vee$. So we have $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{E}^\vee \rightarrow 0$ exact. Hence $0 \rightarrow E \rightarrow C \rightarrow C(\mathcal{G}) \rightarrow 0$ exact. So by ??? we know that $D = C(\mathcal{G})$ which is an abelian cone.

In the general case, we have $C \subset A(C)$ defined by a homogeneous ideal $\mathcal{I} \subset \mathrm{Sym} \mathcal{S}^1$ where $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ and $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$. Now we have an exact sequence $0 \rightarrow \mathcal{G} \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$. Let $\mathcal{J} = \mathcal{I} \cap \mathrm{Sym} \mathcal{G}$ and let $C' = \underline{\mathrm{Spec}}_X ((\mathrm{Sym} \mathcal{G})/\mathcal{J})$ which is the scheme-theoretic image of C in $C(\mathcal{G})$. Hence $\overline{C'}$ is the quotient of C by E . \square

3.2 Cone Stack

3.3 A Stack of Special Type

3.4 A Brief of Cotangent Complexes

3.5 Intrinsic Normal Cone

3.6 Obstruction Theory and Virtual Class

3.7 Examples

4 Atiyah-Bott Localization

We will follow [AB84].

5 Localization of Virtual Fundamental Class

We will follow [GP99].

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