Lecture Notes on Commutative Algebra

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Contents

Preface

Here we will mainly follows [1]. We will assume all rings are commutative with unit. We assume the reader know the basic algebra an some homological algebra, including basic theory of groups, rings, modules, basic things of spectrum of rings and its basic properties, abelian categories, derived categories and derived functors.

Chapter 1

Rings, Ideals and Modules

1.1 Basic Properties

Lemma 1.1.1. *Let R be a ring and let M be an R-module. Then there exists a directed system of finitely presented R-modules M_i such that $M \cong \varinjlim M_i$.*

Proof. Consider any finite subset $S \subset M$ and any finite collection of relations E among the elements of S. So each $s \in S$ corresponds to $x_s \in M$ and each $e \in E$ consists of a vector of elements $f_{e,s} \in R$ such that $\sum f_{e,s} x_s = 0$. Let $M_{S,E}$ be the cokernel of the map

$$R^{\#E} \longrightarrow R^{\#S}, \quad (g_e)_{e \in E} \longmapsto \left(\sum g_e f_{e,s} \right)_{s \in S}.$$

There are canonical maps $M_{S,E} \to M$. If $S \subset S'$ and if the elements of E correspond, via this map, to relations in E', then there is an obvious map $M_{S,E} \to M_{S',E'}$ commuting with the maps to M. Let I be the set of pairs (S, E) with ordering by inclusion as above. It is clear that the colimit of this directed system is M. $\square$

Proposition 1.1.2. *Let R be a ring. Let N be an R-module. The following are equivalent*

(1) N is a finitely generated (finitely presented) R-module.

(2) for any filtered colimit $M = \varinjlim M_i$ of R-modules the map

$$\varinjlim \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M)$$

is injective (bijective).

Proof. Consider the case of finitely generated: Assume (1) and choose generators $x_1, \cdots, x_m$ for N. If $N \to M_i$ is a module map and the composition $N \to M_i \to M$ is zero, then because $M = \varinjlim_{i' \geq i} M_{i'}$ for each $j \in \{1, \ldots, m\}$ we can find a $i' \geq i$ such that x_j maps to zero in $M_{i'}$. Since there are finitely many x_j we can find a single i' which works for all of them. Then the composition $N \to M_i \to M_{i'}$ is zero and we conclude the map is injective, i.e., part (2) holds.

Assume (2). For a finite subset $E \subset N$ denote $N_E \subset N$ the R-submodule generated by the elements of E. Then $0 = \varinjlim N/N_E$ is a filtered colimit. Hence we see that $\mathrm{id} : N \to N$ maps into N_E for some E, i.e., N is finitely generated.

Consider the case of finitely presented: Assume (1) and choose an exact sequence $F_{-1} \to F_0 \to N \to 0$ with F_i finite free. Then we have an exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(F_0, M) \to \mathrm{Hom}_R(F_{-1}, M)$$

functorial in the R-module M. The functors $\mathrm{Hom}_R(F_i, M)$ commute with filtered colimits as $\mathrm{Hom}_R(R^{\oplus n}, M) = M^{\oplus n}$. Since filtered colimits are exact, we see that (2) holds.

Assume (2). By Lemma 1.1.1 we can write $N = \varinjlim N_i$ as a filtered colimit such that N_i is of finite presentation for all i. Thus id_N factors through N_i for some i. This means that N is a direct summand of a finitely presented R-module (namely N_i) and hence finitely presented. □

Proposition 1.1.3. *Let R be a ring, and let M be a finitely generated R-module. There exists a filtration by R-submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each quotient M_i/M_{i-1} is isomorphic to R/I_i for some ideal $I_i \subset R$.

Proof. By induction on the number of generators of M. Let $x_1, \cdots, x_r \in M$ be a minimal number of generators. Let $M' := Rx_1 \subset M$. Then M/M' has $r-1$ generators and the induction hypothesis applies. And clearly $M' \cong R/\operatorname{ann}(x_1)$, well done. □

1.2 Localizations

Definition 1.2.1. *Let R be a ring, S a subset of R. We say S is a multiplicative subset of R if $1 \in S$ and S is closed under multiplication, i.e., $s, s' \in S \Rightarrow ss' \in S$.*

Definition 1.2.2. *Given a ring A and a multiplicative subset S, we define a relation on $A \times S$ as follows:*

$$(x, s) \sim (y, t) \Leftrightarrow \exists u \in S \text{ such that } (xt - ys)u = 0.$$

It is easily checked that this is an equivalence relation. Let x/s be the equivalence class of (x, s) and $S^{-1}A$ be the set of all equivalence classes. Define addition and multiplication in $S^{-1}A$ as follows:

$$x/s + y/t = (xt + ys)/st, \quad x/s \cdot y/t = xy/st.$$

One can check that $S^{-1}A$ becomes a ring under these operations. Then this ring is called the localization of A with respect to S .

We have a natural ring map from A to its localization $S^{-1}A$,

$$A \longrightarrow S^{-1}A, \quad x \longmapsto x/1$$

which is sometimes called the localization map. In general the localization map is not injective, unless S contains no zerodivisors.

The localization of a ring has the following universal property.

Proposition 1.2.3. *Let $f : A \to B$ be a ring map that sends every element in S to a unit of B. Then there is a unique homomorphism $g : S^{-1}A \to B$ such that the following diagram commutes.*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, dashed, "\exists! g"'] & \end{tikzcd}$$

Proof. Existence. We define a map g as follows. For $x/s \in S^{-1}A$, let $g(x/s) = f(x)f(s)^{-1} \in B$. It is easily checked from the definition that this is a well-defined ring map. And it is also clear that this makes the diagram commutative.

Uniqueness. We now show that if $g' : S^{-1}A \to B$ satisfies $g'(x/1) = f(x)$, then $g = g'$. Hence $f(s) = g'(s/1)$ for $s \in S$ by the commutativity of the diagram. But then $g'(1/s)f(s) = 1$ in B, which implies that $g'(1/s) = f(s)^{-1}$ and hence $g'(x/s) = g'(x/1)g'(1/s) = f(x)f(s)^{-1} = g(x/s)$. $\square$

Lemma 1.2.4. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. The category of $S^{-1}R$-modules is equivalent to the category of R-modules N with the property that every $s \in S$ acts as an automorphism on N.*

Proof. The functor which defines the equivalence associates to an $S^{-1}R$-module M the same module but now viewed as an R-module via the localization map $R \to S^{-1}R$. Conversely, if N is an R-module, such that every $s \in S$ acts via an automorphism s_N, then we can think of N as an $S^{-1}R$-module by letting x/s act via $x_N \circ s_N^{-1}$. We omit the verification that these two functors are quasi-inverse to each other. $\square$

The notion of localization of a ring can be generalized to the localization of a module.

Definition 1.2.5. *Let A be a ring, S a multiplicative subset of A and M an A-module. We define a relation on $M \times S$ as follows*

$$(m, s) \sim (n, t) \Leftrightarrow \exists u \in S \text{ such that } (mt - ns)u = 0.$$

This is clearly an equivalence relation. Denote by m/s be the equivalence class of (m, s) and $S^{-1}M$ be the set of all equivalence classes. Define the addition and scalar multiplication as follows

$$m/s + n/t = (mt + ns)/st, \quad m/s \cdot n/t = mn/st.$$

It is clear that this makes $S^{-1}M$ an $S^{-1}A$-module. The $S^{-1}A$-module $S^{-1}M$ is called the localization *of M at S.*

Note that there is an A-module map $M \to S^{-1}M$, $m \mapsto m/1$ which is also called the localization map. It satisfies the following similar universal property.

Lemma 1.2.6. *Let R be a ring. Let $S \subset R$ a multiplicative subset. Let M, N be R-modules. Assume all the elements of S act as automorphisms on N. Then we have*

$$\begin{tikzcd} M \arrow[rr, "\beta"] \arrow[dr] & & N \\ & S^{-1}M \arrow[ur, dashed, "\exists!\alpha"'] & \end{tikzcd}$$

Moroever, the canonical map

$$\mathrm{Hom}_R(S^{-1}M, N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map, is an isomorphism.

Proof. It is clear that the map is well-defined and R-linear. Injectivity: Let $\alpha \in \mathrm{Hom}_R(S^{-1}M, N)$ and take an arbitrary element $m/s \in S^{-1}M$. Then, since $s \cdot \alpha(m/s) = \alpha(m/1)$, we have $\alpha(m/s) = s^{-1}(\alpha(m/1))$, so α is completely determined by what it does on the image of M in $S^{-1}M$. Surjectivity: Let $\beta : M \to N$ be a given R-linear map. We need to show that it can be "extended" to $S^{-1}M$. Define a map of sets

$$M \times S \to N, \quad (m, s) \mapsto s^{-1}\beta(m).$$

Clearly, this map respects the equivalence relation from above, so it descends to a well-defined map $\alpha : S^{-1}M \to N$. It remains to show that this map is R-linear, so take $r, r' \in R$ as well as $s, s' \in S$ and $m, m' \in M$. Then

$$\begin{aligned} \alpha(r \cdot m/s + r' \cdot m'/s') &= \alpha((r \cdot s' \cdot m + r' \cdot s \cdot m')/(ss')) \\ &= (ss')^{-1}\beta(r \cdot s' \cdot m + r' \cdot s \cdot m') \\ &= (ss')^{-1}(r \cdot s'\beta(m) + r' \cdot s\beta(m')) \\ &= r\alpha(m/s) + r'\alpha(m'/s') \end{aligned}$$

and we win. $\square$

Example 1.2.1. *Let A be a ring and let M be an A-module. Here are some important examples of localizations.*

1. *Given $\mathfrak{p}$ a prime ideal of A consider $S = A \setminus \mathfrak{p}$. It is immediately checked that S is a multiplicative set. In this case we denote $A_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ the localization of A and M with respect to S respectively. These are called the localization of A, resp. M at $\mathfrak{p}$.*

2. *Let $f \in A$. Consider $S = \{1, f, f^2, \ldots\}$. This is clearly a multiplicative subset of A. In this case we denote A_f (resp. M_f) the localization $S^{-1}A$ (resp. $S^{-1}M$). This is called the localization of A, resp. M with respect to f. Note that $A_f = 0$ if and only if f is nilpotent in A.*

3. *Let $S = \{f \in A : f \text{ is not a zerodivisor in } A\}$. This is a multiplicative subset of A. In this case the ring $Q(A) = S^{-1}A$ is called either the total quotient ring of A.*

4. *If A is a domain, then the total quotient ring $Q(A)$ is the field of fractions of A.*

Lemma 1.2.7. *Let R be a ring. Let $S \subset R$ be a multiplicative subset. Let M be an R-module. Then*

$$S^{-1}M = \varinjlim_{f \in S} M_f$$

where the preorder on S is given by $f \geq f' \Leftrightarrow f = f'f''$ for some $f'' \in R$ in which case the map $M_{f'} \to M_f$ is given by $m/(f')^e \mapsto m(f'')^e/f^e$.

Proof. Omitted. Just need to check the universal property. □

Proposition 1.2.8. *Let A denote a ring, and M, N denote modules over A. If S and S' are multiplicative sets of A, then it is clear that*

$$SS' = \{ss' : s \in S,\ s' \in S'\}$$

is also a multiplicative set of A. Then the following holds.

(1) *Let $\overline{S}$ be the image of S in $S'^{-1}A$, then $(SS')^{-1}A$ is isomorphic to $\overline{S}^{-1}(S'^{-1}A)$.*

(2) *View $S'^{-1}M$ as an A-module, then $S^{-1}(S'^{-1}M)$ is isomorphic to $(SS')^{-1}M$.*

(3) *Let $L \xrightarrow{u} M \xrightarrow{v} N$ be an exact sequence of R-modules. Then $S^{-1}L \to S^{-1}M \to S^{-1}N$ is also exact.*

(4) *If N is a submodule of M, then $S^{-1}(M/N) \simeq (S^{-1}M)/(S^{-1}N)$.*

(5) *Let I be an ideal of A, S a multiplicative set of A. Then $S^{-1}I$ is an ideal of $S^{-1}A$ and $\overline{S}^{-1}(A/I)$ is isomorphic to $S^{-1}A/S^{-1}I$, where $\overline{S}$ is the image of S in A/I.*

(6) *Any submodule N' of $S^{-1}M$ is of the form $S^{-1}N$ for some $N \subset M$. Indeed one can take N to be the inverse image of N' in M. In particular, each ideal I' of $S^{-1}A$ takes the form $S^{-1}I$, where one can take I to be the inverse image of I' in A.*

Proof. For (1), the map sending $x \in A$ to $x/1 \in (SS')^{-1}A$ induces a map sending $x/s \in S'^{-1}A$ to $x/s \in (SS')^{-1}A$, by universal property. The image of the elements in $\overline{S}$ are invertible in $(SS')^{-1}A$. By the universal property we get a map $f : \overline{S}^{-1}(S'^{-1}A) \to (SS')^{-1}A$ which maps $(x/t')/(s/s')$ to $(x/t')\cdot(s/s')^{-1}$. On the other hand, the map from A to $\overline{S}^{-1}(S'^{-1}A)$ sending $x \in A$ to $(x/1)/(1/1)$ also induces a map $g : (SS')^{-1}A \to \overline{S}^{-1}(S'^{-1}A)$ which sends x/ss' to $(x/s')/(s/1)$, by the universal property again. It is immediately checked that f and g are inverse to each other, hence they are both isomorphisms.

For (2), note that given a A-module M, we have not proved any universal property for $S^{-1}M$. Hence we cannot reason as in the preceding proof; we have to construct the isomorphism explicitly. We define the maps as follows

$$
\begin{aligned}
&f : S^{-1}(S'^{-1}M) \longrightarrow (SS')^{-1}M, \quad \frac{x/s'}{s} \mapsto x/ss' \\
&g : (SS')^{-1}M \longrightarrow S^{-1}(S'^{-1}M), \quad x/t \mapsto \frac{x/s'}{s} \text{ for some } s \in S, s' \in S', \text{ and } t = ss'
\end{aligned}
$$

We have to check that these homomorphisms are well-defined, that is, independent the choice of the fraction. This is easily checked and it is also straightforward to show that they are inverse to each other.

For (3), first it is clear that $S^{-1}L \to S^{-1}M \to S^{-1}N$ is a complex since localization is a functor. Next suppose that x/s maps to zero in $S^{-1}N$ for some $x/s \in S^{-1}M$. Then by definition there is a $t \in S$ such that $v(xt) = v(x)t = 0$ in M, which means $xt \in \ker(v)$. By the exactness of $L \to M \to N$ we have $xt = u(y)$ for some y in L. Then x/s is the image of y/st. This proves the exactness.

For (4), from the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/N) \longrightarrow 0$$

The corollary then follows.

For (5), The fact that $S^{-1}I$ is an ideal is clear since I itself is an ideal. Define

$$f : S^{-1}A \longrightarrow \overline{S}^{-1}(A/I), \quad x/s \mapsto \overline{x}/\overline{s}$$

where $\overline{x}$ and $\overline{s}$ are the images of x and s in A/I. We shall keep similar notations in this proof. This map is well-defined by the universal property of $S^{-1}A$, and $S^{-1}I$ is contained in the kernel of it, therefore it induces a map

$$\overline{f} : S^{-1}A/S^{-1}I \longrightarrow \overline{S}^{-1}(A/I), \quad \overline{x/s} \mapsto \overline{x}/\overline{s}$$

On the other hand, the map $A \to S^{-1}A/S^{-1}I$ sending x to $\overline{x/1}$ induces a map $A/I \to S^{-1}A/S^{-1}I$ sending $\overline{x}$ to $\overline{x/1}$. The image of $\overline{S}$ is invertible in $S^{-1}A/S^{-1}I$, thus induces a map

$$g : \overline{S}^{-1}(A/I) \longrightarrow S^{-1}A/S^{-1}I, \quad \frac{\overline{x}}{\overline{s}} \mapsto \overline{x/s}$$

by the universal property. It is then clear that $\overline{f}$ and g are inverse to each other, hence are both isomorphisms.

For (6), Let N be the inverse image of N' in M. Then one can see that $S^{-1}N \supset N'$. To show they are equal, take x/s in $S^{-1}N$, where $s \in S$ and $x \in N$. This yields that $x/1 \in N'$. Since N' is an $S^{-1}R$-submodule we have $x/s = x/1 \cdot 1/s \in N'$. This finishes the proof. □

1.3 Tensor Products

1.3.1 Tensor Products

Proposition 1.3.1. *Let M, N be R-modules. Then there exists a pair $(M \otimes_R N, g)$ where $M \otimes_R N$ is an R-module, and $g : M \times N \to T$ an R-bilinear mapping, with the following universal property: For any R-module P and any R-bilinear mapping $f : M \times N \to P$, there exists a unique R-linear mapping $\tilde{f} : M \otimes_R N \to P$ such that $f = \tilde{f} \circ g$. In other words, the following diagram commutes:*

$$\begin{tikzcd}
M \times N \arrow[rr, "f"] \arrow[dr, "g"'] & & P \\
& M \otimes_R N \arrow[ur, dashed, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

Then $M \otimes_R N$ is called the tensor product *of R-modules M and N*

Proof. We first prove the existence of such R-module T. Let M, N be R-modules. Let T be the quotient module P/Q, where P is the free R-module $R^{(M \times N)}$ and Q is the R-module generated by all elements of the following types: $(x \in M, y \in N)$

$$\begin{aligned}
(x + x', y) - (x, y) - (x', y), \\
(x, y + y') - (x, y) - (x, y'), \\
(ax, y) - a(x, y), \\
(x, ay) - a(x, y)
\end{aligned}$$

Let $\pi : M \times N \to T$ denote the natural map. This map is R-bilinear, as implied by the above relations when we check the bilinearity conditions. Denote the image $\pi(x,y) = x \otimes y$, then these elements generate T. Now let $f : M \times N \to P$ be an R-bilinear map, then we can define $f' : T \to P$ by extending the mapping $f'(x \otimes y) = f(x,y)$. Clearly $f = f' \circ \pi$. Moreover, f' is uniquely determined by the value on the generating sets $\{x \otimes y : x \in M, y \in N\}$. Suppose there is another pair (T', g') satisfying the same properties. Then there is a unique $j : T \to T'$ and also $j' : T' \to T$ such that $g' = j \circ g$, $g = j' \circ g'$. But then both the maps $(j \circ j') \circ g$ and g satisfies the universal properties, so by uniqueness they are equal, and hence $j' \circ j$ is identity on T. Similarly $(j' \circ j) \circ g' = g'$ and $j \circ j'$ is identity on T'. So j is an isomorphism. □

Proposition 1.3.2. *Let R be a ring. Let M and N be R-modules.*

(1) If N and M are finite, then so is $M \otimes_R N$.

(2) If N and M are finitely presented, then so is $M \otimes_R N$.

Proof. Suppose M is finite. Then choose a presentation $0 \to K \to R^{\oplus n} \to M \to 0$. This gives an exact sequence $K \otimes_R N \to N^{\oplus n} \to M \otimes_R N \to 0$. We conclude that if N is finite too then $M \otimes_R N$ is a quotient of a finite module, hence finite. Similarly, if both N and M are finitely presented, then we see that K is finite and that $M \otimes_R N$ is a quotient of the finitely presented module $N^{\oplus n}$ by a finite module, namely $K \otimes_R N$, and hence finitely presented. □

Proposition 1.3.3. *Let M be an R-module. Then the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are canonically isomorphic, and the canonical isomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ is given by*

$$f((a/s) \otimes m) = am/s, \forall a \in R, m \in M, s \in S.$$

Proof. Obviously, the map $f' : S^{-1}R \times M \to S^{-1}M$ given by $f'(a/s, m) = am/s$ is bilinear, and thus by the universal property, this map induces a unique $S^{-1}R$-module homomorphism $f : S^{-1}R \otimes_R M \to S^{-1}M$ as in the statement of the lemma. Actually every element in $S^{-1}M$ is of the form m/s, $m \in M, s \in S$ and every element in $S^{-1}R \otimes_R M$ is of the form $1/s \otimes m$. To see the latter fact, write an element in $S^{-1}R \otimes_R M$ as

$$\sum_k \frac{a_k}{s_k} \otimes m_k = \sum_k \frac{a_k t_k}{s} \otimes m_k = \frac{1}{s} \otimes \sum_k a_k t_k m_k = \frac{1}{s} \otimes m.$$

Where $m = \sum_k a_k t_k m_k$. Then it is obvious that f is surjective, and if $f(\frac{1}{s} \otimes m) = m/s = 0$ then there exists $t' \in S$ with $tm = 0$ in M. Then we have

$$\frac{1}{s} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0.$$

Therefore f is injective. □

Proposition 1.3.4. *Let M, N be R-modules, then there is a canonical $S^{-1}R$-module isomorphism $f : S^{-1}M \otimes_{S^{-1}R} S^{-1}N \to S^{-1}(M \otimes_R N)$, given by*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st.$$

Proof. We may use Proposition 1.3.3 repeatedly to see that these two $S^{-1}R$-modules are isomorphic, noting that $S^{-1}R$ is an $(R, S^{-1}R)$-bimodule:

$$\begin{aligned} S^{-1}(M \otimes_R N) &\cong S^{-1}R \otimes_R (M \otimes_R N) \\ &\cong S^{-1}M \otimes_R N \\ &\cong (S^{-1}M \otimes_{S^{-1}R} S^{-1}R) \otimes_R N \\ &\cong S^{-1}M \otimes_{S^{-1}R} (S^{-1}R \otimes_R N) \\ &\cong S^{-1}M \otimes_{S^{-1}R} S^{-1}N \end{aligned}$$

This isomorphism is easily seen to be the one stated in the lemma. □

1.3.2 Base-Change Properties

We formally introduce base change in algebra as follows.

Definition 1.3.5. *Let $\varphi : R \to S$ be a ring map. Let M be an S-module. Let $R \to R'$ be any ring map. The base change of φ by $R \to R'$ is the ring map $R' \to S \otimes_R R'$. In this situation we often write $S' = S \otimes_R R'$. The base change of the S-module M is the S'-module $M \otimes_R R'$.*

If $S = R[x_i]/(f_j)$ for some collection of variables x_i, $i \in I$ and some collection of polynomials $f_j \in R[x_i]$, $j \in J$, then $S \otimes_R R' = R'[x_i]/(f'_j)$, where $f'_j \in R'[x_i]$ is the image of f_j under the map $R[x_i] \to R'[x_i]$ induced by $R \to R'$. This simple remark is the key to understanding base change.

Proposition 1.3.6. *The finite generatedness/finite presentation of modules and rings are stable under base change.*

Proof. Trivial since the tensor product is right exact. □

Definition 1.3.7. *Let $\varphi : R \to S$ be a ring map. Given an S-module N we obtain an R-module N_R by the rule $r \cdot n = \varphi(r)n$. This is sometimes called the restriction of N to R.*

Proposition 1.3.8. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto M \otimes_R S$ (base change) are adjoint functors. In a formula*

$$\mathrm{Hom}_R(M, N_R) = \mathrm{Hom}_S(M \otimes_R S, N)$$

Proof. If $\alpha : M \to N_R$ is an R-module map, then we define $\alpha' : M \otimes_R S \to N$ by the rule $\alpha'(m \otimes s) = s\alpha(m)$. If $\beta : M \otimes_R S \to N$ is an S-module map, we define $\beta' : M \to N_R$ by the rule $\beta'(m) = \beta(m \otimes 1)$. We omit the verification that these constructions are mutually inverse. □

The lemma above tells us that restriction has a left adjoint, namely base change. It also has a right adjoint.

Proposition 1.3.9. *Let $R \to S$ be a ring map. The functors $Mod_S \to Mod_R$, $N \mapsto N_R$ (restriction) and $Mod_R \to Mod_S$, $M \mapsto \mathrm{Hom}_R(S, M)$ are adjoint functors. In a formula*

$$\mathrm{Hom}_R(N_R, M) = \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M))$$

Proof. If $\alpha : N_R \to M$ is an R-module map, then we define $\alpha' : N \to \mathrm{Hom}_R(S, M)$ by the rule $\alpha'(n) = (s \mapsto \alpha(sn))$. If $\beta : N \to \mathrm{Hom}_R(S, M)$ is an S-module map, we define $\beta' : N_R \to M$ by the rule $\beta'(n) = \beta(n)(1)$. We omit the verification that these constructions are mutually inverse. □

Proposition 1.3.10. *Let $R \to S$ be a ring map. Given S-modules M, N and an R-module P we have*

$$\mathrm{Hom}_R(M \otimes_S N, P) = \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))$$

Proof. This can be proved directly, but it is also a consequence of Propositions 1.3.8 and 1.3.9. Namely, we have

$$\begin{aligned}\mathrm{Hom}_R(M \otimes_S N, P) &= \mathrm{Hom}_S(M \otimes_S N, \mathrm{Hom}_R(S, P)) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, P))) \\ &= \mathrm{Hom}_S(M, \mathrm{Hom}_R(N, P))\end{aligned}$$

as desired. □

1.4 Some Radicals

1.4.1 Radical of Rings

Definition 1.4.1. *For any ideal $I \subset R$, define $\sqrt{I} := \{x \in R : x^n \in I$ for some $n\}$.*

Proposition 1.4.2. *For any ideal $I \subset R$, we have*

$$\sqrt{I} = \bigcap_{I \subset \mathfrak{p}, \mathfrak{p} \textit{ prime}} \mathfrak{p}.$$

Proof. The inclusion $\sqrt{I} \subset \bigcap_{I \subset \mathfrak{p}, \mathfrak{p} \text{ primes}} \mathfrak{p}$ is trivial by definitions.

Conversely, take $g \in R \backslash \sqrt{I}$, then $g^n \notin I$ for any n. Let $\bar{\mathfrak{p}} \subset R_g$ be a prime such that $IR_g \subset \bar{\mathfrak{p}} \subset R_g$. Take $\mathfrak{p} \subset R$ be the inverse image of $\bar{\mathfrak{p}}$, then $I \subset \mathfrak{p}$ but $P \cap \{1, g, g^2, ...\} = \emptyset$. Well done. □

1.4.2 Jacobson Radical and Nilradical of Rings

Definition 1.4.3. *Let R be a ring.*

(1) The Jacobson radical *of a ring R is*

$$\mathrm{rad}(R) = \bigcap_{\mathfrak{m}, \mathfrak{m} \text{ maximal}} \mathfrak{m}.$$

(2) The nilradical *of a ring R is*

$$\mathrm{nil}(R) = \sqrt{0} = \bigcap_{\mathfrak{p}, \mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Proposition 1.4.4. *Let R be a ring with Jacobson radical* $\mathrm{rad}(R)$*. Let $I \subset R$ be an ideal. The following are equivalent*

(1) $I \subset \mathrm{rad}(R)$, and

(2) every element of $1 + I$ is a unit in R.

In this case every element of R which maps to a unit of R/I is a unit.

Proof. If $f \in \mathrm{rad}(R)$, then $f \in \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Hence $1 + f \notin \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R. Thus the closed subset $V(1 + f)$ of $\mathrm{Spec}(R)$ is empty. This implies that $1 + f$ is a unit.

Conversely, assume that $1 + f$ is a unit for all $f \in I$. If $\mathfrak{m}$ is a maximal ideal and $I \not\subset \mathfrak{m}$, then $I + \mathfrak{m} = R$. Hence $1 = f + g$ for some $g \in \mathfrak{m}$ and $f \in I$. Then $g = 1 + (-f)$ is not a unit, contradiction.

For the final statement let $f \in R$ map to a unit in R/I. Then we can find $g \in R$ mapping to the multiplicative inverse of $f \bmod I$. Then $fg = 1 \bmod I$. Hence fg is a unit of R by (2) which implies that f is a unit. $\square$

Lemma 1.4.5. *Let $\varphi : R \to S$ be a ring map such that the induced map* $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ *is surjective. Then an element $x \in R$ is a unit if and only if $\varphi(x) \in S$ is a unit.*

Proof. If x is a unit, then so is $\varphi(x)$. Conversely, if $\varphi(x)$ is a unit, then $\varphi(x) \notin \mathfrak{q}$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Hence $x \notin \varphi^{-1}(\mathfrak{q}) = \mathrm{Spec}(\varphi)(\mathfrak{q})$ for all $\mathfrak{q} \in \mathrm{Spec}(S)$. Since $\mathrm{Spec}(\varphi)$ is surjective we conclude that x is a unit. $\square$

1.5 Prime Ideals, some Interesting Things

1.5.1 Prime Avoidance

This is an easy but important result.

Lemma 1.5.1. *Let R be a ring, I and J two ideals and $\mathfrak{p}$ a prime ideal containing the product IJ. Then $\mathfrak{p}$ contains I or J.*

Proof. Assume the contrary and take $x \in I \setminus \mathfrak{p}$ and $y \in J \setminus \mathfrak{p}$. Their product is an element of $IJ \subset \mathfrak{p}$, which contradicts the assumption that $\mathfrak{p}$ was prime. □

Proposition 1.5.2 (Prime Avoidance)**.** *Let R be a ring. Let $I_i \subset R$, $i = 1, \ldots, r$, and $J \subset R$ be ideals. Assume*

(1) $J \not\subset I_i$ for $i = 1, \ldots, r$, and

(2) all but two of I_i are prime ideals.

Then there exists an $x \in J$, $x \notin I_i$ for all i.

Proof. The result is true for $r = 1$. If $r = 2$, then let $x, y \in J$ with $x \notin I_1$ and $y \notin I_2$. We are done unless $x \in I_2$ and $y \in I_1$. Then the element $x + y$ cannot be in I_1 (since that would mean $x + y - y \in I_1$) and it also cannot be in I_2.

For $r \geq 3$, assume the result holds for $r - 1$. After renumbering we may assume that I_r is prime. We may also assume there are no inclusions among the I_i. Pick $x \in J$, $x \notin I_i$ for all $i = 1, \ldots, r - 1$. If $x \notin I_r$ we are done. So assume $x \in I_r$. If $JI_1 \ldots I_{r-1} \subset I_r$ then $J \subset I_r$ (by Lemma 1.5.1) a contradiction. Pick $y \in JI_1 \ldots I_{r-1}$, $y \notin I_r$. Then $x + y$ works. □

1.5.2 Oka Families and Its Applications

Here we introduce a very interesting thing.

Definition 1.5.3. *Let R be a ring. If I is an ideal of R and $a \in R$, we define*

$$(I : a) = \{x \in R : xa \in I\}.$$

More generally, if $J \subset R$ is an ideal, we define

$$(I : J) = \{x \in R : xJ \subset I\}.$$

Definition 1.5.4 (Oka Family)**.** *Let R be a ring. Let $\mathcal{F}$ be a set of ideals of R. We say $\mathcal{F}$ is an Oka family if $R \in \mathcal{F}$ and whenever $I \subset R$ is an ideal and $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$, then $I \in \mathcal{F}$.*

Here is the fundamental property of Oka family:

Proposition 1.5.5. *If $\mathcal{F}$ is an Oka family of ideals, then any maximal element of the complement of $\mathcal{F}$ is prime.*

Proof. Suppose $I \notin \mathcal{F}$ is maximal with respect to not being in $\mathcal{F}$ but I is not prime. Note that $I \neq R$ because $R \in \mathcal{F}$. Since I is not prime we can find $a, b \in R - I$ with $ab \in I$. It follows that $(I, a) \neq I$ and $(I : a)$ contains $b \notin I$ so also $(I : a) \neq I$. Thus $(I : a), (I, a)$ both strictly contain I, so they must belong to $\mathcal{F}$. By the Oka condition, we have $I \in \mathcal{F}$, a contradiction. □

Now we discover some special Oka families which will induce many interesting results! Before that, we introduce a lemma:

Lemma 1.5.6. *Let R be a ring. For a principal ideal $J \subset R$, and for any ideal $I \subset J$ we have $I = J(I : J)$.*

Proof. Say $J = (a)$. Then $(I : J) = (I : a)$. Since $I \subset J$ we see that any $y \in I$ is of the form $y = xa$ for some $x \in (I : a)$. Hence $I \subset J(I : J)$. Conversely, if $x \in (I : a)$, then $xJ = (xa) \subset I$, which proves the other inclusion. □

Corollary 1.5.7. *Let R be a ring and let S be a multiplicative subset of R.*

(1) The family $\mathcal{F} = \{I \subset R \mid I \cap S \neq \emptyset\}$ is an Oka family.

(2) An ideal $I \subset R$ which is maximal with respect to the property that $I \cap S = \emptyset$ is prime.

In particular, we have the following things.

(3) An ideal maximal among the ideals which do not contain a nonzerodivisor is prime.

(4) If R is nonzero and every nonzero prime ideal in R contains a nonzerodivisor, then R is a domain.

Proof. For (1), suppose that $(I : a), (I, a) \in \mathcal{F}$ for some $a \in R$. Then pick $s \in (I, a) \cap S$ and $s' \in (I : a) \cap S$. Then $ss' \in I \cap S$ and hence $I \in \mathcal{F}$. Thus $\mathcal{F}$ is an Oka family.

For (2), this follows directly from (1) and Proposition 1.5.5.

For (3), consider the set S of nonzerodivisors. It is a multiplicative subset of R. Hence any ideal maximal with respect to not intersecting S is prime by (1).

Thus for (4), if every nonzero prime ideal contains a nonzerodivisor, then (0) is prime, i.e., R is a domain. □

Corollary 1.5.8. *Let R be a ring.*

(1) The family of finitely generated ideals is an Oka family.

(2) *An ideal $I \subset R$ maximal with respect to not being finitely generated is prime.*

(3) *If every prime ideal of R is finitely generated, then every ideal of R is finitely generated, that is, R is Noetherian.*

Proof. For (1), Let $I \subset R$ an ideal, and $a \in R$. If $(I : a)$ is generated by $a_1, \dots, a_n$ and (I, a) is generated by $a, b_1, \dots, b_m$ with $b_1, \dots, b_m \in I$, we claim that I is generated by $aa_1, \dots, aa_n, b_1, \dots, b_m$.

Indeed, note that if $x \in I$, then $x \in (I, a)$ is a linear combination of $a, b_1, \dots, b_m$, but the coefficient of a must lie in $(I : a)$. As a result, we deduce that the family of finitely generated ideals is an Oka family.

For (2), this is an immediate consequence of (1) and Proposition 1.5.5.

For (3), suppose that there exists an ideal $I \subset R$ which is not finitely generated. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that are not finitely generated is not finitely generated; indeed, if $I = \bigcup I_\alpha$ were generated by $a_1, \dots, a_n$, then all the generators would belong to some I_α and would consequently generate it. By Zorn's lemma, there is an ideal maximal with respect to being not finitely generated. By (2) this ideal is prime. $\square$

Corollary 1.5.9. *Let R be a ring.*

(1) *The family of principal ideals of R is an Oka family.*

(2) *An ideal $I \subset R$ maximal with respect to not being principal is prime.*

(3) *If every prime ideal of R is principal, then every ideal of R is principal.*

Proof. For (1), suppose $I \subset R$ is an ideal, $a \in R$, and (I, a) and $(I : a)$ are principal. Note that $(I : a) = (I : (I, a))$. Setting $J = (I, a)$, we find that J is principal and $(I : J)$ is too. By Lemma 1.5.6 we have $I = J(I : J)$. Thus we find in our situation that since $J = (I, a)$ and $(I : J)$ are principal, I is principal.

For (2), this follows from (1) and Proposition 1.5.5.

For (3), suppose that there exists an ideal $I \subset R$ which is not principal. The union of a totally ordered chain $\{I_\alpha\}$ of ideals that not principal is not principal; indeed, if $I = \bigcup I_\alpha$ were generated by a, then a would belong to some I_α and a would generate it. By Zorn's lemma, there is an ideal maximal with respect to not being principal. This ideal is necessarily prime by (2). $\square$

Corollary 1.5.10. *Let A be a ring, $I \subset A$ an ideal, and $a \in A$ an element. Let P is a property of A-modules that is stable under extensions and holds for* 0.

(1) *The family of ideals I such that A/I has P is an Oka family.*

(2) *The ideal maximal such that P does not holds is prime.*

Proof. For (1), there is a short exact sequence $0 \to A/(I : a) \to A/I \to A/(I, a) \to 0$ where the first arrow is given by multiplication by a. Thus if P is a property of A-modules that is stable under extensions and holds for 0, then the family of ideals I such that A/I has P is an Oka family.

For (2), this follows from (1) and Proposition 1.5.5. □

1.6 Cayley-Hamilton

Here we introduce Cayley-Hamilton theorem of general rings and its applications.

Proposition 1.6.1 (Cayley-Hamilton)**.** *Let R be a ring. Let $A = (a_{ij})$ be an $n \times n$ matrix with coefficients in R. Let $P(x) \in R[x]$ be the characteristic polynomial of A (defined as $\det(x\mathrm{id}_{n\times n} - A)$). Then $P(A) = 0$ in $Mat(n \times n, R)$.*

Proof. We reduce the question to the well-known Cayley-Hamilton theorem from linear algebra in several steps:

1. If $\phi : S \to R$ is a ring morphism and b_{ij} are inverse images of the a_{ij} under this map, then it suffices to show the statement for S and (b_{ij}) since ϕ is a ring morphism.

2. If $\psi : R \hookrightarrow S$ is an injective ring morphism, it clearly suffices to show the result for S and the a_{ij} considered as elements of S.

3. Thus we may first reduce to the case $R = \mathbb{Z}[X_{ij}]$, $a_{ij} = X_{ij}$ of a polynomial ring and then further to the case $R = \mathbb{Q}(X_{ij})$ where we may finally apply Cayley-Hamilton.

Then well done. □

Corollary 1.6.2. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism. Then there exists a monic polynomial $P \in R[T]$ such that $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Consider

$$\begin{tikzcd} R^{\oplus n} \arrow[r] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^{\oplus n} \arrow[r] & M \end{tikzcd}$$

By Proposition 1.6.1 there exists a monic polynomial P such that $P(A) = 0$. Then it follows that $P(\varphi) = 0$. □

Corollary 1.6.3. *Let R be a ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $\varphi : M \to M$ be an endomorphism such that $\varphi(M) \subset IM$. Then there exists a monic polynomial $P = t^n + a_1 t^{n-1} + \ldots + a_n \in R[T]$ such that $a_j \in I^j$ and $P(\varphi) = 0$ as an endomorphism of M.*

Proof. Consider again

$$\begin{array}{ccc} R^{\oplus n} & \longrightarrow & M \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle \varphi} \\ I^{\oplus n} & \longrightarrow & M \end{array}$$

By Proposition 1.6.1 the polynomial $P(t) = \det(t\mathrm{id}_{n\times n} - A)$ has all the desired properties. □

As a fun example application we prove the following surprising property.

Corollary 1.6.4. *Let R be a ring. Let M be a finite R-module. Let $\varphi : M \to M$ be a surjective R-module map. Then φ is an isomorphism.*

Proof. Write $R' = R[x]$ and think of M as a finite R'-module with x acting via φ. Set $I = (x) \subset R'$. By our assumption that φ is surjective we have $IM = M$. Hence we may apply Corollary 1.6.3 to M as an R'-module, the ideal I and the endomorphism id_M. We conclude that $(1 + a_1 + \ldots + a_n)\mathrm{id}_M = 0$ with $a_j \in I$. Write $a_j = b_j(x)x$ for some $b_j(x) \in R[x]$. Translating back into φ we see that $\mathrm{id}_M = -(\sum_{j=1,\ldots,n} b_j(\varphi))\varphi$, and hence φ is invertible. □

1.7 Nakayama's Lemma

First we recall a lemma:

Lemma 1.7.1. *Let R be a ring. Let $n \geq m$. Let A be an $n \times m$ matrix with coefficients in R. Let $J \subset R$ be the ideal generated by the $m \times m$ minors of A.*

1. *For any $f \in J$ there exists a $m \times n$ matrix B such that $BA = f1_{m\times m}$.*
2. *If $f \in R$ and $BA = f1_{m\times m}$ for some $m \times n$ matrix B, then $f^m \in J$.*

Proof. For $I \subset \{1, \ldots, n\}$ with $|I| = m$, we denote by E_I the $m \times n$ matrix of the projection

$$R^{\oplus n} = \bigoplus\nolimits_{i\in\{1,\ldots,n\}} R \longrightarrow \bigoplus\nolimits_{i\in I} R$$

and set $A_I = E_I A$, i.e., A_I is the $m \times m$ matrix whose rows are the rows of A with indices in I. Let B_I be the adjugate (transpose of cofactor) matrix to A_I, i.e., such that $A_I B_I = B_I A_I = \det(A_I)1_{m\times m}$. The $m \times m$ minors of A are the determinants $\det A_I$

for all the $I \subset \{1, \ldots, n\}$ with $|I| = m$. If $f \in J$ then we can write $f = \sum c_I \det(A_I)$ for some $c_I \in R$. Set $B = \sum c_I B_I E_I$ to see that (1) holds.

If $f 1_{m\times m} = BA$ then by the Cauchy-Binet formula we have $f^m = \sum b_I \det(A_I)$ where b_I is the determinant of the $m \times m$ matrix whose columns are the columns of B with indices in I. □

Theorem 1.7.2 (Nakayama's lemma)**.** *Let R be a ring with Jacobson radical* $\mathrm{rad}(R)$. *Let M be an R-module. Let $I \subset R$ be an ideal.*

(1) If $IM = M$ and M is finite, then there exists an $f \in 1 + I$ such that $fM = 0$.

(2) If $IM = M$, M is finite, and $I \subset \mathrm{rad}(R)$, then $M = 0$.

(3) If $N, N' \subset M$, $M = N + IN'$, and N' is finite, then there exists an $f \in 1+I$ such that $fM \subset N$ and $M_f = N_f$.

(4) If $N, N' \subset M$, $M = N + IN'$, N' is finite, and $I \subset \mathrm{rad}(R)$, then $M = N$.

(5) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, and M is finite, then there exists an $f \in 1 + I$ such that $N_f \to M_f$ is surjective.

(6) If $N \to M$ is a module map, $N/IN \to M/IM$ is surjective, M is finite, and $I \subset \mathrm{rad}(R)$, then $N \to M$ is surjective.

(7) If $x_1, \ldots, x_n \in M$ generate M/IM and M is finite, then there exists an $f \in 1+I$ such that $x_1, \ldots, x_n$ generate M_f over R_f.

(8) If $x_1, \ldots, x_n \in M$ generate M/IM, M is finite, and $I \subset \mathrm{rad}(R)$, then M is generated by $x_1, \ldots, x_n$.

(9) If $IM = M$, I is nilpotent, then $M = 0$.

(10) If $N, N' \subset M$, $M = N + IN'$, and I is nilpotent then $M = N$.

(11) If $N \to M$ is a module map, I is nilpotent, and $N/IN \to M/IM$ is surjective, then $N \to M$ is surjective.

(12) If $\{x_\alpha\}_{\alpha\in A}$ is a set of elements of M which generate M/IM and I is nilpotent, then M is generated by the x_α.

Proof. For (1). Choose generators $y_1, \ldots, y_m$ of M over R. For each i we can write $y_i = \sum z_{ij} y_j$ with $z_{ij} \in I$ since $M = IM$. In other words $\sum_j (\delta_{ij} - z_{ij}) y_j = 0$. Let f be the determinant of the $m \times m$ matrix $A = (\delta_{ij} - z_{ij})$. Note that $f \in 1 + I$. By Lemma 1.7.1 (1), there exists an $m \times m$ matrix B such that $BA = f 1_{m\times m}$. Writing out we see that $\sum_i b_{hi} a_{ij} = f \delta_{hj}$ for all h and j; hence, $\sum_{i,j} b_{hi} a_{ij} y_j = \sum_j f \delta_{hj} y_j = f y_h$ for every h. In other words, $0 = f y_h$ for every h (since each i satisfies $\sum_j a_{ij} y_j = 0$). This implies that f annihilates M.

By Lemma 1.4.4 an element of $1 + \mathrm{rad}(R)$ is invertible element of R. Hence we see that (1) implies (2). We obtain (3) by applying (1) to M/N which is finite as N' is finite. We obtain (4) by applying (2) to M/N which is finite as N' is finite. We obtain (5) by applying (3) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (6) by applying (4) to M and the submodules $\mathrm{Im}(N \to M)$ and M. We obtain (7) by applying (5) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$. We obtain (8) by applying (6) to the map $R^{\oplus n} \to M$, $(a_1, \ldots, a_n) \mapsto a_1x_1 + \ldots + a_nx_n$.

Part (9) holds because if $M = IM$ then $M = I^nM$ for all $n \geq 0$ and I being nilpotent means $I^n = 0$ for some $n \gg 0$. Parts (10), (11), and (12) follow from (9) by the arguments used above. □

Lemma 1.7.3. *Let R be a ring, let $S \subset R$ be a multiplicative subset, let $I \subset R$ be an ideal, and let M be a finite R-module. If $x_1, \ldots, x_r \in M$ generate $S^{-1}(M/IM)$ as an $S^{-1}(R/I)$-module, then there exists an $f \in S + I$ such that $x_1, \ldots, x_r$ generate M_f as an R_f-module.*[1]

Proof. Special case $I = 0$. Let $y_1, \ldots, y_s$ be generators for M over R. Since $S^{-1}M$ is generated by $x_1, \ldots, x_r$, for each i we can write $y_i = \sum(a_{ij}/s_{ij})x_j$ for some $a_{ij} \in R$ and $s_{ij} \in S$. Let $s \in S$ be the product of all of the s_{ij}. Then we see that y_i is contained in the R_s-submodule of M_s generated by $x_1, \ldots, x_r$. Hence $x_1, \ldots, x_r$ generates M_s.

General case. By the special case, we can find an $s \in S$ such that $x_1, \ldots, x_r$ generate $(M/IM)_s$ over $(R/I)_s$. By Nakayama's Lemma 1.7.2 we can find a $g \in 1 + I_s \subset R_s$ such that $x_1, \ldots, x_r$ generate $(M_s)_g$ over $(R_s)_g$. Write $g = 1 + i/s'$. Then $f = ss' + is$ works; details omitted. □

1.8 The Spectrums of a Ring

1.8.1 Basic Facts

Proposition 1.8.1. *Let R be a ring with an ideal $J \subset R$ and s subset $S \subset \mathrm{Spec}(R)$. Define $I(S) := \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.*

(1) We have $\sqrt{I(S)} = I(S)$ and $I(V(J)) = \sqrt{J}$ and $V(I(S)) = \overline{S}$.

(2) Let $f : R \to R'$ is a ring map induce $F : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$, then we have

  (a) For any subset $M \subset R$, we have $F^{-1}(V(M)) = V(f(M))$. In particular $F^{-1}(D(r)) = D(f(r))$ for $r \in R$.

  (b) For any ideal $I \subset R'$, we have $V(f^{-1}(I)) = \overline{F(V(I))}$.

[1]Special cases: (I) $I = 0$. The lemma says if $x_1, \ldots, x_r$ generate $S^{-1}M$, then $x_1, \ldots, x_r$ generate M_f for some $f \in S$. (II) $I = \mathfrak{p}$ is a prime ideal and $S = R \backslash \mathfrak{p}$. The lemma says if $x_1, \ldots, x_r$ generate $M \otimes_R \kappa(\mathfrak{p})$ then $x_1, \ldots, x_r$ generate M_f for some $f \in R$, $f \notin \mathfrak{p}$.

Proof. (1) follows from Proposition 1.4.2. (2)(a) are trivial. For (2)(b), as

$$I(F(V(I))) = \bigcap_{\mathfrak{p}\in V(I)} f^{-1}(\mathfrak{p}) = f^{-1}(\sqrt{I}) = \sqrt{(f^{-1}(I))}.$$

Hence by (1) we have $\overline{F(V(I))} = V(I(F(V(I)))) = V(\sqrt{(f^{-1}(I))}) = V(f^{-1}(I))$. □

Corollary 1.8.2. *Let $f: R \to R'$ is a ring map induce $F: \mathrm{Spec}(R') \to \mathrm{Spec}(R)$, then F has a densed image if and only if* $\ker f$ *consist of nilpotent elements.*

Proof. By Proposition 1.8.1(2)(b), we have $V(\ker f) = \overline{F(V(0))} = \overline{F(\mathrm{Spec}(R))}$. Well done. □

1.8.2 Fundamental Diagram of Ring Maps

Proposition 1.8.3. *A fundamental commutative diagram associated to a ring map $\varphi: R \to S$, a prime $\mathfrak{q} \subset S$ and the corresponding prime $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ of R is the following:*

$$\begin{tikzcd}
\kappa(\mathfrak{q}) = S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} & S_{\mathfrak{q}} \arrow[l] & S \arrow[l] \arrow[r] & S/\mathfrak{q} \arrow[r] & \kappa(\mathfrak{q}) \\
\kappa(\mathfrak{p}) \otimes_R S = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \arrow[u] & S_{\mathfrak{p}} \arrow[l] \arrow[u] & S \arrow[l] \arrow[r] \arrow[u] & S/\mathfrak{p}S \arrow[r] \arrow[u] & (R\backslash\mathfrak{p})^{-1}S/\mathfrak{p}S \arrow[u] \\
\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \arrow[u] & R_{\mathfrak{p}} \arrow[l] \arrow[u] & R \arrow[l] \arrow[r] \arrow[u, "\varphi"'] & R/\mathfrak{p} \arrow[r] \arrow[u] & \kappa(\mathfrak{p}) \arrow[u]
\end{tikzcd}$$

In this diagram the arrows in the outer left and outer right columns are identical. The horizontal maps induce on the associated spectra always a homeomorphism onto the image. The lower two rows of the diagram make sense without assuming $\mathfrak{q}$ exists. The lower squares induce fibre squares of topological spaces. This diagram shows that $\mathfrak{p}$ is in the image of the map on Spec *if and only if $S \otimes_R \kappa(\mathfrak{p})$ is not the zero ring.*

1.8.3 Connected Components and Idempotents

It turns out that open and closed subsets of a spectrum correspond to idempotents of the ring.

Lemma 1.8.4. *Let R be a ring. Let $e \in R$ be an idempotent. In this case*

$$\mathrm{Spec}(R) = D(e) \amalg D(1-e).$$

Proof. Trivial. □

Lemma 1.8.5. *Let R_1 and R_2 be rings. Let $R = R_1 \times R_2$. The maps $R \to R_1$, $(x, y) \mapsto x$ and $R \to R_2$, $(x, y) \mapsto y$ induce continuous maps $\mathrm{Spec}(R_1) \to \mathrm{Spec}(R)$ and $\mathrm{Spec}(R_2) \to \mathrm{Spec}(R)$. The induced map*

$$\mathrm{Spec}(R_1) \amalg \mathrm{Spec}(R_2) \longrightarrow \mathrm{Spec}(R)$$

is a homeomorphism. In other words, the spectrum of $R = R_1 \times R_2$ is the disjoint union of the spectrum of R_1 and the spectrum of R_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. We leave it to the reader to show that $R_1 = R_{e_1}$ is the localization of R at e_1. Similarly for e_2. Thus the statement of the lemma follows from Lemma 1.8.4. □

Proposition 1.8.6. *Let R be a ring. For each $U \subset \mathrm{Spec}(R)$ which is open and closed there exists a unique idempotent $e \in R$ such that $U = D(e)$. This induces a 1-1 correspondence between open and closed subsets $U \subset \mathrm{Spec}(R)$ and idempotents $e \in R$.*

Proof. Let $U \subset \mathrm{Spec}(R)$ be open and closed. Since U is closed it is quasi-compact, and similarly for its complement. Write $U = \bigcup_{i=1}^{n} D(f_i)$ as a finite union of standard opens. Similarly, write $\mathrm{Spec}(R) \setminus U = \bigcup_{j=1}^{m} D(g_j)$ as a finite union of standard opens. Since $\emptyset = D(f_i) \cap D(g_j) = D(f_i g_j)$ we see that $f_i g_j$ is nilpotent by Proposition 1.4.2. Let $I = (f_1, \ldots, f_n) \subset R$ and let $J = (g_1, \ldots, g_m) \subset R$. Note that $V(J)$ equals U, that $V(I)$ equals the complement of U, so $\mathrm{Spec}(R) = V(I) \amalg V(J)$. By the remark on nilpotency above, we see that $(IJ)^N = (0)$ for some sufficiently large integer N. Since $\bigcup D(f_i) \cup \bigcup D(g_j) = \mathrm{Spec}(R)$ we see that $I + J = R$. By raising this equation to the $2N$th power we conclude that $I^N + J^N = R$. Write $1 = x + y$ with $x \in I^N$ and $y \in J^N$. Then $0 = xy = x(1 - x)$ as $I^N J^N = (0)$. Thus $x = x^2$ is idempotent and contained in $I^N \subset I$. The idempotent $y = 1 - x$ is contained in $J^N \subset J$. This shows that the idempotent x maps to 1 in every residue field $\kappa(\mathfrak{p})$ for $\mathfrak{p} \in V(J)$ and that x maps to 0 in $\kappa(\mathfrak{p})$ for every $\mathfrak{p} \in V(I)$.

To see uniqueness suppose that e_1, e_2 are distinct idempotents in R. We have to show there exists a prime $\mathfrak{p}$ such that $e_1 \in \mathfrak{p}$ and $e_2 \notin \mathfrak{p}$, or conversely. Write $e_i' = 1 - e_i$. If $e_1 \neq e_2$, then $0 \neq e_1 - e_2 = e_1(e_2 + e_2') - (e_1 + e_1')e_2 = e_1 e_2' - e_1' e_2$. Hence either the idempotent $e_1 e_2' \neq 0$ or $e_1' e_2 \neq 0$. An idempotent is not nilpotent, and hence we find a prime $\mathfrak{p}$ such that either $e_1 e_2' \notin \mathfrak{p}$ or $e_1' e_2 \notin \mathfrak{p}$. It is easy to see this gives the desired prime. □

Corollary 1.8.7. *Let R be a nonzero ring. Then $\mathrm{Spec}(R)$ is connected if and only if R has no nontrivial idempotents.*

Proof. Obvious from Proposition 1.8.6 and the definition of a connected topological space. □

Lemma 1.8.8. *Let R be a ring. A connected component of* $\mathrm{Spec}(R)$ *is of the form* $V(I)$*, where I is an ideal generated by idempotents such that every idempotent of R either maps to* 0 *or* 1 *in* R/I*.*

Proof. Let $\mathfrak{p}$ be a prime of R. By some general topology, the connected component of $\mathfrak{p}$ in $\mathrm{Spec}(R)$ is the intersection of open and closed subsets of $\mathrm{Spec}(R)$ containing $\mathfrak{p}$. Hence it equals $V(I)$ where I is generated by the idempotents $e \in R$ such that e maps to 0 in $\kappa(\mathfrak{p})$, see Proposition 1.8.6. Any idempotent e which is not in this collection clearly maps to 1 in R/I. □

1.8.4 Glueing Properties

In this section we put a number of standard results of the form: if something is true for all members of a standard open covering then it is true. In fact, it often suffices to check things on the level of local rings as in the following lemma.

Proposition 1.8.9. *Let R be a ring.*

(1) For an element x of an R-module M the following are equivalent

- *(a)* $x = 0$*,*
- *(b) x maps to zero in $M_{\mathfrak{p}}$ for all* $\mathfrak{p} \in \mathrm{Spec}(R)$*,*
- *(c) x maps to zero in $M_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of R.*

In other words, the map $M \to \prod_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective.

(2) Given an R-module M the following are equivalent

- *(a) M is zero,*
- *(b) $M_{\mathfrak{p}}$ is zero for all* $\mathfrak{p} \in \mathrm{Spec}(R)$*,*
- *(c) $M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$ of R.*

(3) Given a complex $M_1 \to M_2 \to M_3$ of R-modules the following are equivalent

- *(a) $M_1 \to M_2 \to M_3$ is exact,*
- *(b) for every prime $\mathfrak{p}$ of R the localization $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$ is exact,*
- *(c) for every maximal ideal $\mathfrak{m}$ of R the localization $M_{1,\mathfrak{m}} \to M_{2,\mathfrak{m}} \to M_{3,\mathfrak{m}}$ is exact.*

(4) Given a map $f : M \to M'$ of R-modules the following are equivalent

- *(a) f is injective,*
- *(b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ is injective for all primes $\mathfrak{p}$ of R,*

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is injective for all maximal ideals* $\mathfrak{m}$ *of* R.

(5) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*

   (a) f *is surjective,*

   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is surjective for all primes* $\mathfrak{p}$ *of* R,

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is surjective for all maximal ideals* $\mathfrak{m}$ *of* R.

(6) *Given a map* $f : M \to M'$ *of* R*-modules the following are equivalent*

   (a) f *is bijective,*

   (b) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to M'_{\mathfrak{p}}$ *is bijective for all primes* $\mathfrak{p}$ *of* R,

   (c) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M'_{\mathfrak{m}}$ *is bijective for all maximal ideals* $\mathfrak{m}$ *of* R.

Proof. Let $x \in M$ as in (1). Let $I = \{f \in R \mid fx = 0\}$. It is easy to see that I is an ideal (it is the annihilator of x). Condition (1)(c) means that for all maximal ideals $\mathfrak{m}$ there exists an $f \in R \setminus \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ does not contain a closed point. Hence I is the unit ideal. Hence x is zero, i.e., (1)(a) holds. This proves (1).

Part (2) follows by applying (1) to all elements of M simultaneously.

Proof of (3). Let H be the homology of the sequence, i.e., $H = \ker(M_2 \to M_3)/\mathrm{Im}(M_1 \to M_2)$. As localization is exact, we have that $H_{\mathfrak{p}}$ is the homology of the sequence $M_{1,\mathfrak{p}} \to M_{2,\mathfrak{p}} \to M_{3,\mathfrak{p}}$. Hence (3) is a consequence of (2).

Parts (4) and (5) are special cases of (3). Part (6) follows formally on combining (4) and (5). □

Proposition 1.8.10. *Let* R *be a ring. Let* M *be an* R*-module. Let* S *be an* R*-algebra. Suppose that* $f_1, \ldots, f_n$ *is a finite list of elements of* R *such that* $\bigcup D(f_i) = \mathrm{Spec}(R)$*, in other words* $(f_1, \ldots, f_n) = R$.

(1) *If each* $M_{f_i} = 0$ *then* $M = 0$.

(2) *If each* M_{f_i} *is a finite* R_{f_i}*-module, then* M *is a finite* R*-module.*

(3) *If each* M_{f_i} *is a finitely presented* R_{f_i}*-module, then* M *is a finitely presented* R*-module.*

(4) *Let* $M \to N$ *be a map of* R*-modules. If* $M_{f_i} \to N_{f_i}$ *is an isomorphism for each* i *then* $M \to N$ *is an isomorphism.*

(5) *Let* $0 \to M'' \to M \to M' \to 0$ *be a complex of* R*-modules. If* $0 \to M''_{f_i} \to M_{f_i} \to M'_{f_i} \to 0$ *is exact for each* i*, then* $0 \to M'' \to M \to M' \to 0$ *is exact.*

(6) *If each* R_{f_i} *is Noetherian, then* R *is Noetherian.*

(7) *If each S_{f_i} is a finite type R-algebra, so is S.*

(8) *If each S_{f_i} is of finite presentation over R, so is S.*

Proof. We prove each of the parts in turn.

(1) By second localization, this implies $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, so we conclude by Proposition 1.8.9.

(2) For each i take a finite generating set X_i of M_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $M \to M_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in M. Let Y be the union of these sets. This is still a finite set. Consider the obvious R-linear map $R^Y \to M$ sending the basis element e_y to y. By assumption this map is surjective after localizing at an arbitrary prime ideal $\mathfrak{p}$ of R, so it is surjective by Proposition 1.8.9 and M is finitely generated.

(3) By (2) we have a short exact sequence

$$0 \to K \to R^n \to M \to 0$$

Since localization is an exact functor and M_{f_i} is finitely presented we see that K_{f_i} is finitely generated for all $1 \leq i \leq n$. By (2) this implies that K is a finite R-module and therefore M is finitely presented.

(4) By second localization, the assumption implies that the induced morphism on localizations at all prime ideals is an isomorphism, so we conclude by Lemma 1.8.9.

(5) By second localization, the assumption implies that the induced sequence of localizations at all prime ideals is short exact, so we conclude by Lemma 1.8.9.

(6) We will show that every ideal of R has a finite generating set: For this, let $I \subset R$ be an arbitrary ideal. As localization is exact, each $I_{f_i} \subset R_{f_i}$ is an ideal. These are all finitely generated by assumption, so we conclude by (2).

(7) For each i take a finite generating set X_i of S_{f_i}. Without loss of generality, we may assume that the elements of X_i are in the image of the localization map $S \to S_{f_i}$, so we take a finite set Y_i of preimages of the elements of X_i in S. Let Y be the union of these sets. This is still a finite set. Consider the algebra homomorphism $R[X_y]_{y \in Y} \to S$ induced by Y. Since it is an algebra homomorphism, the image T is an R-submodule of the R-module S, so we can consider the quotient module S/T. By assumption, this is zero if we localize at the f_i, so it is zero by (1) and therefore S is an R-algebra of finite type.

(8) By the previous item, there exists a surjective R-algebra homomorphism $R[X_1, \ldots, X_n] \to S$. Let K be the kernel of this map. This is an ideal in $R[X_1, \ldots, X_n]$, finitely generated in each localization at f_i. Since the f_i generate the unit ideal in R, they also generate the unit ideal in $R[X_1, \ldots, X_n]$, so an application of (2) finishes the proof.

□

Corollary 1.8.11. *Let $R \to S$ be a ring map. Suppose that $g_1, \ldots, g_n$ is a finite list of elements of S such that $\bigcup D(g_i) = \mathrm{Spec}(S)$ in other words $(g_1, \ldots, g_n) = S$.*

(1) If each S_{g_i} is of finite type over R, then S is of finite type over R.

(2) If each S_{g_i} is of finite presentation over R, then S is of finite presentation over R.

Proof. Choose $h_1, \ldots, h_n \in S$ such that $\sum h_i g_i = 1$.

Proof of (1). For each i choose a finite list of elements $x_{i,j} \in S_{g_i}$, $j = 1, \ldots, m_i$ which generate S_{g_i} as an R-algebra. Write $x_{i,j} = y_{i,j}/g_i^{n_{i,j}}$ for some $y_{i,j} \in S$ and some $n_{i,j} \geq 0$. Consider the R-subalgebra $S' \subset S$ generated by $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ and $y_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Since localization is exact, we see that $S'_{g_i} \to S_{g_i}$ is injective. On the other hand, it is surjective by our choice of $y_{i,j}$. The elements $g_1, \ldots, g_n$ generate the unit ideal in S' as $h_1, \ldots, h_n \in S'$. Thus $S' \to S$ viewed as an S'-module map is an isomorphism by Lemma 1.8.10.

Proof of (2). We already know that S is of finite type. Write $S = R[x_1, \ldots, x_m]/J$ for some ideal J. For each i choose a lift $g'_i \in R[x_1, \ldots, x_m]$ of g_i and we choose a lift $h'_i \in R[x_1, \ldots, x_m]$ of h_i. Then we see that

$$S_{g_i} = R[x_1, \ldots, x_m, y_i]/(J_i + (1 - y_i g'_i))$$

where J_i is the ideal of $R[x_1, \ldots, x_m, y_i]$ generated by J. Small detail omitted. We may choose a finite list of elements $f_{i,j} \in J$, $j = 1, \ldots, m_i$ such that the images of $f_{i,j}$ in J_i and $1 - y_i g'_i$ generate the ideal $J_i + (1 - y_i g'_i)$. Set

$$S' = R[x_1, \ldots, x_m]/\left(\sum h'_i g'_i - 1, f_{i,j}; i = 1, \ldots, n, j = 1, \ldots, m_i\right)$$

There is a surjective R-algebra map $S' \to S$. The classes of the elements $g'_1, \ldots, g'_n$ in S' generate the unit ideal and by construction the maps $S'_{g'_i} \to S_{g_i}$ are injective. Thus we conclude as in part (1). □

1.9 Basic Properties of Flatness

1.9.1 Flat and Faithfully Modules

Definition 1.9.1. *Let R be a ring.*

(1) An R-module M is called flat *if whenever $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact as well.*

(2) An R-module M is called faithfully flat *if the complex of R-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_R N_1 \to M \otimes_R N_2 \to M \otimes_R N_3$ is exact.*

(3) A ring map $R \to S$ is called flat *if S is flat as an R-module.*

(4) A ring map $R \to S$ is called faithfully flat *if S is faithfully flat as an R-module.*

Here is an example of how you can use the flatness condition.

Lemma 1.9.2. *Let R be a ring. Let $I, J \subset R$ be ideals. Let M be a flat R-module. Then $IM \cap JM = (I \cap J)M$.*

Proof. Consider the exact sequence $0 \to I \cap J \to R \to R/I \oplus R/J$. Tensoring with the flat module M we obtain an exact sequence

$$0 \to (I \cap J) \otimes_R M \to M \to M/IM \oplus M/JM$$

Since the kernel of $M \to M/IM \oplus M/JM$ is equal to $IM \cap JM$ we conclude. □

Proposition 1.9.3. *Let R be a ring. Let $\{M_i, \varphi_{ii'}\}$ be a directed system of flat R-modules. Then $\varinjlim_i M_i$ is a flat R-module.*

Proof. This follows as $\otimes$ commutes with colimits and because directed colimits are exact. □

Proposition 1.9.4. *A composition of (faithfully) flat ring maps is (faithfully) flat. If $R \to R'$ is (faithfully) flat, and M' is a (faithfully) flat R'-module, then M' is a (faithfully) flat R-module.*

Proof. The first statement of the lemma is a particular case of the second, so it is clearly enough to prove the latter. Let $R \to R'$ be a flat ring map, and M' a flat R'-module. We need to prove that M' is a flat R-module. Let $N_1 \to N_2 \to N_3$ be an exact complex of R-modules. Then, the complex $R' \otimes_R N_1 \to R' \otimes_R N_2 \to R' \otimes_R N_3$ is exact (since R' is flat as an R-module), and so the complex $M' \otimes_{R'} (R' \otimes_R N_1) \to M' \otimes_{R'} (R' \otimes_R N_2) \to M' \otimes_{R'} (R' \otimes_R N_3)$ is exact (since M' is a flat R'-module). Since $M' \otimes_{R'} (R' \otimes_R N) \cong (M' \otimes_{R'} R') \otimes_R N \cong M' \otimes_R N$ for any R-module N functorially, this complex is isomorphic to the complex $M' \otimes_R N_1 \to M' \otimes_R N_2 \to M' \otimes_R N_3$, which is therefore also exact. This shows that M' is a flat R-module. Tracing this argument backwards, we can show that if $R \to R'$ is faithfully flat, and if M' is faithfully flat as an R'-module, then M' is faithfully flat as an R-module. □

Proposition 1.9.5. *Let M be an R-module. The following are equivalent:*

(1) M is flat over R.

(2) for every injection of R-modules $N \subset N'$ the map $N \otimes_R M \to N' \otimes_R M$ is injective.

(3) for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

(4) for every finitely generated ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

Proof. We prove (4) implies (1). Suppose that $N_1 \to N_2 \to N_3$ is exact. Let $K = \ker(N_2 \to N_3)$ and $Q = \mathrm{Im}(N_2 \to N_3)$. Then we get maps

$$N_1 \otimes_R M \to K \otimes_R M \to N_2 \otimes_R M \to Q \otimes_R M \to N_3 \otimes_R M$$

Observe that the first and third arrows are surjective. Thus if we show that the second and fourth arrows are injective, then we are done by some chase. Hence it suffices to show that $- \otimes_R M$ transforms injective R-module maps into injective R-module maps.

Assume $K \to N$ is an injective R-module map and let $x \in \ker(K \otimes_R M \to N \otimes_R M)$. We have to show that x is zero. The R-module K is the union of its finite R-submodules; hence, $K \otimes_R M$ is the colimit of R-modules of the form $K_i \otimes_R M$ where K_i runs over all finite R-submodules of K (because tensor product commutes with colimits). Thus, for some i our x comes from an element $x_i \in K_i \otimes_R M$. Thus we may assume that K is a finite R-module. Assume this. We regard the injection $K \to N$ as an inclusion, so that $K \subset N$.

The R-module N is the union of its finite R-submodules that contain K. Hence, $N \otimes_R M$ is the colimit of R-modules of the form $N_i \otimes_R M$ where N_i runs over all finite R-submodules of N that contain K (again since tensor product commutes with colimits). Notice that this is a colimit over a directed system (since the sum of two finite submodules of N is again finite). Hence, the element $x \in K \otimes_R M$ maps to zero in at least one of these R-modules $N_i \otimes_R M$ (since x maps to zero in $N \otimes_R M$). Thus we may assume N is a finite R-module.

Assume N is a finite R-module. Write $N = R^{\oplus n}/L$ and $K = L'/L$ for some $L \subset L' \subset R^{\oplus n}$. For any R-submodule $G \subset R^{\oplus n}$, we have a canonical map $G \otimes_R M \to M^{\oplus n}$ obtained by composing $G \otimes_R M \to R^n \otimes_R M = M^{\oplus n}$. It suffices to prove that $L \otimes_R M \to M^{\oplus n}$ and $L' \otimes_R M \to M^{\oplus n}$ are injective. Namely, if so, then we see that $K \otimes_R M = L' \otimes_R M / L \otimes_R M \to M^{\oplus n}/L \otimes_R M$ is injective too.

Thus it suffices to show that $L \otimes_R M \to M^{\oplus n}$ is injective when $L \subset R^{\oplus n}$ is an R-submodule. We do this by induction on n. The base case $n = 1$ we handle below.

For the induction step assume $n > 1$ and set $L' = L \cap R \oplus 0^{\oplus n-1}$. Then $L'' = L/L'$ is a submodule of $R^{\oplus n-1}$. We obtain a diagram

$$\begin{tikzcd}
 & L' \otimes_R M \arrow[r] \arrow[d] & L \otimes_R M \arrow[r] \arrow[d] & L'' \otimes_R M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & M^{\oplus n} \arrow[r] & M^{\oplus n-1} \arrow[r] & 0
\end{tikzcd}$$

By induction hypothesis and the base case the left and right vertical arrows are injective. The rows are exact. It follows that the middle vertical arrow is injective too.

The base case of the induction above is when $L \subset R$ is an ideal. In other words, we have to show that $I \otimes_R M \to M$ is injective for any ideal I of R. We know this is true when I is finitely generated. However, $I = \bigcup I_\alpha$ is the union of the finitely generated ideals I_α contained in it. In other words, $I = \varinjlim I_\alpha$. Since $\otimes$ commutes with colimits we see that $I \otimes_R M = \varinjlim I_\alpha \otimes_R M$ and since all the morphisms $I_\alpha \otimes_R M \to M$ are injective by assumption, the same is true for $I \otimes_R M \to M$. $\square$

Proposition 1.9.6. *Let $\{R_i, \varphi_{ii'}\}$ be a system of rings over the directed set I. Let $R = \varinjlim_i R_i$.*

(1) If M is an R-module such that M is flat as an R_i-module for all i, then M is flat as an R-module.

(2) For $i \in I$ let M_i be a flat R_i-module and for $i' \geq i$ let $f_{ii'} : M_i \to M_{i'}$ be a $\varphi_{ii'}$-linear map such that $f_{i'i''} \circ f_{ii'} = f_{ii''}$. Then $M = \varinjlim_{i \in I} M_i$ is a flat R-module.

Proof. Part (1) is a special case of part (2) with $M_i = M$ for all i and $f_{ii'} = \mathrm{id}_M$. Proof of (2). Let $\mathfrak{a} \subset R$ be a finitely generated ideal. By Lemma 1.9.5 it suffices to show that $\mathfrak{a} \otimes_R M \to M$ is injective. We can find an $i \in I$ and a finitely generated ideal $\mathfrak{a}' \subset R_i$ such that $\mathfrak{a} = \mathfrak{a}'R$. Then $\mathfrak{a} = \varinjlim_{i' \geq i} \mathfrak{a}'R_{i'}$. Since $\otimes$ commutes with colimits the map $\mathfrak{a} \otimes_R M \to M$ is the colimit of the maps

$$\mathfrak{a}'R_{i'} \otimes_{R_{i'}} M_{i'} \longrightarrow M_{i'}$$

These maps are all injective by assumption. Since colimits over I are exact, we win. $\square$

Proposition 1.9.7. *Let R be a ring.*

(1) Suppose that M is (faithfully) flat over R, and that $R \to R'$ is a ring map. Then $M \otimes_R R'$ is (faithfully) flat over R'.

(2) Let $R \to R'$ be a faithfully flat ring map. Let M be a module over R, and set $M' = R' \otimes_R M$. Then M is flat over R if and only if M' is flat over R'.

(3) *Let R be a ring. Let $S \to S'$ be a flat map of R-algebras. Let M be a module over S, and set $M' = S' \otimes_S M$. Then If M is flat over R, then M' is flat over R. If $S \to S'$ is faithfully flat, then M is flat over R if and only if M' is flat over R.*

(4) *Let $R \to S$ be a ring map. Let M be an S-module. If M is flat as an R-module and faithfully flat as an S-module, then $R \to S$ is flat.*

Proof. (1) is trivial and we consider (2).

By (1) we see that if M is flat then M' is flat. For the converse, suppose that M' is flat. Let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. We want to show that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We know that $N_1 \otimes_R R' \to N_2 \otimes_R R' \to N_3 \otimes_R R'$ is exact, because $R \to R'$ is flat. Flatness of M' implies that $N_1 \otimes_R R' \otimes_{R'} M' \to N_2 \otimes_R R' \otimes_{R'} M' \to N_3 \otimes_R R' \otimes_{R'} M'$ is exact. We may write this as $N_1 \otimes_R M \otimes_R R' \to N_2 \otimes_R M \otimes_R R' \to N_3 \otimes_R M \otimes_R R'$. Finally, faithful flatness implies that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact.

For (3), let $N \to N'$ be an injection of R-modules. By the flatness of $S \to S'$ we have

$$\ker(N \otimes_R M \to N' \otimes_R M) \otimes_S S' = \ker(N \otimes_R M' \to N' \otimes_R M')$$

If M is flat over R, then the left hand side is zero and we find that M' is flat over R by the second characterization of flatness in Lemma 1.9.5. If M' is flat over R then we have the vanishing of the right hand side and if in addition $S \to S'$ is faithfully flat, this implies that $\ker(N \otimes_R M \to N' \otimes_R M)$ is zero which in turn shows that M is flat over R.

For (4), let $N_1 \to N_2 \to N_3$ be an exact sequence of R-modules. By assumption $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. We may write this as

$$N_1 \otimes_R S \otimes_S M \to N_2 \otimes_R S \otimes_S M \to N_3 \otimes_R S \otimes_S M.$$

By faithful flatness of M over S we conclude that $N_1 \otimes_R S \to N_2 \otimes_R S \to N_3 \otimes_R S$ is exact. Hence $R \to S$ is flat. □

Proposition 1.9.8 (Equational criterion of flatness)**.** *Let R be a ring. Let M be an R-module. Let $\sum f_i x_i = 0$ be a relation in M. We say the relation $\sum f_i x_i$ is* trivial *if there exist an integer $m \geq 0$, elements $y_j \in M$, $j = 1, \ldots, m$, and elements $a_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ such that*

$$x_i = \sum\nolimits_j a_{ij} y_j, \forall i, \quad \text{and} \quad 0 = \sum\nolimits_i f_i a_{ij}, \forall j.$$

Then A module M over R is flat if and only if every relation in M is trivial.

Proof. Assume M is flat and let $\sum f_i x_i = 0$ be a relation in M. Let $I = (f_1, \ldots, f_n)$, and let $K = \ker(R^n \to I, (a_1, \ldots, a_n) \mapsto \sum_i a_i f_i)$. So we have the short exact sequence

$0 \to K \to R^n \to I \to 0$. Then $\sum f_i \otimes x_i$ is an element of $I \otimes_R M$ which maps to zero in $R \otimes_R M = M$. By flatness $\sum f_i \otimes x_i$ is zero in $I \otimes_R M$. Thus there exists an element of $K \otimes_R M$ mapping to $\sum e_i \otimes x_i \in R^n \otimes_R M$ where e_i is the ith basis element of R^n. Write this element as $\sum k_j \otimes y_j$ and then write the image of k_j in R^n as $\sum a_{ij} e_i$ to get the result.

Assume every relation is trivial, let I be a finitely generated ideal, and let $x = \sum f_i \otimes x_i$ be an element of $I \otimes_R M$ mapping to zero in $R \otimes_R M = M$. This just means exactly that $\sum f_i x_i$ is a relation in M. And the fact that it is trivial implies easily that x is zero, because

$$x = \sum f_i \otimes x_i = \sum f_i \otimes \left(\sum a_{ij} y_j\right) = \sum \left(\sum f_i a_{ij}\right) \otimes y_j = 0$$

Well done. $\square$

Proposition 1.9.9. *Suppose that R is a ring.*

(1) Let $0 \to M'' \to M' \to M \to 0$ be a short exact sequence, and N an R-module. If M is flat then $N \otimes_R M'' \to N \otimes_R M'$ is injective, i.e., the sequence

$$0 \to N \otimes_R M'' \to N \otimes_R M' \to N \otimes_R M \to 0$$

is a short exact sequence.

(2) Suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules. If M' and M'' are flat so is M. If M and M'' are flat so is M'.

Proof. For (1), let $R^{(I)} \to N$ be a surjection from a free module onto N with kernel K. The result follows from the snake lemma applied to the following diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
 & M'' \otimes_R N \arrow[r] \arrow[u] & M' \otimes_R N \arrow[r] \arrow[u] & M \otimes_R N \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & (M'')^{(I)} \arrow[r] \arrow[u] & (M')^{(I)} \arrow[r] \arrow[u] & M^{(I)} \arrow[r] \arrow[u] & 0 \\
 & M'' \otimes_R K \arrow[r] \arrow[u] & M' \otimes_R K \arrow[r] \arrow[u] & M \otimes_R K \arrow[r] \arrow[u] & 0 \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

with exact rows and columns. The middle row is exact because tensoring with the free module $R^{(I)}$ is exact.

For (2), we will use the criterion that a module N is flat if for every ideal $I \subset R$ the map $N \otimes_R I \to N$ is injective, see Lemma 1.9.5. Consider an ideal $I \subset R$. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & M' \otimes_R I & \longrightarrow & M \otimes_R I & \longrightarrow & M'' \otimes_R I & \longrightarrow & 0 \end{array}$$

with exact rows. This immediately proves the first assertion. The second follows because if M'' is flat then the lower left horizontal arrow is injective by (1). ☐

Proposition 1.9.10. *Let R be a ring. Let $S \subset R$ be a multiplicative subset.*

(1) The localization $S^{-1}R$ is a flat R-algebra.

(2) If M is an $S^{-1}R$-module, then M is a flat R-module if and only if M is a flat $S^{-1}R$-module.

(3) Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$-module for all primes $\mathfrak{p}$ of R.

(4) Suppose M is an R-module. Then M is a flat R-module if and only if $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of R.

(5) Suppose $R \to A$ is a ring map, M is an A-module, and $g_1, \ldots, g_m \in A$ are elements generating the unit ideal of A. Then M is flat over R if and only if each localization M_{g_i} is flat over R.

(6) Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p}$ the prime of R lying under $\mathfrak{q}$) for all primes $\mathfrak{q}$ of A.

(7) Suppose $R \to A$ is a ring map, and M is an A-module. Then M is a flat R-module if and only if the localization $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module (with $\mathfrak{p} = R \cap \mathfrak{m}$) for all maximal ideals $\mathfrak{m}$ of A.

Proof. Let us prove the last statement. In the proof we will use repeatedly that localization is exact and commutes with tensor product.

Suppose $R \to A$ is a ring map, and M is an A-module. Assume that $M_{\mathfrak{m}}$ is a flat $R_{\mathfrak{p}}$-module for all maximal ideals $\mathfrak{m}$ of A (with $\mathfrak{p} = R \cap \mathfrak{m}$). Let $I \subset R$ be an ideal. We have to show the map $I \otimes_R M \to M$ is injective. We can think of this as a map of A-modules. By assumption the localization $(I \otimes_R M)_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is injective because $(I \otimes_R M)_{\mathfrak{m}} = I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{m}}$. Hence the kernel of $I \otimes_R M \to M$ is zero by Proposition 1.8.9. Hence M is flat over R.

Conversely, assume M is flat over R. Pick a prime $\mathfrak{q}$ of A lying over the prime $\mathfrak{p}$ of R. Suppose that $I \subset R_{\mathfrak{p}}$ is an ideal. We have to show that $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is injective. We can write $I = J_{\mathfrak{p}}$ for some ideal $J \subset R$. Then the map $I \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{q}} \to M_{\mathfrak{q}}$ is just the localization (at $\mathfrak{q}$) of the map $J \otimes_R M \to M$ which is injective. Since localization is exact we see that $M_{\mathfrak{q}}$ is a flat $R_{\mathfrak{p}}$-module.

This proves (7) and (6). The other statements follow in a straightforward way from the last statement (proofs omitted). □

1.9.2 More Faithfully Flatness

Proposition 1.9.11. *Let R be a ring. Let M be an R-module. The following are equivalent*

(1) M is faithfully flat, and

(2) M is flat and for all R-module homomorphisms $\alpha : N \to N'$ we have $\alpha = 0$ if and only if $\alpha \otimes id_M = 0$.

Proof. If M is faithfully flat, then $0 \to \ker(\alpha) \to N \to N'$ is exact if and only if the same holds after tensoring with M. This proves (1) implies (2). For the other, assume (2). Let $N_1 \to N_2 \to N_3$ be a complex, and assume the complex $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Take $x \in \ker(N_2 \to N_3)$, and consider the map $\alpha : R \to N_2/\mathrm{Im}(N_1)$, $r \mapsto rx + \mathrm{Im}(N_1)$. By the exactness of the complex $- \otimes_R M$ we see that $\alpha \otimes \mathrm{id}_M$ is zero. By assumption we get that α is zero. Hence x is in the image of $N_1 \to N_2$. □

Proposition 1.9.12. *Let M be a flat R-module. The following are equivalent:*

(1) M is faithfully flat,

(2) for every nonzero R-module N, then tensor product $M \otimes_R N$ is nonzero,

(3) for all $\mathfrak{p} \in \mathrm{Spec}(R)$ the tensor product $M \otimes_R \kappa(\mathfrak{p})$ is nonzero, and

(4) for all maximal ideals $\mathfrak{m}$ of R the tensor product $M \otimes_R \kappa(\mathfrak{m}) = M/\mathfrak{m}M$ is nonzero.

Proof. Assume M faithfully flat and $N \neq 0$. By Proposition 1.9.11 the nonzero map $1 : N \to N$ induces a nonzero map $M \otimes_R N \to M \otimes_R N$, so $M \otimes_R N \neq 0$. Thus (1) implies (2). The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate.

Assume (4). Suppose that $N_1 \to N_2 \to N_3$ is a complex and suppose that $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M$ is exact. Let H be the cohomology of the complex, so $H = \ker(N_2 \to N_3)/\mathrm{Im}(N_1 \to N_2)$. To finish the proof we will show $H = 0$. By flatness we see that $H \otimes_R M = 0$. Take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. Since $R/I \subset H$ we get $M/IM \subset H \otimes_R M = 0$ by flatness of M. If $I \neq R$ we may choose a maximal ideal $I \subset \mathfrak{m} \subset R$. This immediately gives a contradiction. □

Proposition 1.9.13. *Let $R \to S$ be a flat ring map. The following are equivalent:*

1. *(1) $R \to S$ is faithfully flat,*
2. *(2) the induced map on* Spec *is surjective, and*
3. *(3) any closed point $x \in \mathrm{Spec}(R)$ is in the image of the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$.*

Proof. This follows quickly from Proposition 1.9.12, because we saw in the fundamental diagram that $\mathfrak{p}$ is in the image if and only if the ring $S \otimes_R \kappa(\mathfrak{p})$ is nonzero. □

Corollary 1.9.14. *A flat local ring homomorphism of local rings is faithfully flat.*

Proof. Immediate from Proposition 1.9.13. □

Corollary 1.9.15 (Going down)**.** *Let $R \to S$ be flat. Let $\mathfrak{p} \subset \mathfrak{p}'$ be primes of R. Let $\mathfrak{q}' \subset S$ be a prime of S mapping to $\mathfrak{p}'$. Then there exists a prime $\mathfrak{q} \subset \mathfrak{q}'$ mapping to $\mathfrak{p}$.*

Proof. By Proposition 1.9.10 the local ring map $R_{\mathfrak{p}'} \to S_{\mathfrak{q}'}$ is flat. By Corollary 1.9.14 this local ring map is faithfully flat. By Proposition 1.9.13 there is a prime mapping to $\mathfrak{p}R_{\mathfrak{p}'}$. The inverse image of this prime in S does the job. □

Proposition 1.9.16. *Let R be a ring. Let $\{S_i, \varphi_{ii'}\}$ be a directed system of faithfully flat R-algebras. Then $S = \varinjlim_i S_i$ is a faithfully flat R-algebra.*

Proof. By Proposition 1.9.3 we see that S is flat. Let $\mathfrak{m} \subset R$ be a maximal ideal. By Proposition 1.9.13 none of the rings $S_i/\mathfrak{m}S_i$ is zero. Hence $S/\mathfrak{m}S = \varinjlim S_i/\mathfrak{m}S_i$ is nonzero as well because 1 is not equal to zero. Thus the image of $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ contains $\mathfrak{m}$ and we see that $R \to S$ is faithfully flat by Proposition 1.9.13. □

1.10 Length

Definition 1.10.1. *Let R be a ring. For any R-module M we define the* length of M over R *by the formula*

$$\mathrm{length}_R(M) = \sup\{n : \exists\ 0 = M_0 \subset M_1 \subset \ldots \subset M_n = M,\ M_i \neq M_{i+1}\}.$$

Proposition 1.10.2. *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over R then $\mathrm{length}_R M = \mathrm{length}_R M' + \mathrm{length}_R M''$.*

Proof. Given filtrations of M' and M'' of lengths n', n'' it is easy to make a corresponding filtration of M of length $n' + n''$. Thus we see that $\mathrm{length}_R M \geq \mathrm{length}_R M' + \mathrm{length}_R M''$. Conversely, given a filtration $M_0 \subset M_1 \subset \ldots \subset M_n$ of M consider the induced filtrations $M_i' = M_i \cap M'$ and $M_i'' = \mathrm{Im}(M_i \to M'')$. Let n' (resp. n'') be the number of steps in the filtration $\{M_i'\}$ (resp. $\{M_i''\}$). If $M_i' = M_{i+1}'$ and $M_i'' = M_{i+1}''$ then $M_i = M_{i+1}$. Hence we conclude that $n' + n'' \geq n$. Combined with the earlier result we win. □

Proposition 1.10.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$. If M is an R-module and $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$, then $\text{length}_R(M) = \infty$. In other words, if M has finite length then $\mathfrak{m}^n M = 0$ for some n.*

Proof. Assume $\mathfrak{m}^n M \neq 0$ for all $n \geq 0$. Choose $x \in M$ and $f_1, \ldots, f_n \in \mathfrak{m}$ such that $f_1 f_2 \ldots f_n x \neq 0$. The first n steps in the filtration

$$0 \subset R f_1 \ldots f_n x \subset R f_1 \ldots f_{n-1} x \subset \ldots \subset Rx \subset M$$

are distinct. For example, if $Rf_1 x = Rf_1 f_2 x$, then $f_1 x = g f_1 f_2 x$ for some g, hence $(1 - g f_2) f_1 x = 0$ hence $f_1 x = 0$ as $1 - g f_2$ is a unit which is a contradiction with the choice of x and $f_1, \ldots, f_n$. Hence the length is infinite. □

Lemma 1.10.4. *Let $R \to S$ be a ring map. Let M be an S-module. We always have $\text{length}_R(M) \geq \text{length}_S(M)$. If $R \to S$ is surjective then equality holds.*

Proof. A filtration of M by S-submodules gives rise a filtration of M by R-submodules. This proves the inequality. And if $R \to S$ is surjective, then any R-submodule of M is automatically an S-submodule. Hence equality in this case. □

Proposition 1.10.5. *Let R be a ring with maximal ideal $\mathfrak{m}$. Suppose that M is an R-module with $\mathfrak{m}M = 0$. Then $\text{length}_R M = \dim_{R/\mathfrak{m}} M$. The length is finite if and only if M is a finite R-module.*

Proof. The first part is a special case of Lemma 1.10.4. Thus the length is finite if and only if M has a finite basis as a $R/\mathfrak{m}$-vector space if and only if M has a finite set of generators as an R-module. □

Proposition 1.10.6. *Let R be a ring. Let M be an R-module. Let $S \subset R$ be a multiplicative subset. Then $\text{length}_R(M) \geq \text{length}_{S^{-1}R}(S^{-1}M)$.*

Proof. Any submodule $N' \subset S^{-1}M$ is of the form $S^{-1}N$ for some R-submodule $N \subset M$. The lemma follows. □

Proposition 1.10.7. *Let R be a ring with finitely generated maximal ideal $\mathfrak{m}$. (For example R Noetherian.) Suppose that M is a finite R-module with $\mathfrak{m}^n M = 0$ for some n. Then $\text{length}_R(M) < \infty$.*

Proof. Consider the filtration $0 = \mathfrak{m}^n M \subset \mathfrak{m}^{n-1} M \subset \ldots \subset \mathfrak{m}M \subset M$. All of the subquotients are finitely generated R-modules to which Proposition 1.10.5 applies. We conclude by additivity, see Proposition 1.10.2. □

Definition 1.10.8. *Let R be a ring. Let M be an R-module. We say M is simple if $M \neq 0$ and every submodule of M is either equal to M or to 0.*

Proposition 1.10.9. *Let R be a ring. Let M be an R-module. The following are equivalent:*

1. *(1) M is simple,*
2. *(2) $\text{length}_R(M) = 1$, and*
3. *(3) $M \cong R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R. By Proposition 1.10.5 the module $R/\mathfrak{m}$ has length 1. The equivalence of the first two assertions is tautological. Suppose that M is simple. Choose $x \in M$, $x \neq 0$. As M is simple we have $M = R \cdot x$. Let $I \subset R$ be the annihilator of x, i.e., $I = \{f \in R \mid fx = 0\}$. The map $R/I \to M$, $f \bmod I \mapsto fx$ is an isomorphism, hence R/I is a simple R-module. Since $R/I \neq 0$ we see $I \neq R$. Let $I \subset \mathfrak{m}$ be a maximal ideal containing I. If $I \neq \mathfrak{m}$, then $\mathfrak{m}/I \subset R/I$ is a nontrivial submodule contradicting the simplicity of R/I. Hence we see $I = \mathfrak{m}$ as desired. □

We now show that the simple modules are the building blocks of modules.

Proposition 1.10.10. *Let R be a ring. Let M be a finite length R-module. Choose any maximal chain of submodules*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with $M_i \neq M_{i-1}$, $i = 1, \ldots, n$. Then

1. *(1) $n = \text{length}_R(M)$,*
2. *(2) each M_i/M_{i-1} is simple,*
3. *(3) each M_i/M_{i-1} is of the form $R/\mathfrak{m}_i$ for some maximal ideal $\mathfrak{m}_i$,*
4. *(4) given a maximal ideal $\mathfrak{m} \subset R$ we have*

$$\sharp\{i \mid \mathfrak{m}_i = \mathfrak{m}\} = \text{length}_{R_\mathfrak{m}}(M_\mathfrak{m}).$$

Proof. If M_i/M_{i-1} is not simple then we can refine the filtration and the filtration is not maximal. Thus we see that M_i/M_{i-1} is simple. By Proposition 1.10.9 the modules M_i/M_{i-1} have length 1 and are of the form $R/\mathfrak{m}_i$ for some maximal ideals $\mathfrak{m}_i$. By additivity of length, Lemma 1.10.2, we see $n = \text{length}_R(M)$. Since localization is exact, we see that

$$0 = (M_0)_\mathfrak{m} \subset (M_1)_\mathfrak{m} \subset (M_2)_\mathfrak{m} \subset \ldots \subset (M_n)_\mathfrak{m} = M_\mathfrak{m}$$

is a filtration of $M_\mathfrak{m}$ with successive quotients $(M_i/M_{i-1})_\mathfrak{m}$. Thus the last statement follows directly from the fact that given maximal ideals $\mathfrak{m}$, $\mathfrak{m}'$ of R we have

$$(R/\mathfrak{m}')_\mathfrak{m} \cong \begin{cases} 0 & \text{if } \mathfrak{m} \neq \mathfrak{m}', \\ R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} & \text{if } \mathfrak{m} = \mathfrak{m}' \end{cases}$$

This we leave to the reader. □

Proposition 1.10.11. *Let A be a local ring with maximal ideal $\mathfrak{m}$. Let B be a semi-local ring with maximal ideals $\mathfrak{m}_i$, $i = 1, \ldots, n$. Suppose that $A \to B$ is a homomorphism such that each $\mathfrak{m}_i$ lies over $\mathfrak{m}$ and such that*

$$[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] < \infty.$$

Let M be a B-module of finite length. Then

$$\text{length}_A(M) = \sum_{i=1,\ldots,n} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \, \text{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}),$$

in particular $\text{length}_A(M) < \infty$.

Proof. Choose a maximal chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_m = M$$

by B-submodules as in Proposition 1.10.10. Then each quotient M_j/M_{j-1} is isomorphic to $\kappa(\mathfrak{m}_{i(j)})$ for some $i(j) \in \{1, \ldots, n\}$. Moreover $\text{length}_A(\kappa(\mathfrak{m}_i)) = [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ by Proposition 1.10.5. The lemma follows by additivity of lengths (Proposition 1.10.2). □

Proposition 1.10.12. *Let $A \to B$ be a flat local homomorphism of local rings. Then for any A-module M we have*

$$\text{length}_A(M) \, \text{length}_B(B/\mathfrak{m}_A B) = \text{length}_B(M \otimes_A B).$$

In particular, if $\text{length}_B(B/\mathfrak{m}_A B) < \infty$ *then M has finite length if and only if $M \otimes_A B$ has finite length.*

Proof. The ring map $A \to B$ is faithfully flat by Corollary 1.9.14. Hence if $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ is a chain of length n in M, then the corresponding chain $0 = M_0 \otimes_A B \subset M_1 \otimes_A B \subset \ldots \subset M_n \otimes_A B = M \otimes_A B$ has length n also. This proves $\text{length}_A(M) = \infty \Rightarrow \text{length}_B(M \otimes_A B) = \infty$. Next, assume $\text{length}_A(M) < \infty$. In this case we see that M has a filtration of length $\ell = \text{length}_A(M)$ whose quotients are $A/\mathfrak{m}_A$. Arguing as above we see that $M \otimes_A B$ has a filtration of length ℓ whose quotients are isomorphic to $B \otimes_A A/\mathfrak{m}_A = B/\mathfrak{m}_A B$. Thus the lemma follows. □

Proposition 1.10.13. *Let $A \to B \to C$ be flat local homomorphisms of local rings. Then*

$$\text{length}_B(B/\mathfrak{m}_A B) \, \text{length}_C(C/\mathfrak{m}_B C) = \text{length}_C(C/\mathfrak{m}_A C).$$

Proof. Follows from Proposition 1.10.12 applied to the ring map $B \to C$ and the B-module $M = B/\mathfrak{m}_A B$ □

1.11 Noetherian and Artinian Rings

1.11.1 Basic Facts of Noetherian Rings

Proposition 1.11.1. *Any finitely generated ring over a Noetherian ring is Noetherian. Any localization of a Noetherian ring is Noetherian.*

Proof. The statement on localizations follows from the fact that any ideal $J \subset S^{-1}R$ is of the form $I \cdot S^{-1}R$. Any quotient R/I of a Noetherian ring R is Noetherian because any ideal $\overline{J} \subset R/I$ is of the form J/I for some ideal $I \subset J \subset R$. Thus it suffices to show that if R is Noetherian so is $R[X]$. Suppose $J_1 \subset J_2 \subset \dots$ is an ascending chain of ideals in $R[X]$. Consider the ideals $I_{i,d}$ defined as the ideal of elements of R which occur as leading coefficients of degree d polynomials in J_i. Clearly $I_{i,d} \subset I_{i',d'}$ whenever $i \leq i'$ and $d \leq d'$. By the ascending chain condition in R there are at most finitely many distinct ideals among all of the $I_{i,d}$. (Hint: Any infinite set of elements of $\mathbf{N} \times \mathbf{N}$ contains an increasing infinite sequence.) Take i_0 so large that $I_{i,d} = I_{i_0,d}$ for all $i \geq i_0$ and all d. Suppose $f \in J_i$ for some $i \geq i_0$. By induction on the degree $d = \deg(f)$ we show that $f \in J_{i_0}$. Namely, there exists a $g \in J_{i_0}$ whose degree is d and which has the same leading coefficient as f. By induction $f - g \in J_{i_0}$ and we win. □

Proposition 1.11.2. *If R is a Noetherian ring, then so is the formal power series ring $R[[x_1, \dots, x_n]]$.*

Proof. Since $R[[x_1, \dots, x_{n+1}]] \cong R[[x_1, \dots, x_n]][[x_{n+1}]]$ it suffices to prove the statement that $R[[x]]$ is Noetherian if R is Noetherian. Let $I \subset R[[x]]$ be a ideal. We have to show that I is a finitely generated ideal. For each integer d denote $I_d = \{a \in R \mid ax^d + \text{h.o.t.} \in I\}$. Then we see that $I_0 \subset I_1 \subset \dots$ stabilizes as R is Noetherian. Choose d_0 such that $I_{d_0} = I_{d_0+1} = \dots$. For each $d \leq d_0$ choose elements $f_{d,j} \in I \cap (x^d)$, $j = 1, \dots, n_d$ such that if we write $f_{d,j} = a_{d,j}x^d + \text{h.o.t}$ then $I_d = (a_{d,j})$. Denote $I' = (\{f_{d,j}\}_{d=0,\dots,d_0, j=1,\dots,n_d})$. Then it is clear that $I' \subset I$. Pick $f \in I$. First we may choose $c_{d,i} \in R$ such that

$$f - \sum c_{d,i} f_{d,i} \in (x^{d_0+1}) \cap I.$$

Next, we can choose $c_{i,1} \in R$, $i = 1, \dots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we can choose $c_{i,2} \in R$, $i = 1, \dots, n_{d_0}$ such that

$$f - \sum c_{d,i} f_{d,i} - \sum c_{i,1} x f_{d_0,i} - \sum c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

And so on. In the end we see that

$$f = \sum c_{d,i} f_{d,i} + \sum\nolimits_i (\sum\nolimits_e c_{i,e} x^e) f_{d_0,i}$$

is contained in I' as desired. □

Proposition 1.11.3. *Let R be a Noetherian ring.*

(1) Any finite R-module is of finite presentation.

(2) Any submodule of a finite R-module is finite.

(3) Any finite type R-algebra is of finite presentation over R.

Proof. Let M be a finite R-module. By Proposition 1.1.3 we can find a finite filtration of M whose successive quotients are of the form R/I. Since any ideal is finitely generated, each of the quotients R/I is finitely presented. Hence M is finitely presented. This proves (1).

Let $N \subset M$ be a submodule. As M is finite, the quotient M/N is finite. Thus M/N is of finite presentation by part (1). Thus we see that N is finite. This proves part (2).

To see (3) note that any ideal of $R[x_1, \ldots, x_n]$ is finitely generated by Proposition 1.11.1. □

Proposition 1.11.4. *Let $R \to S$ be a ring map. Let $R \to R'$ be of finite type. If S is Noetherian, then the base change $S' = R' \otimes_R S$ is Noetherian.*

Proof. Now finite type is stable under base change. Thus $S \to S'$ is of finite type. Since S is Noetherian we can apply Lemma 1.11.1. □

1.11.2 More on Noetherian Rings

Proposition 1.11.5 (Artin-Rees)**.** *Suppose that R is Noetherian, $I \subset R$ an ideal. Let $N \subset M$ be finite R-modules. There exists a constant $c > 0$ such that $I^n M \cap N = I^{n-c}(I^c M \cap N)$ for all $n \geq c$.*

Proof. Consider the ring $S = R \oplus I \oplus I^2 \oplus \ldots = \bigoplus_{n \geq 0} I^n$. Convention: $I^0 = R$. Multiplication maps $I^n \times I^m$ into I^{n+m} by multiplication in R. Note that if $I = (f_1, \ldots, f_t)$ then S is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$. The map just sends the monomial $X_1^{e_1} \ldots X_t^{e_t}$ to $f_1^{e_1} \ldots f_t^{e_t}$. Thus S is Noetherian. Similarly, consider the module $M \oplus IM \oplus I^2M \oplus \ldots = \bigoplus_{n \geq 0} I^n M$. This is a finitely generated S-module. Namely, if $x_1, \ldots, x_r$ generate M over R, then they also generate $\bigoplus_{n \geq 0} I^n M$ over S. Next, consider the submodule $\bigoplus_{n \geq 0} I^n M \cap N$. This is an S-submodule, as is easily verified. Hence it is finitely generated as an S-module, say by $\xi_j \in \bigoplus_{n \geq 0} I^n M \cap N$, $j = 1, \ldots, s$. We may assume by decomposing each ξ_j into its homogeneous pieces that each $\xi_j \in I^{d_j} M \cap N$ for some d_j. Set $c = \max\{d_j\}$. Then for all $n \geq c$ every element in $I^n M \cap N$ is of the form $\sum h_j \xi_j$ with $h_j \in I^{n-d_j}$. The lemma now follows from this and the trivial observation that $I^{n-d_j}(I^{d_j} M \cap N) \subset I^{n-c}(I^c M \cap N)$. □

Corollary 1.11.6. *Suppose that $0 \to K \to M \xrightarrow{f} N$ is an exact sequence of finitely generated modules over a Noetherian ring R. Let $I \subset R$ be an ideal. Then there exists a c such that*

$$f^{-1}(I^n N) = K + I^{n-c} f^{-1}(I^c N) \quad \text{and} \quad f(M) \cap I^n N \subset f(I^{n-c} M)$$

for all $n \geq c$.

Proof. Apply Proposition 1.11.5 to $\mathrm{Im}(f) \subset N$ and note that $f : I^{n-c}M \to I^{n-c}f(M)$ is surjective. □

Corollary 1.11.7 (Krull's intersection theorem)**.** *Let R be a Noetherian local ring. Let $I \subset R$ be a proper ideal. Let M be a finite R-module. Then $\bigcap_{n \geq 0} I^n M = 0$.*

Proof. Let $N = \bigcap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By the Artin-Rees Lemma 1.11.5 we see that $N = I^n M \cap N \subset IN$ for some suitably large n. By Nakayama's Lemma 1.7.2 we see that $N = 0$. □

Corollary 1.11.8. *Let R be a Noetherian ring. Let $I \subset R$ be an ideal. Let M be a finite R-module. Let $N = \bigcap_n I^n M$.*

1. *(1) For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ there exists a $f \in R$, $f \notin \mathfrak{p}$ such that $N_f = 0$.*
2. *(2) If I is contained in the Jacobson radical of R, then $N = 0$.*

Proof. Proof of (1). Let $x_1, \ldots, x_n$ be generators for the module N. For every prime $\mathfrak{p}$, $I \subset \mathfrak{p}$ we see that the image of N in the localization $M_{\mathfrak{p}}$ is zero, by Corollary 1.11.7. Hence we can find $g_i \in R$, $g_i \notin \mathfrak{p}$ such that x_i maps to zero in N_{g_i}. Thus $N_{g_1 g_2 \ldots g_n} = 0$.

Part (2) follows from (1) and Proposition 1.8.9. □

Lemma 1.11.9 (Artin-Tate Lemma)**.** *Let R be a Noetherian ring. Let S be a finitely generated R-algebra. If $T \subset S$ is an R-subalgebra such that S is finitely generated as a T-module, then T is of finite type over R.*

Proof. Choose elements $x_1, \ldots, x_n \in S$ which generate S as an R-algebra. Choose $y_1, \ldots, y_m$ in S which generate S as a T-module. Thus there exist $a_{ij} \in T$ such that $x_i = \sum a_{ij} y_j$. There also exist $b_{ijk} \in T$ such that $y_i y_j = \sum b_{ijk} y_k$. Let $T' \subset T$ be the sub R-algebra generated by a_{ij} and b_{ijk}. This is a finitely generated R-algebra, hence Noetherian. Consider the algebra

$$S' = T'[Y_1, \ldots, Y_m]/(Y_i Y_j - \sum b_{ijk} Y_k).$$

Note that S' is finite over T', namely as a T'-module it is generated by the classes of $1, Y_1, \ldots, Y_m$. Consider the T'-algebra homomorphism $S' \to S$ which maps Y_i to y_i. Because $a_{ij} \in T'$ we see that x_j is in the image of this map. Thus $S' \to S$ is surjective. Therefore S is finite over T' as well. Since T' is Noetherian and we conclude that $T \subset S$ is finite over T' and we win. □

1.11.3 Artinian Rings

Proposition 1.11.10. *Suppose R is a finite dimensional algebra over a field. Then R is Artinian.*

Proof. The descending chain condition for ideals obviously holds. □

Proposition 1.11.11. *Let R is Artinian.*

(1) Then R has only finitely many maximal ideals.

(2) The Jacobson radical $\mathrm{rad}(R)$ *is a nilpotent ideal.*

Proof. For (1). Suppose that $\mathfrak{m}_i$, $i = 1, 2, 3, \ldots$ are pairwise distinct maximal ideals. Then $\mathfrak{m}_1 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \supset \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \supset \ldots$ is an infinite descending sequence (because by the Chinese remainder theorem all the maps $R \to \oplus_{i=1}^n R/\mathfrak{m}_i$ are surjective).

For (2). Let $I = \mathrm{rad}(R)$ be the Jacobson radical. Note that $I \supset I^2 \supset I^3 \supset \ldots$ is a descending sequence. Thus $I^n = I^{n+1}$ for some n. Set $J = \{x \in R \mid xI^n = 0\}$. We have to show $J = R$. If not, choose an ideal $J' \neq J$, $J \subset J'$ minimal (possible by the Artinian property). Then $J' = J + Rx$ for some $x \in R$. By Nakayama's Lemma 1.7.2, we have $IJ' \subset J$. Hence $xI^{n+1} \subset xI \cdot I^n \subset J \cdot I^n = 0$. Since $I^{n+1} = I^n$ we conclude $x \in J$. Contradiction. □

Lemma 1.11.12. *Any ring with finitely many maximal ideals and locally nilpotent Jacobson radical is the product of its localizations at its maximal ideals. Also, all primes are maximal.*

Proof. Let R be a ring with finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Let $I = \bigcap_{i=1}^n \mathfrak{m}_i$ be the Jacobson radical of R. Assume I is locally nilpotent. Let $\mathfrak{p}$ be a prime ideal of R. Since every prime contains every nilpotent element of R we see $\mathfrak{p} \supset \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$. Since $\mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$ we conclude $\mathfrak{p} \supset \mathfrak{m}_1 \ldots \mathfrak{m}_n$. Hence $\mathfrak{p} \supset \mathfrak{m}_i$ for some i, and so $\mathfrak{p} = \mathfrak{m}_i$. By the Chinese remainder theorem we have $R/I \cong \bigoplus R/\mathfrak{m}_i$ which is a product of fields. Hence by Proposition 1.8.6 there are idempotents e_i, $i = 1, \ldots, n$ with $e_i \bmod \mathfrak{m}_j = \delta_{ij}$. Hence $R = \prod Re_i$, and each Re_i is a ring with exactly one maximal ideal. □

Proposition 1.11.13. *A ring R is Artinian if and only if it has finite length as a module over itself. Any such ring R is both Artinian and Noetherian, any prime ideal of R is a maximal ideal, and R is equal to the (finite) product of its localizations at its maximal ideals.*

Proof. If R has finite length over itself then it satisfies both the ascending chain condition and the descending chain condition for ideals. Hence it is both Noetherian and Artinian. Any Artinian ring is equal to product of its localizations at maximal ideals by Propositions 1.11.11(1), 1.11.11(2), and Lemma 1.11.12.

Suppose that R is Artinian. We will show R has finite length over itself. It suffices to exhibit a chain of submodules whose successive quotients have finite length. By what we said above we may assume that R is local, with maximal ideal $\mathfrak{m}$. By Proposition 1.11.11(2) we have $\mathfrak{m}^n = 0$ for some n. Consider the sequence $0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \ldots \subset \mathfrak{m} \subset R$. By Proposition 1.10.5 the length of each subquotient $\mathfrak{m}^j/\mathfrak{m}^{j+1}$ is the dimension of this as a vector space over $\kappa(\mathfrak{m})$. This has to be finite since otherwise we would have an infinite descending chain of sub vector spaces which would correspond to an infinite descending chain of ideals in R. $\square$

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