Kuznetsov components, Stability, and Moduli Spaces

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Contents

1 Derived Category and Kuznetsov Components **7**

1.1 Basic Definitions 7

1.1.1 Exceptional Sequences and S.O.Ds 7

1.1.2 Example I – Projective Bundles 8

1.1.3 Example II – Blow-Ups 10

1.2 Kuznetsov Components 14

2 Examples of Fano Manifolds of Calabi-Yau Type **15**

3 Examples of Derived Equivalences of Kuznetsov Components with K3s **17**

4 Stability Conditions on K3 Categories **19**

5 Applications: Mukai's program **21**

6 Application to Cubic Fourfolds and Gushel-Mukai Manifolds **23**

Index **25**

Bibliography **27**

Preface

[1][2]

Chapter 1

Derived Category and Kuznetsov Components

1.1 Basic Definitions

Here we follows some definitions and results in [3]. Note that when I working in the derived category, we will omit the **R** or **L** of the derived functors.

1.1.1 Exceptional Sequences and S.O.Ds

Definition 1.1.1. *A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called* admissible *if the inclusion has a right adjoint $\pi : \mathscr{D} \to \mathscr{D}'$*

The orthogonal complement *of a(an admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory ${\mathscr{D}'}^{\perp}$ of all objects $C \in \mathscr{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{D}'$.*

Definition 1.1.2. *An object $E \in \mathscr{D}$ in a k-linear triangulated category $\mathscr{D}$ is called* exceptional *if*

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k, \text{if } \ell = 0, \\ 0, \text{if } \ell \neq 0. \end{cases}$$

An exceptional sequence *is a sequence $E_1, ..., E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.*

An exceptional sequence is full *if $\mathscr{D}$ is generated by $\{E_i\}$.*

An exceptional collection $E_1, ..., E_n$ is strong *if in addition $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \neq 0$.*

Definition 1.1.3. *A sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is* semi-orthogonal *if for all $i > j$ we have $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$. Such a sequence defines a* semi-orthogonal decomposition *(S.O.D.) of $\mathscr{D}$ if $\mathscr{D}$ is generated by the $\mathscr{D}_i$.*

Remark 1.1.4. *Some remarks:*

(a) *If $E \in \mathscr{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle \subset \mathscr{D}$.*

(b) *Let $E_1, ..., E_n$ be an exceptional sequence in $\mathscr{D}$. Then the admissible triangulated subcategories $\langle E_1 \rangle, ..., \langle E_n \rangle$ form a semi-orthogonal sequence. If this sequence is a full exceptional sequence, then this forms an S.O.D. of $\mathscr{D}$.*

(c) *Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ defines an S.O.D. of $\mathscr{D}$, if and only if any object $A \in \mathscr{D}$ with $A \in \mathscr{D}_i^{\perp}$ for all $i = 1, ..., n$ is trivial. See Lemma 1.61 in [3].*

(d) *If $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is an S.O.D., then $D_1 \subset \langle \mathscr{D}_2, ..., \mathscr{D}_n \rangle^{\perp}$ is an equivalence. See Exercise 1.62 in [3].*

1.1.2 Example I – Projective Bundles

Proposition 1.1.5. *For a smooth projective variety Y we consider the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to Y$ of locally free sheaf $\mathscr{E}$ of rank r on Y, in the sense of Grothendieck. Then for any $a \in \mathbb{Z}$ we claim that $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a+r-1)$ is an S.O.D. of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.*

This combined by the following two things:

Step 1. For any $E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(m), F \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(n)$, we have $\mathrm{Hom}(E, F) = 0$ for any $r - 1 \geq m - n > 0$.

Indeed, we can let $m = 0$ and hence $-r + 1 \leq n < 0$. Let $E = \pi^*E'$ and $F = \pi^*F' \otimes \mathscr{O}(n)$, hence

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(E', \pi_*(\pi^*F' \otimes \mathscr{O}(n))) = \mathrm{Hom}(E', F' \otimes \pi_*\mathscr{O}(n)).$$

It's well-known that $\mathbf{R}^i\pi_*\mathscr{O}(n) = \begin{cases} \mathrm{Sym}^n\mathscr{E}, \text{for } i = 0, \\ 0, \text{for } 0 < i < r-1, \text{ Well done.} \\ \mathrm{Sym}^{-n-r}\mathscr{E}^{\vee}, \text{for } i = r-1. \end{cases}$

Step 2. Categories $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a+r-1)$ generates $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.

Here we generalize the proof for $\mathbb{P}^n$ in [3] Corollary 8.29. Consider

$$\begin{tikzcd}
 & \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}) \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\wedge" near start] & \\
\mathbb{P}(\mathscr{E}) \arrow[dr, "\pi_1"] & & \mathbb{P}(\mathscr{E}) \arrow[dl, "\pi_2"'] \\
 & Y &
\end{tikzcd}$$

then by the canonical identification

$$\begin{aligned}&H^0(\mathbb{P}(\mathscr{E})\times_Y\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\boxtimes\mathscr{Q}^\vee)\\&=H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\otimes p_*q^*\mathscr{Q}^\vee)\\&=H^0(\mathbb{P}(\mathscr{E}),\mathscr{O}(1)\otimes\pi_1^*\pi_{2,*}\mathscr{Q}^\vee)\\&=H^0(Y,\pi_{1,*}\mathscr{O}(1)\otimes\pi_{2,*}\mathscr{Q}^\vee)\\&=H^0(Y,\mathscr{E}\otimes\mathscr{E}^\vee)\end{aligned}$$

where $0\to\mathscr{Q}\to\pi^*\mathscr{E}\to\mathscr{O}(1)\to 0$ is the universal exact sequence. Let s correspond to the $\mathrm{id}_{\mathscr{E}}$, then $Z(s)=\Delta\subset\mathbb{P}(\mathscr{E})\times_Y\mathbb{P}(\mathscr{E})$. By the Koszul resolution of $\mathscr{O}_\Delta$ respect to the s, we have an exact sequence:

$$\begin{aligned}&0\to\bigwedge^{r-1}(\mathscr{O}(-1)\boxtimes\mathscr{Q})\to\bigwedge^{r-2}(\mathscr{O}(-1)\boxtimes\mathscr{Q})\\&\to\cdots\to\mathscr{O}(-1)\boxtimes\mathscr{Q}\to\mathscr{O}\boxtimes\mathscr{O}\to\mathscr{O}_\Delta\to 0.\end{aligned}$$

(you can also use the Euler exact sequence instead of the universal exact sequence, just as in [3] Corollary 8.29)

Now there is to way to slove this.

The First Way: for any coherent sheaf $\mathscr{F}\in\mathrm{Coh}(\mathbb{P}(\mathscr{E}))$, tensoring $q^*\mathscr{F}$ we have

$$\begin{aligned}&0\to\mathscr{O}(-r+1)\boxtimes\bigwedge^{r-1}\mathscr{Q}\otimes\mathscr{F}\to\mathscr{O}(-r+2)\boxtimes\bigwedge^{r-2}\mathscr{Q}\otimes\mathscr{F}\\&\to\cdots\to\mathscr{O}(-1)\boxtimes(\mathscr{Q}\otimes\mathscr{F})\to\mathscr{O}\boxtimes\mathscr{F}\to q^*\mathscr{F}|_\Delta\to 0.\end{aligned}$$

Consider a spectral sequence

$$\begin{aligned}E_1^{ij}&=\mathbf{R}^ip_*(\mathscr{O}(j)\boxtimes\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F})=\mathscr{O}(j)\otimes\mathbf{R}^ip_*q^*\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F}\\&=\mathscr{O}(j)\otimes\pi_1^*\mathbf{R}^i\pi_{2,*}\bigwedge^{-j}\mathscr{Q}\otimes\mathscr{F}\Rightarrow\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta.\end{aligned}$$

We know that $\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta=0$ if $i+j\neq 0$ and $\mathbf{R}^{i+j}p_*q^*\mathscr{F}|_\Delta=\mathscr{F}$ if $i+j=0$. Since any E_1^{ij} contained in

$$\left\langle\pi^*\mathbf{D}^b(Y)\otimes\mathscr{O}(-r+1),...,\pi^*\mathbf{D}^b(Y)\otimes\mathscr{O}(0)\right\rangle,$$

so is $\mathscr{F}$. Hence well done (if you use the Euler exact sequence instead of the universal exact sequence, the similar spectral sequence called the generalized Beilinson spectral sequence as Proposition 8.28 in [3]).

The Second Way: Consider again the Koszul resolution

$$\begin{aligned} 0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \\ \to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Split it into short exact sequences

$$\begin{aligned} &0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-2} \to 0, \\ &0 \to M_{r-2} \to \bigwedge^{r-3}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-3} \to 0, \\ &\cdots, \\ &0 \to M_1 \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0. \end{aligned}$$

Tensor product with q^*F and direct image under the first projection p yields distinguished triangles of Fourier-Mukai transforms:

$$\Phi_{M_{i+1}}(\mathscr{F}) \to \Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \to \Phi_{M_i}(\mathscr{F}) \to \Phi_{M_{i+1}}(\mathscr{F})[1].$$

Easy to see that

$$\Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-i) \right\rangle.$$

By induction we get $F = \Phi_{\mathscr{O}_\Delta} F \in \langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O} \rangle$. Well done.

Fully Exceptional Sequence. By the discussed above, we know that pick any fully exceptional sequence $E_1, ..., E_n$ of Y, the set

$$\{\pi^*E_1 \otimes \mathscr{O}(a), ..., \pi^*E_n \otimes \mathscr{O}(a), ..., \pi^*E_1 \otimes \mathscr{O}(a+r-1), ..., \pi^*E_n \otimes \mathscr{O}(a+r-1)\}$$

is a fully exceptional sequence of $\mathbb{P}(\mathscr{E})$ for any $a \in \mathbb{Z}$.

Example 1.1.1. *More general case, such as Grassmann bundle and even the flag bundle has the similar things. We refer [4].*

1.1.3 Example II – Blow-Ups

Here we follows section 11.1 in [3]. First we need some results about closed immersions.

Lemma 1.1.6. *Suppose $j : Y \hookrightarrow X$ of codimension C with normal bundle $\mathscr{N}$ is the zero locus of a regular section of a locally free sheaf $\mathscr{E}$ of rank c. Then for any $F \in \mathbf{D}^b(Y)$*

there exists the following canonical isomorphisms:

$$
\begin{aligned}
&\text{(i)} j^* j_* \mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{N}^\vee[k],\\
&\text{(ii)} j_* j^* j_* F \simeq j_* \mathscr{O}_Y \otimes j_* F \simeq j_* \left(\bigoplus \bigwedge^k \mathscr{N}^\vee[k] \otimes F \right),\\
&\text{(iii)} \mathscr{H}om_X(j_* \mathscr{O}_Y, j_* F) \simeq j_* \left(\bigoplus \bigwedge^k \mathscr{N}[-k] \otimes F \right).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
\mathcal{H}^\ell(j^* j_* F) &\simeq \bigoplus_{s-r=\ell} \bigwedge^r \mathscr{N}^\vee \otimes \mathcal{H}^s(F)\\
\mathscr{E}xt^\ell_X(j_* \mathscr{O}_Y, j_* F) &\simeq j_* \left(\bigoplus_{r+s=\ell} \bigwedge^r \mathscr{N} \otimes \mathcal{H}^s(F) \right).
\end{aligned}
$$

Proof. For (i), by Koszul resolution we get $j^* j_* \mathscr{O}_Y \simeq \bigwedge^* \mathscr{E}^\vee|_Y$. As the differentials in the Koszul complex $\bigwedge^* \mathscr{E}^\vee$ are given by contraction with the defining section, they become trivial on Y. Hence $j^* j_* \mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{E}^\vee[k]|_Y$. As $\mathscr{E}|_Y \cong \mathscr{N}$, well done.

For (ii), we split the Koszul resolution into the following short exact sequences:

$$
\begin{tikzcd}
 & & & M_i \arrow[dr] & \\
\cdots \arrow[r] & \bigwedge^{i+1} \mathscr{E}^\vee \arrow[rr] \arrow[dr] & & \bigwedge^i \mathscr{E}^\vee \arrow[r] \arrow[u] & \bigwedge^{i-1} \mathscr{E}^\vee \arrow[r] & \cdots \\
 & & M_{i+1} \arrow[ur] & & &
\end{tikzcd}
$$

Again all these morphisms vanish on Y, we have

$$
M_i \otimes j_* F \simeq \left(\bigwedge^i \mathscr{E}^\vee \otimes j_* F \right) \oplus \left(M_{i+1}[1] \otimes j_* F \right).
$$

Putting these togetherand we get the result.

For (iii), as we have $\mathscr{H}om_X(j_* \mathscr{O}_Y, j_* F) \simeq \left(\bigwedge^i \mathscr{E}^\vee \right)^\vee \otimes j_* F$, then by the similar argument of (ii) we get the result.

The final part follows from (ii)(iii) and the fact that j_* is exact and tensor product with the locally free sheaf commutes with taking cohomology. $\square$

Corollary 1.1.7. *Let $j : Y \hookrightarrow X$ be a smooth hypersurface. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following distinguished triangle*

$$F \otimes \mathscr{O}_Y(-Y)[1] \to j^*j_*F \to F \to .$$

Proof. We omit it and refer [3] Corollary 11.4. ☐

Lemma 1.1.8. *Let $j : Y \hookrightarrow X$ be an arbitrary closed embedding of smooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^*j_*\mathscr{O}_Y) \simeq \bigwedge^{-i} \mathscr{N}^\vee_{Y/X}, \quad \mathscr{E}xt^i_X(j_*\mathscr{O}_Y, j_*\mathscr{O}_Y) \simeq \bigwedge^{i} \mathscr{N}_{Y/X}.$$

Proof. Here we just give an idea, the detail we refer Proposition 11.8 in [3]. Here we first pick a global resolution of locally free sheaves $\mathscr{G}^* \to \mathscr{O}_Y$ and get the free resolution $\mathscr{G}^*_y \to \mathscr{O}_{Y,y}$. Also we can let Y defined by a section of a vector bundle near y, hence we get a local Koszul resolution. Hence at the point y we can get the result from before. Easy to see that this is independent of any choice, we get the result. ☐

Proposition 1.1.9. *Let $q : \widetilde{X} \to X$ be the blow-up along a smooth subvariety $Y \subset X$. Then for the structure sheaf $\mathscr{O}_Z$ of a subvariety $Z \subset Y$ considered as an object in $\mathbf{D}^b(X)$ one has*

$$\mathcal{H}^k(q^*\mathscr{O}_Z) \simeq (\omega_\pi^{\otimes -k} \otimes \mathscr{O}_\pi(-k))|_{\pi^{-1}(Z)}$$

where $\pi : \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor.

Proof. We will only show the case that $Y \subset X$ is given as the zero set of a regular section $s \in H^0(X, \mathscr{E})$ of a locally free sheaf $\mathscr{E}$ of rank c. The general case follows from this and the similar argument of Lemma 1.1.8, we refer [3] Proposition 11.12 for details.

Consider $g : \mathbb{P}(\mathscr{E}) \to X$ and consider the Euler sequence

$$0 \to \mathscr{O}_g(-1) \to g^*\mathscr{E} \xrightarrow{\phi} \mathscr{T}_g \otimes \mathscr{O}_g(-1) \to 0.$$

Let $t := \phi(g^*(s)) \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{T}_g \otimes \mathscr{O}_g(-1))$ and consider the zero scheme $Z(t) \subset \mathbb{P}(\mathscr{E})$.

BLABLABLA

Hence g induced $Z(t) \to X$ can be identified with the blow-up $q : \widetilde{X} \to X$. Pick the Koszul resolution $\bigwedge^*(\mathscr{O}_g(1) \otimes \Omega_g) \to \mathscr{O}_{\widetilde{X}} \to 0$ of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules, hence

$$\begin{aligned} \iota_*(\mathcal{H}^k(q^*\mathscr{O}_Z)) &\simeq \iota_*(\mathcal{H}^k(\iota^*g^*\mathscr{O}_Z)) \simeq \mathcal{H}^k(\iota_*\iota^*g^*\mathscr{O}_Z) \\ &\simeq \mathcal{H}^k(g^*\mathscr{O}_Z \otimes \mathscr{O}_{\widetilde{X}}) \simeq \mathcal{H}^k(\bigwedge^*(\mathscr{O}_g(1) \otimes \Omega_g)|_{g^{-1}(Z)}) \end{aligned}$$

where $\iota : \widetilde{X} = Z(t) \hookrightarrow \mathbb{P}(\mathscr{E})$. If Z is contained in Y , the differentials, which are given by contraction with the section t, vanish and, therefore

$$\mathcal{H}^k(q^*\mathscr{O}_Z) \simeq (\omega_g^{\otimes -k} \otimes \mathscr{O}_g(-k))|_{g^{-1}(Z)}.$$

Well done. ☐

Lemma 1.1.10. *Suppose $f : S \to T$ is a projective morphism of smooth projective varieties such that $f_* : \mathbf{D}^b(S) \to \mathbf{D}^b(T)$ sends $\mathscr{O}_S$ to $\mathscr{O}_T$. Then $f^* : \mathbf{D}^b(T) \to \mathbf{D}^b(S)$ is fully faithful and thus describes an equivalence of $\mathbf{D}^b(T)$ with an admissible triangulated subcategory of $\mathbf{D}^b(S)$.*

Proof. Trivial by the projection formula and $f^* \dashv f_*$, which shows directly $\mathrm{id} \simeq f_* f^*$, hence fully faithful. □

Lemma 1.1.11. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Then the functor*

$$\Phi_k = i_*(\mathscr{O}_E(kE) \otimes \pi^*(-)) : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X})$$

is fully faithful for any k. Moreover, Φ_k admits a right adjoint functor.

Proof. The functor Φ_k is a Fourier-Mukai transform with kernel $\mathscr{O}_E(kE)$ considered as on object in $\mathbf{D}^b(Y \times \widetilde{X})$. As such, Φ_k admits in particular right and left adjoint. Now we will use a result due to Bondal-Orlov (Proposition 7.1 in [3]):

- Consider the Fourier-Mukai transform $\Phi_{\mathscr{P}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ between the derived categories of two smooth projective varieties X and Y given by an object $\mathscr{P} \in \mathbf{D}^b(X \times Y)$. Then the functor $\Phi_{\mathscr{P}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\mathrm{Hom}(\Phi_{\mathscr{P}}(\kappa(x)), \Phi_{\mathscr{P}}(\kappa(y))[i]) = \begin{cases} k, \text{if } x = y \text{ and } i = 0; \\ 0, \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

For any j and $x \neq y$, this follows from the fact that the result objects have disjoint supports.

Now we let $x = y \in Y$. We need to show that $\mathrm{Ext}^i_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})$ is trivial for $i \notin [0, d = \dim Y]$ and of dimension one for $i = 0$. By Lemma 1.1.8 we get the spectral sequence

$$\begin{aligned} E_2^{p,q} &= H^p(\widetilde{X}, \mathscr{E}xt^q_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})) = H^p\left(E_x, \bigwedge^q \mathscr{N}_{E_x/\widetilde{X}}\right) \\ &\Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x}). \end{aligned}$$

Hence we need to determine $\mathscr{N}_{E_x/\widetilde{X}}$. Consider the exact sequence

$$0 \to \mathscr{N}_{E_x/E} \to \mathscr{N}_{E_x/\widetilde{X}} \to \mathscr{N}_{E/\widetilde{X}}|_{E_x} \to 0,$$

as $\mathscr{N}_{E/\widetilde{X}} = \mathscr{O}_E(E)$ and $\mathscr{N}_{E_x/E} = \mathscr{O}_{E_x}^{\oplus d}$ and since $E_x \cong \mathbb{P}^{c-1}$ one get

$$\mathscr{N}_{E_x/\widetilde{X}} \cong \mathscr{O}_{E_x}(-1) \oplus \mathscr{O}_{E_x}^{\oplus d}$$

by computing the Ext^1. Hence we can directly get the result. □

Proposition 1.1.12. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Define*

$$\mathscr{D}_k := \mathrm{Im}(\Phi_{-k} : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X}))$$

for $k = -c+1, ..., -1$ and $\mathscr{D}_0 := q^*\mathbf{D}^b(X)$.

Then $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ forms an S.O.D of $\mathbf{D}^b(\widetilde{X})$.

Proof. □

1.2 Kuznetsov Components

Chapter 2

Examples of Fano Manifolds of Calabi-Yau Type

Chapter 3

Examples of Derived Equivalences of Kuznetsov Components with K3s

Chapter 4

Stability Conditions on K3 Categories

Chapter 5

Applications: Mukai's program

Chapter 6

Application to Cubic Fourfolds and Gushel-Mukai Manifolds

Index

admissible, 7

exceptional element, 7
exceptional sequence, 7

orthogonal complement, 7

semi-orthogonal, 7
semi-orthogonal decomposition, 7

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