Dualizing Complexes Using Derived Categories

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1 Introduction

The theory of duality is beautiful theory in the commutative algebra and algebraic geometry. In the book [BH98], the authors introduce the duality of Cohen-Macaulay rings. But it can not work for more general rings. In this note, we will introduce the general theory of duality of more general rings using derived category. We will generalize the notion of canonical modules in [BH98] into dualizing complexes and we will see that in the case of Cohen-Macaulay rings, this complex can be concentrated in the one place. So we again have the canonical modules.

Now we give the outline of the note:

- In the section 2, we will give a quike introduction of local cohomology using derived category which is a basic tool.
- In the section 3, we will introduce the basic theory of dualizing complexes of Noetherian rings and introduce the local duality theorem.
- In the section 4, we will consider the special case of Cohen-Macaulay and Gorenstein Rings and to find some special properties.
- In the section 5, we will consider the more cases of rings which have dualizing complexes.
- In the final section 6, we will give a glimpse of the global theory of dualizing complexes in algebraic geometry.

Note that we will mainly follows the chapter 47 in [Pro24] and Chapter 25 in [GW23]. The basic theory of derived categopries, injective hulls and Matlis duality will be omitted and we refer to [BH98] or the beginning of chapter 47 in [Pro24].

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2 Local Cohomology of Noetherian Rings

Here we will follows section 47.8–47.11 in [Pro24] and summary some results about several definitions of local cohomology using derived categories which we will use later. The reader can begin to read the Section 3 and omit this whole section for now.

More general theory we refer [Har67] or [Gro68] or chapter51 of [Pro24].

2.1 More on Čech Complex and Koszul Complex

Lemma 2.1. *Let R be a ring. Let $\phi : E \to R$ be an R-module map. Let $e \in E$ with image $f = \phi(e)$ in R. Then $f = de + ed$ as endomorphisms of $\mathbf{K}_\bullet(\phi)$. In particular, multiplication by f_i on $\mathbf{K}_\bullet(f_1, ..., f_r)$ is homotopic to zero.*

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 2.2. *Let R be a ring. Let $f_1, \ldots, f_r \in R$. The (extended alternating) Čech complex of R is the cochain complex*

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}$$

where R is in degree 0, the term $\bigoplus_{i_0} R_{f_{i_0}}$ is in degre 1, and so on. The maps are defined as follows

1. *The map $R \to \bigoplus_{i_0} R_{f_{i_0}}$ is given by the canonical maps $R \to R_{f_{i_0}}$.*
2. *Given $1 \le i_0 < \ldots < i_{p+1} \le r$ and $0 \le j \le p+1$ we have the canonical localization map*

$$R_{f_{i_0}\ldots\hat{f}_{i_j}\ldots f_{i_{p+1}}} \to R_{f_{i_0}\ldots f_{i_{p+1}}}$$

3. *The differentials use the canonical maps of (2) with sign $(-1)^j$.*

Then

$$R \to \bigoplus\nolimits_{i_0} R_{f_{i_0}} \to \bigoplus\nolimits_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}$$

is a colimit of the Koszul complexes $\mathbf{K}(R, f_1^n, \ldots, f_r^n)$; see proof for a precise statement.

Proof. We have

$$\mathbf{K}(R, f_1^n, \ldots, f_r^n) : 0 \to \wedge^r(R^{\oplus r}) \to \wedge^{r-1}(R^{\oplus r}) \to \ldots \to R^{\oplus r} \to R \to 0$$

with the term $\wedge^r(R^{\oplus r})$ sitting in degree 0. Let $e_1^n, \ldots, e_r^n$ be the standard basis of $R^{\oplus r}$. Then the elements $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ for $1 \le j_1 < \ldots < j_{r-p} \le r$

form a basis for the term in degree p of the Koszul complex. Further, observe that

$$d(e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}}) = \sum (-1)^{a+1} f^n_{j_a} e^n_{j_1} \wedge \ldots \wedge \hat{e}^n_{j_a} \wedge \ldots \wedge e^n_{j_{r-p}}.$$

The transition maps of our system

$$\mathbf{K}(R, f_1^n, \ldots, f_r^n) \to \mathbf{K}(R, f_1^{n+1}, \ldots, f_r^{n+1})$$

are given by the rule

$$e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}} \longmapsto f_{i_0} \ldots f_{i_{p-1}} e^{n+1}_{j_1} \wedge \ldots \wedge e^{n+1}_{j_{r-p}}$$

where the indices $1 \leq i_0 < \ldots < i_{p-1} \leq r$ are such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to $f_1 \ldots f_r$ in degree r.

Denote $\mathbf{K}^p(R, f_1^n, \ldots, f_r^n)$ the term of degree p in the Koszul complex. Observe that for any $f \in R$ we have

$$R_f = \varinjlim (R \xrightarrow{f} R \xrightarrow{f} R \to \ldots)$$

Hence we see that in degree p we obtain

$$\varinjlim K^p(R, f_1^n, \ldots f_r^n) = \bigoplus\nolimits_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

Here the element $e^n_{j_1} \wedge \ldots \wedge e^n_{j_{r-p}}$ of the Koszul complex above maps in the colimit to the element $(f_{i_0} \ldots f_{i_{p-1}})^{-n}$ in the summand $R_{f_{i_0} \ldots f_{i_{p-1}}}$ where the indices are chosen such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}) = \sum_{i \notin \{i_0, \ldots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0} \ldots f_{i_t} f_i f_{i_{t+1}} \ldots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree p by the map

$$\bigoplus_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}} \longrightarrow \bigoplus_{1 \leq i_0 < \ldots < i_{p-1} \leq r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}} \longmapsto (-1)^{i_0 + \ldots + i_{p-1} + p} \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from $\varinjlim \mathbf{K}(R, f_1^n, \ldots, f_r^n)$ to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out. $\square$

2.2 Deriving Torsion

Definition 2.3. *Let R be a ring with an ideal $I \subset R$. Fix $M \in \mathbf{Mod}_R$.*

(a) We define

$$M[I^n] := \{m \in M : I^n m = 0\}, \quad M[I^\infty] = \bigoplus_n M[I^n].$$

(b) We call M is I^∞-torsion if $M = M[I^\infty]$. We let I^∞**-torsion** *be the subcategory of $\mathbf{Mod}_R$ consist of I^∞-torsion modules*

Here we give some easy but important properties of this notion.

Proposition 2.4. *Let R be a ring with an ideal $I \subset R$.*

(a) Let $M \in$ I^∞**-torsion***, then M admits a resolution*

$$\cdots \to K_2 \to K_1 \to K_0 \to M \to 0$$

with each K_i a direct sum of copies of R/I^n for n variable. In particular, the category I^∞**-torsion** *is a Grothendieck abelian category.*

(b) Let I be a finitely generated ideal of R, then for any $M \in \mathbf{Mod}_R$ we have $(M/M[I^\infty])[I] = 0$.

(c) Let I be a finitely generated ideal of R, then I^∞**-torsion** *is a Serre subcategory of the abelian category $\mathbf{Mod}_R$, that is, an extension of I^∞-torsion modules is I^∞-torsion.*

(d) Let I be a finitely generated ideal of R and $M \in \mathbf{Mod}_R$, then we have an exact sequence

$$0 \to M[I^\infty] \to M \to \prod_{\mathfrak{p} \notin V(I)} M_{\mathfrak{p}}.$$

In particular, we have $M \in$ I^∞**-torsion** *if and only if $\mathrm{supp}(M) \subset V(I)$. Hence the subcategory* I^∞**-torsion** *$\subset \mathbf{Mod}_R$ depends only on the closed subset $V(I) \subset \mathrm{Spec}(R)$.*

Proof. For (a), there is a canonical surjection $\bigoplus_{m \in M} R/I^{n_m} \to M \to 0$ where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I^∞-torsion module. Proceeding inductively, we construct the desired resolution of M.

For (b), Let $m \in M$. If m maps to an element of $(M/M[I^\infty])[I]$ then $Im \subset M[I^\infty]$. Write $I = (f_1, ..., f_t)$. Then we see that $f_i m \in M[I^\infty]$. Thus we see that $I^N m = 0$ for some large $N \gg 0$. Hence m maps to zero in $(M/M[I^\infty])$.

For (c), suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules with M' and M'' both I^∞-torsion modules. Then $M' \subset M[I^\infty]$ and hence $M/M[I^\infty]$ is a quotient of M'' and therefore I^∞-torsion. Combined with (b) this implies that it is zero.

For (d), let $M \in \mathbf{Mod}_R$ and let $x \in M$. If $x \in M[I^\infty]$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $M_\mathfrak{p}$ for all $I \not\subset \mathfrak{p}$. Conversely, if x maps to zero in $M_\mathfrak{p}$ for all $I \not\subset \mathfrak{p}$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, ..., f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geq 1$. It follows that $x \in M[I^N]$ for $N = \sum_i n_i$. Thus $M[I^\infty]$ is the kernel of $M \to \prod_{\mathfrak{p} \notin Z} M_\mathfrak{p}$. □

Definition 2.5 (Fake Local Cohomology)**.** *Let R be a ring and let I be a finitely generated ideal. By Proposition 2.4(a), the category* I^∞**-torsion** *is a Grothendieck abelian category and hence the derived category* $\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion})$ *exists by some homological algebra, as Tag 079Q. Hence we have the derived functor*

$$\mathbf{R}\Gamma_I : \mathbf{D}(R) \to \mathbf{D}(\mathrm{I}^\infty\textbf{-torsion})$$

of $\Gamma_I : \mathbf{Mod}_R \to \mathrm{I}^\infty\textbf{-torsion}$ given by $M \mapsto M[I^\infty]$ which is left exact.

Moreover, we define $H^q_I(K) := H^q(\mathbf{R}\Gamma_I(K))$ for any $K \in \mathbf{D}(R)$.

Remark 2.6. *Note that this functor does not deserve the name local cohomology unless the ring R is Noetherian.*

Now we discuss some basic properties of the functor.

Proposition 2.7. *Let R be a ring and let I be a finitely generated ideal.*

(a) The functor $\mathbf{R}\Gamma_I$ is right adjoint to the functor $\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion}) \to \mathbf{D}(R)$.

(b) For any object K of $\mathbf{D}(R)$ we have

$$\mathbf{R}\Gamma_I(K) = \mathrm{hocolim} \mathbf{R}\operatorname{Hom}_R(R/I^n, K)$$

in $\mathbf{D}(R)$ and hence

$$H^q_I(K) := \mathbf{R}^q\Gamma_I(K) = \varinjlim \operatorname{Ext}^q_R(R/I^n, K)$$

as modules for all $q \in \mathbb{Z}$.

(c) Let $K^\bullet$ be a complex of A-modules such that $f : K^\bullet \to K^\bullet$ is an isomorphism for some $f \in I$, i.e., $K^\bullet$ is a complex of R_f-modules. Then $\mathbf{R}\Gamma_I(K^\bullet) = 0$.

Proof. For (a), this follows from the fact that taking I^∞-torsion submodules is the right adjoint to the inclusion functor $\mathrm{I}^\infty\textbf{-torsion} \to \mathbf{Mod}_R$.

For (b), let $J^\bullet$ be a K-injective resolution of K. Then we have

$$\begin{aligned}\mathbf{R}\Gamma_I(K) &= \Gamma_I(J^\bullet) = J^\bullet[I^\infty] = \varinjlim_n J^\bullet[I^n] \\ &= \varinjlim_n \operatorname{Hom}_R(R/I^n, J^\bullet) = \operatorname{hocolim} \mathbf{R}\operatorname{Hom}_R(R/I^n, K).\end{aligned}$$

Well done.

For (c), in this case the cohomology modules of $\mathbf{R}\Gamma_I(K^\bullet)$ are both f^∞-torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

In the end of this small section we consider another category. Let R be a ring and let I be a finitely generated ideal. By Proposition 2.4(c), I^∞**-torsion** is a Serre subcategory of the abelian category $\mathbf{Mod}_R$. This shows that I^∞**-torsion** $\subset \mathbf{Mod}_R$ exact which induce the functor $\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion}) \to \mathbf{D}(R)$ which factor through

$$\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion}) \to \mathbf{D}_{\mathrm{I}^\infty\textbf{-torsion}}(\mathbf{Mod}_R).$$

Proposition 2.8. *Let R be a ring and let I be a finitely generated ideal. Let $M, N \in \mathrm{I}^\infty$**-torsion**.*

(a) $\operatorname{Hom}_{\mathbf{D}(R)}(M, N) = \operatorname{Hom}_{\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion})}(M, N)$

(b) $\operatorname{Ext}^2_{\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion})}(M, N) \to \operatorname{Ext}^2_{\mathbf{D}(R)}(M, N)$ *is not surjective in general. In particular,* $\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion}) \to \mathbf{D}_{\mathrm{I}^\infty\textbf{-torsion}}(\mathbf{Mod}_R)$ *is not an equivalence in general.*

Proof. (a) is trivial and the counterexample of (b) we refer Tag 0A6P. □

Remark 2.9. *However in the Noetherian case this will be true. We will see this later.*

2.3 Basic Theory of Local Cohomology

Now we will introduce some true local cohomologies.

Theorem 2.10 (Real Local Cohomology, I)**.** *Let R be a ring and let $I \subset R$ be a finitely generated ideal and $Z = V(I) \subset \operatorname{Spec}(R)$. There exists a right adjoint $\mathbf{R}\Gamma_Z$ to the inclusion functor $\mathbf{D}_{\mathrm{I}^\infty\textbf{-torsion}}(R) \to \mathbf{D}(R)$. In fact, if I is generated by $f_1, \ldots, f_r \in R$, then we have*

$$\mathbf{R}\Gamma_Z(K) = \left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0}f_{i_1}} \to \ldots \to R_{f_1\ldots f_r}\right) \otimes_R^{\mathbf{L}} K$$

functorially in $K \in \mathbf{D}(R)$.

Proof. Say $I = (f_1, \dots, f_r)$ is an ideal. Let $K^\bullet$ be a complex of R-modules. There is a canonical map of complexes

$$\left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r} \right) \longrightarrow R.$$

from the extended Čech complex to R. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\mathrm{Tot}\left(K^\bullet \otimes_R (R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r}) \right) \longrightarrow K^\bullet$$

in $\mathbf{D}(R)$. We claim the cohomology modules of the complex on the left are I^∞-torsion, i.e., the LHS is an object of $D_{\mathrm{I^\infty}\textbf{-torsion}}(R)$. Namely, we have

$$\left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r} \right) = \varinjlim \mathbf{K}(R, f_1^n, \dots, f_r^n)$$

by Lemma 2.2. Moreover, multiplication by f_i^n on the complex $\mathbf{K}(R, f_1^n, \dots, f_r^n)$ is homotopic to zero by Lemma 2.1. Since

$$H^q\,(LHS) = \varinjlim H^q(\mathrm{Tot}(K^\bullet \otimes_A \mathbf{K}(R, f_1^n, \dots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{\mathrm{I^\infty}\textbf{-torsion}}(R)$, then the complexes $K^\bullet \otimes_R R_{f_{i_0} \dots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $LHS \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$\mathbf{R}\Gamma_Z(K^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_R \left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r} \right) \right)$$

is functorial in $K^\bullet$ and defines an exact functor $\mathbf{D}(R) \to \mathbf{D}_{\mathrm{I^\infty}\textbf{-torsion}}(R)$ between triangulated categories. It follows formally from the existence of the natural transformation $\mathbf{R}\Gamma_Z \to \mathrm{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $\mathbf{R}\Gamma_Z$ is the desired right adjoint. □

Hence now we have the functor

$$\mathbf{R}\Gamma_Z : \mathbf{D}(R) \to \mathbf{D}_{\mathrm{I^\infty}\textbf{-torsion}}(R).$$

As we have seen, we construct the functor using Čech complex. Is there some relation between this and the functor in algebraic geometry?

Definition 2.11 (Real Local Cohomology, II). *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Consider the functor $\Gamma_Z : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X(X))$ given by*

$$\Gamma_Z(\mathscr{F}) := \{s \in \Gamma(X, \mathscr{F}) : \mathrm{supp}(s) \subset Z\}.$$

Using K-injective resolutions, we obtain the right derived functor

$$\mathbf{R}\Gamma_Z(X, -) : \mathbf{D}(\mathscr{O}_X) \to \mathbf{D}(\mathscr{O}_X(X)).$$

The group $H^q_Z(X, K) = H^q(\mathbf{R}\Gamma_Z(X, K))$ the cohomology module with support in Z.

We now show that they are the same! Indeed, we can use Čech complex to rebuild $\mathbf{R}\Gamma_Z(X, -)$, and then we can connected to $\mathbf{R}\Gamma_Z : \mathbf{D}(R) \to \mathbf{D}_{\mathrm{I}^\infty\text{-}\mathbf{torsion}}(R)$ as before.

Proposition 2.12. *Let R be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(R)$. For $K \in \mathbf{D}(A)$ corresponding to $\widetilde{K} \in \mathbf{D}_{\mathrm{QCoh}}(\mathscr{O}_X)$, there is a functorial isomorphism*

$$\mathbf{R}\Gamma_Z(K) = \mathbf{R}\Gamma_Z(X, \widetilde{K}).$$

Proof. Note that there exists a distinguished triangle

$$\mathbf{R}\Gamma_Z(X, \widetilde{K}) \to \mathbf{R}\Gamma(X, \widetilde{K}) \to \mathbf{R}\Gamma(U, \widetilde{K}) \to \mathbf{R}\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $\mathbf{R}\Gamma(X, \widetilde{K}) = K$. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. As affine schemes are separated, the alternating Čech complex $\mathrm{Tot}(\check{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of R-modules representing K. Working through the definitions we find

$$\mathbf{R}\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_R (\prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = \mathbf{R}\Gamma(X, \widetilde{K^\bullet}) \to \mathbf{R}\Gamma(U, \widetilde{K}^\bullet)$ is induced by the diagonal map from A into $\prod R_{f_i}$. Hence we conclude that

$$\mathbf{R}\Gamma_Z(X, \mathcal{F}^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_R \left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}\right)\right)$$

Well dominate. □

Finally we will introduce the noetherian case and compare the fake local cohomology and the real cohomology.

Proposition 2.13. *Let R be a Noetherian ring and let $I \subset R$ be an ideal.*

1. *The adjunction $\mathbf{R}\Gamma_I(K) \to K$ is an isomorphism for $K \in \mathbf{D}_{\mathrm{I}^\infty\textbf{-torsion}}(R)$.*
2. *The functor $\mathbf{D}(\mathrm{I}^\infty\textbf{-torsion}) \to \mathbf{D}_{\mathrm{I}^\infty\textbf{-torsion}}(R)$ is an equivalence.*
3. *$\mathbf{R}\Gamma_I(K) = \mathbf{R}\Gamma_Z(K)$ for $K \in \mathbf{D}(R)$.*

Proof. Boring proof, we refer Tag 0955. □

So in the Noetherian case (so in the whole theory we consider) we will use $\mathbf{R}_I(-)$.

2.4 Local Cohomology and Depth

In this small section we will introduce a result about the depth and local cohomology. Note that $\mathrm{depth}_I(M)$ here is the grade $\mathrm{Grade}(I, M)$ in [BH98].

Theorem 2.14. *Let R be a Noetherian ring, let $I \subset R$ be an ideal, and let M be a finite A-module such that $IM \neq M$. Then the following integers are equal:*

(1) $\mathrm{depth}_I(M)$*,*

(2) the smallest integer i such that $\mathrm{Ext}^i_A(A/I, M)$ is nonzero, and

(3) the smallest integer i such that $H^i_I(M)$ is nonzero.

Moreover, we have $\mathrm{Ext}^i_A(N, M) = 0$ for $i < \mathrm{depth}_I(M)$ for any finite A-module N annihilated by a power of I.

Proof. We prove the equality of (1) and (2) by induction on $\mathrm{depth}_I(M)$ which is allowed since $\mathrm{depth}_I(M) < \infty$ now.

If $\mathrm{depth}_I(M) = 0$, then I is contained in the union of the associated primes of M. By prime avoidance we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\mathrm{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\mathrm{depth}_I(M) > 0$. Let $f \in I$ be a nonzerodivisor on M such that $\mathrm{depth}_I(M/fM) = \mathrm{depth}_I(M) - 1$. Consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

and the associated long exact sequence for $\mathrm{Ext}^*_A(A/I, -)$. Note that $\mathrm{Ext}^i_A(A/I, M)$ is a finite A/I-module. Hence we obtain

$$\mathrm{Hom}_A(A/I, M/fM) = \mathrm{Ext}^1_A(A/I, M)$$

and short exact sequences

$$0 \to \mathrm{Ext}^i_A(A/I, M) \to \mathrm{Ext}^i_A(A/I, M/fM) \to \mathrm{Ext}^{i+1}_A(A/I, M) \to 0$$

Thus the equality of (1) and (2) by induction.

Observe that $\mathrm{depth}_I(M) = \mathrm{depth}_{I^n}(M)$ for all $n \geq 1$ for example by the fact that the sequence $(f_1, ..., f_r)$ is regular if and only if $(f_1^{e_1}, \ldots, f_r^{e_r})$ is regular for any fixed $e_i > 0$ (see Tag 07DV for the proof). Hence by the equality of (1) and (2) we see that $\mathrm{Ext}^i_A(A/I^n, M) = 0$ for all n and $i < \mathrm{depth}_I(M)$. Let N be a finite A-module annihilated by a power of I. Then we can choose a short exact sequence

$$0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0$$

for some $n, m \geq 0$. Then $\mathrm{Hom}_A(N, M) \subset \mathrm{Hom}_A((A/I^n)^{\oplus m}, M)$ and $\mathrm{Ext}^i_A(N, M) \subset \mathrm{Ext}^{i-1}_A(N', M)$ for $i < \mathrm{depth}_I(M)$. Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have $H^p_I(M) = \varinjlim \mathrm{Ext}^p_A(A/I^n, M)$ by Proposition 2.7(b). Thus we see that $H^i_I(M) = 0$ for $i < \mathrm{depth}_I(M)$. For $i = \mathrm{depth}_I(M)$, using the vanishing of $\mathrm{Ext}^{i-1}_A(I/I^n, M)$ we see that the map $\mathrm{Ext}^i_A(A/I, M) \to H^i_I(M)$ is injective which proves nonvanishing in the correct degree. $\square$

3 Dualizing Complexes

4 Cohen-Macaulay and Gorenstein Rings

5 The Ubiquity of Dualizing Complexes

6 A Glimpse of Duality in Algebraic Geometry

[Illar]

[Har66] or [Con00]

[LH09] and [Ver69]

[Nee96] and [GW23]

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