Notes on 2023 SCMS Summer School Derived Categories, Kuznetsov components and Bridgeland Stability *

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Abstract

We will introduce the derived category, Bridgeland moduli spaces for K3 category, for example the Kuznetsov components of cubic fourfolds, and stability conditions in families.

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*This Course taught by Zili Zhang and Xiaolei Zhao in SCMS at 2023/07/17–2023/07/21, see `https://scms.fudan.edu.cn/info/4503/5820.htm`.

1 Introduction

First you need to read [1].

2 Stablility of Coherent Sheaves

Fix X be a smooth projective variety over $\mathbb{C}$ and $\dim X = d$ with ample bundle $H = \mathscr{O}(1)$. Our goal is to construct the moduli space of (semi-)stable torsion-free coherent sheaves $\mathscr{F}$ on X with fixed topological invariant (such as $\mathrm{ch}(\mathscr{F})$ or $\mathrm{rank}(\mathscr{F}), c_i(\mathscr{F})$ or Hilbert polynomial $P_H(\mathscr{F})$).

Example 2.1. *If we not consider (semi-)stability, we may not have the bounded family. Consider $\{\mathscr{O}(n) \oplus \mathscr{O}(-n)\}$ on $\mathbb{P}^1$, then this can not parametrized by a scheme of finite type.*

Definition 2.1. *Fix (X, H) as above and $\mathscr{F}$ be a coherent torsion-free sheaf on it.*

(i) We define the slope $\mu_H(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot H^{d-1}}{\mathrm{rank}(\mathscr{F})}$;

(ii) we call $\mathscr{F}$ is μ_H-(semi)stable if for any $0 \subset \mathscr{E} \subset \mathscr{F}$ with $0 < \mathrm{rank}\,\mathscr{E} < \mathrm{rank}\,\mathscr{F}$ we have $\mu_H(\mathscr{E}) < (\leq)\mu_H(\mathscr{F})$;

(iii) we consider the Hilbert polynomial $P(\mathscr{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathscr{F})\frac{m^i}{i!}$, then we have $\alpha_d(\mathscr{F}) = \mathrm{rank}(\mathscr{F}) \cdot H^d$ and $\alpha_{d-1}(\mathscr{F}) = \frac{1}{2}\mathrm{rank}(\mathscr{F})\deg T_X + \deg \mathscr{F}$. We define the reduced Hilbert polynomial is

$$p(\mathscr{F}, m) = \frac{P(\mathscr{F}, m)}{\alpha_d(\mathscr{F})} = \frac{m^d}{d!} + \frac{1}{H^d}\left(\frac{1}{2}\deg \mathscr{F} + \mu_H(\mathscr{F})\right)\frac{m^{d-1}}{(d-1)!} + \text{lower terms.}$$

We define $\mathscr{F}$ is (Gieseker-) H-(semi)stable if $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$, then $p(\mathscr{E}, m) < (\leq)p(\mathscr{F}, m)$.

Remark 2.2.

- *Easy to see that μ_H-stable $\Rightarrow$ H-stable $\Rightarrow$ H-ss $\Rightarrow$ μ_H-ss;*
- *if $\dim X = 1$, then μ_H-(semi)stable iff H-(semi)stable.*

Lemma 2.3. *We have the following (easy but important) statements:*

(i) If $\mathscr{F}, \mathscr{G}$ is H-ss with $p(\mathscr{F}) > p(\mathscr{E})$, then

$$\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0;$$

(ii) if $\mathscr{F}, \mathscr{G}$ is H-stable with $p(\mathscr{F}) = p(\mathscr{E})$, then any $\phi : \mathscr{F} \to \mathscr{G}$ either zero or isomorphism.

Proof. Trivial. □

Definition 2.4. *Fix $\mathscr{E} \in \mathrm{Coh}_{\mathrm{tf}}(X)$.*

*(i) A **Harder-Narasimhan filtration** (HN filtration) of $\mathscr{E}$ is*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ are H-ss and $p(\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})) > p(\mathrm{gr}_{i+1}^{\mathrm{HN}}(\mathscr{E}))$ for all i. We define $p_{\max}(\mathscr{E}) := p(\mathrm{gr}_1^{\mathrm{HN}}(\mathscr{E}))$ and $p_{\min}(\mathscr{E}) := p(\mathrm{gr}_l^{\mathrm{HN}}(\mathscr{E}))$;

*(ii) let $\mathscr{E}$ is H-ss, a **Jordan-Hölder filtration** (JH filtration) of $\mathscr{E}$ is*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ are H-stable and $p(\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})) = p(\mathrm{gr}_j^{\mathrm{JH}}(\mathscr{E}))$ for all i, j. We define $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) := \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$;

*(iii) if $\mathscr{E}$ is H-ss, we call $\mathscr{E}$ is H-**polystable** if $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \mathscr{E}$.*

Theorem 2.5. *Fix $\mathscr{E} \in \mathrm{Coh}_{\mathrm{tf}}(X)$.*

(i) There exists unique HN filtration of $\mathscr{E}$;

(ii) if $\mathscr{E}$ is H-ss, then there exists JH filtration of $\mathscr{E}$ but may not unique. In this case $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$ is unique.

Proof. See [2] Chapter 1. □

Remark 2.6. *All of these are similar for μ_H-(semi)stable except for the uniqueness of $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$, this is right for up to codimension ≥ 2.*

Theorem 2.7. *Let $\mathscr{E}_1, \mathscr{E}_2 \in \mathrm{Coh}_{\mathrm{tf}}(X)$ are μ_H-ss, then so is $\mathscr{E}_1 \otimes \mathscr{E}_2$/torsion.*

Theorem 2.8 (Bogomolov Inequality)**.** *If $\dim X = 2$, $\mathscr{E} \in \mathrm{Coh}_{\mathrm{tf}}(X)$ with μ_H-ss and $r = \mathrm{rank}(\mathscr{E})$, then*

$$\Delta(\mathscr{E}) := 2rc_2(\mathscr{E}) - (r-1)c_1^2(\mathscr{E}) \geq 0.$$

Proof. Consider the following steps:

- **Step 1.** We can assume $\mathscr{E}$ is locally free.
- **Step 2.** We can assume $c_1(\mathscr{E}) = 0$.
- **Step 3.**
- **Step 4.** Show $\chi(X, S^n\mathscr{E}) \leq cn^r$.
- **Step 5.** Finish the proof.

□

3 Moduli of Coherent Sheaves

4 Set Up for Bridgeland Stability

5 Calabi-Yau Category and Bridgeland Stability

6 Moduli of Objects in Derived Categories

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References

[1] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer, 1977.

[2] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.