

# Dualizing Complexes Using Derived Categories

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May 11, 2024

## 1 Introduction

The theory of duality is beautiful theory in the commutative algebra and algebraic geometry. In the book [BH98], the authors introduce the duality of Cohen-Macaulay rings. But it can not work for more general rings. In this note, we will introduce the general theory of duality of more general rings using derived category. We will generalize the notion of canonical modules in [BH98] into dualizing complexes and we will see that in the case of Cohen-Macaulay rings, this complex can be concentrated in the one place. So we again have the canonical modules.

Now we give the outline of the note:

- In the section 2, we will give a quike introduction of local cohomology using derived category which is a basic tool.
- In the section 3, we will introduce the basic theory of dualizing complexes of Noetherian rings and introduce the local duality theorem.
- In the section 4, we will consider the special case of Cohen-Macaulay and Gorenstein Rings and to find some special properties. We will also consider the more cases of rings which have dualizing complexes
- In the final section 5, we will give a glimpse of the global theory of dualizing complexes in algebraic geometry.

Note that we will mainly follows the chapter 47 in [Pro24] and Chapter 25 in [GW23]. The basic theory of derived categories, injective hulls and Matlis duality will be omitted and we refer to [BH98] or the beginning of chapter 47 in [Pro24].

**Acknowledgement.** Here I am grateful to Prof. Manolis C.Tsakiris for this interesting course of advanced commutative algebra in the Chinese Academy of Sciences. This course gave me an opportunity to learn these wonderful theories.

## 2 Local Cohomology of Noetherian Rings

Here we will follow section 47.8–47.11 in [Pro24] and summarize some results about several definitions of local cohomology using derived categories which we will use later. The reader can begin to read the Section 3 and omit this whole section for now.

More general theory we refer [Har67] or [Gro68] or chapter 51 of [Pro24].

### 2.1 More on Čech Complex and Koszul Complex

**Lemma 2.1.** *Let  $R$  be a ring. Let  $\phi : E \rightarrow R$  be an  $R$ -module map. Let  $e \in E$  with image  $f = \phi(e)$  in  $R$ . Then  $f = de + ed$  as endomorphisms of  $\mathbf{K}_\bullet(\phi)$ . In particular, multiplication by  $f_i$  on  $\mathbf{K}_\bullet(f_1, \dots, f_r)$  is homotopic to zero.*

*Proof.* This is true because  $d(ea) = d(e)a - ed(a) = fa - ed(a)$ .  $\square$

**Lemma 2.2.** *Let  $R$  be a ring. Let  $f_1, \dots, f_r \in R$ . The (extended alternating) Čech complex of  $R$  is the cochain complex*

$$R \rightarrow \bigoplus_{i_0} R_{f_{i_0}} \rightarrow \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r}$$

where  $R$  is in degree 0, the term  $\bigoplus_{i_0} R_{f_{i_0}}$  is in degree 1, and so on. The maps are defined as follows

1. The map  $R \rightarrow \bigoplus_{i_0} R_{f_{i_0}}$  is given by the canonical maps  $R \rightarrow R_{f_{i_0}}$ .
2. Given  $1 \leq i_0 < \dots < i_{p+1} \leq r$  and  $0 \leq j \leq p+1$  we have the canonical localization map

$$R_{f_{i_0} \dots \hat{f}_{i_j} \dots f_{i_{p+1}}} \rightarrow R_{f_{i_0} \dots f_{i_{p+1}}}$$

3. The differentials use the canonical maps of (2) with sign  $(-1)^j$ .

Then

$$R \rightarrow \bigoplus_{i_0} R_{f_{i_0}} \rightarrow \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r}$$

is a colimit of the Koszul complexes  $\mathbf{K}(R, f_1^n, \dots, f_r^n)$ ; see proof for a precise statement.

*Proof.* We have

$$\mathbf{K}(R, f_1^n, \dots, f_r^n) : 0 \rightarrow \wedge^r(R^{\oplus r}) \rightarrow \wedge^{r-1}(R^{\oplus r}) \rightarrow \dots \rightarrow R^{\oplus r} \rightarrow R \rightarrow 0$$

with the term  $\wedge^r(R^{\oplus r})$  sitting in degree 0. Let  $e_1^n, \dots, e_r^n$  be the standard basis of  $R^{\oplus r}$ . Then the elements  $e_{j_1}^n \wedge \dots \wedge e_{j_{r-p}}^n$  for  $1 \leq j_1 < \dots < j_{r-p} \leq r$

form a basis for the term in degree  $p$  of the Koszul complex. Further, observe that

$$d(e_{j_1}^n \wedge \dots \wedge e_{j_{r-p}}^n) = \sum (-1)^{a+1} f_{j_a}^n e_{j_1}^n \wedge \dots \wedge \hat{e}_{j_a}^n \wedge \dots \wedge e_{j_{r-p}}^n.$$

The transition maps of our system

$$\mathbf{K}(R, f_1^n, \dots, f_r^n) \rightarrow \mathbf{K}(R, f_1^{n+1}, \dots, f_r^{n+1})$$

are given by the rule

$$e_{j_1}^n \wedge \dots \wedge e_{j_{r-p}}^n \mapsto f_{i_0} \dots f_{i_{p-1}} e_{j_1}^{n+1} \wedge \dots \wedge e_{j_{r-p}}^{n+1}$$

where the indices  $1 \leq i_0 < \dots < i_{p-1} \leq r$  are such that  $\{1, \dots, r\} = \{i_0, \dots, i_{p-1}\} \amalg \{j_1, \dots, j_{r-p}\}$ . We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to  $f_1 \dots f_r$  in degree  $r$ .

Denote  $\mathbf{K}^p(R, f_1^n, \dots, f_r^n)$  the term of degree  $p$  in the Koszul complex. Observe that for any  $f \in R$  we have

$$R_f = \varinjlim (R \xrightarrow{f} R \xrightarrow{f} R \rightarrow \dots)$$

Hence we see that in degree  $p$  we obtain

$$\varinjlim \mathbf{K}^p(R, f_1^n, \dots, f_r^n) = \bigoplus_{1 \leq i_0 < \dots < i_{p-1} \leq r} R_{f_{i_0} \dots f_{i_{p-1}}}$$

Here the element  $e_{j_1}^n \wedge \dots \wedge e_{j_{r-p}}^n$  of the Koszul complex above maps in the colimit to the element  $(f_{i_0} \dots f_{i_{p-1}})^{-n}$  in the summand  $R_{f_{i_0} \dots f_{i_{p-1}}}$  where the indices are chosen such that  $\{1, \dots, r\} = \{i_0, \dots, i_{p-1}\} \amalg \{j_1, \dots, j_{r-p}\}$ . Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0} \dots f_{i_{p-1}}}) = \sum_{i \notin \{i_0, \dots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0} \dots f_{i_t} f_i f_{i_{t+1}} \dots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree  $p$  by the map

$$\bigoplus_{1 \leq i_0 < \dots < i_{p-1} \leq r} R_{f_{i_0} \dots f_{i_{p-1}}} \longrightarrow \bigoplus_{1 \leq i_0 < \dots < i_{p-1} \leq r} R_{f_{i_0} \dots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0} \dots f_{i_{p-1}}} \mapsto (-1)^{i_0 + \dots + i_{p-1} + p} \text{ in } R_{f_{i_0} \dots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from  $\varinjlim \mathbf{K}(R, f_1^n, \dots, f_r^n)$  to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out.  $\square$

## 2.2 Deriving Torsion

**Definition 2.3.** Let  $R$  be a ring with an ideal  $I \subset R$ . Fix  $M \in \mathbf{Mod}_R$ .

(a) We define

$$M[I^n] := \{m \in M : I^n m = 0\}, \quad M[I^\infty] = \bigoplus_n M[I^n].$$

(b) We call  $M$  is  $I^\infty$ -torsion if  $M = M[I^\infty]$ . We let  $I^\infty$ -torsion be the subcategory of  $\mathbf{Mod}_R$  consist of  $I^\infty$ -torsion modules

Here we give some easy but important properties of this notion.

**Proposition 2.4.** Let  $R$  be a ring with an ideal  $I \subset R$ .

(a) Let  $M \in I^\infty$ -torsion, then  $M$  admits a resolution

$$\cdots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow M \rightarrow 0$$

with each  $K_i$  a direct sum of copies of  $R/I^n$  for  $n$  variable. In particular, the category  $I^\infty$ -torsion is a Grothendieck abelian category.

(b) Let  $I$  be a finitely generated ideal of  $R$ , then for any  $M \in \mathbf{Mod}_R$  we have  $(M/M[I^\infty])[I] = 0$ .

(c) Let  $I$  be a finitely generated ideal of  $R$ , then  $I^\infty$ -torsion is a Serre subcategory of the abelian category  $\mathbf{Mod}_R$ , that is, an extension of  $I^\infty$ -torsion modules is  $I^\infty$ -torsion.

(d) Let  $I$  be a finitely generated ideal of  $R$  and  $M \in \mathbf{Mod}_R$ , then we have an exact sequence

$$0 \rightarrow M[I^\infty] \rightarrow M \rightarrow \prod_{\mathfrak{p} \notin V(I)} M_{\mathfrak{p}}.$$

In particular, we have  $M \in I^\infty$ -torsion if and only if  $\text{supp}(M) \subset V(I)$ . Hence the subcategory  $I^\infty$ -torsion  $\subset \mathbf{Mod}_R$  depends only on the closed subset  $V(I) \subset \text{Spec}(R)$ .

*Proof.* For (a), there is a canonical surjection  $\bigoplus_{m \in M} R/I^{n_m} \rightarrow M \rightarrow 0$  where  $n_m$  is the smallest positive integer such that  $I^{n_m} \cdot m = 0$ . The kernel of the preceding surjection is also an  $I^\infty$ -torsion module. Proceeding inductively, we construct the desired resolution of  $M$ .

For (b), Let  $m \in M$ . If  $m$  maps to an element of  $(M/M[I^\infty])[I]$  then  $Im \subset M[I^\infty]$ . Write  $I = (f_1, \dots, f_t)$ . Then we see that  $f_i m \in M[I^\infty]$ . Thus we see that  $I^N m = 0$  for some large  $N \gg 0$ . Hence  $m$  maps to zero in  $(M/M[I^\infty])$ .

For (c), suppose that  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is a short exact sequence of modules with  $M'$  and  $M''$  both  $I^\infty$ -torsion modules. Then  $M' \subset M[I^\infty]$  and hence  $M/M[I^\infty]$  is a quotient of  $M''$  and therefore  $I^\infty$ -torsion. Combined with (b) this implies that it is zero.

For (d), let  $M \in \mathbf{Mod}_R$  and let  $x \in M$ . If  $x \in M[I^\infty]$ , then  $x$  maps to zero in  $M_f$  for all  $f \in I$ . Hence  $x$  maps to zero in  $M_{\mathfrak{p}}$  for all  $I \not\subset \mathfrak{p}$ . Conversely, if  $x$  maps to zero in  $M_{\mathfrak{p}}$  for all  $I \not\subset \mathfrak{p}$ , then  $x$  maps to zero in  $M_f$  for all  $f \in I$ . Hence if  $I = (f_1, \dots, f_r)$ , then  $f_i^{n_i} x = 0$  for some  $n_i \geq 1$ . It follows that  $x \in M[I^N]$  for  $N = \sum_i n_i$ . Thus  $M[I^\infty]$  is the kernel of  $M \rightarrow \prod_{\mathfrak{p} \notin Z} M_{\mathfrak{p}}$ .  $\square$

**Definition 2.5** (Fake Local Cohomology). *Let  $R$  be a ring and let  $I$  be a finitely generated ideal. By Proposition 2.4(a), the category  $I^\infty$ -torsion is a Grothendieck abelian category and hence the derived category  $\mathbf{D}(I^\infty\text{-torsion})$  exists by some homological algebra, as Tag 079Q. Hence we have the derived functor*

$$\mathbf{R}\Gamma_I : \mathbf{D}(R) \rightarrow \mathbf{D}(I^\infty\text{-torsion})$$

of  $\Gamma_I : \mathbf{Mod}_R \rightarrow I^\infty\text{-torsion}$  given by  $M \mapsto M[I^\infty]$  which is left exact.

Moreover, we define  $H_I^q(K) := H^q(\mathbf{R}\Gamma_I(K))$  for any  $K \in \mathbf{D}(R)$ .

**Remark 2.6.** *Note that this functor does not deserve the name local cohomology unless the ring  $R$  is Noetherian.*

Now we discuss some basic properties of the functor.

**Proposition 2.7.** *Let  $R$  be a ring and let  $I$  be a finitely generated ideal.*

- (a) *The functor  $\mathbf{R}\Gamma_I$  is right adjoint to the functor  $\mathbf{D}(I^\infty\text{-torsion}) \rightarrow \mathbf{D}(R)$ .*
- (b) *For any object  $K$  of  $\mathbf{D}(R)$  we have*

$$\mathbf{R}\Gamma_I(K) = \text{hocolim} \mathbf{R}\text{Hom}_R(R/I^n, K)$$

*in  $\mathbf{D}(R)$  and hence*

$$H_I^q(K) := \mathbf{R}^q\Gamma_I(K) = \varinjlim \text{Ext}_R^q(R/I^n, K)$$

*as modules for all  $q \in \mathbb{Z}$ .*

- (c) *Let  $K^\bullet$  be a complex of  $A$ -modules such that  $f : K^\bullet \rightarrow K^\bullet$  is an isomorphism for some  $f \in I$ , i.e.,  $K^\bullet$  is a complex of  $R_f$ -modules. Then  $\mathbf{R}\Gamma_I(K^\bullet) = 0$ .*

*Proof.* For (a), this follows from the fact that taking  $I^\infty$ -torsion submodules is the right adjoint to the inclusion functor  $I^\infty\text{-torsion} \rightarrow \mathbf{Mod}_R$ .

For (b), let  $J^\bullet$  be a  $K$ -injective resolution of  $K$ . Then we have

$$\begin{aligned}\mathbf{R}\Gamma_I(K) &= \Gamma_I(J^\bullet) = J^\bullet[I^\infty] = \varinjlim_n J^\bullet[I^n] \\ &= \varinjlim_n \mathrm{Hom}_R(R/I^n, J^\bullet) = \mathrm{hocolim} \mathbf{R} \mathrm{Hom}_R(R/I^n, K).\end{aligned}$$

Well done.

For (c), in this case the cohomology modules of  $\mathbf{R}\Gamma_I(K^\bullet)$  are both  $f^\infty$ -torsion and  $f$  acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero.  $\square$

In the end of this small section we consider another category. Let  $R$  be a ring and let  $I$  be a finitely generated ideal. By Proposition 2.4(c),  $\mathbf{I}^\infty\text{-torsion}$  is a Serre subcategory of the abelian category  $\mathbf{Mod}_R$ . This shows that  $\mathbf{I}^\infty\text{-torsion} \subset \mathbf{Mod}_R$  exact which induce the functor  $\mathbf{D}(\mathbf{I}^\infty\text{-torsion}) \rightarrow \mathbf{D}(R)$  which factor through

$$\mathbf{D}(\mathbf{I}^\infty\text{-torsion}) \rightarrow \mathbf{D}_{\mathbf{I}^\infty\text{-torsion}}(\mathbf{Mod}_R).$$

**Proposition 2.8.** *Let  $R$  be a ring and let  $I$  be a finitely generated ideal. Let  $M, N \in \mathbf{I}^\infty\text{-torsion}$ .*

- (a)  $\mathrm{Hom}_{\mathbf{D}(R)}(M, N) = \mathrm{Hom}_{\mathbf{D}(\mathbf{I}^\infty\text{-torsion})}(M, N)$
- (b)  $\mathrm{Ext}_{\mathbf{D}(\mathbf{I}^\infty\text{-torsion})}^2(M, N) \rightarrow \mathrm{Ext}_{\mathbf{D}(R)}^2(M, N)$  is not surjective in general. In particular,  $\mathbf{D}(\mathbf{I}^\infty\text{-torsion}) \rightarrow \mathbf{D}_{\mathbf{I}^\infty\text{-torsion}}(\mathbf{Mod}_R)$  is not an equivalence in general.

*Proof.* (a) is trivial and the counterexample of (b) we refer Tag 0A6P.  $\square$

**Remark 2.9.** *However in the Noetherian case this will be true. We will see this later.*

## 2.3 Basic Theory of Local Cohomology

Now we will introduce some true local cohomologies.

**Theorem 2.10** (Real Local Cohomology, I). *Let  $R$  be a ring and let  $I \subset R$  be a finitely generated ideal and  $Z = V(I) \subset \mathrm{Spec}(R)$ . There exists a right adjoint  $\mathbf{R}\Gamma_Z$  to the inclusion functor  $\mathbf{D}_{\mathbf{I}^\infty\text{-torsion}}(R) \rightarrow \mathbf{D}(R)$ . In fact, if  $I$  is generated by  $f_1, \dots, f_r \in R$ , then we have*

$$\mathbf{R}\Gamma_Z(K) = \left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0}f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \otimes_R^{\mathbf{L}} K$$

*functorially in  $K \in \mathbf{D}(R)$ .*

*Proof.* Say  $I = (f_1, \dots, f_r)$  is an ideal. Let  $K^\bullet$  be a complex of  $R$ -modules. There is a canonical map of complexes

$$\left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \longrightarrow R.$$

from the extended Čech complex to  $R$ . Tensoring with  $K^\bullet$ , taking associated total complex, we get a map

$$\text{Tot} \left( K^\bullet \otimes_R \left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \right) \longrightarrow K^\bullet$$

in  $\mathbf{D}(R)$ . We claim the cohomology modules of the complex on the left are  $I^\infty$ -torsion, i.e., the LHS is an object of  $D_{I^\infty\text{-torsion}}(R)$ . Namely, we have

$$\left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) = \varinjlim \mathbf{K}(R, f_1^n, \dots, f_r^n)$$

by Lemma 2.2. Moreover, multiplication by  $f_i^n$  on the complex  $\mathbf{K}(R, f_1^n, \dots, f_r^n)$  is homotopic to zero by Lemma 2.1. Since

$$H^q(LHS) = \varinjlim H^q(\text{Tot}(K^\bullet \otimes_A \mathbf{K}(R, f_1^n, \dots, f_r^n)))$$

we obtain our claim. On the other hand, if  $K^\bullet$  is an object of  $D_{I^\infty\text{-torsion}}(R)$ , then the complexes  $K^\bullet \otimes_R R_{f_{i_0} \dots f_{i_p}}$  have vanishing cohomology. Hence in this case the map  $LHS \rightarrow K^\bullet$  is an isomorphism in  $D(A)$ . The construction

$$\mathbf{R}\Gamma_Z(K^\bullet) = \text{Tot} \left( K^\bullet \otimes_R \left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \right)$$

is functorial in  $K^\bullet$  and defines an exact functor  $\mathbf{D}(R) \rightarrow \mathbf{D}_{I^\infty\text{-torsion}}(R)$  between triangulated categories. It follows formally from the existence of the natural transformation  $\mathbf{R}\Gamma_Z \rightarrow \text{id}$  given above and the fact that this evaluates to an isomorphism on  $K^\bullet$  in the subcategory, that  $\mathbf{R}\Gamma_Z$  is the desired right adjoint.  $\square$

Hence now we have the functor

$$\mathbf{R}\Gamma_Z : \mathbf{D}(R) \rightarrow \mathbf{D}_{I^\infty\text{-torsion}}(R).$$

As we have seen, we construct the functor using Čech complex. Is there some relation between this and the functor in algebraic geometry?

**Definition 2.11** (Real Local Cohomology, II). *Let  $(X, \mathcal{O}_X)$  be a ringed space. Let  $Z \subset X$  be a closed subset. Consider the functor  $\Gamma_Z : \mathbf{Mod}(\mathcal{O}_X) \rightarrow \mathbf{Mod}(\mathcal{O}_X(X))$  given by*

$$\Gamma_Z(\mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) : \text{supp}(s) \subset Z\}.$$

*Using  $K$ -injective resolutions, we obtain the right derived functor*

$$\mathbf{R}\Gamma_Z(X, -) : \mathbf{D}(\mathcal{O}_X) \rightarrow \mathbf{D}(\mathcal{O}_X(X)).$$

*The group  $H_Z^q(X, K) = H^q(\mathbf{R}\Gamma_Z(X, K))$  the cohomology module with support in  $Z$ .*

We now show that they are the same! Indeed, we can use Čech complex to rebuild  $\mathbf{R}\Gamma_Z(X, -)$ , and then we can connected to  $\mathbf{R}\Gamma_Z : \mathbf{D}(R) \rightarrow \mathbf{D}_{\text{I}^\infty\text{-torsion}}(R)$  as before.

**Proposition 2.12.** *Let  $R$  be a ring and let  $I$  be a finitely generated ideal. Set  $Z = V(I) \subset X = \text{Spec}(R)$ . For  $K \in \mathbf{D}(A)$  corresponding to  $\tilde{K} \in \mathbf{D}_{\text{QCoh}}(\mathcal{O}_X)$ , there is a functorial isomorphism*

$$\mathbf{R}\Gamma_Z(K) = \mathbf{R}\Gamma_Z(X, \tilde{K}).$$

*Proof.* Note that there exists a distinguished triangle

$$\mathbf{R}\Gamma_Z(X, \tilde{K}) \rightarrow \mathbf{R}\Gamma(X, \tilde{K}) \rightarrow \mathbf{R}\Gamma(U, \tilde{K}) \rightarrow \mathbf{R}\Gamma_Z(X, \tilde{K})[1]$$

where  $U = X \setminus Z$ . We know that  $\mathbf{R}\Gamma(X, \tilde{K}) = K$ . Say  $I = (f_1, \dots, f_r)$ . Then we obtain a finite affine open covering  $\mathcal{U} : U = D(f_1) \cup \dots \cup D(f_r)$ . As affine schemes are separated, the alternating Čech complex  $\text{Tot}(\tilde{\mathcal{C}}_{\text{alt}}^\bullet(\mathcal{U}, \tilde{K}^\bullet))$  computes  $\mathbf{R}\Gamma(U, \tilde{K})$  where  $K^\bullet$  is any complex of  $R$ -modules representing  $K$ . Working through the definitions we find

$$\mathbf{R}\Gamma(U, \tilde{K}) = \text{Tot} \left( K^\bullet \otimes_R \left( \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \right)$$

It is clear that  $K^\bullet = \mathbf{R}\Gamma(X, \tilde{K}^\bullet) \rightarrow \mathbf{R}\Gamma(U, \tilde{K}^\bullet)$  is induced by the diagonal map from  $A$  into  $\prod R_{f_i}$ . Hence we conclude that

$$\mathbf{R}\Gamma_Z(X, \mathcal{F}^\bullet) = \text{Tot} \left( K^\bullet \otimes_R \left( R \rightarrow \prod_{i_0} R_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \rightarrow \dots \rightarrow R_{f_1 \dots f_r} \right) \right)$$

Well dominate.  $\square$



Finally we will introduce the noetherian case and compare the fake local cohomology and the real cohomology.

**Proposition 2.13.** *Let  $R$  be a Noetherian ring and let  $I \subset R$  be an ideal.*

1. *The adjunction  $\mathbf{R}\Gamma_I(K) \rightarrow K$  is an isomorphism for  $K \in \mathbf{D}_{I^\infty\text{-torsion}}(R)$ .*
2. *The functor  $\mathbf{D}(I^\infty\text{-torsion}) \rightarrow \mathbf{D}_{I^\infty\text{-torsion}}(R)$  is an equivalence.*
3.  *$\mathbf{R}\Gamma_I(K) = \mathbf{R}\Gamma_Z(K)$  for  $K \in \mathbf{D}(R)$ .*

*Proof.* Boring proof, we refer Tag 0955.  $\square$

So in the Noetherian case (so in the whole theory we consider) we will use  $\mathbf{R}_I(-)$ .

## 2.4 Local Cohomology and Depth

In this small section we will introduce a result about the depth and local cohomology. Note that  $\text{depth}_I(M)$  here is the grade  $\text{Grade}(I, M)$  in [BH98].

**Theorem 2.14.** *Let  $R$  be a Noetherian ring, let  $I \subset R$  be an ideal, and let  $M$  be a finite  $A$ -module such that  $IM \neq M$ . Then the following integers are equal:*

- (1)  $\text{depth}_I(M)$ ,
- (2) *the smallest integer  $i$  such that  $\text{Ext}_A^i(A/I, M)$  is nonzero, and*
- (3) *the smallest integer  $i$  such that  $H_I^i(M)$  is nonzero.*

*Moreover, we have  $\text{Ext}_A^i(N, M) = 0$  for  $i < \text{depth}_I(M)$  for any finite  $A$ -module  $N$  annihilated by a power of  $I$ .*

*Proof.* We prove the equality of (1) and (2) by induction on  $\text{depth}_I(M)$  which is allowed since  $\text{depth}_I(M) < \infty$  now.

If  $\text{depth}_I(M) = 0$ , then  $I$  is contained in the union of the associated primes of  $M$ . By prime avoidance we see that  $I \subset \mathfrak{p}$  for some associated prime  $\mathfrak{p}$ . Hence  $\text{Hom}_A(A/I, M)$  is nonzero. Thus equality holds in this case.

Assume that  $\text{depth}_I(M) > 0$ . Let  $f \in I$  be a nonzerodivisor on  $M$  such that  $\text{depth}_I(M/fM) = \text{depth}_I(M) - 1$ . Consider the short exact sequence

$$0 \rightarrow M \rightarrow M \rightarrow M/fM \rightarrow 0$$

and the associated long exact sequence for  $\text{Ext}_A^*(A/I, -)$ . Note that  $\text{Ext}_A^i(A/I, M)$  is a finite  $A/I$ -module. Hence we obtain

$$\text{Hom}_A(A/I, M/fM) = \text{Ext}_A^1(A/I, M)$$

and short exact sequences

$$0 \rightarrow \text{Ext}_A^i(A/I, M) \rightarrow \text{Ext}_A^i(A/I, M/fM) \rightarrow \text{Ext}_A^{i+1}(A/I, M) \rightarrow 0$$

Thus the equality of (1) and (2) by induction.

Observe that  $\text{depth}_I(M) = \text{depth}_{I^n}(M)$  for all  $n \geq 1$  for example by the fact that the sequence  $(f_1, \dots, f_r)$  is regular if and only if  $(f_1^{e_1}, \dots, f_r^{e_r})$  is regular for any fixed  $e_i > 0$  (see Tag 07DV for the proof). Hence by the equality of (1) and (2) we see that  $\text{Ext}_A^i(A/I^n, M) = 0$  for all  $n$  and  $i < \text{depth}_I(M)$ . Let  $N$  be a finite  $A$ -module annihilated by a power of  $I$ . Then we can choose a short exact sequence

$$0 \rightarrow N' \rightarrow (A/I^n)^{\oplus m} \rightarrow N \rightarrow 0$$

for some  $n, m \geq 0$ . Then  $\text{Hom}_A(N, M) \subset \text{Hom}_A((A/I^n)^{\oplus m}, M)$  and  $\text{Ext}_A^i(N, M) \subset \text{Ext}_A^{i-1}(N', M)$  for  $i < \text{depth}_I(M)$ . Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have  $H_I^p(M) = \varinjlim \text{Ext}_A^p(A/I^n, M)$  by Proposition 2.7(b). Thus we see that  $H_I^i(M) = 0$  for  $i < \text{depth}_I(M)$ . For  $i = \text{depth}_I(M)$ , using the vanishing of  $\text{Ext}_A^{i-1}(I/I^n, M)$  we see that the map  $\text{Ext}_A^i(A/I, M) \rightarrow H_I^i(M)$  is injective which proves nonvanishing in the correct degree.  $\square$

### 3 Dualizing Complexes

In this section we will consider the general theory of dualizing complexes of Noetherian rings.

#### 3.1 General Properties of Dualizing Complexes

**Definition 3.1.** *Let  $A$  be a Noetherian ring. A dualizing complex is a complex of  $A$ -modules  $\omega_A^\bullet$  such that*

- (a)  $\omega_A^\bullet$  has finite injective dimension.
- (b)  $H^i(\omega_A^\bullet)$  is a finite  $A$ -module for all  $i$ .
- (c)  $A \rightarrow \mathbf{R}\text{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$  is a quasi-isomorphism.

Next we consider some most basic properties.

**Proposition 3.2.** *Let  $A$  be a Noetherian ring. If  $\omega_A^\bullet$  is a dualizing complex, then the functor*

$$D : K \mapsto \mathbf{R}\text{Hom}_A(K, \omega_A^\bullet)$$

*is an anti-equivalence  $\mathbf{D}_{\text{Coh}}(A) \rightarrow \mathbf{D}_{\text{Coh}}(A)$  which exchanges  $\mathbf{D}_{\text{Coh}}^+(A)$  and  $\mathbf{D}_{\text{Coh}}^-(A)$  and induces an anti-equivalence  $\mathbf{D}_{\text{Coh}}^b(A) \rightarrow \mathbf{D}_{\text{Coh}}^b(A)$ . Moreover  $D \circ D$  is isomorphic to the identity functor.*

*Proof.* Note that  $\mathbf{R}\mathrm{Hom}_A(K, \omega_A^\bullet) \in \mathbf{D}_{\mathrm{Coh}}(A)$  follows from the fact that  $\omega_A^\bullet$  has finite injective dimension and consider the exact triangle  $\tau_{\leq n}K \rightarrow K \rightarrow \tau_{\geq n+1}K \rightarrow \tau_{\leq n}K[1]$ .

Now we know that there is a canonical morphism

$$K = \mathbf{R}\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(K, \omega_A^\bullet), \omega_A^\bullet)$$

by taking K-injective resolutions. We can show that this is an isomorphism when  $K$  is pseudo-coherent (see Tag 0A68, since we can choose finite projective modules as resolution). Consider exact triangle  $\tau_{\leq n}K \rightarrow K \rightarrow \tau_{\geq n+1}K \rightarrow \tau_{\leq n}K[1]$  again, then this is an isomorphism for  $\tau_{\leq n}K$ . So it suffices to show that both  $\tau_{\geq n+1}K$  and  $\mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(\tau_{\geq n+1}K, \omega_A^\bullet), \omega_A^\bullet)$  have vanishing cohomology in degrees  $\leq n - c$  for some  $c$ . But  $\omega_A^\bullet$  has finite injective dimension, this is trivial.  $\square$

Let  $R$  be a ring. Recall that an object  $L$  of  $\mathbf{D}(R)$  is invertible if it is an invertible object for the symmetric monoidal structure on  $\mathbf{D}(R)$  given by derived tensor product.

**Proposition 3.3.** *Let  $A$  be a Noetherian ring.*

- (a) *Let  $F : \mathbf{D}_{\mathrm{Coh}}^b(A) \rightarrow \mathbf{D}_{\mathrm{Coh}}^b(A)$  be an  $A$ -linear equivalence of categories. Then  $F(A)$  is an invertible object of  $\mathbf{D}(A)$ .*
- (b) *[Uniqueness] If  $\omega_A^\bullet$  and  $(\omega'_A)^\bullet$  are dualizing complexes, then  $(\omega'_A)^\bullet$  is quasi-isomorphic to  $\omega_A^\bullet \otimes_A^{\mathbf{L}} L$  for some invertible object  $L$  of  $\mathbf{D}(A)$ .*
- (c) *Let  $B = S^{-1}A$  be a localization. If  $\omega_A^\bullet$  is a dualizing complex, then  $\omega_A^\bullet \otimes_A B$  is a dualizing complex for  $B$ .*
- (d) *Let  $f_1, \dots, f_n \in A$  generate the unit ideal. If  $\omega_A^\bullet$  is a complex of  $A$ -modules such that  $(\omega_A^\bullet)_{f_i}$  is a dualizing complex for  $A_{f_i}$  for all  $i$ , then  $\omega_A^\bullet$  is a dualizing complex for  $A$ .*
- (e) *Let  $A \rightarrow B$  be a finite ring map of Noetherian rings. Let  $\omega_A^\bullet$  be a dualizing complex. Then  $\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet)$  is a dualizing complex for  $B$ . In particular, this is right for any surjective ring map.*

*Proof.* For (a), this is not about the dualizing complex and complicated. We refer Tag 0A7E.

For (b), By Proposition 3.2 and (a) the functor

$$K \mapsto \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$$

maps  $A$  to an invertible object  $L$ . In other words, there is an isomorphism

$$L \longrightarrow \mathbf{R}\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet)$$

Since  $L$  has finite tor dimension, this means that we can apply the similar proof in Proposition 3.2 to see that

$$\mathbf{R}\mathrm{Hom}_A(\omega_A^\bullet, (\omega'_A)^\bullet) \otimes_A^L K \longrightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(K, \omega_A^\bullet), (\omega'_A)^\bullet)$$

is an isomorphism for  $K$  in  $\mathbf{D}_{\mathrm{Coh}}^b(A)$ . In particular, setting  $K = \omega_A^\bullet$  finishes the proof.

For (c), let  $\omega_A^\bullet \rightarrow I^\bullet$  be a quasi-isomorphism with  $I^\bullet$  a bounded complex of injectives. Then  $S^{-1}I^\bullet$  is a bounded complex of injective  $B = S^{-1}A$ -modules representing  $\omega_A^\bullet \otimes_A B$ . Thus  $\omega_A^\bullet \otimes_A B$  has finite injective dimension. Since  $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$  by flatness of  $A \rightarrow B$  we see that  $\omega_A^\bullet \otimes_A B$  has finite cohomology modules. Finally, the map

$$B \longrightarrow R\mathrm{Hom}_A(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as internal hom commutes with flat base change in this case (Tag 0A6A). Well done.

For (d), consider the double complex

$$\prod_{i_0} (\omega_A^\bullet)_{f_{i_0}} \rightarrow \prod_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \rightarrow \dots$$

The associated total complex is quasi-isomorphic to  $\omega_A^\bullet$  for example by the descent theory.

By assumption the complexes  $(\omega_A^\bullet)_{f_i}$  have finite injective dimension as complexes of  $A_{f_i}$ -modules. This implies that each of the complexes  $(\omega_A^\bullet)_{f_{i_0} \dots f_{i_p}}$ ,  $p > 0$  has finite injective dimension over  $A_{f_{i_0} \dots f_{i_p}}$ . This in turn implies that each of the complexes  $(\omega_A^\bullet)_{f_{i_0} \dots f_{i_p}}$ ,  $p > 0$  has finite injective dimension over  $A$  as injectivity can be descent via flat maps. Hence  $\omega_A^\bullet$  has finite injective dimension as a complex of  $A$ -modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since  $H^n(\omega_A^\bullet)_{f_i}$  is a finite  $A_{f_i}$  module for each  $i$  we see that  $H^i(\omega_A^\bullet)$  is a finite  $A$ -module. Finally, the (derived) base change of the map  $A \rightarrow \mathbf{R}\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$  to  $A_{f_i}$  is the map  $A_{f_i} \rightarrow \mathbf{R}\mathrm{Hom}_A((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$ . Hence we deduce that  $A \rightarrow \mathbf{R}\mathrm{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$  is an isomorphism and the proof is complete.

For (e), let  $\omega_A^\bullet \rightarrow I^\bullet$  be a quasi-isomorphism with  $I^\bullet$  a bounded complex of injectives. Then  $\mathrm{Hom}_A(B, I^\bullet)$  is a bounded complex of injective  $B$ -modules representing  $\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet)$ . Thus  $\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet)$  has finite injective dimension. By spectral sequence  $\mathrm{Ext}_A^p(B, H^q(\omega_A^\bullet)) \Rightarrow \mathrm{Ext}_A^{p+q}(B, \omega_A^\bullet)$  that  $\omega_A^\bullet$  is an object of  $\mathbf{D}_{\mathrm{Coh}}(B)$ . Finally, we compute

$$\mathrm{Hom}_{\mathbf{D}(B)}(\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet), \mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet)) = \mathrm{Hom}_{\mathbf{D}(A)}(\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for  $n \neq 0$  we compute

$$\mathrm{Hom}_{\mathbf{D}(B)}(\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet), \mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet)[n]) = \mathrm{Hom}_{\mathbf{D}(A)}(\mathbf{R}\mathrm{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the second equality holds by Proposition 3.2.  $\square$

Note that not all rings have dualizing complexes, here we give some way to construct new rings with this property from the old one.

**Proposition 3.4.** *Let  $A$  be a Noetherian ring which has a dualizing complex  $\omega_A^\bullet$ .*

- (a) *Then  $\omega_A^\bullet \otimes_A A[x]$  is a dualizing complex for  $A[x]$ .*
- (b) *Then any  $A$ -algebra essentially of finite type over  $A$  has a dualizing complex.*
- (c) *Let  $\mathfrak{m} \subset A$  be a maximal ideal and set  $\kappa = A/\mathfrak{m}$ . Then  $\mathbf{R}\mathrm{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$  for some  $n \in \mathbb{Z}$ .*

*Proof.* For (a), set  $B = A[x]$  and  $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$ . We know that in this case  $\omega_B^\bullet$  has finite injective dimension. Since  $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$  by flatness of  $A \rightarrow B$  we see that  $\omega_A^\bullet \otimes_A B$  has finite cohomology modules. Finally, the map

$$B \longrightarrow R\mathrm{Hom}_B(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case. Well done.

For (b), this follows from a combination of (a) and Proposition 3.3(c)(e).

For (c), this is true because  $\mathbf{R}\mathrm{Hom}_A(\kappa, \omega_A^\bullet)$  is a dualizing complex over  $\kappa$  by Proposition 3.3(e), because dualizing complexes over  $\kappa$  are unique up to shifts (Proposition 3.3(b)), and because  $\kappa$  is a dualizing complex over  $\kappa$ .  $\square$

### 3.2 Dualizing Complexes over Local Rings

Let  $(A, \mathfrak{m}, \kappa)$  be a Noetherian local ring. By Proposition 3.3(b) we know that if  $R$  has a dualizing complex, then it is unique up to an invertible objects. Now we define a canonical choice this these complexes:

**Definition 3.5.** *Let  $(A, \mathfrak{m}, \kappa)$  be a Noetherian local ring. Then a dualizing complex  $\omega_A^\bullet$  is said to be normalized, if*

$$\mathbf{R}\mathrm{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[0]$$

*by Proposition 3.4(c).*

### 3.3 Grothendieck's Local Duality Theorem

## 4 Cohen-Macaulay, Gorenstein Rings and More

### 4.1 Cohen-Macaulay Rings

### 4.2 Gorenstein Rings

### 4.3 The Ubiquity of Dualizing Complexes

## 5 A Glimpse of Duality in Algebraic Geometry

[Illar]

[Har66] or [Con00]

[LH09] and [Ver69]

[Nee96] and [GW23]

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