Note for the Virtual Fundamental Class

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Contents

1 Introduction

We will follows [BF97][AB84][GP99] and we will also use [Ric22].

We need [Har77][Ful98][EH16].

Here we will consider $\mathbb{P}(-) = \mathbf{Proj}\,\mathrm{Sym}(-)^\vee$ for bundles and the vector bundle is both space and sheaf via $\mathbf{Spec}\,\mathrm{Sym}(-)^\vee$. For a cone $C = \mathbf{Spec}_X \mathscr{S}^*$, we define $\mathbb{P}(C) := \mathbf{Proj}_X \mathscr{S}^*$ and $\mathbb{P}(C \oplus \mathscr{O}) := \mathbf{Proj}_X \mathscr{S}^*[z]$ which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

2 Review of Basic Intersection Theory

We will follows [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field k.

2.1 Basic Facts of Refined Gysin Pullback

Here we will follows Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

Definition 2.1 (Intersection Product)**.** *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $\mathscr{I}$ be the ideal of i and $\mathscr{J}$ be the ideal of j, then we have surjection

$$\bigoplus_n f^*(\mathscr{I}^n/\mathscr{I}^{n+1}) \to \bigoplus_n \mathscr{J}^n/\mathscr{J}^{n+1} \to 0$$

which induce embedding $C_{W/V} \hookrightarrow g^*N_{X/Y}$. Note that $C_{W/V}$ is also a scheme of pure dimension k since $\mathbb{P}(C_{W/V} \oplus \mathscr{O})$ is the exceptional divisor of $\mathrm{Bl}_W(Y \times \mathbb{A}^1)$. Let $0 : W \to g^*N_{X/Y}$ be the zero-section of $\pi : g^*N_{X/Y} \to W$, then we define

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \mathsf{CH}_{k-d}(W)$$

as the intersection class.

Proposition 2.2. *Consider the situation of Definition 2.1.*

(a) We have $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$.

(b) Let $\mathscr{Q}$ be the universal quotient bundle of $q : \mathbb{P}(g^*N_{X/Y} \oplus \mathscr{O}) \to W$, then

$$X \cdot V = q_*(c_d(\mathscr{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathscr{O})]).$$

(c) If $j : W \hookrightarrow V$ is a regular embedding of codimension d', then $X \cdot V = c_{d-d'}(g^*N_{X/Y}/N_{W/V}) \cap [W]$.

Proof. Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98]. □

Definition 2.3 (Refined Gysin Pullback)**.** *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick $f : Y' \to Y$ be a morphism. Consider the cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Then we define $i^! : \mathsf{Z}_k\, Y' \to \mathsf{CH}_{k-d}\, X'$ as $\sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i$. Now $i^!$ can be decomposed as:

$$i^! : \mathsf{Z}_k\, Y' \xrightarrow{\sigma} \mathsf{Z}_k\, C_{X'/Y'} \to \mathsf{CH}_k(g^* N_{X/Y}) \xrightarrow{0^*} \mathsf{CH}_{k-d}\, X'$$

where $\sigma : \mathsf{Z}_k\, Y' \to \mathsf{Z}_k\, C_{X'/Y'}$ given by $[V] \mapsto [C_{V \cap X'/V}]$. By the technique of deformation to the normal cone, this can be descend to the Chow-group level as $\sigma : \mathsf{CH}_k\, Y' \to \mathsf{CH}_k\, C_{X'/Y'}$ (see Proposition 5.2 in [Ful98]) which is called the specialization to the normal cone. *Hence this induce the* refined Gysin pullback

$$i^! : \mathsf{CH}_k\, Y' \to \mathsf{CH}_{k-d}\, X', \quad \sum_i n_i[V_i] \mapsto \sum_i n_i X \cdot V_i.$$

Proposition 2.4. *Consider the situation of Definition 2.3. Consider*

$$\begin{tikzcd} X'' \arrow[r, hook, "i''"] \arrow[d, "q"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y'' \arrow[d, "p"] \\ X' \arrow[r, hook, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

(a) If p proper and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathsf{CH}_{k-d}(X')$.

(b) If p is flat of relative dimension n and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathsf{CH}_{k+n-d}(X'')$.

(c) If i' is also a regular embedding of codimension d and $\alpha \in \mathsf{CH}_k(Y'')$, then $i^! \alpha = (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'')$.

(d) If i' is a regular embedding of codimension d', then for $\alpha \in \mathsf{CH}_k(Y'')$ we have

$$i^!(\alpha) = c_{d-d'}(q^*(g^* N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathsf{CH}_{k-d}(X'').$$

We call $g^* N_{X/Y}/N_{X'/Y'}$ the excess normal bundle.

(e) *Let F be any vector bundle on Y', then for $\alpha \in \mathsf{CH}_k(Y'')$ we have*

$$i^!(c_m(F) \cap \alpha) = c_m((i')^*F) \cap i^!(\alpha) \in \mathsf{CH}_{k-d-m}(X').$$

Proof. See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98]. □

Corollary 2.5. *Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d, then*

$$i^* i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathsf{CH}_*(X).$$

Proof. By Proposition 2.4(d) directly. □

Proposition 2.6. *The refined Gysin pullback have the following properties.*

(a) *Let $i : X \hookrightarrow Y$ and $j : S \hookrightarrow T$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X'' \arrow[r, hook] \arrow[d, hook] & Y'' \arrow[r] \arrow[d, hook, "j'"'] & S \arrow[d, hook, "j"] \\
X' \arrow[r, hook, "i'"] \arrow[d] & Y' \arrow[r, "g"'] \arrow[d, "f"'] & T \\
X \arrow[r, hook, "i"] & Y &
\end{tikzcd}$$

Then for any $\alpha \in \mathsf{CH}_k(Y'')$, we have

$$j^! i^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X'').$$

(b) *Let $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are regular embeddings of codimension d, e, respectively. Consider cartesians:*

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d, "h"] & Y' \arrow[r, hook, "j'"] \arrow[d, "g"'] & Z' \arrow[d, "f"'] \\
X \arrow[r, hook, "i"] & Y \arrow[r, hook, "j"] & Z
\end{tikzcd}$$

Then ji is a regular embedding of codimension $d + e$ and for all $\alpha \in \mathsf{CH}_k(Z')$ we have

$$(ji)^!(\alpha) = i^! j^!(\alpha) \in \mathsf{CH}_{k-d-e}(X').$$

Proof. See Theorem 6.4 and Theorem 6.5 in [Ful98]. □

Proposition 2.7. *Consider cartesians:*

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "h"] \arrow[dr, phantom, "\ulcorner", very near start] & Y' \arrow[r, "p'"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z' \arrow[d, "f"'] \\
X \arrow[r, "i"] & Y \arrow[r, "p"] & Z
\end{tikzcd}$$

(a) If i is a regular embedding of codimension d and p and pi are flat of relative dimension $n, n-d$, respectively. Then i' is a regular embedding of codimension d and $p', p'i'$ are flat, and for $\alpha \in \mathsf{CH}_k(Z')$ we have

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i^!((p')^*\alpha).$$

(b) If i is a regular embedding of codimension d and p is smooth of relative dimension n, and pi is a regular embedding of codimension $d-n$ Then for $\alpha \in \mathsf{CH}_k(Z')$ we have

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

Proof. See Proposition 6.5 in [Ful98]. □

Remark 2.8. *Some remarks.*

(a) For local complete intersection morphism $f : X \to Y$, we can decompose it into $f : X \xrightarrow{i} P \xrightarrow{p} Y$ where i is a closed regular embedding of constant codimension and p is smooth of constant relative dimension. Then we can define $f^! := i^!(p')^*$. See Section 6.6 in [Ful98] for more properties.

(b) If Y is nonsingular of dimension n, then we can define the following intersection product: Let $f : X \to Y$ and $p : X' \to X$ and $q : Y' \to Y$. Let $x \in \mathsf{CH}_k(X')$ and $y \in \mathsf{CH}_l(Y')$, consider the cartesian

$$\begin{tikzcd}
X' \times_Y Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X' \times Y' \arrow[d, "p\times q"] \\
X \arrow[r, "\gamma_f"] & X \times Y
\end{tikzcd}$$

and define $x \cdot_f y := \gamma_f^!(x \times y) \in \mathsf{CH}_{k+l-n}(X' \times_Y Y')$.

So when $x, y \in \mathsf{CH}_*(Y)$, then let $X = Y$ and $X' = |x|, Y' = |y|$, then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case $CH_*(Y)$ is a ring which is called Chow ring.

Finally we will discuss something about equivalence and supportness.

Definition 2.9. *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k. Consider the cartesian diagram*

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near start] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

Let $C_1, ..., C_r$ be the irreducible components of $C_{W/V}$, then $[C_{W/V}] = \sum_{i=1}^r m_i[C_i]$. Let $Z_i = \pi(C_i)$ where $\pi : g^* N_{X/Y} \to W$ and we call them the distinguished varieties *of the intersection of V by X. Let $N_i := (g^* N_{X/Y})|_{Z_i}$ and let $0_i : Z_i \to N_i$ be the zero-sections. Let $\alpha_i := 0_i^*[C_i] \in \mathsf{CH}_{k-d}(Z_i)$ and hence we have $X \cdot V = \sum_{i=1}^r m_i \alpha_i \in \mathsf{CH}_{k-d}(W)$.*

Pick any closed set $S \subset W$, we define

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i \alpha_i \in \mathsf{CH}_{k-d}(S)$$

as the part of $X \cdot V$ supported on S.

Definition 2.10. *Let $X_i \hookrightarrow Y$ be closed regular embeddings of codimension d_i. Let $V \subset Y$ be a k-dimensional subvariety. Consider*

$$\begin{tikzcd} \bigcap_i X_i \cap V \arrow[rr, hook] \arrow[d, hook] \arrow[drr, phantom, "\ulcorner", very near start] & & V \arrow[d, hook, "\delta"] \\ X_1 \times \cdots \times X_r \arrow[rr, hook] & & Y \times \cdots \times Y \end{tikzcd}$$

Then we can get $X_1 \cdot \ldots \cdot x_r \cdot V \in \mathsf{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$.

Let Z be a connected component of $\bigcap_i X_i \cap V$, we will consider

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z \in \mathsf{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

Proposition 2.11. *As in the previous situation, we have*

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cap s(Z, V) \right\}_{\dim V - \sum_i d_i}.$$

If $Z \hookrightarrow V$ is a regular embedding, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If V, Z are both non-singular, then

$$(X_1 \cdot \ldots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^{r} c(N_{X_i/Y}|_Z) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

Proof. See Proposition 9.1.1 in [Ful98]. □

2.2 Localized Chern Class

Here we will follows Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

Definition 2.12. *Let $E \to X$ be a vector bundle of rank e over a purely n-dimensional scheme X. Let $s : X \to E$ be a section, consider the cartesian*

$$\begin{tikzcd} Z(s) \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, hook, "s"'] \\ X \arrow[r, hook, "0"] & E \end{tikzcd}$$

with zero-section $0 : X \to E$ which is a regular section by trivial reason. We define

$$c_{\text{loc}}(E, s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \mathsf{CH}_{n-e}(Z(s))$$

be the **localized (top) Chern class of E with respect to s**.

Proposition 2.13. *Consider the situation of Definition 2.12.*

(a) We have $i_*(c_{\text{loc}}(E, s)) = c_e(E) \cap [X]$.

(b) Each irreducible component of $Z(s)$ has codimension at most e in X. If $\operatorname{codim}_{Z(s)} X = e$, then $c_{\text{loc}}(E, s)$ is a positive cycle whose support is $Z(s)$.

(c) If s is a regular section, then $c_{\text{loc}}(E, s) = [Z(s)]$.

(d) Let $f : X' \to X$ be a morphism, $s' = f^*s$ be a induced section of f^*E. Let $g : Z(s') \to Z(s)$ be the induced morphism.

(d1) If f flat, then $g^* c_{\text{loc}}(E, s) = c_{\text{loc}}(f^*E, s')$.

(d2) If f is proper of varieties, then $g_* c_{\text{loc}}(f^*E, s') = \deg(X'/X) c_{\text{loc}}(E, s)$.

Proof. For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_* 0^! [X] = 0^* s_* [X] = s^* s_* [X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{tikzcd} Z(s') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d, "s'"'] \\ X' \arrow[r, hook, "0_{f^*E}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f^*E \arrow[d] \\ X \arrow[r, hook, "0_E"] & E \end{tikzcd}$$

and Proposition 2.4. □

3 Fundations of Virtual Fundamental Class

We will follows [BF97]. Here an algebraic stack (or Artin stack) over a field k is assumed to be quasi-separated and locally of finite type over k.

3.1 A Brief of Cotangent Complexes

3.2 About Cones

We will let X be a Deligne-Mumford stack now.

Definition 3.1. *Let X be a DM-stack.*

(a) We call an affine X-scheme $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ is a cone over X *if the quasi-coherent algebra $\mathscr{S}$ is graded as $\mathscr{S} = \bigoplus_{i\geq 0} \mathscr{S}^i$ with $\mathscr{S}^0 = \mathscr{O}_X$ and $\mathscr{S}^1$ is coherent and $\mathscr{S}$ is generated by $\mathscr{S}^1$.*

(b) A morphism of cones over X *is an X-morphism induced by a graded morphism of graded sheaves of $\mathscr{O}_X$-algebras. A* closed subcone *is the image of a closed immersion of cones.*

Remark 3.2. *(a) The fiber product of cones over X is still a cone over X.*

(b) For every cone $C \to X$, it has a zero section $0 : X \to C$ induced by $\mathscr{S} \to \mathscr{S}^0$.

(c) For every cone $C \to X$, the grade induce a $\mathbb{G}_m$-action $\mathbb{G}_m \times C = \underline{\mathrm{Spec}}_X \mathscr{S}[t, t^{-1}] \to C$ induced by $\mathscr{S} \to \mathscr{S}[t, t^{-1}]$ via $s_0 + \cdots s_d \mapsto \sum_i a_i t^i$ where $s_i \in \mathscr{S}^i$. Since no negative power of t occurs, we can in fact replace $\mathbb{G}_m$ by $\mathbb{A}^1$. So we have the $\mathbb{A}^1$-action $\gamma : \mathbb{A}^1 \times C \to C$ induced by $\mathscr{S} \to \mathscr{S}[x]$ via $\mathscr{S}^i \ni s \mapsto sx^i$. Note that here $\mathbb{A}^1$ is not a

group scheme and the action *here, as expected, to be the commutativity of the following diagrams:*

$$\begin{tikzcd} C \arrow[r, "{(1,\mathrm{id})/(0,\mathrm{id})}"] \arrow[dr, "\mathrm{id}/0"'] & \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"'] & \mathbb{A}^1 \times \mathbb{A}^1 \times C \arrow[d, "\gamma"] \\ \mathbb{A}^1 \times C \arrow[r, "\gamma"'] & C \end{tikzcd}$$

where $m(x,y) = xy$.

(d) *So a morphism of cones $f : C \to D$ over X is just the $\mathbb{A}^1$-equivariant X-morphism respecting the zero section, that is, the following commutativity of the diagram:*

$$\begin{tikzcd} \mathbb{A}^1 \times C \arrow[r] \arrow[d, "\mathrm{id}\times f"'] & C \arrow[d, "f"'] \arrow[r, shift right] & X \arrow[l, shift right, "0_C"'] \arrow[dl, shift left, "0_D"] \\ \mathbb{A}^1 \times D \arrow[r] & D \arrow[ur, shift left] & \end{tikzcd}$$

Definition 3.3. *Let $\mathscr{F}$ be a coherent sheaf of X, then we can define $C(\mathscr{F}) := \underline{\mathrm{Spec}}_X \operatorname{Sym}(\mathscr{F})$ which is a group scheme over X since it can be represented as $C(\mathscr{F})(T) = \mathrm{Hom}(\mathscr{F}_T, \mathscr{O}_T)$. We call a cone of this form is an* abelian cone over *X.*

Remark 3.4. (a) *A fibered product of abelian cones is an abelian cone.*

(b) *A vector bundle $E = \underline{\mathrm{Spec}}_X \operatorname{Sym}(\mathscr{E}^\vee)$ is a special case.*

(c) *Any cone $C = \underline{\mathrm{Spec}}_X \bigoplus_{i\geq 0} \mathscr{S}^i$ is canonically a closed subcone of an abelian cone $A(C) = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{S}^1$, called the* abelian hull of *C. The abelian hull is a vector bundle if and only if $\mathscr{S}^1$ is locally free.*

(d) *The* abelianization *$C \mapsto A(C)$ is a functor has the forgetful functor as a right adjoint. So we have*

$$\mathrm{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \mathrm{Hom}_{\mathbf{Cone}_X}(C, A).$$

(e) *Let $\mathbf{Alg}_X^o$ as the category of quasicoherent graded $\mathscr{O}_X$-algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:*

$$\begin{tikzcd} \mathbf{Alg}_X^o \arrow[r, "\underline{\mathrm{Spec}}_X"] & \mathbf{Cone}_X^{\mathrm{op}} \\ \mathbf{LocFree}_X \arrow[u, "\mathrm{Sym}"] \arrow[r, "\underline{\mathrm{Spec}}_X \operatorname{Sym}(-)^\vee"] \arrow[d, hook] & \mathbf{Vect}_X^{\mathrm{op}} \arrow[u, hook] \arrow[d, hook] \\ \mathbf{Coh}_X \arrow[r, "\underline{\mathrm{Spec}}_X \operatorname{Sym}"] & \mathbf{AbCone}_X^{\mathrm{op}} \end{tikzcd}$$

Example 3.5. *Tow important examples. Let $X \hookrightarrow Y$ be a closed immersion of ideal $\mathscr{I}$. Then $C_{X/Y} := \underline{\mathrm{Spec}}_X \bigoplus_{n\geq 0} \mathscr{I}^n/\mathscr{I}^{n+1}$ is called the* normal cone of X in Y. *The associated abelian cone $N_{X/Y} = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{I}/\mathscr{I}^2$ is called the* normal sheaf of X in Y.

Lemma 3.6. *About smoothness:*

(a) *Let $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ be a cone over X. Then $C_{X/C} \cong \mathscr{S}^1 \cong 0^*\Omega_{C/X}$.*

(b) *A cone C over X is a vector bundle if and only if it is smooth over X.*

(c) *Let $C \to D$ be a smooth morphism of cones of relative dimension n over X. Then the induced morphism $A(C) \to A(D)$ is also smooth of relative dimension n.*

Proof. For (a), note that $C_{X/C} \cong \mathscr{S}^1$ is trivial by definition. Morever, $0 : X \to C$ is the zero section and we have $0 \to C_{X/C} \to 0^*\Omega_{C/X} \to \Omega_{X/X} = 0$ exact (see Tag 0474). Well done.

For (b), let $C = \underline{\mathrm{Spec}}_X \bigoplus_{i\geq 0} \mathscr{S}^i$ and assume that $C \to X$ has constant relative dimension r. Then $\mathscr{S}^1 = 0^*\Omega_{C/X}$ is locally free of rank r. As $C \hookrightarrow A(C)$ where $A(C)$ is a vector bundle and $\dim C = \dim A(C)$, we know that C is a vector bundle.

For (c), apply the exact triangle of cotangent complex to $X \to C \to D$ and (a), we have an exact sequence

$$0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to 0_C^*\Omega_{C/D} \to 0$$

where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. So locally we have $A(C) = A(D) \times_X \underline{\mathrm{Spec}}_X \operatorname{Sym}(0_C^*\Omega_{C/D})$. Well done. □

Definition 3.7. *A sequence of cone morphisms*

$$0 \to E \xrightarrow{i} C \to D \to 0$$

is called exact *if E is a vector bundle and locally over X there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \cong E \times_X D$.*

Remark 3.8. *As $E \to X$ is smooth and surjective by Lemma 3.6, if $0 \to E \xrightarrow{i} C \to D \to 0$ then locally we have $C \cong E \times_X D$ which force that $C \to D$ is smooth and surjective! Similarly $i : E \to C$ is a closed embedding.*

Lemma 3.9. *We have the following useful results.*

(a) *Given a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0$ of coherent sheaves on X, with $\mathscr{E}$ locally free, then $0 \to C(\mathscr{E}) \to C(\mathscr{F}) \to C(\mathscr{F}') \to 0$ is exact, and conversely is also true.*

(b) *Let $0 \to E \to F \xrightarrow{f} G \to 0$ be an exact sequence of abelian cones over X with E a vector bundle. Assume that $D \subset G$ is a closed subcone, then the induced sequence $0 \to E \to f^{-1}(D) =: C \to D \to 0$ is exact.*

(c) *Let $f : C \to D$ be a morphisms of cones over X which is smooth surjective, then the induced diagram*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, hook] & D \arrow[d, hook] \\ A(C) \arrow[r, "A(f)"] & A(D) \end{tikzcd}$$

is cartesian. Moreover, we have $D = [C/E]$ and $A(D) = [A(C)/E]$, where $E := C \times_{D,0} X = A(C) \times_{A(D)} X$.

(d) *Let E be a vector bundle over X and then the sequence $0 \to E \to C \to D \to 0$ is exact if and only if the abelian hulls $0 \to E \to A(C) \to A(D) \to 0$ is exact and $C \to D$ is smooth and surjective.*

Proof. For (a), we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. As exactness is local, we may assume $\mathscr{E}$ is free. Then the first sequence is exact if and only if $\mathscr{F}' \oplus \mathscr{E} = \mathscr{F}$ if and only if the second sequence is exact as cones, since $\operatorname{Sym}(\mathscr{F}' \oplus \mathscr{E}) = \operatorname{Sym}(\mathscr{F}') \otimes \operatorname{Sym}(\mathscr{E}) = \operatorname{Sym}(\mathscr{F})$.

For (b), note that this can be checked locally, so we can let we can assume that $\mathscr{F} = \mathscr{G} \oplus \mathscr{E}^\vee$ where $E = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and $F = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{F}$ and $G = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{G}$. Let $D = \underline{\operatorname{Spec}}_X \mathscr{T}$, then we have surjection $\operatorname{Sym}(\mathscr{G}) \to \mathscr{T}$. By definition, we have

$$\begin{aligned} C = F \times_G D &= \underline{\operatorname{Spec}}_X(\operatorname{Sym}(\mathscr{F}) \otimes_{\operatorname{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\operatorname{Spec}}_X((\operatorname{Sym}(\mathscr{G}) \otimes \operatorname{Sym} \mathscr{E}^\vee) \otimes_{\operatorname{Sym}(\mathscr{G})} \mathscr{T}) \\ &= \underline{\operatorname{Spec}}_X(\operatorname{Sym} \mathscr{E}^\vee \otimes \mathscr{T}). \end{aligned}$$

This means locally $C = E \oplus D$ and the splitting $C \to E$ is induced by $F \to E$. Well done.

For (c), let $E := C \times_{D,0} X$ and $E' := A(C) \times_{A(D)} D$ with embedding $E \hookrightarrow E'$, then both of them are vector bundles by Lemma 3.6(b)(c) and hence $E = E'$. We have cartesians

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, hook] \\ C \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, hook] \\ A(C) \arrow[r] & A(D) \end{tikzcd}$$

By the properties of commutative affine group schemes, we have $A(D) =$

$[A(C)/E]$. But how about $[C/E]$? Now we have

$$\begin{tikzcd}
 & & D \arrow[ddl, hook] \\
C \arrow[r] \arrow[urr] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & {[C/E]} \arrow[d, hook] & \\
A(C) \arrow[r] & A(D) &
\end{tikzcd}$$

Since $C \to [C/E]$ and $C \to D$ are both smooth and surjective, we know that $[C/E] \to D$ is flat and surjective. But by closed embeddings $[C/E] \to A(D)$ and $D \to A(D)$, we know that $D' \to D$ is also a closed embedding. Thus $D = [C/E]$, well done.

For (d), note that all the question is locally on X. First we assume $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ is exact. Then by (a), to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, we only need to show that $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact where $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$ and $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $D = \underline{\mathrm{Spec}}_X \mathscr{T}$. First since f is faithfully flat and quasi-compact, we know that $\mathscr{T}^1 \to \mathscr{S}^1$ is injective. And since i is a closed embedding, $\mathscr{S}^1 \to \mathscr{E}^\vee$ is surjective. Now by local splitting, we know that locally we have $\operatorname{Sym}(E^\vee) \otimes \mathscr{T} = \mathscr{S}$. In particular, we have $\mathscr{T}^1 \oplus \mathscr{E}^\vee = \mathscr{S}^1$. Thus the exactness of $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is obtained. Conversely we assume that after taking abelian hull, the sequence is exact. Now the result follows from (a) and (c). $\square$

Proposition 3.10. *Let $C \to D$ be a smooth, surjective morphism of cones. If we let $E = C \times_{D,0} X$, then the sequence*

$$0 \to E \to C \to D \to 0$$

Conversely if $0 \to E \to C \to D \to 0$ is exact, then $E \cong C \times_{D,0} X$.

Proof. Let $C = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathscr{S}^i$ and $D = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathscr{T}^i$.

Let $E = C \times_{D,0} X = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$, by Lemma 3.9(d) we just need to show that $0 \to E \to A(C) \to A(D) \to 0$ is exact, that is, $0 \to \mathscr{T}^1 \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$ is exact by Lemma 3.9(a). Note that $\operatorname{Sym} \mathscr{E}^\vee = \mathscr{S} \otimes_{\mathscr{T}} (\mathscr{T}/\mathscr{T}^{\geq 1})$ which force $\mathscr{E}^{\vee,1} \cong \mathscr{S}^1/\mathscr{T}^1$. Well done.

Conversely, assume that the sequence $0 \to E \to C \to D \to 0$ is exact and $F = C \times_{D,0} X$. Then by the universal property of fibre product, we get a morphism $E \to F$. From the construction, it is easy to see that $\mathscr{F}^\vee \to \mathscr{E}^\vee$ surjective. Since they are both bundles of the same rank over X, we know that $E = F$. $\square$

Definition 3.11. *(a) If E is a vector bundle and $f : E \to C(\mathscr{F})$ a morphism of abelian cones. The there is an E-action as $E \times_X C(\mathscr{F}) \to C(\mathscr{F})$ as $(\nu, \gamma) \mapsto f\nu + \gamma$.*

(b) If E is a vector bundle and $d : E \to C$ a morphism of cones, we say that C is an E-cone, if C is invariant under the action of E on $A(C)$.

(c) A morphism ϕ from an E-cone C to an F-cone D is a commutative diagram of cones

$$\begin{tikzcd} E \arrow[r, "d_E"] \arrow[d, "\phi"] & C \arrow[d, "\phi"] \\ F \arrow[r, "d_F"] & D \end{tikzcd}$$

(d) If $\phi : (E, d_E, C) \to (F, d_F, D)$ and $\psi : (E, d_E, C) \to (F, d_F, D)$ are morphisms, we call them homotopic, if there exists a morphism of cones $k : C \to F$, such that $kd_E = \psi - \phi = d_F k$.

Lemma 3.12. *Some useful lemmas:*

(a) Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D = [C/E] \to 0$ be a sequence of algebraic X-spaces with E a bundle, C is a E-cone, i a closed embedding and $f : C \to D = [C/E]$ is the universal family. Then locally on X, there is a $j : C \to E$ split i and induces an isomorphism $(f, j) : C \to D \oplus E$.

(b) Let $0 \to E \xrightarrow{i} C \xrightarrow{f} D \to 0$ be a sequence of algebraic X-spaces with E a bundle, C is a E-cone, i a closed embedding. Then D is a cone with the sequence exact if and only if $D \cong [C/E]$.

Proof. For (a), since the question is local we can assume that E is a trivial bundle and X is a scheme. Let $i' : E \to A(C)$ and $C = \underline{\operatorname{Spec}}_X \mathscr{S}$ and $E = \underline{\operatorname{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Then the surjection $\mathscr{S}^1 \twoheadrightarrow \mathscr{E}^\vee$ has a splitting $\mathscr{E}^\vee \hookrightarrow \mathscr{S}^1$, which gives $j' : A(C) \to E$ such that $j' \circ i' = \mathrm{id}_E$. Then we just define $j : C \to E$ as composition with $C \to A(C)$ and j'. Hence $j \circ i = \mathrm{id}_E$.

Now since $C \to D$ is also a principal E-bundle, and we have a E-equivariant D-morphism $(f, j) : C \to D \oplus E$ from C to the trivial principal bundle. Since they are both E-principal bundle, we know that (f, j) is an isomorphism.

For (b) □

Proposition 3.13. *Let X be a DM-stack.*

(a) Let E be a vector bundle. Consider the sequence of cone morphisms $0 \to E \xrightarrow{i} C \xrightarrow{\phi} D \to 0$ with i a closed embedding. Then it is exact if

and only if C is a E-cone, $\phi : C \to D$ is faithfully flat and the diagram

$$\begin{tikzcd} E \times C \arrow[r, "\sigma"] \arrow[d, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "\phi"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

is cartesian with projection p and action σ.

(b) *Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions and let $\phi : C \to D$ be an $\mathbb{A}^1$-equivariant X-morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Assume that E is a vector bundle. Then C is an E-cone (resp. abelian cone, vector bundle) over X if and only if D is a cone (resp. abelian cone, vector bundle) over X.*

Proof. For (a), if it is exact, locally we have $C \cong E \times_X D$. So E act on C locally as $E \times E \times_X D \to E \times_X D$ given by $(f, (e, d)) \mapsto (i(f) + e, d)$. So C is a E-cone. Now $\phi : C \to D$ is surjective is trivial. The cartesian diagram follows from local case.

Conversely, we need to find that $C \cong E \times_X D$ in local.

For (b), let C is an E-cone over X. By faithfully flat descent, we know that D is a relative spectrum over X. As $(D, 0, \gamma)$ be algebraic X-spaces with sections and $\mathbb{A}^1$-actions, we know that D is a relative spectrum of graded algebra with zero section and $\phi : C \to D$ induced by graded morphism and preserving zero sections which is smooth and surjective.

We first assume C is abelian, that is, $C = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{F}$. Let $E = \underline{\mathrm{Spec}}_X \operatorname{Sym} \mathscr{E}^\vee$. Hence we have surjection $\mathscr{F} \to \mathscr{E}^\vee$. So we have $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{E}^\vee \to 0$ exact. Hence $0 \to E \to C \to C(\mathscr{G}) \to 0$ exact. So by ??? we know that $D = C(\mathscr{G})$ which is an abelian cone.

In the general case, we have $C \subset A(C)$ defined by a homogeneous ideal $\mathscr{I} \subset \operatorname{Sym} \mathscr{S}^1$ where $C = \underline{\mathrm{Spec}}_X \mathscr{S}$ and $\mathscr{S} = \bigoplus_{i \geq 0} \mathscr{S}^i$. Now we have an exact sequence $0 \to \mathscr{G} \to \mathscr{S}^1 \to \mathscr{E}^\vee \to 0$. Let $\mathscr{J} = \mathscr{I} \cap \operatorname{Sym} \mathscr{G}$ and let $C' = \underline{\mathrm{Spec}}_X((\operatorname{Sym} \mathscr{G})/\mathscr{J})$ which is the scheme-theoric image of C in $C(\mathscr{G})$. Hence C' is the quotient of C by E. □

Remark 3.14. *In the original paper [BF97] they claim (a) is enough for the surjectivity of f.*

3.3 Cone Stack

Let X be a Deligne-Mumford stack.

Definition 3.15. *Let $\mathfrak{C}$ be an algebraic stack over X, together with a section $0 : X \to \mathfrak{C}$. An $\mathbb{A}^1$-action on $(\mathfrak{C}, 0)$ is given by a morphism of X-stacks*

$\gamma : \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$ *and three* 2*-isomorphisms* θ_1, θ_0 *and* θ_γ *between the* 1*-morphisms in the following diagrams.*

$$\begin{tikzcd} \mathfrak{C} \arrow[rr, "(1,\mathrm{id})/(0,\mathrm{id})"] \arrow[dr, "\mathrm{id}/0"'] \arrow[rr, phantom, shift right=3ex, "\Rightarrow \theta_1/\theta_0"] & & \mathbb{A}^1 \times \mathfrak{C} \arrow[dl, "\gamma"] \\ & \mathfrak{C} & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\gamma"] \arrow[d, "m\times\mathrm{id}"] \arrow[dr, phantom, "\Rightarrow \theta_\gamma"] & \mathbb{A}^1 \times \mathfrak{C} \arrow[d, "\gamma"] \\ \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\gamma"'] & \mathfrak{C} \end{tikzcd}$$

The 2*-isomorphisms* θ_1, θ_0 *and* θ_γ *are required to satisfy certain compatibilities.*

Definition 3.16. *Let* $(\mathfrak{C}, 0, \gamma)$ *and* $(\mathfrak{D}, 0, \gamma)$ *be* X*-stacks with sections and* $\mathbb{A}^1$*-actions. Then an* $\mathbb{A}^1$-equivariant morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ *is a triple* $(\phi, \eta_0, \eta_\gamma)$*, where* $\phi : \mathfrak{C} \to \mathfrak{D}$ *is a morphism of algebraic* X*-stacks and* η_0 *and* η_γ *are* 2*-isomorphisms between the morphisms in the following diagrams.*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "0"'] \arrow[dr, phantom, shift left=2ex, "\Rightarrow \eta_0"] & \mathfrak{C} \arrow[d, "\phi"] \\ & \mathfrak{D} \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^1 \times \mathfrak{C} \arrow[r, "\mathrm{id}\times\phi"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\Rightarrow \eta_\gamma"] & \mathbb{A}^1 \times \mathfrak{D} \arrow[d, "\gamma"] \\ \mathfrak{C} \arrow[r, "\phi"] & \mathfrak{D} \end{tikzcd}$$

Again, the 2*-isomorphisms have to satisfy certain compatibilities.*

Definition 3.17. *Let* $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \to \mathfrak{D}$ *and* $(\psi, \eta_0', \eta_\gamma') : \mathfrak{C} \to \mathfrak{D}$ *be two* $\mathbb{A}^1$*-equivariant morphisms. An* $\mathbb{A}^1$-equivariant isomorphism $\zeta : \phi \to \psi$ *is a* 2*-isomorphism* $\zeta : \phi \to \psi$ *such that the diagrams*

$$\begin{tikzcd} 0 \arrow[r, "\eta_0"] \arrow[dr, "\eta_0'"'] & \phi \circ 0 \arrow[d, "\zeta\circ 0"] \\ & \psi \circ 0 \end{tikzcd} \qquad \begin{tikzcd} \phi \circ \gamma \arrow[r] \arrow[d, "\zeta\circ\gamma"] & \gamma \circ (\mathrm{id} \times \phi) \arrow[d, "\gamma\circ(\mathrm{id}\times\zeta)"] \\ \psi \circ \gamma \arrow[r] & \gamma \circ (\mathrm{id} \times \psi) \end{tikzcd}$$

commute.

Example 3.18. *Let* C *be a* E*-cone, then consider the quotient stack* $[C/E]$*. We claim that* $[C/E]$ *a zero section and an* $\mathbb{A}^1$*-action.*

Indeed, the zero section $0 : X \to [C/E]$ *given by* $X \leftarrow E \to C$*.*

Proposition 3.19.

3.4 A Stack of Special Type

3.5 Intrinsic Normal Cone

3.6 Obstruction Theory and Virtual Class

3.7 Examples

4 Atiyah-Bott Localization

We will follows [AB84].

5 Localization of Virtual Fundamental Class

We will follows [GP99].

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