Lecture Notes on Commutative Algebra

Xiaolong Liu

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Contents

Preface

Here we will mainly follows [pc23]. We will assume all rings are commutative with unit. We assume the reader know the basic algebra an some homological algebra, including basic theory of groups, rings, modules, basic things of spectrum of rings and its basic properties, abelian categories, derived categories and derived functors.

Chapter 1

Rings, Ideals and Modules

1.1 Basic Properties

Lemma 1.1.1. *Let R be a ring and let M be an R-module. Then there exists a directed system of finitely presented R-modules M_i such that $M \cong \varinjlim M_i$.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto \left(\sum g_e f_{e,s}\right)_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. $\square$

Proposition 1.1.2. *Let R be a ring. Let N be an R-module. The following are equivalent*

(1) N is a finitely generated (finitely presented) R-module.

(2) for any filtered colimit $M = \varinjlim M_i$ of R-modules the map

$$\varinjlim \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$$

is injective (bijective).

Proof. Consider the case of finitely generated: Assume (1) and choose generators $x_1, \cdots, x_m$ for N. If $N \to M_i$ is a module map and the composition $N \to M_i \to M$ is zero, then because $M = \varinjlim_{i' \geq i} M_{i'}$ for each $j \in \{1, \ldots, m\}$ we can find a $i' \geq i$ such that x_j maps to zero in $M_{i'}$. Since there are finitely many x_j we can find a single i' which works for all of them. Then the composition $N \to M_i \to M_{i'}$ is zero and we conclude the map is injective, i.e., part (2) holds.

Assume (2). For a finite subset $E \subset N$ denote $N_E \subset N$ the R-submodule generated by the elements of E. Then $0 = \varinjlim N/N_E$ is a filtered colimit. Hence we see that $\mathrm{id} : N \to N$ maps into N_E for some E, i.e., N is finitely generated.

Consider the case of finitely presented: Assume (1) and choose an exact sequence $F_{-1} \to F_0 \to N \to 0$ with F_i finite free. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_{-1}, M)$$

functorial in the R-module M. The functors $\mathrm{Hom}_R(F_i, M)$ commute with filtered colimits as $\mathrm{Hom}_R(R^{\oplus n}, M) = M^{\oplus n}$. Since filtered colimits are exact, we see that (2) holds.

Assume (2). By Lemma 1.1.1 we can write $N = \varinjlim N_i$ as a filtered colimit such that N_i is of finite presentation for all i. Thus id_N factors through N_i for some i. This means that N is a direct summand of a finitely presented R-module (namely N_i) and hence finitely presented. $\square$

Proposition 1.1.3. *Let R be a ring, and let M be a finitely generated R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal $I_i \subset R$.

Proof. By induction on the number of generators of M. Let $x_1, \cdots, x_r \in M$ be a minimal number of generators. Let $M' := Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/\mathrm{Ann}(x_1)$, well done. $\square$

1.2 Localizations

Definition 1.2.1. *Let R be a ring, S a subset of R. We say S is a multiplicative subset of R if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.*

Definition 1.2.2. *Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:*

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0.$$

It is easily checked that this is an equivalence relation. Let x/s be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st.$$

One can check that $S^{-1}A$ becomes a ring under these operations. Then this ring is called the **localization of A with respect to S** .

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the **localization map**. In general the localization map is not injective, unless S contains no zerodivisors.

The localization of a ring has the following universal property.

Proposition 1.2.3. *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. □

Lemma 1.2.4. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. □

The notion of localization of a ring can be generalized to the localization of a module.

Definition 1.2.5. *Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows*

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0.$$

This is clearly an equivalence relation. Denote by m/s be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st.$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$-module. The $S^{-1}A$-module $S^{-1}M$ is called the localization *of M at S.*

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is also called the localization map. It satisfies the following similar universal property.

Lemma 1.2.6. *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then we have*

$$\begin{tikzcd} M \arrow[rr, "\beta"] \arrow[dr] & & N \\ & S^{-1}M \arrow[ur, dashed, "\exists!\alpha"'] & \end{tikzcd}$$

Moroever, the canonical map

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m).$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. $\square$

Example 1.2.1. *Let A be a ring and let M be an A-module. Here are some important examples of localizations.*

1. *Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_\mathfrak{p}$ and $M_\mathfrak{p}$ the localization of A and M with respect to S respectively. These are called the* localization of A, resp. M at $\mathfrak{p}$.

2. *Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the* localization of A, resp. M with respect to f. *Note that $A_f = 0$ if and only if f is nilpotent in A.*

3. *Let $S = \{f \in A : f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the* total quotient ring *of A.*

4. *If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A.*

Lemma 1.2.7. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \varinjlim_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Just need to check the universal property. □

Proposition 1.2.8. *Let A denote a ring, and M, N denote modules over A. If S and S' are multiplicative sets of A, then it is clear that*

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

(1) Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.

(2) View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.

(3) Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.

(4) If N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.

(5) Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.

(6) *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M. In particular, each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

Proof. For (1), the map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/t')/(s/s')$ to $(x/t')\cdot(s/s')^{-1}$. On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms.

For (2), note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly. We define the maps as follows

$$f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss'$$
$$g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other.

For (3), first it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \ker(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness.

For (4), from the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows.

For (5), The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms.

For (6), Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. $\square$

1.3 Tensor Products

1.3.1 Tensor Products

Proposition 1.3.1. *Let M, N be R-modules. Then there exists a pair $(M \otimes_R N, g)$ where $M \otimes_R N$ is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : M \otimes_R N \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\
& M \otimes_R N \arrow[ur, dashed, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

Then $M \otimes_R N$ is called the tensor product *of R-modules M and N*

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M \times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$\begin{aligned}
(x + x', y) - (x, y) - (x', y), \\
(x, y + y') - (x, y) - (x, y'), \\
(ax, y) - a(x, y), \\
(x, ay) - a(x, y)
\end{aligned}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x, y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x, y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. □

Proposition 1.3.2. *Let R be a ring. Let M and N be R-modules.*

(1) If N and M are finite, then so is $M \otimes_R N$.

(2) If N and M are finitely presented, then so is $M \otimes_R N$.

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented. □

Proposition 1.3.3. *Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S.$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f'(a/s, m) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m.$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0.$$

Therefore f is injective. □

Proposition 1.3.4. *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st.$$

Proof. We may use Proposition 1.3.3 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. □

1.3.2 Base-Change Properties

We formally introduce base change in algebra as follows.

Definition 1.3.5. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The* base change *of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The* base change *of the S-module M is the S'-module $M \otimes_R R'$.*

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

Proposition 1.3.6. *The finite generatedness/finite presentation of modules and rings are stable under base change.*

Proof. Trivial since the tensor product is right exact. □

Definition 1.3.7. *Let $\varphi : R \to S$ be a ring map. Given an S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the* restriction *of N to R.*

Proposition 1.3.8. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$\mathrm{Hom}_R(M, N_R) = \mathrm{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. □

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

Proposition 1.3.9. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto \mathrm{Hom}_R(S, M)$ are adjoint functors. In a formula*

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \mathrm{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \mathrm{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. □

Proposition 1.3.10. *Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have*

$$\mathrm{Hom}_R(M \otimes_S N, P) = \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Propositions 1.3.8 and 1.3.9. Namely, we have

$$\begin{aligned}\mathrm{Hom}_R(M \otimes_S N, P) &= \mathrm{Hom}_S(M \otimes_S N, \mathrm{Hom}_R(S, P)) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, P))) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))\end{aligned}$$

as desired. □

1.4 Some Radicals

1.4.1 Radical of Rings

Definition 1.4.1. *For any ideal $I \subset R$, define $\sqrt{I} := \{x \in R : x^n \in I \text{ for some } n\}$.*

Proposition 1.4.2. *For any ideal $I \subset R$, we have*

$$\sqrt{I} = \bigcap_{I \subset \mathfrak{p}, \mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Proof. The inclusion $\sqrt{I} \subset \bigcap_{I \subset \mathfrak{p}, \mathfrak{p} \text{ primes}} \mathfrak{p}$ is trivial by definitions.

Conversely, take $g \in R \backslash \sqrt{I}$, then $g^n \notin I$ for any n. Let $\bar{\mathfrak{p}} \subset R_g$ be a prime such that $IR_g \subset \bar{\mathfrak{p}} \subset R_g$. Take $\mathfrak{p} \subset R$ be the inverse image of $\bar{\mathfrak{p}}$, then $I \subset \mathfrak{p}$ but $P \cap \{1, g, g^2, ...\} = \emptyset$. Well done. □

1.4.2 Jacobson Radical and Nilradical of Rings

Definition 1.4.3. *Let R be a ring.*

(1) *The* Jacobson radical *of a ring R is*

$$\mathrm{rad}(R) = \bigcap_{\mathfrak{m}, \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

(2) *The* nilradical *of a ring R is*

$$\mathrm{nil}(R) = \sqrt{0} = \bigcap_{\mathfrak{p}, \mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Proposition 1.4.4. *Let R be a ring with Jacobson radical $\mathrm{rad}(R)$. Let $I \subset R$ be an ideal. The following are equivalent*

(1) *$I \subset \mathrm{rad}(R)$, and*

(2) *every element of $1 + I$ is a unit in R.*

In this case every element of R which maps to a unit of R/I is a unit.

Proof. If $f \in \mathrm{rad}(R)$, then $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence $1 + f \notin \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Thus the closed subset $V(1 + f)$ of $\mathrm{Spec}(R)$ is empty. This implies that $1 + f$ is a unit.

Conversely, assume that $1 + f$ is a unit for all $f \in I$. If $\mathfrak{m}$ is a maximal ideal and $I \not\subset \mathfrak{m}$, then $I + \mathfrak{m} = R$. Hence $1 = f + g$ for some $g \in \mathfrak{m}$ and $f \in I$. Then $g = 1 + (-f)$ is not a unit, contradiction.

For the final statement let $f \in R$ map to a unit in R/I. Then we can find $g \in R$ mapping to the multiplicative inverse of f mod I. Then $fg = 1$ mod I. Hence fg is a unit of R by (2) which implies that f is a unit. □

Lemma 1.4.5. *Let $\varphi : R \to S$ be a ring map such that the induced map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective. Then an element $x \in R$ is a unit if and only if $\varphi(x) \in S$ is a unit.*

Proof. If x is a unit, then so is $\varphi(x)$. Conversely, if $\varphi(x)$ is a unit, then $\varphi(x) \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Hence $x \notin \varphi^{-1}(\mathfrak{q}) = \mathrm{Spec}(\varphi)(\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Since $\mathrm{Spec}(\varphi)$ is surjective we conclude that x is a unit. □

1.5 Prime Ideals, some Interesting Things

1.5.1 Prime Avoidance

This is an easy but important result.

Lemma 1.5.1. *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in J \setminus \mathfrak{p}$. Their product is an element of $IJ \subset \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

Proposition 1.5.2 (Prime Avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

(1) $J \not\subset I_i$ for $i = 1, \ldots, r$, and

(2) all but two of I_i are prime ideals.

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 1.5.1) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

1.5.2 Oka Families and Its Applications

Here we introduce a very interesting thing.

Definition 1.5.3. *Let R be a ring. If I is an ideal of R and $a \in R$, we define*

$$(I : a) = \{x \in R : xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R : xJ \subset I\}.$$

Definition 1.5.4 (Oka Family)**.** *Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an* Oka family *if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.*

Here is the fundamental property of Oka family:

Proposition 1.5.5. *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. It follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. □

Now we discover some special Oka families which will induce many interesting results! Before that, we introduce a lemma:

Lemma 1.5.6. *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. □

Corollary 1.5.7. *Let R be a ring and let S be a multiplicative subset of R.*

(1) The family $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family.

(2) An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.

In particular, we have the following things.

(3) An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.

(4) If R is nonzero and every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.

Proof. For (1), suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

For (2), this follows directly from (1) and Proposition 1.5.5.

For (3), consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime by (1).

Thus for (4), if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. □

Corollary 1.5.8. *Let R be a ring.*

(1) The family of finitely generated ideals is an Oka family.

(2) *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*

(3) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated, that is, R is Noetherian.*

Proof. For (1), Let $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \dots, a_n$ and (I, a) is generated by $a, b_1, \dots, b_m$ with $b_1, \dots, b_m \in I$, we claim that I is generated by $aa_1, \dots, aa_n, b_1, \dots, b_m$.

Indeed, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \dots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

For (2), this is an immediate consequence of (1) and Proposition 1.5.5.

For (3), suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \dots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By (2) this ideal is prime. $\square$

Corollary 1.5.9. *Let R be a ring.*

(1) *The family of principal ideals of R is an Oka family.*

(2) *An ideal $I \subset R$ maximal with respect to not being principal is prime.*

(3) *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. For (1), suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 1.5.6 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

For (2), this follows from (1) and Proposition 1.5.5.

For (3), suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by (2). $\square$

Corollary 1.5.10. *Let A be a ring, $I \subset A$ an ideal, and $a \in A$ an element. Let P is a property of A-modules that is stable under extensions and holds for* 0.

(1) *The family of ideals I such that A/I has P is an Oka family.*

(2) *The ideal maximal such that P does not holds is prime.*

Proof. For (1), there is a short exact sequence $0 \to A/(I:a) \to A/I \to A/(I,a) \to 0$ where the first arrow is given by multiplication by a. Thus if P is a property of A-modules that is stable under extensions and holds for 0, then the family of ideals I such that A/I has P is an Oka family.

For (2), this follows from (1) and Proposition 1.5.5. □

1.6 Cayley-Hamilton

Here we introduce Cayley-Hamilton theorem of general rings and its applications.

Proposition 1.6.1 (Cayley-Hamilton)**.** *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(x\mathrm{id}_{n\times n} - A)$). Then $P(A) = 0$ in Mat$(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

1. If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.

2. If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.

3. Thus we may first reduce to the case $R = \mathbb{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbb{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

Then well done. □

Corollary 1.6.2. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Consider

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^{\oplus n} \arrow[r] & M \end{tikzcd}$$

By Proposition 1.6.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

Corollary 1.6.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Consider again

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ I^{\oplus n} & \longrightarrow & M \end{array}$$

By Proposition 1.6.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. $\square$

As a fun example application we prove the following surprising property.

Corollary 1.6.4. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

Proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Corollary 1.6.3 to M as an R'-module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi))\varphi$, and hence φ is invertible. $\square$

1.7 Nakayama's Lemma

First we recall a lemma:

Lemma 1.7.1. *Let R be a ring. Let $n \geq m$. Let A be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.*

1. *For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f1_{m\times m}$.*
2. *If $f \in R$ and $BA = f1_{m\times m}$ for some $m \times n$ matrix B, then $f^m \in J$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$, we denote by E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i\in\{1,\ldots,n\}} R \longrightarrow \bigoplus\nolimits_{i\in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I)1_{m\times m}$. The $m \times m$ minors of A are the determinants $\det A_I$

for all the $I \subset \{1, \ldots, n\}$ with $|I| = m$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f 1_{m \times m} = BA$ then by the Cauchy-Binet formula we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. □

Theorem 1.7.2 (Nakayama's lemma)**.** *Let R be a ring with Jacobson radical* $\mathrm{rad}(R)$*. Let M be an R-module. Let $I \subset R$ be an ideal.*

(1) If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.

(2) If $IM = M$, M is finite, and $I \subset \mathrm{rad}(R)$, then $M = 0$.

(3) If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1 + I$ such that $fM \subset N$ and $M_f = N_f$.

(4) If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \mathrm{rad}(R)$, then $M = N$.

(5) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.

(6) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \mathrm{rad}(R)$, then $N \to M$ is surjective.

(7) If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1 + I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.

(8) If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \mathrm{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.

(9) If $IM = M$, I is nilpotent, then $M = 0$.

(10) If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.

(11) If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.

(12) If $\{x_\alpha\}_{\alpha \in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.

Proof. For (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ since $M = IM$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma 1.7.1 (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m \times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f \delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f \delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Proposition 1.4.4 an element of $1 + \mathrm{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. $\square$

Lemma 1.7.3. *Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S + I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.*[1]

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum(a_{ij}/s_{ij})x_j$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Let $s \in S$ be the product of all of the s_{ij}. Then we see that y_i is contained in the R_s-submodule of M_s generated by $x_1, \ldots, x_r$. Hence $x_1, \ldots, x_r$ generates M_s.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Nakayama's Lemma 1.7.2 we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. $\square$

1.8 The Spectrums of a Ring

1.8.1 Basic Facts

Proposition 1.8.1. *Let R be a ring with an ideal $J \subset R$ and s subset $S \subset \mathrm{Spec}(R)$. Define $I(S) := \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.*

(1) We have $\sqrt{I(S)} = I(S)$ and $I(V(J)) = \sqrt{J}$ and $V(I(S)) = \overline{S}$.

(2) Let $f : R \to R'$ is a ring map induce $F : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$, then we have

   (a) For any subset $M \subset R$, we have $F^{-1}(V(M)) = V(f(M))$. In particular $F^{-1}(D(r)) = D(f(r))$ for $r \in R$.

   (b) For any ideal $I \subset R'$, we have $V(f^{-1}(I)) = \overline{F(V(I))}$.

[1]Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \backslash \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \notin \mathfrak{p}$.

Proof. (1) follows from Proposition 1.4.2. (2)(a) are trivial. For (2)(b), as

$$I(F(V(I))) = \bigcap_{\mathfrak{p}\in V(I)} f^{-1}(\mathfrak{p}) = f^{-1}(\sqrt{I}) = \sqrt{(f^{-1}(I))}.$$

Hence by (1) we have $\overline{F(V(I))} = V(I(F(V(I)))) = V(\sqrt{(f^{-1}(I))}) = V(f^{-1}(I))$. □

Corollary 1.8.2. *Let $f : R \to R'$ is a ring map induce $F : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$, then F has a densed image if and only if $\ker f$ consist of nilpotent elements.*

Proof. By Proposition 1.8.1(2)(b), we have $V(\ker f) = \overline{F(V(0))} = \overline{F(\mathrm{Spec}(R))}$. Well done. □

1.8.2 Fundamental Diagram of Ring Maps

Proposition 1.8.3. *A fundamental commutative diagram associated to a ring map $\varphi : R \to S$, a prime $\mathfrak{q} \subset S$ and the corresponding prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R is the following:*

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} & S_{\mathfrak{q}} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \arrow[u] & S_{\mathfrak{p}} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R\backslash\mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[u] & R_{\mathfrak{p}} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u, "\varphi"'] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the arrows in the outer left and outer right columns are identical. The horizontal maps induce on the associated spectra always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$ exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec *if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.*

1.8.3 Connected Components and Idempotents

It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

Lemma 1.8.4. *Let R be a ring. Let $e \in R$ be an idempotent. In this case*

$$\mathrm{Spec}(R) = D(e) \amalg D(1-e).$$

Proof. Trivial. □

Lemma 1.8.5. *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. The induced map*

$$\mathrm{Spec}(R_1) \amalg \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 1.8.4. □

Proposition 1.8.6. *Let R be a ring. For each $U \subset \mathrm{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \mathrm{Spec}(R)$ and idempotents $e \in R$.*

Proof. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact, and similarly for its complement. Write $U = \bigcup_{i=1}^n D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^m D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Proposition 1.4.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \amalg V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $0 = xy = x(1 - x)$ as $I^N J^N = (0)$. Thus $x = x^2$ is idempotent and contained in $I^N \subset I$. The idempotent $y = 1 - x$ is contained in $J^N \subset J$. This shows that the idempotent x maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that x maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \notin \mathfrak{p}$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1 e_2' \notin \mathfrak{p}$ or $e_1' e_2 \notin \mathfrak{p}$. It is easy to see this gives the desired prime. □

Corollary 1.8.7. *Let R be a nonzero ring. Then $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents.*

Proof. Obvious from Proposition 1.8.6 and the definition of a connected topological space. □

Lemma 1.8.8. *Let R be a ring. A connected component of* $\mathrm{Spec}(R)$ *is of the form $V(I)$, where I is an ideal generated by idempotents such that every idempotent of R either maps to 0 or 1 in R/I.*

Proof. Let $\mathfrak{p}$ be a prime of R. By some general topology, the connected component of $\mathfrak{p}$ in $\mathrm{Spec}(R)$ is the intersection of open and closed subsets of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Proposition 1.8.6. Any idempotent e which is not in this collection clearly maps to 1 in R/I. □

1.8.4 Glueing Properties

In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

Proposition 1.8.9. *Let R be a ring.*

(1) For an element x of an R-module M the following are equivalent

  (a) $x = 0$,

  (b) x maps to zero in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$,

  (c) x maps to zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.

  In other words, the map $M \to \prod_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective.

(2) Given an R-module M the following are equivalent

  (a) M is zero,

  (b) $M_{\mathfrak{p}}$ is zero for all $\mathfrak{p} \in \mathrm{Spec}(R)$,

  (c) $M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$ of R.

(3) Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent

  (a) $M_1 \to M_2 \to M_3$ is exact,

  (b) for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,

  (c) for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.

(4) Given a map $f : M \to M'$ of R-modules the following are equivalent

  (a) f is injective,

  (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is injective for all primes $\mathfrak{p}$ of R,

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is injective for all maximal ideals* $\mathfrak{m}$ *of* R.

(5) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*

   (a) f *is surjective,*

   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is surjective for all primes* $\mathfrak{p}$ *of* R,

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is surjective for all maximal ideals* $\mathfrak{m}$ *of* R.

(6) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*

   (a) f *is bijective,*

   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is bijective for all primes* $\mathfrak{p}$ *of* R,

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is bijective for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. Hence I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \ker(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. As localization is exact, we have that $H_{\mathfrak{p}}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). □

Proposition 1.8.10. *Let* R *be a ring. Let* M *be an* R*-module. Let* S *be an* R*-algebra. Suppose that* $f_1, \ldots, f_n$ *is a finite list of elements of* R *such that* $\bigcup D(f_i) = \mathrm{Spec}(R)$*, in other words* $(f_1, \ldots, f_n) = R$.

(1) *If each* $M_{f_i} = 0$ *then* $M = 0$.

(2) *If each* M_{f_i} *is a finite* R_{f_i}*-module, then* M *is a finite* R*-module.*

(3) *If each* M_{f_i} *is a finitely presented* R_{f_i}*-module, then* M *is a finitely presented* R*-module.*

(4) *Let* $M \to N$ *be a map of* R*-modules. If* $M_{f_i} \to N_{f_i}$ *is an isomorphism for each* i *then* $M \to N$ *is an isomorphism.*

(5) *Let* $0 \to M'' \to M \to M' \to 0$ *be a complex of* R*-modules. If* $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ *is exact for each* i*, then* $0 \to M'' \to M \to M' \to 0$ *is exact.*

(6) *If each* R_{f_i} *is Noetherian, then* R *is Noetherian.*

(7) *If each S_{f_i} is a finite type R-algebra, so is S.*

(8) *If each S_{f_i} is of finite presentation over R, so is S.*

Proof. We prove each of the parts in turn.

(1) By second localization, this implies $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, so we conclude by Proposition 1.8.9.

(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it is surjective by Proposition 1.8.9 and M is finitely generated.

(3) By (2) we have a short exact sequence

$$0 \to K \to R^n \to M \to 0$$

Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$. By (2) this implies that K is a finite R-module and therefore M is finitely presented.

(4) By second localization, the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 1.8.9.

(5) By second localization, the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 1.8.9.

(6) We will show that every ideal of R has a finite generating set: For this, let $I \subset R$ be an arbitrary ideal. As localization is exact, each $I_{f_i} \subset R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).

(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by (1) and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, \ldots, X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, \ldots, X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, \ldots, X_n]$, so an application of (2) finishes the proof.

□

Corollary 1.8.11. *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_n$ is a finite list of elements of S such that $\bigcup D(g_i) = \mathrm{Spec}(S)$ in other words $(g_1, \ldots, g_n) = S$.*

(1) If each S_{g_i} is of finite type over R, then S is of finite type over R.

(2) If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.

Proof. Choose $h_1, \ldots, h_n \in S$ such that $\sum h_i g_i = 1$.

Proof of (1). For each i choose a finite list of elements $x_{i,j} \in S_{g_i}$, $j = 1, \ldots, m_i$ which generate S_{g_i} as an R-algebra. Write $x_{i,j} = y_{i,j}/g_i^{n_{i,j}}$ for some $y_{i,j} \in S$ and some $n_{i,j} \geq 0$. Consider the R-subalgebra $S' \subset S$ generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ and $y_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Since localization is exact, we see that $S'_{g_i} \to S_{g_i}$ is injective. On the other hand, it is surjective by our choice of $y_{i,j}$. The elements $g_1, \ldots, g_n$ generate the unit ideal in S' as $h_1, \ldots, h_n \in S'$. Thus $S' \to S$ viewed as an S'-module map is an isomorphism by Lemma 1.8.10.

Proof of (2). We already know that S is of finite type. Write $S = R[x_1, \ldots, x_m]/J$ for some ideal J. For each i choose a lift $g_i' \in R[x_1, \ldots, x_m]$ of g_i and we choose a lift $h_i' \in R[x_1, \ldots, x_m]$ of h_i. Then we see that

$$S_{g_i} = R[x_1, \ldots, x_m, y_i]/(J_i + (1 - y_i g_i'))$$

where J_i is the ideal of $R[x_1, \ldots, x_m, y_i]$ generated by J. Small detail omitted. We may choose a finite list of elements $f_{i,j} \in J$, $j = 1, \ldots, m_i$ such that the images of $f_{i,j}$ in J_i and $1 - y_i g_i'$ generate the ideal $J_i + (1 - y_i g_i')$. Set

$$S' = R[x_1, \ldots, x_m]/\left(\sum h_i' g_i' - 1, f_{i,j}; i = 1, \ldots, n, j = 1, \ldots, m_i\right)$$

There is a surjective R-algebra map $S' \to S$. The classes of the elements $g_1', \ldots, g_n'$ in S' generate the unit ideal and by construction the maps $S'_{g_i'} \to S_{g_i}$ are injective. Thus we conclude as in part (1). □

1.8.5 More on Images

This is an important theorem in commutative algebra and algebraic geometry.

Theorem 1.8.12 (Chevalley's Theorem)**.** *Suppose that $R \to S$ is of finite presentation. The image of a constructible subset of $\operatorname{Spec}(S)$ in $\operatorname{Spec}(R)$ is constructible.*

Proof. Omitted, we refer Tag 00FE. □

Next, we collect a few additional lemmas concerning the image on Spec for ring maps.

Lemma 1.8.13. *Let $R \subset S$ be an inclusion of domains. Assume that $R \to S$ is of finite type. There exists a nonzero $f \in R$, and a nonzero $g \in S$ such that $R_f \to S_{fg}$ is of finite presentation.*

Proof. By induction on the number of generators of S over R. During the proof we may replace R by R_f and S by S_f for some nonzero $f \in R$.

Suppose that S is generated by a single element over R. Then $S = R[x]/\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset R[x]$. If $\mathfrak{q} = (0)$ there is nothing to prove. If $\mathfrak{q} \neq (0)$, then let $h \in \mathfrak{q}$ be a nonzero element with minimal degree in x. Write $h = fx^d + a_{d-1}x^{d-1} + \ldots + a_0$ with $a_i \in R$ and $f \neq 0$. After inverting f in R and S we may assume that h is monic. We obtain a surjective R-algebra map $R[x]/(h) \to S$. We have $R[x]/(h) = R \oplus Rx \oplus \ldots \oplus Rx^{d-1}$ as an R-module and by minimality of d we see that $R[x]/(h)$ maps injectively into S. Thus $R[x]/(h) \cong S$ is finitely presented over R.

Suppose that S is generated by $n > 1$ elements over R. Say $x_1, \ldots, x_n \in S$ generate S. Denote $S' \subset S$ the subring generated by $x_1, \ldots, x_{n-1}$. By induction hypothesis we see that there exist $f \in R$ and $g \in S'$ nonzero such that $R_f \to S'_{fg}$ is of finite presentation. Next we apply the induction hypothesis to $S'_{fg} \to S_{fg}$ to see that there exist $f' \in S'_{fg}$ and $g' \in S_{fg}$ such that $S'_{fgf'} \to S_{fgf'g'}$ is of finite presentation. We leave it to the reader to conclude. □

Proposition 1.8.14. *Let $R \to S$ be a finite type ring map. Denote $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$. Write $f : Y \to X$ the induced map of spectra. Let $E \subset Y = \operatorname{Spec}(S)$ be a constructible set. If a point $\xi \in X$ is in $f(E)$, then $\overline{\{\xi\}} \cap f(E)$ contains an open dense subset of $\overline{\{\xi\}}$.*

Proof. Let $\xi \in X$ be a point of $f(E)$. Choose a point $\eta \in E$ mapping to ξ. Let $\mathfrak{p} \subset R$ be the prime corresponding to ξ and let $\mathfrak{q} \subset S$ be the prime corresponding to η. Consider the diagram

$$\begin{tikzcd}
\eta \arrow[r] \arrow[d, mapsto] & E \cap Y' \arrow[r] \arrow[d] & Y' = \operatorname{Spec}(S/\mathfrak{q}) \arrow[r] \arrow[d] & Y \arrow[d] \\
\xi \arrow[r] & f(E) \cap X' \arrow[r] & X' = \operatorname{Spec}(R/\mathfrak{p}) \arrow[r] & X
\end{tikzcd}$$

As affine map is quasi-compact, the set $E \cap Y'$ is constructible in Y'. It follows that we may replace X by X' and Y by Y'. Hence we may assume that $R \subset S$ is an inclusion of domains, ξ is the generic point of X, and η is the generic point of Y. By Lemma 1.8.13 combined with Chevalley's theorem 1.8.12 we see that there exist dense opens $U \subset X$, $V \subset Y$ such that $f(V) \subset U$ and such that $f : V \to U$ maps constructible sets to constructible sets. Note that $E \cap V$ is constructible in V. Hence $f(E \cap V)$ is constructible in U and contains ξ. By the basic property of constructible sets (see Tag 005K) we see that $f(E \cap V)$ contains a dense open $U' \subset U$. $\square$

At the end of this section we present a few more results on images of maps on Spectra that have nothing to do with constructible sets.

Proposition 1.8.15. *Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

(1) The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.

(2) For any ideal $I \subset R$ the inverse image of $\sqrt{IS}$ in R is equal to $\sqrt{I}$.

(3) For any radical ideal $I \subset R$ the inverse image of IS in R is equal to I.

(4) For every prime $\mathfrak{p}$ of R the inverse image of $\mathfrak{p}S$ in R is $\mathfrak{p}$.

In this case the same is true after any base change: Given a ring map $R \to R'$ the ring map $R' \to R' \otimes_R S$ has the equivalent properties (1), (2), (3) as well.

Proof. If $J \subset S$ is an ideal, then $\sqrt{\varphi^{-1}(J)} = \varphi^{-1}(\sqrt{J})$. This shows that (2) and (3) are equivalent. The implication (3) $\Rightarrow$ (4) is immediate. If $I \subset R$ is a radical ideal, then $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$. Hence (4) $\Rightarrow$ (2). We have $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$ if and only if $\mathfrak{p}$ is in the image. Hence (1) $\Leftrightarrow$ (4). Thus (1), (2), (3), and (4) are equivalent.

Assume (1) holds. Let $R \to R'$ be a ring map. Let $\mathfrak{p}' \subset R'$ be a prime ideal lying over the prime $\mathfrak{p}$ of R. To see that $\mathfrak{p}'$ is in the image of $\mathrm{Spec}(R' \otimes_R S) \to \mathrm{Spec}(R')$ we have to show that $(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}')$ is not zero. But we have

$$(R' \otimes_R S) \otimes_{R'} \kappa(\mathfrak{p}') = S \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

which is not zero as $S \otimes_R \kappa(\mathfrak{p})$ is not zero by assumption and $\kappa(\mathfrak{p}) \to \kappa(\mathfrak{p}')$ is an extension of fields. $\square$

Lemma 1.8.16. *Let R be a domain. Let $\varphi : R \to S$ be a ring map. The following are equivalent:*

(1) The ring map $R \to S$ is injective.

(2) The image $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains a dense set of points.

(3) There exists a prime ideal $\mathfrak{q} \subset S$ whose inverse image in R is (0).

Proof. Let K be the field of fractions of the domain R. Assume that $R \to S$ is injective. Since localization is exact we see that $K \to S \otimes_R K$ is injective. Hence there is a prime mapping to (0).

Note that (0) is dense in $\mathrm{Spec}(R)$, so that the last condition implies the second.

Suppose the second condition holds. Let $f \in R$, $f \neq 0$. As R is a domain we see that $V(f)$ is a proper closed subset of R. By assumption there exists a prime $\mathfrak{q}$ of S such that $\varphi(f) \notin \mathfrak{q}$. Hence $\varphi(f) \neq 0$. Hence $R \to S$ is injective. □

Lemma 1.8.17. *Let $R \subset S$ be an injective ring map. Then $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ hits all the minimal primes.*

Proof. Let $\mathfrak{p} \subset R$ be a minimal prime. In this case $R_{\mathfrak{p}}$ has a unique prime ideal. Hence it suffices to show that $S_{\mathfrak{p}}$ is not zero. And this follows from the fact that localization is exact. □

Lemma 1.8.18. *Let $R \to S$ be a ring map. If a minimal prime $\mathfrak{p} \subset R$ is in the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$, then it is the image of a minimal prime.*

Proof. Say $\mathfrak{p} = \mathfrak{q} \cap R$. Then choose a minimal prime $\mathfrak{r} \subset S$ with $\mathfrak{r} \subset \mathfrak{q}$. By minimality of $\mathfrak{p}$ we see that $\mathfrak{p} = \mathfrak{r} \cap R$. □

1.9 Basic Properties of Flatness

1.9.1 Flat and Faithfully Modules

Definition 1.9.1. *Let R be a ring.*

(1) *An R-module M is called* flat *if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.*

(2) *An R-module M is called* faithfully flat *if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.*

(3) *A ring map $R \to S$ is called* flat *if S is flat as an R-module.*

(4) *A ring map $R \to S$ is called* faithfully flat *if S is faithfully flat as an R-module.*

Here is an example of how you can use the flatness condition.

Lemma 1.9.2. *Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.*

Proof. Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Since the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$ we conclude. □

Proposition 1.9.3. *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\varinjlim_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact. □

Proposition 1.9.4. *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R'-module, then M' is a (faithfully) flat R-module.*

Proof. The first statement of the lemma is a particular case of the second, so it is clearly enough to prove the latter. Let $R \to R'$ be a flat ring map, and M' a flat R'-module. We need to prove that M' is a flat R-module. Let $N_1 \to N_2 \to N_3$ be an exact complex of R-modules. Then, the complex $R' \otimes_R N_1 \to R' \otimes_R N_2 \to R' \otimes_R N_3$ is exact (since R' is flat as an R-module), and so the complex $M' \otimes_{R'} (R' \otimes_R N_1) \to M' \otimes_{R'} (R' \otimes_R N_2) \to M' \otimes_{R'} (R' \otimes_R N_3)$ is exact (since M' is a flat R'-module). Since $M' \otimes_{R'} (R' \otimes_R N) \cong (M' \otimes_{R'} R') \otimes_R N \cong M' \otimes_R N$ for any R-module N functorially, this complex is isomorphic to the complex $M' \otimes_R N_1 \to M' \otimes_R N_2 \to M' \otimes_R N_3$, which is therefore also exact. This shows that M' is a flat R-module. Tracing this argument backwards, we can show that if $R \to R'$ is faithfully flat, and if M' is faithfully flat as an R'-module, then M' is faithfully flat as an R-module. □

Proposition 1.9.5. *Let M be an R-module. The following are equivalent:*

1. *M is flat over R.*
2. *for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.*
3. *for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*
4. *for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.*

Proof. We prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \ker(N_2 \to N_3)$ and $Q = \mathrm{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done by some chase. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

Assume $K \to N$ is an injective R-module map and let $x \in \ker(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes_R M$ is the colimit of R-modules of the form $K_i \otimes_R M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes_R M$. Thus we may assume that K is a finite R-module. Assume this. We regard the injection $K \to N$ as an inclusion, so that $K \subset N$.

The R-module N is the union of its finite R-submodules that contain K. Hence, $N \otimes_R M$ is the colimit of R-modules of the form $N_i \otimes_R M$ where N_i runs over all finite R-submodules of N that contain K (again since tensor product commutes with colimits). Notice that this is a colimit over a directed system (since the sum of two finite submodules of N is again finite). Hence, the element $x \in K \otimes_R M$ maps to zero in at least one of these R-modules $N_i \otimes_R M$ (since x maps to zero in $N \otimes_R M$). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subset L' \subset R^{\oplus n}$. For any R-submodule $G \subset R^{\oplus n}$, we have a canonical map $G \otimes_R M \to M^{\oplus n}$ obtained by composing $G \otimes_R M \to R^n \otimes_R M = M^{\oplus n}$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M/L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too.

Thus it suffices to show that $L \otimes_R M \to M^{\oplus n}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below. For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{tikzcd} & L' \otimes_R M \arrow[r] \arrow[d] & L \otimes_R M \arrow[r] \arrow[d] & L'' \otimes_R M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & M^{\oplus n} \arrow[r] & M^{\oplus n-1} \arrow[r] & 0 \end{tikzcd}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of the finitely generated ideals I_α contained in it. In other words, $I = \varinjlim I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \varinjlim I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. $\square$

Proposition 1.9.6. *Let R be a ring. Let M be an R-module. The following are equivalent:*

(1) *The module M is flat over R.*

(2) *For all $i > 0$ the functor $\mathrm{Tor}_i^R(M, -)$ is zero.*

(3) *The functor $\mathrm{Tor}_1^R(M, -)$ is zero.*

(4) *For all ideals $I \subset R$ we have $\mathrm{Tor}_1^R(M, R/I) = 0$.*

(5) *For all finitely generated ideals $I \subset R$ we have $\mathrm{Tor}_1^R(M, R/I) = 0$.*

Proof. Suppose M is flat. Let N be an R-module. Let $F_\bullet$ be a free resolution of N. Then $F_\bullet \otimes_R M$ is a resolution of $N \otimes_R M$, by flatness of M. Hence all higher Tor groups vanish.

It now suffices to show that the last condition implies that M is flat. Let $I \subset R$ be an ideal. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. We get an exact sequence

$$\mathrm{Tor}_1^R(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0$$

Since obviously $M \otimes_R R = M$ we conclude that the last hypothesis implies that $M \otimes_R I \to M$ is injective for every finitely generated ideal I. Thus M is flat by Proposition 1.9.5. □

Proposition 1.9.7. *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings over the directed set I. Let $R = \varinjlim_i R_i$.*

(1) *If M is an R-module such that M is flat as an R_i-module for all i, then M is flat as an R-module.*

(2) *For $i \in I$ let M_i be a flat R_i-module and for $i' \geq i$ let $f_{ii'} : M_i \to M_{i'}$ be a $\varphi_{ii'}$-linear map such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$. Then $M = \varinjlim_{i \in I} M_i$ is a flat R-module.*

Proof. Part (1) is a special case of part (2) with $M_i = M$ for all i and $f_{ii'} = \mathrm{id}_M$. Proof of (2). Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 1.9.5 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = \varinjlim_{i' \geq i} \mathfrak{a}'R_{i'}$. Since $\otimes$ commutes with colimits the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}'R_{i'} \otimes_{R_{i'}} M_{i'} \longrightarrow M_{i'}$$

These maps are all injective by assumption. Since colimits over I are exact, we win. □

Proposition 1.9.8. *Let R be a ring.*

(1) *Suppose that M is (faithfully) flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.*

(2) *Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.*

(3) *Let R be a ring. Let $S \to S'$ be a flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$. Then If M is flat over R, then M' is flat over R. If $S \to S'$ is faithfully flat, then M is flat over R if and only if M' is flat over R.*

(4) *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. (1) is trivial and we consider (2).

By (1) we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact.

For (3), let $N \to N'$ be an injection of R-modules. By the flatness of $S \to S'$ we have

$$\ker(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \ker(N \otimes_R M' \to N' \otimes_R M')$$

If M is flat over R, then the left hand side is zero and we find that M' is flat over R by the second characterization of flatness in Lemma 1.9.5. If M' is flat over R then we have the vanishing of the right hand side and if in addition $S \to S'$ is faithfully flat, this implies that $\ker(N \otimes_R M \to N' \otimes_R M)$ is zero which in turn shows that M is flat over R.

For (4), let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. ☐

Proposition 1.9.9 (Equational criterion of flatness). *Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is trivial if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that*

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \textit{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

Then A module M over R is flat if and only if every relation in M is trivial.

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation in M. Let $I = (f_1, \dots, f_n)$, and let $K = \ker(R^n \to I, (a_1, \dots, a_n) \mapsto \sum_i a_i f_i)$. So we have the short exact sequence $0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes x_i \in R^n \otimes_R M$ where e_i is the ith basis element of R^n. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum f_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j \right) = \sum \left(\sum f_i a_{ij} \right) \otimes y_j = 0$$

Well done. $\square$

Proposition 1.9.10. *Suppose that R is a ring.*

1. *Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence*

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

   is a short exact sequence.

2. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.*

Proof. For (1), let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows from the snake lemma applied to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R N & \longrightarrow & M' \otimes_R N & \longrightarrow & M \otimes_R N & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (M'')^{(I)} & \longrightarrow & (M')^{(I)} & \longrightarrow & M^{(I)} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & M'' \otimes_R K & \longrightarrow & M' \otimes_R K & \longrightarrow & M \otimes_R K & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact.

For (2), we will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 1.9.5. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \\
 & M' \otimes_R I \arrow[r] \arrow[u] & M \otimes_R I \arrow[r] \arrow[u] & M'' \otimes_R I \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by (1). $\square$

Proposition 1.9.11. *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

1. *The localization $S^{-1}R$ is a flat R-algebra.*
2. *If M is an $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.*
3. *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak p}$ is a flat $R_{\mathfrak p}$-module for all primes $\mathfrak p$ of R.*
4. *Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak m}$ is a flat $R_{\mathfrak m}$-module for all maximal ideals $\mathfrak m$ of R.*
5. *Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.*
6. *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak q}$ is a flat $R_{\mathfrak p}$-module (with $\mathfrak p$ the prime of R lying under $\mathfrak q$) for all primes $\mathfrak q$ of A.*
7. *Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak m}$ is a flat $R_{\mathfrak p}$-module (with $\mathfrak p = R \cap \mathfrak m$) for all maximal ideals $\mathfrak m$ of A.*

Proof. Let us prove the last statement. In the proof we will use repeatedly that localization is exact and commutes with tensor product.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_{\mathfrak m}$ is a flat $R_{\mathfrak p}$-module for all maximal ideals $\mathfrak m$ of A (with $\mathfrak p = R \cap \mathfrak m$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_{\mathfrak m} \to M_{\mathfrak m}$ is injective because

$(I \otimes_R M)_{\mathfrak{m}} = I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{m}}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Proposition 1.8.9. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_{\mathfrak{p}}$ is an ideal. We have to show that $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is injective. We can write $I = J_{\mathfrak{p}}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

1.9.2 More Faithfully Flatness

Proposition 1.9.12. *Let R be a ring. Let M be an R-module. The following are equivalent*

(1) M is faithfully flat, and

(2) M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.

Proof. If M is faithfully flat, then $0 \to \ker(\alpha) \to N \to N'$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \ker(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. □

Proposition 1.9.13. *Let M be a flat R-module. The following are equivalent:*

(1) M is faithfully flat,

(2) for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,

(3) for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and

(4) for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.

Proof. Assume M faithfully flat and $N \neq 0$. By Proposition 1.9.12 the nonzero map $1 : N \to N$ induces a nonzero map $M \otimes_R N \to M \otimes_R N$, so $M \otimes_R N \neq 0$. Thus (1) implies (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate.

Assume (4). Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \ker(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. To finish the proof we will show $H = 0$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. □

Proposition 1.9.14. *Let $R \to S$ be a flat ring map. The following are equivalent:*

(1) *$R \to S$ is faithfully flat,*

(2) *the induced map on* Spec *is surjective, and*

(3) *any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*

Proof. This follows quickly from Proposition 1.9.13, because we saw in the fundamental diagram that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

Corollary 1.9.15. *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Proposition 1.9.14. □

Corollary 1.9.16 (Going down)**.** *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. By Proposition 1.9.11 the local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat. By Corollary 1.9.15 this local ring map is faithfully flat. By Proposition 1.9.14 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

Proposition 1.9.17. *Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \varinjlim_i S_i$ is a faithfully flat R-algebra.*

Proof. By Proposition 1.9.3 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Proposition 1.9.14 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \varinjlim S_i/\mathfrak{m}S_i$ is nonzero as well because 1 is not equal to zero. Thus the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Proposition 1.9.14. □

1.10 Length

Definition 1.10.1. *Let R be a ring. For any R-module M we define the* length of M over R *by the formula*

$$\mathrm{length}_R(M) = \sup\{n : \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

Proposition 1.10.2. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then $\mathrm{length}_R M = \mathrm{length}_R M' + \mathrm{length}_R M''$.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\mathrm{length}_R M \geq \mathrm{length}_R M' + \mathrm{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \mathrm{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. □

Proposition 1.10.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$. If M is an R-module and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $length_R(M) = \infty$. In other words, if M has finite length then $\mathfrak{m}^n M = 0$ for some n.*

Proof. Assume $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. Choose $x \in M$ and $f_1, \ldots, f_n \in \mathfrak{m}$ such that $f_1 f_2 \ldots f_n x \neq 0$. The first n steps in the filtration

$$0 \subset R f_1 \ldots f_n x \subset R f_1 \ldots f_{n-1} x \subset \ldots \subset R x \subset M$$

are distinct. For example, if $R f_1 x = R f_1 f_2 x$, then $f_1 x = g f_1 f_2 x$ for some g, hence $(1 - g f_2) f_1 x = 0$ hence $f_1 x = 0$ as $1 - g f_2$ is a unit which is a contradiction with the choice of x and $f_1, \ldots, f_n$. Hence the length is infinite. $\square$

Lemma 1.10.4. *Let $R \to S$ be a ring map. Let M be an S-module. We always have $length_R(M) \geq length_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. $\square$

Proposition 1.10.5. *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m} M = 0$. Then $\text{length}_R M = \dim_{R/\mathfrak{m}} M$. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 1.10.4. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. $\square$

Proposition 1.10.6. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$. The lemma follows. $\square$

Proposition 1.10.7. *Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1} M \subset \ldots \subset \mathfrak{m} M \subset M$. All of the subquotients are finitely generated R-modules to which Proposition 1.10.5 applies. We conclude by additivity, see Proposition 1.10.2. $\square$

Definition 1.10.8. *Let R be a ring. Let M be an R-module. We say M is* simple *if $M \neq 0$ and every submodule of M is either equal to M or to 0.*

Proposition 1.10.9. *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *(1) M is simple,*
2. *(2) $\text{length}_R(M) = 1$, and*
3. *(3) $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Proposition 1.10.5 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

We now show that the simple modules are the building blocks of modules.

Proposition 1.10.10. *Let R be a ring. Let M be a finite length R-module. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

1. *(1) $n = \text{length}_R(M)$,*
2. *(2) each M_i/M_{i-1} is simple,*
3. *(3) each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
4. *(4) given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\sharp\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = \text{length}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Proposition 1.10.9 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 1.10.2, we see $n = \text{length}_R(M)$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_\mathfrak{m}$ with successive quotients $(M_i/M_{i-1})_\mathfrak{m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R we have

$$(R/\mathfrak{m}')_\mathfrak{m} \cong \begin{cases} 0 & \text{if } \mathfrak{m} \neq \mathfrak{m}', \\ R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} & \text{if } \mathfrak{m} = \mathfrak{m}' \end{cases}$$

This we leave to the reader. □

Proposition 1.10.11. *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$\text{length}_A(M) = \sum_{i=1,\ldots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \, \text{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $\text{length}_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$$

by B-submodules as in Proposition 1.10.10. Then each quotient M_j/M_{j-1} is isomorphic to $\kappa(\mathfrak{m}_{i(j)})$ for some $i(j) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak{m}_i)) = [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ by Proposition 1.10.5. The lemma follows by additivity of lengths (Proposition 1.10.2). □

Proposition 1.10.12. *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$\text{length}_A(M) \, \text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(M \otimes_A B).$$

In particular, if $\text{length}_B(B/\mathfrak{m}_A B) < \infty$ *then M has finite length if and only if $M \otimes_A B$ has finite length.*

Proof. The ring map $A \to B$ is faithfully flat by Corollary 1.9.15. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_A B$. Thus the lemma follows. □

Proposition 1.10.13. *Let $A \to B \to C$ be flat local homomorphisms of local rings. Then*

$$\text{length}_B(B/\mathfrak{m}_A B) \, \text{length}_C(C/\mathfrak{m}_B C) = \text{length}_C(C/\mathfrak{m}_A C).$$

Proof. Follows from Proposition 1.10.12 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak{m}_A B$ □

1.11 Noetherian and Artinian Rings

1.11.1 Basic Facts of Noetherian Rings

Proposition 1.11.1. *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \ldots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. $\square$

Proposition 1.11.2. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \ldots, x_n]]$.*

Proof. Since $R[[x_1, \ldots, x_{n+1}]] \cong R[[x_1, \ldots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \ldots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \ldots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \ldots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\ldots,d_0, j=1,\ldots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \ldots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. $\square$

Proposition 1.11.3. *Let R be a Noetherian ring.*

(1) Any finite R-module is of finite presentation.

(2) Any submodule of a finite R-module is finite.

(3) Any finite type R-algebra is of finite presentation over R.

Proof. Let M be a finite R-module. By Proposition 1.1.3 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite. This proves part (2).

To see (3) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Proposition 1.11.1. $\square$

Proposition 1.11.4. *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

Proof. Now finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma 1.11.1. $\square$

1.11.2 More on Noetherian Rings

Proposition 1.11.5 (Artin-Rees)**.** *Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. Hence it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. $\square$

Corollary 1.11.6. *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that*

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \text{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Proposition 1.11.5 to $\mathrm{Im}(f) \subset N$ and note that $f : I^{n-c}M \to I^{n-c}f(M)$ is surjective. ☐

Corollary 1.11.7 (Krull's intersection theorem)**.** *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 1.11.5 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 1.7.2 we see that $N = 0$. ☐

Corollary 1.11.8. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.*

1. *For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*
2. *If I is contained in the Jacobson radical of R, then $N = 0$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_{\mathfrak{p}}$ is zero, by Corollary 1.11.7. Hence we can find $g_i \in R$, $g_i \notin \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Proposition 1.8.9. ☐

Lemma 1.11.9 (Artin-Tate Lemma)**.** *Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is of finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. ☐

1.11.3 Artinian Rings

Here we give some useful things about Artinian rings, see also Proposition 3.2.2.

Proposition 1.11.10. *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. □

Proposition 1.11.11. *Let R is Artinian.*

(1) Then R has only finitely many maximal ideals.

(2) The Jacobson radical $\mathrm{rad}(R)$ is a nilpotent ideal.

Proof. For (1). Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are pairwise distinct maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak{m}_i$ are surjective).

For (2). Let $I = \mathrm{rad}(R)$ be the Jacobson radical. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By Nakayama's Lemma 1.7.2, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. □

Lemma 1.11.12. *Any ring with finitely many maximal ideals and locally nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $I = \bigcap_{i=1}^n \mathfrak{m}_i$ be the Jacobson radical of R. Assume I is locally nilpotent. Let $\mathfrak{p}$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak{p} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. Since $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$ we conclude $\mathfrak{p} \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$. Hence $\mathfrak{p} \supset \mathfrak{m}_i$ for some i, and so $\mathfrak{p} = \mathfrak{m}_i$. By the Chinese remainder theorem we have $R/I \cong \bigoplus R/\mathfrak{m}_i$ which is a product of fields. Hence by Proposition 1.8.6 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak{m}_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. □

Proposition 1.11.13. *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Propositions 1.11.11(1), 1.11.11(2), and Lemma 1.11.12.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak{m}$. By Proposition 1.11.11(2) we have $\mathfrak{m}^n = 0$ for some n. Consider the sequence $0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Proposition 1.10.5 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of sub vector spaces which would correspond to an infinite descending chain of ideals in R. □

1.12 Supports and Annihilators

Here are some very basic definitions and properties.

Definition 1.12.1. *Let R be a ring and let M be an R-module. The* support of M *is the set*

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$$

Lemma 1.12.2. *Let R be a ring. Let M be an R-module. Then*

$$M = (0) \Leftrightarrow \mathrm{Supp}(M) = \emptyset.$$

Proof. Follows from Proposition 1.8.9. □

Definition 1.12.3. *Let R be a ring. Let M be an R-module.*

(1) Given an element $m \in M$ the annihilator of m *is the ideal*

$$\mathrm{Ann}_R(m) = \{f \in R \mid fm = 0\}.$$

(2) The annihilator of M *is the ideal*

$$\mathrm{Ann}_R(M) = \{f \in R \mid fm = 0 \ \forall m \in M\}.$$

Proposition 1.12.4. *Let $R \to S$ be a flat ring map. Let M be an R-module and $m \in M$. Then $\mathrm{Ann}_R(m)S = \mathrm{Ann}_S(m \otimes 1)$. If M is a finite R-module, then $\mathrm{Ann}_R(M)S = \mathrm{Ann}_S(M \otimes_R S)$.*

Proof. Set $I = \mathrm{Ann}_R(m)$. By definition there is an exact sequence $0 \to I \to R \to M$ where the map $R \to M$ sends f to fm. Using flatness we obtain an exact sequence $0 \to I \otimes_R S \to S \to M \otimes_R S$ which proves the first assertion. If $m_1, \ldots, m_n$ is a set of generators of M then $\mathrm{Ann}_R(M) = \bigcap \mathrm{Ann}_R(m_i)$. Similarly $\mathrm{Ann}_S(M \otimes_R S) = \bigcap \mathrm{Ann}_S(m_i \otimes 1)$. Set $I_i = \mathrm{Ann}_R(m_i)$. Then it suffices to show that $\bigcap_{i=1,\ldots,n}(I_i S) = (\bigcap_{i=1,\ldots,n} I_i)S$. This is Lemma 1.9.2. □

Proposition 1.12.5. *Let R be a ring and let M be an R-module. If M is finite, then $\mathrm{Supp}(M)$ is closed. More precisely, we have $V(\mathrm{Ann}(M)) = \mathrm{Supp}(M)$.*

Proof. We will show that $V(I) = \mathrm{Supp}(M)$ where $I = \mathrm{Ann}(M)$.

Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$. Then $M_\mathfrak{p} \neq 0$. Choose an element $m \in M$ whose image in $M_\mathfrak{p}$ is nonzero. Then the annihilator of m is contained in $\mathfrak{p}$ by construction of the localization $M_\mathfrak{p}$. Hence a fortiori $I = \mathrm{Ann}(M)$ must be contained in $\mathfrak{p}$.

Conversely, suppose that $\mathfrak{p} \notin \mathrm{Supp}(M)$. Then $M_\mathfrak{p} = 0$. Let $x_1, \ldots, x_r \in M$ be generators. Then there exists an $f \in R$, $f \notin \mathfrak{p}$ such that $x_i/1 = 0$ in M_f. Hence $f^{n_i} x_i = 0$ for some $n_i \geq 1$. Hence $f^n M = 0$ for $n = \max\{n_i\}$ as desired. □

Proposition 1.12.6. *Let $R \to R'$ be a ring map and let M be a finite R-module. Then $\mathrm{Supp}(M \otimes_R R')$ is the inverse image of $\mathrm{Supp}(M)$.*

Proof. Let $\mathfrak{p} \in \mathrm{Supp}(M)$. By Nakayama's lemma 1.7.2 we see that

$$M \otimes_R \kappa(\mathfrak{p}) = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$$

is a nonzero $\kappa(\mathfrak{p})$ vector space. Hence for every prime $\mathfrak{p}' \subset R'$ lying over $\mathfrak{p}$ we see that

$$(M \otimes_R R')_{\mathfrak{p}'}/\mathfrak{p}'(M \otimes_R R')_{\mathfrak{p}'} = (M \otimes_R R') \otimes_{R'} \kappa(\mathfrak{p}') = M \otimes_R \kappa(\mathfrak{p}) \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}')$$

is nonzero. This implies $\mathfrak{p}' \in \mathrm{Supp}(M \otimes_R R')$. For the converse, if $\mathfrak{p}' \subset R'$ is a prime lying over an arbitrary prime $\mathfrak{p} \subset R$, then

$$(M \otimes_R R')_{\mathfrak{p}'} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}.$$

Hence if $\mathfrak{p}' \in \mathrm{Supp}(M \otimes_R R')$ lies over the prime $\mathfrak{p} \subset R$, then $\mathfrak{p} \in \mathrm{Supp}(M)$. □

Lemma 1.12.7. *Let R be a ring, let M be an R-module, and let $m \in M$. Then $\mathfrak{p} \in V(\mathrm{Ann}(m))$ if and only if m does not map to zero in $M_\mathfrak{p}$.*

Proof. We may replace M by $Rm \subset M$. Then (1) $\mathrm{Ann}(m) = \mathrm{Ann}(M)$ and (2) m does not map to zero in $M_\mathfrak{p}$ if and only if $\mathfrak{p} \in \mathrm{Supp}(M)$. The result now follows from Proposition 1.12.5. □

Proposition 1.12.8. *Let R be a ring and let M be an R-module. If M is a finitely presented R-module, then $\mathrm{Supp}(M)$ is a closed subset of $\mathrm{Spec}(R)$ whose complement is quasi-compact.*

Proof. Choose a presentation

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \to 0$$

Let $A \in \mathrm{Mat}(n \times m, R)$ be the matrix of the first map. By Nakayama's Lemma 1.7.2 we see that

$$M_{\mathfrak{p}} \neq 0 \Leftrightarrow M \otimes \kappa(\mathfrak{p}) \neq 0 \Leftrightarrow \mathrm{rank}(A \bmod \mathfrak{p}) < n.$$

Hence, if I is the ideal of R generated by the $n \times n$ minors of A, then $\mathrm{Supp}(M) = V(I)$. Since I is finitely generated, say $I = (f_1, \ldots, f_t)$, we see that $\mathrm{Spec}(R) \setminus V(I)$ is a finite union of the standard opens $D(f_i)$, hence quasi-compact. □

Proposition 1.12.9. *Let R be a ring and let M be an R-module.*

(1) If M is finite then the support of M/IM is $\mathrm{Supp}(M) \cap V(I)$*.*

(2) If $N \subset M$, then $\mathrm{Supp}(N) \subset \mathrm{Supp}(M)$*.*

(3) If Q is a quotient module of M then $\mathrm{Supp}(Q) \subset \mathrm{Supp}(M)$*.*

(4) If $0 \to N \to M \to Q \to 0$ is a short exact sequence then $\mathrm{Supp}(M) = \mathrm{Supp}(Q) \cup \mathrm{Supp}(N)$*.*

Proof. The functors $M \mapsto M_{\mathfrak{p}}$ are exact. This immediately implies all but the first assertion. For the first assertion we need to show that $M_{\mathfrak{p}} \neq 0$ and $I \subset \mathfrak{p}$ implies $(M/IM)_{\mathfrak{p}} = M_{\mathfrak{p}}/IM_{\mathfrak{p}} \neq 0$. This follows from Nakayama's Lemma 1.7.2. □

1.13 Hilbert Nullstellensatz and Jacobson Rings

1.13.1 Hilbert Nullstellensatz

This is the heart in the algebraic gepmetry.

Lemma 1.13.1. *Let E/F be a finite or more generally an algebraic extension of fields. Any subring $F \subset R \subset E$ is a field.*

Proof. Let $\alpha \in R$ be nonzero. Then $1, \alpha, \alpha^2, ...$ are contained in R. Then we find a nontrivial relation $a_0 + a_1\alpha + ... + a_d\alpha^d = 0$. We may assume $a_0 \neq 0$ because if not we can divide the relation by α to decrease d. Then we see that $a_0 = \alpha(-a_1 - ... - a_d\alpha^{d-1})$ which proves that α has inverse in R. □

Theorem 1.13.2 (Hilbert Nullstellensatz)**.** *Let k be a field and R br s finite type k-algebra.*

(1) For any maximal ideal $\mathfrak{m} \subset R$ the field extension $\kappa(\mathfrak{m})/k$ is finite.

(2) For any ideal $I \subset R$, we have

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I, \mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{m} \supset I, \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

Proof. It is enough to prove part (1) of the theorem for the case of a polynomial algebra $k[x_1, \ldots, x_n]$, because any finitely generated k-algebra is a quotient of such a polynomial algebra. We prove this by induction on n. The case $n = 0$ is clear. Suppose that $\mathfrak{m}$ is a maximal ideal in $k[x_1, \ldots, x_n]$. Let $\mathfrak{p} \subset k[x_n]$ be the intersection of $\mathfrak{m}$ with $k[x_n]$.

If $\mathfrak{p} \neq (0)$, then $\mathfrak{p}$ is maximal and generated by an irreducible monic polynomial P (because of the Euclidean algorithm in $k[x_n]$). Then $k' = k[x_n]/\mathfrak{p}$ is a finite field extension of k and contained in $\kappa(\mathfrak{m})$. In this case we get a surjection

$$k'[x_1, \ldots, x_{n-1}] \to k'[x_1, \ldots, x_n] = k' \otimes_k k[x_1, \ldots, x_n] \longrightarrow \kappa(\mathfrak{m})$$

and hence we see that $\kappa(\mathfrak{m})$ is a finite extension of k' by induction hypothesis. Thus $\kappa(\mathfrak{m})$ is finite over k as well.

If $\mathfrak{p} = (0)$ we consider the ring extension $k[x_n] \subset k[x_1, \ldots, x_n]/\mathfrak{m}$. This is a finitely generated ring extension, hence of finite presentation. Thus the image of $\mathrm{Spec}(k[x_1, \ldots, x_n]/\mathfrak{m})$ in $\mathrm{Spec}(k[x_n])$ is constructible by Theorem 1.8.12. Since the image contains (0) we conclude that it contains a standard open $D(f)$ for some $f \in k[x_n]$ nonzero. Since clearly $D(f)$ is infinite we get a contradiction with the assumption that $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a field (and hence has a spectrum consisting of one point).

Proof of (2). WLOG we let $I \subset R$ be a radical ideal. Let $f \in R$, $f \notin I$. We have to find a maximal ideal $\mathfrak{m} \subset R$ with $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. The ring $(R/I)_f$ is nonzero, since $1 = 0$ in this ring would mean $f^n \in I$ and since I is radical this would mean $f \in I$ contrary to our assumption on f. Thus we may choose a maximal ideal $\mathfrak{m}'$ in $(R/I)_f$. Let $\mathfrak{m} \subset R$ be the inverse image of $\mathfrak{m}'$ in R. We see that $I \subset \mathfrak{m}$ and $f \notin \mathfrak{m}$. If we show that $\mathfrak{m}$ is a maximal ideal of R, then we are done. We clearly have

$$k \subset R/\mathfrak{m} \subset \kappa(\mathfrak{m}').$$

By part (1) the field extension $\kappa(\mathfrak{m}')/k$ is finite. Hence $R/\mathfrak{m}$ is a field by Lemma 1.13.1. Thus $\mathfrak{m}$ is maximal and the proof is complete. □

Remark 1.13.3. *We will give a more simple proof using Noether normalization theorem. There is another interesting proof using Rabinowitsch's trick in [Kem11].*

Corollary 1.13.4. *Let R be a ring. Let K be a field. If $R \subset K$ and K is of finite type over R, then there exists an $f \in R$ such that R_f is a field, and K/R_f is a finite field extension.*

Proof. By Lemma 1.8.14 there exist a nonempty open $U \subset \mathrm{Spec}(R)$ contained in the image $\{(0)\}$ of $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$. Choose $f \in R$, $f \neq 0$ such that $D(f) \subset U$, i.e., $D(f) = \{(0)\}$. Then R_f is a domain whose spectrum has exactly one point and R_f is a field. Then K is a finitely generated algebra over the field R_f and hence a finite field extension of R_f by the Hilbert Nullstellensatz (Theorem 1.13.2). □

Corollary 1.13.5 (Special Hilbert Nullstellensatz)**.** *Let k be an algebraically closed field.*

(1) Let A be a finite type k-algebra, then $A/\mathfrak{m} = k$ for maximal ideal $\mathfrak{m}$.

(2) Let $\mathfrak{m} \subset k[x_1, ..., x_n]$ be a maximal ideal, then we have

$$\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n).$$

Proof. Follows directly from Hilbert Nullstellensatz 1.13.2. □

1.13.2 Jacobson Rings

Jacobson rings motivated from Hilbert nullstellensatz.

Definition 1.13.6. *Let R be a ring. We say that R is a* Jacobson ring *if for any ideal $I \subset R$, we have*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I, \mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{m} \supset I, \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

Corollary 1.13.7. *Any algebra of finite type over a field is Jacobson.*

Proof. This follows from Hilbert nullstellensatz Theorem 1.13.2. □

Lemma 1.13.8. *Let R be a ring. Then R is Jacobson if and only if for any prime ideal $\mathfrak{p} \subset R$ we have*

$$\sqrt{p} = \bigcap_{\mathfrak{m} \supset p, \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

Proof. This is immediately clear from the fact that every radical ideal $I \subset R$ is the intersection of the primes containing it. □

Proposition 1.13.9. *A ring R is Jacobson if and only if $\mathrm{Spec}(R)$ is Jacobson, that is, every closed subset $Z \subset \mathrm{Spec}(R)$ is the closure of $Z \cap X_0$ where $X_0 \subset \mathrm{Spec}(R)$ is the subset of closed points.*

Proof. Suppose R is Jacobson. Let $Z \subset \text{Spec}(R)$ be a closed subset. We have to show that the set of closed points in Z is dense in Z. Let $U \subset \text{Spec}(R)$ be an open such that $U \cap Z$ is nonempty. We have to show $Z \cap U$ contains a closed point of $\text{Spec}(R)$. We may assume $U = D(f)$ as standard opens form a basis for the topology on $\text{Spec}(R)$ and $Z = V(I)$, where I is a radical ideal. We see also that $f \notin I$. By assumption, there exists a maximal ideal $\mathfrak{m} \subset R$ such that $I \subset \mathfrak{m}$ but $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f) \cap V(I) = U \cap Z$ as desired.

Conversely, suppose that $\text{Spec}(R)$ is Jacobson. Let $I \subset R$ be a radical ideal. Let $J = \cap_{I \subset \mathfrak{m}} \mathfrak{m}$ be the intersection of the maximal ideals containing I. Clearly J is a radical ideal, $V(J) \subset V(I)$, and $V(J)$ is the smallest closed subset of $V(I)$ containing all the closed points of $V(I)$. By assumption we see that $V(J) = V(I)$. But there is a bijection between Zariski closed sets and radical ideals, hence $I = J$ as desired. □

Proposition 1.13.10. *More generally, let R be a ring such that*

(1) R is a domain,

(2) R is Noetherian,

(3) any nonzero prime ideal is a maximal ideal, and

(4) R has infinitely many maximal ideals.

Then R is a Jacobson ring. In particular, the ring $\mathbb{Z}$ is a Jacobson ring.

Proof. Let R satisfy (1), (2), (3) and (4). The statement means that $(0) = \bigcap_{\mathfrak{m} \subset R} \mathfrak{m}$. Since R has infinitely many maximal ideals it suffices to show that any nonzero $x \in R$ is contained in at most finitely many maximal ideals, in other words that $V(x)$ is finite. We know that $V(x)$ is homeomorphic to $\text{Spec}(R/xR)$. By assumption (3) every prime of R/xR is minimal and hence corresponds to an irreducible component of $\text{Spec}(R/xR)$. As R/xR is Noetherian, the topological space $\text{Spec}(R/xR)$ is Noetherian and has finitely many irreducible components. Thus $V(x)$ is finite as desired. □

Proposition 1.13.11. *Let R be a Jacobson ring.*

(1) Let $f \in R$. The ring R_f is Jacobson and maximal ideals of R_f correspond to maximal ideals of R not containing f.

(2) Let $I \subset R$ be an ideal. The ring R/I is Jacobson and maximal ideals of R/I correspond to maximal ideals of R containing I.

Proof. For (1), by Proposition 1.13.9 we see that $D(f) = \text{Spec}(R_f)$ is Jacobson and that closed points of $D(f)$ correspond to closed points in $\text{Spec}(R)$ which happen to lie in $D(f)$. Thus R_f is Jacobson.

The proof of (2) is similar as (1). □

Here we will consider a general version of Hilbert nullstellensatz. But before doing that, we will consider some lemmas.

Lemma 1.13.12. *Let R be a ring. If R is not Jacobson there exist a prime $\mathfrak{p} \subset R$, an element $f \in R$ such that the following hold*

(1) $\mathfrak{p}$ is not a maximal ideal,

(2) $f \notin \mathfrak{p}$,

(3) $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, and

(4) $(R/\mathfrak{p})_f$ is a field.

On the other hand, if R is Jacobson, then for any pair $(\mathfrak{p}, f)$ such that (1) and (2) hold the set $V(\mathfrak{p}) \cap D(f)$ is infinite.

Proof. Assume R is not Jacobson. By Proposition 1.13.9 this means there exists an closed subset $T \subset \operatorname{Spec}(R)$ whose set $T_0 \subset T$ of closed points is not dense in T. Choose an $f \in R$ such that $T_0 \subset V(f)$ but $T \not\subset V(f)$. Note that $T \cap D(f)$ is homeomorphic to $\operatorname{Spec}((R/I)_f)$ if $T = V(I)$. As any ring has a maximal ideal we can choose a closed point t of space $T \cap D(f)$. Then t corresponds to a prime ideal $\mathfrak{p} \subset R$ which is not maximal (as $t \notin T_0$). Thus (1) holds. By construction $f \notin \mathfrak{p}$, hence (2). As t is a closed point of $T \cap D(f)$ we see that $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}\}$, i.e., (3) holds. Hence we conclude that $(R/\mathfrak{p})_f$ is a domain whose spectrum has one point, hence (4) holds.

Conversely, suppose that R is Jacobson and $(\mathfrak{p}, f)$ satisfy (1) and (2). If $V(\mathfrak{p}) \cap D(f) = \{\mathfrak{p}, \mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ then $\mathfrak{p} \neq \mathfrak{q}_i$ implies there exists an element $g \in R$ such that $g \notin \mathfrak{p}$ but $g \in \mathfrak{q}_i$ for all i. Hence $V(\mathfrak{p}) \cap D(fg) = \{\mathfrak{p}\}$ which is impossible since each locally closed subset of $\operatorname{Spec}(R)$ contains at least one closed point as $\operatorname{Spec}(R)$ is a Jacobson topological space. □

Lemma 1.13.13. *Let $R \to S$ be a ring map. Let $\mathfrak{m} \subset R$ be a maximal ideal. Let $\mathfrak{q} \subset S$ be a prime ideal lying over $\mathfrak{m}$ such that $\kappa(\mathfrak{q})/\kappa(\mathfrak{m})$ is an algebraic field extension. Then $\mathfrak{q}$ is a maximal ideal of S.*

Proof. Consider the diagram

$$\begin{tikzcd} S \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\ R \arrow[u] \arrow[r] & R/\mathfrak{m} \arrow[u] & \end{tikzcd}$$

We see that $\kappa(\mathfrak{m}) \subset S/\mathfrak{q} \subset \kappa(\mathfrak{q})$. Because the field extension $\kappa(\mathfrak{m}) \subset \kappa(\mathfrak{q})$ is algebraic, any ring between $\kappa(\mathfrak{m})$ and $\kappa(\mathfrak{q})$ is a field Thus $S/\mathfrak{q}$ is a field, and a posteriori equal to $\kappa(\mathfrak{q})$. □

Lemma 1.13.14. *Let R be a Jacobson ring. Let K be a field. Let $R \subset K$ and K is of finite type over R. Then R is a field and K/R is a finite field extension.*

Proof. First note that R is a domain. By Lemma 1.13.4 we see that R_f is a field and K/R_f is a finite field extension for some nonzero $f \in R$. Hence (0) is a maximal ideal of R_f and by Proposition 1.13.11 we conclude (0) is a maximal ideal of R. □

Here is our main result:

Theorem 1.13.15 (General Hilbert Nullstellensatz)**.** *Let R be a Jacobson ring. Let $R \to S$ be a ring map of finite type. Then*

(1) The ring S is Jacobson.

(2) The map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ transforms closed points to closed points.

(3) For $\mathfrak{m}' \subset S$ maximal lying over $\mathfrak{m} \subset R$ the field extension $\kappa(\mathfrak{m}')/\kappa(\mathfrak{m})$ is finite.

Proof. Let $\mathfrak{m}' \subset S$ be a maximal ideal and $R \cap \mathfrak{m}' = \mathfrak{m}$. Then $R/\mathfrak{m} \to S/\mathfrak{m}'$ satisfies the conditions of Lemma 1.13.14 by Proposition 1.13.11(2). Hence $R/\mathfrak{m}$ is a field and $\mathfrak{m}$ a maximal ideal and the induced residue field extension is finite. This proves (2) and (3).

If S is not Jacobson, then by Lemma 1.13.12 there exists a non-maximal prime ideal $\mathfrak{q}$ of S and an $g \in S$, $g \notin \mathfrak{q}$ such that $(S/\mathfrak{q})_g$ is a field. To arrive at a contradiction we show that $\mathfrak{q}$ is a maximal ideal. Let $\mathfrak{p} = \mathfrak{q} \cap R$. Then $R/\mathfrak{p} \to (S/\mathfrak{q})_g$ satisfies the conditions of Lemma 1.13.14 by Proposition 1.13.11(2). Hence $R/\mathfrak{p}$ is a field and the field extension $\kappa(\mathfrak{p}) \to (S/\mathfrak{q})_g = \kappa(\mathfrak{q})$ is finite, thus algebraic. Then $\mathfrak{q}$ is a maximal ideal of S by Lemma 1.13.13. Contradiction. □

Remark 1.13.16. *There is another generalization of Hilbert nullstellensatz:*

- *Let k be a field. Let S be a k-algebra generated over k by the elements $\{x_i\}_{i \in I}$. Assume the cardinality of I is smaller than the cardinality of k. Then*

  (1) for all maximal ideals $\mathfrak{m} \subset S$ the field extension $\kappa(\mathfrak{m})/k$ is algebraic, and

  (2) S is a Jacobson ring.

See the proof for Tag 00FU.

1.14 Zerodivisors and Total Rings of Fractions

Lemma 1.14.1. *Let $\mathfrak{p}$ be a minimal prime of a ring R. Every element of the maximal ideal of $R_{\mathfrak{p}}$ is nilpotent. If R is reduced then $R_{\mathfrak{p}}$ is a field.*

Proof. If some element x of $\mathfrak{p}R_{\mathfrak{p}}$ is not nilpotent, then $D(x) \neq \emptyset$. This contradicts the minimality of $\mathfrak{p}$. If R is reduced, then $\mathfrak{p}R_{\mathfrak{p}} = 0$ and hence it is a field. □

Proposition 1.14.2. *Let R be a reduced ring. Then*

1. *R is a subring of a product of fields,*
2. *$R \to \prod_{\mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$ is an embedding into a product of fields,*
3. *$\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$ is the set of zerodivisors of R.*

Proof. By Lemma 1.14.1 each of the rings $R_{\mathfrak{p}}$ is a field. In particular, the kernel of the ring map $R \to R_{\mathfrak{p}}$ is $\mathfrak{p}$. We have $\bigcap_{\mathfrak{p}} \mathfrak{p} = (0)$. Hence (2) and (1) are true. If $xy = 0$ and $y \neq 0$, then $y \notin \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Hence $x \in \mathfrak{p}$. Thus every zerodivisor of R is contained in $\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p}$. Conversely, suppose that $x \in \mathfrak{p}$ for some minimal prime $\mathfrak{p}$. Then x maps to zero in $R_{\mathfrak{p}}$, hence there exists $y \in R$, $y \notin \mathfrak{p}$ such that $xy = 0$. In other words, x is a zerodivisor. This finishes the proof of (3) and the lemma. □

Lemma 1.14.3. *Let R be a ring. Let $S \subset R$ be a multiplicative subset consisting of nonzerodivisors. Then $Q(R) \cong Q(S^{-1}R)$. In particular $Q(R) \cong Q(Q(R))$.*

Proof. If $x \in S^{-1}R$ is a nonzerodivisor, and $x = r/f$ for some $r \in R$, $f \in S$, then r is a nonzerodivisor in R. Whence the lemma. □

Proposition 1.14.4. *Let R be a ring. Assume that R has finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_t$, and that $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$ is the set of zerodivisors of R. Then the total ring of fractions $Q(R)$ is equal to $R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$.*

Proof. There are natural maps $Q(R) \to R_{\mathfrak{q}_i}$ since any nonzerodivisor is contained in $R \setminus \mathfrak{q}_i$. Hence a natural map $Q(R) \to R_{\mathfrak{q}_1} \times \ldots \times R_{\mathfrak{q}_t}$. For any nonminimal prime $\mathfrak{p} \subset R$ we see that $\mathfrak{p} \not\subset \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_t$. Hence $\mathrm{Spec}(Q(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}$ (as subsets of $\mathrm{Spec}(R)$). Therefore $\mathrm{Spec}(Q(R))$ is a finite discrete set and it follows that $Q(R) = A_1 \times \ldots \times A_t$ with $\mathrm{Spec}(A_i) = \{q_i\}$. Moreover A_i is a local ring, which is a localization of R. Hence $A_i \cong R_{\mathfrak{q}_i}$. □

Chapter 2

Projective, Injective and Flat Modules

2.1 Projective and Locally Free Modules

2.1.1 General Properties

Definition 2.1.1. *Let R be a ring. An R-module P is* projective *if and only if the functor* $\mathrm{Hom}_R(P, -) : \mathsf{Mod}_R \to \mathsf{Mod}_R$ *is an exact functor.*

The functor $\mathrm{Hom}_R(M, -)$ is left exact for any R-module M. Hence the condition for P to be projective really signifies that given a surjection of R-modules $N \to N'$ the map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, N')$ is surjective.

Proposition 2.1.2. *Let R be a ring. Let P be an R-module. The following are equivalent*

(1) P is projective,

(2) P is a direct summand of a free R-module, and

(3) $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M.

In particular, projective modules are flat.

Proof. Assume P is projective. Choose a surjection $\pi : F \to P$ where F is a free R-module. As P is projective there exists a $i \in \mathrm{Hom}_R(P, F)$ such that $\pi \circ i = \mathrm{id}_P$. In other words $F \cong \ker(\pi) \oplus i(P)$ and we see that P is a direct summand of F.

Conversely, assume that $P \oplus Q = F$ is a free R-module. Note that the free module $F = \bigoplus_{i \in I} R$ is projective as $\mathrm{Hom}_R(F, M) = \prod_{i \in I} M$ and the functor $M \mapsto \prod_{i \in I} M$ is exact. Then $\mathrm{Hom}_R(F, -) = \mathrm{Hom}_R(P, -) \times \mathrm{Hom}_R(Q, -)$ as functors, hence both P and Q are projective.

Assume $P \oplus Q = F$ is a free R-module. Then we have a free resolution $F_\bullet$ of the form

$$\dots F \xrightarrow{a} F \xrightarrow{b} F \to P \to 0$$

where the maps a, b alternate and are equal to the projector onto P and Q. Hence the complex $\mathrm{Hom}_R(F_\bullet, M)$ is split exact in degrees ≥ 1, whence we see the vanishing in (3).

Assume $\mathrm{Ext}^1_R(P, M) = 0$ for every R-module M. Pick a free resolution $F_\bullet \to P$. Set $M = \mathrm{Im}(F_1 \to F_0) = \ker(F_0 \to P)$. Consider the element $\xi \in \mathrm{Ext}^1_R(P, M)$ given by the class of the quotient map $\pi : F_1 \to M$. Since ξ is zero there exists a map $s : F_0 \to M$ such that $\pi = s \circ (F_1 \to F_0)$. Clearly, this means that

$$F_0 = \ker(s) \oplus \ker(F_0 \to P) = P \oplus \ker(F_0 \to P)$$

and we win. □

Proposition 2.1.3. *A direct sum of projective modules is projective.*

Proof. This is true by the characterization of projectives as direct summands of free modules in Proposition 2.1.2. □

Proposition 2.1.4. *Let R be a Noetherian ring. Let P be a finite R-module. If $\mathrm{Ext}^1_R(P, M) = 0$ for every finite R-module M, then P is projective.*

This proposition can be strengthened: There is a version for finitely presented R-modules if R is not assumed Noetherian. There is a version with M running through all finite length modules in the Noetherian case.

Proof. Choose a surjection $R^{\oplus n} \to P$ with kernel M. Since $\mathrm{Ext}^1_R(P, M) = 0$ this surjection is split and we conclude by Proposition 2.1.2. □

Proposition 2.1.5. *Let $R \to S$ be a ring map. Let P be a projective R-module, then $P \otimes_R S$ is a projective S-module.*

Proof. As P is a direct summand of some free R-module by Proposition 2.1.2, so is $P \otimes_R S$. □

Here we give a extension theorem of nilpotent ideals without proof.

Proposition 2.1.6. *Let R be a ring.*

(1) Let $I \subset R$ be a nilpotent ideal. Let $\overline{P}$ be a projective R/I-module. Then there exists a projective R-module P such that $P/IP \cong \overline{P}$.

(2) Let $I \subset R$ be a locally nilpotent ideal. Let $\overline{P}$ be a finite projective R/I-module. Then there exists a finite projective R-module P such that $P/IP \cong \overline{P}$.

(3) *Let $I \subset R$ be an ideal. Let M be an R-module. Assume*

(a) *I is nilpotent,*

(b) *M/IM is a projective R/I-module,*

(c) *M is a flat R-module.*

Then M is a projective R-module.

Proof. See Tag 07LV, Tag 0D47 and Tag 05CG. □

2.1.2 Finite Projective Modules

Definition 2.1.7. *Let R be a ring and M an R-module.*

(1) *We say that M is locally free if we can cover $\operatorname{Spec}(R)$ by standard opens $D(f_i)$, $i \in I$ such that M_{f_i} is a free R_{f_i}-module for all $i \in I$.*

(2) *We say that M is finite locally free if we can choose the covering such that each M_{f_i} is finite free.*

(3) *We say that M is finite locally free of rank r if we can choose the covering such that each M_{f_i} is isomorphic to $R_{f_i}^{\oplus r}$.*

Here is an important thing about finite projective modules and locally free modules:

Proposition 2.1.8. *Let R be a ring and let M be an R-module. The following are equivalent*

(1) *M is finitely presented and R-flat,*

(2) *M is finite projective,*

(3) *M is a direct summand of a finite free R-module,*

(4) *M is finitely presented and for all $\mathfrak{p} \in \operatorname{Spec}(R)$ the localization $M_{\mathfrak{p}}$ is free,*

(5) *M is finitely presented and for all maximal ideals $\mathfrak{m} \subset R$ the localization $M_{\mathfrak{m}}$ is free,*

(6) *M is finite and locally free,*

(7) *M is finite locally free, and*

(8) *M is finite, for every prime $\mathfrak{p}$ the module $M_{\mathfrak{p}}$ is free, and the function*

$$\rho_M : \operatorname{Spec}(R) \to \mathbb{Z}, \quad \mathfrak{p} \longmapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$$

is locally constant in the Zariski topology.

If R is reduced and let M be an R-module. Then the above equivalent conditions are also equivalent to

(9) *M is finite and the function $\rho_M : \mathrm{Spec}(R) \to \mathbb{Z}$, $\mathfrak{p} \mapsto \dim_{\kappa(\mathfrak{p})} M \otimes_R \kappa(\mathfrak{p})$ is locally constant in the Zariski topology.*

Proof. First suppose M is finite projective, i.e., (2) holds. Take a surjection $R^n \to M$ and let K be the kernel. Since M is projective, $0 \to K \to R^n \to M \to 0$ splits. Hence (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from the fact that a direct summand of a projective is projective, see Lemma 2.1.2.

Assume (3), so we can write $K \oplus M \cong R^{\oplus n}$. So K is a direct summand of R^n and thus finitely generated. This shows $M = R^{\oplus n}/K$ is finitely presented. In other words, (3) $\Rightarrow$ (1).

Assume M is finitely presented and flat, i.e., (1) holds. We will prove that (7) holds. Pick any prime $\mathfrak{p}$ and $x_1, \ldots, x_r \in M$ which map to a basis of $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's lemma (in the form of Lemma 1.7.3) these elements generate M_g for some $g \in R$, $g \notin \mathfrak{p}$. The corresponding surjection $\varphi : R_g^{\oplus r} \to M_g$ has the following two properties: (a) $\ker(\varphi)$ is a finite R_g-module and (b) $\ker(\varphi) \otimes \kappa(\mathfrak{p}) = 0$ by flatness of M_g over R_g (see Proposition1.9.10(1)). Hence by Nakayama's lemma again there exists a $g' \in R_g$ such that $\ker(\varphi)_{g'} = 0$. In other words, $M_{gg'}$ is free.

A finite locally free module is a finite module, see Proposition 1.8.10, hence (7) $\Rightarrow$ (6). It is clear that (6) $\Rightarrow$ (7) and that (7) $\Rightarrow$ (8).

A finite locally free module is a finitely presented module, see Proposition 1.8.10, hence (7) $\Rightarrow$ (4). Of course (4) implies (5). Since we may check flatness locally (see Lemma 1.9.11) we conclude that (5) implies (1). At this point we have

$$\begin{tikzcd}
(2) \arrow[r, leftrightarrow] & (3) \arrow[r] & (1) \arrow[r] & (7) \arrow[r, leftrightarrow] \arrow[d] \arrow[dr] & (6) \\
 & & (5) \arrow[u] & (4) \arrow[l] & (8)
\end{tikzcd}$$

Suppose that M satisfies (1), (4), (5), (6), and (7). We will prove that (3) holds. It suffices to show that M is projective. We have to show that $\mathrm{Hom}_R(M, -)$ is exact. Let $0 \to N'' \to N \to N' \to 0$ be a short exact sequence of R-module. We have to show that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact. As M is finite locally free there exist a covering $\mathrm{Spec}(R) = \bigcup D(f_i)$ such that M_{f_i} is finite free. We see that

$$0 \to \mathrm{Hom}_R(M, N'')_{f_i} \to \mathrm{Hom}_R(M, N)_{f_i} \to \mathrm{Hom}_R(M, N')_{f_i} \to 0$$

is equal to $0 \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N''_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N_{f_i}) \to \mathrm{Hom}_{R_{f_i}}(M_{f_i}, N'_{f_i}) \to 0$ which is exact as M_{f_i} is free and as the localization $0 \to N''_{f_i} \to N_{f_i} \to N'_{f_i} \to 0$ is exact (as localization is exact). Whence we see that $0 \to \mathrm{Hom}_R(M, N'') \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N') \to 0$ is exact by Proposition 1.8.10.

Finally, assume that (8) holds. Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 1.7.2 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. To see the claim it suffices to show that the induced map on localizations $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in D(fg)$, see Lemma 1.8.9. By our choice of f the map $\Psi_\mathfrak{p}$ is surjective. By assumption (8) we have $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus \rho_M(\mathfrak{p})}$ and by our choice of g we have $\rho_M(\mathfrak{p}) = r$. Hence $\Psi_\mathfrak{p}$ determines a surjection $R_\mathfrak{p}^{\oplus r} \to M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus r}$ whence is an isomorphism by Corollary 1.6.4.

Finally let R reduced, we consider (9). Pick a maximal ideal $\mathfrak{m} \subset R$. Pick $x_1, \ldots, x_r \in M$ which map to a $\kappa(\mathfrak{m})$-basis of $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$. In particular $\rho_M(\mathfrak{m}) = r$. By Nakayama's Lemma 1.7.2 there exists an $f \in R$, $f \notin \mathfrak{m}$ such that $x_1, \ldots, x_r$ generate M_f over R_f. By the assumption that ρ_M is locally constant there exists a $g \in R$, $g \notin \mathfrak{m}$ such that ρ_M is constant equal to r on $D(g)$. We claim that

$$\Psi : R_{fg}^{\oplus r} \longrightarrow M_{fg}, \quad (a_1, \ldots, a_r) \longmapsto \sum a_i x_i$$

is an isomorphism. This claim will show that M is finite locally free, i.e., that (7) holds. Since Ψ is surjective, it suffices to show that Ψ is injective. Since R_{fg} is reduced, it suffices to show that Ψ is injective after localization at all minimal primes $\mathfrak{p}$ of R_{fg}, see Lemma 1.14.2. However, we know that $R_\mathfrak{p} = \kappa(\mathfrak{p})$ by Lemma 1.14.1 and $\rho_M(\mathfrak{p}) = r$ hence $\Psi_\mathfrak{p} : R_\mathfrak{p}^{\oplus r} \to M \otimes_R \kappa(\mathfrak{p})$ is an isomorphism as a surjective map of finite dimensional vector spaces of the same dimension. □

Remark 2.1.9. *It is not true that a finite R-module which is R-flat is automatically projective. A counter example is where $R = \mathcal{C}^\infty(\mathbf{R})$ is the ring of infinitely differentiable functions on $\mathbf{R}$, and $M = R_\mathfrak{m} = R/I$ where $\mathfrak{m} = \{f \in R \mid f(0) = 0\}$ and $I = \{f \in R \mid \exists \epsilon, \epsilon > 0 : f(x) = 0\ \forall x, |x| < \epsilon\}$.*

Proposition 2.1.10. *Suppose R is a local ring, and M is a finite flat R-module. Then M is finite free.*

Proof. Follows from the equational criterion of flatness, see Lemma 1.9.9. Namely, suppose that $x_1, \ldots, x_r \in M$ map to a basis of $M/\mathfrak{m}M$. By Nakayama's Lemma 1.7.2 these elements generate M. We want to show there is no relation among the x_i. Instead, we will show by induction on n that if $x_1, \dots, x_n \in M$ are linearly independent in the vector space $M/\mathfrak{m}M$ then they are independent over R.

The base case of the induction is where we have $x \in M$, $x \notin \mathfrak{m}M$ and a relation $fx = 0$. By the equational criterion there exist $y_j \in M$ and $a_j \in R$ such that $x = \sum a_j y_j$ and $fa_j = 0$ for all j. Since $x \notin \mathfrak{m}M$ we see that at least one a_j is a unit and hence $f = 0$.

Suppose that $\sum f_i x_i$ is a relation among $x_1, \dots, x_n$. By our choice of x_i we have $f_i \in \mathfrak{m}$. According to the equational criterion of flatness there exist $a_{ij} \in R$ and $y_j \in M$ such that $x_i = \sum a_{ij} y_j$ and $\sum f_i a_{ij} = 0$. Since $x_n \notin \mathfrak{m}M$ we see that $a_{nj} \notin \mathfrak{m}$ for at least one j. Since $\sum f_i a_{ij} = 0$ we get $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. The relation $\sum f_i x_i = 0$ now can be rewritten as $\sum_{i=1}^{n-1} f_i (x_i + (-a_{ij}/a_{nj}) x_n) = 0$. Note that the elements $x_i + (-a_{ij}/a_{nj}) x_n$ map to $n-1$ linearly independent elements of $M/\mathfrak{m}M$. By induction assumption we get that all the f_i, $i \leq n-1$ have to be zero, and also $f_n = \sum_{i=1}^{n-1} (-a_{ij}/a_{nj}) f_i$. This proves the induction step. $\square$

Remark 2.1.11. *This holds for any projective modules over local rings, we refer Tag 0593.*

Proposition 2.1.12. *Let $R \to S$ be a flat local homomorphism of local rings. Let M be a finite R-module. Then M is finite projective over R if and only if $M \otimes_R S$ is finite projective over S.*

Proof. By Proposition 2.1.8 being finite projective over a local ring is the same thing as being finite free. Suppose that $M \otimes_R S$ is a finite free S-module. Pick $x_1, \dots, x_r \in M$ whose images in $M/\mathfrak{m}_R M$ form a basis over $\kappa(\mathfrak{m})$. Then we see that $x_1 \otimes 1, \dots, x_r \otimes 1$ are a basis for $M \otimes_R S$. This implies that the map $R^{\oplus r} \to M, (a_i) \mapsto \sum a_i x_i$ becomes an isomorphism after tensoring with S. By faithful flatness of $R \to S$, see Corollary 1.9.15 we see that it is an isomorphism. $\square$

Remark 2.1.13. *For more faithfully flat descent for projectivity of modules, we refer Tag 058B.*

2.1.3 Projective Ideals and Invertible Ideals of Domains

Definition 2.1.14. *An ideal I of a domain R is called invertible if there exists an R-submodule M of the quotient field K of R such that $IM = R$.*

Proposition 2.1.15. *A nonzero ideal I of a domain R is invertible if and only if it is projective as an R-module.*

Proof. Let I be an invertible ideal with $IM = R$. Then $1 = \sum_{i=1}^n a_i q_i$ with $a_i \in I, q_i \in M$. Consider the free R-module $F = \bigoplus_{i=1}^n Re_i$. Let $f : F \to I$ be the homomorphism such that $f(e_i) = a_i$. Then $s : I \to F$ given by $s(a) = \sum_{i=1}^n a q_i e_i$ is a section of f. Thus I is a direct summand of F, hence projective.

Conversely, let I be an ideal of R that is projective as an R-module. Then there exists a free R-module $F = \bigoplus_\alpha Re_\alpha$ and homomorphisms $f : F \to I$ and $s : I \to F$ with $fs = \mathrm{id}_I$. Put $a_\alpha = f(e_\alpha)$. Let $a \in I$ with $a \neq 0$. Then by Lemma 2.1.16 we have $s(a) = \sum_\alpha aq_\alpha e_\alpha$ where $q_\alpha \in K$ and $q_\alpha I \subset R$. Let $M := \sum_\alpha Rq_\alpha$, then $a = fs(a) = \sum_\alpha aq_\alpha a_\alpha$. Hence $1 = \sum_\alpha q_\alpha a_\alpha$ and well done. □

Lemma 2.1.16. *Let $s : I \to R$ be a homomorphism of R-modules. Then there exists $q \in K = \mathrm{Frac}(R)$ such that $s(a) = qa$ for all $a \in I$.*

Proof. Note that $bs(a) = s(ab) = as(b)$. Hence $q := s(a)/a$ and well done. □

2.2 Injective Modules

Definition 2.2.1. *Let R be a ring. An R-module J is* injective *if and only if the functor $\mathrm{Hom}_R(-, J) : \mathsf{Mod}_R \to \mathsf{Mod}_R$ is an exact functor.*

The functor $\mathrm{Hom}_R(-, M)$ is left exact for any R-module M. Hence the condition for J to be injective really signifies that given an injection of R-modules $M \to M'$ the map $\mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M, J)$ is surjective.

2.2.1 General Properties

Proposition 2.2.2. *Let R be a ring. Let J be an R-module. The following are equivalent*

(1) J is injective,

(2) $\mathrm{Ext}^1_R(M, J) = 0$ for every R-module M.

(3) $\mathrm{Ext}^1_R(R/I, J) = 0$ for every ideal $I \subset R$, and

(4) for an ideal $I \subset R$ and module map $I \to J$ there exists an extension $R \to J$.

Proof. Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence of R-modules. Consider the long exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_R(M, J) \to \mathrm{Hom}_R(M', J) \to \mathrm{Hom}_R(M'', J) \\ &\to \mathrm{Ext}^1_R(M, J) \to \mathrm{Ext}^1_R(M', J) \to \mathrm{Ext}^1_R(M'', J) \to \dots \end{aligned}$$

Thus we see that (2) implies (1). Conversely, if J is injective then the Ext-group is zero similar as projective modules. Thus (1) and (2) are equivalent.

If $I \subset R$ is an ideal, then the short exact sequence $0 \to I \to R \to R/I \to 0$ gives an exact sequence

$$\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J) \to \mathrm{Ext}^1_R(R/I, J) \to 0$$

and the fact that $\mathrm{Ext}^1_R(R, J) = 0$ as R is projective (Algebra, Proposition 2.1.2). Thus (3) and (4) are equivalent. In this proof we will show that (1) $\Leftrightarrow$ (4) which is known as Baer's criterion.

Assume (1). Given a module map $I \to J$ as in (3) we find the extension $R \to J$ because the map $\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(I, J)$ is surjective by definition.

Assume (4). Let $M \subset N$ be an inclusion of R-modules. Let $\varphi : M \to J$ be a homomorphism. We will show that φ extends to N which finishes the proof of the lemma. Consider the set of homomorphisms $\varphi' : M' \to J$ with $M \subset M' \subset N$ and $\varphi'|_M = \varphi$. Define $(M', \varphi') \geq (M'', \varphi'')$ if and only if $M' \supset M''$ and $\varphi'|_{M''} = \varphi''$. If $(M_i, \varphi_i)_{i \in I}$ is a totally ordered collection of such pairs, then we obtain a map $\bigcup_{i \in I} M_i \to J$ defined by $a \in M_i$ maps to $\varphi_i(a)$. Thus Zorn's lemma applies. To conclude we have to show that if the pair (M', φ') is maximal then $M' = N$. In other words, it suffices to show, given any subgroup $M \subset N$, $M \neq N$ and any $\varphi : M \to J$, then we can find $\varphi' : M' \to J$ with $M \subset M' \subset N$ such that (a) the inclusion $M \subset M'$ is strict, and (b) the morphism φ' extends φ.

To prove this, pick $x \in N$, $x \notin M$. Let $I = \{f \in R \mid fx \in M\}$. This is an ideal of R. Define a homomorphism $\psi : I \to J$ by $f \mapsto \varphi(fx)$. Extend to a map $\tilde{\psi} : R \to J$ which is possible by assumption (4). By our choice of I the kernel of $M \oplus R \to J$, $(y, f) \mapsto y - \tilde{\psi}(f)$ contains the kernel of the map $M \oplus R \to N$, $(y, f) \mapsto y + fx$. Hence this homomorphism factors through the image $M' = M + Rx$ and this extends the given homomorphism as desired. $\square$

Here we give more properties of injective modules:

Proposition 2.2.3. *Let R be a ring.*

1. *Any product of injective R-modules is injective.*
2. *Let $R \to S$ be a flat ring map. If E is an injective S-module, then E is injective as an R-module.*
3. *Let $R \to S$ be an epimorphism of rings. Let E be an S-module. If E is injective as an R-module, then E is an injective S-module.*
4. *Let $R \to S$ be a ring map. If E is an injective R-module, then $\mathrm{Hom}_R(S, E)$ is an injective S-module.*
5. *Every direct summand of an injective module is injective.*

Proof. (1) is almost trivial and can be directly from diagram chase. For (2) this is true because $\mathrm{Hom}_R(M, E) = \mathrm{Hom}_S(M \otimes_R S, E)$ by Proposition 1.3.8 and the fact that tensoring with S is exact. For (3), this is true because $\mathrm{Hom}_R(N, E) = \mathrm{Hom}_S(N, E)$ for any S-module N. For (4), this is true because $\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, E)) = \mathrm{Hom}_R(N, E)$ by Proposition 1.3.9. For (5), this is directly from diagram chase. $\square$

For noetherian rings, we have more properties:

Proposition 2.2.4. *Let R be a Noetherian ring.*

(1) A direct sum of injective modules is injective.

(2) Let $S \subset R$ be a multiplicative subset. If E is an injective R-module, then $S^{-1}E$ is an injective $S^{-1}R$-module.

Proof. For (1), let E_i be a family of injective modules parametrized by a set I. Set $E = \bigoplus E_i$. To show that E is injective we use Proposition 2.2.3(3). Thus let $\varphi : I \to E$ be a module map from an ideal of R into E. As I is a finite R-module (because R is Noetherian) we can find finitely many elements $i_1, \ldots, i_r \in I$ such that φ maps into $\bigoplus_{j=1,\ldots,r} E_{i_j}$. Then we can extend φ into $\bigoplus_{j=1,\ldots,r} E_{i_j}$ using the injectivity of the modules E_{i_j}.

For (2), since $R \to S^{-1}R$ is an epimorphism of rings, it suffices to show that $S^{-1}E$ is injective as an R-module, see Proposition 2.2.3(3). To show this we use Baer criteria 2.2.2(4). Thus let $I \subset R$ be an ideal and let $\varphi : I \to S^{-1}E$ be an R-module map. As I is a finitely presented R-module (because R is Noetherian) we can find an $f \in S$ and an R-module map $I \to E$ such that $f\varphi$ is the composition $I \to E \to S^{-1}E$. Then we can extend $I \to E$ to a homomorphism $R \to E$. Then the composition

$$R \to E \to S^{-1}E \xrightarrow{f^{-1}} S^{-1}E$$

is the desired extension of φ to R. □

2.2.2 Divisible Modules over Domains

Definition 2.2.5. *Let M be an R-module over a domain R. If $r \in R$ and $m \in M$, then we say that m is* divisible by *r if there is some $m' \in M$ with $m = rm'$. We say that M is a* divisible module *if each $m \in M$ is divisible by every nonzero $r \in R$.*

Some trivial properties:

Proposition 2.2.6. *Let R be a domain.*

(1) Direct sums and direct products of divisible R-modules are divisible. It follows that every vector space over $\mathrm{Frac}(R)$ is a divisible R-module.

(2) Every quotient of a divisible R-module is divisible. It follows that every direct summand of a divisible R-module is divisible.

Proof. Trivial and omit. □

Proposition 2.2.7. *Let R be a domain.*

(1) Every injective R-module E is a divisible module.

(2) $Q := \mathrm{Frac}(R)$ is an injective R-module. Every vector space E over Q is an injective R-module.

Proof. For (1), assume that E is injective. Let $e \in E$ and let $r_0 \in R$ be nonzero; we must find $x \in E$ with $e = r_0 x$. Define $f : Rr_0 \to E$ by $f(rr0) = re$ (note that f is well-defined because R is a domain). Since E is injective, there exists $h : R \to E$ extending f. In particular, $e = f(r_0) = h(r_0) = r_0 h(1)$, so that $x = h(1)$ is the element in E required by the definition of divisible.

For (2), by Baer's Criterion, it suffices to extend an R-map $f : I \to Q$, where I is an ideal in R, to all of R. Note first that if $a, b \in I$ are nonzero, then $af(b) = f(ab) = bf(a)$, so that $f(a)/a = f(b)/b$ in Q for all nonzero $a, b \in I$; let $c \in Q$ denote their common value. Define $g : R \to Q$ by $g(r) = rc$ for all $r \in R$. It is obvious that g is an R-map. Easy to see that g extends f, well done. The second statement is similar and we omit it. We refer Proposition 3.34.(ii) in [JR09]. □

Corollary 2.2.8. *Let R be a PID.*

(1) An R-module E is injective if and only if it is divisible.

(2) Every quotient of an injective R-module E is itself injective.

Proof. For (1), we use Baer's Criterion. Assume that $f : I \to E$ is an R-map, where I is a nonzero ideal; by hypothesis, $I = Ra$ for some nonzero $a \in I$. Since E is divisible, there is $e \in E$ with $f(a) = ae$. Define $h : R \to E$ by $h(s) = se$ for all $s \in R$. It is easy to check that h is an R-map; moreover, h extends f. Therefore, E is injective.

For (2), since E is injective, it is divisible; hence, if $M \subset E$ is any submodule, then E/M is divisible. By (1), E/M is injective. □

Remark 2.2.9. *This is also right for Dedekind domain. Need to add later.*

2.2.3 Character Module

Here we introduce an interesting thing which give us some important results.

Definition 2.2.10. *If M is an R-module, define its* character module *B^* as the R-module $B^* = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})$.*

Proposition 2.2.11. *Let R be a ring.*

(1) A sequence of R-modules $A \to B \to C$ is exact if and only if $C^* \to B^* \to A^*$ is exact.

(2) *An R-module B is flat if and only if its character module B^* is an injective R-module.*

Proof. For (1), since $\mathbb{Q}/\mathbb{Z}$ is a divisible $\mathbb{Z}$-module. Hence it is an injective $\mathbb{Z}$-module by Corollary 2.2.8. Conversely this is a boring diagram chase. We refer to Lemma 3.53 in [JR09].

For (2), if R-module B is flat, by adjointness we can find that B^* is an injective R-module. Conversely, assume that B^* is an injective R-module and $A' \to A$ is an injection of R-modules. Now we have the following commutative diagram with isomorphic vertical maps and exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_R(A, B^*) \arrow[r] \arrow[d] & \mathrm{Hom}_R(A, B^*) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}_{\mathbb{Z}}(A \otimes_R B, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(A' \otimes_R B, \mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
(A \otimes_R B)^* \arrow[r] & (A' \otimes_R B)^* \arrow[r] & 0
\end{tikzcd}$$

Hence by (1) $0 \to A' \otimes_R B \to A \otimes_R B$ is exact. Hence B is flat. $\square$

Finally we will introduce an important result follows from the construction of character modules.

There is a canonical map $\mathrm{ev} : M \to M^{**}$ given by evaluation: given $x \in M$ we let $\mathrm{ev}(x) \in M^{**} = \mathrm{Hom}(M^*, \mathbb{Q}/\mathbb{Z})$ be the map $\varphi \mapsto \varphi(x)$.

Lemma 2.2.12. *For any R-module M the evaluation map* $\mathrm{ev} : M \to M^{**}$ *is injective.*

Proof. You can check this using that $\mathbb{Q}/\mathbb{Z}$ is an injective abelian group. Namely, if $x \in M$ is not zero, then let $M' \subset M$ be the cyclic group it generates. There exists a nonzero map $M' \to \mathbb{Q}/\mathbb{Z}$ which necessarily does not annihilate x. This extends to a map $\varphi : M \to \mathbb{Q}/\mathbb{Z}$ and then $\mathrm{ev}(x)(\varphi) = \varphi(x) \neq 0$. $\square$

The canonical surjection $F(M) \to M$ of R-modules turns into a canonical injection, see above, of R-modules

$$M^{**} \longrightarrow (F(M^*))^*.$$

Set $J(M) = (F(M^*))^*$. The composition of ev with this the displayed map gives $M \to J(M)$ functorially in M.

Theorem 2.2.13. *Let R be a ring.*

(1) *For every R-module M the R-module $J(M)$ is injective.*

(2) The construction above defines a covariant functor $M \mapsto (M \to J(M))$ from the category of R-modules to the category of arrows of R-modules such that for every module M the output $M \to J(M)$ is an injective map of M into an injective R-module $J(M)$.

Proof. Note that (2) follows directly from (1), so we just consider (1). For (1), Note that $J(M) \cong \prod_{\varphi \in M^*} R^*$ as an R-module. As the product of injective modules is injective, it suffices to show that R^* is injective. For this we use that

$$\mathrm{Hom}_R(N, R^*) = \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})) = N^*$$

and the fact that $(-)^*$ is an exact functor. □

In particular, for any map of R-modules $M \to N$ there is an associated morphism $J(M) \to J(N)$ making the following diagram commute:

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ J(M) & \longrightarrow & J(N) \end{array}$$

Well done.

2.2.4 Self-Injective Rings and Quasi-Frobenius Rings

Definition 2.2.14. *We say that a ring R is* self-injective *if the R is injective as R-module.*

Theorem 2.2.15 (Faith-Walker)**.** *Let R be a ring. The following conditions are equivalent.*

(1) R is Noetherian and self-injective.

(2) Every projective R-module is injective.

(3) Every injective R-module is projective.

In this case we call R a quasi-Frobenius ring*.*

Proof. Some directions are easy, but some of them are not. We omit it and we refer Chapter 24 in [Fai76]. □

Here we give some examples.

Proposition 2.2.16. *Let R be a PID and let $a \in R$ such that $a \neq 0$. Then R/aR is quasi-Frobenius.*

Proof. We apply Baer's criterion to show that it is self-injective. Let $A = bR/aR$ be an ideal of R/aR and let $h : A \to R/aR$ be a homomorphism. We need to extend h to R/aR. Set $h(\bar{b}) = \bar{r}$ for some $r \in R$. We have $Ra \subset Rb$, so that $a = cb$ for some nonzero $c \in R$. We have $0 = h(\bar{a}) = h(\bar{c}\bar{b}) = \bar{c}\bar{r}$. Thus $cr = as$ for some $s \in R$. Canceling c, we get $r = bs$, so $h(\bar{b}) = \bar{b}\bar{s}$. Thus h have been extended. □

Example 2.2.1. *The group algebra $k[G]$ is a quasi-Frobenius ring for any field k and any finite group G. See Proposition 3.14, Example 3.15E in [Lam99].*

2.3 More on Flatness

In this section we discuss criteria for flatness.

Proposition 2.3.1. *Let M be an R-module. The following are equivalent:*

(1) M is flat.

(2) If $f : R^n \to M$ is a module map and $x \in \ker(f)$, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $x \in \ker(h)$.

(3) Suppose $f : R^n \to M$ is a module map, $N \subset \ker(f)$ any submodule, and $h : R^n \to R^m$ a map such that $N \subset \ker(h)$ and f factors through h. Then given any $x \in \ker(f)$ we can find a map $h' : R^n \to R^{m'}$ such that $N + Rx \subset \ker(h')$ and f factors through h'.

(4) If $f : R^n \to M$ is a module map and $N \subset \ker(f)$ is a finitely generated submodule, then there are module maps $h : R^n \to R^m$ and $g : R^m \to M$ such that $f = g \circ h$ and $N \subset \ker(h)$.

Proof. That (1) is equivalent to (2) is just a reformulation of the equational criterion for flatness[1].

To show (2) implies (3), let $g : R^m \to M$ be the map such that f factors as $f = g \circ h$. By (2) find $h'' : R^m \to R^{m'}$ such that h'' kills $h(x)$ and $g : R^m \to M$ factors through h''. Then taking $h' = h'' \circ h$ works. (3) implies (4) by induction on the number of generators of $N \subset \ker(f)$ in (4). Clearly (4) implies (2). □

Proposition 2.3.2. *Let M be an R-module. Then M is flat if and only if the following condition holds: if P is a finitely presented R-module and $f : P \to M$ a module map,*

[1] In fact, a module map $f : R^n \to M$ corresponds to a choice of elements $x_1, x_2, \ldots, x_n$ of M (namely, the images of the standard basis elements $e_1, e_2, \ldots, e_n$); furthermore, an element $x \in \ker(f)$ corresponds to a relation between these $x_1, x_2, \ldots, x_n$ (namely, the relation $\sum_i f_i x_i = 0$, where the f_i are the coordinates of x). The module map h (represented as an $m \times n$-matrix) corresponds to the matrix (a_{ij}) from Lemma 1.9.9, and the y_j of Proposition 1.9.9 are the images of the standard basis vectors of R^m under g.

then there is a free finite R-module F and module maps $h : P \to F$ and $g : F \to M$ such that $f = g \circ h$.

Proof. This is just a reformulation of condition (4) from Proposition 2.3.1. □

Proposition 2.3.3. *Let M be an R-module. Then M is flat if and only if the following condition holds: for every finitely presented R-module P, if $N \to M$ is a surjective R-module map, then the induced map $\mathrm{Hom}_R(P, N) \to \mathrm{Hom}_R(P, M)$ is surjective.*

Proof. First suppose M is flat. We must show that if P is finitely presented, then given a map $f : P \to M$, it factors through the map $N \to M$. By Proposition 2.3.2 the map f factors through a map $F \to M$ where F is free and finite. Since F is free, this map factors through $N \to M$. Thus f factors through $N \to M$.

Conversely, suppose the condition of the Proposition holds. Let $f : P \to M$ be a map from a finitely presented module P. Choose a free module N with a surjection $N \to M$ onto M. Then f factors through $N \to M$, and since P is finitely generated, f factors through a free finite submodule of N. Thus M satisfies the condition of Proposition 2.3.2, hence is flat. □

Theorem 2.3.4 (Lazard's theorem)**.** *Let M be an R-module. Then M is flat if and only if it is the colimit of a directed system of free finite R-modules.*

Proof. A colimit of a directed system of flat modules is flat, as taking directed colimits is exact and commutes with tensor product. Hence if M is the colimit of a directed system of free finite modules then M is flat.

For the converse, first recall that any module M can be written as the colimit of a directed system of finitely presented modules, in the following way. Choose a surjection $f : R^I \to M$ for some set I, and let K be the kernel. Let E be the set of ordered pairs (J, N) where J is a finite subset of I and N is a finitely generated submodule of $R^J \cap K$. Then E is made into a directed partially ordered set by defining $(J, N) \leq (J', N')$ if and only if $J \subset J'$ and $N \subset N'$. Define $M_e = R^J/N$ for $e = (J, N)$, and define $f_{ee'} : M_e \to M_{e'}$ to be the natural map for $e \leq e'$. Then $(M_e, f_{ee'})$ is a directed system and the natural maps $f_e : M_e \to M$ induce an isomorphism $\varinjlim_{e \in E} M_e \xrightarrow{\cong} M$.

Now suppose M is flat. Let $I = M \times \mathbf{Z}$, write (x_i) for the canonical basis of R^I, and take in the above discussion $f : R^I \to M$ to be the map sending x_i to the projection of i onto M. To prove the theorem it suffices to show that the $e \in E$ such that M_e is free form a cofinal subset of E. So let $e = (J, N) \in E$ be arbitrary. By Lemma 2.3.2 there is a free finite module F and maps $h : R^J/N \to F$ and $g : F \to M$ such that the natural map $f_e : R^J/N \to M$ factors as $R^J/N \xrightarrow{h} F \xrightarrow{g} M$. We are going to realize F as $M_{e'}$ for some $e' \geq e$.

Let $\{b_1, \ldots, b_n\}$ be a finite basis of F. Choose n distinct elements $i_1, \ldots, i_n \in I$ such that $i_\ell \notin J$ for all ℓ, and such that the image of x_{i_ℓ} under $f : R^I \to M$ equals the image of b_ℓ under $g : F \to M$. This is possible since every element of M can be written as $f(x_i)$ for infinitely many distinct $i \in I$ (by our choice of I). Now let $J' = J \cup \{i_1, \ldots, i_n\}$, and define $R^{J'} \to F$ by $x_i \mapsto h(x_i)$ for $i \in J$ and $x_{i_\ell} \mapsto b_\ell$ for $\ell = 1, \ldots, n$. Let $N' = \ker(R^{J'} \to F)$. Observe:

1. The square

$$\begin{tikzcd} R^{J'} \arrow[r] \arrow[d, hook] & F \arrow[d, "g"] \\ R^I \arrow[r, "f"'] & M \end{tikzcd}$$

   is commutative, hence $N' \subset K = \ker(f)$;

2. $R^{J'} \to F$ is a surjection onto a free finite module, hence it splits and so N' is finitely generated;

3. $J \subset J'$ and $N \subset N'$.

By (1) and (2) $e' = (J', N')$ is in E, by (3) $e' \geq e$, and by construction $M_{e'} = R^{J'}/N' \cong F$ is free. □

2.4 Several Homology Dimensions

2.4.1 Projective Dimensions

The following lemma is often used to compare different projective resolutions of a given module.

Lemma 2.4.1 (Schanuel's lemma). *Let R be a ring. Let M be an R-module. Suppose that*

$$0 \to K \xrightarrow{c_1} P_1 \xrightarrow{p_1} M \to 0 \quad \textit{and} \quad 0 \to L \xrightarrow{c_2} P_2 \xrightarrow{p_2} M \to 0$$

are two short exact sequences, with P_i projective. Then $K \oplus P_2 \cong L \oplus P_1$. More precisely, there exist a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \oplus P_2 \arrow[r, "(c_1, \mathrm{id})"] \arrow[d] & P_1 \oplus P_2 \arrow[r, "(p_1, 0)"] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & P_1 \oplus L \arrow[r, "(\mathrm{id}, c_2)"] & P_1 \oplus P_2 \arrow[r, "(0, p_2)"] & M \arrow[r] & 0 \end{tikzcd}$$

whose vertical arrows are isomorphisms.

Proof. Boring diagram chase, we refer Tag 00O3 or [JR09]. □

Definition 2.4.2. *Let R be a ring and M be an R-module. We say M has finite projective dimension if it has a finite length resolution by projective R-modules. The minimal length* $\text{proj.}\dim_R(M)$ *of such a resolution is called the projective dimension of M.*

It is clear that the projective dimension of M is 0 if and only if M is a projective module. The following lemma explains to what extent the projective dimension is independent of the choice of a projective resolution.

Lemma 2.4.3. *Let R be a ring. Suppose* $\text{proj.}\dim_R(M) \leq d$. *Suppose that $F_e \to F_{e-1} \to \ldots \to F_0 \to M \to 0$ is exact with F_i projective and $e \geq d-1$. Then the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective in case $e = 0$).*

Proof. We prove this by induction on d. If $d = 0$, then M is projective. In this case there is a splitting $F_0 = \ker(F_0 \to M) \oplus M$, and hence $\ker(F_0 \to M)$ is projective. This finishes the proof if $e = 0$, and if $e > 0$, then replacing M by $\ker(F_0 \to M)$ we decrease e.

Next assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ be a minimal length finite resolution with P_i projective. According to Schanuel's Lemma 2.4.1 we have $P_0 \oplus \ker(F_0 \to M) \cong F_0 \oplus \ker(P_0 \to M)$. This proves the case $d = 1$, $e = 0$, because then the right hand side is $F_0 \oplus P_1$ which is projective. Hence now we may assume $e > 0$. The module $F_0 \oplus \ker(P_0 \to M)$ has the finite projective resolution

$$0 \to P_d \to P_{d-1} \to \ldots \to P_2 \to P_1 \oplus F_0 \to \ker(P_0 \to M) \oplus F_0 \to 0$$

of length $d-1$. By induction applied to the exact sequence

$$F_e \to F_{e-1} \to \ldots \to F_2 \to P_0 \oplus F_1 \to P_0 \oplus \ker(F_0 \to M) \to 0$$

of length $e-1$ we conclude $\ker(F_e \to F_{e-1})$ is projective (if $e \geq 2$) or that $\ker(F_1 \oplus P_0 \to F_0 \oplus P_0)$ is projective. This implies the lemma. $\square$

Proposition 2.4.4. *Let R be a ring. Let M be an R-module. Let $d \geq 0$. The following are equivalent.*

(1) $\text{proj.}\dim_R(M) \leq d$.

(2) there exists a resolution $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ with P_i projective.

(3) for some resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\ker(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\ker(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$.

(4) for any resolution $\ldots \to P_2 \to P_1 \to P_0 \to M \to 0$ with P_i projective we have $\ker(P_{d-1} \to P_{d-2})$ is projective if $d \geq 2$, or $\ker(P_0 \to M)$ is projective if $d = 1$, or M is projective if $d = 0$.

(5) $\mathrm{Ext}^i_R(M, N) = 0$ *for all* R*-modules* N *and all* $i \geq d + 1$.

(6) $\mathrm{Ext}^{d+1}_R(M, N) = 0$ *for all* R*-modules* N.

If R *be a local ring, then the equivalent conditions (1) – (6) are also equivalent to*

(7) *there exists a resolution* $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ *with* P_i *free.*

If R *be a Noetherian ring and* M *be a finite* R*-module, then the equivalent conditions (1) – (6) are also equivalent to*

(8) *there exists a resolution* $0 \to P_d \to P_{d-1} \to \ldots \to P_0 \to M \to 0$ *with* P_i *finite projective.*

If R *be a Noetherian local ring and* M *be a finite* R*-module, then the equivalent conditions (1) – (8) are also equivalent to*

(9) *there exists a resolution* $0 \to F_d \to F_{d-1} \to \ldots \to F_0 \to M \to 0$ *with* F_i *finite free.*

Proof. The equivalence of (1) and (2) is the definition of projective dimension. We have (2) $\Rightarrow$ (4) by Lemma 2.4.3. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) are immediate.

Assume (1). Choose a free resolution $F_\bullet \to M$ of M. Denote $d_e : F_e \to F_{e-1}$. By Lemma 2.4.3 we see that $P_e = \ker(d_e)$ is projective for $e \geq n - 1$. This implies that $F_e \cong P_e \oplus P_{e-1}$ for $e \geq n$ where d_e maps the summand P_{e-1} isomorphically to P_{e-1} in F_{e-1}. Hence, for any R-module N the complex $\mathrm{Hom}_R(F_\bullet, N)$ is split exact in degrees $\geq n + 1$. Whence (2) holds. The implication (5) $\Rightarrow$ (6) is trivial.

Assume (6) holds. If $n = 0$ then M is projective by Lemma 2.1.2 and we see that (1) holds. If $n > 0$ choose a free R-module F and a surjection $F \to M$ with kernel K. By the long exact sequence of Ext and the vanishing of $\mathrm{Ext}^i_R(F, N)$ for all $i > 0$ by part (1) we see that $\mathrm{Ext}^n_R(K, N) = 0$ for all R-modules N. Hence by induction we see that K has projective dimension $\leq n - 1$. Then M has projective dimension $\leq n$ as any finite projective resolution of K gives a projective resolution of length one more for M by adding F to the front. $\square$

Proposition 2.4.5. *Let* R *be a ring. Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* R*-modules.*

(1) *If* M *has projective dimension* $\leq n$ *and* M'' *has projective dimension* $\leq n + 1$*, then* M' *has projective dimension* $\leq n$.

(2) *If* M' *and* M'' *have projective dimension* $\leq n$ *then* M *has projective dimension* $\leq n$.

(3) *If* M' *has projective dimension* $\leq n$ *and* M *has projective dimension* $\leq n + 1$ *then* M'' *has projective dimension* $\leq n + 1$.

Proof. Combine the characterization of projective dimension in Proposition 2.4.4 with the long exact sequence of Ext groups. $\square$

2.4.2 Injective Dimensions

Need to add.

2.4.3 Tor-Dimensions

Definition 2.4.6. *Let R be a ring with a R-module M, we define its* Tor-dimension *is*

$$\operatorname{tor.dim}_R(M) := \sup\{n \in \mathbb{Z} : \operatorname{Tor}_n^R(M, N) \neq 0 \text{ for some } N\}.$$

2.4.4 Global Dimensions

Definition 2.4.7. *Let R be a ring. The ring R is said to have* finite global dimension *if there exists an integer n such that every R-module has a resolution by projective R-modules of length at most n. The minimal such $n =: \operatorname{gl.dim}(R)$ is then called the* global dimension *of R.*

First we consider a lemma:

Lemma 2.4.8. *Let R be a ring. Suppose we have a module $M = \bigcup_{e \in E} M_e$ where the M_e are submodules well-ordered by inclusion. Assume the quotients $M_e / \bigcup_{e' < e} M_{e'}$ have projective dimension $\leq n$. Then M has projective dimension $\leq n$.*

Proof. We will prove this by induction on n.

Base case: $n = 0$. Then $P_e = M_e / \bigcup_{e' < e} M_{e'}$ is projective. Thus we may choose a section $P_e \to M_e$ of the projection $M_e \to P_e$. We claim that the induced map $\psi : \bigoplus_{e \in E} P_e \to M$ is an isomorphism. Namely, if $x = \sum x_e \in \bigoplus P_e$ is nonzero, then we let $e_{\max}$ be maximal such that $x_{e_{\max}}$ is nonzero and we conclude that $y = \psi(x) = \psi(\sum x_e)$ is nonzero because $y \in M_{e_{\max}}$ has nonzero image $x_{e_{\max}}$ in $P_{e_{\max}}$. On the other hand, let $y \in M$. Then $y \in M_e$ for some e. We show that $y \in \operatorname{Im}(\psi)$ by transfinite induction on e. Let $x_e \in P_e$ be the image of y. Then $y - \psi(x_e) \in \bigcup_{e' < e} M_{e'}$. By induction hypothesis we conclude that $y - \psi(x_e) \in \operatorname{Im}(\psi)$ hence $y \in \operatorname{Im}(\psi)$. Thus the claim is true and ψ is an isomorphism. We conclude that M is projective as a direct sum of projectives, see Proposition 2.1.3.

If $n > 0$, then for $e \in E$ we denote F_e the free R-module on the set of elements of M_e. Then we have a system of short exact sequences

$$0 \to K_e \to F_e \to M_e \to 0$$

over the well-ordered set E. Note that the transition maps $F_{e'} \to F_e$ and $K_{e'} \to K_e$ are injective too. Set $F = \bigcup F_e$ and $K = \bigcup K_e$. Then

$$0 \to K_e / \bigcup\nolimits_{e' < e} K_{e'} \to F_e / \bigcup\nolimits_{e' < e} F_{e'} \to M_e / \bigcup\nolimits_{e' < e} M_{e'} \to 0$$

is a short exact sequence of R-modules too and $F_e/\bigcup_{e'<e} F_{e'}$ is the free R-module on the set of elements in M_e which are not contained in $\bigcup_{e'<e} M_{e'}$. Hence by Proposition 2.4.5 we see that the projective dimension of $K_e/\bigcup_{e'<e} K_{e'}$ is at most $n-1$. By induction we conclude that K has projective dimension at most $n-1$. Whence M has projective dimension at most n and we win. □

Proposition 2.4.9. *Let R be a ring. The following are equivalent.*

(1) R has finite global dimension $\mathrm{gl.\,dim}(R) \leq n$.

(2) Every finite R-module M satisfies $\mathrm{proj.\,dim}_R(M) \leq n$.

(3) Every cyclic R-module R/I satisfies $\mathrm{proj.\,dim}_R(R/I) \leq n$.

Proof. It is clear that (1) ⇒ (2) and (2) ⇒ (3). Assume (3). Choose a set $E \subset M$ of generators of M. Choose a well ordering on E. For $e \in E$ denote M_e the submodule of M generated by the elements $e' \in E$ with $e' \leq e$. Then $M = \bigcup_{e\in E} M_e$. Note that for each $e \in E$ the quotient

$$M_e/\bigcup_{e'<e} M_{e'}$$

is either zero or generated by one element, hence has projective dimension $\leq n$ by (3). By Lemma 2.4.8 this means that M has projective dimension $\leq n$. □

Proposition 2.4.10. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset.*

(1) If $\mathrm{proj.\,dim}_R(M) \leq n$, *then* $\mathrm{proj.\,dim}_{S^{-1}R}(S^{-1}M) \leq n$.

(2) If $\mathrm{gl.\,dim}(R) \leq n$, *then* $\mathrm{gl.\,dim}(S^{-1}R) \leq n$.

Proof. Let $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ be a projective resolution. As localization is exact and as each $S^{-1}P_i$ is a projective $S^{-1}R$-module, we see that $0 \to S^{-1}P_n \to \ldots \to S^{-1}P_0 \to S^{-1}M \to 0$ is a projective resolution of $S^{-1}M$. This proves (1). Let M' be an $S^{-1}R$-module. Note that $M' = S^{-1}M'$. Hence we see that (2) follows from (1). □

2.4.5 Weak Dimensions

Definition 2.4.11. *Let R be a ring and we define its weak dimension is*

$$\mathrm{w.\,dim}(R) := \sup\{n \in \mathbb{Z} : \mathrm{Tor}_n^R(M,N) \neq 0 \textit{ for some } M,N\} = \sup_M \mathrm{tor.\,dim}_R(M).$$

Chapter 3

Dimension Theory

3.1 Hilbert Functions and Polynomials of Noetherian Local Rings

In all of this section $(R, \mathfrak{m}, \kappa)$ is a Noetherian local ring.

Definition 3.1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. An ideal $I \subset R$ such that $\sqrt{I} = \mathfrak{m}$ is called an ideal of definition of R.*

Definition 3.1.2. *Let $I \subset R$ be an ideal of definition. Because R is Noetherian this means that $\mathfrak{m}^r \subset I$ for some r. Hence any finite R-module annihilated by a power of I has a finite length, see Proposition 1.10.7. Thus it makes sense to define the Hilbert function*

$$\varphi_{I,M}(n) = \text{length}_R(I^n M / I^{n+1} M) \quad \textit{and} \quad \chi_{I,M}(n) = \text{length}_R(M / I^{n+1} M)$$

for all $n \geq 0$. By by Proposition 1.10.2 We have that

$$\chi_{I,M}(n) = \sum\nolimits_{i=0}^{n} \varphi_{I,M}(i).$$

Lemma 3.1.3. *Suppose that $M' \subset M$ are finite R-modules with finite length quotient. Then there exists a constants c_1, c_2 such that for all $n \geq c_2$ we have*

$$c_1 + \chi_{I,M'}(n - c_2) \leq \chi_{I,M}(n) \leq c_1 + \chi_{I,M'}(n)$$

Proof. Since M/M' has finite length there is a $c_2 \geq 0$ such that $I^{c_2} M \subset M'$. Let $c_1 = \text{length}_R(M/M')$. For $n \geq c_2$ we have

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\leq c_1 + \text{length}_R(M'/I^{n+1}M') \\ &= c_1 + \chi_{I,M'}(n) \end{aligned}$$

On the other hand, since $I^{c_2}M \subset M'$, we have $I^nM \subset I^{n-c_2}M'$ for $n \geq c_2$. Thus for $n \geq c_2$ we get

$$\begin{aligned} \chi_{I,M}(n) &= \text{length}_R(M/I^{n+1}M) \\ &= c_1 + \text{length}_R(M'/I^{n+1}M) \\ &\geq c_1 + \text{length}_R(M'/I^{n+1-c_2}M') \\ &= c_1 + \chi_{I,M'}(n - c_2) \end{aligned}$$

which finishes the proof. □

Lemma 3.1.4. *Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finite R-modules. Then there exists a submodule $N \subset M'$ with finite colength l and $c \geq 0$ such that*

$$\chi_{I,M}(n) = \chi_{I,M''}(n) + \chi_{I,N}(n - c) + l$$

and

$$\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n - c)$$

for all $n \geq c$.

Proof. Note that $M/I^nM \to M''/I^nM''$ is surjective with kernel $M'/M' \cap I^nM$. By the Artin-Rees Lemma 1.11.5 there exists a constant c such that $M' \cap I^nM = I^{n-c}(M' \cap I^cM)$. Denote $N = M' \cap I^cM$. Note that $I^cM' \subset N \subset M'$. Hence $\text{length}_R(M'/M' \cap I^nM) = \text{length}_R(M'/N) + \text{length}_R(N/I^{n-c}N)$ for $n \geq c$. From the short exact sequence

$$0 \to M'/M' \cap I^nM \to M/I^nM \to M''/I^nM'' \to 0$$

and additivity of lengths (Proposition 1.10.2) we obtain the equality

$$\chi_{I,M}(n-1) = \chi_{I,M''}(n-1) + \chi_{I,N}(n-c-1) + \text{length}_R(M'/N)$$

for $n \geq c$. We have $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ and similarly for the modules M'' and N. Hence we get $\varphi_{I,M}(n) = \varphi_{I,M''}(n) + \varphi_{I,N}(n-c)$ for $n \geq c$. □

Lemma 3.1.5. *Suppose that I, I' are two ideals of definition for the Noetherian local ring R. Let M be a finite R-module. There exists a constant a such that $\chi_{I,M}(n) \leq \chi_{I',M}(an)$ for $n \geq 1$.*

Proof. There exists an integer $c \geq 1$ such that $(I')^c \subset I$. Hence we get a surjection $M/(I')^{c(n+1)}M \to M/I^{n+1}M$. Whence the result with $a = 2c - 1$. □

Proposition 3.1.6. *Let R be a Noetherian local ring. Let M be a finite R-module. Let $I \subset R$ be an ideal of definition. The Hilbert function $\varphi_{I,M}$ and the function $\chi_{I,M}$ are numerical polynomials.*

Proof. Boring things, we refer Tag 00K1 and Tag 00K8. □

Definition 3.1.7. *Let R be a Noetherian local ring. Let M be a finite R-module. The Hilbert polynomial of M over R is the element $P(t) \in \mathbb{Q}[t]$ such that $P(n) = \varphi_M(n)$ for $n \gg 0$.*

By Proposition 3.1.6 we see that the Hilbert polynomial exists.

Proposition 3.1.8. *Let R be a Noetherian local ring. Let M be a finite R-module.*

1. *The degree of the numerical polynomial $\varphi_{I,M}$ is independent of the ideal of definition I.*
2. *The degree of the numerical polynomial $\chi_{I,M}$ is independent of the ideal of definition I.*

Proof. Part (2) follows immediately from Lemma 3.1.5. Part (1) follows from (2) because $\varphi_{I,M}(n) = \chi_{I,M}(n) - \chi_{I,M}(n-1)$ for $n \geq 1$. □

Definition 3.1.9. *Let R be a local Noetherian ring and M a finite R-module. We denote $d(M)$ the element of $\{-\infty, 0, 1, 2, \ldots\}$ defined as follows:*

1. *If $M = 0$ we set $d(M) = -\infty$,*
2. *if $M \neq 0$ then $d(M)$ is the degree of the numerical polynomial χ_M.*

If $\mathfrak{m}^n M \neq 0$ for all n, then we see that $d(M)$ is the degree $+1$ of the Hilbert polynomial of M.

Lemma 3.1.10. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let M be a finite R-module which does not have finite length. If $M' \subset M$ is a submodule with finite colength, then $\chi_{I,M} - \chi_{I,M'}$ is a polynomial of degree $<$ degree of either polynomial.*

Proof. Follows from Lemma 3.1.3 by elementary calculus. □

Lemma 3.1.11. *Let R be a Noetherian local ring. Let $I \subset R$ be an ideal of definition. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finite R-modules. Then*

(1) *if M' does not have finite length, then $\chi_{I,M} - \chi_{I,M''} - \chi_{I,M'}$ is a numerical polynomial of degree $<$ the degree of $\chi_{I,M'}$,*

(2) $\max\{\deg(\chi_{I,M'}), \deg(\chi_{I,M''})\} = \deg(\chi_{I,M})$, *and*

(3) $\max\{d(M'), d(M'')\} = d(M)$,

Proof. We first prove (1). Let $N \subset M'$ be as in Lemma 3.1.4. By Lemma 3.1.10 the numerical polynomial $\chi_{I,M'} - \chi_{I,N}$ has degree < the common degree of $\chi_{I,M'}$ and $\chi_{I,N}$. By Lemma 3.1.4 the difference

$$\chi_{I,M}(n) - \chi_{I,M''}(n) - \chi_{I,N}(n-c)$$

is constant for $n \gg 0$. By elementary calculus the difference $\chi_{I,N}(n) - \chi_{I,N}(n-c)$ has degree < the degree of $\chi_{I,N}$ which is bigger than zero (see above). Putting everything together we obtain (1).

Note that the leading coefficients of $\chi_{I,M'}$ and $\chi_{I,M''}$ are nonnegative. Thus the degree of $\chi_{I,M'} + \chi_{I,M''}$ is equal to the maximum of the degrees. Thus if M' does not have finite length, then (2) follows from (1). If M' does have finite length, then $I^nM \to I^nM''$ is an isomorphism for all $n \gg 0$ by Artin-Rees (Lemma 1.11.5). Thus $M/I^nM \to M''/I^nM''$ is a surjection with kernel M' for $n \gg 0$ and we see that $\chi_{I,M}(n) - \chi_{I,M''}(n) = \text{length}(M')$ for all $n \gg 0$. Thus (2) holds in this case also.

Proof of (3). This follows from (2) except if one of M, M', or M'' is zero. We omit the proof in these special cases. □

3.2 Dimension Theory of Noetherian Local Rings

Definition 3.2.1. *The Krull dimension* $\dim(R)$ *of the ring R is the supremum of the integers $n \geq 0$ such that R has a chain of prime ideals*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n, \quad \mathfrak{p}_i \neq \mathfrak{p}_{i+1}.$$

of length n.

Here first we consider the case of $\dim(R) = 0, 1$ as the preparation of the induction for the general case.

Proposition 3.2.2. *Let R be a ring. The following are equivalent:*

(1) R is Artinian,

(2) R is Noetherian and $\dim(R) = 0$*,*

(3) R has finite length as a module over itself,

(4) R is a finite product of Artinian local rings,

(5) R is Noetherian and $\text{Spec}(R)$ *is a finite discrete topological space,*

(6) R is a finite product of Noetherian local rings of dimension 0*,*

(7) R is a finite product of Noetherian local rings R_i with $d(R_i) = 0$,

(8) *R is a finite product of Noetherian local rings R_i whose maximal ideals are nilpotent,*

(9) *R is Noetherian, has finitely many maximal ideals and its Jacobson radical ideal is nilpotent, and*

(10) *R is Noetherian and there are no strict inclusions among its primes.*

Proof. This is a combination of Proposition 1.11.12, 1.11.13 and the following claims:

First we claim that a Noetherian ring of dimension 0 is Artinian. Indeed, assume R is a Noetherian ring of dimension 0. Now $\mathrm{Spec}(R)$ has finitely many irreducible components, say $\mathrm{Spec}(R) = Z_1 \cup \ldots \cup Z_r$ and $Z_i = V(\mathfrak{p}_i)$ with $\mathfrak{p}_i$ a minimal ideal. Since the dimension is 0 these $\mathfrak{p}_i$ are also maximal. Thus $\mathrm{Spec}(R)$ is the discrete topological space with elements $\mathfrak{p}_i$. All elements f of the Jacobson radical $\bigcap_i \mathfrak{p}_i$ are nilpotent since otherwise R_f would not be the zero ring and we would have another prime. By Lemma 1.11.12 R is equal to $\prod R_{\mathfrak{p}_i}$. Since $R_{\mathfrak{p}_i}$ is also Noetherian and dimension 0, the previous arguments show that its radical $\mathfrak{p}_i R_{\mathfrak{p}_i}$ is locally nilpotent. Hence $\mathfrak{p}_i^n R_{\mathfrak{p}_i} = 0$ for some $n \geq 1$. By Proposition 1.10.7 we conclude that $R_{\mathfrak{p}_i}$ has finite length over R. Hence we conclude that R is Artinian by Proposition 1.11.13.

Next we claim that any Artinian ring is Noetherian of dimension zero. Indeed, if R is an Artinian ring then by Proposition 1.11.13 it is Noetherian. All of its primes are maximal by a combination of Proposition 1.11.11(1), 1.11.11(2) and Lemma 1.11.12.

Finally we claim that for a Noetherian local ring R, we have $\dim(R) = 0 \Leftrightarrow d(R) = 0$. Indeed, this is because $d(R) = 0$ if and only if R has finite length as an R-module. See Proposition 1.11.13. □

Proposition 3.2.3. *Let R be a local Noetherian ring. The following are equivalent:*

(1) $\dim(R) = 1$,

(2) $d(R) = 1$,

(3) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $V(x) = \{\mathfrak{m}\}$,*

(4) *there exists an $x \in \mathfrak{m}$, x not nilpotent such that $\mathfrak{m} = \sqrt{(x)}$, and*

(5) *there exists an ideal of definition generated by 1 element, and no ideal of definition is generated by 0 elements.*

Proof. First, assume that $\dim(R) = 1$. Let $\mathfrak{p}_i$ be the minimal primes of R. Because the dimension is 1 the only other prime of R is $\mathfrak{m}$. Now there are finitely many irreducible components. Hence we can find $x \in \mathfrak{m}$ but $x \notin \mathfrak{p}_i$. Thus the only prime containing x is $\mathfrak{m}$ and hence (3) holds.

If (3) holds, then $\mathfrak{m} = \sqrt{(x)}$, and hence (4) holds. The converse is clear as well. The equivalence of (4) and (5) follows from directly the definitions.

Assume (5). Let $I = (x)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of R/I via multiplication by x^n and hence $\text{length}_R(I^n/I^{n+1})$ is bounded. Thus $d(R) = 0$ or $d(R) = 1$, but $d(R) = 0$ is excluded by the assumption that 0 is not an ideal of definition. Hence (2) holds.

Assume (2) holds and we claim (1) holds. To get a contradiction, assume there exist primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{m}$, with both inclusions strict. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 3.1.11. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq 1$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree < 1. In other words, $d(R/(xR + \mathfrak{p})) = 0$, and hence $\dim(R/(xR + \mathfrak{p})) = 0$, by Proposition 3.2.2. But $R/(xR+\mathfrak{p})$ has the distinct primes $\mathfrak{q}/(xR+\mathfrak{p})$ and $\mathfrak{m}/(xR+\mathfrak{p})$ which gives the desired contradiction. □

Here is the main property for general dimension:

Proposition 3.2.4. *Let R be a local Noetherian ring. Let $d \geq 0$ be an integer. The following are equivalent:*

(1) $\dim(R) = d$,

(2) $d(R) = d$,

(3) there exists an ideal of definition generated by d elements, and no ideal of definition is generated by fewer than d elements.

Proof. This proof is really just the same as the proof of Proposition 3.2.3. We will prove the proposition by induction on d. By Propositions 3.2.2 and 3.2.3 we may assume that $d > 1$. Denote the minimal number of generators for an ideal of definition of R by $d'(R)$. We will prove the inequalities $\dim(R) \geq d'(R) \geq d(R) \geq \dim(R)$, and hence they are all equal.

First, assume that $\dim(R) = d$. Let $\mathfrak{p}_i$ be the minimal primes of R. Note that there are finitely many irreducible components. Hence we can find $x \in \mathfrak{m}$, $x \notin \mathfrak{p}_i$. Note that every maximal chain of primes starts with some $\mathfrak{p}_i$, hence the dimension of R/xR is at most $d - 1$. By induction there are $x_2, \ldots, x_d$ which generate an ideal of definition in R/xR. Hence R has an ideal of definition generated by (at most) d elements.

Assume $d'(R) = d$. Let $I = (x_1, \ldots, x_d)$ be an ideal of definition. Note that I^n/I^{n+1} is a quotient of a direct sum of $\binom{d+n-1}{d-1}$ copies R/I via multiplication by all degree n monomials in $x_1, \ldots, x_d$. Hence $\text{length}_R(I^n/I^{n+1})$ is bounded by a polynomial of degree $d - 1$. Thus $d(R) \leq d$.

Assume $d(R) = d$. Consider a chain of primes $\mathfrak{p} \subset \mathfrak{q} \subset \mathfrak{q}_2 \subset \ldots \subset \mathfrak{q}_e = \mathfrak{m}$, with all inclusions strict, and $e \geq 2$. Pick some ideal of definition $I \subset R$. We will repeatedly use Lemma 3.1.11. First of all it implies, via the exact sequence $0 \to \mathfrak{p} \to R \to R/\mathfrak{p} \to 0$, that $d(R/\mathfrak{p}) \leq d$. But it clearly cannot be zero. Pick $x \in \mathfrak{q}$, $x \notin \mathfrak{p}$. Consider the short exact sequence

$$0 \to R/\mathfrak{p} \to R/\mathfrak{p} \to R/(xR + \mathfrak{p}) \to 0.$$

This implies that $\chi_{I,R/\mathfrak{p}} - \chi_{I,R/\mathfrak{p}} - \chi_{I,R/(xR+\mathfrak{p})} = -\chi_{I,R/(xR+\mathfrak{p})}$ has degree $< d$. In other words, $d(R/(xR + \mathfrak{p})) \leq d - 1$, and hence $\dim(R/(xR + \mathfrak{p})) \leq d - 1$, by induction. Now $R/(xR+\mathfrak{p})$ has the chain of prime ideals $\mathfrak{q}/(xR+\mathfrak{p}) \subset \mathfrak{q}_2/(xR+\mathfrak{p}) \subset \ldots \subset \mathfrak{q}_e/(xR+\mathfrak{p})$ which gives $e-1 \leq d-1$. Since we started with an arbitrary chain of primes this proves that $\dim(R) \leq d(R)$.

Reading back the reader will see we proved the circular inequalities as desired. □

Definition 3.2.5. *Let R be a Noetherian local ring. Let* $\dim(R) = d$.

(1) *A system of parameters of R is a sequence of elements $x_1, ..., x_d \in \mathfrak{m}$ which generates an ideal of definition of R.*

(2) *By Nakayama's lemma we have $\mathfrak{m}$ has $\dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$. Hence $d \leq \dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2)$. We call R is a regular local ring if $\dim_{\kappa(\mathfrak{m})}(\mathfrak{m}/\mathfrak{m}^2) = d$. Here $\mathfrak{m} = (x_1, ..., x_d)$ and we call $x_1, ..., x_d$ here is a regular system of parameters.*

3.3 Krull's Height Theorem and Other Applications

Here are some important results about height.

Theorem 3.3.1 (Krull's Principal Ideal Theorem)**.** *Let R be a Noetherian ring. Let $x \in R$.*

(1) *If $\mathfrak{p}$ is minimal over (x) then* $\operatorname{height}(\mathfrak{p}) \leq 1$.

(2) *If $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ and $\mathfrak{q}$ is minimal over $(\mathfrak{p}, x)$, then there is no prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$.*

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over x, then the only prime ideal of $R_\mathfrak{p}$ containing x is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence Proposition 3.2.3 shows $\dim(R_\mathfrak{p}) = 1$ if x is not nilpotent in $R_\mathfrak{p}$. Of course, if x is nilpotent in $R_\mathfrak{p}$ the argument gives that $\mathfrak{p}R_\mathfrak{p}$ is the only prime ideal and we see that the height is 0.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height 1 or 0 in $R/\mathfrak{p}$. This immediately implies there cannot be a prime strictly between $\mathfrak{p}$ and $\mathfrak{q}$. □

Theorem 3.3.2 (Krull's Height Theorem)**.** *Let R be a Noetherian ring. Let $f_1, \ldots, f_r \in R$.*

(1) If $\mathfrak{p}$ is minimal over $(f_1, \ldots, f_r)$ then $\mathrm{height}(\mathfrak{p}) \leq r$.

(2) If $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$ and $\mathfrak{q}$ is minimal over $(\mathfrak{p}, f_1, \ldots, f_r)$, then every chain of primes between $\mathfrak{p}$ and $\mathfrak{q}$ has length at most r.

Proof. Proof of (1). If $\mathfrak{p}$ is minimal over $f_1, \ldots, f_r$, then the only prime ideal of $R_\mathfrak{p}$ containing $f_1, \ldots, f_r$ is the maximal ideal $\mathfrak{p}R_\mathfrak{p}$. Hence Proposition 3.2.4 shows $\dim(R_\mathfrak{p}) \leq r$.

Proof of (2). By part (1) we see that $\mathfrak{q}/\mathfrak{p}$ is a prime of height $\leq r$. This immediately implies the statement about chains of primes between $\mathfrak{p}$ and $\mathfrak{q}$. □

Other applications as follows.

Proposition 3.3.3. *Suppose that R is a Noetherian local ring and $x \in \mathfrak{m}$ an element of its maximal ideal. Then $\dim R \leq \dim R/xR + 1$. If x is not contained in any of the minimal primes of R then equality holds. (For example if x is a nonzerodivisor.)*

Proof. If $x_1, \ldots, x_{\dim R/xR} \in R$ map to elements of R/xR which generate an ideal of definition for R/xR, then $x, x_1, \ldots, x_{\dim R/xR}$ generate an ideal of definition for R. Hence the inequality by Proposition 3.2.4. On the other hand, if x is not contained in any minimal prime of R, then the chains of primes in R/xR all give rise to chains in R which are at least one step away from being maximal. □

Proposition 3.3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Suppose $x_1, \ldots, x_d \in \mathfrak{m}$ generate an ideal of definition and $d = \dim(R)$. Then $\dim(R/(x_1, \ldots, x_i)) = d - i$ for all $i = 1, \ldots, d$.*

Proof. Follows either from the proof of Proposition 3.2.4, or by using induction on d and Proposition 3.3.3. □

Remark 3.3.5. *Other applications we refer Tag 02IF.*

3.4 Homomorphisms and Dimension

Here are some results about the ring maps.

Proposition 3.4.1. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Then*

$$\dim(S_\mathfrak{q}) \leq \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}).$$

Proof. We use the characterization of dimension of Proposition 3.2.4. Let $x_1, \ldots, x_d$ be elements of $\mathfrak{p}$ generating an ideal of definition of $R_\mathfrak{p}$ with $d = \dim(R_\mathfrak{p})$. Let $y_1, \ldots, y_e$ be elements of $\mathfrak{q}$ generating an ideal of definition of $S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}$ with $e = \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$. It is clear that $S_\mathfrak{q}/(x_1, \ldots, x_d, y_1, \ldots, y_e)$ has a nilpotent maximal ideal. Hence $x_1, \ldots, x_d, y_1, \ldots, y_e$ generate an ideal of definition of $S_\mathfrak{q}$. □

Definition 3.4.2. *A ring R is said to be* catenary *if for any pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$, there exists an integer bounding the lengths of all finite chains of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_e = \mathfrak{q}$ and all maximal such chains have the same length.*

A Noetherian ring R is said to be universally catenary *if every R-algebra of finite type is catenary.*

Proposition 3.4.3. *Let $R \to S$ be a ring map. Let $\mathfrak{q}$ be a prime of S lying over the prime $\mathfrak{p}$ of R. Assume that*

(1) R is Noetherian,

(2) $R \to S$ is of finite type,

(3) R, S are domains, and

(4) $R \subset S$.

Then we have

$$\text{height}(\mathfrak{q}) \leq \text{height}(\mathfrak{p}) + \text{trdeg}_R(S) - \text{height}_{\kappa(\mathfrak{p})}\, \kappa(\mathfrak{q})$$

with equality if R is universally catenary.

Proof. Suppose that $R \subset S' \subset S$ is a finitely generated R-subalgebra of S. In this case set $\mathfrak{q}' = S' \cap \mathfrak{q}$. The result for the ring maps $R \to S'$ and $S' \to S$ implies the result for $R \to S$ by additivity of transcendence degree in towers of fields. Hence we can use induction on the number of generators of S over R and reduce to the case where S is generated by one element over R.

Case I: $S = R[x]$ is a polynomial algebra over R. In this case we have $\text{trdeg}_R(S) = 1$. Also $R \to S$ is flat and hence

$$\dim(S_\mathfrak{q}) = \dim(R_\mathfrak{p}) + \dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q})$$

see Proposition 4.3.1. Let $\mathfrak{r} = \mathfrak{p}S$. Then $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 1$ is equivalent to $\mathfrak{q} = \mathfrak{r}$, and implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$. In the same vein $\text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q}) = 0$ is equivalent to having a strict inclusion $\mathfrak{r} \subset \mathfrak{q}$, which implies that $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 1$. Thus we are done with case I with equality in every instance.

Case II: $S = R[x]/\mathfrak{n}$ with $\mathfrak{n} \neq 0$. In this case we have $\text{trdeg}_R(S) = 0$. Denote $\mathfrak{q}' \subset R[x]$ the prime corresponding to $\mathfrak{q}$. Thus we have

$$S_\mathfrak{q} = (R[x])_{\mathfrak{q}'}/\mathfrak{n}(R[x])_{\mathfrak{q}'}$$

By the previous case we have $\dim((R[x])_{\mathfrak{q}'}) = \dim(R_\mathfrak{p}) + 1 - \text{trdeg}_{\kappa(\mathfrak{p})}\kappa(\mathfrak{q})$. Since $\mathfrak{n} \neq 0$ we see that the dimension of $S_\mathfrak{q}$ decreases by at least one, see Proposition 3.3.3, which proves the inequality of the lemma. To see the equality in case R is universally catenary note that $\mathfrak{n} \subset R[x]$ is a height one prime as it corresponds to a nonzero prime in $F[x]$ where F is the fraction field of R. Hence any maximal chain of primes in $S_{\mathfrak{q}} = R[x]_{\mathfrak{q}'}/\mathfrak{n}R[x]_{\mathfrak{q}'}$ corresponds to a maximal chain of primes with length 1 greater between $\mathfrak{q}'$ and (0) in $R[x]$. If R is universally catenary these all have the same length equal to the height of $\mathfrak{q}'$. This proves that $\dim(S_{\mathfrak{q}}) = \dim(R[x]_{\mathfrak{q}'}) - 1$ and this implies equality holds as desired. $\square$

Chapter 4

Integral and Going up and down

4.1 Finite and Integral Ring Maps

Definition 4.1.1. *Let $\varphi : R \to S$ be a ring map.*

(1) An element $s \in S$ is integral over *R if there exists a monic polynomial $P(x) \in R[x]$ such that $P^{\varphi}(s) = 0$, where $P^{\varphi}(x) \in S[x]$ is the image of P under $\varphi : R[x] \to S[x]$.*

(2) The ring map φ is integral *if every $s \in S$ is integral over R.*

(3) We say $\varphi : R \to S$ is finite *if S is finite as an R-module.*

Here we introduce some basic properties of these things.

Proposition 4.1.2. *Let $\varphi : R \to S$ be a ring map. Let $y \in S$. If there exists a finite R-submodule M of S such that $1 \in M$ and $yM \subset M$, then y is integral over R.*

Proof. Consider the map $\varphi : M \to M$, $x \mapsto y \cdot x$. By Corollary 1.6.2 there exists a monic polynomial $P \in R[T]$ with $P(\varphi) = 0$. In the ring S we get $P(y) = P(y) \cdot 1 = P(\varphi)(1) = 0$. □

Corollary 4.1.3. *A finite ring extension is integral.*

Proof. Let $R \to S$ be finite. Let $y \in S$. Apply Proposition 4.1.2 to $M = S$ to see that y is integral over R. □

Lemma 4.1.4. *Let $\varphi : R \to S$ be a ring map. Let $s_1, \ldots, s_n$ be a finite set of elements of S. In this case s_i is integral over R for all $i = 1, \ldots, n$ if and only if there exists an R-subalgebra $S' \subset S$ finite over R containing all of the s_i.*

Proof. If each s_i is integral, then the subalgebra generated by $\varphi(R)$ and the s_i is finite over R. Namely, if s_i satisfies a monic equation of degree d_i over R, then this subalgebra is generated as an R-module by the elements $s_1^{e_1} \ldots s_n^{e_n}$ with $0 \leq e_i \leq d_i - 1$. Conversely, suppose given a finite R-subalgebra S' containing all the s_i. Then all of the s_i are integral by Corollary 4.1.3. □

Proposition 4.1.5. *Let $R \to S$ be a ring map. The following are equivalent*

(1) $R \to S$ is finite,

(2) $R \to S$ is integral and of finite type, and

(3) there exist $x_1, \ldots, x_n \in S$ which generate S as an algebra over R such that each x_i is integral over R.

Proof. Clear from Lemma 4.1.4. □

Corollary 4.1.6. *Suppose that $R \to S$ and $S \to T$ are integral ring maps. Then $R \to T$ is integral.*

Proof. Let $t \in T$. Let $P(x) \in S[x]$ be a monic polynomial such that $P(t) = 0$. Apply Lemma 4.1.4 to the finite set of coefficients of P. Hence t is integral over some subalgebra $S' \subset S$ finite over R. Apply Lemma 4.1.4 again to find a subalgebra $T' \subset T$ finite over S' and containing t. As finite map stable under composition by trivial reason, applied this to $R \to S' \to T'$ shows that T' is finite over R. The integrality of t over R now follows from Corollary 4.1.3. □

Proposition 4.1.7. *Let $\varphi : R \to S$ be a ring map. Let $x \in S$. The following are equivalent:*

(1) x is integral over R, and

(2) for every prime ideal $\mathfrak{p} \subset R$ the element $x \in S_\mathfrak{p}$ is integral over $R_\mathfrak{p}$.

Proof. It is clear that (1) implies (2). Assume (2). Consider the R-algebra $S' \subset S$ generated by $\varphi(R)$ and x. Let $\mathfrak{p}$ be a prime ideal of R. Then we know that $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in $S_\mathfrak{p}$ for some $a_i \in R_\mathfrak{p}$. Hence we see, by looking at which denominators occur, that for some $f \in R$, $f \notin \mathfrak{p}$ we have $a_i \in R_f$ and $x^d + \sum_{i=1,\ldots,d} \varphi(a_i)x^{d-i} = 0$ in S_f. This implies that S'_f is finite over R_f. Since $\mathfrak{p}$ was arbitrary and $\operatorname{Spec}(R)$ is quasi-compact, we can find finitely many elements $f_1, \ldots, f_n \in R$ which generate the unit ideal of R such that S'_{f_i} is finite over R_{f_i}. Hence we conclude from Proposition 1.8.10 that S' is finite over R. Hence x is integral over R by Lemma 4.1.4. □

Proposition 4.1.8. *Let $R \to S$ and $R \to R'$ be ring maps. Set $S' = R' \otimes_R S$.*

(1) *If $R \to S$ is integral so is $R' \to S'$.*

(2) *If $R \to S$ is finite so is $R' \to S'$.*

Proof. We prove (1). Let $s_i \in S$ be generators for S over R. Each of these satisfies a monic polynomial equation P_i over R. Hence the elements $1 \otimes s_i \in S'$ generate S' over R' and satisfy the corresponding polynomial P_i' over R'. Since these elements generate S' over R' we see that S' is integral over R'. Proof of (2) omitted. □

Proposition 4.1.9. *Let $R \to S$ be a ring map. Let $f_1, \dots, f_n \in R$ generate the unit ideal.*

(1) *If each $R_{f_i} \to S_{f_i}$ is integral, so is $R \to S$.*

(2) *If each $R_{f_i} \to S_{f_i}$ is finite, so is $R \to S$.*

Proof. Proof of (1). Let $s \in S$. Consider the ideal $I \subset R[x]$ of polynomials P such that $P(s) = 0$. Let $J \subset R$ denote the ideal of leading coefficients of elements of I. By assumption and clearing denominators we see that $f_i^{n_i} \in J$ for all i and certain $n_i \geq 0$. Hence J contains 1 and we see s is integral over R. Proof of (2) omitted. □

Proposition 4.1.10. *Let $A \to B \to C$ be ring maps.*

(1) *If $A \to C$ is integral so is $B \to C$.*

(2) *If $A \to C$ is finite so is $B \to C$.*

Proof. Omitted. □

Next we introduce an important definition, the intergral closure, and its several applications. But first we consider some properties.

Lemma 4.1.11. *Let $R \to S$ be a ring homomorphism. The set*

$$S' = \{s \in S \mid s \text{ is integral over } R\}$$

is an R-subalgebra of S.

Proof. This is clear from Lemma 4.1.4 and Corollary 4.1.3. □

Lemma 4.1.12. *Let $R_i \to S_i$ be ring maps $i = 1, \dots, n$. Let R and S denote the product of the R_i and S_i respectively. Then an element $s = (s_1, \dots, s_n) \in S$ is integral over R if and only if each s_i is integral over R_i.*

Proof. Trivial. □

Definition 4.1.13. *Let $R \to S$ be a ring map. The ring $S' \subset S$ of elements integral over R, see Lemma 4.1.11, is called the* integral closure *of R in S. If $R \subset S$ we say that R is* integrally closed *in S if $R = S'$.*

In particular, we see that $R \to S$ is integral if and only if the integral closure of R in S is all of S.

Lemma 4.1.14. *Let $R_i \to S_i$ be ring maps $i = 1, \ldots, n$. Denote the integral closure of R_i in S_i by S'_i. Further let R and S denote the product of the R_i and S_i respectively. Then the integral closure of R in S is the product of the S'_i. In particular $R \to S$ is integrally closed if and only if each $R_i \to S_i$ is integrally closed.*

Proof. This follows immediately from Lemma 4.1.12. □

Proposition 4.1.15 (Integral closure commutes with localization)**.** *If $A \to B$ is a ring map, and $S \subset A$ is a multiplicative subset, then the integral closure of $S^{-1}A$ in $S^{-1}B$ is $S^{-1}B'$, where $B' \subset B$ is the integral closure of A in B.*

Proof. Since localization is exact we see that $S^{-1}B' \subset S^{-1}B$. Suppose $x \in B'$ and $f \in S$. Then $x^d + \sum_{i=1,\ldots,d} a_i x^{d-i} = 0$ in B for some $a_i \in A$. Hence also

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} a_i/f^i (x/f)^{d-i} = 0$$

in $S^{-1}B$. In this way we see that $S^{-1}B'$ is contained in the integral closure of $S^{-1}A$ in $S^{-1}B$. Conversely, suppose that $x/f \in S^{-1}B$ is integral over $S^{-1}A$. Then we have

$$(x/f)^d + \sum\nolimits_{i=1,\ldots,d} (a_i/f_i)(x/f)^{d-i} = 0$$

in $S^{-1}B$ for some $a_i \in A$ and $f_i \in S$. This means that

$$(f' f_1 \ldots f_d x)^d + \sum\nolimits_{i=1,\ldots,d} f^i (f')^i f_1^i \ldots f_i^{i-1} \ldots f_d^i a_i (f' f_1 \ldots f_d x)^{d-i} = 0$$

for a suitable $f' \in S$. Hence $f' f_1 \ldots f_d x \in B'$ and thus $x/f \in S^{-1}B'$ as desired. □

Proposition 4.1.16. *Let $A \to B \to C$ be ring maps. Let B' be the integral closure of A in B, let C' be the integral closure of B' in C. Then C' is the integral closure of A in C.*

Proof. Omitted. □

Proposition 4.1.17 (Lying Over)**.** *Suppose that $R \to S$ is an integral ring extension with $R \subset S$. Then $\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proof. Let $\mathfrak{p} \subset R$ be a prime ideal. We have to show $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$. The localization $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is injective (as localization is exact) and integral by Proposition 4.1.15 or 4.1.8. Hence we may replace R, S by $R_{\mathfrak{p}}$, $S_{\mathfrak{p}}$ and we may assume R is local with maximal ideal $\mathfrak{m}$ and it suffices to show that $\mathfrak{m}S \neq S$. Suppose $1 = \sum f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$ in order to get a contradiction. Let $R \subset S' \subset S$ be such that $R \to S'$ is finite and $s_i \in S'$, see Lemma 4.1.4. The equation $1 = \sum f_i s_i$ implies that the finite R-module S' satisfies $S' = \mathfrak{m}S'$. Hence by Nakayama's Lemma 1.7.2 we see $S' = 0$. Contradiction. $\square$

Here are two applications between ring and fields in two situations:

Corollary 4.1.18. *Let R be a ring. Let K be a field. If $R \subset K$ and K is integral over R, then R is a field and K is an algebraic extension. If $R \subset K$ and K is finite over R, then R is a field and K is a finite algebraic extension.*

Proof. Assume that $R \subset K$ is integral. By Proposition 4.1.17 we see that $\mathrm{Spec}(R)$ has 1 point. Since clearly R is a domain we see that $R = R_{(0)}$ is a field (Lemma 1.14.1). The other assertions are immediate from this. $\square$

Corollary 4.1.19. *Let k be a field. Let S be a k-algebra over k.*

(1) If S is a domain and finite dimensional over k, then S is a field.

(2) If S is integral over k and a domain, then S is a field.

(3) If S is integral over k then every prime of S is a maximal ideal.

Proof. The statement on primes follows from the statement "integral + domain $\Rightarrow$ field". Let S integral over k and assume S is a domain, Take $s \in S$. By Lemma 4.1.4 we may find a finite dimensional k-subalgebra $k \subset S' \subset S$ containing s. Hence S is a field if we can prove the first statement. Assume S finite dimensional over k and a domain. Pick $s \in S$. Since S is a domain the multiplication map $s : S \to S$ is surjective by dimension reasons. Hence there exists an element $s_1 \in S$ such that $ss_1 = 1$. So S is a field. $\square$

Lemma 4.1.20. *Suppose $R \to S$ is integral. Let $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(S)$ be distinct primes having the same image in $\mathrm{Spec}(R)$. Then neither $\mathfrak{q} \subset \mathfrak{q}'$ nor $\mathfrak{q}' \subset \mathfrak{q}$.*

Proof. Let $\mathfrak{p} \subset R$ be the image. By fundamental diagram, the primes $\mathfrak{q}, \mathfrak{q}'$ correspond to ideals in $S \otimes_R \kappa(\mathfrak{p})$. Thus the lemma follows from Corollary 4.1.19. $\square$

Proposition 4.1.21. *Suppose $R \to S$ is finite. Then the fibres of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ are finite.*

Proof. By the discussion in fundamental diagram, the fibres are the spectra of the rings $S \otimes_R \kappa(\mathfrak{p})$. As $R \to S$ is finite, these fibre rings are finite over $\kappa(\mathfrak{p})$ hence Noetherian by Proposition 1.11.1. By Lemma 4.1.20 every prime of $S \otimes_R \kappa(\mathfrak{p})$ is a minimal prime. Hence there are at most finitely many. $\square$

Proposition 4.1.22 (Going Up)**.** *Let $R \to S$ be a ring map such that S is integral over R. Let $\mathfrak{p} \subset \mathfrak{p}' \subset R$ be primes. Let $\mathfrak{q}$ be a prime of S mapping to $\mathfrak{p}$. Then there exists a prime $\mathfrak{q}'$ with $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}'$.*

Proof. We may replace R by $R/\mathfrak{p}$ and S by $S/\mathfrak{q}$. This reduces us to the situation of having an integral extension of domains $R \subset S$ and a prime $\mathfrak{p}' \subset R$. By Proposition 4.1.17 we win. □

4.2 Normal Rings

We first introduce the notion of a normal domain, and then we introduce the (very general) notion of a normal ring.

Definition 4.2.1. *A domain R is called normal if it is integrally closed in its field of fractions.*

Lemma 4.2.2. *Let $R \to S$ be a ring map. If S is a normal domain, then the integral closure of R in S is a normal domain.*

Proof. Trivial. □

Here are some useful properties. But before that we need two lemmas.

Lemma 4.2.3. *Let R be a domain with fraction field K. If $u, v \in K$ are almost integral over R, that is, there exists an element $r, s \in R$, $r, s \neq 0$ such that $ru^n, rv^n \in R$ for all $n \geq 0$, then so are $u + v$ and uv. Any element $g \in K$ which is integral over R is almost integral over R. If R is Noetherian then the converse holds as well.*

Proof. If $ru^n \in R$ for all $n \geq 0$ and $v^n r' \in R$ for all $n \geq 0$, then $(uv)^n rr'$ and $(u+v)^n rr'$ are in R for all $n \geq 0$. Hence the first assertion. Suppose $g \in K$ is integral over R. In this case there exists an $d > 0$ such that the ring $R[g]$ is generated by $1, g, \ldots, g^d$ as an R-module. Let $r \in R$ be a common denominator of the elements $1, g, \ldots, g^d \in K$. It is follows that $rR[g] \subset R$, and hence g is almost integral over R.

Suppose R is Noetherian and $g \in K$ is almost integral over R. Let $r \in R$, $r \neq 0$ be as in the definition. Then $R[g] \subset \frac{1}{r}R$ as an R-module. Since R is Noetherian this implies that $R[g]$ is finite over R. Hence g is integral over R, see Corollary 4.1.3. □

Lemma 4.2.4. *Let R be a domain with fraction field K. Suppose $f = \sum \alpha_i x^i$ is an element of $K[x]$.*

(1) If f is integral over $R[x]$ then all α_i are integral over R, and

(2) If f is almost integral over $R[x]$ then all α_i are almost integral over R.

Proof. We first prove the second statement. Write $f = \alpha_0 + \alpha_1 x + \ldots + \alpha_r x^r$ with $\alpha_r \neq 0$. By assumption there exists $h = b_0 + b_1 x + \ldots + b_s x^s \in R[x]$, $b_s \neq 0$ such that $f^n h \in R[x]$ for all $n \geq 0$. This implies that $b_s \alpha_r^n \in R$ for all $n \geq 0$. Hence α_r is almost integral over R. Since the set of almost integral elements form a subring (Lemma 4.2.3) we deduce that $f - \alpha_r x^r = \alpha_0 + \alpha_1 x + \ldots + \alpha_{r-1} x^{r-1}$ is almost integral over $R[x]$. By induction on r we win.

In order to prove the first statement we will use absolute Noetherian reduction. Namely, write $\alpha_i = a_i/b_i$ and let $P(t) = t^d + \sum_{j<d} f_j t^j$ be a polynomial with coefficients $f_j \in R[x]$ such that $P(f) = 0$. Let $f_j = \sum f_{ji} x^i$. Consider the subring $R_0 \subset R$ generated by the finite list of elements a_i, b_i, f_{ji} of R. It is a domain; let K_0 be its field of fractions. Since R_0 is a finite type $\mathbf{Z}$-algebra it is Noetherian. It is still the case that $f \in K_0[x]$ is integral over $R_0[x]$, because all the identities in R among the elements a_i, b_i, f_{ji} also hold in R_0. By Lemma 4.2.3 the element f is almost integral over $R_0[x]$. By the second statement of the lemma, the elements α_i are almost integral over R_0. And since R_0 is Noetherian, they are integral over R_0, see Lemma 4.2.3. Of course, then they are integral over R. $\square$

Proposition 4.2.5. *For domains, we have:*

(1) Any localization of a normal domain is normal.

(2) A principal ideal domain is normal.

(3) Let R be a normal domain. Then $R[x]$ is a normal domain.

(4) Let R be a Noetherian normal domain. Then $R[[x]]$ is a Noetherian normal domain.

Proof. For (1), let R be a normal domain and let $S \subset R$ be a multiplicative subset. Suppose g is an element of the fraction field of R which is integral over $S^{-1}R$. Let $P = x^d + \sum_{j<d} a_j x^j$ be a polynomial with $a_i \in S^{-1}R$ such that $P(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for all i. Then sg satisfies the monic polynomial $x^d + \sum_{j<d} s^{d-j} a_j x^j$ which has coefficients $s^{d-j} a_j$ in R. Hence $sg \in R$ because R is normal. Hence $g \in S^{-1}R$.

For (2), let R be a principal ideal domain. Let $g = a/b$ be an element of the fraction field of R integral over R. Because R is a principal ideal domain we may divide out a common factor of a and b and assume $(a, b) = R$. In this case, any equation $(a/b)^n + r_{n-1}(a/b)^{n-1} + \ldots + r_0 = 0$ with $r_i \in R$ would imply $a^n \in (b)$. This contradicts $(a, b) = R$ unless b is a unit in R.

For (3), the result is true if R is a field K because $K[x]$ is a euclidean domain and hence a principal ideal domain and hence normal by (2). Let g be an element of the fraction field of $R[x]$ which is integral over $R[x]$. Because g is integral over $K[x]$ where K is the fraction field of R we may write $g = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \ldots + \alpha_0$ with $\alpha_i \in K$. By Lemma 4.2.4 the elements α_i are integral over R and hence are in R.

For (4), the power series ring is Noetherian by Proposition 1.11.2. Let $f, g \in R[[x]]$ be nonzero elements such that $w = f/g$ is integral over $R[[x]]$. Let K be the fraction field of R. Since the ring of Laurent series $K((x)) = K[[x]][1/x]$ is a field, we can write $w = a_n x^n + a_{n+1}x^{n+1} + \ldots$ for some $n \in \mathbf{Z}$, $a_i \in K$, and $a_n \neq 0$. By Lemma 4.2.3 we see there exists a nonzero element $h = b_m x^m + b_{m+1}x^{m+1} + \ldots$ in $R[[x]]$ with $b_m \neq 0$ such that $w^e h \in R[[x]]$ for all $e \geq 1$. We conclude that $n \geq 0$ and that $b_m a_n^e \in R$ for all $e \geq 1$. Since R is Noetherian this implies that $a_n \in R$ by the same lemma. Now, if $a_n, a_{n+1}, \ldots, a_{N-1} \in R$, then we can apply the same argument to $w - a_n x^n - \ldots - a_{N-1}x^{N-1} = a_N x^N + \ldots$. In this way we see that all $a_i \in R$ and the lemma is proved. □

Next we will show that the normality is a local property. This is important since we can use this to define a more general version of normal rings, not only for domains!

Proposition 4.2.6. *Let R be a domain. The following are equivalent:*

(1) The domain R is a normal domain,

(2) for every prime $\mathfrak{p} \subset R$ the local ring $R_{\mathfrak{p}}$ is a normal domain, and

(3) for every maximal ideal $\mathfrak{m}$ the ring $R_{\mathfrak{m}}$ is a normal domain.

Proof. This follows easily from the fact that for any domain R we have

$$R = \bigcap\nolimits_{\mathfrak{m}} R_{\mathfrak{m}}$$

inside the fraction field of R. Namely, if g is an element of the right hand side then the ideal $I = \{x \in R \mid xg \in R\}$ is not contained in any maximal ideal $\mathfrak{m}$, whence $I = R$. □

Definition 4.2.7. *A ring R is called* normal *if for every prime $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is a normal domain.*

Note that a normal ring is a reduced ring, as R is a subring of the product of its localizations at all primes (see for example Proposition 1.8.9).

Proposition 4.2.8. *We have:*

(1) A normal ring is integrally closed in its total ring of fractions.

(2) A localization of a normal ring is a normal ring.

(3) Let R be a normal ring. Then $R[x]$ is a normal ring.

(4) A finite product of normal rings is normal.

(5) Let $(R_i, \varphi_{ii'})$ be a directed system of rings. If each R_i is a normal ring so is $R = \varinjlim_i R_i$.

Proof. For (1), let R be a normal ring. Let $x \in Q(R)$ be an element of the total ring of fractions of R integral over R. Set $I = \{f \in R, fx \in R\}$. Let $\mathfrak{p} \subset R$ be a prime. As $R \to R_\mathfrak{p}$ is flat we see that $R_\mathfrak{p} \subset Q(R) \otimes_R R_\mathfrak{p}$. As $R_\mathfrak{p}$ is a normal domain we see that $x \otimes 1$ is an element of $R_\mathfrak{p}$. Hence we can find $a, f \in R$, $f \notin \mathfrak{p}$ such that $x \otimes 1 = a \otimes 1/f$. This means that $fx - a$ maps to zero in $Q(R) \otimes_R R_\mathfrak{p} = Q(R)_\mathfrak{p}$, which in turn means that there exists an $f' \in R$, $f' \notin \mathfrak{p}$ such that $f'fx = f'a$ in R. In other words, $ff' \in I$. Thus I is an ideal which isn't contained in any of the prime ideals of R, i.e., $I = R$ and $x \in R$.

(2) is trivial. (3) is also trivial. For example, let $\mathfrak{q}$ be a prime of $R[x]$. Set $\mathfrak{p} = R \cap \mathfrak{q}$. Then we see that $R_\mathfrak{p}[x]$ is a normal domain by Proposition4.2.5(3). Hence $(R[x])_\mathfrak{q}$ is a normal domain by Proposition4.2.5(1).

For (4), It suffices to show that the product of two normal rings, say R and S, is normal. By Proposition 1.8.6 the prime ideals of $R \times S$ are of the form $\mathfrak{p} \times S$ and $R \times \mathfrak{q}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are primes of R and S respectively. Localization yields $(R \times S)_{\mathfrak{p} \times S} = R_\mathfrak{p}$ which is a normal domain by assumption. Similarly for S.

For (5), let $\mathfrak{p} \subset R$ be a prime ideal. Set $\mathfrak{p}_i = R_i \cap \mathfrak{p}$ (usual abuse of notation). Then we see that $R_\mathfrak{p} = \varinjlim_i (R_i)_{\mathfrak{p}_i}$. Since each $(R_i)_{\mathfrak{p}_i}$ is a normal domain we reduce to proving the statement of the lemma for normal domains. If $a, b \in R$ and a/b satisfies a monic polynomial $P(T) \in R[T]$, then we can find a (sufficiently large) $i \in I$ such that a, b come from objects a_i, b_i over R_i, P comes from a monic polynomial $P_i \in R_i[T]$ and $P_i(a_i/b_i) = 0$. Since R_i is normal we see $a_i/b_i \in R_i$ and hence also $a/b \in R$. □

Corollary 4.2.9. *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

(1) R is a normal ring,

(2) R is integrally closed in its total ring of fractions, and

(3) R is a finite product of normal domains.

Proof. The implications (1) ⇒ (2) and (3) ⇒ (1) hold in general, see Proposition4.2.8(1) and Proposition4.2.8(4).

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. By Proposition 1.14.2 and Proposition 1.14.4 we have $Q(R) = R_{\mathfrak{p}_1} \times \ldots \times R_{\mathfrak{p}_n}$, and by Lemma 1.14.1 each factor is a field. Denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the ith idempotent of $Q(R)$.

If R is integrally closed in $Q(R)$, then it contains in particular the idempotents e_i, and we see that R is a product of n domains. Each factor is of the form $R/\mathfrak{p}_i$ with field of fractions $R_{\mathfrak{p}_i}$. By Lemma 4.1.14 each map $R/\mathfrak{p}_i \to R_{\mathfrak{p}_i}$ is integrally closed. Hence R is a finite product of normal domains. □

4.3 Going Up and Going Down Properties

Add some dimension theory as in Tag 00OG.

Proposition 4.3.1. *Let $R \to S$ be a homomorphism of Noetherian rings. Let $\mathfrak{q} \subset S$ be a prime lying over the prime $\mathfrak{p}$. Assume the going down property holds for $R \to S$ (for example if $R \to S$ is flat, see Corollary 1.9.16). Then*

$$\dim(S_{\mathfrak{q}}) = \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}}).$$

Proof. By Proposition 3.4.1 we have an inequality $\dim(S_{\mathfrak{q}}) \leq \dim(R_{\mathfrak{p}}) + \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. To get equality, choose a chain of primes $\mathfrak{p}S \subset \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_d = \mathfrak{q}$ with $d = \dim(S_{\mathfrak{q}}/\mathfrak{p}S_{\mathfrak{q}})$. On the other hand, choose a chain of primes $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_e = \mathfrak{p}$ with $e = \dim(R_{\mathfrak{p}})$. By the going down theorem we may choose $\mathfrak{q}_{-1} \subset \mathfrak{q}_0$ lying over $\mathfrak{p}_{e-1}$. And then we may choose $\mathfrak{q}_{-2} \subset \mathfrak{q}_{e-1}$ lying over $\mathfrak{p}_{e-2}$. Inductively we keep going until we get a chain $\mathfrak{q}_{-e} \subset \ldots \subset \mathfrak{q}_d$ of length $e + d$. □

Corollary 4.3.2. *Let $R \to S$ be a local homomorphism of local Noetherian rings. Assume*

1. *R is regular,*
2. *$S/\mathfrak{m}_R S$ is regular, and*
3. *$R \to S$ is flat.*

Then S is regular.

Proof. By Proposition 4.3.1 we have $\dim(S) = \dim(R) + \dim(S/\mathfrak{m}_R S)$. Pick generators $x_1, \ldots, x_d \in \mathfrak{m}_R$ with $d = \dim(R)$, and pick $y_1, \ldots, y_e \in \mathfrak{m}_S$ which generate the maximal ideal of $S/\mathfrak{m}_R S$ with $e = \dim(S/\mathfrak{m}_R S)$. Then we see that $x_1, \ldots, x_d, y_1, \ldots, y_e$ are elements which generate the maximal ideal of S and $e + d = \dim(S)$. □

4.4 Noether Normalization

4.4.1 Dimension of Finite Type Algebras over Fields-I

4.4.2 Noether Normalization

4.4.3 Dimension of Finite Type Algebras over Fields-II

4.5 Special Rings over Fields

Chapter 5

Completion of Rings

5.1 General Cases

5.2 Noetherian Cases

Chapter 6

Some Basic Rings, Ideals and Modules

6.1 Valuation Rings

6.2 UFDs

6.3 One-Dimensional Rings

6.4 Pure Ideals

6.5 Torsion Free Modules

6.6 Reflexive Modules

Chapter 7

Associated Primes

7.1 Support and Dimension of Modules

7.2 Associated Primes and Embedded Primes

7.3 Primary Decompositions

Chapter 8

Regular Sequences and Depth

8.1 Several Regular Sequences

8.1.1 Regular Sequences

8.1.2 Koszul Complex and Koszul Regular Sequences

8.2 Depth

8.3 Projective Dimension and Global Dimension

8.4 Auslander-Buchsbaum

Chapter 9

Serre's Conditions and Regular Local Rings

9.1 Serre's Criterions and Its Applications

9.2 Regular Local Rings

9.2.1 Basic Things

9.2.2 Why UFD?

9.2.3 Regular Rings and Global Dimensions

Chapter 10

Cohen-Macaulay Rings

Chapter 11

More Flatness Criteria

We will consider:

(1) Local criterion, see section (B.9) in [Gortz 1].
(2) Slicing criterion, see Lemma 14.23 in [Gortz 1].
(3) Fiber criterion, see Lemma 14.26 in [Gortz 1].
(4) The valuation criterion of flatness, see section (14.6) in [Gortz 1].
(5) Moreover, consider Tag 00R3 Tag 00MD, Tag 051D and Tag 051E.

Chapter 12

Differentials, Naive Cotangent Complex and Smoothness

12.1 Differentials

12.2 The Naive Cotangent Complex

12.3 Local Complete Intersections

12.4 Smoothness, Étaleness and Unramified maps

Chapter 13

Derived Categories of Modules

Chapter 14

Dualizing Complex and Gorenstein Rings

14.1 Projective Covers and Injective Hulls

14.2 Deriving Torsion and Local Cohomology

14.2.1 Deriving Torsion

14.2.2 Local Cohomology

14.2.3 Relation to the Depth

14.3 Dualizing Complexes

14.4 Cohen-Macaulay Rings and Gorenstein Rings

A theorem by Foxby says that R possesses a module of both finite proj. and inj. dim. if and only if R is Gorenstein.

Chapter 15

Others

15.1 Krull-Akizuki

15.2 The Cohen Structure Theorem

Index

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