Note for the Virtual Fundamental Class

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Contents

1 Introduction

We will follows [BF97][AB84][GP99] and we will also use [Ric22].

We need [Har77][Ful98][EH16].

Here we will consider $\mathbb{P}(-) = \mathbf{Proj}\,\mathrm{Sym}(-)^\vee$ for bundles and the vector bundle is both space and sheaf via $\mathbf{Spec}\,\mathrm{Sym}(-)^\vee$. For a cone $C = \mathbf{Spec}_X \mathscr{S}^*$, we define $\mathbb{P}(C) := \mathbf{Proj}_X \mathscr{S}^*$ and $\mathbb{P}(C \oplus \mathscr{O}) := \mathbf{Proj}_X \mathscr{S}^*[z]$ which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

2 Review of Basic Intersection Theory

We will follows [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field k.

2.1 Basic Facts of Refined Gysin Pullback

Here we will follows Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

Definition 2.1 (Intersection Product)**.** *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme* *of pure dimension k. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $\mathscr{I}$ be the ideal of i and $\mathscr{J}$ be the ideal of j, then we have surjection

$$\bigoplus_n f^*(\mathscr{I}^n/\mathscr{I}^{n+1}) \to \bigoplus_n \mathscr{J}^n/\mathscr{J}^{n+1} \to 0$$

which induce embedding $C_{W/V} \hookrightarrow g^*N_{X/Y}$. Note that $C_{W/V}$ is also a scheme of pure dimension k since $\mathbb{P}(C_{W/V} \oplus \mathscr{O})$ is the exceptional divisor of $\mathrm{Bl}_W(Y \times \mathbb{A}^1)$. Let $0 : W \to g^*N_{X/Y}$ be the zero-section of $\pi : g^*N_{X/Y} \to W$, then we define

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \mathsf{CH}_{k-d}(W)$$

as the intersection class.

Proposition 2.2. *Consider the situation of Definition 2.1.*

(a) *We have $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$.*

(b) *Let $\mathscr{Q}$ be the universal quotient bundle of $q : \mathbb{P}(g^*N_{X/Y} \oplus \mathscr{O}) \to W$, then*
$$X \cdot V = q_*(c_d(\mathscr{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathscr{O})]).$$

(c) *If $j : W \hookrightarrow V$ is a regular embedding of codimension d', then $X \cdot V = c_{d-d'}(g^*N_{X/Y}/N_{W/V}) \cap [W]$.*

Proof. Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98]. □

Definition 2.3 (Refined Gysin Pullback)**.** *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick $f : Y' \to Y$ be a morphism. Consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Then we define $i^! : \mathsf{Z}_k\, Y' \to \mathsf{CH}_{k-d}\, X'$ as $\sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i$. Now $i^!$ can be decomposed as:

$$i^! : \mathsf{Z}_k\, Y' \xrightarrow{\sigma} \mathsf{Z}_k\, C_{X'/Y'} \to \mathsf{CH}_k(g^*N_{X/Y}) \xrightarrow{0^*} \mathsf{CH}_{k-d}\, X'$$

where $\sigma : \mathsf{Z}_k Y' \to \mathsf{Z}_k C_{X'/Y'}$ given by $[V] \mapsto [C_{V\cap X'/V}]$. By the technique of deformation to the normal cone, this can be descend to the Chow-group level as $\sigma : \mathsf{CH}_k Y' \to \mathsf{CH}_k C_{X'/Y'}$ (see Proposition 5.2 in [Ful98]) which is called the specialization to the normal cone. *Hence this induce the* refined Gysin pullback

$$i^! : \mathsf{CH}_k Y' \to \mathsf{CH}_{k-d} X', \quad \sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i.$$

Proposition 2.4. *Consider the situation of Definition 2.3. Consider*

$$\begin{tikzcd}
X'' \arrow[r, hook, "i''"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y'' \arrow[d, "p"] \\
X' \arrow[r, hook, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\
X \arrow[r, hook, "i"] & Y
\end{tikzcd}$$

(a) If p proper and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathsf{CH}_{k-d}(X')$.

(b) If p is flat of relative dimension n and $\alpha \in \mathsf{CH}_k(Y')$, then $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathsf{CH}_{k+n-d}(X'')$.

(c) If i' is also a regular embedding of codimension d and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^!\alpha = (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'')$.

(d) If i' is a regular embedding of codimension d', then for $\alpha \in \mathsf{CH}_k(Y'')$ we have

$$i^!(\alpha) = c_{d-d'}(q^*(g^* N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'').$$

We call $g^* N_{X/Y}/N_{X'/Y'}$ the excess normal bundle.

(e) Let F be any vector bundle on Y', then for $\alpha \in \mathsf{CH}_k(Y')$ we have

$$i^!(c_m(F) \cap \alpha) = c_m((i')^* F) \cap i^!(\alpha) \in \mathsf{CH}_{k-d-m}(X').$$

Proof. See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98]. □

Corollary 2.5. *Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d, then*

$$i^* i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathsf{CH}_*(X).$$

Proof. By Proposition 2.4(d) directly. □

Proposition 2.6. *The refined Gysin pullback have the following properties.*

(a) *Let $i : X \hookrightarrow Y$ and $j : S \hookrightarrow T$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & Y'' \arrow[r] \arrow[d, hook, "j'"'] \arrow[dr, phantom, "\ulcorner", very near start] & S \arrow[d, hook, "j"] \\
X' \arrow[r, hook, "i'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r, "g"'] \arrow[d, "f"'] & T \\
X \arrow[r, hook, "i"] & Y &
\end{tikzcd}$$

Then for any $\alpha \in \mathsf{CH}_k(Y'')$, we have

$$j^! i^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X'').$$

(b) *Let $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r, hook, "j'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z' \arrow[d, "f"'] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & Z
\end{tikzcd}$$

Then ji is a regular embedding of codimension $d + e$ and for all $\alpha \in \mathsf{CH}_k(Z')$ we have

$$(ji)^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X').$$

Proof. See Theorem 6.4 and Theorem 6.5 in [Ful98]. $\square$

Proposition 2.7. *Consider cartesians:*

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r, "p'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z' \arrow[d, "f"'] \\
X \arrow[r, "i"] & Y \arrow[r, "p"] & Z
\end{tikzcd}$$

(a) *If i is a regular embedding of codimension d and p and pi are flat of relative dimension $n, n-d$, respectively. Then i' is a regular embedding of codimension d and $p', p'i'$ are flat, and for $\alpha \in \mathsf{CH}_k(Z')$ we have*

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i^!((p')^*\alpha).$$

(b) *If i is a regular embedding of codimension d and p is smooth of relative dimension n, and pi is a regular embedding of codimension $d - n$ Then for $\alpha \in \mathsf{CH}_k(Z')$ we have*

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

Proof. See Proposition 6.5 in [Ful98]. □

Remark 2.8. *Some remarks.*

(a) *For local complete intersection morphism $f : X \to Y$, we can decompose it into $f : X \xrightarrow{i} P \xrightarrow{p} Y$ where i is a closed regular embedding of constant codimension and p is smooth of constant relative dimension. Then we can define $f^! := i^!(p')^*$. See Section 6.6 in [Ful98] for more properties.*

(b) *If Y is nonsingular of dimension n, then we can define the following intersection product: Let $f : X \to Y$ and $p : X' \to X$ and $q : Y' \to Y$. Let $x \in \mathsf{CH}_k(X')$ and $y \in \mathsf{CH}_l(Y')$, consider the cartesian*

$$\begin{tikzcd} X' \times_Y Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \times Y' \arrow[d, "p\times q"] \\ X \arrow[r, "\gamma_f"] & X \times Y \end{tikzcd}$$

and define $x \cdot_f y := \gamma_f^!(x \times y) \in \mathsf{CH}_{k+l-n}(X' \times_Y Y')$.

So when $x, y \in \mathsf{CH}_*(Y)$, then let $X = Y$ and $X' = |x|, Y' = |y|$, then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case $CH_*(Y)$ is a ring which is called Chow ring.

Finally we will discuss something about equivalence and supportness.

Definition 2.9. *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $C_1, ..., C_r$ be the irreducible components of $C_{W/V}$, then $[C_{W/V}] = \sum_{i=1}^r m_i[C_i]$. Let $Z_i = \pi(C_i)$ where $\pi : g^*N_{X/Y} \to W$ and we call them the distinguished varieties *of the intersection of V by X. Let $N_i := (g^*N_{X/Y})|_{Z_i}$ and let $0_i : Z_i \to N_i$ be the zero-sections. Let $\alpha_i := 0_i^*[C_i] \in \mathsf{CH}_{k-d}(Z_i)$ and hence we have $X \cdot V = \sum_{i=1}^r m_i\alpha_i \in \mathsf{CH}_{k-d}(W)$.*

Pick any closed set $S \subset W$, we define

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i\alpha_i \in \mathsf{CH}_{k-d}(S)$$

as the part of $X \cdot V$ supported on S.

Definition 2.10. *Let $X_i \hookrightarrow Y$ be closed regular embeddings of codimension d_i. Let $V \subset Y$ be a k-dimensional subvariety. Consider*

$$\begin{tikzcd}
\bigcap_i X_i \cap V \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, hook, "\delta"] \\
X_1 \times \cdots \times X_r \arrow[r, hook] & Y \times \cdots \times Y
\end{tikzcd}$$

Then we can get $X_1 \cdot \ldots \cdot x_r \cdot V \in \mathsf{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$.

Let Z be a connected component of $\bigcap_i X_i \cap V$, we will consider

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z \in \mathsf{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

Proposition 2.11. *As in the previous situation, we have*

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cap s(Z,V) \right\}_{\dim V - \sum_i d_i}.$$

If $Z \hookrightarrow V$ is a regular embedding, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If V, Z are both non-singular, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

Proof. See Proposition 9.1.1 in [Ful98]. □

2.2 Localized Chern Class

Here we will follows Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

Definition 2.12. *Let $E \to X$ be a vector bundle of rank e over a purely n-dimensional scheme X. Let $s : X \to E$ be a section, consider the cartesian*

$$\begin{tikzcd}
Z(s) \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, hook, "s"'] \\
X \arrow[r, hook, "0"] & E
\end{tikzcd}$$

with zero-section $0 : X \to E$ *which is a regular section by trivial reason. We define*

$$c_{\mathrm{loc}}(E,s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \mathsf{CH}_{n-e}(Z(s))$$

be the ***localized (top) Chern class of*** E ***with respect to*** s.

Proposition 2.13. *Consider the situation of Definition 2.12.*

(a) *We have* $i_*(c_{\mathrm{loc}}(E,s)) = c_e(E) \cap [X]$.

(b) *Each irreducible component of* $Z(s)$ *has codimension at most* e *in* X. *If* $\mathrm{codim}_{Z(s)} X = e$, *then* $c_{\mathrm{loc}}(E,s)$ *is a positive cycle whose support is* $Z(s)$.

(c) *If* s *is a regular section, then* $c_{\mathrm{loc}}(E,s) = [Z(s)]$.

(d) *Let* $f : X' \to X$ *be a morphism,* $s' = f^*s$ *be a induced section of* f^*E. *Let* $g : Z(s') \to Z(s)$ *be the induced morphism.*

  (d1) *If* f *flat, then* $g^*c_{\mathrm{loc}}(E,s) = c_{\mathrm{loc}}(f^*E, s')$.

  (d2) *If* f *is proper of varieties, then* $g_*c_{\mathrm{loc}}(f^*E,s') = \deg(X'/X)c_{\mathrm{loc}}(E,s)$.

Proof. For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_*0^![X] = 0^*s_*[X] = s^*s_*[X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{tikzcd}
Z(s') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d, "s'"] \\
X' \arrow[r, hook, "0_{f^*E}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f^*E \arrow[d] \\
X \arrow[r, hook, "0_E"] & E
\end{tikzcd}$$

and Proposition 2.4. □

3 A Brief of Cotangent Complexes

Here we will give a quike introduction of cotangent complexes. We will consider Deligne-Mumford stacks locally of finite type over k. Morphisms are quasicompact and quasiseparated. We work over étale site.

Theorem 3.1. *For every morphism $f : X \to Y$ of DM-stacks (resp. finite type morphism of noetherian DM-stacks), there exists a complex*

$$\mathbb{L}_{X/Y} : \cdots \to \mathbb{L}^{-1}_{X/Y} \to \mathbb{L}^{0}_{X/Y} \to 0$$

of flat $\mathscr{O}_X$-modules with quasi-coherent (resp., coherent) cohomology, whose image $\mathbf{D}^{-}_{\mathrm{Qcoh}}(X_{\acute{e}t})$ (resp. $\mathbf{D}^{-}_{\mathrm{Coh}}(X_{\acute{e}t})$) is also denoted by $\mathbb{L}_{X/Y}$. This is called the cotangent complex of f. *It satisfies the following properties.*

(a) *$H^0(X, \mathbb{L}_{X/Y}) = \Omega^1_{X/Y}$.*

(b) *The morphism f is smooth if and only if f is locally of finite presentation and $\mathbb{L}_{X/Y}$ is a perfect complex supported in degree 0. In this case, there is a quasi-isomorphism $\mathbb{L}_{X/Y} \cong \Omega^1_{X/Y}[0]$.*

(c) *If f factors as $X \hookrightarrow Z$ defined by a sheaf of ideals $\mathscr{I}$ and a smooth morphism $Z \to Y$, then*

$$\mathbb{L}_{X/Y} \cong [0 \to \mathscr{I}/\mathscr{I}^2 \to \Omega^1_{Z/Y}|_X \to 0]$$

in $\mathbf{D}^{-}_{\mathrm{Qcoh}}(X_{\acute{e}t})$ with $\Omega^1_{X/Y}$ in degree 0. If in addition f is generically smooth, then $\mathbb{L}_{X/Y} \cong \Omega^1_{X/Y}[0]$. Moreover, if f is lci, then $\mathbb{L}_{X/Y}$ is perfect of perfect amplitude contained in $[-1, 0]$.

(d) *If we have a cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

then there is a morphism $(\mathbf{L}g')^*\mathbb{L}_{X/Y} \to \mathbb{L}_{X'/Y'}$. When f or g is flat, then it is a quasi-isomorphism.

(e) *If $X \xrightarrow{f} Y \to Z$ is a composition of morphisms of DM-stacks, then there is an exact triangle*

$$\mathbf{L}f^*\mathbb{L}_{Y/Z} \to \mathbb{L}_{X/Z} \to \mathbb{L}_{X/Y} \to \mathbf{L}f^*\mathbb{L}_{Y/Z}[1]$$

in $\mathbf{D}^{-}_{\mathrm{Qcoh}}(X_{\acute{e}t})$. This induces a long exact sequence on cohomology

$$\cdots \to H^{-1}(\mathbb{L}_{X/Z}) \to H^{-1}(\mathbb{L}_{X/Y}) \to f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0.$$

Proof. In the level of ring maps $A \to B$, this constructed by standard simplicial free A-resolution $B \to P(B)_*$ where $P(B)_n = A[\cdots[A[B]]\cdots]$ as

$$\mathbb{L}_{B/A} := \Omega_{P(B)_*/A} \otimes_{P(B)_*} B.$$

See Tag 08UV Tag 0D0N Tag 0FK3 Tag 08QQ Tag 08T4. □

Remark 3.2. *For the general algebraic stacks, any quasicompact and quasiseparated* 1*-morphism* $f : \mathscr{X} \to \mathscr{Y}$ *there exists a relative cotangent complex*

$$\mathbb{L}_f \in \mathbf{D}^{\leq 1}_{\mathrm{Coh}}(\mathscr{X}_{\textit{lis-ét}})$$

over lisse-étale site of $\mathscr{X}$*. Existence is good, but the fact that the cotangent complex trespasses to positive degree forces one to pay more attention when performing the cutoff. If the diagonal of* f *is unramified (as we consider now), then this problem goes away, in the sense that* $\mathbb{L}_f \in \mathbf{D}^{\leq 0}_{\mathrm{Coh}}(\mathscr{X}_{\textit{lis-ét}})$*. We refer section C.3 in [Ric22] for more comments about this and the generalization of the properties as above.*

4 Fundations of Virtual Fundamental Class

We will follows [BF97]. Here an algebraic stack (or Artin stack) over a field k is assumed to be quasi-separated and locally of finite type over k.

4.1 About Cones

We will let X be a Deligne-Mumford stack now.

Definition 4.1. *Let* X *be a DM-stack.*

(a) We call an affine X*-scheme* $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ *is a* cone over X *if the quasi-coherent algebra* $\mathscr{S}$ *is graded as* $\mathscr{S} = \bigoplus_{i \geq 0} \mathscr{S}^i$ *with* $\mathscr{S}^0 = \mathscr{O}_X$ *and* $\mathscr{S}^1$ *is coherent and* $\mathscr{S}$ *is generated by* $\mathscr{S}^1$*.*

(b) A morphism of cones over X *is an* X*-morphism induced by a graded morphism of graded sheaves of* $\mathscr{O}_X$*-algebras. A* closed subcone *is the image of a closed immersion of cones.*

Remark 4.2. *(a) The fiber product of cones over* X *is still a cone over* X*.*

(b) For every cone $C \to X$*, it has a zero section* $0 : X \to C$ *induced by* $\mathscr{S} \to \mathscr{S}^0$*.*

(c) For every cone $C \to X$*, the grade induce a* $\mathbb{G}_m$*-action* $\mathbb{G}_m \times C = \underline{\mathrm{Spec}}_X \mathscr{S}[t, t^{-1}] \to C$ *induced by* $\mathscr{S} \to \mathscr{S}[t, t^{-1}]$ *via* $s_0 + \cdots s_d \mapsto \sum_i a_i t^i$ *where* $s_i \in \mathscr{S}^i$*. Since no negative power of* t *occurs, we can in fact replace* $\mathbb{G}_m$ *by* $\mathbb{A}^1$*. So we have the* $\mathbb{A}^1$*-action* $\gamma : \mathbb{A}^1 \times C \to C$ *induced by* $\mathscr{S} \to \mathscr{S}[x]$ *via* $\mathscr{S}^i \ni s \mapsto sx^i$*. Note that here* $\mathbb{A}^1$ *is not a group scheme and the* action *here, as expected, to be the commutativity*

of the following diagrams:

$$\begin{tikzcd} C \arrow[r, "{(1,\mathrm{id})/(0,\mathrm{id})}"] \arrow[dr, "\mathrm{id}/0"'] & \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"'] & \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ \mathbb{A}^1 \times C \arrow[r, "\gamma"'] & C \end{tikzcd}$$

where $m(x, y) = xy$.

(d) *So a morphism of cones $f : C \to D$ over X is just the $\mathbb{A}^1$-equivariant X-morphism respecting the zero section, that is, the following commutativity of the diagram:*

$$\begin{tikzcd} \mathbb{A}^1 \times C \arrow[r] \arrow[d, "\mathrm{id}\times f"'] & C \arrow[r, shift left] \arrow[d, "f"'] & X \arrow[l, shift left, "0_C"'] \arrow[dl, shift left, "0_D"] \\ \mathbb{A}^1 \times D \arrow[r] & D \arrow[ur, shift left] \end{tikzcd}$$

Definition 4.3. *Let $\mathscr{F}$ be a coherent sheaf of X, then we can define $C(\mathscr{F}) := \underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathscr{F})$ which is a group scheme over X since it can be represented as $C(\mathscr{F})(T) = \mathrm{Hom}(\mathscr{F}_T, \mathscr{O}_T)$. We call a cone of this form is an abelian cone over X.*

Remark 4.4. (a) *A fibered product of abelian cones is an abelian cone.*

(b) *A vector bundle $E = \underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathscr{E}^\vee)$ is a special case.*

(c) *Any cone $C = \underline{\mathrm{Spec}}_X \bigoplus_{i\geq 0} \mathscr{S}^i$ is canonically a closed subcone of an abelian cone $A(C) = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{S}^1$, called the abelian hull of C. The abelian hull is a vector bundle if and only if $\mathscr{S}^1$ is locally free.*

(d) *The abelianization $C \mapsto A(C)$ is a functor has the forgetful functor as a right adjoint. So we have*

$$\mathrm{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \mathrm{Hom}_{\mathbf{Cone}_X}(C, A).$$

(e) *Let $\mathbf{Alg}_X^o$ as the category of quasicoherent graded $\mathscr{O}_X$-algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:*

$$\begin{tikzcd} \mathbf{Alg}_X^o \arrow[r, "\underline{\mathrm{Spec}}_X"] & \mathbf{Cone}_X^{\mathrm{op}} \\ \mathbf{LocFree}_X \arrow[u, "\mathrm{Sym}"] \arrow[r, "\underline{\mathrm{Spec}}_X \mathrm{Sym}(-)^\vee"] \arrow[d, hook] & \mathbf{Vect}_X^{\mathrm{op}} \arrow[u, hook] \arrow[d, hook] \\ \mathbf{Coh}_X \arrow[r, "\underline{\mathrm{Spec}}_X \mathrm{Sym}"] & \mathbf{AbCone}_X^{\mathrm{op}} \end{tikzcd}$$

Example 4.5. *Tow important examples. Let $X \hookrightarrow Y$ be a closed immersion of ideal $\mathscr{I}$. Then $C_{X/Y} := \underline{\mathrm{Spec}}_X \bigoplus_{n\geq 0} \mathscr{I}^n/\mathscr{I}^{n+1}$ is called the* normal cone of X in Y. *The associated abelian cone $N_{X/Y} = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{I}/\mathscr{I}^2$ is called the* normal sheaf of X in Y.

Lemma 4.6. *About smoothness:*

(a) Let $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ be a cone over X. Then $C_{X/C} \cong \mathscr{S}^1 \cong 0^*\Omega_{C/X}$.

(b) A cone C over X is a vector bundle if and only if it is smooth over X.

(c) Let $C \to D$ be a smooth morphism of cones of relative dimension n over X. Then the induced morphism $A(C) \to A(D)$ is also smooth of relative dimension n.

Proof. For (a), note that $C_{X/C} \cong \mathscr{S}^1$ is trivial by definition. Morever, $0 : X \to C$ is the zero section and we have $0 \to C_{X/C} \to 0^*\Omega_{C/X} \to \Omega_{X/X} = 0$ exact (see Tag 0474). Well done.

For (b), let $C = \underline{\mathrm{Spec}}_X \bigoplus_{i\geq 0} \mathscr{S}^i$ and assume that $C \to X$ has constant relative dimension r. Then $\mathscr{S}^1 = 0^*\Omega_{C/X}$ is locally free of rank r. As $C \hookrightarrow A(C)$ where $A(C)$ is a vector bundle and $\dim C = \dim A(C)$, we know that C is a vector bundle.

For (c), apply the exact triangle of cotangent complex to $X \to C \to D$ and (a), we have an exact sequence

$$0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to 0_C^*\Omega_{C/D} \to 0$$

where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. So locally we have $A(C) = A(D) \times_X \underline{\mathrm{Spec}}_X \operatorname{Sym}(0_C^*\Omega_{C/D})$. Well done. □

Definition 4.7. *A sequence of cone morphisms*

$$0 \to E \xrightarrow{i} C \to D \to 0$$

is called exact *if E is a vector bundle and locally over X there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \cong E \times_X D$.*

Remark 4.8. *As $E \to X$ is smooth and surjective by Lemma 4.6, if $0 \to E \xrightarrow{i} C \to D \to 0$ then locally we have $C \cong E \times_X D$ which force that $C \to D$ is smooth and surjective! Similarly $i : E \to C$ is a closed embedding.*

Lemma 4.9. *We have the following useful results.*

(a) Given a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0$ of coherent sheaves on X, with $\mathscr{E}$ locally free, then $0 \to C(\mathscr{E}) \to C(\mathscr{F}) \to C(\mathscr{F}') \to 0$ is exact, and conversely is also true.

(b) *Let* $0 \to E \to F \xrightarrow{f} G \to 0$ *be an exact sequence of abelian cones over* X *with* E *a vector bundle. Assume that* $D \subset G$ *is a closed subcone, then the induced sequence* $0 \to E \to f^{-1}(D) =: C \to D \to 0$ *is exact.*

(c) *Let* $f : C \to D$ *be a morphisms of cones over* X *which is smooth surjective, then the induced diagram*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, hook] & D \arrow[d, hook] \\ A(C) \arrow[r, "A(f)"] & A(D) \end{tikzcd}$$

is cartesian. Moreover, we have $D = [C/E]$ *(see Lemma 4.12(a)) and* $A(D) = [A(C)/E]$*, where* $E := C \times_{D,0} X = A(C) \times_{A(D),0} X$*.*

(d) *Let* E *be a vector bundle over* X *and then the sequence* $0 \to E \to C \to D \to 0$ *is exact if and only if the abelian hulls* $0 \to E \to A(C) \to A(D) \to 0$ *is exact and* $C \to D$ *is smooth and surjective.*

Proof. For (a), we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. As exactness is local, we may assume $\mathscr{E}$ is free. Then the first sequence is exact if and only if $\mathscr{F}' \oplus \mathscr{E} = \mathscr{F}$ if and only if the second sequence is exact as cones, since $\mathrm{Sym}(\mathscr{F}' \oplus \mathscr{E}) = \mathrm{Sym}(\mathscr{F}') \otimes \mathrm{Sym}(\mathscr{E}) = \mathrm{Sym}(\mathscr{F})$.

For (b), note that this can be checked locally, so we can let we can assume that $\mathscr{F} = \mathscr{G} \oplus \mathscr{E}^\vee$ where $E = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{E}^\vee$ and $F = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{F}$ and $G = \underline{\mathrm{Spec}}_X \mathrm{Sym}\, \mathscr{G}$. Let $D = \underline{\mathrm{Spec}}_X \mathscr{T}$, then we have surjection $\mathrm{Sym}(\mathscr{G}) \to \mathscr{T}$. By definition, we have

$$\begin{aligned} C = F \times_G D &= \underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathscr{F}) \otimes_{\mathrm{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\mathrm{Spec}}_X((\mathrm{Sym}(\mathscr{G}) \otimes \mathrm{Sym}\, \mathscr{E}^\vee) \otimes_{\mathrm{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\mathrm{Spec}}_X(\mathrm{Sym}\, \mathscr{E}^\vee \otimes \mathscr{T}). \end{aligned}$$

This means locally $C = E \oplus D$ and the splitting $C \to E$ is induced by $F \to E$. Well done.

For (c), let $E := C \times_{D,0} X$ and $E' := A(C) \times_{A(D)} D$ with embedding $E \hookrightarrow E'$, then both of them are vector bundles by Lemma 4.6(b)(c) and hence $E = E'$. We have cartesians

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook] \\ A(C) \arrow[r] & A(D) \end{tikzcd}$$

By the properties of commutative affine group schemes, we have $A(D) =$

$[A(C)/E]$. But how about $[C/E]$? Now we have

$$\begin{tikzcd}
 & & D \arrow[ddl, hook] \\
C \arrow[r] \arrow[urr] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & {[C/E]} \arrow[d, hook] & \\
A(C) \arrow[r] & A(D) &
\end{tikzcd}$$

Since $C \to [C/E]$ and $C \to D$ are both smooth and surjective, we know that $[C/E] \to D$ is flat and surjective. But by closed embeddings $[C/E] \to A(D)$ and $D \to A(D)$, we know that $[C/E] \to D$ is also a closed embedding. Thus $D = [C/E]$, well done.

For (d), note that all the question is locally on X. First we assume $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ is exact. Then by (a), to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, we only need to show that $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact where $E = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and $C = \underline{\operatorname{Spec}}_X \mathscr{S}$ and $D = \underline{\operatorname{Spec}}_X \mathscr{T}$. First since f is faithfully flat and quasi-compact, we know that $\mathscr{T}^1 \to \mathscr{S}^1$ is injective. And since i is a closed embedding, $\mathscr{S}^1 \to \mathscr{E}^\vee$ is surjective. Now by local splitting, we know that locally we have $\operatorname{Sym}(E^\vee) \otimes \mathscr{T} = \mathscr{S}$. In particular, we have $\mathscr{T}^1 \oplus \mathscr{E}^\vee = \mathscr{S}^1$. Thus the exactness of $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is obtained. Conversely we assume that after taking abelian hull, the sequence is exact. Now the result follows from (a) and (c). $\square$

Proposition 4.10. *Let $C \to D$ be a smooth, surjective morphism of cones. If we let $E = C \times_{D,0} X$, then the sequence*

$$0 \to E \to C \to D \to 0$$

is exact. Conversely if $0 \to E \to C \to D \to 0$ is exact, then $E \cong C \times_{D,0} X$.

Proof. Let $C = \underline{\operatorname{Spec}}_X \bigoplus_{i \geq 0} \mathscr{S}^i$ and $D = \underline{\operatorname{Spec}}_X \bigoplus_{i \geq 0} \mathscr{T}^i$.

Let $E = C \times_{D,0} X = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$, by Lemma 4.9(d) we just need to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, that is, $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact by Lemma 4.9(a). Note that $\operatorname{Sym} \mathscr{E}^\vee = \mathscr{S} \otimes_{\mathscr{T}} (\mathscr{T}/\mathscr{T}^{\geq 1})$ which force $\mathscr{E}^\vee \cong \mathscr{S}^1/\mathscr{T}^1$. Well done.

Conversely, assume that the sequence $0 \to E \to C \to D \to 0$ is exact and $F = C \times_{D,0} X$. Then by the universal property of fibre product, we get a morphism $E \to F$. From the construction, it is easy to see that $\mathscr{F}^\vee \to \mathscr{E}^\vee$ surjective. Since they are both bundles of the same rank over X, we know that $E = F$. $\square$

Definition 4.11. *(a) If E is a vector bundle and $f : E \to C(\mathscr{F})$ a morphism of abelian cones. The there is an E-action as $E \times_X C(\mathscr{F}) \to C(\mathscr{F})$ as $(\nu, \gamma) \mapsto f\nu + \gamma$.*

(b) If E is a vector bundle and $d : E \to C$ a morphism of cones, we say that C is an E-cone, if C is invariant under the action of E on $A(C)$.

(c) A morphism ϕ from an E-cone C to an F-cone D is a commutative diagram of cones

$$\begin{tikzcd} E \arrow[r, "d_E"] \arrow[d, "\phi"] & C \arrow[d, "\phi"] \\ F \arrow[r, "d_F"] & D \end{tikzcd}$$

(d) If $\phi : (E, d_E, C) \to (F, d_F, D)$ and $\psi : (E, d_E, C) \to (F, d_F, D)$ are morphisms, we call them homotopic, if there exists a morphism of cones $k : C \to F$, such that $kd_E = \psi - \phi = d_F k$.

Lemma 4.12. *Some useful lemmas:*

(a) Let $f : C \to D$ be a smooth surjective cone morphism with $E = C \times_{D,0} X$, then C is an E-cone.

(b) Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D = [C/E] \to 0$ be a sequence of algebraic X-spaces with E a bundle, C is a E-cone, i a closed embedding and $f : C \to D = [C/E]$ is the universal family. Then locally on X, there is a $j : C \to E$ split i and induces an isomorphism $(f, j) : C \to D \times_X E$.

(c) Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ be a sequence of algebraic X-spaces with sections and $\mathbb{A}^1$-actions such that E a bundle, C is a E-cone, i is a closed embedding and f is $\mathbb{A}^1$-equivariant. Then D is a cone with the sequence exact if and only if $D \cong [C/E]$.

Proof. For (a), this follows from directly check. We omit it.

For (b), since the question is local we can assume that E is a trivial bundle and X is a scheme. Let $i' : E \to A(C)$ and $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Then the surjection $\mathscr{S}^1 \twoheadrightarrow \mathscr{E}^\vee$ has a splitting $\mathscr{E}^\vee \hookrightarrow \mathscr{S}^1$, which gives $j' : A(C) \to E$ such that $j' \circ i' = \mathrm{id}_E$. Then we just define $j : C \to E$ as composition with $C \to A(C)$ and j'. Hence $j \circ i = \mathrm{id}_E$.

Now since $C \to D$ is also a principal E-bundle, and we have a E-equivariant D-morphism $(f, j) : C \to D \oplus E$ from C to the trivial principal bundle. Since they are both E-principal bundle, we know that (f, j) is an isomorphism.

For (c), let $D = [C/E]$. We know that $D \to X$ is affine since locally on X we have $C \cong D \times_X E \to E$ is affine and (b) and faithfully flat descent. By construction we have $E = C \times_{D,0} X$, hence by Proposition 4.10 we just need to show D is a cone. Now as $D \to X$ affine we have $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. If $C = \underline{\mathrm{Spec}}_X \mathscr{S}$, then $\mathscr{T} \subset \mathscr{S}$ as $C \to D$ is faithfully flat. Hence it has graded structure $\mathscr{T} = \bigoplus_{i \geq 0} \mathscr{T} \cap \mathscr{S}^i$ as f is $\mathbb{A}^1$-equivariant. As it have zero section, we have $\mathscr{T}^0 = \mathscr{O}_X$. Finally we have $\mathbb{A}^1$-equivariant embedding $D \hookrightarrow [A(C)/E]$ and $[A(C)/E]$ is a cone by Lemma 4.9(c). Hence $\mathscr{T}$ generated by the coherent sheaf $\mathscr{T}^1$.

Conversely, we assume D is a cone and that sequence is exact. Let $D' = [C/E]$. By the universal property of quotient, we have a natural map $g : D' \to D$. Since $0 \to E \to C \to D' \to 0$ is also exact by the first case, by exactness we have locally $C \cong E \times_X D \cong E \times_X D'$. Note that these isomorphisms compatible with $g : D' \to D$, hence by faithfully flat descent we have g is an isomorphism. □

Proposition 4.13. *Let X be a DM-stack.*

(a) Let E be a vector bundle. Consider the sequence of cone morphisms $0 \to E \xrightarrow{i} C \xrightarrow{\phi} D \to 0$ with i a closed embedding. Then it is exact if and only if C is a E-cone, $\phi : C \to D$ is faithfully flat and the diagram

$$\begin{tikzcd} E \times C \arrow[r, "\sigma"] \arrow[d, "p"] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d, "\phi"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

is cartesian with projection p and action σ.

(b) Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions and let $\phi : C \to D$ be an $\mathbb{A}^1$-equivariant X-morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Assume that E is a vector bundle. Then C is an E-cone (resp. abelian cone, vector bundle) over X if and only if D is a cone (resp. abelian cone, vector bundle) over X and C is affine over X.

Proof. For (a), if it is exact, locally we have $C \cong E \times_X D$. So E act on C locally as $E \times E \times_X D \to E \times_X D$ given by $(f, (e, d)) \mapsto (i(f) + e, d)$. So C is a E-cone. Now $\phi : C \to D$ is trivially faithfully flat. The cartesian diagram follows from Lemma 4.12(c).

Conversely, since ϕ is fppf, this diagram is also cocartesian by Proposition V.1.3.1 in [Li18] which force $D = [C/E]$. Hence the results follows from Lemma 4.12(c).

For (b), let C is an E-cone over X. Then we have $g : [C/E] \to D$. We claim that g is an isomorphism. Indeed, by the diagram in (a), we know that g induces an isomorphism $g' : E \times_X C = C \times_{[C/E]} C \to C \times_D C$. Note that we have a cartesian diagram:

$$\begin{tikzcd} C \times_{[C/E]} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C \times_D C \arrow[d] \\ {[C/E]} \arrow[r, hook] & {[C/E] \times_D [C/E]} \end{tikzcd}$$

where $C \times_D C \to [C/E] \times_D [C/E]$ is faithfully flat, hence $[C/E] \hookrightarrow [C/E] \times_D [C/E]$ is an isomorphism. So g is a monomorphism. But since $C \to [C/E]$ and $C \to D$ are faithfully flat, hence epimorphism. Thus g is also an epimorphism, hence an isomorphism. Thus $D \cong [C/E]$ and the result follows from Lemma 4.12(c).

Now assume that $C = A(C)$ is an abelian cone, then taking hull to $0 \to E \to C \to D = [C/E] \to 0$. By Lemma 4.9(c)(d) we have $A(D) = [A(C)/E] = [C/E] = D$. Hence D is also an abelian cone.

Finally assume that C is a bundle. Then by the previous case we know that D is an abelian cone. The $\mathscr{T}^1 = \ker(\mathscr{S}^1 \twoheadrightarrow \mathscr{E}^\vee)$ is clearly locally free since $\mathscr{C}^1$ and $\mathscr{E}$ are where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$, $D = \underline{\mathrm{Spec}}_X \mathscr{T}$ and $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$.

Conversely we let D is a cone and C is affine over X. Hence we have $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ where $\mathscr{S} = \bigoplus_{i \geq 0} \mathscr{S}^i$ and $\mathscr{S}^1 = \mathscr{O}_X$. By the same reason E is affine over X. Hence we have $C = \underline{\mathrm{Spec}}_X \mathscr{F}$ where $\mathscr{F} = \bigoplus_{i \geq 0} \mathscr{F}^i$ and $\mathscr{F}^1 = \mathscr{O}_X$. If we let $D = \underline{\mathrm{Spec}}_X \mathscr{T}$, then $\mathscr{F} = \mathscr{S}/(\mathscr{T}^{\geq 1}\mathscr{S})$.

Apply the exact triangle of cotangent complex to $X \overset{0_C}{\to} C \to D$, we have an exact sequence

$$0 \to \mathscr{T}^1 \to \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 = C_{X/C} \to \mathscr{E}^\vee := 0_C^* \Omega_{C/D} \to 0.$$

As $\mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 = \mathscr{S}^1 \oplus \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2$, we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathscr{T}^1 \arrow[r] \arrow[d, "="] & \mathscr{S}^1 \arrow[r] \arrow[d] & \mathscr{F}^1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{T}^1 \arrow[r] & \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2 \arrow[r] \arrow[d] & \mathscr{E}^\vee \arrow[r] \arrow[d] & 0 \\ & & \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2 \arrow[r, "="] \arrow[d] & \mathscr{E}^\vee/\mathscr{F} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Locally on X we can assume that $\mathscr{E}$ is free and $\mathscr{T}^1 \oplus \mathscr{E}^\vee = \mathscr{S}^{\geq 1}/(\mathscr{S}^{\geq 1})^2$. Then as $\mathscr{F}^1 \subset \mathscr{E}^\vee$, we know that $\mathscr{F}^1$. Since $\mathscr{T}^1$ is also coherent, we know that so is $\mathscr{S}^1$. Finally we just need to show $\mathscr{S}$ generated by $\mathscr{S}^1$ as by Lemma 4.12(a) here C will be an E-cone.

Then locally on X we can choose generators of $\mathscr{T}^1, \mathscr{F}^1, \mathscr{E}^\vee/\mathscr{F}^1 = \mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2$ such that gives a surjective $\mathscr{O}_X$-algebra morphism $\phi : \mathscr{T} \oplus \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{S}$ which induce $\mathscr{T} \oplus \operatorname{Sym} \mathscr{F}^1 \to \mathscr{T} \oplus \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{S}$ is graded. Tensoring $(-) \otimes_{\mathscr{T}} \mathscr{O}_X$ with ϕ we get surjection $\phi' : \operatorname{Sym} \mathscr{E}^\vee \twoheadrightarrow \mathscr{F}$. This induce the closed immersion $E \hookrightarrow \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Since they are both smooth of a same relative dimension over X and $\underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ is a vector bundle, hence $E \cong \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and ϕ' is an isomorphism. Hence $\mathscr{F} = \operatorname{Sym}(\mathscr{F}^1)$ and $\mathscr{F}^1$ is locally free. As $\operatorname{Sym}(\mathscr{F}^1) \subset \operatorname{Sym} \mathscr{E}^\vee \xrightarrow{\phi'} \mathscr{F} = \operatorname{Sym}(\mathscr{F}^1)$ is identity, this force $\mathscr{E}^\vee = \mathscr{F}^1$. As this can be check locally, we have $\mathscr{E}^\vee = \mathscr{F}^1$ in whole X. By the diagram above, we have $\mathscr{S}^{\geq 2}/(\mathscr{S}^{\geq 1})^2 = \mathscr{E}^\vee/\mathscr{F}^1 = 0$. This means $\mathscr{S}$ generated by $\mathscr{S}^1$. Well done. $\square$

Remark 4.14. *In the original paper [BF97] they claim (a) is enough for the surjectivity of f.*

4.2 Cone Stack

Let X be a Deligne-Mumford stack.

Definition 4.15. *Let $\mathfrak{C}$ be an algebraic stack over X, together with a section $0 : X \to \mathfrak{C}$. An $\mathbb{A}^1$-action on $(\mathfrak{C}, 0)$ is given by a morphism of X-stacks $\gamma : \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$ and three 2-isomorphisms θ_1, θ_0 and θ_γ between the 1-morphisms in the following diagrams.*

$$\begin{tikzcd}
\mathfrak{C} \arrow[rr, "(1,\mathrm{id})/(0,\mathrm{id})"] \arrow[dr, "\mathrm{id}/0"'] & \arrow[d, phantom, "\Rightarrow \theta_1/\theta_0"] & \mathbb{A}^1 \times \mathfrak{C} \arrow[dl, "\gamma"] \\
& \mathfrak{C} &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"] \arrow[dr, phantom, "\Rightarrow \theta_\gamma"] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\
\mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\gamma"'] & \mathfrak{C}
\end{tikzcd}$$

The 2-isomorphisms θ_1, θ_0 and θ_γ are required to satisfy certain compatibilities.

Definition 4.16. *Let $(\mathfrak{C}, 0, \gamma)$ and $(\mathfrak{D}, 0, \gamma)$ be X-stacks with sections and $\mathbb{A}^1$-actions. Then an $\mathbb{A}^1$-equivariant morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ is a triple $(\phi, \eta_0, \eta_\gamma)$,* *where $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of algebraic X-stacks and η_0 and η_γ are 2-isomorphisms between the morphisms in the following diagrams.*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "0"'] & \mathfrak{C} \arrow[d, "\phi"] \arrow[dl, phantom, "\Uparrow \eta_0"] \\ & \mathfrak{D} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\phi"] \arrow[d, "\gamma"] & \mathbb{A}^1 \times \mathfrak{D} \arrow[d, "\gamma"] \\ \mathfrak{C} \arrow[r, "\phi"] \arrow[ur, phantom, "\eta_\gamma"] & \mathfrak{D} \end{tikzcd}$$

Again, the 2-isomorphisms have to satisfy certain compatibilities.

Definition 4.17. *Let $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \to \mathfrak{D}$ and $(\psi, \eta'_0, \eta'_\gamma) : \mathfrak{C} \to \mathfrak{D}$ be two $\mathbb{A}^1$-equivariant morphisms. An $\mathbb{A}^1$-*equivariant isomorphism* $\zeta : \phi \to \psi$ is a 2-isomorphism $\zeta : \phi \to \psi$ such that the diagrams*

$$\begin{tikzcd} 0 \arrow[r, "\eta_0"] \arrow[dr, "\eta'_0"'] & \phi \circ 0 \arrow[d, "\zeta\circ 0"] \\ & \psi \circ 0 \end{tikzcd} \qquad \begin{tikzcd} \phi\circ\gamma \arrow[r] \arrow[d, "\zeta\circ\gamma"] & \gamma\circ(\mathrm{id}\times\phi) \arrow[d, "\gamma\circ(\mathrm{id}\times\zeta)"] \\ \psi\circ\gamma \arrow[r] & \gamma\circ(\mathrm{id}\times\psi) \end{tikzcd}$$

commute.

Example 4.18. *Let C be a E-cone, then consider the quotient stack $[C/E]$. We claim that $[C/E]$ a zero section and an $\mathbb{A}^1$-action.*

Indeed, the zero section $0 : X \to [C/E]$ given by $X \leftarrow E \to C$. The $\mathbb{A}^1$-action of $\alpha \in \mathbb{A}^1(T)$ on $(P, f) \in [C/E](T)$ defined by $(\alpha P, \alpha f)$ where $\alpha P = P \times^{E,\alpha} E$ and $\alpha f : P \times^{E,\alpha} E \to C$ given by $[p, v] \mapsto \alpha f(p) + d(v)$ where $d : E \to C$.

Moreover, if $\phi : (E, C) \to (F, D)$ is a morphism of vector bundle cones we get an induced $\mathbb{A}^1$-equivariant morphism $\tilde{\phi} : [C/E] \to [D/F]$.

Lemma 4.19. *Some useful results.*

(a) A homotopy $k : \phi \to \psi$ of two morphisms of vector bundle cones $\phi, \psi : (E, C) \to (F, D)$ gives rise to an $\mathbb{A}^1$-equivariant 2-isomorphism $\tilde{k} : \tilde{\phi} \to \tilde{\psi}$ of $\mathbb{A}^1$-equivariant morphisms of stacks with $\mathbb{A}^1$-action.

(b) Conversely, let two morphisms of vector bundle cones $\phi, \psi : (E, C) \to (F, D)$ with an $\mathbb{A}^1$-equivariant 2-isomorphism $\zeta : \tilde{\phi} \to \tilde{\psi}$ of $\mathbb{A}^1$-equivariant morphisms of stacks with $\mathbb{A}^1$-action. Then $\zeta = \tilde{k}$ for unique homotopy $k : \phi \to \psi$.

Proof. For (a), samilar to Proposition 4.29. For (b) TBC... □

Proposition 4.20. *Let C be an E-cone and D an F-cone and let $\phi : (E, C) \to (F, D)$ be a morphism. If the diagram*

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "\phi"] \\ F \arrow[r, "d"'] & D \end{tikzcd}$$

is cartesian and $F \times_X C \to D$ by $(\mu, \gamma) \mapsto d\mu + \phi(\gamma)$ is surjective, then $[C/E] \to [D/F]$ is an isomorphism of algebraic X-stacks with $\mathbb{A}^1$-action.

Proof. For the same proof of Proposition 4.30. □

Definition 4.21. *(a) We call an algebraic stack $(\mathfrak{C}, 0, \gamma)$ over X with section and $\mathbb{A}^1$-action a* cone stack*, if, étale locally on X, there exists a cone C over X and an $\mathbb{A}^1$-equivariant morphism $C \to \mathfrak{C}$ that is smooth and surjective and such that $E = C \times_{\mathfrak{C},0} X$ is a vector bundle over X.*

(b) The morphism $C \to \mathfrak{C}$ is called a local presentation of *$\mathfrak{C}$. The section $0 : X \to \mathfrak{C}$ is called the vertex of $\mathfrak{C}$.*

(c) Let $\mathfrak{C}$ and $\mathfrak{D}$ be cone stacks over X. A morphism of cone stacks *$\phi : \mathfrak{C} \to \mathfrak{D}$ is an $\mathbb{A}^1$-equivariant morphism of algebraic X-stacks. A 2-*isomorphism of cone stacks *is just an $\mathbb{A}^1$-equivariant 2-isomorphism.*

(d) A cone stack $\mathfrak{C}$ over X is called abelian cone stack *(resp.* vector bundle stack*), if, locally in X, one can find presentations $C \to \mathfrak{C}$, where C is an abelian cone (resp. vector bundle).*

Remark 4.22. *Some basic properties of cone stacks.*

(a) If $C \to \mathfrak{C}$ is a global presentation with $E = C \times_{\mathfrak{C},0} X$, then C is an E-cone with $\mathfrak{C} \cong [C/E]$ as stacks with $\mathbb{A}^1$-action. This follows from Proposition 4.10 and 4.13 and Lemma 4.12.

(b) If $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of cone stacks, then, étale locally on X, ϕ is $\mathbb{A}^1$-equivariantly isomorphic to $[C/E] \to [D/F]$, where $E \to F$ is a morphism of vector bundles over X and $C \to D$ is a morphism from the E-cone C to the F-cone D.

(c) A 2-isomorphism of cone stacks $\zeta : \phi \to \psi$, where $\phi, \psi : \mathfrak{C} \to \mathfrak{D}$, is étale locally over X given by a homotopy of morphisms of vector bundle cones. This follows from Lemma 4.19(b).

(d) Let $C \to \mathfrak{C}$ and $D \to \mathfrak{D}$ be two local presentation of a cone stack $\mathfrak{C}$ over X, then so is $C \times_{\mathfrak{C}} D \to \mathfrak{C}$.

Indeed, we only need to show that $C \times_{\mathfrak{C}} D$ is a cone. Since $C \to \mathfrak{C}$ and $D \to X$ are affine, we know that $C \times_{\mathfrak{C}} D \to D \to X$ is also affine. Then $C \times_{\mathfrak{C}} D$ is a cone a by Proposition 4.13(b) and the result follows.

(e) *Every fibered product of cone stacks is a cone stack.*

(f) *If $\mathfrak{C}$ is a representable cone stack over X, then it is a cone.*

Indeed, locally on X, $\mathfrak{C} \to X$ is $\mathbb{A}^1$-isomorphic to a cone. In particular, as $\mathfrak{C} \to X$ is representable, it is affine. Then we assume that $C = \underline{\mathrm{Spec}}_X \mathscr{S}$. Since there is a non-trivial $\mathbb{A}^1$-action on C and has a section, we know that $\mathscr{S}$ is a graded algebra with $\mathscr{S}^0 = \mathscr{O}_X$. To show C is a cone, we only need to show that $\mathscr{S}^1$ is coherent and $\mathscr{S}$ is locally generated by $\mathscr{S}^1$. These are both local property, then they hold since locally $\mathfrak{C} \to X$ is $\mathbb{A}^1$-isomorphic to a cone.

(g) *If $\mathfrak{C}$ is abelian (a vector bundle stack), then for every local presentation $C \to \mathfrak{C}$ the cone C will be abelian (a vector bundle).*

Example 4.23. *Note that all cones are cone stacks and all morphisms of cones are morphisms of cone stacks. For a vector bundle E on X, the classifying stack $\mathbf{B}_X E$ is a cone stack. Every homomorphism of vector bundles $\phi : E \to F$ gives rise to a morphism of cone stacks.*

Proposition 4.24. *Every cone stack is a closed subcone stack of an abelian cone stack. There exists a universal such abelian cone stack. It is called the abelian hull.*

Proof. Just glue the stacks obtained from the abelian hulls of local presentations. □

Definition 4.25. (a) *Let $\mathfrak{E}$ be a vector bundle stack and $\mathfrak{E} \to \mathfrak{C}$ a morphism of cone stacks. We say that $\mathfrak{C}$ is an $\mathfrak{E}$-cone stack, if $\mathfrak{E} \to \mathfrak{C}$ is locally isomorphic (as a morphism of cone stacks) to the morphism $[E_1/E_0] \to [C/F]$ coming from a commutative diagram*

$$\begin{array}{ccc} E_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & C \end{array}$$

where C is both E_1- and F-cone. The natural action $\mathfrak{E} \times_X \mathfrak{C} \to \mathfrak{C}$ induced by $E_1 \times C \to C$.

(b) *Let $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ be a sequence of morphisms of cone stacks where $\mathfrak{C}$ is an $\mathfrak{E}$-cone stack. If*

(b1) *$\mathfrak{C} \to \mathfrak{D}$ is a smooth epimorphism.*

(b2) *The diagram*

$$\begin{tikzcd} \mathfrak{E} \times_X \mathfrak{C} \arrow[r, "\sigma"] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{C} \arrow[d] \\ \mathfrak{C} \arrow[r] & \mathfrak{D} \end{tikzcd}$$

is cartesian where σ is action and p is projection.

Then we call $0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ is a short exact sequence of cone stacks. As before, this is equivalent to $\mathfrak{C}$ being locally isomorphic to $\mathfrak{E} \times_X \mathfrak{D}$.

Proposition 4.26. *The sequence $0 \to \mathfrak{E} \to \mathfrak{C} \to \mathfrak{D} \to 0$ of morphisms of cone stacks is exact if and only if locally in X there exist commutative diagrams*

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_1 \arrow[r] & C \arrow[r] & D \arrow[r] & 0 \end{tikzcd}$$

where the top row is a short exact sequence of vector bundles and the bottom row is a short exact sequence of cones, such that $\mathfrak{E} \to \mathfrak{C} \to \mathfrak{D}$ is isomorphic to $[E_1/E_0] \to [C/F] \to [D/G]$.

Proof. The statement is local on X. To prove the only if part we can assume $\mathfrak{C} = \mathfrak{E} \times_X \mathfrak{D}$, and then it is trivial. To prove the if part, note that both short exact sequences are locally split. $\square$

4.3 A Picard Stack of Special Type

General Theory

First we will consider the case of complex of two terms.

Definition 4.27. *Let X be a topos.*

(a) *Let $d : E^0 \to E^1$ a homomorphism of abelian sheaves on X, which we shall consider as a complex of abelian sheaves on X. Via d, the abelian sheaf E^0 acts on E^1 and we may consider the quotient stack of this action, denoted*

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := [E^1/E^0]$$

which is a Picard stack over X.

(b) *Now if $d : F^0 \to F^1$ is another homomorphism of abelian sheaves on X and $\phi : E^\bullet \to F^\bullet$ is a homomorphism of complexes, then we get an induced morphism of Picard stacks*

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

given by $(P, f) \mapsto (P \times^{E^0, \phi^0} F^0, \phi^1(f))$ *where* $\phi^1(f)$ *is the map*

$$\phi^1(f) : P \times^{E^0, \phi^0} F^0 \to F^1, \quad [p, \nu] \mapsto \phi^1(f(p) + d(\nu)).$$

(c) *Now, if* $\psi : E^\bullet \to F^\bullet$ *is another homomorphism of complexes, then the* homotopy $k : \phi \to \psi$ *is a homomorphism of abelian sheaves* $k : E^1 \to F^0$*, such that* $kd = \psi^0 - \phi^0$ *and* $dk = \psi^1 - \phi^1$*.*

Remark 4.28. *Note that roughly speaking, a Picard stack is a stack together with an 'addition' operation, that is both associative and commutative. For the precise definition of Picard stack see Sect. 1.4 of Exposé XVIII in [AGV73].*

Here the quotient stack is similar as before: the groupoid $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$ *is the category of pairs* (P, f)*, where* P *is an* E^0*-torsor over* U *and* $f : P \to E^1|_U$ *is an* E^0*-equivariant morphism of sheaves on* U*.*

Proposition 4.29. *As in the considtion of definition, if we have a homotopy* $k : \phi \to \psi$*, then this can induce isomorphism* $\theta : \mathcal{H}^1/\mathcal{H}^0(\phi) \to \mathcal{H}^1/\mathcal{H}^0(\psi)$ *of morphisms of Picard stacks from* $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ *to* $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)$*.*

Proof. Pick object $U \in \mathrm{ob}(X)$ and $(P, f) \in \mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$, then $\theta(U)(P, f) : \mathcal{H}^1/\mathcal{H}^0(\phi)(U)(P, f) \to \mathcal{H}^1/\mathcal{H}^0(\psi)(U)(P, f)$ in $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)(U)$ is the isomorphism of $F^0|_U$-torsors

$$\theta(U)(P, f) : P \times^{E^0, \phi^0} F^0 \to P \times^{E^0, \psi^0} F^0$$

given by $[p, \nu] \mapsto [p, kf(p) + \nu]$ such that the diagram of $F^0|_U$-sheaves

$$\begin{tikzcd} P \times^{E^0,\phi^0} F^0 \arrow[d, "\theta(U)(P,f)"'] \arrow[dr, "\phi^1(f)"] & \\ P \times^{E^0,\psi^0} F^0 \arrow[r, "\psi^1(f)"'] & F^1 \end{tikzcd}$$

commutes. ☐

Proposition 4.30. *Let* $\phi : E^\bullet \to F^\bullet$ *is a homomorphism of complexes of abelian sheaves in the topos* X*. If* ϕ *induces isomorphisms on kernels and cokernels (i.e. if* ϕ *is a quasi-isomorphism), then*

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

is an isomorphism of Picard stacks over X*.*

Proof. First let us treat the case that ϕ is a homotopy equivalence, that is, there is a homotopy inverse of ϕ such that compositions are homotopic to $\mathrm{id}_{E^\bullet}$ and $\mathrm{id}_{F^\bullet}$, respectively. By Proposition 4.29 well done.

Next we assume ϕ is an epimorphism. In this case $E^1 \to [F^1/F^0]$ is an epimorphism, so we just need to prove the diagram

$$\begin{tikzcd} E^0 \times E^1 \arrow[r, "d+\mathrm{id}"] \arrow[d, "p"] & E^1 \arrow[d] \\ E^1 \arrow[r] & {[F^1/F^0]} \end{tikzcd}$$

is cartesian as in this case this will be a cocartesian diagram! This quickly reduces to proving that

$$\begin{tikzcd} E^1 \times E^0 \arrow[r] \arrow[d] & E^1 \arrow[d] \\ E^1 \times F^0 \arrow[r] & F^1 \end{tikzcd}$$

is cartesian, which, in turn, is equivalent to

$$\begin{tikzcd} E^0 \arrow[r] \arrow[d] & E^1 \arrow[d] \\ F^0 \arrow[r] & F^1 \end{tikzcd}$$

being cartesian, which is a consequence of the assumptions.

Finally in general case, let us note that a general ϕ factors as a homotopy equivalence followed by an epimorphism, then well done. Indeed, consider $E^\bullet \oplus F^0$, which is homotopy equivalent to $E^\bullet$. Define a homomorphism $\psi : E^\bullet \oplus F^0 \to F^\bullet$ by $\psi^0(\nu, \mu) = \phi^0(\nu) + \mu$ and $\psi^1(\xi, \mu) = \phi^1(\xi) + \mu$. Then ψ is surjective and $\phi = \psi \circ i$ where $i : E^\bullet \hookrightarrow E^\bullet \oplus F^0$ is the canonical embedding. $\square$

Now we consider the general case.

Definition 4.31. *Let X be a topos and $E^\bullet$ be a complex of abelian sheaves on X, then we define*

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := \mathcal{H}^1/\mathcal{H}^0(\tau^{[0,1]}E^\bullet).$$

Lemma 4.32. *Let X be a ringed topos with structure sheaf of rings $\mathscr{O}_X$.*

(a) We can define $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ and homomorphisms can defined over $\mathbf{D}(\mathscr{O}_X)$.

(b) *Let $\phi, \psi : E^\bullet \to F^\bullet$ be two morphisms in $\mathbf{D}(\mathscr{O}_X)$. Then, if for some choice of $\mathcal{H}^1/\mathcal{H}^0(\phi)$ and $\mathcal{H}^1/\mathcal{H}^0(\psi)$ we have $\mathcal{H}^1/\mathcal{H}^0(\phi) \cong \mathcal{H}^1/\mathcal{H}^0(\psi)$ as morphisms of Picard stacks, then $\phi = \psi$.*

(c) *Consider the zero morphism $0(E, F) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \to \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$. Then $\mathrm{Aut}(0(E, F)) = \mathrm{Hom}^{-1}_{\mathbf{D}(\mathscr{O}_X)}(E^\bullet, F^\bullet)$.*

Proof. For (b)(c), see Sect. 1.4 of Exposé XVIII in [AGV73]. For (a), the quasi-isomorphism induce an isomorphism of Picard stacks, see Proposition 4.30. □

Example 4.33. *Consider $E^\bullet$ an we focus on $d^0 : E^0 \to E^1$.*

(1) *If d^0 is a monomorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \mathrm{coker}(d^0)$ is a sheaf.*

(2) *If d^0 is a epimorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \mathbf{B}_X \ker(d^0)$ is a gerbe.*

Application

Come back to our case, let X be a DM-stack over a field k, then consider the big fppf topos X_{fppf} and small étale topos $X_{\text{ét}}$. Then we have the morphism of topoi

$$v : X_{\mathrm{fppf}} \to X_{\text{ét}}.$$

(a) Then we we can get $\mathbf{L}v^* : \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}}) \to \mathbf{D}^-(\mathscr{O}_{X_{\mathrm{fppf}}})$. We may let $M^\bullet_{\mathrm{fppf}} := \mathbf{L}v^* M^\bullet$ for any $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}})$.

(b) We also have $\mathbf{R}\mathscr{H}om(-, \mathscr{O}_{X_{\mathrm{fppf}}}) : \mathbf{D}^-(\mathscr{O}_{X_{\mathrm{fppf}}}) \to \mathbf{D}^+(\mathscr{O}_{X_{\mathrm{fppf}}})$. We may let $M^{\bullet,\vee} := \mathbf{R}\mathscr{H}om(M^\bullet, \mathscr{O}_{X_{\mathrm{fppf}}})$ for any $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\mathrm{fppf}}})$.

Remark 4.34. *We will consider the stack $\mathcal{H}^1/\mathcal{H}^0(M^{\bullet,\vee}_{\mathrm{fppf}})$ for $M^\bullet \in \mathbf{D}^-(\mathscr{O}_{X_{\text{ét}}})$. Note that in this case*

$$\mathcal{H}^1/\mathcal{H}^0(M^{\bullet,\vee}_{\mathrm{fppf}}) = \mathcal{H}^1/\mathcal{H}^0((\tau^{\geq -1} M^\bullet_{\mathrm{fppf}})^\vee).$$

Remark 4.35. *For a complex $E^\bullet$, we define $Z^i(E^\bullet) = \ker(E^i \to E^{i+1})$ and $C^i(E^\bullet) = \mathrm{coker}(E^{i-1} \to E^i)$.*

Definition 4.36. *We call an object $L^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies Condition (*) if*

(1) *$H^i(L^\bullet) = 0$ for all $i > 0$.*

(2) *$H^i(L^\bullet)$ is coherent for all $i = 0, -1$.*

Here are some fundamental results:

Proposition 4.37. *Let X be a DM-stack.*

(a) Let $L^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfy Condition (*). Then the X-stack $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet,\vee})$ is an abelian cone stack over X. Moreover, if $L^\bullet$ is of perfect amplitude contained in $[-1,0]$, then $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet,\vee})$ is a vector bundle stack.

(b) If $\phi : E^\bullet \to L^\bullet$ is a homomorphism in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $L^\bullet$ satisfy (*), then we get an induced morphism of algebraic stacks

$$\phi^\vee : \mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet,\vee}) \to \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee}).$$

Then $\phi^\vee$ is a morphism of abelian cone stacks. Moreover, $H^0(\phi)$ is surjective if and only if $\phi^\vee$ is representable.

(c) The morphism $\phi^\vee$ is a closed immersion if and only if $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. Moreover, $\phi^\vee$ is an isomorphism if and only if $H^0(\phi)$ and $H^{-1}(\phi)$ are isomorphisms.

(d) Let $E^\bullet \to F^\bullet \to G^\bullet \to E^\bullet[1]$ be a distinguished triangle in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$, where $E^\bullet$ and $F^\bullet$ satisfy (*) and $G^\bullet$ is of perfect amplitude contained in $[-1,0]$. Then the induced sequence

$$\mathcal{H}^1/\mathcal{H}^0(G_{\text{fppf}}^{\bullet,\vee}) \to \mathcal{H}^1/\mathcal{H}^0(F_{\text{fppf}}^{\bullet,\vee}) \to \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee})$$

is a short exact sequence of abelian cone stacks over X.

Proof. For (a), as the claim is étale local, we may assume $L^\bullet$ consists of free $\mathscr{O}_X$-modules with $L^i = 0$ for $i > 0$ and L^0, L^{-1} have finite rank. Then $L_{\text{fppf}}^\bullet = v^* L^\bullet$ and $L_{\text{fppf}}^{\bullet,\vee}$ is taking dual of $L_{\text{fppf}}^\bullet$ component-wise. Hence we have

$$\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet,\vee}) = [Z^1(L^{\vee,\bullet})/L^{\vee,0}]$$

which is an abelian cone stack given by $L^{\vee,0} \to Z^1(L^{\vee,\bullet}) = C(C^{-1}L^\bullet)$.

When $L^\bullet$ is of perfect amplitude contained in $[-1,0]$, then $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet,\vee})$ is a vector bundle stack since étale locally as above we have $Z^1(L^{\vee,\bullet}) = L^{\vee,1}$.

For (b), the fact that $\phi^\vee$ is a morphism of abelian cone stacks is immediate from the definition. The second question is étale local in X, so we may assume that $E^\bullet$ and $L^\bullet$ are complexes of free $\mathscr{O}_X$-modules and that $E^i = L^i = 0$, for $i > 0$, and that L^0, L^{-1}, E^0 and E^{-1} are of finite rank. Consider the commutative diagram

$$\begin{tikzcd}
C^{-1}(E^\bullet) \arrow[drr] \arrow[ddr] & & \\
& F \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E^0 \arrow[d] \\
& B^{-1}(L^\bullet) \arrow[r] & L^0
\end{tikzcd}$$

of coherent sheaves with fiber product F. This force $0 \to F \to E^0 \oplus C^{-1}(L^\bullet) \to L^0$ exact. Then its easy to see that $H^0(\phi)$ is surjective if and only if $0 \to F \to E^0 \oplus C^{-1}(L^\bullet) \to L^0 \to 0$ exact. Hence taking duality we get $0 \to L^{\vee,0} \to E^{\vee,0} \times_X Z^1(L^{\vee,\bullet}) \to C(F) \to 0$ exact. Then by Proposition 4.20 we get

$$[Z^1(L^{\vee,\bullet})/L^{\vee,0}] \cong [C(F)/E^{\vee,0}].$$

This force the following cartesians

$$\begin{tikzcd}
C(F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z^1(E^{\vee,\bullet}) \arrow[d] \\
\mathcal{H}^1/\mathcal{H}^0(L^{\bullet,\vee}_{\mathrm{fppf}}) \arrow[r, "\phi^\vee"] & \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\mathrm{fppf}})
\end{tikzcd}$$

hence $\phi^\vee$ is representable.

For the converse, note that $\phi^\vee : [Z^1(L^{\vee,\bullet})/L^{\vee,0}] \to [Z^1(E^{\vee,\bullet})/E^{\vee,0}]$ representable implies that $[Z^1(L^{\vee,\bullet})/L^{\vee,0}] = [W/E^{\vee,0}]$. Then we have the commutative diagram:

$$\begin{tikzcd}
Z^1(L^{\vee,\bullet}) \times_X L^{\vee,0} \arrow[r] \arrow[d] & Z^1(L^{\vee,\bullet}) \arrow[d] \\
Z^1(L^{\vee,\bullet}) \times_X E^{\vee,0} \arrow[r] \arrow[d] & W \arrow[d] \\
Z^1(L^{\vee,\bullet}) \arrow[r] & {[W/E^{\vee,0}]}
\end{tikzcd}$$

such that the the whole diagram and the lower diagram are cartesians, then this force the upper square is cartesian. So we get cartesians

$$\begin{tikzcd}
L^{\vee,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z^1(L^{\vee,\bullet}) \times_X L^{\vee,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z^1(L^{\vee,\bullet}) \arrow[d] \\
E^{\vee,0} \arrow[r] & Z^1(L^{\vee,\bullet}) \times_X E^{\vee,0} \arrow[r] & W
\end{tikzcd}$$

Hence $L^{\vee,0} \cong E^{\vee,0} \times_W Z^1(L^{\vee,\bullet}) \to E^{\vee,0} \times_X Z^1(L^{\vee,\bullet})$ is a closed immersion. This implies that $E^0 \oplus C^{-1}(L^\bullet) \to L^0$ is an epimorphism.

For (c), following the previous argument in (b), $\phi^\vee$ is a closed immersion if and only if $C(F) \to Z^1(E^{\vee,\bullet})$ is. This is equivalent to $C^{-1}(E^\bullet) \to F$ being surjective. A simple diagram chase shows that this is equivalent to $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. The 'moreover' follows similarly.

For (d), the question is étale local, so assume that E^i and F^i are 0 for $i > 0$ and vector bundles for $i = 0, -1$, and that $G^i = E^{i+1} \oplus F^i$, that is, $G^\bullet = \mathrm{cone}(E^\bullet \to F^\bullet)$. If we consider the small enough étale locally, we may let $G^i = 0$ for $i \leq -2$ as $G^\bullet$ is of perfect amplitude contained in $[-1, 0]$. Now we have to prove that

$$0 \to [Z^1(G^{\vee,\bullet})/G^{\vee,0}] \to [Z^1(F^{\vee,\bullet})/F^{\vee,0}] \to [Z^1(E^{\vee,\bullet})/E^{\vee,0}] \to 0$$

is a short exact sequence of cone stacks. Now by directly check, we have the exact sequence of sheaves

$$0 \to C^{-1}(E^\bullet) \to C^{-1}(F^\bullet) \oplus E^0 \to C^{-1}(G^\bullet) \to 0.$$

Hence consider

$$\begin{tikzcd}
0 \arrow[r] & C^{-1}(E^\bullet) \arrow[r] \arrow[d] & C^{-1}(F^\bullet) \oplus E^0 \arrow[r] \arrow[d] & C^{-1}(G^\bullet) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^0 \arrow[r] & F^0 \oplus E^0 \arrow[r] & G^0 = F^0 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Finally by Proposition 4.26 we get the result. □

4.4 About Normal Cones

Here we will consider some useful results about normal cones of DM-stacks.

Consider the commutative diagram of algebraic stacks

$$\begin{tikzcd}
X' \arrow[r, "j"] \arrow[d, "u"] & Y' \arrow[d, "v"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}$$

with where i and j are local immersions. Then there is a natural morphism of cones over X'

$$\alpha : C_{X'/Y'} \to C_{X/Y}.$$

If the diagram is cartesian, then α is a closed immersion. If, moreover, v is flat, then α is an isomorphism.

Proposition 4.38. *Consider a commutative diagram of DM-stacks*

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[dr, "i"'] & Y' \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

where i and i' are local immersions and f is smooth. Then the sequence of morphisms of cones over X

$$(i')^*T_{Y'/Y} \to C_{X/Y'} \to C_{X/Y}$$

is exact.

Proof. The question is local, so we can assume them are schemes and that i' and i are immersions. This is then Example 4.2.6 in [Ful98]. $\square$

Lemma 4.39. *Let $f : U \to M$ be a local immersion of affine k-schemes of finite type, where M is smooth over k. Then the normal cone $C_{U/M} \hookrightarrow N_{U/M}$ is invariant under the action of f^*T_M on $N_{U/M}$. In other words, $C_{U/M}$ is an f^*T_M-cone.*

Proof. Consider projections $p_i : M \times M \to M$, we consider two diagrams:

$$\begin{tikzcd} U \arrow[r, "\Delta f"] \arrow[dr, "f"'] & M \times M \arrow[d, "p_i"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\Delta f"'] & M \arrow[d, "\Delta"] \\ & M \times M \end{tikzcd}$$

The first one give us exact sequence of abelian cones on U:

$$0 \to f^*T_M \overset{j_i}{\ } N_{U/M\times M} \overset{p_{i,*}}{\to} N_{U/M} \to 0$$

and the second one give us a homomorphism of abelian cones $s : N_{U/M} \to N_{U/M\times M}$ which is a section of both $p_{i,*}$.

Using $(j_1, p_{1,*})$ we make the identification $N_{U/M\times M} = f^*T_M \times N_{U/M}$ and $p_{2,*}$ is identified with the action of f^*T_M on $N_{U/M}$. Since the same functorialities of normal sheaves used so far are enjoyed by normal cones, we get that under the identification above the subcone $C_{U/M\times M} \subset N_{U/M\times M}$ corresponds to $f^*T_M \times C_{U/M}$ and the action $p_{2,*} : f^*T_M \times N_{U/M} \to N_{U/M}$ restricts to $p_{2,*} : f^*T_M \times C_{U/M} \to C_{U/M}$. $\square$

4.5 Intrinsic Normal Cone

Now let X be a Deligne-Mumford stack, locally of finite type over k. Now we will construct the intrinsic normal cone and intrinsic normal sheaf of X and their basic properties.

Definition 4.40. *We denote the abelian cone stack*

$$\mathfrak{N}_X := \mathcal{H}^1/\mathcal{H}^0((\mathbb{L}^\bullet_{X,\mathrm{fppf}})^\vee)$$

and call it the intrinsic normal sheaf of X where $\mathbb{L}^\bullet_X \in \mathbf{D}^{\leq 0}(\mathscr{O}_{X_{\text{ét}}})$ is the cotangent complex which satisfies the condition (*).

Definition 4.41. *(a) A local embedding of X is a pair (U, M) with a diagram $X \xleftarrow{i} U \xrightarrow{f} M$ where*

(a1) U is an affine k-scheme of finite type;

(a2) $i : U \to X$ is an étale morphism;

(a3) M is a smooth affine k-scheme of finite type;

(a4) $f : U \to M$ is a local immersion.

(b) A morphism of local embeddings $\phi : (U', M') \to (U, M)$ is a pair of morphisms $\phi_U : U' \to U$ and $\phi_M : M' \to M$ such that

(b1) ϕ_U is an étale X-morphism;

(b2) ϕ_M is smooth morphism such that

$$\begin{tikzcd} U' \arrow[r, "f'"] \arrow[d, "\phi_U"] & M' \arrow[d, "\phi_M"] \\ U \arrow[r, "f"] & M \end{tikzcd}$$

commutes.

Remark 4.42. *If (U', M') and (U, M) are local embeddings of X, then $(U' \times_X U, M' \times M)$ is naturally a local embedding of X which we call the product of local embeddings, even though it may not be the direct product in the category of local embeddings of X.*

Now we consider the local presentation of intrinsic normal sheaf $\mathfrak{N}_X$. Indeed, consider a local embedding $X \xleftarrow{i} U \xrightarrow{f} M$ of X, then we have a natural homomorphism

$$\phi : \mathbb{L}_X^\bullet|_U \to [\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_M^1]$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U_{\text{ét}}})$ where $\mathscr{I}$ be the ideal correspond to f and $[\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_M^1] \in \mathbf{D}^{[-1,0]}(\mathscr{O}_{U_{\text{ét}}})$. Moreover, by Theorem 3.1(c) we know that ϕ induces an isomorphism on H^{-1} and H^0. By Proposition 4.30 we get an induced isomorphism of cone stacks

$$\phi^\vee : [N_{U/M}/f^*T_M] \cong i^*\mathfrak{N}_X.$$

In other words, $N_{U/M}$ is a local presentation of the abelian cone stack $\mathfrak{N}_X$.

Theorem 4.43. *There exists a unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ such that for every local embedding (U, M) of X we have $\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]$, that is, the diagram*

$$\begin{tikzcd} C_{U/M} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & N_{U/M} \arrow[d] \\ \mathfrak{C}_X|_U \arrow[r, hook] & \mathfrak{N}_X|_U \end{tikzcd}$$

Proof. If $\chi : (U', M') \to (U, M)$ is a morphism of local embeddings, we have a commutative diagram

$$\begin{tikzcd}
\mathbb{L}^\bullet_X|_{U'} \arrow[d, "\phi|_{U'}"] \arrow[drr, "\phi'"] & & \\
{[\mathscr{I}/\mathscr{I}^2 \to f^*\Omega^1_M]|_{U'}} \arrow[rr, "\tilde{\chi}"] & & {[\mathscr{I}'/(\mathscr{I}')^2 \to (f')^*\Omega^1_{M'}]}
\end{tikzcd}$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U'_{\text{ét}}})$ because of the naturality of ϕ and thus induce the commutative diagram

$$\begin{tikzcd}
{[N_{U'/M'}/(f')^*T_{M'}]} \arrow[r, "\tilde{\chi}^\vee"] \arrow[d, "(\phi')^\vee{,}\cong"'] & {[N_{U/M}/f^*T_M]|_{U'}} \arrow[dl, "\phi^\vee|_{U'}{,}\cong"] \\
\mathfrak{N}_X|_{U'} &
\end{tikzcd}$$

in $\mathbf{D}^{\leq 0}(\mathscr{O}_{U'_{\text{ét}}})$. In particular, $\tilde{\chi}^\vee$ is an isomorphism of cone stacks over U'.

Now by Lemma 4.39 χ induce a morphism from the $(f')^*T_{M'}$-cone $C_{U'/M'}$ to the $f^*T_M|_{U'}$-cone $C_{U/M}|_{U'}$. By Proposition 4.26 the pair $(C_{U/M} \hookrightarrow N_{U/M})|_{U'}$ is the quotient of $(C_{U'/M'} \hookrightarrow N_{U'/M'})$ by the action of $(f')^*T_{M'/M}$ since the kernel of $(f')^*T_{M'} \to f^*T_M|_{U'}$ is $(f')^*T_{M'/M}$. This implies that the isomorphism above

$$\tilde{\chi}^\vee : [N_{U'/M'}/(f')^*T_{M'}] \cong [N_{U/M}/f^*T_M]|_{U'}$$

identifies the closed subcone stack $[C_{U'/M'}/(f')^*T_{M'}]$ with the closed subcone stack $[C_{U/M}/f^*T_M]|_{U'}$. This give us the unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ with the properties above. $\square$

Definition 4.44. *This unique closed subcone stack $\mathfrak{C}_X$ is called the intrinsic normal cone of X.*

Theorem 4.45. *The intrinsic normal cone $\mathfrak{C}_X$ is of pure dimension zero which abelian hull is the intrinsic normal sheaf $\mathfrak{N}_X$.*

Proof. The second claim follows because the normal sheaf is the abelian hull of the normal cone, for any local embedding.

To prove the claim about the dimension of $\mathfrak{C}_X$, consider a local embedding (U, M) of X, giving rise to the local presentation $C_{U/M}$ of $\mathfrak{C}_X$. Assume that M is of pure dimension. We then have a cartesian diagram of U-stacks

$$\begin{tikzcd}
C_{U/M} \times f^*T_M \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C_{U/M} \arrow[d] \\
C_{U/M} \arrow[r] & {[C_{U/M}/f^*T_M]}
\end{tikzcd}$$

Thus $C_{U/M} \to [C_{U/M}/f^*T_M]$ is a smooth epimorphism of relative dimension $\dim M$. So since $C_{U/M}$ is of pure dimension $\dim M$ (see the comments on the Definition 2.1), the stack $[C_{U/M}/f^*T_M]$ has pure dimension $\dim M - \dim M = 0$.Well done. □

Finally, we discuss some basic properties of them.

Proposition 4.46. *Let X be a DM-stack.*

(a) The following are equivalent.

(a1) X is a local complete intersection.

(a2) $\mathfrak{C}_X$ is a vector bundle stack.

(a3) $\mathfrak{C}_X = \mathfrak{N}_X$.

If X is smooth, we have $\mathfrak{C}_X = \mathfrak{N}_X = \mathbf{B}_X(T_X)$.

(b) We have $\mathfrak{N}_{X\times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y$ and $\mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$.

(c) Let $f : X \to Y$ be a local complete intersection morphism. Then we have a natural short exact sequence of cone stacks

$$\mathfrak{N}_{X/Y} := \mathcal{H}^1/\mathcal{H}^0(\mathbb{T}^\bullet_{X/Y}) \to \mathfrak{C}_X \to f^*\mathfrak{C}_Y.$$

Proof. (a) is trivial. (b) follows from the fact that if C is an E-cone and D is an F-cone, then $C\times D$ is an $E\times F$-cone and there is a canonical isomorphism of cone stacks $[C/E] \times [D/F] \to [C \times D/E \times F]$.

For (c), by Theorem 3.1(c)(e) we have an exact triangle

$$\mathbf{L}f^*\mathbb{L}_Y \to \mathbb{L}_X \to \mathbb{L}_{X/Y} \to \mathbf{L}f^*\mathbb{L}_Y[1]$$

in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ and $\mathbb{L}_{X/Y}$ is of perfect amplitude contained in $[-1, 0]$. By Proposition 4.37(d) we have a short exact sequence of abelian cone stacks

$$\mathfrak{N}_{X/Y} \to \mathfrak{N}_X \to f^*\mathfrak{N}_Y.$$

So the claim is local in X and we may assume that we have a diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & M'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M' \arrow[d] \\ & Y \arrow[r] & M \end{tikzcd}$$

where the square is cartesian, the vertical maps are smooth, the horizontal maps are local immersions, i is regular and M is smooth. Then we have a morphism of short exact sequences of cones on X

$$\begin{array}{ccccc} i^*T_{M''/Y} & \longrightarrow & T_{M'}|_X & \longrightarrow & T_M|_X \\ \downarrow & & \downarrow & & \downarrow \\ N_{X/M''} & \longrightarrow & C_{X/M'} & \longrightarrow & C_{Y/M}|_X \end{array}$$

Hence by Proposition 4.26 we get the result. □

4.6 About Obstruction Theories

Intrinsic Normal Sheaf as Obstruction

Let X be a DM-stack with intrinsic normal sheaf $\mathfrak{N}_X$. Let $T \hookrightarrow \overline{T}$ be a closed immersion with ideal $\mathscr{J}$ such that $\mathscr{J}^2 = 0$. If we have $g : T \to X$, may we have the extension $\overline{g} : \overline{T} \to X$ of g? What is the obstruction of this deformation?

First, by Theorem 3.1(e) we have a composition of canonical morphisms

$$\mathbf{L}g^*\mathbb{L}^\bullet_X \to \mathbb{L}^\bullet_T \to \mathbb{L}^\bullet_{T/\overline{T}}.$$

Since $\tau^{\geq -1}\mathbb{L}^\bullet_{T/\overline{T}} = \mathscr{J}[1]$, this homomorphism may be considered as an element $\omega(g) \in \mathrm{Ext}^1(g^*\mathbb{L}^\bullet_X, \mathscr{J})$. Then the basic deformation theory find that an extension $\overline{g} : \overline{T} \to X$ of g exists if and only if $\omega(g) = 0$ and if $\omega(g) = 0$ the extensions form a torsor under $\mathrm{Ext}^0(g^*\mathbb{L}^\bullet_X, \mathscr{J}) = \mathrm{Hom}(g^*\Omega_X, \mathscr{J})$.

Here we will use the intrinsic normal sheaf $\mathfrak{N}_X$ to interpret this. Recall the morphism as above

$$\mathbf{L}g^*\mathbb{L}^\bullet_X \to \mathbb{L}^\bullet_T \to \mathbb{L}^\bullet_{T/\overline{T}}.$$

This induce a morphism

$$\mathbf{ob}(g) : C(\mathscr{J}) = \mathcal{H}^1/\mathcal{H}^0(\mathbb{L}^{\bullet,\vee}_{T/\overline{T},\mathrm{fppf}}) \to \mathcal{H}^1/\mathcal{H}^0(\mathbf{L}g^*\mathbb{L}^{\bullet,\vee}_{X,\mathrm{fppf}}) = g^*\mathfrak{N}_X$$

since $\tau^{\geq -1}\mathbb{L}^\bullet_{T/\overline{T}} = \mathscr{J}[1]$ by Theorem 3.1(c). Consider another morphism $\mathbf{0}(g) : C(\mathscr{J}) \to T \xrightarrow{0} g^*\mathfrak{N}_X$.

- Consider a sheaf $\mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$ of 2-isomorphisms of cone stacks from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$, restricted to $T_{\text{ét}}$.
- Denote the sheaf of extensions $\overline{T} \to X$ of g by $\mathscr{E}xt(g, T)$ on $T_{\text{ét}}$.

Proposition 4.47. *There is a canonical isomorphism*

$$\mathscr{E}xt(g,T) \xrightarrow{\cong} \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$$

of sheaves on $T_{\text{ét}}$. Hence in particular, extensions of g to $\overline{T}$ exist if and only if $\mathbf{ob}(g)$ is $\mathbb{A}^1$-equivariantly isomorphic to $\mathbf{0}(g)$.

Proof. Locally we can have an embedding $i : X \hookrightarrow M$ where M is smooth of ideal $\mathscr{I}$. Then by the formally-smoothness of M we have the lifting:

$$\begin{tikzcd} X \arrow[r, hook, "i"] & M \arrow[dr] & \\ T \arrow[u, "g"] \arrow[r, hook] & \overline{T} \arrow[u, dashed, "h"'] \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

and such extensions is a $\mathrm{Hom}(g^*i^*\Omega_M, \mathscr{J})$-torsor. Now, any such h induce $h^\sharp : g^*\mathscr{I}/\mathscr{I}^2 \to \mathscr{J}$. By the local description before the Theorem 4.43, $\mathbf{ob}(g)$ induced by

$$h^\sharp : \mathbf{L}g^*[\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_M] \to [\mathscr{J} \to 0].$$

Now the torsor structure induce the following homotopy

$$\begin{tikzcd} 0 \arrow[r] & g^*\mathscr{I}/\mathscr{I}^2 \arrow[r] \arrow[d, "{h^\sharp,(\tilde{h})^\sharp}"'] & g^*i^*\Omega_M \arrow[r] \arrow[d, "{h^\sharp,(\tilde{h})^\sharp}"] \arrow[dl, dashed] & 0 \\ 0 \arrow[r] & \mathscr{J} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

of extensions $h^\sharp, (\tilde{h})^\sharp$.

Now let $\overline{g} : \overline{T} \to X$ be an extension of g. Then easy to see that $(i \circ g)^\sharp = 0$, so that we get a homotopy from any local $h^\sharp$ as above to 0, or in other words a local $\mathbb{A}^1$-equivariant isomorphism from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$ by Proposition 4.29. Since these local isomorphisms glue, we get the required map

$$\mathscr{E}xt(g, T) \to \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g)).$$

Now we consider the inverse. Let $\theta : \mathbf{ob}(g) \to \mathbf{0}(g)$ be a 2-isomorphism of cone stacks. By Lemma 4.19(a), θ defines for every local h as above an extension of $h^\sharp$ to $\overline{h}^\sharp : g^*i^*\Omega_M \to \mathscr{J}$. So we can get $h' : \overline{T} \to M$ such that $(h')^\sharp = 0$ by the changing via homotopy $\overline{h}^\sharp$. So h' factor through X and we get $h' : \overline{T} \to X$. Gluing them we get the inverse. $\square$

Proposition 4.48. *There is a canonical isomorphism*

$$\mathscr{A}ut(\mathbf{0}(g)) \xrightarrow{\cong} \mathscr{H}om(g^*\Omega_X, \mathscr{J})$$

of sheaves on $T_{\text{ét}}$.

Proof. Again similar as above, Lemma 4.19(a) shows that the automorphisms of $\mathbf{0}(g)$ are (locally) the homomorphisms from $g^*i^*\Omega_M$ to $\mathscr{J}$ vanishing on $g^*\mathscr{I}/\mathscr{I}^2$. The exact sequence

$$\mathscr{I}/\mathscr{I}^2 \to i^*\Omega_M \to \Omega_X \to 0$$

give the result. $\square$

Remark 4.49. *This shows that the sheaf* $\mathscr{E}xt(g,T) \cong \mathscr{I}som(\mathbf{ob}(g), \mathbf{0}(g))$ *is a formal* $\mathscr{H}om(g^*\Omega_X, \mathscr{J})$*-torsor. So if* $\mathbf{ob}(g) \cong \mathbf{0}(g)$*, the set* $\mathrm{Hom}(\mathbf{ob}(g), \mathbf{0}(g))$ *is a torsor under the group* $\mathrm{Hom}(g^*\Omega_X, \mathscr{J})$*.*

Obstruction Theories

Here we consider more general setting.

Definition 4.50. *Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies condition (*). Then a homomorphism $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ is called an* obstruction theory for X *if $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective.*

Now we will give some equivalent conditions of obstruction theory which is connected to the obstruction of extensions as above.

Situation 1. *Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies condition (*) with a homomorphism $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$. Let $\mathfrak{E} := \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee})$ with induced morphism of cone stacks $\phi^\vee : \mathfrak{N}_X \to \mathfrak{E}$.*

Let $T \hookrightarrow \overline{T}$ be a closed immersion with ideal $\mathscr{J}$ such that $\mathscr{J}^2 = 0$ with a morphism $g : T \to X$. Then we can consider the obstruction class $\omega(g) \in \mathrm{Ext}^1(g^*\mathbb{L}_X^\bullet, \mathscr{J})$ Define $\phi^*\omega(g) \in \mathrm{Ext}^1(g^*E^\bullet, \mathscr{J})$ be the pullback and $\phi^\vee(\mathbf{ob}(g))$ be the composition

$$C(\mathscr{J}) \overset{\mathbf{ob}(g)}{\to} g^*\mathfrak{N}_X \overset{g^*\phi^\vee}{\to} g^*\mathfrak{E}$$

of cone stacks over T. Moreover $\mathbf{0} : C(\mathscr{J}) \to T \to g^*\mathfrak{E}$ is the vertex.

Theorem 4.51. *Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ satisfies condition (*) with a homomorphism $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$. Let $\mathfrak{E} := \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee})$ with induced morphism of cone stacks $\phi^\vee : \mathfrak{N}_X \to \mathfrak{E}$. Then the following are equivalent.*

- *(a) $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ is an obstruction theory for X.*
- *(b) The induced $\phi^\vee : \mathfrak{N}_X \to \mathfrak{E}$ is a closed immersion of cone stacks over X. In this case we call $\phi^\vee(\mathfrak{C}_X) \subset \mathfrak{E}$ the* obstruction cone *of the obstruction theory for X where $\mathfrak{C}_X \subset \mathfrak{N}_X$ is the intrinsic normal cone.*

(c) *Consider any choice of Situation 1, then the obstruction $\phi^*\omega(g) \in \mathrm{Ext}^1(g^*E^\bullet, \mathscr{J})$ vanishes if and only if an extension $\overline{g}$ of g to $\overline{T}$ exists; and if $\phi^*\omega(g) = 0$, then the extensions form a torsor under $\mathrm{Ext}^0(g^*E^\bullet, \mathscr{J}) = \mathrm{Hom}(g^*H^0E^\bullet, \mathscr{J})$.*

(d) *Consider any choice of Situation 1, then we have isomorphism*

$$\mathscr{E}xt(g, T) \xrightarrow{\cong} \mathscr{I}som(\phi^\vee \mathbf{ob}(g), \mathbf{0})$$

of sheaves on $T_{\text{ét}}$.

Proof. Note that (1)⇔(2) follows from Proposition 4.37(c). By the similar proof as Proposition 4.47 we know that (2)⇒(4). By Lemma 4.32(b) we know that (4)⇒(3). So we just need to consider (3)⇒(1).

We show $H^0(\phi)$ is an isomorphism. Let $X = \operatorname{Spec} R$ as this is local. For any R-algebra A and R-module M, let $T := \operatorname{Spec} A$ and $\overline{T} := \operatorname{Spec}(A \oplus M)$ for the nilpotent extension. Easy to see that there is extension $\overline{g} : \overline{T} \to X$. So we have a bijection

$$\mathrm{Hom}(H^0(\mathbb{L}_X^\bullet) \otimes A, M) \to \mathrm{Hom}(H^0(E^\bullet) \otimes A, M).$$

This implies easily that $H^0(\phi)$ is an isomorphism.

We show $H^{-1}(\phi)$ is surjective. As this is étale local and only depends on $\tau^{\geq -1}E^\bullet$, we may assume X is an affine scheme, $i : X \to W$ a closed embedding in a smooth affine scheme W, and let $\mathscr{I}$ be the ideal of X in W. Also $E^\bullet = [E^{-1} \to f^*\Omega_W]$ as a complex of coherent sheaves (see the proof of Proposition 4.37(b)). As in this case $\mathbb{L}_X^\bullet = [\mathscr{I}/\mathscr{I}^2 \to f^*\Omega_W]$, we claim that $E^{-1} \to \mathscr{I}/\mathscr{I}^2$ is surjective.

Indeed, let $M := \mathrm{Im}(E^{-1} \to \mathscr{I}/\mathscr{I}^2)$. Let $T = X$ and $M' \subset \mathscr{I}$ be the primage of M and let $\overline{T} \subset W$ defined by M'. So we can extend $g = \mathrm{id}_X$ to the inclusion $\overline{g} : \overline{T} \to W$. Let $\pi : \mathscr{I}/\mathscr{I}^2 \to \mathscr{I}/M'$ be the natural projection. By assumption π factors via E^0 if and only if g extends to a map $\overline{T} \to X$, if and only if $\pi \circ \phi^{-1}$ factors via E^0. As $\pi \circ \phi^{-1}$ is the zero map, it certainly factors. Therefore π also factors via E^0. Consider now the commutative diagram with exact rows

$$\begin{tikzcd}
E^{-1} \arrow[r] \arrow[d, "\phi^{-1}"] & E^0 \arrow[r] \arrow[d, "="] & H^0(E^\bullet) \arrow[r] \arrow[d, "="] & 0 \\
\mathscr{I}/\mathscr{I}^2 \arrow[r] & E^0 \arrow[r] & H^0(E^\bullet) \arrow[r] & 0
\end{tikzcd}$$

By an easy diagram chasing argument, the fact that π factors via E^0 together with $\pi \circ \phi^{-1} = 0$ implies $\pi = 0$, hence well done. □

Remark 4.52. *See more things about the obstruction ofsmall extensions we refer the final part of Section 4 in [BF97].*

4.7 Vistoli's Rational Equivalence

Before starting the theory of virtual class, we need some results of Vistoli. We will follows something in [Vis89].

Definition 4.53. *Let X be a stack.*

(a) The group $Z_k(X)$ of cycles of dimension k *is generated by all integral closed substacks of dimension k. And $Z_*(X) := \bigoplus_k Z_k(X)$.*

(b) The group of rational equivalences *on X is*

$$W_k(X) := \bigoplus_G K(G)^*, \quad W_*(X) := \bigoplus_k W_k(X)$$

where the direct sum is taken over all integral substacks G of X of dimension $k+1$.

(c) If X is a scheme, there is a canonical homorphism

$$\partial_X : W_*(X) \to Z_*(X).$$

This is commute with proper pushforward and flat pullback.

Remark 4.54. *Note that when X be a DM-stack, we can restricting Z_* and W_* to the étale site of X, we get two sheaves $\mathscr{Z}_*$ and $\mathscr{W}_*$ on X. As Z_* and W_* commute with proper pushforward and flat pullback, $\partial : \mathscr{W}_* \to \mathscr{Z}_*$ is a morphism of sheaves, so we get a homomorphism $\partial_X : W_*(X) \to Z_*(X)$.*

Recall that we consider again the cartesian diagram of algebraic stacks

$$\begin{tikzcd} X' \arrow[r, hook, "i"] \arrow[d, "u"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, hook, "v"] \\ X \arrow[r, hook, "j"] & Y \end{tikzcd}$$

with i and j are local immersions and v is a regular local immersion and Y is smooth of constant dimension. Then this induce the cartesians

$$\begin{tikzcd} N_{Y'/Y} \times_Y C_{X/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & u^* C_{X/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C_{X/Y} \arrow[d] \\ j^* N_{Y'/Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \arrow[r, "u"] \arrow[d, hook, "j"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook, "i"] \\ N_{Y'/Y} \arrow[r, "\rho"] & Y' \arrow[r, hook, "v"] & Y \end{tikzcd}$$

Theorem 4.55 (Vistoli)**.** *Consider the above situation, if Y is a scheme, then there is a canonical rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ such that*

$$\partial\beta(Y', X) = [C_{u^*C_{X/Y}/C_{X/Y}}] - [\rho^* C_{X'/Y'}].$$

Proof. See Lemma 4.6 in [Vis89]. □

Corollary 4.56. *In this case we have $v^![C_{X/Y}] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$.*

Proof. Let $0 : u^*C \to N \times_Y C$ be the zero section, then by definition of refined Gysin pullback

$$0^*[C_{u^*C_{X/Y}/C_{X/Y}}] = v^![C] \in \mathsf{CH}_*(u^*C_{X/Y}).$$

Moreover

$$0^*[\rho^* C_{X'/Y'}] = 0^*\rho^![C_{X'/Y'}] = C_{X'/Y'}.$$

By Theorem 4.55 we get $v^![C] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$. □

But now we need to consider the Vistoli rational equivalence at the level of stacks. So we need some base-change result about this:

Proposition 4.57. *Vistoli's rational equivalence commutes with any smooth base change $\phi : Y_1 \to Y$.*

Proof. If ϕ is étale, this is Lemma 4.6(ii) in [Vis89]. Vistoli's proof is based on the fact that the following commute with étale base change: blowing up a scheme along a closed subscheme; normalization; order of a Cartier divisor along an irreducible Weil divisor on a reduced, equidimensional scheme. But all these operations do in fact commute with smooth base change. Hence well done. □

Corollary 4.58. *We have Vistoli's rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ for any algebraic stacks. Moreover, if Y is a DM-stack, then $v^![C_{X/Y}] = [C_{X'/Y'}] \in \mathsf{CH}_*(u^*C_{X/Y})$ holds.*

Proof. Follows directly from the previous Proposition. □

Now we again consider the general case. We assume $i : X \to Y$ can factor as $X \xrightarrow{\tilde{i}} \tilde{Y} \xrightarrow{\pi} Y$ where $\tilde{i}$ is another local immersion and π is of relative Deligne-Mumford type (i.e. has unramified diagonal) and is smooth of constant fiber dimension.

Then the previous diagram can be fused into a large diagram of cartesians:

$$\begin{tikzcd}
N_{Y'/Y} \times_Y C_{X/\tilde{Y}} \arrow[r] \arrow[d] & u^*C_{X/\tilde{Y}} \arrow[r] \arrow[d] & C_{X/\tilde{Y}} \arrow[d, "\alpha"] \\
N_{Y'/Y} \times_Y C_{X/Y} \arrow[r] \arrow[d] & u^*C_{X/Y} \arrow[r] \arrow[d] & C_{X/Y} \arrow[d] \\
j^*N_{Y'/Y} \arrow[r] \arrow[d] & X' \arrow[r, "u"] \arrow[d, "\tilde{j}"] & X \arrow[d, hook, "\tilde{i}"] \\
\pi^*N_{Y'/Y} \arrow[r, "\tilde{\rho}"] \arrow[d] & \tilde{Y}' \arrow[r, "\tilde{v}"] \arrow[d] & \tilde{Y} \arrow[d, "\pi"] \\
N_{Y'/Y} \arrow[r, "\rho"] & Y' \arrow[r, "v"] & Y
\end{tikzcd}$$

Hence by Proposition 4.38 $\alpha : C_{X/\tilde{Y}} \to C_{X/Y}$ is a $T_{\tilde{Y}/Y} \times_{\tilde{Y}} C_{X/Y}$-bundle.

Proposition 4.59. *We have $\alpha^*(\beta(Y', X)) = \beta(\tilde{Y}', X) \in W_*(N_{Y'/Y} \times_Y C_{X/\tilde{Y}})$.*

Proof. In the compatibilities of β proved in [Vis89] we reduce to the case that $\tilde{Y} = \mathbb{A}^n_Y$. Then one checks that Vistoli's construction in the case directly. □

Proposition 4.60. *Back to the original diagram, assume that Y is of Deligne-Mumford type. Vistoli's rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ is invariant under the natural action of $j^*N_{Y'/Y} \times_Y T_Y$ on $N_{Y'/Y} \times_Y C_{X/Y}$.*

Proof. The vector bundle i^*T_Y acts on the X-cone $C_{X/Y}$ by Lemma 4.39. Pulling back from X to $j^*N_{Y'/Y}$ gives the natural action of $j^*N_{Y'/Y} \times_Y T_Y$ on $N_{Y'/Y} \times_Y C_{X/Y}$. Using the construction of the proof of Lemma 4.39 the claim follows from Proposition 4.59 applied to $\tilde{Y} = Y \times Y$ and $\tilde{i} = \Delta \circ i : X \to Y \times Y$. □

4.8 Virtual Fundamental Classes

Definition 4.61. *Let X be a DM-stack and an obstruction theory $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ for X. We say ϕ is a* perfect obstruction theory *if $E^\bullet$ is of perfect amplitude contained in $[-1, 0]$.*

We will construct the virtual fundamental class associated to a perfect obstruction theory. But before that, we will discuss a toy version which is a local model of the general theory.

Local Model, an Intuition

Consider Y be a smooth variety of dimension d with a vector bundle $E = \underline{\mathrm{Spec}}_Y \operatorname{Sym} \mathscr{E}^\vee$ over it of rank r. Let $s : Y \to E$ be a section and $0 : Y \to E$ be the zero section, then we cnsider the zero locus of s as:

$$\begin{tikzcd} X = Z(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, hook, "s"'] \\ Y \arrow[r, hook, "0"] & E \end{tikzcd}$$

Note that $X = Z(s)$ defined by the ideal $\mathscr{I} := \operatorname{Im}(s^\vee : \mathscr{E}^\vee \to \mathscr{O}_Y)$.

Now the most nice condition is that when s is a regular section, then $\dim Z(s) = d - r$. But in general this might not be true! So we define $d^{\mathrm{vir}}(X) = d - r$ to be the virtual dimension of X.

Moreover, we define the virtual fundamental class of X is

$$[X]^{\mathrm{vir}} := c_{\mathrm{loc}}(E, s) := 0^!([Y]) = 0^*(C_{Z(s)/Y}) \in \mathsf{CH}_{d^{\mathrm{vir}}(X)}(X)$$

which is the localized (top) Chern class of E with respect to s.

In this case the perfect obstruction theory is

$$\begin{tikzcd} E^\bullet : & \mathscr{E}^\vee|_X \arrow[r] \arrow[d, "s^\vee|_X"] & \Omega_Y|_X \arrow[d, "\mathrm{id}"] \\ \mathbb{L}_X^\bullet : & \mathscr{I}/\mathscr{I}^2 \arrow[r, "d_X"] & \Omega_Y|_X \end{tikzcd}$$

Remark 4.62. *Any perfect obstruction theory on a DM-stack is locally of this form. See the Remark 1.7 in [Tod21].*

Consider a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ for X. Locally, we embed X into a smooth scheme M with defining ideal $\mathscr{I}$.

$$\begin{tikzcd} \mathrm{cone}(\psi) = \mathscr{E}^\vee[1] & & & \\ E^\bullet \arrow[u] \arrow[r, "\phi"] & \tau^{\geq -1}\mathbb{L}_X^\bullet \arrow[r] & \mathrm{cone}(\phi) = P[2] \arrow[r] & E^\bullet[1] \\ \Omega_Y|_X \arrow[u, dashed, "\psi"] \arrow[ur] & & & \\ \mathscr{E}^\vee \arrow[u] & & & \end{tikzcd}$$

Here $\mathscr{E}$ locally free. Moreover wecan represent the morphism ϕ as a morphism of complexes from $[\mathscr{E}^\vee \to \Omega_Y|_X]$ to $[\mathscr{I}/\mathscr{I}^2 \to \Omega_Y|_X]$. Then extend $\mathscr{E}$ to a *vector bundle $\mathscr{F}$ on Y and use the surjection $\mathscr{E}^\vee \to \mathscr{I}/\mathscr{I}^2$ and projectivity of $\mathscr{E}^\vee$ to lift it to $\mathscr{F}^\vee \to \mathscr{I}$. This defines a section of $\mathscr{F}$ cutting out $X \subset Y$.*

Remark 4.63. *This also shows why we consider intrinsic normal cone and the obstruction theory. We need a global version of the previous case.*

General Construction

In the global version of a DM-stack X, fix a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}_X^\bullet$. We want to intersect the intrinsic normal cone $\mathfrak{C}_X$ with the vertex of $\mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\text{fppf}})$ to get the virtual fundamental class $[X]^{\text{vir}}_\phi$. Here we begin our story.

Remark 4.64. *Actually by the general theory of the intersection theory of general algebraic stacks had been developed in [Kre99] after our [BF97], so we can define $[X]^{\text{vir}}_\phi = 0^*[\mathfrak{C}_X]$ as the local case. But there we will follows the original method in [BF97] instead of this by assume we have global resolution.*

Definition 4.65. *Let a DM-stack X and a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}_X^\bullet$. We define the* virtual dimension of X with respect to the ϕ *is*

$$d^{\text{vir}}_\phi(X) := \operatorname{rank}(E^\bullet) = \dim E^0 - \dim E^{-1}$$

if locally $E^\bullet$ is written as a complex of vector bundles $[E^{-1} \to E^0]$.

Definition 4.66. *Let $F^\bullet = [F^{-1} \to F^0]$ be a homomorphism of vector bundles on X considered as a complex of $\mathscr{O}_X$-modules concentrated in degrees -1 and 0. An isomorphism $F^\bullet \to E^\bullet$ in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ is called a* global resolution of $E^\bullet$.

Note also that the intersection theory in [Vis89] holds when we consider the rational coefficient Chow group with there is a scheme F and a proper surjective morphism $F \to X$. In our case, if $X \to \operatorname{Spec} k$ is separaed, then this condition holds, see Theorem 4.12 in [DM69].

So in this whole section we will assume X to be a separated DM-stack locally of finite type over a field k.

Definition 4.67. *In this condition, consider a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ and admits a global resolution $F^\bullet \to E^\bullet$ with $F^\bullet = [F^{-1} \to F^0]$. Then $\mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\text{fppf}}) = [F^{-1,\vee}/F^{0,\vee}]$. Consider the cartesian*

$$\begin{tikzcd}
C(F^\bullet) \arrow[r, hook] \arrow[d] & F^{-1,\vee} \arrow[d] \\
\mathfrak{C}_X \arrow[r, hook] & \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\text{fppf}}) = [F^{-1,\vee}/F^{0,\vee}]
\end{tikzcd}$$

Then define **virtual fundamental class of X associated to the perfect obstruction theory ϕ** *is:*

$$[X]^{\mathrm{vir}}_{\phi} := 0^!(C(F^\bullet)) \in \mathsf{CH}_{d^{\mathrm{vir}}_{\phi}(X)}(X)_{\mathbb{Q}}$$

where $0 : X \to F^{-1,\vee}$ be the zero section with refined Gysin pullback $0^!$.

Proposition 4.68. *The virtual fundamental class $[X]^{\mathrm{vir}}_{\phi}$ of X associated to the perfect obstruction theory ϕ is independent of the global resolution $F^\bullet$ used to construct it.*

Proof. Give another global resolution $H^\bullet$. WLOG assume that $H^\bullet \to E^\bullet$ and $F^\bullet \to E^\bullet$ are given by morphisms of complexes. Then we get an induced homomorphism $H^0 \oplus F^0 \to E^0$. Consider cartesian diagram

$$\begin{tikzcd} K^{-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & H^0 \oplus F^0 \arrow[d] \\ E^{-1} \arrow[r] & E^0 \end{tikzcd}$$

Let $K^0 = H^0 \oplus F^0$ and hence we get another global resolution $K^\bullet$. So we just need to consider $F^\bullet$ and $K^\bullet$.

Now $K^{-1,\vee} \to E^{-1,\vee}$ is an epimorphism. Consider cartesians

$$\begin{tikzcd} X \arrow[r, hook, "0"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C(K^\bullet) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & C(F^\bullet) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{C}_X \arrow[d, hook] \\ X \arrow[r, hook, "0"] & K^{-1,\vee} \arrow[r, "\alpha"] & F^{-1,\vee} \arrow[r] & \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\mathrm{fppf}}) \end{tikzcd}$$

Now α is smooth surjective, the virtual fundamental class using $F^\bullet$ is equal to

$$(\alpha \circ 0)^![C(F^\bullet)] = 0^!\alpha^![C(F^\bullet)] = 0^![C(K^\bullet)]$$

which is also the virtual fundamental class using $K^\bullet$. □

Remark 4.69 (Virtual Structure Sheaf)**.** *Let a perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}^\bullet_X$. Let $\mathfrak{E} := \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\mathrm{fppf}})$ be the abelian cone stack.*

Consider the presentation $\mathfrak{E} = [E^{-1,\vee}/E^{0,\vee}]$, then $\mathfrak{C}_X \hookrightarrow \mathfrak{E}$ induce a subcone $C \subset E^{-1,\vee}$. We then set

$$[\mathscr{O}^{\mathrm{vir}}_{X,\phi}] := [\mathscr{O}_C \otimes^{\mathbb{L}}_{\mathscr{O}_{E^{-1,\vee}}} \mathscr{O}_X] = \sum_{i \geq 0} (-1)^i [\mathscr{T}or_i^{\mathscr{O}_{E^{-1,\vee}}}(\mathscr{O}_C, \mathscr{O}_X)] \in K_0(X)$$

which is called the **virtual structure sheaf of X relative to the perfect obstruction theory ϕ.**

Now assume that there is a Chern character $\mathfrak{c} : K_0(X) \to \mathsf{CH}_*(X)_{\mathbb{Q}}$ *and* $E^\bullet$ *admits a global resolution, then we can define the virtual fundamental class as:*

$$[X]^{\mathrm{vir}}_{\phi} = \mathrm{td}(E^\bullet) \cap \mathfrak{c}([\mathscr{O}^{\mathrm{vir}}_{X,\phi}]) \in \mathsf{CH}_{d^{\mathrm{vir}}_{\phi}(X)}(X)_{\mathbb{Q}}.$$

These two constructions agree when they are both possible.

Some Examples

Example 4.70 (Trivial Obstructions)**.** *Let* $\mathbb{L}^\bullet_X$ *is of perfect amplitude contained in* $[-1, 0]$ *(such as* X *is a complete intersection) then* $\mathbb{L}^\bullet_X$ *itself is a perfect obstruction theory. Any embedding of* X *into a smooth DM-stack gives rise to a global resolution of* $\mathbb{L}^\bullet_X$*. The virtual fundamental class* $[X]^{\mathrm{vir}}$ *thus obtained is equal to* $[X]$*, the 'usual' fundamental class.*

Example 4.71 (No Obstructions)**.** *If* $E^\bullet$ *is perfect,* $H^0(E^\bullet)$ *is locally free and* $H^1(E^\bullet) = 0$*, then* X *is smooth and* $d^{\mathrm{vir}}_{E^\bullet}(X) = \dim X$ *and the virtual fundamental class* $[X]^{\mathrm{vir}}_{E^\bullet} = [X]$*, the usual fundamental class.*

Example 4.72 (Locally Free Obstructions)**.** *If* X *is smooth,* $E^\bullet$ *is perfect and* $H^1(E^{\bullet,\vee})$ *is locally free, then* $d^{\mathrm{vir}}_{E^\bullet}(X) = \mathrm{rank}(H^1(E^{\bullet,\vee}))$ *and the virtual fundamental class*

$$[X]^{\mathrm{vir}}_{E^\bullet} = c_{\mathrm{rank}(H^1(E^{\bullet,\vee}))}(H^1(E^{\bullet,\vee})) \cap [X].$$

Example 4.73 (Products)**.** *Consider two perfect obstruction theories* $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ *and* $\psi : F^\bullet \to \mathbb{L}^\bullet_Y$*. Then* $\mathbb{L}^\bullet_{X \times Y} = \mathbb{L}^\bullet_X \boxplus \mathbb{L}^\bullet_Y$*, then we have also a perfect obstruction theory* $\phi \boxplus \psi : E^\bullet \boxplus F^\bullet \to \mathbb{L}^\bullet_{X \times Y}$*. If both* $E^\bullet, F^\bullet$ *have global resolutions, then so is* $E^\bullet \boxplus F^\bullet$*. By Proposition 4.46(b) we have*

$$[X \times Y]^{\mathrm{vir}}_{\phi \boxplus \psi} = [X]^{\mathrm{vir}}_{\phi} \times [Y]^{\mathrm{vir}}_{\psi} \in \mathsf{CH}_{d^{\mathrm{vir}}_{\phi}(X) + d^{\mathrm{vir}}_{\psi}(Y)}(X \times Y).$$

Pullback of Virtual Fundamental Class

We will show the pullback formula via local complete intersection morphism for now as in [BF97]. For the general case, we refer [Man12].

Cnosider a cartesian diagram of DM-stacks

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

where v is a local complete intersection morphism. Let $\Phi : E^\bullet \to \mathbb{L}^\bullet_X$ and $\Psi : F^\bullet \to \mathbb{L}^\bullet_{X'}$ be two perfect obstruction theories.

Definition 4.74. *A compatibility datum (relative to v) for $E^\bullet$ and $F^\bullet$ is a triple (ϕ,ψ,χ) of morphisms in $\mathbf{D}(\mathscr{O}_{X_{\text{ét}}})$ giving rise to a morphism of distinguished triangles*

$$\begin{tikzcd}
u^*E^\bullet \arrow[r, "\phi"] \arrow[d] & F^\bullet \arrow[r, "\psi"] \arrow[d] & g^*\mathbb{L}_{Y'/Y} \arrow[r, "\chi"] \arrow[d] & u^*E^\bullet[1] \arrow[d] \\
u^*\mathbb{L}^\bullet_X \arrow[r] & \mathbb{L}^\bullet_{X'} \arrow[r] & \mathbb{L}^\bullet_{X'/X} \arrow[r] & u^*\mathbb{L}^\bullet_X[1]
\end{tikzcd}$$

Given a compatibility datum, we call $E^\bullet$ and $F^\bullet$ compatible (over v).

Assume that $E^\bullet$ and $F^\bullet$ are endowed with such a compatibility datum. Then we get (Proposition 4.37(d)) a short exact sequence of vector bundle stacks

$$g^*\mathfrak{N}_{Y'/Y} = g^*\mathcal{H}^1/\mathcal{H}^0(\mathbb{T}^\bullet_{Y'/Y,\text{fppf}}) \to \mathfrak{F} = \mathcal{H}^1/\mathcal{H}^0(\mathbb{F}^{\bullet,\vee}_{\text{fppf}}) \xrightarrow{\phi} u^*\mathfrak{E} = u^*\mathcal{H}^1/\mathcal{H}^0(\mathbb{E}^{\bullet,\vee}_{\text{fppf}}).$$

Lemma 4.75. *If Y and Y' are smooth and v a regular local immersion, then $\mathfrak{N}_{Y'/Y} = N_{Y'/Y}$ is the normal bundle and we denote $N := g^*N_{Y'/Y}$. Then there is a (canonical) rational equivalence $\beta(Y',X) \in W_*(N\times\mathfrak{F})$ such that*

$$\partial\beta(Y',X) = [\phi^*C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}] - [N\times\mathfrak{C}_{X'}].$$

Proof. Let $X \to M$ be a local embedding, where M is smooth. We get an induced cartesian diagrams

$$\begin{tikzcd}
N\times_X C_{X/Y\times M} \arrow[r] \arrow[d] & u^*C_{X/Y\times M} \arrow[r] \arrow[d] & C_{X/Y\times M} \arrow[d] \\
N \arrow[r] \arrow[d] & X' \arrow[r, "u"] \arrow[d, "j"'] & X \arrow[d, "i"'] \\
N_{Y'/Y}\times M \arrow[r, "\rho"] & Y'\times M \arrow[r, "v"] & Y\times M
\end{tikzcd}$$

Now we have Vistoli's rational equivalence $\beta(Y'\times M,X) \in W_*(N\times_X C_{X/Y\times M})$ such that

$$\partial\beta(Y'\times M,X) = [C_{u^*C_{X/Y\times M}/C_{X/Y\times M}}] - [N\times C_{X'/Y'\times M}].$$

By Proposition 4.60, $\beta(Y'\times M,X)$ is invariant under the action of $N\times u^*i^*T_{Y\times M}$ on $N\times_X C_{X/Y\times M}$. Hence in particular, $\beta(Y'\times M,X)$ is invariant under the subsheaf $N\times j^*T_{Y'\times M}$ and thus descends to $N\times[u^*C_{X/Y\times M}/j^*T_{Y'\times M}] = N\times\mathfrak{F}\times_{\mathfrak{E}}\mathfrak{C}_X$ which is a closed subcone stack of $\mathfrak{F}$. So pushing forward via this closed immersion, we get a rational equivalence on $N \times \mathfrak{F}$ which we denote it by $\beta(Y', X)$. Now we have

$$\partial\beta(Y', X) = [\phi^* C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}] - [N \times \mathfrak{C}_{X'}]$$

as we need. By Proposition 4.59, we can glue them up and well done. $\square$

Theorem 4.76 (Pullback)**.** *Let $E^\bullet$ and $F^\bullet$ be compatible perfect obstruction theories, as above. If $E^\bullet$ and $F^\bullet$ have global resolutions then*

$$v^![X]^{\mathrm{vir}}_{E^\bullet} = [X']^{\mathrm{vir}}_{F^\bullet}$$

holds in the following cases:

(1) v is smooth.

(2) Y' and Y are smooth.

Proof. First note that one may choose global resolutions $[E_0 \to E_1]$ of $E^{\bullet,\vee}$ and $[F_0 \to F_1]$ of $F^{\bullet,\vee}$ together with a pair of epimorphisms $\phi_0 : F_0 \to u^*E_0$ and $\phi_1 : F_1 \to u^*E_1$ with kernels G_i such that the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_0 & \longrightarrow & F_0 & \xrightarrow{\phi_0} & u^*E_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & u^*E_1 & \longrightarrow & 0 \end{array}$$

The induced short exact sequence of vector bundle stacks

$$[G_1/G_0] \to [F_1/F_0] \to [u^*E_1/u^*E_0]$$

is isomorphic to $g^*\mathfrak{N}_{Y'/Y} \to \mathfrak{F} \to u^*\mathfrak{E}$. Let $C_1 := \mathfrak{C}_X \times_{\mathfrak{E}} E_1$ and $D_1 := \mathfrak{C}_{X'} \times_{\mathfrak{F}} F_1$ and by definition we have $[X]^{\mathrm{vir}}_{E^\bullet} = 0^!_{E_1}[C_1]$ and $[X']^{\mathrm{vir}}_{F^\bullet} = 0^!_{F_1}[D_1]$.

For case (1), let v is smooth. Then by Proposition 4.46(c) we get the cartesian of the left following diagram:

$$\begin{array}{ccc} \mathfrak{C}_{X'} & \longrightarrow & u^*\mathfrak{C}_X \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{F} & \longrightarrow & u^*\mathfrak{E} \end{array} \qquad\qquad \begin{array}{ccc} D_1 & \longrightarrow & u^*C_1 \\ \downarrow & \ulcorner & \downarrow \\ F_1 & \longrightarrow & u^*E_1 \end{array}$$

which imply the right one is also cartesian. This shows $0^!_{u^*E_1}[u^*C_1] = 0^!_{F_1}[D_1]$. Hence

$$v^![X]^{\mathrm{vir}}_{E^\bullet} = v^! 0^!_{E_1}[C_1] = 0^!_{u^*E_1}[u^*C_1] = 0^!_{F_1}[D_1] = [X']^{\mathrm{vir}}_{F^\bullet}.$$

Well done.

Now we consider case (2) where Y' and Y are smooth. First we claim this is true for the case where v is a regular local immersion. Indeed, in this case we may choose F_1 as the fibered product

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & u^*E_1 \arrow[d] \\ \mathfrak{F} \arrow[r] & u^*\mathfrak{E} \end{tikzcd}$$

Now lifting the rational equivalence in Lemma 4.75 to $N \times F_1$ we get $[N \times D_1] = \phi^*[C_{u^*C_1/C_1}]$. Hence we have

$$\begin{aligned} [X']^{\mathrm{vir}}_{F^\bullet} &= 0^!_{F_1}[D_1] = 0^!_{N\times F_1}[N \times D_1] = 0^!_{N\times F_1}\phi^*[C_{u^*C_1/C_1}] \\ &= 0^!_{N\times u^*E_1}[C_{u^*C_1/C_1}] = 0^!_{u^*E_1}v^![C_1] = v^!0^!_{E_1}[C_1] = v^![X]^{\mathrm{vir}}_{E^\bullet}. \end{aligned}$$

Well done. Now we back to the general case.

In the general case factor v as $v : Y' \xrightarrow{\Gamma_v} Y' \times Y \xrightarrow{p} Y$. Since Y' is smooth it has a canonical obstruction theory $\Omega_{Y'}$ and an obstruction theory on $Y' \times X$ is $\Omega_{Y'} \boxplus E^\bullet$. Then combine the previous two cases and the fact that these obstruction theories are trivially compactible, well done. $\square$

Remark 4.77. *See the relative version of this theory in Section 7 of [BF97] and we will not consider them here.*

4.9 Examples

Basic Case of Gysin Pullback

Example 4.78 (Basic Case of Gysin Pullback)**.** *Consider a cartesian diagram of schemes*

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "g"] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "f"] \\ Y \arrow[r, hook, "i"] & W \end{tikzcd}$$

that V and W are smooth and that i is a regular embedding.

Consider complex $E^\bullet \in \mathbf{D}^{[-1,0]}(X)$ be the composition

$$g^*N^\vee_{Y/W} \to g^*i^*\Omega_W = j^*f^*\Omega_W \to j^*\Omega_V.$$

Then the morphism $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ defined by $g^*\mathbb{L}^\bullet_Y \to \mathbb{L}^\bullet_X$ and $j^*\mathbb{L}^\bullet_V \to \mathbb{L}^\bullet_X$. This defines a perfect obstruction theory for X.

Now in this case we have cartesian

$$\begin{tikzcd} C_{X/V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & g^*N_{Y/W} \arrow[d] & \\ \mathfrak{C}_X \arrow[r, hook] & \mathcal{H}^1/\mathcal{H}^0(E^{\bullet,\vee}_{\mathrm{fppf}}) \arrow[r, "="] & [g^*N_{Y/W}/j^*T_V] \end{tikzcd}$$

and hence

$$[X]^{\mathrm{vir}}_{\phi} = 0^*[C_{X/V}] = i^![V]$$

be the refined Gysin pullback! This is also work if we consider DM-stacks.

An application of this:

Example 4.79 (Fibres of Morphism Between Smooth Stacks)**.** *Let $f : V \to W$ be a morphism of algebraic stacks. We shall assume that V and W are smooth over algebraically closed k and that f has unramified diagonal, so that V is a relative Deligne-Mumford stack over W. Let $w : \operatorname{Spec} k \to W$ be a k-valued point of W and let X be the fiber of f over w.*

Now we define an obstruction theory. Consider smooth cover $\tilde{W} \to W$ with fiber product $\tilde{V}$ which is a smooth DM-stack:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \tilde{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \tilde{W} \arrow[r] & W \end{tikzcd}$$

Then using the previous example and well done. This is a straightforward verification to check that the obstruction theory so defined does not depend on the choices of coverings.

Moduli Stacks of Projective Varieties

Let M and X be Deligne-Mumford stacks.

Definition 4.80. *A morphism $f : M \to X$ is called a relatively Gorenstein morphism if f has constant relative dimension and that the relative dualizing complex $\omega^\bullet_{M/X} = \omega_{M/X}[-\dim M - \dim X]$ for some invertible sheaf $\omega_{M/X}$.*

Example 4.81 (Moduli Stacks of Projective Varieties)**.** *Let X be a moduli (sub-)stack of some flat, relatively Gorenstein projective morphism families which is a DM-stack. Let M be its universal family.*

Lemma 4.82. *Consider $p : M \to X$, then for any cartesian*

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "p"] \\ T \arrow[r, "f"] & X \end{tikzcd}$$

Then for any $F^\bullet \in \mathbf{D}^-_{\mathrm{Qcoh}}(M)$ and $G^\bullet \in \mathbf{D}^b_{\mathrm{Qcoh}}(X)$ we have

$$\mathrm{Ext}^k_N(\mathbf{L}g^*F^\bullet, q^*G^\bullet) \cong \mathrm{Ext}^k_T(\mathbf{L}f^*(\mathbf{R}p_*(F^\bullet \otimes^{\mathbf{L}} \omega^\bullet_{M/X})), G^\bullet).$$

Proof. From flat base-change and Grothenieck duality directly. □

Now we define $E^\bullet := \mathbf{R}p_*(\mathbb{L}^\bullet_{M/X} \otimes^{\mathbf{L}} \omega^\bullet_{M/X})[-1]$. The distinguished triangle of cotangent complexes induce $\mathbb{L}^\bullet_{M/X} \to p^*\mathbb{L}^\bullet_X[1]$ (the derived Kodaira-Spencer map) which induce $\phi : E^\bullet \to \mathbb{L}^\bullet_X$.

Proposition 4.83. *In this case, $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ is an obstruction theory. If moreover p is smooth of relative dimension ≤ 2, then it is a perfect obstruction theory.*

Proof. The fact that M is a universal family and X is DM-stack implies that the fibers of p have finite and reduced automorphism group, hence $E^\bullet$ satisfies (*).

Next, let T be a scheme, $f : T \to X$ a morphism, and consider the cartesian

$$\begin{array}{ccc} N & \xrightarrow{g} & M \\ \big\downarrow{\scriptstyle q} & \ulcorner & \big\downarrow{\scriptstyle p} \\ T & \xrightarrow{f} & X \end{array}$$

If $T \to \overline{T}$ is a square zero extension with ideal sheaf $\mathscr{J}$, the obstruction to extending N to a flat family over $\overline{T}$ lies in $\mathrm{Ext}^2(\mathbb{L}^\bullet_{N/T}, q^*\mathscr{J})$, and the extensions, if they exist, are a torsor under $\mathrm{Ext}^1(\mathbb{L}^\bullet_{N/T}, q^*\mathscr{J})$ (see Tag 08V5).

The map $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ induces morphisms

$$\phi_k : \mathrm{Ext}^{k-1}_T(\mathbf{L}f^*\mathbb{L}^\bullet_X, \mathscr{J}) \to \mathrm{Ext}^{k-1}_T(\mathbf{L}f^*E^\bullet, \mathscr{J}) = \mathrm{Ext}^k_N(\mathbb{L}^\bullet_{N/T}, q^*\mathscr{J})$$

by Lemma 4.82. The universality of M means that extending N to a family over $\overline{T}$ is equivalent to extending f to a morphism to X defined on $\overline{T}$. Hence by Theorem 4.51, ϕ is an obstruction theory for X. The final statement is trivial. □

5 About Atiyah-Bott Localization Formula

The original paper is [AB84]. The basic theory of equivariant cohomology we refer Section 2 of [AB84] or Chapter 7 in [Ric22].

Here WLOG we consider the homology/cohomology groups of $\mathbb{Q}$-coefficients.

5.1 Approximation Spaces

Let us now assume that X is a space and G is a Lie group. The fact that the spaces involved, like $\mathbf{E}G$ and $\mathbf{B}G$, are infinite-dimensional, is not quite an obstacle to the computation of the equivariant cohomology groups. This is the case because of the following "approximation" result.

Theorem 5.1 (Approximation)**.** *Let $(E_m)_{m\geq 0}$ be a family of connected spaces on which G acts freely on the right. Let $k : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function such that $\pi_i(E_m) = 0$ for $0 < i < k(m)$ and such that $\lim_{m\to\infty} k(m) = \infty$. Then, for any left G-action on a space X, there are natural isomorphisms*

$$H^i_G(X) \cong H^i(E_m \times^G X), \quad \textit{for } i < k(m).$$

Proof. Here we give a sketch. Consider the following:

$$\begin{array}{ccccc}
E_m \times X & \longleftarrow & \mathbf{E}G \times E_m \times X & \longrightarrow & \mathbf{E}G \times X \\
\downarrow & & \downarrow & & \downarrow \\
E_m \times^G X & \longleftarrow & (\mathbf{E}G \times E_m) \times^G X & \longrightarrow & \mathbf{E}G \times^G X
\end{array}$$

where the vertical maps are the (free) quotient maps, and the horizontal maps are locally trivial fibre bundles with fibre indicated on top of the corresponding arrow. As a consequence of the Leray-Hirsch Lemma, since $k(m)$ goes to infinity as m grows, we can apply it to $k(m)$ directly, showing that for all $i < k(m)$ we have isomorphisms

$$H^i(E_m \times^G X) \cong H^i((\mathbf{E}G \times E_m) \times^G X) \cong H^i(\mathbf{E}G \times^G X) = H^i_G(X).$$

Well done. $\square$

Remark 5.2. *In the smooth category, if G is a compact Lie group, then $\mathbf{E}G \to \mathbf{B}G$ is a colimit of smooth principal G-bundles $E_m \to B_m$ where E_m is m-connected.*

5.2 Equivariant Pullback and Chern Classes

Here and the next small section we will introduce some functorial things in the category of equivariant objects.

Definition 5.3 (Equivariant Pullback)**.** *Let G be a group acting on X and H a group acting on Y. Suppose there are maps $\phi : G \to H$ and $f : X \to Y$ such that $f(gx) = \phi(g)f(x)$, then induce $\mathbf{E}G \times^G X \to \mathbf{E}H \times^H Y$ which induce the equivariant pullback:*

$$f^* : H^*_H(Y) = H^*(\mathbf{E}H \times^H Y) \to H^*(\mathbf{E}G \times^G X) = H^*_G(X).$$

Definition 5.4 (Equivariant Chern Classes)**.** *Consider a G-equivariant vector bundle $\pi : E \to X$, then this induce a now vector bundle $V_E := \mathbf{E}G \times^G E \to \mathbf{E}G \times^G X$ of the same rank as E. Then the* equivariant Chern classes *of $E \to X$ are the characteristic classes*

$$c_i^G(E) := c_i(V_E) \in H_G^{2i}(X).$$

Moreover, the equivariant Euler class *of $E \to X$ is the characteristic class*

$$e^G(E) := c_{\text{top}}^G(E) = c_{\text{top}}(V_E) \in H_G^{2\cdot\text{top}}(X).$$

Remark 5.5. *These classes can clearly be computed through approximation spaces. Indeed, the vector bundle $V_E \to \mathbf{E}G \times^G X$ can be approximated by vector bundles*

$$V_{E,m} := E_m \times^G E \to E_m \times^G X$$

and $c_i(V_{E,m}) \in H^{2i}(E_m \times^G X) = H_G^{2i}(X)$ for $m \gg 0$.

5.3 Equivariant Pushforward

Now we consider G be a compact Lie group. Let $f : X \to Y$ be a G-equivariant map of compact manifolds. Set $\dim X = n, \dim Y = m$ and $d = m - n$.

Definition 5.6 (Equivariant Pushforward)**.** *Fix a directed system of principal G-bundles $\{E_i \to B_i\}_{i\geq 0}$ whose limit recovers the classifying space $\mathbf{E}G \to \mathbf{B}G$. Let $X_G^i := E_i \times^G X$ and $Y_G^i := E_i \times^G Y$, then*

$$H_G^p(X) \cong H^p(X_G^i) \text{ and } H_G^p(Y) \cong H^p(Y_G^i), \quad \text{for } p \leq i.$$

Now let $\dim B_i = \ell_i$, then $\dim X_G^i = n + \ell_i$ and $\dim Y_G^i = m + \ell_i$. Then for $p \leq i$ we define

$$\begin{tikzcd}
H_G^p(X) = H^p(X_G^i) \arrow[r, "f_*^{G,p}"] \arrow[d, "\mathsf{PD},\cong"] & H_G^{p+d}(Y) = H^{p+d}(Y_G^i) \\
H_{\ell_i+n-p}(X_G^i) \arrow[r, "f_*^i"] & H_{\ell_i+n-p}(Y_G^i) \arrow[u, "(\mathsf{PD})^{-1},\cong"']
\end{tikzcd}$$

This yields a system of maps which compatible with the structure of inverse system:

$$f_*^{G,p} : H_G^p(X) \to H_G^{p+d}(Y), \quad f_*^G : H_G^*(X) \to H_G^{*+d}(Y)$$

which is the equivariant pushforward.

5.4 Torus Fixed Loci

Here we give something about fixed locus in the algebraic settings. The classical topological setting is more easier.

Definition 5.7 (Fixed Locus)**.** *Let X be a scheme over a field k with an action of an affine algebraic group G, we define the* fixed locus *is*

$$X^G := \underline{\mathsf{Hom}}^G(\operatorname{Spec} k, X) : \mathbf{Sch}/k \to \mathbf{Sets},$$

the set of G-equivariant maps from S to X where S with trivial action.

Before we consider the geometric properties of X^G, we need an important lemma:

Lemma 5.8. *Let X be an algebraic space locally of finite type over an algebraically closed field k with affine diagonal. Suppose that X has an action of an affine algebraic group G. If $x \in X(k)$ has linearly reductive stabilizer, then there exists a G-equivariant étale neighborhood* $(\operatorname{Spec} A, u) \to (X, x)$ *inducing an isomorphism of stabilizer groups at u.*

If G is a torus, then every point has a G-invariant étale neighborhood $(\operatorname{Spec} A, u) \to (X, x)$ *inducing an isomorphism of stabilizer groups at u.*

Proof. By J. Alper's theory (see the book [Alp24]) of the local structure of algebraic stacks, there is an étale neighborhood $([\operatorname{Spec} A/G_x], u) \to ([X/G], x)$ such that w is a closed point and f induces an isomorphism of stabilizer groups at w and such that $[\operatorname{Spec} A/G_x] \to [X/G] \to \mathbf{B}G$ is affine. Therefore, $W := [\operatorname{Spec} A/G_x] \times_{[X/G]} X$ is an affine scheme and $W \to X$ is a G-equivariant étale neighborhood of x. When G is a torus, then any subgroup of G and in particular each stabilizer group is linearly reductive. □

Theorem 5.9. *Let X be a scheme of finite type over an algebraically closed field k with affine diagonal and with an action of a linearly reductive algebraic group G.*

(a) The fixed locus X^G is represented by a subscheme of X.

(b) If G is a torus, then $X^G \subset X$ be a closed subscheme.

(c) If X is smooth, so is X^G.

Proof. If G is connected and $U \to X$ is a G-equivariant étale morphism, we claim that $X^G \times_X U \cong U^G$. Indeed, suppose $S \to U$ is a map such that $S \to U \to X$ is G-invariant. Let $U_S \to S$ be the base change of $U \to X$ by $S \to X$. Since $U_S \to S$ is G-equivariant, it suffices to show that the section $j : S \to U_S$ is G-invariant. As $U \to X$ is étale, $j : S \to U_S$ is an open immersion. Because G is connected, for each point $s \in S$, the G-orbit $Gj(s) \subset U_S$ is connected and thus contained in S.

For (a), given a fixed point $x \in X^G(k)$, Lemma 5.8 produces a G-equivariant étale neighborhood $(U, u) \to (X, x)$ with U affine and $u \in U^G(k)$. If G is connected, then $U^G \to X^G$ is étale and representable by the previous argument. Thus it suffices to show that U^G is representable. Since U is affine, we can choose a G-equivariant embedding $U \hookrightarrow \mathbb{A}(V)$ into a finite dimensional G-representation. In this case, $\mathbb{A}(V)^G = \mathbb{A}(V^G)$ and thus $U^G = U \cap \mathbb{A}(V)^G$ is representable. In general, let $G_0 \subset G$ be the connected component of the identity, and let $g_1, ..., g_n \in G(k)$ be representatives of the finitely many cosets $G(k)/G_0(k)$. Then G/G_0 acts on X^{G_0} and $X^G = \bigcap_i (X^{G_0})^{g_i}$, where $(X^{G_0})^{g_i}$ is identified with the fiber product of the diagonal $X^G \to X^G \times X^G$ and the map $X^G \to X^G \times X^G$ given by $x \mapsto (x, gx)$.

For (b), every subgroup of G is linearly reductive and Lemma 5.8 therefore produces a G-equivariant étale surjective morphism $U \to X$ from an affine scheme. As G is connected, the argument above shows that $U^G \subset U$ is a closed immersion and thus by étale descent so is $X^G \subset X$.

For (c), if $x \in X^G(k)$, there is a G-equivariant étale morphism $(U, u) \to (X, x)$ from an affine scheme and a G-invariant étale morphism $U \to T_{U,u}$ via Luna map. Since $T_{U,u}^G$ is a linear subspace, it is smooth. Since $U^G \to X^G$ and $U^G \to T_{U,u}^G$ are étale at u, the statement follows from étale descent. □

Remark 5.10. *So this theorem is right for algebraic spaces. Note also that we have the Białynicki-Birula stratification for the more results of $\mathbb{G}_m$-action (see Theorem 6.7.13 in [Alp24]).*

5.5 The Localization Formula

Consider a compact Lie group G and $\mathbb{T} \subset G$ be a maximal torus. The case $G = \mathbb{T}$ is the most important case, essentially by the following proposition:

Proposition 5.11. *Let $\mathbb{W} = N(\mathbb{T})/Z(\mathbb{T}) = N(\mathbb{T})/\mathbb{T}$ be the Weyl group. Then*

$$H_G^*(X) \cong H_{\mathbb{T}}^*(X)^{\mathbb{W}}$$

where $H_{\mathbb{T}}^*(X)^{\mathbb{W}}$ is the $\mathbb{W}$-invariant elements of the equivariant cohomology.

So we focus on the torus $\mathbb{T}$ and we need a key lemma.

Lemma 5.12. *If there is a $\mathbb{T}$-equivariant map $V \to \mathbb{T}/K$ for a closed subgroup $K \subset \mathbb{T}$, then*

$$\mathrm{supp}(H_{\mathbb{T}}^*(V)) \subset \mathfrak{k}_{\mathbb{C}}$$

where $\mathfrak{k}_{\mathbb{C}}$ be the complexification of the Lie algebra of K and we view $H_{\mathbb{T}}^*(V)$ as $H_{\mathbb{T}}^*$-module. Note that since $H_{\mathbb{T}}^* = \mathbb{C}[u_1, ..., u_\ell]$ the supports of $H_{\mathbb{T}}^*$-modules lying over the affine space $\mathrm{Spec}\, H_{\mathbb{T}}^* = \mathfrak{t}_{\mathbb{C}}$.

Proof. The morphism $V \to \mathbb{T}/K \to \{\mathrm{pt}\}$ induce $H^*_{\mathbb{T}} \to H^*_{\mathbb{T}}(\mathbb{T}/K) \to H^*_{\mathbb{T}}(V)$. Now

$$H^*_{\mathbb{T}}(\mathbb{T}/K) = H^*(\mathbf{E}\mathbb{T} \times^{\mathbb{T}} \mathbb{T}/K) = H^*(\mathbf{E}\mathbb{T}/K) = H^*(\mathbf{B}K) = H^*_K = H^*_{K_0}.$$

So $H^*_{\mathbb{T}} \to H^*_{\mathbb{T}}(V)$ factor through $H^*_{K_0}$ and hence $\mathrm{supp}(H^*_{\mathbb{T}}(V)) \subset \mathfrak{k}_{\mathbb{C}}$. $\square$

Here are our main results:

Theorem 5.13 (Atiyah-Bott, 1984)**.** *Let X be a compact smooth manifold equipped with an action of $\mathbb{T}$. Let $\iota : X^{\mathbb{T}} \hookrightarrow X$ be the inclusion of the fixed point locus. For $i^* : H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(X^{\mathbb{T}})$ and $i^{\mathbb{T}}_* : H^*_{\mathbb{T}}(X^{\mathbb{T}}) \to H^*_{\mathbb{T}}(X)$, support of their kernels and cokernels lie in $\bigcup_K \mathfrak{k}_{\mathbb{C}}$ where K ranges for all proper isotropy subgroups $K \subset \mathbb{T}$.*

Proof. We stratify X by $\mathbb{T}$-orbits of varying isotropy groups. By the compactness of $\mathbb{T}$ we can construct $\mathbb{T}$-invariant tubular neighborhoods of these orbits. Take U to be the neighborhood of $X^{\mathbb{T}}$ in X, and $X\backslash U$ by compactness is covered by finitely many such neighborhoods of orbits.

By the Mayer-Vietoris on the finite cover of $X\backslash U$ by neighborhoods of orbits, we see that $H^*_{\mathbb{T}}(X\backslash U)$ is torsion over what we want by Lemma 5.12. Now since U has a $\mathbb{T}$-equivariantly deformation retracts onto $X^{\mathbb{T}}$, $H^*_{\mathbb{T}}(X\backslash X^{\mathbb{T}})$ is torsion similarly. The same proof shows that the result holds for all $\mathbb{T}$-invariant subspaces of $X\backslash X^{\mathbb{T}}$, and consequently for all pairs of such subspaces in $X\backslash X^{\mathbb{T}}$.

In particular, $H^*_{\mathbb{T}}(X, X^{\mathbb{T}}) \cong H^*_{\mathbb{T}}(X\backslash U, \partial(X\backslash U))$ (by excision) is torsion. The long exact sequence for the pair $(X, X^{\mathbb{T}})$ now shows the desired result for the pullback $i^* : H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(X^{\mathbb{T}})$.

The results for pushforward $i^{\mathbb{T}}_* : H^*_{\mathbb{T}}(X^{\mathbb{T}}) \to H^*_{\mathbb{T}}(X)$ is similar as using Thom isomorphism

$$\begin{tikzcd}
H^*_{\mathbb{T}} X^{\mathbb{T}} \arrow[r, "i^{\mathbb{T}}_*"] \arrow[d, "\mathsf{Thom}"] & H^{*+d}_{\mathbb{T}} X \\
H^{*+d}_{\mathbb{T}}(N_{X^{\mathbb{T}}/X}, N_{X^{\mathbb{T}}/X}\backslash X^{\mathbb{T}}) \arrow[r, "\cong"] & H^{*+d}_{\mathbb{T}}(X, X\backslash X^{\mathbb{T}}) \arrow[u]
\end{tikzcd}$$

Now the fact that $H^*_{\mathbb{T}}(X\backslash X^{\mathbb{T}})$ is torsion with the long exact sequence for the pair $(X, X\backslash X^{\mathbb{T}})$ implies the desired result. $\square$

This show that if we invert the certain polynomials in the ring $H^*_{\mathbb{T}}$ that vanish on all of $\mathfrak{k}_{\mathbb{C}}$ for the proper isotropy subgroups $K \subset \mathbb{T}$, the equivariant cohomology of X becomes isomorphic to the equivariant cohomology of $X^{\mathbb{T}}$! This is of course true if we invert the whole ring and we can get an integration formula:

Theorem 5.14 (Atiyah-Bott Localization Formula)**.** *Let X be a compact smooth manifold equipped with an action of $\mathbb{T}$. Let $\iota : X^{\mathbb{T}} \hookrightarrow X$ be the inclusion of the fixed point locus and $X^{\mathbb{T}} := \coprod_\alpha F_\alpha$ be the decomposition of connected components with inclusion $\iota_\alpha : F_\alpha \hookrightarrow X$. In this case $H^*_{\mathbb{T}}(X^{\mathbb{T}}) = \bigoplus_\alpha H^*_{\mathbb{T}}(F_\alpha)$. Let $\mathscr{H}_{\mathbb{T}} := \mathrm{Frac}(H^*_{\mathbb{T}})$. Then the equivariant pushforward along ι induces an isomorphism*

$$\iota^{\mathbb{T}}_* : H^*_{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}} \xrightarrow{\cong} H^*_{\mathbb{T}}(X) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}}$$

with inverse morphism

$$\psi \mapsto \sum_\alpha \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_{\iota_\alpha})}.$$

In particular, very class $\psi \in H^*_{\mathbb{T}}(X) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}}$ writes uniquely as

$$\psi = \sum_\alpha \iota^{\mathbb{T}}_{\alpha,*} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_{\iota_\alpha})}.$$

Proof. The fact that

$$\iota^{\mathbb{T}}_* : H^*_{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}} \xrightarrow{\cong} H^*_{\mathbb{T}}(X) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}}$$

is an isomorphism follows from Theorem 5.13. Then the $\iota^*_\alpha \iota^{\mathbb{T}}_{\alpha,*}(-) = e^{\mathbb{T}}(N_{\iota_\alpha}) \cap (-)$ implies the theorem. $\square$

5.6 Some Applications of Localization Formula

First we consider a direct corollary.

Corollary 5.15. *Let X be a compact smooth manifold equipped with an action of $\mathbb{T}$. Let $\iota : X^{\mathbb{T}} \hookrightarrow X$ be the inclusion of the fixed point locus and $X^{\mathbb{T}} := \coprod_\alpha F_\alpha$ be the decomposition of connected components with inclusion $\iota_\alpha : F_\alpha \hookrightarrow X$. Let $\mathscr{H}_{\mathbb{T}} := \mathrm{Frac}(H^*_{\mathbb{T}})$. Then we have an integration*

$$\int_X : H^*_{\mathbb{T}}(X) \otimes_{H^*_{\mathbb{T}}} \mathscr{H}_{\mathbb{T}} \to \mathscr{H}_{\mathbb{T}}.$$

Moreover, we have

$$\int_X \psi = \sum_\alpha \int_{F_\alpha} \frac{\iota^*_\alpha \psi}{e^{\mathbb{T}}(N_{\iota_\alpha})} \in \mathscr{H}_{\mathbb{T}}$$

for all $\psi \in H^*_{\mathbb{T}}(X)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[d, "q"'] & F_\alpha \arrow[l, hook', "\iota_\alpha"'] \arrow[dl, "q_\alpha"] \\ \mathrm{pt} & \end{tikzcd}$$

Then q_* induce this integration. Moreover, we have

$$\int_X \psi = q_*^{\mathbb{T}}\psi = \sum_\alpha q_{\alpha,*}^{\mathbb{T}} \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_{\iota_\alpha})} = \sum_\alpha \int_{F_\alpha} \frac{\iota_\alpha^*\psi}{e^{\mathbb{T}}(N_{\iota_\alpha})} \in \mathscr{H}_{\mathbb{T}}$$

and well done. $\square$

Then there are some easy but interesting applications.

Proposition 5.16. *Let M be a smooth oriented compact manifold with a torus $\mathbb{T}$ action having finitely many fixed points $p_1, ..., p_s$. Then*

$$\chi(M) = s.$$

Proof. By Gauss-Bonnet we have

$$\begin{aligned} \chi(M) &= \int_M e(T_M) = \int_M e^{\mathbb{T}}(T_M) \\ &= \sum_{1\le i\le s} \frac{e^{\mathbb{T}}(T_M)|_{p_i}}{e^{\mathbb{T}}(N_{p_i/M})} = \sum_{1\le i\le s} 1 = s. \end{aligned}$$

Well done. $\square$

Proposition 5.17. *Let $\mathbb{T}$ act on a complex smooth projective variety X, then we have*

$$\chi(X) = \chi(X^{\mathbb{T}}).$$

Proof. By Theorem 5.9(c) we know that $X^{\mathbb{T}}$ smooth. Now we have a $\mathbb{T}$-equivariant exact sequence

$$0 \to T_{X^{\mathbb{T}}} \to T_X|_{X^{\mathbb{T}}} \to N_{X^{\mathbb{T}}/X} \to 0.$$

This implies $e^{\mathbb{T}}(T_X|_{X^{\mathbb{T}}}) = e^{\mathbb{T}}(T_{X^{\mathbb{T}}})e^{\mathbb{T}}(N_{X^{\mathbb{T}}/X})$. Then we have

$$\begin{aligned} \chi(X) &= \int_X e(T_X) = \int_X e^{\mathbb{T}}(T_X) = \int_X \frac{e^{\mathbb{T}}(T_X|_{X^{\mathbb{T}}})}{e^{\mathbb{T}}(N_{X^{\mathbb{T}}/X})} \\ &= \int_X e^{\mathbb{T}}(T_{X^{\mathbb{T}}}) = \int_X e(T_{X^{\mathbb{T}}}) = \chi(X^{\mathbb{T}}). \end{aligned}$$

Well done. $\square$

5.7 To Solve Some Classical Enumerative Problems

Here we compute two classical toy enumerative problems. But before that we need consider an example.

As the localization formula told us that we just need to consider the integral over fixed loci. Here our example is that there are only finitely many fixed points. So we need to calculate some equivariant Chern classes of bundles over point, that is, the equivariant Chern classes of vector spaces.

Example 5.18. *Consider a vector bundle E on Y and the frame bundle $\mathsf{Fr}(d,E) \to Y$. Then $\mathsf{Fr}(d,E) \times^{\mathrm{GL}_d} \mathbb{C}^d$ is just the tautological subbundle $S \subset \pi^*E$ of rank d where $\pi : \mathsf{Grass}(d,E) \to Y$ be the Grassman bundle.*

Example 5.19. *For each integer a, $\mathbb{G}_m$ has the 1-dimensional representation $\mathbb{C}_a$, where $\mathbb{G}_m$ acts on $\mathbb{C}$ by $z \cdot v = z^a v$. Consider a approximation with*

$$\begin{array}{ccc} (\mathbb{C}^m \backslash 0) \times^{\mathbb{G}_m} \mathbb{C}_a & \xrightarrow{\cong} & \mathscr{O}(-a) \\ \downarrow & & \downarrow \\ (\mathbb{C}^m \backslash 0) \times^{\mathbb{G}_m} \mathsf{pt} & \xrightarrow{\cong} & \mathbb{P}^{m-1} \end{array}$$

Hence $c_1^{\mathbb{G}_m}(\mathbb{C}_a) = a c_1^{\mathbb{G}_m}(\mathbb{C}_1)$.

Example 5.20. *Consider $\mathbb{T} := (\mathbb{G}_m)^n$ acting on $\mathbb{C}^n = V$ by the standard action scaling coordinates. For i, we have one-dimensional representations $\mathbb{C}_{t_i}$. Then $c_i^{\mathbb{T}}(V) = \mathsf{elesym}_i(t_1, ..., t_n)$ where $t_i = c_1^{\mathbb{T}}(\mathbb{C}_{t_i})$ and elesym_i is the elementary symmetric polynomial. Using $E_m = (\mathbb{C}^m \backslash 0)^n$ and $B_m = (\mathbb{P}^{m-1})^n$, the class t_i is identified with the Chern class of the tautological bundle from the i-th factor of B_m.*

In general case we can use characters to compute them.

Two Lines in a Plane

Example 5.21. *Two lines in $\mathbb{P}^2$ intersect at unique point.*

Proof. Consider $\mathbb{G}_m$ act at $V := H^0(\mathbb{P}^2, \mathscr{O}(1)) = \mathrm{span}\{x_0, x_1, x_2\}$ with weight w_0, w_1, w_2, that is, $t \cdot x_i = t^{w_i} x_i$. We let $w_i \neq w_j$ for $i \neq j$. So we have fixed points

$$(\mathbb{P}^2)^{\mathbb{G}_m} = \{p_0 = (1:0:0), p_1 = (0:1:0), p_2 = (0:0:1)\}.$$

So the number we need is

$$\int_{\mathbb{P}^2} c_1(\mathscr{O}(1))^2 = \int_{\mathbb{P}^2} c_1^{\mathbb{G}_m}(\mathscr{O}(1))^2 = \sum_{i=0}^{2} \frac{c_1^{\mathbb{G}_m}(\mathscr{O}(1)|_{p_i})}{e^{\mathbb{G}_m}(T_{p_i}\mathbb{P}^2)}.$$

Consider the universal sequence

$$0 \to \mathscr{S} \to V \otimes \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_{\mathbb{P}^2}(1) \to 0.$$

Then restrict it to p_i we get

$$0 \to V_{jk} := \operatorname{span}(x_j, x_k) \to V \to V/V_{jk} = \mathbb{C} \cdot x_i \to 0.$$

This force $T_{p_i}\mathbb{P}^2 = V_{jk}^* \otimes V/V_{jk} = \operatorname{span}(x_j^* \otimes x_i, x_k^* \otimes x_i)$ with weight $w_i - w_j$ and $w_i - w_k$. And $\mathscr{O}(1)|_{p_i}$ has weight w_i. This shows $e^{\mathbb{G}_m}(T_{p_i}\mathbb{P}^2) = (w_i - w_j)(w_i - w_k)$ and $c_1^{\mathbb{G}_m}(\mathscr{O}(1)|_{p_i}) = w_i$. Hence

$$\begin{aligned}\sum_{i=0}^{2} \frac{c_1^{\mathbb{G}_m}(\mathscr{O}(1)|_{p_i})}{e^{\mathbb{G}_m}(T_{p_i}\mathbb{P}^2)} &= \frac{w_0^2}{(w_0 - w_1)(w_0 - w_2)} + \frac{w_1^2}{(w_1 - w_0)(w_1 - w_2)} \\ &\quad + \frac{w_2^2}{(w_2 - w_0)(w_2 - w_1)} \equiv 1\end{aligned}$$

and well done. □

The 27 Lines on a Smooth Cubic Surface

Example 5.22. *A general cubic surface $S \subset \mathbb{P}^3$ contains exactly 27 lines.*

Proof. Let $\mathbb{G}_m$ be a torus acting on $\mathbb{P}^3$ with distinct weights (w_0, w_1, w_2, w_3), this means, $t \cdot x_i = t^{w_i} x_i$. By the properties of Fano scheme of lines, our number is

$$\int_{\mathsf{Grass}(2,4)} e(\operatorname{Sym}^3 S^*), \quad \text{where } S \text{ is the tautological subbundle.}$$

The torus action has four fixed points $p_0, ..., p_3 \in \mathbb{P}^3$ and six invariant lines $\ell_{ij} \subset \mathbb{P}^3$ which are the lines joining the fixed points. These correspond to the fixed points of the Grassmannian $\mathsf{Grass}(2,4)$ under the lifted $\mathbb{G}_m$-action. By localization formula we have

$$\int_{\mathsf{Grass}(2,4)} e(\operatorname{Sym}^3 S^*) = \sum_{\ell_{ij}} \frac{e^{\mathbb{G}_m}(\operatorname{Sym}^3 S^*)|_{[\ell_{ij}]}}{e^{\mathbb{G}_m}(T_{[\ell_{ij}]}\, \mathsf{Grass}(2,4))}.$$

Now as before we consider universal sequence

$$0 \to S \to V^* \otimes \mathscr{O}_{\mathsf{Grass}(2,4)} \to Q \to 0.$$

Restrict it to $[\ell_{ij}]$, we get

$$0 \to \ell_{ij} = \operatorname{span}(x_i^*, x_j^*) \to V^* \to Q|_{[\ell_{ij}]} = \operatorname{span}(x_h^*, x_k^*) \to 0$$

for $\{i,j,h,k\} = \{0,1,2,3\}$. Hence we have

$$
\begin{aligned}
T_{[\ell_{ij}]}\,\mathsf{Grass}(2,4) &= S^*|_{[\ell_{ij}]} \otimes Q|_{[\ell_{ij}]} = \mathrm{span}(x_i \otimes x_h^*, x_j \otimes x_h^*, x_i \otimes x_k^*, x_j \otimes x_k^*) \\
\mathrm{Sym}^3 S^*|_{[\ell_{ij}]} &= \mathrm{Sym}^3(\mathbb{C}x_i \oplus \mathbb{C}x_j) = \mathrm{span}(x_i^3, x_i^2 x_j, x_i x_j^2, x_j^3).
\end{aligned}
$$

Hence we have

$$
\sum_{\ell_{ij}} \frac{e^{\mathbb{G}_m}(\mathrm{Sym}^3 S^*)|_{[\ell_{ij}]}}{e^{\mathbb{G}_m}(T_{[\ell_{ij}]}\,\mathsf{Grass}(2,4))} = \sum_{0 \le i < j \le 3} \frac{(3w_i)(2w_i + w_j)(w_i + 2w_j)(3w_j)}{(w_i - w_h)(w_j - w_h)(w_i - w_k)(w_j - w_k)} \equiv 27.
$$

Well done. $\square$

More examples we refer Chapter 9 in [Ric22].

6 Localization of Virtual Fundamental Class

We will mainly follows the original paper [GP99] and book [Ric22]. Here our Chow groups are all $\mathbb{Q}$-coefficients.

6.1 Equivariant Sheaves and Complexes

Let X be a noetherian separated scheme over $\mathbb{C}$, equipped with an action of a complex group scheme G.

Definition 6.1. *Now we have the following commutative diagram:*

$$
\begin{tikzcd}
G \times X & G \times G \times X \arrow[l, "p_{23}"'] \arrow[r, "m \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"'] & G \times X \arrow[r, "p_2"] \arrow[d, "\sigma"'] & X \\
& G \times X \arrow[r, "\sigma"] & X &
\end{tikzcd}
$$

(a) A G-equivariant quasicoherent sheaf on X is a pair $(\mathscr{F}, \vartheta)$ where $\mathscr{F} \in \mathrm{Qcoh}(X)$ and $\vartheta : p_2^*\mathscr{F} \cong \sigma^*\mathscr{F}$ and $(m \times \mathrm{id}_X)^*\vartheta = (\mathrm{id}_G \times \sigma)^*\vartheta \circ p_{23}^*\vartheta$.

(b) A morphism $(\mathscr{F}, \vartheta) \to (\mathscr{F}', \vartheta')$ of G-equivariant quasicoherent sheaves is a morphism $\phi : \mathscr{F} \to \mathscr{F}'$ in $\mathrm{Qcoh}(X)$ such that the diagram

$$
\begin{tikzcd}
p_2^*\mathscr{F} \arrow[r, "p_2^*\phi"] \arrow[d, "\vartheta"] & p_2^*\mathscr{F}' \arrow[d, "\vartheta'"] \\
\sigma^*\mathscr{F} \arrow[r, "\sigma_2^*\phi"] & \sigma^*\mathscr{F}'
\end{tikzcd}
$$

commutes in $\mathrm{Qcoh}(G \times X)$. Let $\mathrm{Qcoh}^G(X)$ denote the category of G-equivariant quasicoherent sheaves.

Remark 6.2. *We have*

$$\mathrm{Hom}_{\mathrm{Qcoh}^G(X)}((\mathscr{F},\vartheta),(\mathscr{F}',\vartheta')) = \mathrm{Hom}_X(\mathscr{F},\mathscr{F}')^G.$$

Note also that there is an equivalence of abelian categories $\mathrm{Qcoh}^G(X) \cong \mathrm{Qcoh}([X/G])$.

Now $\mathrm{Qcoh}^G(X)$ is a $\mathbb{C}$-linear Grothendieck abelian category and its derived category will be denoted $\mathbf{D}(\mathrm{Qcoh}^G(X))$. Every object in $\mathbf{D}(\mathrm{Qcoh}^G(X))$ has a K-injective resolution and a K-flat resolution.

6.2 Brief of Equivariant Intersection Theory

We first define the equivariant Chow groups follows [EG98]. We will see that this definition of a suitable approximation of $\mathbf{E}G \times^G X$ in the definition of equivariant cohomology. Here all spaces and groups are quasi-separated and of finite type over an algebraically closed field k. We will working over algebraic spaces which is naturally appear in the quotient.

Definition 6.3 (Equivariant Chow Groups)**.** *Let G be a smooth affine algebraic group over k of dimension g, and let X be an n-dimensional algebraic space over k. For each i, choose an r-dimensional G-representation V such that there is a nonempty open subscheme $U \subset \mathbb{A}(V)$ such that*

- *(a) G acts freely on U.*
- *(b) The quotient U/G is a scheme (see Lemma 9 in [EG98]).*
- *(c)* $\mathrm{codim}(\mathbb{A}(V) \backslash U) > n - i$.

Such representations exist. The we define the i-th equivariant Chow group of *X is*

$$\mathsf{CH}_i^G(X) := \mathsf{CH}_{i+r-g}(X \times^G U).$$

Note that this group is independent of the representation, see Definition-Proposition 1 in [EG98].

Proposition 6.4. *Let $f : X \to Y$ is G-equivariant, then if f is proper, flat, smooth, regular embedding or lci, then so is $f^G : X \times^G U \to Y \times^G U$. Moreover, wquivariant Chow groups have the same functoriality, such as proper push forward, flat pullback, refined Gysin pullback of regular embedding or lci maps, as ordinary Chow groups for equivariant morphisms with the corresponding properties.*

Proof. See Proposition 2 and 3 in [EG98]. □

Proposition 6.5. *If $\alpha \in \mathsf{CH}^G_m(X)$, then there exists a representation V such that*

$$\alpha = \sum_i a_i [S_i]_G,$$

where S_i are $m+l$ invariant subvarieties of $X \times V$, where $l = \dim V$.

Proof. See Proposition 1 in [EG98]. $\square$

Definition 6.6. *Let X be an algebraic space with a G-action, and let E be an equivariant vector bundle (in the category of algebraic spaces) Consider the quotient $E_G := E \times^G U$. Then $E_G \to X_G := X \times^G U$ is also a vector bundle. Now we define the* equivariant Chern classes *as*

$$c_j^G(E) \cap (-) : \mathsf{CH}^G_i(X) \to \mathsf{CH}^G_{i-j}(X), \quad \alpha \mapsto c_j(E_G) \cap \alpha.$$

6.3 Equivariant Virtual Class and Localization Formula

In this section, we let X be a separated algebraic scheme over $\mathbb{C}$ with an action by an algebraic group G.

Definition 6.7. *A* G-equivariant perfect obstruction theory *$\phi : E^\bullet \to \mathbb{L}^\bullet_X$ is a perfect obstruction theory which can be lifted to $\mathbf{D}(\mathrm{QCoh}^G(X))$. The G-structure on $\mathbb{L}^\bullet_X$ is induced by the G-action on X.*

Let X equipped with a G-equivariant perfect obstruction theory, and carrying a G-equivariant closed embedding $X \hookrightarrow Y$ where Y is a nonsingular scheme with ideal $\mathscr{I}$ (for the general case is similar and we can glue them up). Then

$$\mathbb{L}^\bullet_X = [\mathscr{I}/\mathscr{I}^2 \to \Omega_Y|_X] \in \mathbf{D}^{[-1,0]}(\mathrm{QCoh}^G(X)).$$

Let the G-equivariant perfect obstruction theory $\phi : E^\bullet \to \mathbb{L}^\bullet_X$ with global resolution, hence we can let $E^\bullet = [E^{-1} \to E^0]$ with locally free sheaves E^i. Then as by the construction of virtual class before the cones we need are all G-equivariant! Hence we can get

$$[X]^{\mathrm{vir},G}_\phi \in \mathsf{CH}^G_{d^{\mathrm{vir}}_\phi(X)}(X),$$

the G-equivariant virtual fundamental class of X respect to ϕ.

Now we consider the virtual localization formula. Now we set $G = \mathbb{G}_m = \mathbb{C}^\times$. The same formula holds for an arbitrary torus $\mathbb{T} = \mathbb{G}_m^r$.

Let $X^{\mathsf{fix}} := X^{\mathbb{G}_m} \subset X$ and $Y^{\mathsf{fix}} := Y^{\mathbb{G}_m} \subset Y$. Hence $X^{\mathsf{fix}} = X \cap Y^{\mathsf{fix}}$. Then by Theorem 5.9(c), Y^{fix} is smooth. Let $Y^{\mathsf{fix}} = \coprod_i Y_i$ the decomposition of irreducible components and let $X_i = X \cap Y_i$.

Let $\mathscr{S} \in \mathrm{Coh}^{\mathbb{G}_m}(X_i)$, then $\mathbb{G}_m$-action induce a decomposition $\mathscr{S} = \bigoplus_{k\in\mathbb{Z}} \mathscr{S}^k$ where we are identitying the character group of $\mathbb{G}_m$ with $\mathbb{Z}$. Define the fixed part is $\mathscr{S}^{\mathsf{fix}} := \mathscr{S}^0$ and moving part is $\mathscr{S}^{\mathsf{mov}} := \bigoplus_{k\neq 0} \mathscr{S}^k$. Of course, the construction of fixed and moving part of a sheaf extends to complexes in $\mathbf{D}(\mathrm{Coh}^{\mathbb{G}_m} X_i)$.

Lemma 6.8. *There is an identity*

$$\Omega_X|_{X_i}^{\mathsf{fix}} = \Omega_{X_i}.$$

Proof. Note that we have

$$\Omega_Y|_{Y_i}^{\mathsf{fix}} = \Omega_{Y_i}.$$

Then this follows from $X_i = X \cap Y_i$. $\square$

Lemma 6.9. *Let $E_i^\bullet := E^\bullet|_{X_i}$ on X_i, define*

$$\psi_i : E_i^{\bullet,\mathsf{fix}} \xrightarrow{\phi_i^{\mathsf{fix}}} (\mathbb{L}_X^\bullet|_{X_i})^{\mathsf{fix}} \xrightarrow{\delta_i^{\mathsf{fix}}} \mathbb{L}_{X_i}^{\bullet,\mathsf{fix}} = \mathbb{L}_{X_i}^\bullet.$$

Then ψ_i is a perfect obstruction theory on X_i. This is often called the $\mathbb{G}_m$-fixed obstruction theory on X_i.

Proof. Note that as $E_i^{\bullet,\mathsf{fix}}$ is perfect in $[-1,0]$, we just need to show that $H^0(\phi_i^{\mathsf{fix}})$ and $H^0(\delta_i^{\mathsf{fix}})$ are isomorphisms and $H^1(\phi_i^{\mathsf{fix}})$ and $H^1(\delta_i^{\mathsf{fix}})$ are surjective.

By some simple diagram chase, we know that $v : A^\bullet \to B^\bullet$ satisfies these two conditions if and only if $A^{-1} \oplus B^{-2} \to A^0 \oplus B^{-1} \to B^0 \to 0$ is exact! Hence since $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ is an obstruction theory, and because the restriction $(-)|_{X_i}$ is right exact, this ensures that ϕ_i satisfies both conditions. Since taking invariants is exact (as $\mathbb{G}_m$ reductive), the same holds for ϕ_i^{fix}!

Now we consider δ_i^{fix}. Now as Y_i smooth, we have

$$\mathbb{L}_{X_i}^\bullet = [\mathscr{I}_{X_i/Y_i}/\mathscr{I}_{X_i/Y_i}^2 \to \Omega_{Y_i}|_{X_i}].$$

As $H^0(\delta_i^{\mathsf{fix}}) : H^0((\mathbb{L}_X^\bullet|_{X_i})^{\mathsf{fix}}) = (\Omega_X|_{X_i})^{\mathsf{fix}} = \Omega_{X_i} = H^0(\mathbb{L}_{X_i}^\bullet)$ by Lemma 6.8, hence $H^0(\delta_i^{\mathsf{fix}})$ is an isomorphism.

Consider $H^{-1}(\delta_i^{\mathsf{fix}})$, it is the following diagram:

$$\begin{tikzcd}
(\mathscr{I}_{X/Y}/\mathscr{I}_{X/Y}^2)|_{X_i}^{\mathsf{fix}} \arrow[r] \arrow[d, two heads, "d^{-1}"] & \Omega_Y|_{X_i}^{\mathsf{fix}} \arrow[d, "{d^0,\cong}"] \\
\mathscr{I}_{X_i/Y_i}/\mathscr{I}_{X_i/Y_i}^2 \arrow[r] & \Omega_{Y_i}|_{X_i}
\end{tikzcd}$$

where d^{-1} is surjective follows from $X_i = X \cap Y_i$. Moreover d^0 is an isomorphism since $\Omega_Y|_{Y_i}^{\mathsf{fix}} = \Omega_{Y_i}$. This shows $H^{-1}(\delta_i^{\mathsf{fix}})$ is surjective. Well done. $\square$

Finally we give the statement of the virtual localization formula.

Definition 6.10. *The virtual normal bundle to X_i is the moving part of the derived dual:*

$$N_i^{\mathrm{vir}} := E_i^{\bullet,\vee,\mathsf{mov}} \in \mathbf{D}^{[0,1]}(\mathrm{Coh}^{\mathbb{G}_m}(X)).$$

Theorem 6.11 (Virtual Localization Formula, Graber-Pandharipande)**.** *Let $\iota : X^{\mathsf{fix}} \hookrightarrow X$ be the inclusion, then*

$$[X]_\phi^{\mathrm{vir},\mathbb{G}_m} = \iota_* \sum_i \frac{[X_i]_{\psi_i}^{\mathrm{vir},\mathbb{G}_m}}{e(N_i^{\mathrm{vir}})} \in \mathsf{CH}_*^{\mathbb{G}_m}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t]$$

where t is the generator of $\mathsf{CH}_*^{\mathbb{G}_m}(\mathsf{pt}) = \mathsf{CH}^*(\mathbf{B}\mathbb{G}_m) = \mathbb{Q}[t]$.

Remark 6.12. *The Euler class of $[B^0 \to B^1]$ is $e(B^0)/e(B^1)$.*

6.4 Basic Case of Virtual Localization Formula

We first prove the simple but fundamental case:

Example 6.13. *Consider Y be a smooth variety of dimension d with $\mathbb{G}_m$-action and with a vector bundle $F = \underline{\mathrm{Spec}}_Y \mathrm{Sym}\, \mathscr{F}^\vee$ over it of rank r. Let $s : Y \to F$ be a $\mathbb{G}_m$-invariant section (that is, $s \in H^0(Y, \mathscr{F})^{\mathbb{G}_m}$) and $0 : Y \to F$ be the zero section, then we consider the zero locus of s as:*

$$\begin{tikzcd} X = Z(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, hook, "s"'] \\ Y \arrow[r, hook, "0"] & F \end{tikzcd}$$

Note that $X = Z(s)$ defined by the ideal $\mathscr{I} := \mathrm{Im}(s^\vee : \mathscr{F}^\vee \to \mathscr{O}_Y)$. Now the perfect obstruction theory is $\phi : E^\bullet \to \mathbb{L}_X^\bullet$

$$\begin{tikzcd} E^\bullet : & \mathscr{F}^\vee|_X \arrow[r] \arrow[d, "s^\vee|_X"] & \Omega_Y|_X \arrow[d, "\mathrm{id}"] \\ \mathbb{L}_X^\bullet : & \mathscr{I}/\mathscr{I}^2 \arrow[r, "d_X"] & \Omega_Y|_X \end{tikzcd}$$

which is naturally $\mathbb{G}_m$-equivariant. In this case the $\mathbb{G}_m$-equivariant virtual fundamental class of X is

$$[X]_\phi^{\mathrm{vir},\mathbb{G}_m} := 0^!([Y]) = 0^*(C_{X/Y}) \in \mathsf{CH}_*^{\mathbb{G}_m}(X).$$

Again we have $j_i : Y_i \hookrightarrow Y$ be the components of fixed locus Y^{fix} and $\iota_i : X_i = Y_i \cap X \hookrightarrow X$. Hence $F|_{Y_i} = F|_{Y_i}^{\mathsf{fix}} \oplus F|_{Y_i}^{\mathsf{mov}}$. Let $s_i := s|_{Y_i}$ be the *section of $F|_{Y_i}$ which is also $\mathbb{G}_m$-invariant, this become a section $\tilde{s}_i$ of $F|_{Y_i}^{\mathsf{fix}}$! Now $X_i = Z(s_i) = Z(\tilde{s}_i)$.*

We have defined the perfect obstruction theory on X_i as

$$\begin{tikzcd} E_i^{\bullet,\mathsf{fix}}: \arrow[d, "\psi_i"] & \mathscr{F}^\vee|_{X_i}^{\mathsf{fix}} \arrow[r] \arrow[d, "s^\vee|_X"] & \Omega_Y|_{X_i}^{\mathsf{fix}} = \Omega_{Y_i}|_{X_i} \arrow[d, "\mathrm{id}"] \\ \mathbb{L}_{X_i}^\bullet: & \mathscr{I}_{X_i/Y_i}/\mathscr{I}_{X_i/Y_i}^2 \arrow[r, "d_X"] & \Omega_{Y_i}|_{X_i} \end{tikzcd}$$

where $E_i^\bullet := E^\bullet|_{X_i}$.

Now we begin the prove the virtual localization formula in this basic situation.

Lemma 6.14. *In the situation as above.*

(a) *We have $0_i^![Y_i] = (0_i^{\mathsf{fix}})^![Y_i] \cap e(F_i^{\mathsf{mov}}) \in \mathsf{CH}_*^{\mathbb{G}_m}(X_i)$ where 0_i and 0_i^{fix} be the zero sections of $F|_{Y_i}$ and $F|_{Y_i}^{\mathsf{fix}}$, respectively.*

(b) *The K-theory virtual normal normal bundle to X_i has the following expression:*

$$[N_i^{\mathrm{vir}}] = \varepsilon_i^*([N_{Y_i/Y}] - [\mathscr{F}_i^{\mathsf{mov}}]) \in K_0^{\mathbb{G}_m}(X_i).$$

Proof. For (a), we have the cartesians

$$\begin{tikzcd} X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y_i \arrow[d, hook, "\tilde{s}_i"] \\ Y_i \arrow[r, hook, "0_i^{\mathsf{fix}}"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\ulcorner"] & F|_{Y_i}^{\mathsf{fix}} \arrow[d, hook] \\ Y_i \arrow[r, hook, "0_i"] & F|_{Y_i} \end{tikzcd}$$

Hence by excess intersection formula we have

$$0_i^![Y_i] = (0_i^{\mathsf{fix}})^![Y_i] \cap e(F_i^{\mathsf{mov}}) \in \mathsf{CH}_*^{\mathbb{G}_m}(X_i).$$

For (b), we have

$$\begin{aligned} [N_i^{\mathrm{vir}}] &= [E_i^\bullet] - [E_i^{\bullet,\mathsf{mov}}] \\ &= [T_Y|_{X_i}] - [\mathscr{F}|_{X_i}] - [T_{Y_i}|_{X_i}] + [\mathscr{F}|_{X_i}^{\mathsf{fix}}] \\ &= ([T_Y] - [T_{Y_i}])|_{X_i} - [\mathscr{F}|_{X_i}^{\mathsf{mov}}] \\ &= \varepsilon_i^*([N_{Y_i/Y}] - [\mathscr{F}_i^{\mathsf{mov}}]) \in K_0^{\mathbb{G}_m}(X_i). \end{aligned}$$

Well done. $\square$

Proof of Theorem 6.11 for basic case. Now by Lemma 6.14(b) we have

$$e^{\mathbb{G}_m}(N_i^{\mathrm{vir}}) = \varepsilon_i^* \left(\frac{e^{\mathbb{G}_m}(N_{Y_i/Y})}{e(\mathscr{F}_i^{\mathsf{mov}})} \right).$$

By the Chow-version of Atiyah-Bott theorem we have

$$[Y] = \sum_i j_{i,*} \frac{[Y_i]}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \in \mathsf{CH}_*^{\mathbb{G}_m}(Y) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].$$

So we by Lemma 6.14(a) and these things we have

$$\begin{aligned}
[X]_\phi^{\mathrm{vir},\mathbb{G}_m} &= 0^![Y] = \sum_i 0^! \left(j_{i,*} \frac{[Y_i]}{e^{\mathbb{G}_m}(N_{Y_i/Y})} \right) \\
&= \sum_i \iota_{i,*} 0_i^! \frac{[Y_i]}{e^{\mathbb{G}_m}(N_{Y_i/Y})} = \sum_i \iota_{i,*} \frac{(0_i^{\mathsf{fix}})^![Y_i] \cap e(\mathscr{F}_i^{\mathsf{mov}})}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}}) e(\mathscr{F}_i^{\mathsf{mov}})} \\
&= \sum_i \iota_{i,*} \frac{(0_i^{\mathsf{fix}})^![Y_i]}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})} = \sum_i \iota_{i,*} \frac{[X_i]_{\psi_i}^{\mathrm{vir}}}{e^{\mathbb{G}_m}(N_i^{\mathrm{vir}})} \in \mathsf{CH}_*^{\mathbb{G}_m}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].
\end{aligned}$$

Well done. □

6.5 General Case of Virtual Localization Formula

To prove the general case where $X \hookrightarrow Y$ be an embedding, we recall the construction of virtual fundamental class in our case. Now pick a ($\mathbb{G}_m$-equivariant) obstruction theory $\phi : E^\bullet \to \mathbb{L}_X^\bullet$ with global resolution.

As in the proof of Proposition 4.37(b), we have the following commutative diagram with exact rows of cones

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{L}_X^{\vee,0} = f^*T_Y \arrow[r] \arrow[d, "\mathrm{id}"] & E^{\vee,0} \times_X C_{X/Y} \arrow[r] \arrow[d, hook] & D^{\mathrm{vir}} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{L}_X^{\vee,0} = f^*T_Y \arrow[r] & E^{\vee,0} \times_X C(\mathscr{I}_{X/Y}/\mathscr{I}_{X/Y}^2) \arrow[r] & C(Q) \arrow[r] & 0
\end{tikzcd}$$

with two embeddings of the right square. By Proposition 4.20 we have $\mathfrak{C}_X = [C_{X/Y}/f^*T_Y] \cong [D^{\mathrm{vir}}/E^{\vee,0}]$ which force

$$\begin{tikzcd}
D^{\mathrm{vir}} \arrow[r, hook] \arrow[d] & C(Q) \arrow[r, hook] \arrow[d] & E^{-1,\vee} \arrow[d] \\
\mathfrak{C}_X \cong [D^{\mathrm{vir}}/E^{\vee,0}] \arrow[r, hook] & {[C(\mathscr{I}_{X/Y}/\mathscr{I}_{X/Y}^2)/f^*T_Y] \cong [C(Q)/E^{\vee,0}]} \arrow[r, hook] & {[E^{-1,\vee}/E^{\vee,0}]}
\end{tikzcd}$$

Hence $[X]^{\mathrm{vir}}_{\phi} = 0^!_{E^{-1,\vee}}[D^{\mathrm{vir}}]$.

Here is another description, by Proposition 4.10 we have cartesians

$$\begin{tikzcd}
f^*T_Y \arrow[r, hook] \arrow[d, "p"] \arrow[dr, phantom, "\ulcorner"] & E^{0,\vee} \times_X C_{X/Y} \arrow[d, "q"] \\
X \arrow[r, hook] \arrow[d, "\mathrm{id}_X"] \arrow[dr, phantom, "\ulcorner"] & D^{\mathrm{vir}} \arrow[d, hook] \\
X \arrow[r, hook] & E^{-1,\vee}
\end{tikzcd}$$

Then we have

$$\begin{aligned}
[X]^{\mathrm{vir}}_{\phi} &= 0^*_{f^*T_Y} p^* 0^!_{E^{-1,\vee}}[D^{\mathrm{vir}}] = 0^*_{f^*T_Y} 0^!_{E^{-1,\vee}} q^*[D^{\mathrm{vir}}] \\
&= 0^*_{f^*T_Y} 0^!_{E^{-1,\vee}}[E^{0,\vee} \times_X C_{X/Y}].
\end{aligned}$$

Now we have $X^{\mathsf{fix}} = X \cap Y^{\mathsf{fix}}$ and $Y^{\mathsf{fix}} = \coprod_i Y_i$ the decomposition of irreducible components and let $X_i = X \cap Y_i$. Here we need some notations for convenience:

$$\begin{tikzcd}
 & X_i \arrow[r, hook, "\varepsilon_i"] \arrow[dl, hook] \arrow[d, hook, "\iota_i"'] & Y_i \arrow[d, hook, "j_i"'] \arrow[dr, hook] & \\
X^{\mathsf{fix}} \arrow[r, hook, "\iota"] & X \arrow[r, hook, "f"] & Y & Y^{\mathsf{fix}} \arrow[l, hook', "j"']
\end{tikzcd}$$

Now we have the perfect obstruction theory

$$\psi_i : E_i^{\bullet,\mathsf{fix}} \xrightarrow{\phi_i^{\mathsf{fix}}} (\mathbb{L}^{\bullet}_X|_{X_i})^{\mathsf{fix}} \xrightarrow{\delta_i^{\mathsf{fix}}} \mathbb{L}^{\bullet,\mathsf{fix}}_{X_i} = \mathbb{L}^{\bullet}_{X_i}$$

of X_i where $E_i^{\bullet} := E^{\bullet}|_{X_i}$ as before. Since we consider all $\mathbb{G}_m$-fixed cones, the constructions as above is right for our case, that is, we have exact sequence of cones

$$0 \to \varepsilon_i^* T_{Y_i} \to C_{X_i/Y_i} \times_{X_i} (E_i^{0,\vee})^{\mathsf{fix}} \to D_i^{\mathrm{vir}} \to 0$$

with embedding $D_i^{\mathrm{vir}} \hookrightarrow (E_i^{-1,\vee})^{\mathsf{fix}}$ with $[X_i]^{\mathrm{vir}}_{\psi_i} = 0^!_{(E_i^{-1,\vee})^{\mathsf{fix}}}[D_i^{\mathrm{vir}}]$.

Remark 6.15. *Since X_i is possibly disconnected, it should be noted that the ranks of the bundles $(E_i^{0,\vee})^{\mathsf{fix}}$ and $(E_i^{-1,\vee})^{\mathsf{fix}}$ may vary on the connected components. The Euler classes of these bundles on X_i are taken with respect to their ranks on each component.*

Now apply the localization formula to Y, we get

$$[Y] = j_* \sum_i \frac{[Y_i]}{e(T_Y^{\mathsf{mov}})} \in \mathsf{CH}_*^{\mathbb{G}_m}(Y) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].$$

Hence by refined intersection with $[X]^{\mathrm{vir}}_{\phi}$, we have

$$[X]^{\mathrm{vir}}_{\phi} = \iota_* \sum_i \frac{[X]^{\mathrm{vir}}_{\phi} \cdot [Y_i]}{e(f^* T_Y^{\mathsf{mov}})} \in \mathsf{CH}^{\mathbb{G}_m}_*(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].$$

So to show Theorem 6.11, we just need to show that

$$\frac{[X]^{\mathrm{vir}}_{\phi} \cdot [Y_i]}{e(f^* T_Y^{\mathsf{mov}})} = \frac{[X_i]^{\mathrm{vir}}_{\psi_i} \cap e((E_i^{-1,\vee})^{\mathsf{mov}})}{e((E_i^{0,\vee})^{\mathsf{mov}})} \in \mathsf{CH}^{\mathbb{G}_m}_*(X_i) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].$$

Lemma 6.16. *In this case, we have the following:*

(a) Write $D = C_{X/Y} \times_X E^{0,\vee}$ and $D_i = C_{X_i/Y_i} \times_{X_i} (E_i^{0,\vee})^{\mathsf{fix}}$, then we have

$$j_i^![D] = [D_i \times_X (E_i^{0,\vee})^{\mathsf{mov}}] \in \mathsf{CH}^{\mathbb{G}_m}_*(j_i^* D).$$

(b) Let B_0 and B_1 are $\mathbb{G}_m$-equivariant bundles on X_i with equivariant inclusions:

$$\begin{tikzcd} Z \arrow[r, hook, "j_1"] \arrow[d, hook, "j_0"] & B_1 \arrow[d] \\ B_0 \arrow[r] & X_i \end{tikzcd}$$

Then for any $\zeta \in \mathsf{CH}^{\mathbb{G}_m}_*(Z)$, we have

$$0^*_{B_0} j_{0,*}(\zeta) \cap e^{\mathbb{G}_m}(B_1) = 0^*_{B_1} j_{1,*}(\zeta) \cap e^{\mathbb{G}_m}(B_0) \in \mathsf{CH}^{\mathbb{G}_m}_*(X_i).$$

Proof. For (a), by $\mathbb{G}_m$-equivariant Vistoli's rational equivalence (Corollary 4.58) we have $j_i^![C_{X/Y}] = [C_{X_i/Y_i}] \in \mathsf{CH}^{\mathbb{G}_m}_*(\iota_i^* C_{X/Y})$. Then via this condition we can get

$$j_i^![D] = [D_i \times_X (E_i^{0,\vee})^{\mathsf{mov}}] \in \mathsf{CH}^{\mathbb{G}_m}_*(j_i^* D).$$

For (b), consider the family of inclusions $j_t : Z \hookrightarrow B_0 \times_X B_1$ as $j_t = (1-t)j_0 + t j_1$. This shows

$$0^*_{B_0 \times_X B_1} j_{0,*}(\zeta) = 0^*_{B_0 \times_X B_1} j_{1,*}(\zeta).$$

Hence the results follows from Proposition 2.4(d). □

Proof of Theorem 6.11. As we have told, we just need to show that

$$\frac{[X]^{\mathrm{vir}}_{\phi} \cdot [Y_i]}{e(f^* T_Y^{\mathsf{mov}})} = \frac{[X_i]^{\mathrm{vir}}_{\psi_i} \cap e((E_i^{-1,\vee})^{\mathsf{mov}})}{e((E_i^{0,\vee})^{\mathsf{mov}})} \in \mathsf{CH}^{\mathbb{G}_m}_*(X_i) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, 1/t].$$

First, by Lemma 6.16(a) we have

$$\begin{aligned}[X]^{\mathrm{vir}}_\phi \cdot [Y_i] &= j_i^! 0^*_{f^*T_Y} 0^!_{E^{-1,\vee}}[D] = 0^*_{\iota_i^* f^* T_Y} 0^!_{E_i^{-1,\vee}} j_i^! [D] \\ &= 0^*_{\iota_i^* f^* T_Y} 0^!_{E_i^{-1,\vee}} [D_i \times_X (E_i^{0,\vee})^{\mathsf{mov}}] \in \mathsf{CH}^{\mathbb{G}_m}_*(X_i)\end{aligned}$$

Easy to see that we have cartesians

$$\begin{tikzcd} \iota_i^* f^* T_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & j_i^* D \arrow[d] \\ (\iota_i^* f^* T_Y)/(\varepsilon_i^* T_{Y_i}) = (\iota_i^* f^* T_Y)^{\mathsf{mov}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (j_i^* D)/(\varepsilon_i^* T_{Y_i}) \arrow[d] \\ X_i \arrow[r, hook, "0_{E_i^{-1,\vee}}"] & E_i^{-1,\vee} \end{tikzcd}$$

Hence

$$0^*_{\iota_i^* f^* T_Y} 0^!_{E_i^{-1,\vee}} [D_i \times_X (E_i^{0,\vee})^{\mathsf{mov}}] = 0^*_{(\iota_i^* f^* T_Y)^{\mathsf{mov}}} 0^!_{E_i^{-1,\vee}} [D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}].$$

Now the scheme-theoretic intersection $0^{-1}_{E_i^{-1,\vee}}(D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}) \subset (\iota_i^* f^* T_Y)^{\mathsf{mov}}$ and the morphism $D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}} \to E_i^{-1,\vee}$ is the product of $D_i^{\mathrm{vir}} \subset (E_i^{-1,\vee})^{\mathsf{fix}}$ and $(E_i^{0,\vee})^{\mathsf{mov}} \to (E_i^{-1,\vee})^{\mathsf{mov}}$. Hence $0^{-1}_{E_i^{-1,\vee}}(D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}})$ also lies in $(E_i^{0,\vee})^{\mathsf{mov}}$. Hence we have

$$\begin{tikzcd} 0^{-1}_{E_i^{-1,\vee}}(D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}) \arrow[r, hook] \arrow[d, hook] & (E_i^{0,\vee})^{\mathsf{mov}} \arrow[d] \\ (\iota_i^* f^* T_Y)^{\mathsf{mov}} \arrow[r] & X_i \end{tikzcd}$$

commutes! By Lemma 6.16(b) to $0^!_{E_i^{-1,\vee}}[D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}]$ we have

$$[X]^{\mathrm{vir}}_\phi \cdot [Y_i] = 0^!_{(E_i^{0,\vee})^{\mathsf{mov}}} 0^!_{E_i^{-1,\vee}} [D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}] \cdot \frac{e((\iota_i^* f^* T_Y)^{\mathsf{mov}})}{e((E_i^{0,\vee})^{\mathsf{mov}})},$$

where $0^!_{E_i^{-1,\vee}}[D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}]$ considered in $\mathsf{CH}^{\mathbb{G}_m}_*((E_i^{0,\vee})^{\mathsf{mov}})$.

As this class does not depend on the bundle map $(E_i^{0,\vee})^{\mathsf{mov}} \to (E_i^{-1,\vee})^{\mathsf{mov}}$, we may assume it is trivial (why???)! Then by the following cartesian

$$\begin{tikzcd} (E_i^{0,\vee})^{\mathsf{mov}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & D_i^{\mathrm{vir}} \times (E_i^{0,\vee})^{\mathsf{mov}} \arrow[d] \\ X_i \arrow[r, hook] & (E_i^{-1,\vee})^{\mathsf{fix}} \times (E_i^{-1,\vee})^{\mathsf{mov}} \end{tikzcd}$$

we have $0^!_{E_i^{-1,\vee}}[D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}] = [X_i]^{\mathrm{vir}}_{\psi_i} \times_X 0^!_{(E_i^{-1,\vee})^{\mathsf{mov}}}((E_i^{0,\vee})^{\mathsf{mov}})$. Moreover as map is trivial we also have the following cartesian

$$\begin{tikzcd}
(E_i^{0,\vee})^{\mathsf{mov}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (E_i^{0,\vee})^{\mathsf{mov}} \arrow[d] \\
X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_i \arrow[d] \\
X_i \arrow[r, hook] & (E_i^{-1,\vee})^{\mathsf{mov}}
\end{tikzcd}$$

Hence by excess intersection theorem, Proposition 2.4(d), we have

$$0^!_{(E_i^{0,\vee})^{\mathsf{mov}}} 0^!_{E_i^{-1,\vee}}[D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}] = [X_i]^{\mathrm{vir}}_{\psi_i} \cap e((E_i^{-1,\vee})^{\mathsf{mov}}).$$

This give us

$$\begin{aligned}
\frac{[X]^{\mathrm{vir}}_{\phi} \cdot [Y_i]}{e(f^*T_Y^{\mathsf{mov}})} &= \frac{0^!_{(E_i^{0,\vee})^{\mathsf{mov}}} 0^!_{E_i^{-1,\vee}}[D_i^{\mathrm{vir}} \times_X (E_i^{0,\vee})^{\mathsf{mov}}] \cdot e((\iota_i^* f^* T_Y)^{\mathsf{mov}})}{e(f^*T_Y^{\mathsf{mov}}) e((E_i^{0,\vee})^{\mathsf{mov}})} \\
&= \frac{[X_i]^{\mathrm{vir}}_{\psi_i} \cap e((E_i^{-1,\vee})^{\mathsf{mov}})}{e((E_i^{0,\vee})^{\mathsf{mov}})}.
\end{aligned}$$

This is what we want. Well done. □

6.6 Localization Formula for DM-stacks

6.7 Need to add

7 Cosection Localization Principle

8 Torus Localization for Cosection Localized Virtual Cycles

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