Varieties of Minimal Rational Tangents on the Fano Varieties

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Contents

Preface

Note that $\mathbb{P}(-)$ is in the sense of Grothendieck and $\mathbf{P}(-)$ is in the geometric sense and $\mathsf{Grass}(s, V)$ is in the sense of geometry.

Chapter 1

Basic Theory of Rational Curves

The main results here we follows the famous book [26].

1.1 Hilbert Schemes and Chow Schemes

1.1.1 Hilbert Schemes, a Basic Introduction

Definition 1.1.1. *Let X be an S-scheme, we define the Hilbert functor $\mathscr{H}ilb_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z.*

Fix a Polynomial P and a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{H}ilb^P_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z with Hilbert Polynomial P.

Theorem 1.1.2 (Grothendieck)**.** *Let S be a noetherian scheme, let $X \to S$ be a projective morphism, and $\mathscr{L}$ a relatively very ample line bundle on X. Then for any polynomial P, the Hilbert functor $\mathscr{H}ilb^P_{X/S}$ is representable by a projective S-scheme $\mathsf{Hilb}^P_{X/S}$. We also have $\mathsf{Hilb}_{X/S} = \coprod_P \mathsf{Hilb}^P_{X/S}$.*

Proof. Note that this notion of projectivity is much general than [15], but is the same when $S = \operatorname{Spec} k$. The proof is to embed it into Grassmannian. The original proof in [13] and we also refer [33], [26] and [10]. $\square$

Remark 1.1.3. *In [4] we can remove the noetherian hypothesis, by instead assuming strong (quasi-)projectivity of $X \to S$. So also [1].*

Example 1.1.1. *Some examples and interesting results:*

(a) We have $\mathsf{Hilb}^1_{X/S} = X/S$.

(b) *Let C be a curve over a field k, then*

$$\mathsf{Hilb}^m_{C/k} \cong S^m C := \underbrace{C \times \cdots \times C}_{m} / \mathfrak{S}_m.$$

Hence if C smooth, so is $\mathsf{Hilb}^m_{C/k}$. See also [10] Theorem 7.2.3(1) and Proposition 7.3.3.

(c) *Let S be a smooth surface over a field k, then $\mathsf{Hilb}^m_{S/k}$ is also smooth of dimension $2m$ and hence $\mathsf{Hilb}^m_{S/k} \to S^m X$ (we will see this later for general settings) is a resolution of singularities. Note that $S^m X$ is smooth if and only if X is smooth and $\dim X = 1$ or $m < 2$. See [10] Theorem 7.2.3(2) and Theorem 7.3.4.*

(d) *Let X be a nonsingular variety. Then $\mathsf{Hilb}^m_{X/k}$ is nonsingular for $m \leq 3$. Moreover, for any nonsingular 3-fold the scheme $\mathsf{Hilb}^4_{X/k}$ is singular. See [10] Remark 7.2.5 and 7.2.6.*

(e) *Let $\mathscr{E}$ be a vector bundle of rank $m+1$ over S and let $P_d(n) = \binom{m+n}{m} - \binom{m+n-d}{m}$, then*

$$\mathsf{Hilb}^{P_d}_{\mathbb{P}(\mathscr{E})/S} \cong \mathbb{P}((\mathrm{Sym}^d \mathscr{E})^\vee).$$

(f) *Let $Z \to S$, we have $\mathsf{Hilb}_{X \times_S Z/Z} \cong \mathsf{Hilb}_{X/S} \times_S Z$.*

(g) Hartshorne's Connectedness Theorem: *for every connected noetherian scheme S, $\mathsf{Hilb}^P_{\mathbb{P}^n_S/S}$ is connected.*

(h) *Let X be a connected variety over k, then $\mathsf{Hilb}^n_{X/k}$ is connected for all $n > 0$.*

(i) Murphy's Law: *It has many singularities, that is, for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q \in \mathsf{Hilb}^P_{\mathbb{P}^n/k}$ of some Hilbert scheme and an isomorphism*

$$\widehat{\mathscr{O}}_{X,p}[[x_1, ..., x_s]] \cong \widehat{\mathscr{O}}_{\mathsf{Hilb}^P_{\mathbb{P}^n/k},q}[[y_1, ..., y_t]].$$

See [38]. In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some n. It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

(j) *In [37] they gave a full classification of the situation where $\mathsf{Hilb}^P_{\mathbb{P}^n/k}$ smooth.*

Definition 1.1.4. *Let $X/S, Y/S$ are S-schemes, then we have a functor $\mathscr{H}om_S(X, Y)$ send S-scheme T into a set of T-morphisms $X \times_S T \to Y \times_S T$.*

For a subscheme $B \subset X$ proper over S and $g : B \to Y$, we have a functor $\mathscr{H}om_S(X, Y; g)$ send S-scheme T into a set of T-morphisms $X \times_S T \to Y \times_S T$ such that $f|_{B \times_S T} = g \times_S \mathrm{id}_T$.

Proposition 1.1.5. *If X/S and Y/S are both projective over S and X is flat over S, then $\mathscr{H}om_S(X,Y)$ represented by an open subscheme $\mathsf{Hom}_S(X,Y) \subset \mathsf{Hilb}_{X\times_S Y/S}$.*

Proof. Any $X \times_S T \to Y \times_S T$ correspond to its graph which is a closed immersion $\Gamma : X \times_S T \to X \times_S Y \times_S T$. As X is flat over S, then $X \times_S T$ is flat over T. Hence we get a morphism $\mathsf{Hom}_S(X,Y) \to \mathsf{Hilb}_{X\times_S Y/S}$. We omit the more details and refer Theorem I.1.10 in [26]. □

Proposition 1.1.6. *If X/S and Y/S are both projective over S and X, B are both flat over S, then $\mathscr{H}om_S(X,Y;g)$ represented by a subscheme $\mathsf{Hom}_S(X,Y;g) \subset \mathrm{Hom}_S(X,Y)$.*

Proof. Consider the restriction map $R : \mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_S(B,Y)$, then $g : B \to Y$ gives a section $G : S \to \mathrm{Hom}_S(B,Y)$. Hence $\mathsf{Hom}_S(X,Y;g) := R^{-1}(G(S)) \subset \mathsf{Hom}_S(X,Y)$ represents $\mathscr{H}om_S(X,Y;g)$. □

Now we state the deformation theory of Hilbert schemes. We only consider the simpler case that all schemes over a field k. For general case we refer Section 1.2 in [26].

Theorem 1.1.7. *Let Y be a projective scheme over a field k and $Z \subset Y$ is a subscheme. Then*

(a) We have

$$T_{[Z]}\mathsf{Hilb}_Y \cong \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

(b) The dimension of every irreducible components of Hilb_Y at $[Z]$ is at least

$$\dim \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) - \dim \mathrm{Ext}^1_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

Proof. See Theorem I.2.8 in [26]. For family case we refer Theorem I.2.15 in [26]. □

Corollary 1.1.8. *Let X, Y are projective varieties over a field k with a morphism $f : X \to Y$. Let Y is smooth over k. Then*

(a) We have

$$T_{[f]}\mathsf{Hom}_k(X,Y) \cong \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

(b) The dimension of every irreducible components of $\mathsf{Hom}_k(X,Y)$ at $[f]$ is at least

$$\dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

Proof. Let $Z \subset X \times_k Y$ be the graph of f, we claim that $\mathscr{I}_Z/\mathscr{I}_Z^2 \cong f^*\Omega^1_Y$. Indeed we have an exact sequence $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega^1_{X\times_k Y}|_Z \to \Omega^1_Z \to 0$. This is split by $\mathscr{O}_Z \cong \mathscr{O}_X \overset{(\mathrm{id}_X,1)}{\to} \mathscr{O}_{X\times_k Y}$. Then we can show the claim. Hence the results follows from Theorem 1.1.7. The family version we refer Theorem I.2.17 in [26]. □

1.1.2 Chow Schemes, a Basic Introduction

Here we only consider the schemes over a field k such that $\mathrm{char}(k) = 0$. The positive characteristic case is very complicated and we refer Section I.4 in [26].

Definition 1.1.9. *Let $g_i : U_i \to W$ be a proper morphism of schemes over W. Assume that W is reduced and U_i is irreducible. By generic flatness there is an open subset $W_i \subset g_i(U_i) \subset W$ such that g_i is flat of relative dimension d over W_i. Let $T = \operatorname{Spec} \Delta$ be the spectrum of a DVR Δ and $h : T \to W$ a morphism such that $h(T_g) \in W_i$ and $h(T_0) = w \in W$. Let $h^*U_i = U_i \times_h T$ and $\mathscr{J} \subset \mathscr{O}_{h^*U_i}$ the ideal of those sections whose support is contained in the special fiber of $h * U_i \to T$. Let $(U_i)'_T := \underline{\operatorname{Spec}}_T \mathscr{O}_{h^*U_i}/\mathscr{J}$ which is flat over T. Then we let $[Z_0]$ be the fundamental cycle of the central fiber of $(U_i)'_T \to T$, and define*

$$\lim_{h \to w} (U_i/U) := [Z_0] \in Z_d(g_i^{-1}(w) \times_{\kappa(w)} T_0)$$

which is called the cycle theoretic fiber *of g_i at w along h.*

Definition 1.1.10. *A* well defined family of d-dimensional proper algebraic cycles over W *is a pair $(g : U \to W)$ satisfying the following properties:*

(a) There is a reduced scheme $\operatorname{supp} U$ with irreducible components U_i such that $U = \sum_i m_i[U_i]$ is an algebraic cycle.

(b) W is a reduced scheme and $g : \operatorname{supp} U \to W$ is a proper morphism.

(c) Let $g_i := g|_{U_i}$. Then every g_i maps onto an irreducible component of W and every fiber of g_i is either empty or has dimension d. In particular there is a dense open subset $W_0 \subset W$ such that every g_i is flat over W_0.

(d) For every $w \in W$ there is a cycle $g^{[-1]}(w) \in Z_d(g^{-1}(w))$ such that for any $h : T \to W$ of spectrum of DVR such that $h(T_0) = w$ and $h(T_g) \in W_0$ we have

$$g^{[-1]}(w) =_{\mathrm{ess}} \sum_i m_i \lim_{h \to w} (U_i/W).$$

That is, both two cycles from a single cycle of $Z_d(g^{-1}(w))$.

Remark 1.1.11. *If W is normal, then (d) can be implied by (a)-(c). See Theorem I.3.17 in [26].*

Definition 1.1.12. *Let X be a scheme over S. A* well defined family of proper algebraic cycles of X/S over W/S *is a pair $(g : U/S \to W/S)$ satisfying the following properties:*

(a) $\operatorname{supp} U$ is a closed subscheme of $X \times_S W$ and g is the natural projection morphism.

(b) *$(g : U \to W)$ is a well defined family of d-dimensional proper algebraic cycles over W for some d.*

Proposition 1.1.13. *Assume that $g : U \to W$ is proper and flat of relative dimension d and W is reduced. Let $\sum_i m_i[U_i]$ be the fundamental cycle of U. Then $g : [U] \to W$ is a well defined family of algebraic cycles over W.*

Proof. See Lemma I.3.14 and Corollary I.3.15 in [26]. □

Definition 1.1.14 (Chow Schemes of Characteristic Zero)**.** *Let X/S and we define a functor $\mathscr{C}how_{X/S}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$.*

Let a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{C}how^{d,d'}_{X/S}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$ which is of dimension d and degree d'.

Theorem 1.1.15. *Let X/S be a scheme, projective over S and $\mathscr{O}(1)$ relatively ample. Then the functor $\mathscr{C}how^{d,d'}_{X/S}$ is representable by a semi-normal and projective S-scheme $\mathsf{Chow}^{d,d'}_{X/S}$. We also have $\mathsf{Chow}_{X/S} = \coprod_{d,d'} \mathsf{Chow}^{d,d'}_{X/S}$.*

Proof. Very complicated, we refer Theorem I.3.21 in [26]. □

Example 1.1.2. *Let X be a semi-normal variety, then $\mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.*

Proposition 1.1.16 (Hilbert-Chow)**.** *Let X, Y be S-schemes.*

(a) *We have a natural morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$. This morphism can be factored by dimensions.*

(b) *If X, Y be projective S-schemes and X/S flat, then we have*

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Chow}_{Y/S}.$$

Proof. For (a), consider $[\mathsf{Univ}^{\mathsf{Hilb}} \times_{\mathsf{Hilb}_{X/S}} \mathsf{Hilb}^{\mathrm{sn}}_{X/S}] \to \mathsf{Hilb}^{\mathrm{sn}}_{X/S}$, then by Proposition 1.1.13 this is a well defined family of algebraic cycles. This gives such morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$.

For (b), by (a) we have

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Hilb}(X \times_S Y/S)^{\mathrm{sn}} \to \mathsf{Chow}_{X \times_S Y/S} \to \mathsf{Chow}_{Y/S}$$

and well done. □

Remark 1.1.17. *Let X be a semi-normal variety, hence we have $(\mathsf{Hilb}^m_{X/k})^{\mathrm{sn}} \to \mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.*

1.1.3 Small Applications to Curves

For more applications we refer Section II.1 in [26]. Here we only need some easy case. We assume over a field k.

Theorem 1.1.18. *Let C be a proper curve and $f : C \to Y$ a morphism to a projective variety Y of dimension n such that Y is smooth along $f(C)$. Then*

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + n\chi(\mathscr{O}_C).$$

And equality holds if $H^1(C, f * T_Y) = 0$, in this case it is smooth at $[f]$.

Proof. By Corollary 1.1.8(b) we have

$$\begin{aligned}\dim_{[f]} \mathsf{Hom}(C, Y) &\geq \dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X) \\ &= h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y) \\ &= \deg f^*T_Y + n\chi(\mathscr{O}_C)\end{aligned}$$

by Riemann-Roch theorem. The final statement follows from Corollary 1.1.8(a). □

Proposition 1.1.19. *Assume that X/S is flat, B/S is flat and finite of degree m and Y/S is smooth of relative dimension n. Then* $\dim \mathsf{Hom}(X, Y; g) \geq \dim \mathsf{Hom}(X, Y) - kn$.

Proof. Let $p : B \to S$ be the projection. By Corollary 1.1.8 we find that $\mathsf{Hom}(B, Y)$ is smooth over S of relative dimension rank kn. Thus $g(S) \subset \mathrm{Hom}(B, Y)$ is locally defined by kn equations. Pulling back these equations by R we obtain local defining equations. □

Lemma 1.1.20. *Let $0 \in T$ be the spectrum of a local ring and let U/T be a flat and proper and V/T be a variety. Let $p : U \to V$ as a T-morphism. If $p_0 : U_0 \to V_0$ is a closed immersion (resp. an isomorphism), then so is p.*

Proof. See Lemma I.1.10.1 and Proposition I.7.4.1.2 in [26]. We omit this. □

Theorem 1.1.21. *Let C be a projective curve over k and Y a smooth variety over k. Let $B \subset C$ be a closed subscheme which is finite over k. Assume that C is smooth along B. Let $g : B \to Y$ be a morphism. Then*

(a) We have

$$T_{[f]}\mathsf{Hom}(C, Y; g) \cong H^0(C, f^*T_Y \otimes \mathscr{I}_B).$$

(b) The dimension of every irreducible component of $\mathsf{Hom}(C, Y; g)$ *at* $[f]$ *is at least*

$$h^0(C, f^*T_Y \otimes \mathscr{I}_B) - h^1(C, f^*T_Y \otimes \mathscr{I}_B).$$

Proof. The original proof we refer [31]. A simple case of family version we refer Theorem II.1.7 in [26]. Here we assume k is algebraically closed. Here $\mathscr{I}_B = \mathscr{O}_C(-s_1 - ... - s_m)$.

Let $X_0 := C \times_k Y$ and let $\gamma_0 : C \cong \Gamma_0 \subset X_0$ be the graph of f. Let $\pi_1 : X_1 := \mathrm{Bl}_{\{s_1\}} X_0 \to X_0$ and Γ_1 be the strict transform of Γ_0. Let $\gamma_1 : C \cong \Gamma_1 \subset X_1$ as C is smooth at s_1. Repeat the process and finally we get $\pi_m : X_m := \mathrm{Bl}_{\{s_m\}} X_{m-1} \to X_{m-1}$ and Γ_m be the strict transform of Γ_{m-1}. Let $\gamma_m : C \cong \Gamma_m \subset X_m$. Then we have $\gamma_0^*(\mathscr{I}_{\Gamma_0}/\mathscr{I}_{\Gamma_0}^2) \cong f^*\Omega_Y^1$ and $\gamma_{i+1}^*(\mathscr{I}_{\Gamma_{i+1}}/\mathscr{I}_{\Gamma_{i+1}}^2) \cong \gamma_i^*(\mathscr{I}_{\Gamma_i}/\mathscr{I}_{\Gamma_i}^2) \otimes \mathscr{O}_C(-s_{i+1})$. Hence we get $\gamma_m^*(\mathscr{I}_{\Gamma_m}/\mathscr{I}_{\Gamma_m}^2) \cong f^*\Omega_Y^1 \otimes \mathscr{I}_B$.

Now we claim that there is an open neighborhood $[\Gamma_m] \in U \subset \mathsf{Hilb}_{X_m}$ such that $\mathsf{Hom}(C, Y; g) \cong U$. Indeed, let $U \subset \mathsf{Hilb}_{X_m}$ be the open set parametrizing those 1-cycles D for which the projection $D \to C$ is an isomorphism. This is open by Lemma 1.1.20.

First, the universal family of U is contained in $\mathsf{Hom}(C, Y; g)(U)$. Conversely consider $[p_0 : C \times R \to Y \times R] \in \mathsf{Hom}(C, Y; g)(R)$. Let its graph is $G_0 \subset X_0 \times R$. As $\{s_1\} \times R \subset G_0$ and $G_0 \to R$ smooth along $\{s_1\} \times R$, we let $G_1 \subset X_1 \times R$ be the strict transform of G_0. Then $G_1 \cong G_0 \cong C \times R$. Repeat the process and finally we get $X_m \times R \supset C \times R \cong G_m \in \mathsf{Hilb}_{X_m}(R)$. Hence this give the isomorphism $\mathsf{Hom}(C, Y; g) \cong U$. Hence by Theorem 1.1.7 and we get the result. □

1.2 Families of Rational Curves

We may assume all schemes over a field k of characteristic zero locally of finite type. Note that there are also have the same results by some small modification in the case of positive characteristic, see Section II.2 in [26].

Proposition 1.2.1. *Let $f : X \to Y$ be a proper morphism of relative dimension one. Assume that if T is the spectrum of a DVR and $h : T \to Y$ a morphism, then every irreducible component of $T \times_Y X$ has dimension two (By Corollary I.3.16 in [26] this is always the case if f is a well defined family of proper algebraic 1-cycles). Then the subset*

$$\{y \in Y : f^{-1}(y) \text{ has geometrically rational components}\} \subset Y$$

is closed in Y.

Proof. See Proposition II.2.2 in [26]. □

Corollary 1.2.2. *Let $g : U \to V$ be a family of proper algebraic 1-cycles of X/S. Let $U' \subset U$ be the set of points $u \in U$ which are contained in a geometrically rational component of $g^{-1}(g(u))$. The image of the natural morphism $U' \to X$ is called the rational locus of g. It is denoted by $\mathsf{RatLocus}(g : U \to V)$.*

Now let $V \to S$ is proper, then $\mathsf{RatLocus}(g : U \to V)$ is proper over S.

Proof. WLOG we let V is irreducible. Let $U = \sum_i a_i U_i$, then we just need to consider every $g_i : U_i \to V$. Consider the generic fiber D_i of g_i which is a irreducible curve, then if D_i rational, then so is whole g_i by Proposition 1.2.1. Hence $\mathsf{RatLocus}(g_i : U_i \to V) = \mathrm{Im}(U_i \to X)$ is proper over S. If D_i is not rational, then there is an open subset $\emptyset \neq W \subset V$ such that the fibers of g_i over Ware irreducible and nonrational. Thus

$$\mathsf{RatLocus}(g_i : U_i \to V) = \mathsf{RatLocus}(g_i : g_i^{-1}(V \backslash W) \to V \backslash W).$$

Hence we can apply Noetherian induction. □

Definition 1.2.3. *Let* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *be a subscheme correspond to the morphisms* $\mathbb{P}^1 \to X$ *birational to its image. By Lemma 1.1.20 since* $\mathbb{P}^1 \to X$ *birational to its image if and only if it is a immersion at its generic point, then* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *is an open subscheme.*

Definition 1.2.4. *Let X/S be a scheme, projective over S.*

(a) Let $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} = \bigcup_i W_i$ *be the decomposition into irreducible subschemes of semi-normalization of* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$. *By Proposition 1.1.16 we have the Hilbert-Chow morphism* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} \to \mathsf{Chow}_{X/S}$. *Let* $V_i' = \overline{\mathrm{Im}(U_i \to \mathsf{Chow}_{X/S})}$. *By Proposition 1.2.1* V_i' *parametrizes 1-cycles with geometrically rational components, and the generic 1-cycle is irreducible. Let* $V_i \subset V_i'$ *be the open subscheme parametrizing irreducible 1-cycles.*

Let $\eta_i \in V_i$ *be the generic points correspond to curves* C_i. *By generic smoothness* C_i *is a smooth rational curve. Let* V_i^{n} *be the normalization of* V_i. *Then we define the family of rational curves on* X *is*

$$\mathsf{RatCurves}^{\mathrm{n}}(X/S) := \coprod_i V_i^{\mathrm{n}}.$$

with a normalization morphism $\mathsf{RatCurves}^{\mathrm{n}}(X/S) \to \mathsf{Chow}_{X/S}$.

If $\mathscr{L}$ *is ample on* X/S, *then we can define* $\mathsf{RatCurves}^{\mathrm{n}}(X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *where* $\mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *is quasi-projective over* S *for any* d. *We define its universal rational curve is*

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) := \left(\mathsf{RatCurves}^{\mathrm{n}}(X/S) \times_{\mathsf{Chow}_{X/S}} \mathsf{Univ}^{\mathsf{Chow}}_{X/S}\right)^{\mathrm{n}}$$

be the normalization.

(b) Fix a section $f : S \to X$. *Similar as (a) we can define* $\mathsf{RatCurves}^{\mathrm{n}}(f, X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(f, X/S)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(f, X/S)$. *This is called family of rational curves passing through* $\mathrm{Im}(f)$.

In particular if $S = \operatorname{Spec} k$ *where* k *is a field and* $f : (\operatorname{Spec} k) = x \in X$, *then we will use the notation* $\mathsf{RatCurves}^{\mathrm{n}}(x, X) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(x, X)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(x, X)$.

Theorem 1.2.5. *(a) Let $f : X \to Y$ be a proper and surjective morphism between irreducible and normal schemes. Assume that the dimension of every fiber is one (hence f is a well defined family of proper* 1*-cycles by Remark 1.1.11). Assume that for every $y \in Y$ the cycle theoretic fiber $f^{[-1]}(y)$ is an irreducible and reduced rational curve, then f is a $\mathbb{P}^1$-bundle.*

(b) In the case of the definition, the universal morphisms

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) \to \mathsf{RatCurves}^{\mathrm{n}}(X/S) \text{ and } \mathsf{Univ}^{\mathsf{rc}}(x, X) \to \mathsf{RatCurves}^{\mathrm{n}}(x, X)$$

are $\mathbb{P}^1$-bundles.

Proof. (b) follows directly from (a), so we just need to prove (a).

One can show that f is smooth at the generic point of every fiber (see Theorem I.6.5 in [26]). For $y \in Y$ pick three different points $x_1, x_2, x_3 \in f^{-1}(y)$ such that f is smooth at x_i. Let $S_i \subset X$ be a Cartier divisor which intersects $f^{[-1]}(y)$ transversally at x_i (there may be other intersection points). Hence $S_i \to Y$ is étale at x_i. Let

$$Z = S_1 \times_Y S_2 \times_Y S_3, \quad z = (x_1, x_2, x_3) \in Z \text{ and } X_Z = X \times_Y Z.$$

So $Z \to Y$ is étale at z, thus X_Z is normal along $f_Z^{-1}(z)$ and f is smooth above y iff f_Z is smooth above z by some commutative algebra. Furthermore, f_Z has three sections $s_i : Z \to X_Z$ corresponding to the S_i. By shrinking Z we may assume that these sections are disjoint.

In $\mathbb{P}^1_Z \to Z$ we have three disjoint sections $p_i : Z \to \mathbb{P}^1_Z$ corresponding to $\{0, 1, \infty\}$. Our aim is the construct an isomorphism $q : \mathbb{P}^1_Z \cong X_Z$ such that $q \circ p_i = s_i$. Let $h : \mathbb{P}^1_Z \times_Z X_Z \to Z$ be the projection. In order to construct the graph of q let $\Gamma \subset \mathsf{Chow}_{\mathbb{P}^1_Z \times_Z X_Z/Z}$ be the closed subvariety parametrizing 1-cycles D with the following properties:

(1) $\deg \mathscr{O}_{\mathbb{P}^1}(1)|_D = 1$;

(2) $\deg \mathscr{O}(s_1(Z))|_D = 1$;

(3) $(p_i(h(D)), s_i(h(D))) \in D$ for $i = 1, 2, 3$.

Let $\mathsf{Univ}^\Gamma \to \Gamma$ be the universal family. We claim that the natural projections $\pi_1 : \mathsf{Univ}^\Gamma \to \mathbb{P}^1_Z$ and $\pi_2 : \mathsf{Univ}^\Gamma \to X_Z$ are isomorphisms.

For any $t \in Z$ consider $h^{-1}(t)$. By construction $(h^{-1}(t))_{\mathrm{red}} \cong \mathbb{P}^1_{\kappa(t)} \times C_t$ where C_t is an irreducible geometrically rational curve, smooth for general t. As D gives a 1-cycle on $(h^{-1}(t))_{\mathrm{red}}$ which has bidegree $(1, 1)$, thus D is either the graph of a birational morphism $q_t : \mathbb{P}^1_{\kappa(t)} \to C_t$ or the union of a vertical and of a horizontal section. In the latter case it can not contain all three points $(p_i(t), s_i(t))$. Hence D is the graph of the unique birational morphism q_t such that $q_t(p_i(t)) = s_i(t)$ for $i = 1, 2, 3$. Thus π_1, π_2 are both one-to-one. If C_t is smooth, then q_t is defined over $\kappa(t)$, thus π_1, π_2 are isomorphisms over the generic point of Z. Since X_Z and $\mathbb{P}^1_Z$ are normal, this implies that π_1, π_2 are isomorphisms. Well done. $\square$

Remark 1.2.6. *In positive characteristic, (a) is right if we assume generic-smoothness.*

Proposition 1.2.7. *Notation as above definitions, then*

(a) Let $m = \min\{d : \mathsf{RatCurves}^n_d(X/S) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(X/S)$ is proper over S for $k < 2m$.

(b) Let S be a field and let $m(x) = \min\{d : \mathsf{RatCurves}^n_d(x, X) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(x, X)$ is proper for $k < m + m(x)$.

Proof. (b) follows from the same proof of (a). For (a), as $\mathsf{Chow}^{1,k}_{X/S}$ is proper over S, we just need to show that $\bigcup_i V_i \subset \mathsf{Chow}^{1,k}_{X/S}$ is closed where $\mathsf{RatCurves}^n_k(X/S) = \bigcup_i V_i^{\mathrm{n}} \to \bigcup_i V_i$ is finite. Let $\sum_i a_i D_i \in \overline{\mathsf{RatCurves}^n_k(X/S)}$, then every D_i is rational by Proposition 1.2.1 and $\sum_i a_i \deg D_i = k < 2m$. By assumption $\deg D_i \geq m$, then $\sum_i a_i D_i$ is an irreducible and reduced rational curve. Hence $\mathsf{RatCurves}^n_k(X/S)$ closed. $\square$

Theorem 1.2.8. *Let $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}$ be the normalization of $\mathsf{Hom}_{\mathsf{bir}}$, then we have the following important results:*

(a) Let X/S projective scheme over S, then there is a natural commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{U} & \mathsf{Univ}^{\mathsf{rc}}(X/S) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{u} & \mathsf{RatCurves}^{\mathrm{n}}(X/S) \end{array}$$

where U and u are smooth of relative dimension 3 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles)

(b) Let X projective scheme over k with a k-point $x \in X(k)$, then there is a natural commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{U} & \mathsf{Univ}^{\mathsf{rc}}(x, X) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{u} & \mathsf{RatCurves}^{\mathrm{n}}(x, X) \end{array}$$

where U and u are smooth of relative dimension 2 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundles)

Proof. These are easy but boring since we consider the characteristic zero. See [26] Theorem II.2.15 and II.2.16. $\square$

Corollary 1.2.9. *Let X projective scheme over k with a k-point $x \in X(k)$, then*

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong H^0(\mathbb{P}^1, N_C), \quad T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(x, X) \cong H^0(\mathbb{P}^1, N_C \otimes \mathfrak{m}_x)$$

for general point $[C]$ where $f : \mathbb{P}^1 \to C \subset X$ is birational and $N_C = f^*T_X/T_{\mathbb{P}^1}$.

Proof. By Theorem 1.2.8, canonical morphism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) \to \mathsf{RatCurves}^{\mathrm{n}}(X/k)$ is a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle which is smooth. Hence we have

$$0 \to u^*\Omega^1_{\mathsf{RatCurves}^{\mathrm{n}}(X/k)} \to \Omega^1_{\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)} \to \Omega^1_u \to 0.$$

As $[C]$ general, we have $T_{[f]}\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) = T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)$. Hence

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)/\mathrm{Aut}(\mathbb{P}^1) \cong H^0(\mathbb{P}^1, N_C)$$

by trivial reason. Similar for $\mathsf{RatCurves}^{\mathrm{n}}(x, X)$. □

1.3 Free and Minimal Rational Curves

We will assume all scheme over a algebraically closed field k of characteristic zero.

1.3.1 Free Rational Curves

Definition 1.3.1. *Let C be a proper curve, X a smooth variety and $f : C \to X$ a morphism. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. We call f is called* free over f *if f is nonconstant and $H^1(C, f^*T_X \otimes \mathscr{I}_B) = 0$ and $f^*T_X \otimes \mathscr{I}_B$ is generated by global sections. Therefore we can define $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ parameterizes the free rational curves.*

Proposition 1.3.2. *Being free is an open. Hence $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ is open.*

Proof. Trivial by definition. □

Theorem 1.3.3. *Let C be a proper curve and X a smooth variety. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. Let $F : C \times \mathsf{Hom}(C, X; g) \to X$ be the universal morphism. Then $T_{\kappa(p,[f]), C\times\mathsf{Hom}(C,X;g)} = T_{\kappa(p),C} \oplus H^0(C, f^*T_X \otimes \mathscr{I}_B)$ if $p \notin B$ Consider the differential $df(s) : T_{\kappa(s),C} \to T_{\kappa(f(s)),X}$ and evaluation map*

$$\phi(p, f) : H^0(C, f^*T_X \otimes \mathscr{I}_B) \to f^*T_X \otimes \kappa(p),$$

then $dF(p, [f]) = df(p) + \phi(p, f)$. Furthermore If $\phi(p, f)$ is surjective, then F is smooth at $(p, [f])$. The converse also holds if $H^0(T_C \otimes \mathscr{I}_B) \to T_{\kappa(p),C}$ is surjective.

Proof. Trivial by definitions. □

Corollary 1.3.4. *If C is smooth and $f : C \to X$ is free over g, then $F : C \times \mathsf{Hom}(C, X; g) \to X$ is smooth along $(C\backslash B) \times [f]$. In particular $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth.*

Proposition 1.3.5. *Assume that $f : \mathbb{P}^1 \to X$, $g = f|_B$, $\mathrm{length} B \leq 2$ and write $f^*T_X \otimes \mathscr{I}_B = \sum_i \mathscr{O}(a_i)$. Then $\sharp\{i : a_i \geq 0\} = \mathrm{rank} dF(p, [f])$ for all $p \in \mathbb{P}^1\backslash B$.*

In particular, if

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is smooth at $(p, [f])$ for some $p \in \mathbb{P}^1$, then f is free over g.

Proof. Note that $\mathrm{length} B \leq 2$ implies $H^0(T_{\mathbb{P}^1} \otimes \mathscr{I}_B) \to T_{\kappa(p),\mathbb{P}^1}$ is surjective for all $p \in \mathbb{P}^1\backslash B$. Then these are trivial by arguments in Theorem 1.3.3. □

Theorem 1.3.6 (Kollár-Miyaoka-Mori, 1992)**.** *Let X be a smooth projective variety over k. Let $B \subset \mathbb{P}^1_k$ be a closed subscheme with $\mathrm{length} B \leq 2$ and $g : B \to X$. There are countably many subvarieties $V_i = V_i(B, g) \subset X$ such that if $f : \mathbb{P}^1 \to X$ is a nonconstant morphism such that $f|_B = g$ and $\mathrm{Im}(f) \nsubseteq \bigcup_i V_i$, then f is free over B.*

Proof. Let Z_i be the irreducible components of $\mathsf{Hom}(\mathbb{P}^1, X; g)$ with universal morphisms $F_i : \mathbb{P}^1 \times Z_i \to X$. Let $V_i = \overline{\mathrm{Im}(F_i)}$ if F_i is not dominant, and $V_i = X\backslash U_{F_i}$ if F_i is dominant, where $U_{F_i} \subset X$ is an open and dense subset such that $F_{i,\mathrm{red}} : \mathbb{P}^1 \times Z_{i,\mathrm{red}} \to X$ is smooth over U_{F_i} (this is where we use the char $= 0$ assumption). Then the result is trivial. □

Theorem 1.3.7. *Let X be a smooth proper variety over k, then the following statements equivalent.*

(1) X is uniruled.

(2) Generic rational curves of X are free.

(3) X has a free rational curve.

Proof. If X is uniruled then since the morphism

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is dominant, it is generic smooth. Hence by Proposition 1.3.5 the generic rational curves of X are free.

If the generic rational curves of X are free, then X has a free rational curve.

If X has a free rational curve, then the morphism $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth by Corollary 1.3.4. Hence it has densed image. Hence X is uniruled. □

Remark 1.3.8. *More properties of uniruled varieties we refer Section IV.1 in [26].*

1.3.2 Minimal Rational Curves

Definition 1.3.9. *Let X be a smooth projective variety over k of dimension n.*

(a) A rational curve $f : \mathbb{P}^1 \to X$ is called standard *(or* unbendable*) if*

$$f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-1-p}$$

where $p + 2 = -\deg f^*K_X$.

(b) Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a minimal free rational curve *if it is a free rational curve such that $-\deg f^*K_X$ is minimal.*

(c) Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a minimal rational curve *if it is a deformation of the minimal free rational curves. An irreducible component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}(X)$ is called a* minimal rational component *if it contains a rational curve of minimal degree.*

Remark 1.3.10. *For any non-constant $f : \mathbb{P}^1 \to X$, it can be factored by $f : \mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} X$ where h is birational to its image, then it is a immersion at generic points. Hence $T_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{R}^1}(2) \subset h^*T_X$. Hence $\mathscr{O}_{\mathbb{P}^1}(2\deg g) \subset f^*T_X$. So if we let $f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathscr{O}_{\mathbb{P}^1}(a_n)$ with $a_1 \geq \cdots \geq a_n$, then $a_1 \geq 2$.*

Proposition 1.3.11. *Let X be a smooth proper variety over k.*

(a) If X has a free rational curve, then generic free rational curves of X are standard.

(b) If X is Fano and $x \in X$ is a general point, let minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}(X)$ and the corresponding component $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(x, X)$ be of minimal degree $p + 2$. Then $\mathcal{K}_x$ is a union of smooth varieties of dimension p and the general points are minimal standard.

Proof. For (a), let that free rational curve is g, pick an irreducible component $V \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ containing $[g]$. Then by Theorem 1.3.7 V is dominated to X. Then by Theorem IV.2.4 and Corollary IV.2.9 in [26] there is a $W \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ such that dominated to X and general points in W is standard.

For (b), WLOG we let $\mathcal{K}_x$ irreducible and let $V \subset \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ be the irreducible component correspond to $\mathcal{K}_x$. Now since x is general, by Theorem 1.3.6 any members of V and hence $\mathcal{K}_x$ are free. Hence for any $[f] \in V$ we have $H^1(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = 0$. Then $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ is smooth at $[f]$ in this case. Hence by Theorem 1.1.21 V is also smooth at $[f]$ and of dimension $H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = p + 2$. Hence by Theorem 1.2.8(b) the morephism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) \to \mathsf{RatCurves}^{\mathrm{n}}(x, X)$ is smooth and is an $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundle, hence so is $V \to \mathcal{K}_x$. So $\mathcal{K}_x$ is smooth variety of dimension p. $\square$

1.4 Bend and Break

Bend and Break is a classical method aiming to find the rational curves over the projective varieties which is first observed by S. Mori in [31]. Here we will give the main results proved in [26]. See also the first chapter in [29] for a brief introduction. Here we assume all schemes over a infinity field k.

1.4.1 Main Results of Bend and Break

Definition 1.4.1. *Let S be a proper surface and $B \subset S$ a proper curve. We say that B is* contractible in *S if there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional.*

Proposition 1.4.2 (Rigidity Lemma)**.** *Let $f : X \to Y$ be a proper morphism such that $f_*\mathscr{O}_X = \mathscr{O}_Y$. Let $g : X \to Z$ be a morphism. Assume that for some $y \in Y$ there is a factorization*

$$
\begin{tikzcd}
 & & Z \\
X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(y) \arrow[l, hook'] \arrow[ur, "g|_{f^{-1}(y)}"] \arrow[d, "f_y"'] & \\
Y & \{y\} \arrow[l, hook'] \arrow[uur, "h_y"'] &
\end{tikzcd}
$$

Then there is an open neighborhood $y \in U \subset Y$ and a factorization

$$
\begin{tikzcd}
 & & Z \\
X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(U) \arrow[l, hook'] \arrow[ur, "g|_{f^{-1}(U)}"] \arrow[d, "f_U"'] & \\
Y & U \arrow[l, hook'] \arrow[uur, "h_U"'] &
\end{tikzcd}
$$

Proof. Let $\Gamma \subset Y \times Z$ be the image of (f, g). Then $p : \Gamma \to Y$ is proper and $p^{-1}(y) = (y, h_y(y))$ is finite over y. Thus there is an open neighborhood $y \in U \subset Y$ such that $p^{-1}(U) \to U$ is finite. Since

$$f_*\mathscr{O}_{f^{-1}(U)} \supset p_*\mathscr{O}_{p^{-1}(U)} \supset \mathscr{O}_U \supset f_*\mathscr{O}_{f^{-1}(U)}$$

which shows that $p^{-1}(U) \to U$ is an isomorphism. $\square$

Corollary 1.4.3. *Let S be a proper surface and $B \subset S$ a contractible curve. Then $B \cdot B < 0$.*

In particular, let D be an irreducible and proper curve and C an arbitrary curve. Let $B_c = B \times \{c\} \subset B \times C$ where $c \in C$ is arbitrary. Then B_c is not contractible in $B \times C$.

Proof. Since $B \subset S$ a contractible, there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional. We prove this only for S smooth and S' projective. The general case works the same once the definition of intersection numbers is established in general.

Since S' projective, then we can find a finite morphism $f : S' \to \mathbb{P}^2$ since k is infinity. Let $\mathscr{O}(H) = f^*\mathscr{O}(1)$ which is ample and $H \cdot H > 0$ and $H \cdot B = 0$. By Hodge index theorem we have $B \cdot B < 0$.

For the final statement, note that $B_c \cdot B_c = 0$ hence B_c is not contractible. □

Theorem 1.4.4 (Fundamental Bend and Break, Mori-Miyaoka 1979-1986)**.** *Let B be a smooth proper and irreducible curve over k and S an irreducible, proper and normal surface. Let $p : S \to B$ be a morphism. Assume that there is an open subset $B^0 \subset B$, a smooth projective curve C and an isomorphism*

$$f : [C \times B^0 \xrightarrow{\pi} B^0] \cong [p^{-1}(B^0) \xrightarrow{p} B^0].$$

We call a section $s : B \to S$ is called flat if $s(B^0) = \{c\} \times B^0$ under the above isomorphism.

(a) If there is a contractible flat section $s_1 : B \to S$, then for some $b \in B\backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve intersecting $s_1(B)$.

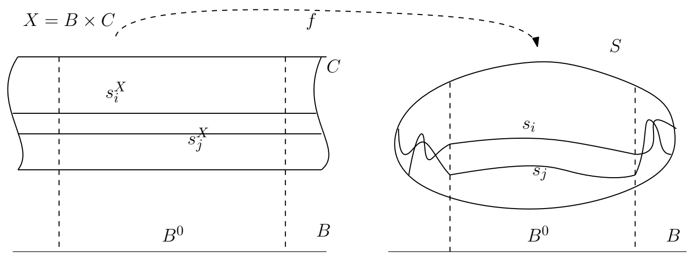
(b) If k algebraically closed, $g(C) = 0$ and there are two contmctible sections $s_1, s_2 : B \to S$, then for some $b \in B\backslash B^0$ the fiber $p^{-1}(b)$ is either reducible or nonreduced.

(c) Let L be a nef $\mathbb{R}$-Cartier divisor on S. If there are $k \geq 1$ contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B\backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $L \cdot D \leq \frac{2}{k} L \cdot C$ where C be the general fiber of p.

(d) Let L be a nef $\mathbb{R}$-Cartier divisor on S with $L^2 > 0$. If there are k contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B\backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $0 < L \cdot D < \frac{2}{k} L \cdot C$ where C be the general fiber of p.

Proof. Let $X := C \times B$ and $\Gamma \subset X \times_B S$ be the closure of the graph of f. Consider projections p_X, p_S and every flat section s_i induces a flat section $s_i^X : B \to X$:

By Corollary 1.4.3 the rational map $f : X \dashrightarrow S$ is not defined some where along $s_i^X(B)$ if s_i contractible. Here we only prove (a) and (b). Actually (c) and (d) including the same idea with complicated computation and we refer Theorem II.5.4 in [26].



For (a), since $s_1 : B \to S$ is a contractible flat section, then $f : X \dashrightarrow S$ is not defined some where along $s_1^X(B)$. So we have a exceptional curve $D' \subset \Gamma$ of p_X. One can show that D' is rational, then take $D = p_S(D')$ and we get (a).

For (b), we assume that every fibres of p are integral, then $h^1(\mathscr{O}_{p^{-1}(b)}) = 1 - \chi(\mathscr{O}_{p^{-1}(b)})$ since k is algebraically closed. Then it is independent of $b \in B$ and every fiber of p is isomorphic to $\mathbb{P}^1$. Since p has sections, then S is a minimal ruled surface over B. Now the matrix of intersection form of $s_1(B), s_2(B)$ and $C \times \{b\}$ is $\mathbf{M} = \begin{pmatrix} -a_1 & c & 1 \\ c & -a_2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ where $-a_i = s_i(B)^2 < 0$ by Corollary 1.4.3 and $c = s_1(B) \cdot s_2(B) \geq 0$. Hence $\det \mathbf{M} = 2c + a_1 a + 2 > 0$ which is impossible since $\dim N_1(S) = 2$ since $N_1(S)$ generated by $s_1(B)$ and $C \times \{b\}$. $\square$

Corollary 1.4.5. *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $p_1, ..., p_k \in C$ be k distinct points and $g : \{p_1, ..., p_k\} \to X$ a morphism. Assume that there is a smooth, irreducible, proper curve B, an open set $B^0 \subset B$ and a morphism*

$$[h^0 : C \times B^0 \to X \times B^0] \in \mathsf{Hom}(C, X; g)(B^0)$$

such that $h^0(C \times \{b\})$ and $p_X \circ h^0(\{c\} \times B^0)$ are one dimensional for some $b \in B^0$ and $c \in C$.

Then there is a unique normal compactification $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$. Let $p : S \to B$.

(a) If $k \geq 1$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D which passes through $g(p_1)$.

(b) If $C \cong \mathbb{P}^1$, $\dim \operatorname{Im}(p_X \circ h^0) = 2$ and $k \geq 2$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ is either reducible or nonreduced.

(c) Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 \leq L \cdot D \leq \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.

(d) *Let L be a nef $\mathbb{R}$-Cartier divisor on X with $h^*L^2 > 0$ and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 < L \cdot D < \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.*

Proof. If $h^0(C \times \{b\})$ is a point for some $b \in B^0$, then by rigidity lemma $h^0(C \times \{b\})$ is a point for any $b \in B^0$, a contradiction. Thus h^0 is finite on every fiber of $C \times B^0 \to B^0$, hence the natural morphism h^0 is quasifinite. $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$.

If $\mathrm{Im}(p_X \circ h^0)$ is of dimension one, this is not hard to see. If $\mathrm{Im}(p_X \circ h^0)$ is of dimension two, then any p_i determines a contractible flat section of S given by $s_i : B^0 \to \{p_i\} \times B^0$. Then this follows from Theorem 1.4.4. □

Theorem 1.4.6 (Bend and Break)**.** *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $f : C \to X$ be a nonconstant morphism.*

(a) *If $\dim_{[f]} \mathsf{Hom}(C, X) \geq \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \mathrm{supp}(\sum_i a_i D_i)$ and*

$$f_*[C] \sim_{\mathrm{alg}} (f_x)_*[C] + \sum_i a_i [D_i].$$

(b) *If $g(C) = 0$ and $\dim_{[f]} \mathsf{Hom}(C, X) \geq 2 \dim X + 2$ (holds if $-K_X \cdot C \geq n + 2$), then for every $x_1, x_2 \in f(C)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x_1, x_2 \in \mathrm{supp}(\sum_i a_i D_i)$ and*

$$f_*[C] \sim_{\mathrm{alg}} \sum_i a_i [D_i], \quad \sum_i a_i \geq 2.$$

(c) *Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. If $\dim_{[f]} \mathsf{Hom}(C, X) \geq k \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ ($a_1 > 0$) whose irreducible components are rational curves such that $x \in D_1$ and*

$$f_*[C] \sim_{\mathrm{alg}} (f_x)_*[C] + \sum_i a_i [D_i], \quad L \cdot D_1 \leq \frac{2}{k} L \cdot f_* C.$$

Proof. Choose $\{p_1, ..., p_k\} \subset C$ with $g = f|_{\{p_1, ..., p_k\}}$, then by Proposition 1.1.19 we have

$$\dim_{[f]} \mathsf{Hom}(C, X; g) \geq \dim_{[f]} \mathsf{Hom}(C, X) - k \dim X.$$

For (a), we assume $k = 1$ and $f(p_1) = x$ then $\dim_{[f]} \mathsf{Hom}(C, X; g) \geq 1$. Let B^0 be the normalization of an irreducible curve in $\mathsf{Hom}(C, X; g)$ containing $[f]$ and $h^0 : C \times B^0 \to$

$X \times B^0$ the natural cycle morphism. By Corollary 1.4.5 we have compactifications B and S. Resolve the indeterminacies of $C \times B \dashrightarrow S$ we get

$$\begin{tikzcd} C \times B \arrow[dr, "q"'] & Y \arrow[l, "\rho_X"'] \arrow[r, "\rho_S"] & S \arrow[dl, "p"] \arrow[r, "h"] & X \times B \\ & B & & \end{tikzcd}$$

Pick $b \in B \backslash B^0$ as before we get $(p \circ \rho_S)^{-1}(b) = (q \circ \rho_X)^{-1}(b) = [C_0] + \sum_j e_j[E_j]$ where $C_0 \cong C$ and E_j rational as the exceptional curves of ρ_X. Set $f_x = (h \circ \rho_S)|_{C_0}$ and $\sum_i a_i D_i = (h \circ \rho_S)_*(\sum_j e_j[E_j])$ and well done.

The proof of (b) is similar as (a) using Corollary 1.4.5(b).

For (c), as before we obtain $D = D_1$ which satisfies all the requirements except that we only know that $D \cap \{f(p_1), ..., f(p_k)\} \neq \emptyset$. By letting the points p_i vary, we conclude that (c) holds except possibly for $k-1$ points of $f(C)$.

Let $W \subset \mathsf{Chow}^1(X)$ be the connected component of $f_*[C]$. Let $V \subset W$ be the set of those points such that the corresponding cycle Z has the form $Z \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i[D_i]$ where the D_i are rational. By Proposition 1.2.1 V is closed in W and hence proper. By Corollary 1.2.2 $\mathsf{RatLocus}(V) \subset X$ is closed. Thus $\mathsf{RatLocus}(V) \cap C$ is a closed subset whose complement has at most $k-1$ points. Therefore $C \subset \mathsf{RatLocus}(V)$ and this completes the proof. □

Theorem 1.4.7 (Smooth Bend and Break, Mori 1979-1982). *Let X be a smooth projective variety.*

(a) Let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism. Then for every $x \in f(\mathbb{P}^1)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and

$$f_*[C] \sim_{\text{alg}} \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

(b) Let C be a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Assume that $\deg_C f^*(-K_X) > g(C) \dim X$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and $\deg_C f_x^*(-K_X) \leq g(C) \dim X$ and

$$f_*[C] \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

Proof. By using Theorem 1.4.6(b) to our (a) and 1.4.6(a) to our (b) and induction on $\deg f^*H$ for some fixed ample divisor H on X, we can get the results. □

1.4.2 Connection of Zero and Positive Characteristics

When we want to find the rational curves on variety X, we need to use the bend and break as Theorem 1.4.6(c). For any $f : C \to X$ passing $x \in X$ we need to make sure that $\dim_{[f]} \mathsf{Hom}(C, X) \geq k \dim X + 1$ for some k. Now by Theorem 1.1.18 we have

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + \dim X \chi(\mathscr{O}_C) = -C \cdot K_Y + \dim X - \dim X g(C).$$

If $-K_X \cdot C > 0$, to make sure the latter number larger, we need to find $C' \to C$ such that $-K_X \cdot C'$ larger but $g(C)$ do not change.

For $g(C) = 0$ we can use the large degree map $\mathbb{P}^1 \to \mathbb{P}^1$; for $g(C) = 1$ we use the $\times n$ morphism. But if $g(C) \geq 2$ we do not have such things. Now that in char $= p$ case we have Frobenius map which satisfies this condition. So we need to make char $= 0$ into char $= p$ case and come back to char $= 0$. This is the magic method due to Mori.

Assume that we are given finitely many schemes of finite type X_i, coherent sheaves $\mathscr{F}_i$ and maps g_i defined over a field k. All of these can be described by a finite number of equations (the schemes are given by affine charts and patching functions, the sheaves by finitely presented modules over the affine charts and patchings and the maps are described by their graphs which are schemes themselves). All these equations involve only finitely many elements a_j of the field k.

Let $\mathbb{F} \subset k$ be a subring which denote $\mathbb{F}_p$ if $\operatorname{char}(k) = p$ and $\mathbb{Z}$ if $\operatorname{char}(k) = 0$. Let $R := \mathbb{F}[a_j]$ is a finite type $\mathbb{F}$-algebra.

Lemma 1.4.8. *Let R be a finitely generated ring over $\mathbb{F}$. Then*

(a) The residue field $R/\mathfrak{m}$ of any maximal ideal $\mathfrak{m} \subset R$ is finite.

(b) The maximal ideals are dense in $\operatorname{Spec} R$.

Proof. (a) is trivial and (b) follows from both cases are Jacobson rings. □

Aftering choose a_j and then R, we may consider X_i, $\mathscr{F}_i$ and g_i defined over $\operatorname{Spec} R$ which we denote them as $X_i^R, \mathscr{F}_i^R$and g_i^R. Hence after base change to $\operatorname{Spec} k$ we again have $X_i, \mathscr{F}_i, g_i$. Hence we constructed data $\{X_i^R, \mathscr{F}_i^R, g_i^R\}$ over $\operatorname{Spec} R$ such that the fibers over $\operatorname{Spec} k$ are the original data $\{X_i, \mathscr{F}_i, g_i\}$. Similarly for maximal ideal $\mathfrak{m} \subset R$ we have data $\{X_i^{\mathfrak{m}}, \mathscr{F}_i^{\mathfrak{m}}, g_i^{\mathfrak{m}}\}$ over $\operatorname{Spec} R/\mathfrak{m}$ which is positive characteristic by the previous Lemma (a).

Definition 1.4.9. *Let (P) be a property of schemes (morphisms etc.) in algebraic geometry. We say that (P) is of* finite type *if:*

Let K/k be a field extension and X_k a k-scheme. Then (P) holds for X_K iff there is a finitely generated subextension $K/F/k$ such that (P) holds for X_L for every L/F.

Remark 1.4.10. *A typical property that is not of finite type is: X_K has only finitely many K-points.*

Theorem 1.4.11 (Meta)**.** *Let $(P_1) \Rightarrow (P_2)$ be a statement in algebraic geometry that we want to prove. Assume the following four conditions:*

(1) (P_1) and (P_2) are of finite type.

(2) If (P_1) holds for the generic fiber of a morphism $X \to Y$, then it holds for every fiber over a nonempty open set.

(3) If (P_2) holds for every fiber of a morphism $X \to Y$ over a (not necessarily open) dense set, then it holds for the generic fiber.

(4) $(P_1) \Rightarrow (P_2)$ holds in positive characteristic.

Then $(P_1) \Rightarrow (P_2)$ always holds.

We may not use this meta-theorem and we will show how to use the propcess before the theorem, that is, a proof of the special (but nice and classical) case of the theorem in the next section.

1.4.3 Applications of General Varieties and Fano Varieties

We assume that all varieties over an algebraically closed field k.

Theorem 1.4.12 (Kollár-Miyaoka-Mori, 1979-1982-1986-1991)**.** *Let X be a projective variety over k, let C a smooth, projective and irreducible curve, $f : C \to X$ a morphism and M any nef $\mathbb{R}$-divisor. Assume that X is smooth along $f(C)$ and $-K_X \cdot C > 0$.*

Then for every $x \in f(C)$ there is a rational curve $L_x \subset X$ containing x such that

$$M \cdot L_x \leq 2 \dim X \frac{M \cdot C}{-K_X \cdot C}.$$

Proof. Fix the condition in the theorem and consider the following proposition:

(P) M any ample $\mathbb{R}$-divisor and $\varepsilon > 0$ there is a rational curve $L_{x,\varepsilon} \subset X$ containing x such that
$$M \cdot L_{x,\varepsilon} \leq (2 \dim X + \varepsilon) \frac{M \cdot C}{-K_X \cdot C}.$$

Now we prove this theorem with several steps:

▶ **Step 1.** Prove the proposition (P) for M is ample divisor and char $= p > 0$.

Consider the Frobenius $F^m : C^m \to C$ of degree p^m and consider $f^m : C^m \to X$, then $-K_X \cdot C^m = p^m(-K_X \cdot C)$. Hence by Theorem 1.1.18 we have

$$\dim_{[f^m]} \mathsf{Hom}(C^m, X) \geq p^m(-K_X \cdot C) + \dim X \chi(\mathscr{O}_C)$$

since X is smooth along $f(C)$. Then for $m \gg 0$ we have $\dim_{[f^m]} \mathsf{Hom}(C^m, X) \geq p^m \frac{-K_X \cdot C}{\dim X + \varepsilon/2} \dim X + 2$. By Theorem 1.4.6(c) and we get the claim.

▶ **Step 2.** Prove the proposition (P) for char $= 0$.

We just need to show the case when M is ample divisor since $\mathbb{R}$-divisor can be approximated by $\mathbb{Q}$-divisors.

Let $f(p) = x$ and we construct R as before such that $p \subset C \xrightarrow{f} X$ and M over $\operatorname{Spec} R$. Hence we have $p^R, x^R, C^R, f^R, X^R, M^R$. By shrinking $\operatorname{Spec} R$ we may assume $C^R \to \operatorname{Spec} R$ is smooth, $X^R \to \operatorname{Spec} R$ is smooth along $f^R(C^R)$ and M^R is locally free (since $K(R)$ is of char $= 0$).

Let $W_\varepsilon \subset \mathsf{Chow}^1(X_R/\operatorname{Spec} R)$ be the subvariety parametrizing those 1-cycles $Z = \sum_i a_i D_i$ which satisfies that every D_i is rational and $Z \cdot M \leq (2\dim X + \varepsilon)\frac{M\cdot C}{-K_X\cdot C}$ and $\operatorname{supp}(Z) \cap f^R(X^R) \neq \emptyset$. Consider $\pi : W_\varepsilon \to \operatorname{Spec} R$. We claim that π is surjective.

Indeed, we know that π is proper by Theorem 1.1.15 and Proposition 1.2.1. Since the closed points dense in $\operatorname{Spec} R$, we just need to show that $\pi(W_\varepsilon)$ contains all closed points of $\operatorname{Spec} R$. Pick a maximal ideal $\mathfrak{m} \subset R$ and $\{p^{\mathfrak{m}}, x^{\mathfrak{m}}, C^{\mathfrak{m}}, f^{\mathfrak{m}}, X^{\mathfrak{m}}, M^{\mathfrak{m}}\}$ as before over $\operatorname{Spec} R/\mathfrak{m}$ of positive characteristic. Hence by Step 1 we have rational curve $L_{x^{\mathfrak{m}},\varepsilon}$ such that $[L_{x^{\mathfrak{m}},\varepsilon}] \in W_\varepsilon$. Hence we get the claim.

By the claim we find that $W_\varepsilon \times_{\operatorname{Spec} R} \operatorname{Spec} k \neq \emptyset$. Hence we finish this step.

▶ **Step 3.** Prove the theorem.

Now come back to our general theorem. Now M be any nef $\mathbb{R}$-divisor and we fix an ample divisor H. Then $kM + H$ is ample for any $k \geq 0$. By Step 1,2, for any$\varepsilon > 0$ there is a rational curve $L_{x,k,\varepsilon} \subset X$ containing x such that

$$(kM + H) \cdot L_{x,k,\varepsilon} \leq (2\dim X + \varepsilon)k\frac{M \cdot C}{-K_X \cdot C} + (2\dim X + \varepsilon)\frac{H \cdot C}{-K_X \cdot C}.$$

Then we have

$$k\left(M \cdot L_{x,k,\varepsilon} - 2\dim X\frac{M \cdot C}{-K_X \cdot C}\right) + H \cdot L_{x,k,\varepsilon} \leq (2\dim X + \varepsilon)\frac{H \cdot C}{-K_X \cdot C} + k\varepsilon\frac{M \cdot C}{-K_X \cdot C}.$$

If $M \cdot L_{x,k_0,\varepsilon} - 2\dim X\frac{M\cdot C}{-K_X\cdot C} \leq 0$ for some k_0, ε, then we take $L_x := L_{x,k_0,\varepsilon}$ and then well done. If not we have

$$H \cdot L_{x,k,\varepsilon} \leq (2\dim X + \varepsilon)\frac{H \cdot C}{-K_X \cdot C} + k\varepsilon\frac{M \cdot C}{-K_X \cdot C}.$$

for every k, ε. Set $\varepsilon = \frac{1}{k}$ and $k \to \infty$. We obtain a sequence of curves $L_{x,k} := L_{x,k,1/k}$. So $H \cdot L_{x,k}$ is uniformly bounded, thus the $L_{x,k}$ form a bounded family. By Theorem 1.1.15 $\mathsf{Chow}^1(X)$ has only finitely many components parametrizing 1-cycles of bounded degree. In particular there is a subsequence $k_i \to \infty$ such that $P := P(i) := M \cdot L_{x,k_i} - 2\dim X\frac{M\cdot C}{-K_X\cdot C}$ is independent of i. Hence

$$k_i P \leq (2\dim X + 1)\frac{H \cdot C}{-K_X \cdot C} + \varepsilon\frac{M \cdot C}{-K_X \cdot C}, \quad k_i \to \infty.$$

Hence $P \leq 0$ and we take $L_x := L_{x,k_i}$ and well done. $\square$

Theorem 1.4.13 (Smooth Case)**.** *Let X be a smooth projective variety, C a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Let M be any nef $\mathbb{R}$-divisor. Assume that $-K_X \cdot C > 0$, then for any $x \in f(C)$ there is a rational curve $D_x \subset X$ containing x such that*

$$M \cdot D_x \leq 2 \dim X \frac{M \cdot C}{-K_X \cdot C}, \quad -K_X \cdot D_x \leq \dim X + 1.$$

Proof. Use Theorem 1.4.7 and Theorem 1.4.12. This is trivial. □

Remark 1.4.14. *Both Theorem 1.4.12 and Theorem 1.4.13 have generalizations with the same proof, see Theorem II.1.3 and Remark II.5.15 in [26].*

Corollary 1.4.15 (Fano Case)**.** *Let X be a smooth Fano variety, then for any x there is a rational curve $D_x \subset X$ containing x such that $-K_X \cdot D_x \leq \dim X + 1$. In particular any smooth Fano variety is uniruled.*

1.5 Application I: Basic Theory of Fano Manifolds

Some general theory of Fano varieties we refer [35]. Here we give some important basic theory of Fano manifolds. We consider any schemes over an algebraically closed field k.

1.5.1 Some General Properties

Theorem 1.5.1. *Let G be a reduced and connected linear algebraic group and X be a proper homogeneous space under the action of G. Pick $x \in X$ and stabilizer $G_x \subset G$. If G_x is reduced (always hold if $\mathrm{char} = 0$), then T_X is generated by global sections and $-K_X$ is very ample.*

Proof. Omitted, we refer Theorem V.1.4 in [26]. □

Proposition 1.5.2. *Let X be a smooth Fano variety over an algebraically closed field k of characteristic zero.*

(a) *We have $\chi(X, \mathscr{O}_X) = 1$ and X is simply connected.*

(b) $\mathrm{Pic}(X)$ *is finite generated and torsion free.*

Proof. For (a), by Kodaira's vanishing theorem we find that $H^m(X, \mathscr{O}_X) = 0$ for all $m > 0$, hence $\chi(X, \mathscr{O}_X) = 1$. If $\pi : X' \to X$ is a connected finite étale cover, then X is also a smooth Fano variety. Hence $\chi(X', \mathscr{O}_{X'}) = 1$. But $\chi(X', \mathscr{O}_{X'}) = \deg \pi \chi(X, \mathscr{O}_X)$. Hence π is an isomorphism.

For (b) we may assume $k = \mathbb{C}$. By exponential sequence one has

$$H^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

By Kodaira's vanishing theorem, we find that $H^m(X, \mathscr{O}_X) = 0$ for all $m > 0$, hence $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. Hence $\mathrm{Pic}(X)$ is finite generated. To show $\mathrm{Pic}(X)$ is torsion free, we just need to show $H^2(X, \mathbb{Z})$ is torsion free. By universal coefficient theorem for cohomology, one has

$$0 \to \mathrm{Ext}^1(H_1(X, \mathbb{Z}), \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

As $\mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$ is torsion free, the only torsion of $H^2(X, \mathbb{Z})$ follows from $H_1(X, \mathbb{Z})$. As $H_1(X, \mathbb{Z}) = \pi_1(X)^{\mathrm{abel}} = 0$ by (a), hence $\mathrm{Pic}(X)$ is torsion free. □

Theorem 1.5.3 (Cone Theorem)**.** *Let X be a smooth Fano variety over an algebraically closed field k. On X there are only finitely many families of rational curves C_μ such that $-K_X \cdot C_\mu \leq \dim X + 1$. Let $C_i : 1 \leq i \leq N$ be a set of representatives, then*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_i \mathbb{R}^+[C_i].$$

Proof. A very special case of Theorem 3.7 in [29]. Omitted. □

Proposition 1.5.4. *Let $f : X \to Y$ be a smooth morphism between smooth projective varieties over an algebraically closed field k.*

(a) If $\dim Y > 0$ then $-K_{X/Y}$ is not (absolutely) ample on X.

(b) If X is Fano, then Y is also Fano.

Proof. For (a), need to add.

For (b), we may assume $\dim Y > 0$. Pick an ample divisor H and $a > 0$ such that $-K_X - af^*H$ is nef. Let $h : C \to Y$ be a non-constant morphism from a smooth projective curve C. Consider $c \xleftarrow{f_C} X_C := X \times_Y C \xrightarrow{g} X$. Now $g^*(-K_X)$ is ample but $-K_{X_C/C}$ is not by (a). Hence for any $\varepsilon > 0$ there exists an irreducible curve $D \subset X_C$ such that $-K_{X_C/C} \cdot D < \varepsilon(-g^*K_X \cdot D)$. As $-K_{X_C/C} = g^*f^*K_Y - g^*K_X$, we have

$$-g^*f^*K_Y \cdot D > (1 - \varepsilon)(-g^*K_X \cdot D) \geq (1 - \varepsilon)(ag^*f^*H \cdot D).$$

One can choose $D \to C$ non-constant, so pushforward to C we have

$$\deg h^*(-K_Y) > (1 - \varepsilon)a \deg h^*H.$$

Hence since $\varepsilon > 0$ and $h : C \to Y$ are arbitrary, we know that $-K_Y - aH$ is nef. Hence $-K_Y$ is ample and Y is Fano. □

Remark 1.5.5. *Noe that if f is only flat, this is not true.*

1.5.2 Classifications Via Fano Index

Definition 1.5.6. *Let X be a smooth Fano variety. The* Fano index *of X is*

$$\mathsf{Index}(X) := \max\{m \in \mathbb{N} : -K_X \sim mH \text{ for some Cartier divisor } H\}.$$

Theorem 1.5.7 (Kobayashi-Ochiai, 1970)**.** *Let X be a smooth Fano variety of dimension n over a field of characteristic zero. Then*

(a) $\mathsf{Index}(X) \le n+1$.

(b) Let $-K_X \sim \mathsf{Index}(X)H$, *then* $\chi(X, \mathscr{O}_X(jH)) = \begin{cases} 1 & j=0 \\ 0 & -\mathsf{Index}(X) < j < 0 \\ (-1)^n & j = -\mathsf{Index}(X) \end{cases}$.

Moreover we have

$$\chi(X, \mathscr{O}_X(tH)) = \begin{cases} \binom{t+n}{n} & \mathsf{Index} = n+1 \\ \binom{t+n+1}{n+1} - \binom{t+n-1}{n+1} & \mathsf{Index} = n \\ H^n\binom{t+n-1}{n} + \binom{t+n-2}{n-2} & \mathsf{Index} = n-1 \\ H^n\binom{2t+n-2}{2n}\binom{t+n-2}{n-1} + \binom{t+n-2}{n-2} + \binom{t+n-3}{n-2} & \mathsf{Index} = n-2 \end{cases}.$$

Hence $H^n = \begin{cases} 1 & \mathsf{Index} = n+1 \\ 2 & \mathsf{Index} = n \end{cases}$ *and* $h^0(X, \mathscr{O}_X(H)) = \begin{cases} n+1 & \mathsf{Index} = n+1 \\ n+2 & \mathsf{Index} = n \\ H^n + n - 1 & \mathsf{Index} = n-1 \\ \frac{1}{2}H^n + n & \mathsf{Index} = n-2 \end{cases}$.

(c) $\mathsf{Index}(X) = n+1$ *if and only if* $X \cong \mathbb{P}^n$.

(d) $\mathsf{Index}(X) = n$ *if and only if* $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$ *be a smooth quadric.*

Proof. For (a), by Corollary 1.4.15 we can find a rational curve C such that $-K_X \cdot C \le n+1$. But $C \cdot H \ge 1$, hence $\mathsf{Index}(X) \le n+1$.

For (b), $\chi(X, \mathscr{O}_X(jH))$ follows from Kodaira vanishing theorem and Serre duality. Then using this we know some roots of $\chi(X, \mathscr{O}_X(tH))$ correspond to t. Hence others are not hard to find. By Kodaira vanishing theorem again we get $h^0(X, \mathscr{O}_X(H))$ and H^n.

For (c), actually one can show that $\mathscr{O}_X(H)$ is base-point free by Claim V.1.11.7 in [26]. Hence by (b) this induce $p : X \to \mathbb{P}^n$. Let $Y := \mathrm{Im}(p)$, then $1 = H^n = \deg p \deg Y$. Hence $\deg p = \deg Y = 1$. As H is ample, p is finite. Hence p is an isomorphism.

For (d), one can show that $\mathscr{O}_X(H)$ is base-point free by Claim V.1.11.7 in [26]. Hence by (b) this induce $p : X \to \mathbb{P}^{n+1}$. Let $Y := \mathrm{Im}(p)$, then $2 = H^n = \deg p \deg Y$. As $\mathsf{Index}(X) = n$, Y is not linear. Hence $\deg p = 1$ and $\deg Y = 2$. As H is ample, p is finite. Hence p is an isomorphism. $\square$

Remark 1.5.8. *Some remarks:*

(1) *If one assumes only that $-K_X \sim mH$ is nef and big, then essentially the same proof gives that $X \cong \mathbb{P}^n$ if $m = n+1$. If $m = n$, then either X is a smooth quadric in $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$ or $p : X \to Y$ is a birational morphism onto a singular quadric of rank 2.*

(2) *Let X be a smooth Fano variety of dimension n (any characteristic) such that $-K_X \sim (n+1)H$, we also have $H^n = 1$.*

Indeed, section of $\mathscr{O}(mH)$ has $\binom{m+n-1}{n}$ conditions vanishing at $x \in X$. So if $H^n > 1$, then $H^0(X, \mathscr{O}_X(mH) \otimes \mathfrak{m}_x^{m+1}) \geq cm^n$ for some $c > 0$ (see also VI.2.15.7 in [26]). Pick a such section D. By Corollary 1.4.15 we can find a rational curve $x \in C \nsubseteq D$ such that $C \cdot D = m$ since $-K_X \sim (n+1)H$. But $C \cdot D \geq m+1$ which is impossible.

Theorem 1.5.9 (Fujita, 1990)**.** *Let X be a smooth Fano variety of dimension $n \geq 3$ over a field of characteristic zero such that $\mathsf{Index}(X) = n-1$. Assume $N^1(X) \cong \mathbb{R}$. Let $-K_X = (n-l)H$. Then one of the following holds:*

(a) *$H^n = 1$ and $X \cong X_6 \subset \mathbb{P}(1^{n-1}, 2, 3)$.*

(b) *$H^n = 2$ and $X \cong X_4 \subset \mathbb{P}(1^n, 2)$.*

(c) *$H^n = 3$ and $X \cong X_3 \subset \mathbb{P}(1^{n+1})$.*

(d) *$H^n = 4$ and $X \cong X_{2,2} \subset \mathbb{P}(1^{n+2})$.*

(e) *$H^n = 5$ and X is a linear space section of the Grassmannian $\mathsf{Grass}(2,5) \subset \mathbb{P}^9$ (thus $n \leq 6$).*

Proof. See 8.11 in [11]. □

1.6 Application II: Boundedness of Fano Manifolds

Here we will give a brief introduction about the boundedness of Fano manifolds using rational curves due to Kollár-Miyaoka-Mori (see Section V.2 in [26] or original paper [28] for details). Then we will give a statement of BAB conjecture which has proved by Birkar. We consider schemes over an algebraically closed field k of characteristic zero.

Theorem 1.6.1 (Kollár-Miyaoka-Mori, 1992)**.** *Let X be a smooth Fano variety of dimension n over k. Then there is a number $d(n)$ (depending only on n) such that any two points of X can be joined by an irreducible rational curve of anticanonical degree at most $d(\dim X)$.*

Proof. This follows from the rational connected varieties, see Section IV.3 and IV.4 and Corollary V.2.14.2 in [26]. □

Proposition 1.6.2. *Let X be a proper variety of dimension n, $x \in X$ a smooth point and $\mathscr{L}$ an nef and big line bundle on X. Choose $d > 0$ such that a general point $x' \in X$ can be connected to x by an irreducible curve $C_{x'}$ such that $\mathscr{L} \cdot C_{x'} \leq d$. Then $\mathscr{L}^n \leq d^n$.*

Proof. Fix $\varepsilon > 0$ and use a classical result (see Corollary VI.2.15.7 in [26], actually with the similar proof of Remark 1.5.8(2)) there is a $k > 0$ and a divisor $D_k \in |k\mathscr{L}|$ such that $\mathrm{mult}_x D_k \geq k\sqrt[n]{\mathscr{L}^n} - k\varepsilon$. Pick a general point $x' \notin \operatorname{supp} D_k$. Then $C_{x'}$ is not contained in D_k hence

$$kd \geq D_k \cdot C_{x'} \geq \mathrm{mult}_x D_k \geq k\sqrt[n]{\mathscr{L}^n} - k\varepsilon.$$

Hence $d \geq \sqrt[n]{\mathscr{L}^n} - \varepsilon$ and let $\varepsilon \to 0$. □

Theorem 1.6.3 (Boundedness of Fano Manifolds, Kollár-Miyaoka-Mori 1992)**.** *All n-dimensional Fano Manifolds over k forms a bounded family.*

Proof. By Theorem 1.6.1 and Proposition 1.6.2, we know that $(-1)^n K_X^n$ is bounded. Using Matsusaka estimate (see Exercise VI.2.15.8 in [26], proved by Kollár-Matsusaka in [27] in 1983) we know that for any nef and big divisor H, the coefficients of polynomial $\chi(X, \mathscr{O}_X(tH))$ can be bounded by H^m and $K_X \cdot H^{m-1}$. So $\chi(X, \mathscr{O}_X(tK_X))$ has bounded coefficients. In 1970, Matsusaka in [30] shows that there are only finitely many deformation types with fixed Hilbert polynomial. So All n-dimensional Fano Manifolds over k forms a bounded family. □

This finish the story of the smooth Fano varieties. If we have some mild singularities, then this problem is the famous conjecture in birational geometry:

Theorem 1.6.4 (BAB-Conjecture, Birkar 2016)**.** *Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Then the set of projective varieties X such that (X, B) is ε-lc of dimension d for some boundary B and $-(K_X + B)$ is nef and big, form a bounded family.*

Some History. This is one of the fundamental result of singular Fano varieties and is one of the most important conjectures in birational geometry and it is related to the termination of flips.

As we have seen, Kollár-Miyaoka-Mori in 1992 showed the boundedness of smooth Fano varieties using rational curves. But this can not be used in the BAB-conjecture.

In 1992 Kawamata showed the boundedness of terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one. In 1992 Borisov-Borisov shows this for toric cases. In 1994 V. Alexeev proved the BAB-conjecture for surfaces. In 2000 Kollár-Miyaoka-Mori-Takagi showed the boundedness of canonical $\mathbb{Q}$-Fano threefolds. Then in 2014 C. Jiang proved the weak BAB-conjecture for 3-fold, which is an important step towards the BAB-conjecture.

Finally BAB-Conjecture (along with the Weak BAB Conjecture) in arbitrary dimension was proved by C. Birkar in 2016 by different and much stronger methods, see his papers [5] and [6]. □

Remark 1.6.5. *The theory of moduli of Fano varieties is an application of J. Alper's theory of good moduli space. Many mathematicians build the whole theory in recent years using K-stability theory.*

In fact, by the theory of Birkar in [5], C. Jiang in 2017 showed that any K-semistable Fano varieties with dimension n and volumn $(-K_X)^n = V$ is bounded. Then there exists $N \gg 0$ such that $|-NK_X|$ gives an embedding to $\mathbb{P}^M$. Fix a Hilbert polynomial and then using the theory of KSBA-moduli space, there is a subspace of that Hilbert space H' correspond what we what. Hence the moduli stack $\mathscr{M}^{\mathrm{Kss}}_{n,V}$ of K-semistable Fano varieties with dimension n and volumn $(-K_X)^n = V$ is $[H'/\mathrm{PGL}]$ which is an algebraic stack of finite type. Then using Alper's theory we construct the separated good moduli space $\mathscr{M}^{\mathrm{Kss}}_{n,V} \to M^{\mathrm{Kps}}_{n,V}$ with ample CM-line bundle.

1.7 Application III: Hartshorne's Conjecture

Hartshorne's Conjecture is first proved by S. Mori in his famous and important paper [31]. This paper is the beginning of the theory of VMRT.

Theorem 1.7.1 (Hartshorne's Conjecture, Mori 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k, if T_X is ample then $X \cong \mathbb{P}^n_k$.*

Proof. By Theorem 1.7.3 directly. □

This conjecture motivated by an important conjecture in complex geometry:

Theorem 1.7.2 (Frankel's Conjecture, Mori 1979 and Siu-Yau 1980)**.** *If X is a compact Kähler manifold of dimension n with everywhere positive holomorphic bisectional curvature, then $X \cong \mathbb{P}^n_{\mathbb{C}}$.*

Proof. By Kodaira embedding theorem to $-K_X$ we know that X is a projective manifold. Then by Theorem 1.7.1 we get the result. □

Our main result in this section is the following due to Mori which is much stronger than the Hartshorne's Conjecture as we mentioned above.

Theorem 1.7.3 (Mori, 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k. If*

(1) $-K_X$ is ample, that is, X is a Fano manifold;

(2) For any non-constant morphism $f : \mathbb{P}^1_k \to X$ the bundle f^*T_X is the sum of line bundles of positive degree.

Then $X \cong \mathbb{P}^n_k$.

Proof. We will use the following lemmas:

- **Lemma A.** For any $f : \mathbb{P}^1_k \to X$ such that bundle f^*T_X is the sum of line bundles of positive degree, we have $\deg f^*T_X \geq n+1$. If equality holds, then f is an closed embedding and is standard, that is, $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$.

  Proof of Lemma A. Let $f^*T_X \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$ where $a_1 \geq \cdots \geq a_n$. Then $a_i \geq 1$ and $a_1 \geq 2$ by Remark 1.3.10. Hence $\deg f^*T_X \geq n+1$. If equality holds, then the only possibility is $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$. To show f is an embedding, first we now that f is unramified by trivial reason. Others are also easy and we refer to Lemma V.3.7.3.2 in [26]. $\square$

- **Lemma B.** In the case of Theorem, any rational curve can be deformed as a cycle to the sum of rational curves C such that $-K_X \cdot C = n+1$.

  Proof of Lemma B. From bend and break directly. $\square$

Back to the theorem. We let $n \geq 2$. Pick $f : \mathbb{P}^1 \to X$ passing a general point $x \in X$ with $0 \mapsto x$ and with minimal degree $n+1$ by Lemma B. By Proposition 1.3.11 the components $V \subset \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ containing $[f]$ is smooth of dimension $n+1$ and the correspond $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{n+1}(x, X)$ is also smooth of dimension $n-1$. Actually $\gamma : V \to \mathcal{K}_x$ is a principal $G := \mathrm{Aut}(\mathbb{P}^1; 0)$-bundle.

▶ **Step 1.** We claim that $\mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

Consider the tangent $\Phi : V \to \mathbb{V}(\Omega^1_{X,x})$ via $v \mapsto (dv)_0(\frac{d}{dt})$ for uniformizer $t \in \mathscr{O}_{\mathbb{P}^1,0}$ by Lemma A. First we claim that Φ is smooth. Easy to see that Φ is flat and we just need to show $\Phi^{-1}(\Phi(v))$ is smooth. Note that for any finite type k-scheme T and for any morphism $T \to V$ over k, it factors through $\Phi^{-1}(\Phi(v)) \to V$ if and only if the morphism $\mathbb{P}^1_T \to X_T$ coincides on $\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})$ with v_T. Hence

$$\Phi^{-1}(\Phi(v)) \cong V \cap \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; v|_{\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})})$$

which is open and hence smooth with the same proof of Proposition 1.3.11.

Hence by Lemma A again we get a smooth morphism $\Phi : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$. Hence it is finite étale. Hence $\mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

▶ **Step 2.** Let $F : V \times \mathbb{P}^1 \to \mathcal{K}_x \times X$ defined by $(v, x) \mapsto (\gamma(v), v(x))$, consider $Z := \underline{\mathrm{Spec}}_{\mathcal{K}_x \times X} F_*\mathscr{O}^G$ which is a geometrically quotient by G (can be checked along the principal bundle $V \to \mathcal{K}_x$). As $\psi : Z \to \mathcal{K}_x$ is a $\mathbb{P}^1$-bundle with a section $S \subset Z$ induced by $V \to V \times \mathbb{P}^1$ as $v \mapsto (v, 0)$, then $Z \cong \mathbb{P}(\psi_*\mathscr{O}_Z(S))$ is a projective bundle. Define a universal cycle map $\pi : Z \to X$ induced by G-invariant cycle morphism $V \times \mathbb{P}^1 \to X$. We claim that $\pi : Z \to X$ is étale on $Z \backslash S$ and $\pi(S) = x$.

Actually $\pi(S) = x$ is trivial, to show $\pi|_{Z\backslash S}$ is étale we just need to show $V \times \mathbb{P}^1 \to X$ is smooth. This follows from Corollary 1.3.4 and Theorem 1.3.6. Hence we get the claim.

▶ **Step 3.** Consider the Stein factorization we have $\pi : Z \xrightarrow{\phi} U \cong \underline{\mathrm{Spec}}_X \pi_* \mathscr{O}_Z \xrightarrow{\eta} X$. We claim that η is étale, $Z \backslash S \cong U \backslash \{r\}$ where $\phi(S) = r$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

In fact by Stein factorization η is étale outside a codimension ≥ 2 locus, by purity of branced locus we know that η is étale. Now $Z \backslash S \cong U \backslash \{r\}$ where $\phi(S) = r$ follows from Zariski main theorem. Finally we show that $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. Indeed, picka hyperplane $L \subset \mathcal{K}_x$ and a line $C \cong \mathbb{P}^1 \subset S$ such that $\psi(C) \nsubseteq L$. Let $D := \psi^{-1}(L)$, then $C \cdot D = 1$. As $r \in \phi(D)$, we have $\phi^{-1}\phi(D) = D + aS$ for some $a > 0$. So $C \cdot \phi^{-1}\phi(D) = \phi(D) \cdot D = 0$. Hence $C \cdot S = -1$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

▶ **Step 4.** We claim that $U \cong \mathbb{P}^n$.

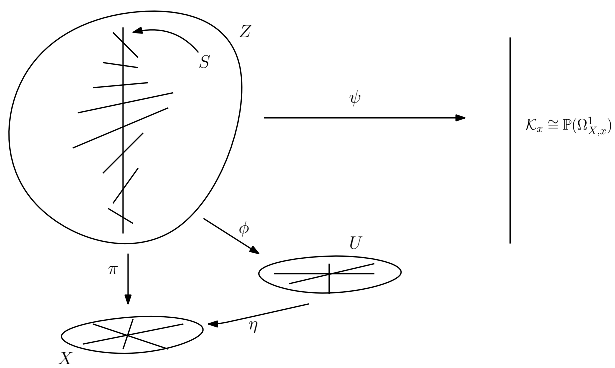
By Step 3 we have $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$, hence

$$0 \to \mathscr{O}_Z \to \mathscr{O}_Z(S) \to \mathscr{O}_S(-1) \to 0$$

exact. Since $R^1\psi_*\mathscr{O}_Z = 0$, we get

$$0 \to \mathscr{O}_{\mathcal{K}_x} \to \psi_*\mathscr{O}_Z(S) \to \mathscr{O}_{\mathcal{K}_x}(-1) \to 0$$

exact. As $\mathrm{Ext}^1_{\mathbb{P}^{n-1}}(\mathscr{O}(-1), \mathscr{O}) = 0$, we get $\psi_*\mathscr{O}_Z(S) \cong \mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)$. Hence by Step 2 we have $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1))$.



Hence $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)) \cong \mathrm{Bl}_O\mathbb{P}^n$. We can have a contraction map $Z \to \mathrm{Bl}_O\mathbb{P}^n$ makes S to a point $O \in \mathbb{P}^n$ (in fact it is induced by $\psi^*\mathscr{O}(1) \otimes \mathscr{O}(S)$). Hence via $\mathbb{P}^n \leftarrow Z \to U$ we have a birational map $\mathbb{P}^n \dashrightarrow U$. This must be an isomorphism since $Z \cong \mathrm{Bl}_O\mathbb{P}^n$ has only two dimensional Mori cone, hence the only birational contraction is this one (another is that $\mathbb{P}^1$-bundle).

▶ **Step 5.** Finish the proof, that is, we have $X \cong \mathbb{P}^n$.

Since $\mathbb{P}^n$ is simply connected, $U \cong \mathbb{P}^n \to X$ is a Galois covering by Step 3 and 4. Thus $X \cong \mathbb{P}^n$ because any automorphism of $\mathbb{P}^n$ has a fixed point. □

Remark 1.7.4. *Note that by the proof this is right if we just consider the rational curves containing a sufficient general point.*

Corollary 1.7.5 (Lazarsfeld, 1984). *Let X be a smooth projective variety over an algebraically closed field k of dimension > 0. Let there is a surjective separable morphism $p : \mathbb{P}^n_k \to X$, then $X \cong \mathbb{P}^n$.*

Proof. By the Chow ring structure of projective space, we know that $\dim X = n$ and p is finite. Hence let R be a ramification divisor of p, we have $p^*(-K_X) = -K_{\mathbb{P}^n} + R$ hence some multiple of $-K_X$ is effective. As p surjective, then $\dim N_1(X) = 1$ Hence $-K_X$ is ample and X is Fano. For a sufficient general point $x \in X$ outside of the ramification divisor, consider $f : \mathbb{P}^1 \to X$ as $0 \mapsto x$. Let C be a normalization of a component in $\mathbb{P} \times_X \mathbb{P}^1$, we have

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "q"] & \mathbb{P}^n \arrow[d, "p"] \\ \mathbb{P}^1 \arrow[r, "f"] & X \end{tikzcd}$$

The natural map $r : h^*T_{\mathbb{P}^n} \to h^*p^*T_X = q^*f^*T_X$ is a local isomorphism $q^{-1}(0) \subset C$ since p is étale above x. Write $f^*T_X = \bigoplus_i \mathscr{O}_{\mathbb{P}^1}(a_i)$. For any j we have

$$\bigoplus h^*\mathscr{O}_{\mathbb{P}^1}(1) \to h^*T_{\mathbb{P}^n} \xrightarrow{r} \bigoplus_i q^*\mathscr{O}_{\mathbb{P}^1}(a_i) \to q^*\mathscr{O}_{\mathbb{P}^1}(a_j)$$

which is surjective over an open subspace $U \subset C$. So $q^*\mathscr{O}_{\mathbb{P}^1}(a_j)$ has a section vanishing at somr point. Hence $a_i > 0$ for any i. So by Theorem 1.7.3 we have $X \cong \mathbb{P}^n$. $\square$

Chapter 2

Varieties of Minimal Rational Tangents

We will assume the base field is $\mathbb{C}$.

2.1 Basic Properties

In this section we will discover some fundamental and important properties of tangent map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x})$ with VMRT $\mathcal{C}_x$ for any smooth Fano variety X. First we need to find some properties of singular rational curves.

Definition 2.1.1. *Let X be a smooth uniruled variety over $\mathbb{C}$ and $x \in X$ is a point. Choose a (dominanted) minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ and the corresponding component $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(x, X)$ be of minimal degree $p+2$. Consider the rational map*

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x}), \quad [i : C \subset X] \mapsto \left.\frac{di}{dt}\right|_{t=0}$$

where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{C,0}$, defined on curves smooth at x. We define the variety of minimal rational tangents *or* VMRT *$\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ at x is the closure of the image of τ_x. Moreover, we define*

$$\mathcal{C} := \overline{\bigcup_{x \text{ general}} \mathcal{C}_x} \subset \mathbb{P}(\Omega^1_X)$$

the total variety of minimal rational tangents *or* total VMRT.

Remark 2.1.2. *Note that there are only finitely many choice of minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$, hence there are only finitely many choice of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$, at least for general point $x \in X$.*

Theorem 2.1.3 (Kebekus [24], 2002)**.** *Let X be a smooth uniruled variety and $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ a (dominanted) minimal rational component. Let $\mathcal{K}'_x \subset \mathcal{K}$ be the locus of curves passing through x where $x \in X$ be a general point (hence $\mathcal{K}_x \to \mathcal{K}'_x$ is a normalization). consider the closed subvarieties*

$$\mathcal{K}_x^{\mathsf{sing}} := \{[C] \in \mathcal{K}'_x : C \text{ singular}\}, \quad \mathcal{K}_x^{\mathsf{sing},x} := \{[C] \in \mathcal{K}'_x : C \text{ singular at } x\}.$$

Then the following holds.

(a) The space $\mathcal{K}_x^{\mathsf{sing}}$ has dimension at most one, and the subspace $\mathcal{K}_x^{\mathsf{sing},x}$ is at most finite. Moreover, if $\mathcal{K}_x^{\mathsf{sing},x}$ is not empty, the associated curves are unramified .

(b) If there exists a line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ that intersects the curves with multiplicity 2, then $\mathcal{K}_x^{\mathsf{sing}}$ is at most finite and $\mathcal{K}_x^{\mathsf{sing},x}$ is empty.

Proof. See the original paper [24] or the sketch in Theorem 2.12 in the survey [25]. □

Remark 2.1.4. *There is another thing about the singular rational curves: if there is a curve parametrized by $\mathcal{K}_x$ singular at x, then there is also a curve parametrized by $\mathcal{K}_x$ with a cuspidal singularity. See V.3.6 in [26].*

Corollary 2.1.5. *By Theorem 2.1.3(a), very curve parametrized by $\mathcal{K}_x$ is unramified at x (i.e., its normalization is unramified at $0 \mapsto x$).*

Theorem 2.1.6 (Kebekus-2002, Hwang-Mok-2004)**.** *Let X be a smooth uniruled variety and $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ a (dominanted) minimal rational component. Let $x \in X$ be a general point, consider the tangent map*

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x}), \quad [f : \mathbb{P}^1 \to X] \mapsto \left.\frac{df}{dt}\right|_{t=0}.$$

(a) τ_x is actually a finite morphism, we can called it tangent morphism*.*

(b) $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a birational morphism, hence

(c) $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is the normalization.

Proof. (a) and (b) implies (c) in this case.

For (a) (proved in [24]), we will first show that $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x})$ actually can be a morphism. We have two arguments with the same result:

(M1) By Theorem 1.2.8(b) we have q as follows

$$\begin{tikzcd} \mathcal{K}_x \arrow[r, "q"] \arrow[dr, "\tau_x"'] & \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \arrow[d, "t_x"] \\ & \mathbb{P}(\Omega^1_{X,x}) \end{tikzcd}$$

where $t_x : \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \to \mathbb{P}(\Omega^1_{X,x})$ sends f to $(df)_0(\frac{d}{dt})$ for uniformizer $t \in \mathscr{O}_{\mathbb{P}^1,0}$ since it is unramified by Corollary 2.1.5.

(M2) Consider the universal morphism and cycle morphism

$$\begin{tikzcd} \mathsf{Univ}^{\mathrm{n}}(x,X) & \mathcal{U}_x^n \arrow[l, hook'] \arrow[r, "\iota_x"] \arrow[d, "\pi_x"] & X \\ \mathsf{RatCurves}^{\mathrm{n}}(x,X) & \mathcal{K}_x \arrow[l, hook'] & \end{tikzcd}$$

We have a section $\mathcal{K}_x \cong \sigma_\infty \subset \mathcal{U}_x^n$ contracted to $x \in X$ via ι_x which is canonical by Theorem 2.1.3(a). By Corollary 2.1.5 again we can consider a nowhere vanishing morphism of vector bundles

$$T_{\mathcal{U}_x^n/\mathcal{K}_x}|_{\sigma_\infty} \to \iota_x^*(T_{X,x})$$

and yields $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathbb{P}(\Omega^1_{X,x})$.

Now we need to show τ_x is finite. If not, we have a curve $C \subset \mathcal{K}_x$ contracted by τ_x. Let the normalization of universal family $U \to C$ is again a $\mathbb{P}^1$-bundle. Let the corresponding section is $s_\infty \subset U$. Consider $N_{s_\infty/U}$. Since s_∞ contracted into a point, its normal bundle is negative. But this is the tangent morphism, the normal bundle nedd to be trivial. This is impossible. Hence τ_x is finite.

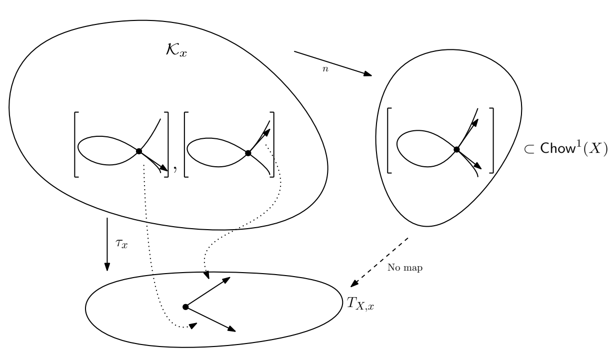
For (b), proved in [23] Theorem 1 and we will omit it. □

Remark 2.1.7. *Note that by the proof of (a) we have $\tau_x^*(\mathscr{O}(1)) \cong \mathscr{O}_{\sigma_\infty}(K_{\mathcal{U}_x^n/\mathcal{K}_x})$.*

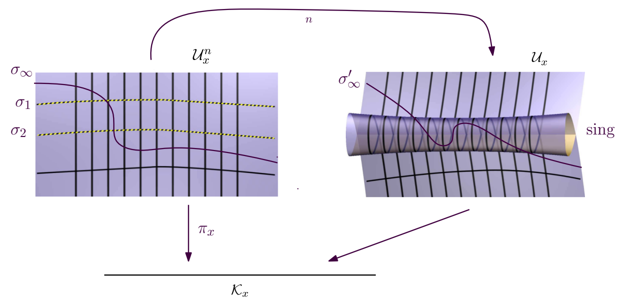
Remark 2.1.8. *Note also that we need to think (M1) and (M2) deeply as follows:*

The fundamental question is that if the minimal rational curve C not smooth at x (however it is unramified at x by Corollary 2.1.5), how to choose the different tangent vectors?

(M1) In this method, since $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \cong \mathcal{K}_x$ we know that there are several curves in $\mathcal{K}_x$ maps to $[C]$ and their tangent vectors separated by the tangent vectors of C at x since C is not smooth at x. The diagram as follows:



(M2) In this method, the section $\sigma_\infty \cong \mathcal{K}_x$ will meet the sections of singular points at finite points. For example in local case where $\sigma_\infty \subset \iota_x^{-1}(x)$ be that section and σ_1, σ_2 are preimage of singular locus sing $\subset \mathcal{U}_x$*:*



Hence the choice of tangent vectors are canonical. Another interesting method is that we can use the universal property of the blow-up:

$$\begin{tikzcd} & & \mathrm{Bl}_x X \arrow[d, "b"] \\ \mathcal{K}_x & \mathcal{U}_x^n \arrow[l] \arrow[r, "\iota_x"'] \arrow[ur, dashed, "\hat{\iota}_x"] & X \end{tikzcd}$$

Then we have $\tau_x = \hat{\iota}_x|_{\sigma_\infty} : \mathcal{K}_x \cong \sigma_\infty \to E = \mathbb{P}(\Omega^1_{X,x})$.

Remark 2.1.9. *In fact in [24] they show that $\iota_x^{-1}(x) = \sigma_\infty \cup \{\text{finite points}\}$. Moreover the tangent morphism $d\iota_x$ has rank one along σ_∞.*

Proposition 2.1.10. *Let X be a smooth uniruled variety and $x \in X$ be a general point, then the morphism $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is unramified at $[f] \in \mathcal{K}_x$ if and only if $[f]$ is standard.*

Proof. We follows Proposition 1.4 in the survey [19] or Proposition 2.7 in [2]. Consider

$$\begin{tikzcd} \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & V_x \arrow[l] \arrow[r, "\phi_x"] \arrow[dr, "\psi_x"'] & \mathcal{K}_x \arrow[d, "\tau_x"] \\ & & \mathbb{P}(\Omega^1_{X,x}) \end{tikzcd}$$

Pick any $[C] \in \mathcal{K}_x$ and its normalization $[f] \in V_x$, then we need to consider $(d\psi_x)_{[f]} : T_{[f]}V_x \to T_{\psi_x[f]}\mathbb{P}(\Omega^1_{X,x})$. Now $T_{[f]}V_x \cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$ and $T_{\psi_x[f]}\mathbb{P}(\Omega^1_{X,x}) \cong T_xX/\hat{\psi}_x[f]$ where $\hat{\psi}_x[f]$ denotes the 1-dimensional subspace of T_xX corresponding to the point $\psi_x[f]$. If $v \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$, then we let a deformation f_s with $f_0 = f$ such that

$\frac{df_s}{ds}|_{t=0} = v$. Then

$$(d\psi_x)_{[f]}(v) = \frac{d}{ds}\bigg|_{s=0} \frac{df_s}{dt}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} \frac{df_s}{ds}\bigg|_{s=0} = \frac{dv}{dt}\bigg|_{t=0} \in T_xX/\hat{\psi}_x[f] = f^*T_X|_0/T_o\mathbb{P}^1$$

where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{\mathbb{P}^1,0}$. For a $v \neq 0$ such that v not be zero after quotient by $T_o\mathbb{P}^1$, we find that $(d\psi_x)_{[f]}(v) = 0$ if and only if $\mathscr{O}(2) \subset f^*T_X|_0/T_o\mathbb{P}^1$ if and only if $[f]$ is standard. $\square$

Remark 2.1.11. *Hence we give another proof of that τ_x is generically finite.*

Corollary 2.1.12. *Let X be a smooth uniruled variety and $x \in X$ be a general point. If every irreducible component of $\mathcal{C}_x$ is smooth, then all curves parametrized by $\mathcal{K}_x$ are smooth at x.*

Proof. Since every irreducible component of $\mathcal{C}_x$ is smooth, τ_x is unramified by Theorem 2.1.6 (in fact, the restriction of τ_x to each irreducible component of $\mathcal{K}_x$ is an isomorphism). Thus, by Proposition 2.1.10, f is standard for every member $[f] \in \mathcal{K}_x$. Hence there is no curve parametrized by $\mathcal{K}_x$ has a cuspidal singularity. Then the result follows from Remark 2.1.4. $\square$

Corollary 2.1.13. *Let X be a smooth uniruled variety and $x \in X$ be a general point. We assume that under the embedding $X \subset \mathbb{P}^N$, any point in X lies in a line on X. Then $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is an embedding, hence $\mathcal{C}_x$ is smooth.*

Proof. Note that the map τ_x is injective, because any line through x is uniquely determined by its tangent direction. Hence we just need to show that τ_x is unramified. By Proposition 2.1.10 we just need to show that any minimal rational curve, that is, these lines C containing x is standard. Indeed, let $T_X|_C \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_1)$ with $a_1 \geq \cdots \geq a_n \geq 0$. Hence $a_i \geq 2$. As $T_X|_C \subset T_{\mathbb{P}^n}|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus N-1}$, we get $a_1 = 2$ and $1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and C is standard. $\square$

Corollary 2.1.14. *If X be a smooth prime Fano variety of Fano index $\mathsf{Index}(X) > \frac{n+1}{2}$ with dimension n, then X satisfies the conditions in Corollary 2.1.13. Hence $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is an embedding for a general point $x \in X$, hence $\mathcal{C}_x$ is smooth.*

Proof. For any minimal rational curve C (let the anticanonical degree is $p+2$), we have

$$n + 1 \geq p + 2 = -K_X \cdot C = \mathsf{Index}(X) C \cdot \mathscr{L}$$

where $\mathscr{L}$ generates $\mathrm{Pic}(X)$. As $\mathsf{Index}(X) > \frac{n+1}{2}$, then C must be a line under the embedding given by $\mathscr{L}$. $\square$

Proposition 2.1.15. *Let X be a smooth uniruled variety and $x \in X$ be a general point. For general $[C] \in \mathcal{K}_x$ with normalization $f : \mathbb{P}^1 \to C \subset X$ with minimal degree $p+2$. Define $T_xX_C^+ \subset T_xX$ be the subspace correspond to the positive part, that is, the stalk of*

$$\mathrm{Im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes \mathscr{O} \to f^*T_X(-1)] \otimes \mathscr{O}(1) \subset f^*T_X$$

at x. Then $\mathbb{P}((T_xX_C^+)^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ is the projective tangent space of $\mathcal{C}_x$ at $\tau_x([f])$.

Proof. As general curve, we just consider the standard one. By proposition 2.1.10, if $v \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$, then the differential sends v to $\frac{dv}{dt}\big|_{t=0}$ where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{\mathbb{P}^1,0}$. Since v lies in the positive part, then so is $\frac{dv}{dt}$. As $\dim \mathcal{C}_x = p = \dim \mathbb{P}(\mathscr{O}(2) \oplus \mathscr{O}(1)^p)$, then well done. □

2.2 Basic Examples of VMRT

2.2.1 Projective Spaces

Proposition 2.2.1. *If $X = \mathbb{P}^n$, then $\tau_x : \mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.*

Proof. By the proof of Theorem 1.7.3 or Corollary 2.1.14. □

Conversely we introduce some characterizations of projective spaces. Some of them we have proved and some of them are easy to prove. We also will to prove some of them using VMRT theory.

Theorem 2.2.2 (Cho-Miyaoka-Barron, 2002)**.** *Let X be a smooth projective variety of dimension n and $x_0 \in X$ be a general point . Then the following fourteen conditions are equivalent:*

- *(a)* $X \cong \mathbb{P}^n$.
- *(b)* Hirzebruch-Kodaira-Yau condition: *X homotopic to $\mathbb{P}^n$.*
- *(c)* Kobayashi-Ochiai condition: *X is Fano and $c_1(X)$ is divisible by $n+1$ in $H_2(X,\mathbb{Z})$.*
- *(d)* Frankel-Siu-Yau condition: *X carries a Kähler metric of positive holomorphic bisectional curvature.*
- *(e)* Hartshorne-Mori condition: *T_X is ample.*
- *(f)* Mori condition: *X is Fano and $T_X|_C$ is ample for any rational curves C.*
- *(g)* Doubly transitive group action: *The action of $\mathrm{Aut}(X)$ on X is doubly transitive.*
- *(h)* Remmert-Vande Ven-Lazarsfeld condition: *There exists a surjective morphism from a suitable projective space onto X.*
- *(i)* Length condition: *X is uniruled and $-K_X \cdot C \geq n+1$ for any curve $C \subset X$.*

(j) *Length condition on rational curves:* X *is uniruled and* $-K_X \cdot C \geq n+1$ *for any rational curve* $C \subset X$.

(k) *Length condition on rational curves with base point:* X *is uniruled and* $-K_X \cdot C \geq n+1$ *for any rational curve* $C \subset X$ *passing through a general point* $x_0 \in X$.

(l) *VMRT condition:* X *is uniruled and* $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

First Comments. Actually there is a much general condition in the original paper [8] implies all of these, but we will omit it. Note that we also omit the proof of $(k) \Rightarrow (a)$ since it use that general condition. But we finally will prove $(i) \Rightarrow (l) \Rightarrow (a)$ by using VMRT theory as in

Here are some trivial implications. We have (a) implies everything. We have $(i) \Rightarrow (j) \Rightarrow (k)$ and $(d) \Rightarrow (e) \Rightarrow (f)$. Morever $(c) \Rightarrow (i)$ and $(f) \Rightarrow (j)$ are also trivial. Note also that $(a) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ are proved in Theorem 1.7.1, Theorem 1.7.2 and Theorem 1.7.3. Note also that $(h) \Rightarrow (k)$ and $(h) \Rightarrow (a)$ is proved also in Corollary 1.7.5. For $(g) \Rightarrow (f)$ we refer Page 45 in [8]. □

Proof of $(b) \Rightarrow (c)$. As X homotopic to $\mathbb{P}^n$, then X is simply connected. By the proof of Proposition 1.5.2(b) we have $\mathrm{Pic}(X) \cong H^2(X,\mathbb{Z}) = H^2(\mathbb{P}^n,\mathbb{Z}) \cong \mathbb{Z}$. Pick an ample generator h and let $c_1(X) = mh$. As $c_1^n(X)$ is homotopic invariant up the sign (see [16]), we have $m = \pm(n+1)$. If $m = n+1$ then well done.

If $m = -(n+1)$ and we will show that this is impossible. In this case K_X is ample, then X has KE-metric by several works [3][40][41]. The Chern number $c_1^{n-2}(2(n+1)c_2 - nc_1^2)$ is again homotopic invariant up the sign. By Chen-Ogiue-Yau's result ([7][40][41]) this would imply that the universal cover of X is the open unit ball, contradicting the assumption that the compact manifold X is simply connected. □

Finally we will prove $(i) \Rightarrow (l) \Rightarrow (a)$ **using VMRT**.

Proof of $(i) \Rightarrow (l)$. By Theorem 2.1.6(a), we have $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathbb{P}(\Omega^1_{X,x})$ is finite. Since $\dim \mathcal{K}_x = n-1 = \dim \mathbb{P}(\Omega^1_{X,x})$, we know that τ_x is surjective. By Theorem 2.1.6(b) we find that τ_x is birational (**Note that the proof of 2.1.6(b) in [23] is to reduce the general case to our case. So we can not use this at all. But for convenience we will use this directly**). Hence by Zariski main theorem we know that $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$ are isomorphisms. □

Proof of $(l) \Rightarrow (a)$. This is the same proof of the final step of Hartshorne's conjecture 1.7.3. As $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$ where by Theorem 2.1.6 τ_x is a normalization, hence $\mathcal{K}_x \cong \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x}) \cong \mathbb{P}^{n-1}$.

By Stein factorization we have $\iota_x : \mathcal{U}^n_x \xrightarrow{A} Y \xrightarrow{B} X$ where $A(\sigma_\infty) = \{\mathrm{pt}\}$ and B finite. Similarly pushforward $0 \to \mathscr{O}_{\mathcal{U}^n_x} \to \mathscr{O}_{\mathcal{U}^n_x}(\sigma_\infty) \to \mathscr{O}_{\sigma_\infty}(\sigma_\infty) \to 0$ to $\mathcal{K}_x$ and consider Ext^1 we have $\mathcal{U}^n_x \cong \mathbb{P}_{\mathcal{K}_x}(\mathscr{O} \oplus \mathscr{O}(-1))$ and get $Y \cong \mathbb{P}^n$. Finally by Corollary 1.7.5 we get $X \cong \mathbb{P}^n$. □

Remark 2.2.3. *Note that the history about the characterizations of projective space is very long and we refer Remark 5.2 in [8]. Noe also that there is an anologue of quadric hypersurfaces, see Remark 5.3 in [8].*

Theorem 2.2.4 (Wahl, 1983)**.** *Let X be a complex projective non-singular variety, let $\mathscr{L}$ be an ample line bundle. If $H^0(X, T_X \otimes \mathscr{L}^{-1}) \neq 0$, then $(X, \mathscr{L})$ is $(\mathbb{P}^n, \mathscr{O}(1))$ or $(\mathbb{P}^n, \mathscr{O}(2))$.*

Proof. See the main theorem in the paper [39]. □

2.2.2 Fano Hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree d where $n \geq 3$. Hence now $d \leq n+1$. We first consider the following general result which will be useful later:

Proposition 2.2.5. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d **over any field** k. If $n \geq 3$ then*
$$\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathscr{O}_X(1).$$

Proof. For the proof over any field we refer XII. Cor 3.6 in [14]. We only prove the case where $k = \mathbb{C}$. By exponential sequence one has

$$H^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

By the Lefschetz hyperplane theorem we have $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$ since $n \geq 3$. Hence $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. By the Lefschetz hyperplane theorem again we have $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathscr{O}_X(1)$. Well done. □

To consider $\mathcal{C}_x$ for $x \in X$, we first consider when does the lines lie over the $X \subset \mathbb{P}^{n+1}$. Let $F(t_0, ..., t_{n+1})$ be the homogeneous polynomial of degree d defining X and let $x = [x_0 : ... : x_{n+1}] \in X$ be a general point.

Proposition 2.2.6. *If $d \leq n$, then $\mathcal{C}_x$ is the smooth complete intersection of multi-degree $(2, 3, ..., d)$.*

Proof. A line through x given by $l = [x_0 + \lambda y_0 : ... : x_{n+1} + \lambda y_{n+1}]$ where $[y_0 : ... : y_{n+1}] \in \mathbb{P}^{n+1}$ be some point. Hence $l \subset X$ if and only if $F(x_0 + \lambda y_0, ..., x_{n+1} + \lambda y_{n+1}) = 0$ for any λ. So this if and only if $\sum_{i=0}^{d} \lambda^i \frac{1}{i!}(\Delta_x(y))^i F(x) = 0$ where $\Delta_x(y) = \sum_i y_i \frac{\partial}{\partial t_i}$. Hence this if and only if

$$\Delta_x(y)F(x) = 0, (\Delta_x(y))^2 F(x) = 0, \cdots, (\Delta_x(y))^d F(x) = 0.$$

Note that the first one is just the defining equation of $\mathbb{P}(\Omega^1_{X,x})$, hence well done. □

Remark 2.2.7. *Some situations:*

(a) *When $d = 2$ then X is the hyperquadric $\mathbb{Q}_n$ which is homogeneous. Hence VMRT $\mathcal{C}_x \cong \mathbb{Q}_{n-2} \subset \mathbb{P}(\Omega^1_{X,x})$.*

(b) *When d is high and $d < n$, then VMRT os Calabi-Yau or of general type.*

(c) *When $d = n$ then VMRT is finite and of cardinality $n!$.*

(d) *When $d = n+1$ there exists no line but has finite conics (see V.4.4.4 in [26]).*

2.2.3 Grassmannians

Let $X = \mathsf{Grass}(s, V)$ is Grassmannian of $s > 0$-dimensional subspaces where $\dim V = r + s$. Pick a general point $x = [W] \in X$.

Proposition 2.2.8. *In this case $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is just the Segre embedding*

$$\tau_x : \mathbb{P}(W) \times \mathbb{P}((V/W)^*) \hookrightarrow \mathbb{P}(W \otimes (V/W)^*).$$

Proof. Via Plücker embedding X covered by lines, hence by Corollary 2.1.13 τ_x is an embedding. Note that a line on $\mathsf{Grass}(s, V)$ through a point $x = [W] \in X = \mathsf{Grass}(s, V)$ is determined by a choice of subspace W' of dimension $s - 1$ contained in W and a subspace W'' of dimension $s + 1$ containing W. Then that line consist of subspaces of dimension s which are containing W' and contained in W''. So $\mathcal{K}_x \cong \mathbb{P}(W) \times \mathbb{P}((V/W)^*)$. Hence easy to see the tangent morphism is just Segre embedding:

$$\tau_x : \mathcal{K}_x \cong \mathbb{P}(W) \times \mathbb{P}((V/W)^*) \hookrightarrow \mathbb{P}(W \otimes (V/W)^*) \cong \mathbb{P}(\Omega^1_{X,x}).$$

Well done. □

2.2.4 Moduli Space of Stable Bundles over curves

Consider a smooth projective curve C of genus $g \geq 2$.

Proposition 2.2.9. *Consider the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of degree d. If d is odd, then $M_{2;\mathscr{D},d}(C)$ is a $(3g-3)$-dimensional Fano manifold of Picard number 1 (it is prime). Moreover $M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},1}(C)$ in this case. In particular, when $g = 2$ the space $M_{2;\mathscr{D},1}(C)$ is a intersection of two quadrics in $\mathbb{P}^5$.*

Proof. We refer [32], omit it. □

Corollary 2.2.10. *When $g = 2$, the VMRT is just four points in $\mathbb{P}(\Omega^1_{X,x})$ given by the intersection of two conics.*

Proof. See the proof of Proposition 2.2.6. □

For $g \geq 3$ we will construct some kind of rational curves on $X = M_{2;\mathscr{D},1}(C)$ which is called the Hecke curves. There are two equivalent constructions:

(M1) Pick $[W] \in X$ which is $(1,1)$-stable, that is, any sub-line-bundle has degree < 0, is dense in X by [32]. Consider $\pi : \mathbf{P}(W) \to C$ and $\eta \in \mathbf{P}(W)$ with $y = \pi(\eta) \in C$.

First we get a new bundle W^η of rank 2:

$$0 \to W^\eta \to W \to \mathscr{O}_y \otimes (W_y/\eta) \to 0.$$

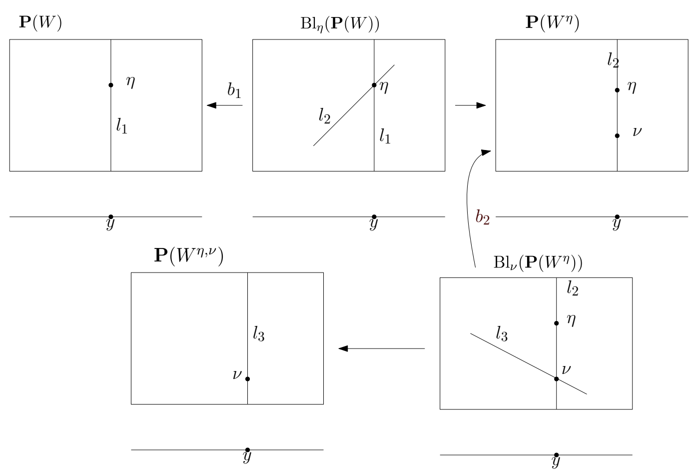
Hence $\deg((W^\eta)^\vee) = \det(W)^{-1} \otimes \mathscr{O}(y)$. Now for any $\nu \in \mathbf{P}((W^\eta)^\vee_y)$ we have another new bundle V^ν of rank 2:

$$0 \to V^\nu \to (W^\eta)^\vee \to \mathscr{O}_y \otimes ((W^\eta)^\vee_y/\nu) \to 0.$$

So $\det(V^\nu)^\vee = \det W$ and V^ν is stable. Then $\{(V^\nu)^\vee : \nu \in \mathbf{P}((W^\eta)^\vee_y)\}$ is a rational curve on X.

Since by dual we have $0 \to W^\vee \xrightarrow{f} (W^\eta)^\vee \to \mathscr{O}_y = \mathrm{Ext}^1 \to 0$. Let $\nu' = \mathrm{coker} f$ and then $W \cong (V^{\nu'})^\vee$. Hence $\{(V^\nu)^\vee : \nu \in \mathbf{P}((W^\eta)^\vee_y)\}$ is a rational curve on X passing through W which is the Hecke curve.

(M2) This is more geometric. Pick $[W] \in X$ which is $(1,1)$-stable and the process as follows:



Consider the blow-up $b_1 : \mathrm{Bl}_\eta(\mathbf{P}(W)) \to \mathbf{P}(W)$ over $\eta \in \mathbf{P}(W)$ over $y \in C$ with fiber $l_1 = \mathbf{P}(W_y)$. The exceptional divisor is $l_2 \cong \mathbf{P}(T_\eta \mathbf{P}(W))$. The strict transform of l_1 is a (-1)-curve since $0 = (l_1 + l_2)^2$. Hence blow-down the l_1 we get a new ruled surface $\mathbf{P}(W^\eta)$. For the choose of tangent direction $\nu \in l_2 = \mathbf{P}(T_\eta \mathbf{P}(W)) = \mathbf{P}(W^\eta_y)$, we blow-up ν again and we get $b_2 : \mathrm{Bl}_\nu(\mathbf{P}(W^\eta)) \to \mathbf{P}(W^\eta)$

and blow-down via (-1)-curve l_2 and we get a new ruled surface $\mathbf{P}(W^{\eta,\nu})$. When ν is tangent to l_1, then we have $W^{\eta,\nu} = W$. Hence $\{W^{\eta,\nu} : \nu \in \mathbf{P}(T_\eta \mathbf{P}(W))\}$ is a rational curve on X passing $[W]$.

Proposition 2.2.11. *Consider a smooth projective curve C of genus $g \geq 3$. and the moduli space $X = M_{2;\mathscr{D},1}(C)$ of stable bundles of rank 2 and degree 1. Let $\mathscr{L}$ be the ample generator of Picard group, then $-K_X = 2\mathscr{L}$ and Hecke curves have degree 2 with respect to $\mathscr{L}$. Hecke curves are smooth in the smooth lucus of X. Moreover, Hecke curves are minimal rational curves on X.*

Proof. We refer [32] for the proof of that fact that $-K_X = 2\mathscr{L}$, Hecke curves have degree 2 with respect to $\mathscr{L}$ and Hecke curves are smooth in the smooth lucus of X.

For the last statement, the original proof is Proposition 8 in [18]. The basic idea is follows. We just need to show that there are no rational curves of degree 1. By Kodaira's stability, if a rational curve of degree 1 exists at a generic point of X for some C, such a curve exists at a generic point of X for any C of the same genus. So if a rational curve of degree 1 exists at a generic point of X for our C, then pick a hyperelliptic curve C' and its X' is also in this case. But in the hyperelliptic case X' is the set of $(g-2)$-dimensional linear subspaces in the intersection of two quadrics in $\mathbb{P}^{2g+1}$ determined by the hyperelliptic curve, see Theorem 1 in [9]. If lines exist through generic points of X', we have at least a $(3g-3)-(g-1) = (2g-2)$-dimensional family of $(g-1)$-dimensional linear subspaces in the intersection of the two quadrics. By Theorem 2 in [9] the set of $(g-1)$-dimensional linear subspaces of the intersection of the two quadrics is equivalent to the Jacobian of C' which has dimension g. Hence this is impossible since $g \geq 3$. □

Proposition 2.2.12. *Consider a smooth projective curve C of genus $g \geq 3$. and the moduli space $X = M_{2;\mathscr{D},1}(C)$ of stable bundles of rank 2 and degree 1.*

(a) Then for any $(1,1)$-stable $[W] \in X$, the Hecke curves associated to two distinct η_1, η_2 are distinct rational curves on X.

(b) We have $\mathcal{K}_{[W]} \cong \mathbf{P}_C(W) = \mathbb{P}(W^\vee)$ and the tangent morphism $\tau_{[W]} : \mathcal{K}_{[W]} \to \mathbb{P}(\Omega^1_{X,[W]})$ is given by the linear system $\pi^* K_C \otimes T_{\mathbf{P}_C(W)/C} = 2\pi^* K_C - K_{\mathbf{P}_C(W)}$. Moreover $\mathcal{C}_{[W]}$ is nondegenerate in $\mathbb{P}(\Omega^1_{X,[W]})$.

Proof. For (a) this is 5.13 in [32] and we omit it.

For (b), we give the main idea and the details we refer Proposition 11 in [18]. By (a) we know that the set of Hecke curves is just $\mathbf{P}_C(W) \subset \mathcal{K}_{[W]}$. As $\dim \mathcal{K}_{[W]} = \dim \mathbf{P}_C(W) = 2$ we have $\mathcal{K}_{[W]} \cong \mathbf{P}_C(W) = \mathbb{P}(W^\vee)$. Moreover, by Euler sequence we have $\pi_* T_{\mathbf{P}_C(W)/C} = \mathrm{ad}(W)^\vee$, then traceless endomorphism bundle of W, and $R^1\pi_* T_{\mathbf{P}_C(W)/C} = 0$ where $\pi : \mathbf{P}_C(W) \to C$. As the tangent space of X is just $H^1(C, \mathrm{ad}(W))$, we have the tangent morphism $\tau_{[W]} : \mathbf{P}_C(W) \to \mathbf{P}H^1(C, \mathrm{ad}(W))$. As

$$H^0(\mathbf{P}_C(W), \pi^* K_C \otimes T_{\mathbf{P}_C(W)/C}) = H^0(C, K_C \otimes \mathrm{ad}(W)^\vee) \cong H^1(C, \mathrm{ad}(W))^\vee,$$

it is not different to see that $\tau_{[W]}$ is given by the linear system $\pi^* K_C \otimes T_{\mathbf{P}_C(W)/C}$. $\square$

2.2.5 Need to add

2.3 Distribution and Its Basic Properties

Definition 2.3.1. *Let X be a smooth uniruled variety with fixed minimal rational component $\mathcal{K}$. For general $x \in X$ we have VMRT $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$. Consider its linear span $W'_x \subset T_x X$. As x varies over an zariski open subset (which is our maining of general) U we have a subbundle $W' \subset T_U$. Define its annihilator $(W')^\perp \subset \Omega^1_X$ and the annihilator $W \subset T_V$ (saturation of W') of $(W')^\perp \subset \Omega^1_X$ where V is a open subset of codimension ≥ 2.*

Lemma 2.3.2. *Given any subset $E \subset X$ of codimension ≥ 2, we can find a standard minimal rational curve disjoint from E.*

Proof. Choose a standard minimal rational curve C through a general point $x \notin E$. Let $N_C \cong \mathscr{O}(1)^{\oplus p} \oplus \mathscr{O}^{\oplus n-1-p}$ and choose sections $\sigma_1, ..., \sigma_p$ of N_C correspond to the independent sections of $\mathscr{O}(1)^{\oplus p}$ vanishing at x, and sections $\sigma_{p+1}, ..., \sigma_{n-1}$ generates $\mathscr{O}^{\oplus n-1-p}$. Since no obstruction, we have a $(n-1)$-dimensional deformation of C whose initial velocities are contained in the linear span of $\sigma_1, ..., \sigma_{n-1}$. If all members meets E, this means we have a 1-dimensional subfamily passing through a given point $y \in E$ since $\operatorname{codim} E \geq 2$. Hence in the linear span of $\sigma_1, ..., \sigma_{n-1}$ there exists a non-zero section vanishing at y. But this is impossible since $\sigma_1, ..., \sigma_{n-1}$ are pairwisely independent outside x. Hence well done. $\square$

2.3.1 Levi Tensor of the Distribution

Definition 2.3.3. *Fix a distribution $\mathcal{D} \subset T_M$ for a complex manifold. For any $x \in M$ and any two vectors $u, v \in \mathcal{D}_x$, let their local sections $\tilde{u}, \tilde{v}$. Then we define the* Levi tensor *of $\mathcal{D}$, which is a section of $\mathscr{H}om\left(\bigwedge^2 \mathcal{D}, T_M/\mathcal{D}\right)$, as*

$$\mathsf{Levi}^{\mathcal{D}}_x(u, v) := [\tilde{u}, \tilde{v}]_x \pmod{} D_x.$$

Remark 2.3.4. *In the old survey [19], this is called the Frobenius bracket tensor.*

Proposition 2.3.5. *Let X be a smooth uniruled variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ associated to a distribution W. If W is a proper distribution, then it is not integrable at general points.*

Proof. For the whole proof we refer Proposition 2.2 in [19]. Here we give some idea. If W is integrable, then by Frobenius theorem it defines a non-trivial foliation on $X \backslash E$ for some codim$E \geq 2$. By some argument one can compactify the leaves of this foliations into algebraic subvarieties.

Pick a Chow schemes Chow_W of compactifications of these leaves. Choosing a hypersurface H in Chow_W generically, we get a hypersurface L in X which is the closure of the codimension 1 part of the union of compactificated leaves corresponding to H. A generic member of $\mathcal{K}$ lies in a leaf of $\mathcal{D}$ but is disjoint from H, hence disjoint from L, a contradiction to the Picard number condition on X. □

Proposition 2.3.6. *Let X be a smooth uniruled variety with fixed minimal rational component $\mathcal{K}$ associated to a distribution W. Let $\mathcal{T}_x \subset \mathsf{Grass}(1, \mathbf{P}(W_x)) \subset \mathbf{P}\left(\bigwedge^2 W_x\right)$ be lines tangent to the smooth locus of $\mathcal{C}_x$. Then $\mathcal{T}_x$ is contained in the projectivization of the kernel of the Levi tensor $\mathsf{Levi}_x^W(-,-) : \bigwedge^2 W_x \to T_xX/W_x$.*

Proof. By Proposition 2.1.15 we just need to show that $\mathsf{Levi}_x^W(\alpha, \beta) = 0$ for any $\alpha \in W_x$ correspond to the general point of $\mathcal{C}_x$ and $\beta \in T_xX_\alpha^+$. So WLOG we let both of them are non-zero. Hence we just need to show that there is a local complex analytic surface through x tangent to W in the neighborhood of x whose tangent space at x containing α, β.

Choose a standard rational curve C through x whose tangent vector is α (as α general) and fix $y \in C$ with $x \neq y$. Now β correspond to the positivr part of $T_X|_C$, thus there exists a non-zero section σ of the normal bundle so that $\sigma(y) = 0$ and $\sigma(x) = \beta$. As $H^1(C, N_C \otimes \mathfrak{m}_y) = 0$, we can find a deformation C_t of C fix y with initial velocity β. This fomrs a local complex analytic surface through x whose tangent space at x spanned by α, β. Moreover its tangent space at z near x spanned by T_zC_t and $\sigma_t(z)$ where $\sigma_t \in H^0(C_t, N_{C_t} \otimes \mathfrak{m}_y)$. By Proposition 2.1.15 again we know that σ_t in the tangent space of $\mathcal{C}_z$, hence in W_z. Hence this surface tangent to W. Well done. □

2.3.2 Nondegeneracy of the Distribution

In this small section we will consider when the VMRT $\mathcal{C}_x$ is nondegenerate.

Proposition 2.3.7. *Let W be a vector space with a non-linear cone $J \subset W$ such that $\dim J > \frac{1}{2}\dim W$ and $\mathbf{P}(J)$ is a smooth subvariety of $\mathbf{P}(W)$. Let $\mathcal{T} \subset \mathbf{P}\left(\bigwedge^2 W\right)$ be the variety of tangent lines of $\mathbf{P}(J)$, then $\mathcal{T}$ is nondegenerate in $\mathbf{P}\left(\bigwedge^2 W\right)$.*

Proof. This is a boring result deduced by dimension-counting and Zak's theorem in the projective geometry about tangencies. We refer the proof of Proposition 2.6 in [19]. □

Theorem 2.3.8. *Let X be a smooth uniruled variety of Picard number 1 and dimension n with $\dim \mathcal{C}_x = p > \frac{n-3}{2}$, then if $\mathcal{C}_x$ is smooth for some general point, then it is nondegenerate.*

Proof. If it is degenerate, defining the non-trivial distribution W of rank $m < n$. Since $\mathcal{C}_x$ is smooth and $\dim \mathcal{C}_x = p > \frac{n-3}{2}$, the Levi tensor of W vanish identically by Proposition 2.3.6 and 2.3.7. But by Proposition 2.3.5 this is impossible! □

Corollary 2.3.9. *Let X be a prime smooth Fano variety of dimension n with $\mathsf{Index}(X) > \frac{n+1}{2}$, then the VMRT is nondegenerate.*

Proof. This follows directly from this Theorem and Corollary 2.1.14. □

2.3.3 Cauchy Characteristic of the Distribution

Definition 2.3.10. *Let a distribution $\mathcal{D}$ on a complex manifold X regarded as a subsheaf of T_X. The Cauchy characteristic of $\mathcal{D}$ is a subsheaf defined as*

$$\mathsf{Ch}(\mathcal{D})(U) := \{f \in \mathcal{D}(U) : \mathsf{Levi}^{\mathcal{D}}(f, g) = 0, \forall g \in \mathcal{D}(U)\}.$$

Remark 2.3.11. *Actually $\mathsf{Ch}(\mathcal{D})$ is a integrable distribution over the open subset where it is locally free.*

Lemma 2.3.12. *Let $g : M \to N$ be a submersion of complex manifolds so that the fibers of g define a distribution $\ker(dg)$ on M. Let $\mathcal{D}$ be a distribution on N, define the pull-back distribution is $(g^*\mathcal{D})_m = (dg)^{-1}(\mathcal{D}_{g(m)})$. Then we have*

$$\ker(dg) \subset \mathsf{Ch}(g^*\mathcal{D}).$$

Proof. Almost trivial. Omitted. □

Proposition 2.3.13. *Let X be a smooth Fano variety of Picard number 1. Consider the total VMRT*

$$\mathcal{C} := \overline{\bigcup_{x \text{ general}} \mathcal{C}_x} \subset \mathbb{P}(\Omega^1_X)$$

and consider the universal cycle morphisms $\mathcal{K} \xleftarrow{\rho} \mathcal{U} \xrightarrow{\mu} X$. Note that the normalization $(\mu^{-1}(x))^n = \mathcal{K}_x$ and the tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ induce a rational map $\tau : \mathcal{U} \dashrightarrow \mathcal{C}$ which is generically finite. The image of τ of fibers of ρ induce a multi-valued foliation $\mathscr{F}$ and the leaf of it is the lift of the minimal rational curve to its tangent vectors.

Define a distribution $\mathcal{P}$ of rank $2p + 1$ on generic part of $\mathcal{C}$ as

$$\mathcal{P}_\alpha := (d\pi)^{-1}(\mathbb{P}((T_x X^+_\alpha)^\vee))$$

where $\pi : \mathcal{C} \to X$ sends $\alpha \mapsto x$.

Now choose an analytic open subspace $O \subset \mathcal{U}$ such that $\tau|_O$ is biholomorphic, we can regard O as an open subset of $\mathcal{C}$ and $\mathscr{F}$ be a univalent foliation on O. If $\mathcal{C}_x$ has generically finite Gauss map for general $x \in O$, then $\mathscr{F} = \mathsf{Ch}(\mathcal{P})$ on O.

Remark 2.3.14. *Let us examine what the condition on Gauss map means in this remark.*

It is perhaps easier to look at the affine case. So let $Z \subset \mathbb{A}^n_{\mathbb{C}}$ be an affine variety of dimension m and let $z \in Z$ be a generic smooth point. Let $z_1, ..., z_m$ be a local coordinate system of Z at z and $w_1, ..., w_n$ be an affine coordinate system. The Gauss map of Z is just associating to z its tangent space $T_z(Z)$. If the Gauss map is not generically finite, its differential has kernel in a neighborhood of z. Let $v \in T_z(Z)$ be in the kernel of the differential of the Gauss map. This means that in the direction of v, the tangent spaces $T_z(Z)$ remain constant to the first order as x varies in a neighborhood of z.

In particular, for any local vector field ω on Z as $\omega = \sum_i a_i(z_1, ..., z_m)\frac{\partial}{\partial w_i}$ and its derivative in the direction of v is $D_v\omega = \sum_i v(a_i(z_1, ..., z_m))\frac{\partial}{\partial w_i}$ also tangent to Z at z.

Conversely, one can see that if v is a tangent vector to Z at z so that $D_v\omega(z) \in T_zZ$ for any local vector field ω on Z, then v is in the kernel of the differential of the Gauss map. This can be applied to a projective subvariety of $\mathbb{P}^{n-1}$ by taking its affine cone.

Sketched proof of Proposition 2.3.13. Now assume all we work are on O.

One one side (without assuming the Gauss map), if we definr the distribution $\mathcal{Q}$ generically on $\mathcal{K}$ as $\mathcal{Q}_{[C]} = H^0(C, \mathscr{O}(1)^{\oplus p}) \subset T_{[C]}\mathcal{K} = H^0(C, N_C)$. then by Proposition 2.1.15 we have $\mathcal{P} = \rho^*\mathcal{Q}$. By Lemma 2.3.12, we have $\mathscr{F} \subset \mathsf{Ch}(\rho^*\mathcal{Q}) = \mathsf{Ch}(\mathcal{P})$.

Conversely, if there exists a vector in $\mathsf{Ch}(\mathcal{P})_\alpha$ not in $\mathscr{F}_\alpha$, then there must a vector v tangent to the fibers of $\pi : \mathcal{C} \to X$, that is, $v \in T_\alpha\mathcal{C}_x$ where $x = \pi(\alpha)$ by Jacobi indentity. The condition $v \in \mathsf{Ch}(\mathcal{P})_\alpha$ as $\mathsf{Levi}^{\mathcal{P}}_\alpha(v, \mathcal{P}) \subset \mathcal{P}$. Hence

$$\mathsf{Levi}^{\mathcal{P}}_\alpha(v, \mathcal{P} \cap T_{\mathbb{P}(\Omega^1_{X,x})}) \subset \mathcal{P} \cap T_{\mathbb{P}(\Omega^1_{X,x})}.$$

As $\mathcal{P}_\alpha \cap T_\alpha\mathbb{P}(\Omega^1_{X,x}) = T_\alpha(\mathcal{C}_x)$, we have $\mathsf{Levi}^{\mathcal{P}}_\alpha(v, T_{\mathcal{C}_x}) \subset T_{\mathcal{C}_x}$. Hence v is must in this kernel of the Gauss map since $v \in T_\alpha\mathcal{C}_x$. Hence well done. □

2.4 Cartan-Fubini Type Extension Theorem

2.4.1 Some History

In this small section we will follows the introduction survey [20]. The beginning of these problems is the following theorem due to Liouville:

Theorem 2.4.1 (Liouville)**.** *For any conformal map $f : U_1 \to U_2$ between two domains in sphere S^n for $n \geq 2$, there is a Möbius transformation $f : S^n \to S^n$ satisfying $f = F|_{U_1}$.*

As a natural extension in the projective geometry, we may ask:

Theorem 2.4.2. *For any holomorphic conformal map $f : U_1 \to U_2$ between two domains in $\mathbb{Q}^n, n \geq 3$, there is a biholomorphic automorphism $F \in \mathrm{Aut}(\mathbb{Q}^n)$ satisfying $f = F|_{U_1}$.*

As a generalization, we consider the following theorems:

Theorem 2.4.3 (Fubini-Cartan-Jensen-Musso)**.** *Let $X_1, X_2 \subset \mathbb{P}^{n+1}$ be two smooth hypersurfaces of degree $d \geq 2$. If a biholomorphic map $f : U_1 \to U_2$ between two domains $U_1 \subset X_1$ and $U_2 \subset X_2$ preserves the structures given by both the second fundamental form and the Fubini cubic form, then there is a biholomorphic morphism $F : X_1 \to X_2$ satisfying $f = F|_{U_1}$.*

In our sense of VMRT, we may consider the following questions:

Problem 2.4.1. *Let X be a smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}$ so that the VMRT $\mathcal{C}_x$ at a general point $x \in X$. Does $\mathcal{C}_x$ determine X in the following sense:*

Let X' be any smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}'$ for which we denote the VMRT $\mathcal{C}'_{x'}$ for general $x' \in X'$. Suppose there exists connected analytic open subsets $U \subset X, U' \subset X'$ and a biholomorphic map $\phi : U \to U'$ with isomorphism $\psi : \mathbf{P}T_U \to \mathbf{P}T_{U'}$ compactible with ϕ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}'_{\phi(x)}$ for general $x \in U$. Do we have a biholomorphic map $X \to X'$?

This question is not right for the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of odd degree d over a smooth projective curve C of genus $g = 2$.

This question is right for $\mathbb{P}^n$ by Cho-Miyaoka and right for any irreducible Hermitian symmetric space by Hwang-Mok.

2.4.2 The Main Result

We will follows the survey [19] and paper [22]. We consider the following theorem due to Hwang-Mok:

Theorem 2.4.4 (Cartan-Fubini Type Extension Theorem)**.** *Let X be a smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}$ so that the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is of positive dimension $p > 0$ and the Gauss map of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ is generically finite.*

Let X' be any smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}'$ for which we denote the VMRT $\mathcal{C}'_{x'}$ for general $x' \in X'$.

Suppose there exists connected analytic open subsets $U \subset X, U' \subset X'$ and a biholomorphic map $\phi : U \to U'$ so that the differential $\phi_* : \mathbf{P}T_U \to \mathbf{P}T_{U'}$ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}'_{\phi(x)}$ for general $x \in U$, then ϕ can extended to a biholomorphic map $X \to X'$.

Remark 2.4.5. *Several remarks:*

(a) *Althrough this theorem is not true for projective space (note that the Gauss map is not generically finite), the Problem 2.4.1 is true for projective space.*

(b) *Actually the Gauss map of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ is generically finite (actually finite by Zak's results) for any non-linear smooth projective variety, see [12]. Hence the theorem is right for any examples we want to see, except projective space, with* $p > 0$.

Sketched proof of Theorem 2.4.4. We will follows the sketch in [19] Theorem 3.2 and we refer the detailed proof in [22]. We will follows the several steps.

▶ **Step 1. Show the map ϕ sends local pieces of $\mathcal{K}$ in U to local pieces of $\mathcal{K}'$ in U'.**

Consider the Proposition 2.3.13, then since ϕ_* sends $\mathcal{C}|_U$ to $\mathcal{C}'|_{U'}$, then it sends $\mathcal{P}$ to $\mathcal{P}'$. Hence it sends $\mathscr{F}$ to $\mathscr{F}'$. Well done.

▶ **Step 2. To extend the domain of ϕ from the analytic open set to an étale open set.**

Suppose C the standard minimal rational curve intersecting U. ϕ is defined on $C \cap U$ and we want to extend it to other points on C. To define the extension at a point $y \in C$, consider a deformation C_t of C fixing the point y since $p > 0$. Now consider the local pieces $U \cap C_t$. By Step 1, $\phi(U \cap C_t)$ is a local piece of some minimal rational curve C'_t belonging to $\mathcal{K}'$. We claim that these curves C'_t have a unique common point y'.

Indeed the common point y' exists because it exists when y is chosen to be inside U. It is unique because C'_t do not have deformations fixing two or more points. In fact, if such a deformation exists, then its initial velocity is a section of the normal bundle of a standard minimal rational curve vanishing at two or more points, a contradiction to the splitting type. Hence we proved the claim. Hence we can define y' as the image of y and then we can extend ϕ along standard minimal rational curves intersecting U (this has some problems, but we have shown in bold font below). This enlarges the domain of definition of ϕ to a bigger open set $\widehat{U}$. Applying the same argument to $\widehat{U}$, we can analytically continue along standard minimal rational curves intersecting $\widehat{U}$.

We can repeat this procedure until the domain of definition covers a Zariski open subset in X. But **there is a gap in this extension argument. A point outside U may belong to different standard minimal rational curves intersecting U. So when we carry out the analytic continuation, we end up with a multi-valued extension of ϕ.** So what we get at the end is a multi-valued extension of ϕ over an étale open subset $\widetilde{U}$ of X, namely a quasi-projective variety $\widetilde{U}$ with an étale morphism $\widetilde{U} \to X$ covering a Zariski open subset of X and a morphism $\widetilde{\phi} : \widetilde{U} \to X'$ extending ϕ. We skipped many technique things and we refer [22].

▶ **Step 3. To extend the domain from the étale open set to a Zariski open set.**

To extend $\widetilde{\phi}$ to a morphism Φ_0, defined on a Zariski open subset X_0 of X, we have to reduce the multi-valuedness of $\widetilde{\phi}$. First of all, we can reduce the multi-valuedness of $\widetilde{\phi}$ by identifying two points $u_1, u_2 \in \widetilde{U}$ if $\nu(u_1) = \nu(u_2)$ and $\widetilde{\phi}(u_1) = \widetilde{\phi}(u_2)$ where $\nu : \widetilde{U} \to X$ be that étale morphism. So let us assume that there is no such two distinct points. Then we claim that ν must be 1-to-1.

Indeed, if not then by Lemma 2.4.7 we can choose a standard minimal rational curve C generically and pick a generic point $x \in C$. Then there exists an irreducible component C' of $\nu^{-1}(C)$ containing a pair of points $u_1, u_2 \in \widetilde{U}$ with $\nu(u_1) = \nu(u_2) = x$ and $\widetilde{\phi}(u_1) \neq \widetilde{\phi}(u_2)$. Now let C_t be a deformation of C with x fixed, which exists by $p > 0$, then their inverse images under ν contains components C'_t which are deformations of C' fixing u_1 and u_2. Then their images under $\widetilde{\phi}$ define a family of standard rational curves in X' fixing two distinct points $\widetilde{\phi}(u_1) \neq \widetilde{\phi}(u_2)$, a contradiction. This finishes Step 3.

▶ **Step 4. To extend the domain from the Zariski open set to the whole Fano manifold X.**

By applying the same extension to $\phi^{-1} : U' \to U$, we see that the rational map Φ_0 in Step 3 is birational. For Step 4, if Φ_0 has exceptional set $E \subset X$ of codimension 1 which is contracted to a set $Z \subset X'$ of codimension 2. From the Picard number condition, all members of $\mathcal{K}$ intersect E. It follows that generic members of $\mathcal{K}$ must intersect Z, a contradiction to Lemma 2.3.2. Hence Φ_0 is a birational map with no exceptional set.

Hence Φ_0 induce the isomorphisms $H^0(X, -mK_X) \cong H^0(X', -mK_{X'})$. Hence Φ_0 induce

$$\Phi : X \cong \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, -mK_X) \cong \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X', -mK_{X'}) \cong X'.$$

Well done. □

Remark 2.4.6. *In the proof, the hypothesis of Gauss map is used only in step 1 and the hypothesis of $p > 0$ is used only in step 2,3.*

Lemma 2.4.7. *Let $\pi : Y \to X$ be a generically finite morphism from a normal variety Y onto a Fano manifold X with Picard number 1. Suppose for a generic standard rational curve $C \subset X$ belonging to a chosen minimal rational component, each component of the inverse image $\pi^{-1}(C)$ is birational to C by π. Then $\pi : Y \to X$ itself is birational.*

Proof. Let π is not birational. By Stein factorization π can be factored into $Y \xrightarrow{g} Y' \xrightarrow{h} X$ where g is birational and h is finite. By Proposition 1.5.2(a) we know that h is not étale. Hence we can choose a ramification divisor $R \subset Y$ such that $\pi(R) \subset X$ is also a divisor.

By genericity of C, we may assume that $\pi^{-1}(C)$ lies on the smooth part of the normal variety Y. Let C_1 be any irreducible component of $\pi^{-1}(C)$. Then C_1 is also a rational curve and deformations of C_1 give deformations of C since $\pi|_{C_1}$ is birational. It follows that the space of deformations of C and the space of deformations of C_1 have equal dimensions. So we have $K_X \cdot C = K_Y \cdot C_1$. This implies C is disjoint from the ramication divisor R. Since this holds for any components of $\pi^{-1}(C)$, we know that C is disjoint from $\pi(R)$. But this is impossible by the assumption that X is of Picard number 1. □

2.4.3 More Comments

We may ask what is the difference between Problem 2.4.1 and Theorem 2.4.4. We will follows [19] and consider the case $X = \mathbb{Q}^n \subset \mathbb{P}^{n+1}$. For more general setting and more detailed computations about conformal differential geometry we refer paper [21].

Actually the VMRT is a hyperquadric in $\mathbb{P}(\Omega^1_{X,x})$. Hence they generate a subbundle of $\mathbb{P}\Omega_X$ with fibers isomorphic to hyperquadrics. This gives a conformal structure on X.

Definition 2.4.8. *A conformal structure on a complex manifold M is vector bundle morphism $\sigma : \mathrm{sym}^2 T_M \to \mathscr{L}$ for some line bundle $\mathscr{L}$ which gives a nondegenerate symmetric bilinear form at each fiber T_xM.*

The null-cone $\mathcal{C} \subset \mathbf{P}T_M$ is the zero locus of bilinear form σ whose fibers are $\mathcal{C}_x \subset \mathbf{P}T_xM$.

After choose a local trivialization of $\mathscr{L}$, we have locally

$$\sigma = \sum_{ij} g_{ij}(z) dz^i \otimes dz^j$$

for local coordinates $z_1, ..., z_n$ and (g_{ij}) are nondegenerate symmetric matrix. Consider the curvature tensor

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial z^k} - \frac{\partial \Gamma^i_{jk}}{\partial z^l} + \sum_{\mu} (\Gamma^{\mu}_{jl}\Gamma^i_{\mu k} - \Gamma^{\mu}_{jk}\Gamma^i_{\mu j}) = \mathsf{Weyl} + m\mathsf{Ric} + n\mathsf{Sca}.$$

Also, the geodesic defined by $\frac{d^2\gamma^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$. Althrough R^i_{jkl} depends on the choice of the trivialization, the Weyl tensor Weyl is not and so is the geodesics which tangent to the null-cone (null-geodesics). If $\mathsf{Weyl} = 0$ we say the conformal structure is flat.

For our X, the conformal structure given by the VMRT is flat which can be seen by the choise of a flattening coordinate system! This is an example of Harish-Chandra coordinate on Hermitian symmetric spaces. In this case the minimal rational curves are null-geodesics.

In the sense of Theorem 2.4.4, $\phi_* \mathcal{C}_x = \mathcal{C}'_{\phi(x)}$ means the conformal structure defined at generic points on X' is flat. Hence the difference between Problem 2.4.1 and Theorem 2.4.4 is just the Weyl tensor Weyl.

Now we give an very special example which shows how to use VMRT to handle the curvature:

Example 2.4.1. *Let X be a Fano manifold of Picard number 1 with VMRT are hyperquadric. Hence we have a conformal structure given on a Zariski open set of X. No we assume that the conformal structure can be extends to the whole X. Then the Weyl tensor $\mathsf{Weyl} \in H^0(X, \bigwedge^2 \Omega_X \otimes \mathscr{E}nd(T_X))$ vanish.*

Proof. Consider a standard minimal rational curve C and $T_X|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-2} \oplus \mathscr{O}$ in this case. We need to show $\mathsf{Weyl}(u \wedge v) \in \mathrm{End} T_x X$ vanish at all $u, v \in T_x X$. By Proposition 2.3.7 we just need to consider $u \wedge v$ for $u \in \mathcal{C}_x$ and $v \in T_u \mathcal{C}_x$. Let u in the $\mathscr{O}(2)$-part vanish at two points and v vanish at one point. Hence $\mathsf{Weyl}(u \wedge v)$ has three zeros. If it is not zero, then since $\mathsf{Weyl}(u \wedge v)$ be a section of $\mathscr{E}nd(T_X|_C) = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus 2n-4} \oplus \cdots$. Hence if it is not zero, then it can not have three zeros! $\square$

Chapter 3

Some Basic Applications of VMRT

3.1 Stability of the Tangent Bundles

3.1.1 Basic Facts about Stability of the Tangent Bundles

Proposition 3.1.1 (Simpleness)**.** *Let X be a smooth uniruled variety. If the VMRT $\mathcal{C}_x$ is irreducible and nondegenerate for some choise of minimal rational component, then T_X is simple.*

Proof. Let $\xi \in \mathrm{End}(T_X)$. Let x general and $v \in T_xX$ be a tangent vector to standard minimal rational curve C through x. Consider the extended vector field $\tilde{v}$ on C having two distinct zeroes. Then $\xi(\tilde{v}) \in \Gamma(T_X|_C)$ vanishing at two distinct points. As C is standard, then either $\xi(\tilde{v}) = 0$ or $\xi(\tilde{v})$ is proportional to $\tilde{v}$. Hence v is the eigenvector of ξ in T_xX. As this is true for any choice of v tangent to some standard minimal rational curve C through x and since $\mathcal{C}_x$ is nondegenerate, then ξ act as scalar multiplication in T_xX. Since $\xi(\tilde{v})$ is the constant multiple of $\tilde{v}$, the eigenvalues must be constant on C. Hence ξ must be a scalar multiplication and T_X is simple. $\square$

Now we consider the stability of tangent bundles. We will follows Section 2.4 in the survey [19] and the paper [17]. This is a standard method developed in [17]. Note that the results in this small section hold for any rational component $\mathcal{K}'$ of Chow schemes but we don not care.

Now we will assume X be an n-dimensional smooth Fano variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ of degree $p+2$. Then to show the stability of T_X we just need to show that for any subsheaf $\mathscr{F} \subset T_X$ of rank $1 \leq k \leq n-1$ we have $\frac{c_1(\mathscr{F})\cdot(-K_X)^{n-1}}{k} < \frac{c_1(T_X)\cdot(-K_X)^{n-1}}{n}$. As Picard number is 1, we can check this over a generic standard minimal rational curve C. Hence for a sheaf $\mathscr{F}$ of rank r, which can be assumed to be locally free over C by Lemma 2.3.2, we can define $\mu(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot C}{r}$. Note that $\mu(\mathscr{F})$ depends only on $\mathscr{F}$ and $\mathcal{K}$ and does not depend on the choice of C. For example $\mu(T_X) = \frac{p+2}{n}$.

Example 3.1.1 (Baby version for $\mathbb{P}^n$)**.** *We will show that $T_{\mathbb{P}^n}$ is stable. For any subsheaf $\mathscr{F} \subset T_{\mathbb{P}^n}$ Choose a generic line C, so that $\mathscr{F}|_C$ is a vector bundle and splits as $\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ where $a_1 \geq \cdots \geq a_r$. Since $T_{\mathbb{P}^n}|_C \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$, if $\mu(\mathscr{F}) \geq \mu(T_X) = \frac{n+1}{n}$, then $a_1 = 2$. This implies that the line C is tangent to the distribution $\mathscr{F}$. But this is true for any generic choice of C. Hence $\mathscr{F}$ must have rank n, and we are done.*

Proposition 3.1.2. *Suppose that T_X is not stable (resp. not semi-stable). Then we can find a subsheaf $\mathscr{F} \subset T_X$ of rank $r, 1 < r < n$ with torsion free quotient $T_X/\mathscr{F}$, satisfying $\mu(\mathscr{F}) \geq \mu(T_X)$ (resp. $\mu(\mathscr{F}) > \mu(T_X)$), whose Levi tensor $\mathsf{Levi}^{\mathscr{F}}$ vanishes for general x.*

Proof. Consider a subsheaf $\mathscr{F} \subset T_X$ of rank r smaller than n with maximal values of $\mu(\mathscr{F}) \geq \mu(T_X) > 0$. Moreover, we can choose such $\mathscr{F}$ so that $T_X/\mathscr{F}$ is torsion free. In fact, if $T_X/\mathscr{F}$ has torsion $(T_X/\mathscr{F})_{\mathrm{Tor}}$ for a such choice of $\mathscr{F} \subset T_X$, the inverse image $\mathscr{F}'$ of $(T_X/\mathscr{F})_{\mathrm{Tor}}$ in T_X under the quotient map is a subsheaf of rank r with $\mu(\mathscr{F}') \geq \mu(\mathscr{F})$, and we may choose $\mathscr{F}'$ as our $\mathscr{F}$.

First we have $r > 1$. Indeed, if $r = 1$ then $\mathscr{F}^{\vee\vee}$ is an ample line subbundle of T_X (since Picard number is 1) and hence X is a projective space by Theorem 2.2.4, a contradiction to the assumption that T_X is not stable.

By the choice, $\mathscr{F}$ is semi-stable and $\bigwedge^2 \mathscr{F}$ is also semi-stable. Let the image of the Levi tensor $\mathsf{Levi}^{\mathscr{F}} : \bigwedge^2 \mathscr{F} \to T_X/\mathscr{F}$ is $\mathscr{G}$. If it has positive rank, by semi-stability, we have $\mu(\mu(G)) \geq \mu\left(\bigwedge^2 \mathscr{F}\right) = 2\mu(\mathscr{F}) > \mu(\mathscr{F})$.

Suppose the rank of $\mathscr{G}$ is equal to the rank of $T_X/\mathscr{F}$. Then $\mu(\mathscr{G}) \leq \mu(T_X/\mathscr{F}) \leq \mu(T_X) \leq \mu(\mathscr{F})$. A contradiction to $\mu(\mu(G)) > \mu(\mathscr{F})$.

Suppose if $\mathscr{G}$ has positive, but strictly smaller rank than that of $T_X/\mathscr{F}$. let $\mathscr{G}' \subset T_X$ be the kernel sheaf of $T_X \to (T_X/\mathscr{F})/\mathscr{G}$. Let m be the rank of $\mathscr{G}'$ with $r < m < n$. Then

$$\mu(\mathscr{G}') = \frac{r}{m}\mu(\mathscr{F}) + \frac{m-r}{m}\mu(\mathscr{G}) \geq \mu(\mathscr{F})$$

which is a contradiction to the choice of $\mathscr{F}$. $\square$

Proposition 3.1.3. *Let $\mathscr{F}$ be any subsheaf of T_X with rank $< n$. If generic curves in $\mathcal{K}$ are tangent to $\mathscr{F}$, then $\mathscr{F}$ cannot be integrable at generic points.*

Proof. Assume that $\mathscr{F}$ is integrable. Let $Z \subset X$ be the singular loci of the foliation defined by $\mathscr{F}$. The codimension of Z is ≥ 2. Thus a generic member of $\mathcal{K}$ is disjoint from Z (Lemma 2.3.2) and lies in a single leaf of $\mathscr{F}$.

For a given point $x \in X \backslash Z$, let D_x be the set of points which can be joined to x by a connected curve each component of which is a member of $\mathcal{K}$ disjoint from Z. Then D_x is a constructible set (see Section IV.4 in [26]) and the collection of D_x's for generic $x \in X$ defines a meromorphic foliation $\mathcal{D}$ on X. Clearly, D_x is contained in the leaf of $\mathscr{F}$ containing x. It follows that $\mathcal{D}$ is a nontrivial foliation of X. Let $\mathsf{Chow}_{\mathcal{D}}$ be the Chow variety whose generic points corresponds to leaves of $\mathcal{D}$. Choosing a hypersurface H in $\mathsf{Chow}_{\mathcal{D}}$ generically, we get a hypersurface L in X which is the closure of the codimension 1 part of the union of $\mathcal{D}$-leaves corresponding to H. A generic member of $\mathcal{K}$ lies in a leaf of $\mathcal{D}$ but is disjoint from H, hence disjoint from L, a contradiction to the Picard number condition on X. □

Corollary 3.1.4. *For the choice of Proposition 3.1.2, we have $\mu(\mathscr{F}) \leq 1$.*

Proof. Let $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ for $a_1 \geq \cdots \geq a_r$. If $\mu(\mathscr{F}) = \sum_{i=1}^r a_i/r > 1$, then $a_1 = 2$ and implying that C is tangent to $\mathscr{F}$. By Proposition 3.1.3 this is impossible. □

Theorem 3.1.5. *If $p = n-1$ or 0, then T_X is stable. If $p = n-2$, then T_X is semi-stable.*

Proof. For $p = n-1, n-2$, this is immediate from $\mu(T_X) = \frac{p+2}{n} \geq 1$ and Corollary 3.1.4. For $p = 0$ assuming that T_X is not stable, choose $\mathscr{F}$ as in Proposition 3.1.2 and choose a generic C from $\mathcal{K}$ so that both $\mathscr{F}$ and $T_X/\mathscr{F}$ are locally free on C. Let $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ for $a_1 \geq \cdots \geq a_r$. As $T_X|_C = \mathscr{O}(2) \oplus \mathscr{O}^{\oplus n-1}$, then $a_1 = 2$ and implying that C is tangent to $\mathscr{F}$. By Proposition 3.1.3 this is impossible. □

Theorem 3.1.6. *Let X be a smooth Fano variety of Picard number 1. Assume that for a general point of the VMRT $\alpha \in \mathcal{C}_x$ and for any $k-1$-dimensional $\mathbb{P}(F_x^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ we have $\dim(\mathbb{P}(F_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) < \frac{k}{n}(p+2) - 1$ where $p = \dim \mathcal{C}_x$. Then T_X is stable.*

Proof. If T_X is not stable, choose $\mathscr{F}$ as in Proposition 3.1.2. For general $C \in \mathcal{K}$ we have $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_k)$ for $a_1 \geq \cdots \geq a_k$. As $\mathscr{F}|_C \subset T_X|_C$ we have $a_1 \leq 2$. If $a_1 = 2$, then C tangent to $\mathscr{F}$ and this is impossible by Proposition 3.1.3. Hence $1 = a_1 = \cdots = a_q > a_{q+1} \geq \cdots$ for some $q \leq k$. As $\mu(\mathscr{F}) = \sum_{i=1}^k \frac{a_i}{k} \geq \mu(T_X) = \frac{p+2}{n}$ and hence $q \geq \frac{k}{n}(p+2)$. Let $x \in C$ general with tangents correspond to $\alpha \in \mathcal{C}_x$, then by definition we have $\dim(\mathbb{P}(\mathscr{F}_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) \geq q - 1 = \frac{k}{n}(p+2) - 1$ which is impossible by the hypothesis. □

Proposition 3.1.7. *Let X be a prime smooth Fano variety of dimension n with $\mathsf{Index}(X) > \frac{n+1}{2}$, then T_X is stable.*

Proof. If not, by Theorem 3.1.6 we have a $k-1$-dimensional $\mathbb{P}(F_x^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ we have $\dim(\mathbb{P}(F_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) \geq \frac{k}{n}(p+2) - 1$ where $p = \dim \mathcal{C}_x$.

Consider the projection $\psi : \mathbb{P}(\Omega^1_{X,x})\backslash\mathbb{P}(F_x^\vee) \to \mathbb{P}^{n-k-1}$ and let q be the dimension of the generic fiber of $\psi|_{\mathcal{C}_x}$. Then $q \geq \frac{k}{n}(p+2)$. Let T be the projective tangent space of $\psi(\mathcal{C}_x)$ at general point $\alpha \in \psi(\mathcal{C}_x)$, then $\dim \psi^{-1}(T) = \dim T + k = p - q + k$. This $\psi^{-1}(T)$ tangent to Y along $(\psi|_{\mathcal{C}_x})^{-1}(\alpha)$. By Corollary 2.1.14 $\mathcal{C}_x$ is smooth, hence by Zak's theorem on tangencies we can find that $q \leq \frac{k}{2}$. As $q \geq \frac{k}{n}(p+2)$ we get $\mathsf{Index}(X) = p + 2 \leq \frac{n}{2}$ which is impossible by the hypothesis. □

3.1.2 For Low Dimensional Fano manifolds

We will follows the paper [17]. As in the previous section, we fix X be an n-dimensional smooth Fano variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ of degree $p+2$.

Recall that as Picard number is 1, we can check this over a generic standard minimal rational curve C. Hence for a sheaf $\mathscr{F}$ of rank r, which can be assumed to be locally free over C by Lemma 2.3.2, we can define $\mu(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot C}{r}$. Note that $\mu(\mathscr{F})$ depends only on $\mathscr{F}$ and $\mathcal{K}$ and does not depend on the choice of C. For example $\mu(T_X) = \frac{p+2}{n}$.

Proposition 3.1.8 (For $p=1$)**.** *If $p = 1$ and $n \leq 6$, then T_X is semi-stable, and stable except possibly when $n = 6$.*

Proof. If T_X is not semi-stable, choose $\mathscr{F}$ as in Proposition 3.1.2. From $\mathscr{F}|_C \subset \mathscr{O}(2) \oplus \mathscr{O}(1) \oplus \mathscr{O}^{\oplus n-2}$ with $T_X/\mathscr{F}|_C$ being locally free and $\mu(F) > 0$, we see $\mathscr{F}|_C = \mathscr{O}(1) \oplus \mathscr{O}^{\oplus r-1}$ by Proposition 3.1.3. From $\frac{1}{r} = \mu(\mathscr{F}) > \mu(T_X) = \frac{3}{n}$ and $r > 1$, we get $n > 6$. If T_X is semi-stable but not stable, we have $\mu(\mathscr{F}) = \mu(T_X)$ and $n = 3r$. □

Proposition 3.1.9 (For $p=2$)**.** *Suppose $p = 2$ and $n > 4$. If T_X is not stable, the for any $\mathscr{F}$ as in Proposition 3.1.2 we have $\mu(F) < 1$.*

Proof. We need several conclusions on surfaces:

- **Lemma A.** Let $W \subset \mathbb{P}^{n-1}$ be an irreducible surface with $n > 4$, which is not necessarily smooth. Suppose there exists a line l in P^{n-1} so that the tangent spaces to W at all generic points of W contain l. Then W is a plane.
- **Lemma B.** Let S be a normal projective surface. Suppose for a generic point $s \in S$, there exists a family $\mathscr{D}_s$ of rational curves through s, parametrized by a complete curve Λ_s, so that each member of the family is irreducible and reduced as a cycle. Then $S \cong \mathbb{P}^2$.

For the proof see also Lemma 1,2 in [17].

By Corollary 3.1.4 for $\mathscr{F} \subset T_X$ in Proposition 3.1.2 we have $\mu(\mathscr{F}) \leq 1$. If $\mu(\mathscr{F}) = 1 > \frac{4}{n} = \mu(T_X)$, we see that the only possible splitting type of $\mathscr{F}$ on a generic member C is $\mathscr{O}(1) \oplus \mathscr{O}(1)$ because the splitting type of $T_X|_C$ and $T_X/\mathscr{F}$ is locally free on C. By

Lemma A and Theorem 3.1.6, $\mathcal{C}_x$ for generic x is a finite union of planes intersecting along the line $\mathbf{P}\mathscr{F}_x$.

By this observation, consider

$$\begin{tikzcd} \mathbb{P}(\Omega_X) & \mathcal{U} \arrow[l, dashed, "\Phi"'] \arrow[r, "\phi"] \arrow[d, "\psi"] & X \\ & \mathcal{K} & \end{tikzcd}$$

where ψ is the universal family with cycle map ϕ and tangent map Φ. One can show that $\psi' : \Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)) \to \mathcal{K}' := \psi(\Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)))$ is a 1-dimensional fibration and $\mathcal{K}' \subset \mathcal{K}$ is codimension 1.

Let $C \subset X$ be the image of a generic fiber of ψ' under ϕ. For a smooth point $y \in C$, let $z \in \Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee))$ be its inverse image under ϕ. Then by the definition of the tangent map, the fibers of ψ' correpond to curves in X tangent to the meromorphic foliation $\mathscr{F}$.

From the minimality of $\mathcal{K}$ and the fact that Φ_x is generically finite on each component of $\mathcal{U}_x$ for a generic x, while $\mathbb{P}(\mathscr{F}_x^\vee)$ is ample on each component of $\mathcal{C}_x$ for a generic x, we can choose a generic point x so that each curve corresponding to a point of $\mathcal{K}_x = \psi(\mathcal{U}_x) = \psi(\phi^{-1}(x))$ is reduced and irreducible and $\mathcal{K}' := \mathcal{K}_x \cap \psi(\Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)))$ consists of 1-dimensional components, and there exists at least one component of $\mathcal{K}'_x$ for each component of $\mathcal{K}_x$.

Let S' be the closure of the $\mathscr{F}$-leaf through x. The 1-dimensional families of curves corresponding to $\mathcal{K}'_x$ lie on the $\mathscr{F}$-leaf through x and their tangents span $\mathscr{F}$ at x. Thus S' is the closure of the union of curves corresponding to $\mathcal{K}'_x$ and is an algebraic surface. For each generic point $s \in S'$, S' is the closure of the $\mathscr{F}$-leaf through s. The families of curves corresponding to $\mathcal{K}'_s$ consist of irreducible and reduced cycles. By Lemma B, the normalization S of S' is $\mathbb{P}^2$. Thus $\mathcal{K}'_x$ is just the set of lines through a generic point on $\mathbb{P}^2$, and is irreducible for a generic choice of x. Hence $\mathcal{K}_x$ and hence $\mathcal{U}_x$ and $\mathcal{C}_x$ are irreducible.

Since $\mathcal{C}_x$ is irreducible, the collection of $\mathcal{C}_x$ in $\mathbb{P}(\Omega^1_{X,x})$ at generic x, defines a meromorphic distribution $\mathscr{F}'$ of rank 3. For a generic member C we have $\mathscr{F}'|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus 2}$ and $T_X/\mathscr{F}'|_C = \mathscr{O}^{\oplus n-3}$. This implies that $\mathscr{F}'$ is integrable. A contradiction to Proposition 3.1.3. $\square$

Lemma 3.1.10 (Reid, 1977)**.** *Let X be a Fano manifold of dimension n. Let $\mathscr{G} \subset T_X$ be a proper reflexive subsheaf. Then $c_1(\mathscr{G}) < c_1(X)$. In particular, T_X is stable if* $\mathsf{Index}(X) = 1$.

Proof. Pick such $\mathscr{G} \subset T_X$ of rank $p < n$. If $c_1(\mathscr{G}) \geq c_1(X)$, then we have a nonzero $\det \mathscr{G} \to \bigwedge^p T_X$. Hence

$$0 \neq H^0(X, \bigwedge^p T_X \otimes \det G^\vee) = H^0(X, \Omega_X^{n-p} \otimes \det T_X \otimes \det G^\vee).$$

If $c_1(\mathscr{G}) > c_1(X)$, then this is impossible by Kodaira-Nakano vanishing theorem. If $c_1(\mathscr{G}) = c_1(X)$, then by Hodge symmetry this is impossible by Kodaira vanishing theorem as $H^{n-p}(X, \mathscr{O}_X) = 0$. □

Theorem 3.1.11. *Fano 5-folds with Picard number 1 have stable tangent bundles.*

Proof. For $p = 0, 1, 4$, the result follows from Theorem 3.1.5 and Proposition 3.1.8. If $p = 3$, the index of X is either 5 or 1. If the index is 5, X is a hyperquadric by Theorem 1.5.7(d). If the index is 1, done by Lemma 3.1.10. If $p = 2$, and T_X is not stable, choose $\mathscr{F}$ with $1 \geq \mu(\mathscr{F}) \geq \frac{4}{5} = \mu(T_X)$. Since $\mu(\mathscr{F})$ is a rational number with denominator $2, 3$, or 4, we get $\mu(\mathscr{F}) = 1$, a contradiction by Proposition 3.1.9. □

Theorem 3.1.12. *Fano 6-folds with Picard number 1 have semi-stable tangent bundles.*

Proof. If $p = 0, 1, 4, 5$, the result follows from Theorem 3.1.5 and Proposition 3.1.8. If $p = 3$, X has index 5 or 1. If the index is 1, done by Lemma 3.1.10. If the index is 5, done by [36] Theorem 3(1).

If $p = 2$ and T_X is not semi-stable, choose $\mathscr{F}$ as in Proposition 3.1.2 and $1 \geq \mu(\mathscr{F}) > \frac{4}{6} = \mu(T_X)$. Hence we have $\mu(\mathscr{F}) = 1, \frac{4}{5}, \frac{3}{4}$. But $\mu(\mathscr{F}) = 1$ is not possible by Proposition 3.1.9. The case $\mu(\mathscr{F}) = \frac{4}{5}$ implies that $\mathscr{F}|_C = \mathscr{O}(1)^{\oplus 4} \oplus \mathscr{O}$, violating the locally freeness of $T_X/\mathscr{F}|_C$. The same contradiction for $\mu(\mathscr{F}) = \frac{3}{4}$. □

3.1.3 For Hecke Curves on Moduli Space of Bundles on Curves

We will follows the paper [18]. For a smooth projective curve C of genus g. Consider the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of degree d. If d is odd (we will assume d odd in whole section), then $M_{2;\mathscr{D},d}(C)$ is a $(3g-3)$-dimensional Fano manifold of Picard number 1 (it is prime). Moreover $M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},1}(C)$ in this case. In particular, when $g = 2$ the space $M_{2;\mathscr{D},1}(C)$ is a intersection of two quadrics in $\mathbb{P}^5$.

Proposition 3.1.13. *Let $g \geq 3$. For a general $[W] \in X$ and tangent morphism $\tau_{[W]} : \mathcal{K}_{[W]} \to \mathbb{P}(\Omega^1_{X,[W]})$ which is given by the linear system $2\pi^* K_C - K_{\mathbf{P}_C(W)}$ (by Proposition 2.2.12). Given any linear subspace $\mathbb{P}(F^\vee) \subset \mathbb{P}(\Omega^1_{X,[W]})$ of dimension $r-1$, its intersection with the projective tangent space at a generic point of $\mathcal{C}_{[W]}$ is either empty or has dimension smaller than $(4r/(3g-3)) - 1$.*

Proof. Let such $\mathbb{P}(F^\vee) \subset \mathbb{P}(\Omega^1_{X,[W]})$ of dimension $r-1$ we have

$$\dim(\mathbb{P}(F^\vee) \cap \mathbb{P}((T_{[W]}X^+_\alpha)^\vee)) \geq \frac{4r}{3g-3} - 1$$

for generic $\alpha \in \mathcal{C}_{[W]}$.

Since the surface $\mathcal{C}_{[W]}$ is nondegenerate in $\mathbb{P}(\Omega^1_{X,[W]})$ (see Proposition 2.2.12(b)), the intersection can have dimension 0 or 1. If the intersection has dimension 1, then the projection from $\mathbb{P}(F^\vee)$ sends the tangent space at a generic point of $\mathcal{C}_{[W]}$ to zero. Thus the projection sends $\mathcal{C}_{[W]}$ to a point. This implies that $\mathcal{C}_{[W]}$ is contained in some linear subspace containing $\mathbb{P}(F^\vee)$, a contradiction to the nondegeneracy of $\mathcal{C}_{[W]}$. It follows that the intersection has dimension 0 and $r \leq \frac{3}{4}(g-1)$. Moreover, the projection from $\mathbb{P}(F^\vee)$ projects $\mathcal{C}_{[W]}$ to a curve $\ell \subset \mathbb{P}^{3g-4-r}$.

Suppose the $\tau_{[W]}$-image of a generic fiber of $\pi : \mathbf{P}_C(W) \to C$ is dominant over ℓ. Since the image of this fiber under $\tau_{[W]}$ is of degree less than or equal to 2, ℓ must be contained in a plane. This implies that $\mathcal{C}_{[W]}$ is contained in some $\mathbb{P}^{r+2}$ containing $\mathbb{P}(F^\vee)$, a contradiction to the nondegeneracy of $\mathcal{C}_{[W]}$ again. Thus the projection to $\mathbb{P}^{3g-4-r}$ contracts generic fibers of π to a point. It follows that the $\tau_{[W]}$-image of a generic fiber of π is contained in some linear subspace $\mathbb{P}^r$ containing $\mathbb{P}(F^\vee)$ as a hyperplane, and it intersects $\mathbb{P}(F^\vee)$.

Let $\Xi \subset |2\pi^* K_C - K_{\mathbf{P}_C(W)}|$ be the subsystem of dimension $3g-4-r$ defining the projection of $\mathbf{P}_C(W)$ to $\mathbb{P}^{3g-4-r}$ from $\mathbb{P}(F^\vee)$. Let $D \subset \mathbf{P}_C(W)$ be the base locus of Ξ. Hence D corresponds to the intersection of $\mathcal{C}_{[W]}$ with $\mathbb{P}(F^\vee)$. Hence generic fibers of $\pi : \mathbf{P}_C(W) \to C$ intersect D twice, counting multiplicity. Using the notation of [15] for ruled surface, we have $D \sim_{\text{num}} 2C_0 + df$ and $2\pi^* K_C - K_{\mathbf{P}_C(W)} \sim_{\text{num}} 2C_0 + (2g-2+e)f$. Thus the moving part of the system Ξ is just the pullback of a linear system on X of degree $2gg-2+e-d$. By Nagata's result about the intersection number of ruled surface in [34], we have $0 < C_0^2 = -e \leq g$. Since C_0 is ample by [15] Proposition 2.21, we have $D \cdot C_0 > 0$ and $-2e + d > 0$. So Ξ is the pullback of a linear system of degree less than or equal to $3g-3$. By the Riemann-Roch theorem and Clifford's theorem (see [15] page 343), we have $\dim \Xi \leq \max((3/2)(g-1), 2g-3) = 2g-3$. Combined with $\dim \Xi = 3g-4-r$, we get $g \leq r+1$, a contradiction to $r \leq (3/4)(g-1)$. □

Theorem 3.1.14. *Let the moduli space $X := M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of odd degree d over a smooth projective curve C of genus g. If $g \geq 2$, then T_X is stable.*

Proof. For $g = 2$, this can be directly deduced by Proposition 2.2.9 and Corollary 2.2.10. For $g \geq 3$, this follows directly from Theorem 3.1.6 and Proposition 3.1.13. □

3.1.4 Need to add

Need to add

3.2 The Remmert-Van de Ven / Lazarsfeld Problem

Need to add

3.3 Deformation Rigidity

Need to add

3.4 Uniqueness of Contact Structures

Need to add

Chapter 4

About Semiample Tangent Bundles

Chapter 5

VMRT of Rational Homogeneous Varieties

Chapter 6

Need to add

Index

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