

A Quick Tour of Derived Algebraic Geometry

Xiaolong Liu

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1 Introduction

We will assume the base ring \mathbb{K} containing \mathbb{Q} . We will follow [EP23] and Joyce's slides as in <https://people.maths.ox.ac.uk/~joyce/DAG2022/index.html>.

2 Commutative Differential Graded Algebras

- Definition 2.1.** (a) A differential graded \mathbb{K} -algebra (dga or dg-algebra for short) $A^\bullet = (A^*, d)$ consists of a chain complex with a unital associative multiplication. Concretely, that is a family of \mathbb{K} -modules $\{A^i\}_{i \in \mathbb{Z}}$, an associative \mathbb{K} -linear multiplication $(-\cdot-) : A^i \times A^j \rightarrow A^{i+j}$ (for all i, j), a unit $1 \in A^0$ and a differential $d : A^i \rightarrow A^{i+1}$ (for all i) which is \mathbb{K} -linear, satisfies $d^2 = 0$ and is a derivation with respect to the multiplication, which means $d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b)$.
- (b) A graded \mathbb{K} -algebra A is graded-commutative if $a \cdot b = (-1)^{\deg(a) \cdot \deg(b)} b \cdot a$.
- (c) A morphism of dg-algebras is a map $f : A^\bullet \rightarrow B^\bullet$ that respects the differentials (i.e. $f d_A = d_B f$), and the multiplication (i.e. $f(a \cdot_A b) = f(a) \cdot_B f(b)$).

In this note we mainly focus on the following:

Definition 2.2. We define the category $\mathbf{cdg}^- \mathbf{Alg}_{\mathbb{K}}$ of the graded-commutative differential graded \mathbb{K} -algebras which are concentrated in non-positive degree. Hence $A^\bullet \in \mathbf{cdg}^- \mathbf{Alg}_{\mathbb{K}}$ we have $A^\bullet = \bigoplus_{k=0}^{-\infty} A^k$. Moreover, for $R^\bullet \in \mathbf{cdg}^- \mathbf{Alg}_{\mathbb{K}}$, we can define $\mathbf{cdg}^- \mathbf{Alg}_{R^\bullet} := R^\bullet \downarrow \mathbf{cdg}^- \mathbf{Alg}_{\mathbb{K}}$ of \mathbf{cdga} R^\bullet -algebras.

Definition 2.3. (a) Let $A^\bullet, B^\bullet \in \mathbf{cdg}^- \mathbf{Alg}_{R^\bullet}$. A morphism $f : A^\bullet \rightarrow B^\bullet$ of R^\bullet - \mathbf{cdga} is a quasi-isomorphism (or weak equivalence) if it induces an isomorphism on cohomology $H^*(A^\bullet) \cong H^*(B^\bullet)$.
(b) We say that R^\bullet - \mathbf{cdga} A^\bullet and B^\bullet are quasi-isomorphic if there exists a diagram $A^\bullet \leftarrow C^\bullet \rightarrow B^\bullet$ of quasi-isomorphisms in $\mathbf{cdg}^- \mathbf{Alg}_{R^\bullet}$.

Remark 2.4. Note that there is a global version of these.

A very important example of \mathbf{cdga} s and derived schemes:

Example 2.5. Let M^\bullet be a graded \mathbb{K} -module. The free graded-commutative \mathbb{K} -algebra generated by M^\bullet is

$$\mathbb{K}[M^\bullet] := \left(\bigoplus \mathrm{Sym}^n M^{\mathrm{even}} \right) \otimes_{\mathbb{K}} \left(\bigoplus \bigwedge^n M^{\mathrm{odd}} \right)$$

with the degree of a product of elements being the sum of the degrees of those elements.

In particular, we consider the free graded-commutative \mathbb{K} -algebra A^\bullet generated by $x_1, \dots, x_m; y_1, \dots, y_n$ where $\deg x_i = 0$ and $\deg y_j = -1$. Hence

$$A^k = \mathbb{K}[x_1, \dots, x_m] \otimes_{\mathbb{K}} \bigwedge^{-k} \mathbb{K}^n$$

for $k = 0, -1, \dots, -n$ and $A^k = 0$ for otherwise. Pick $p_1, \dots, p_n \in \mathbb{K}[x_1, \dots, x_m]$, as A^\bullet is free there are unique maps $d : A^k \rightarrow A^{k+1}$ satisfying the Leibnitz rule, such that $d(y_i) = p_i(x_1, \dots, x_m)$ for $i = 1, \dots, n$. Also $d^2 = 0$ and $A^\bullet \in \mathbf{cdg}^- \mathbf{Alg}_{\mathbb{K}}$.

Now $H^0(A^\bullet) = \mathbb{K}[x_1, \dots, x_m]/(p_1, \dots, p_n)$ and hence $\mathrm{Spec} H^0(A^\bullet)$ is a subscheme defined by these polynomials. Now the derived scheme $\mathbf{Spec} A^\bullet$ remembers information about the dependencies between p_1, \dots, p_n which have more information than the truncated classical scheme $\mathrm{Spec} H^0(A^\bullet)$.

3 Basic Concepts of Infinity Categories

In this lecture ‘ ∞ -category’ always means ‘ $(\infty, 1)$ -category’, that is, all n -morphisms are invertible for $n \geq 2$. (Although ‘ n -morphism’ may not make sense, depending on your model for ∞ -categories.)

There are a bunch of different but related structures which are more-or-less kinds of ∞ -category:

- Model categories.
- Categories enriched in topological spaces.
- Simplicial categories; simplicial model categories.
- Quasicategories.

Of these, model categories are the oldest (Quillen 1967), and look least like an ∞ -category (they have no visible higher morphisms). But most of the other kinds of ∞ -category use model categories under the hood. Toën-Vezzosi’s DAG is written in terms of model categories and simplicial categories. Lurie works with quasicategories, which may be the best and coolest version.

- If you start with an ordinary category \mathcal{C} and invert some class of morphisms \mathcal{W} in \mathcal{C} (‘weak equivalences’), the result $\mathcal{C}[\mathcal{W}^{-1}]$ should really be an ∞ -category with homotopy category $\mathrm{Ho}(\mathcal{C}[\mathcal{W}^{-1}])$ an ordinary category.

This idea is similar as derived category $\mathbf{D}(\mathcal{A})$ construct from $\mathrm{Ho}(\mathrm{Com}(\mathcal{A}))$ by inverting the class \mathcal{W} of quasi-isomorphisms. Note that $\mathbf{D}(\mathcal{A}) = \mathrm{Ho}(\mathbf{D}(\mathcal{A}))$ for a stable ∞ -category $\mathbf{D}(\mathcal{A})$.

Here we give some idea how to consider the ∞ -category. We know that a $(2, 1)$ -category \mathcal{C} is a category enriched in groupoids. A $(3, 1)$ -category \mathcal{C} is a category enriched in 2-groupoids. So similarly, an $(\infty, 1)$ -category is really a ‘category enriched in ∞ -groupoids’. But what is an ∞ -groupoid?

Two models for the (model/ ∞ -)category of ∞ -groupoids are topological spaces \mathbf{Top} (up to homotopy), and simplicial sets \mathbf{sSets} . Note that \mathbf{Top} and \mathbf{sSets} are Quillen equivalent as model categories, theories of ∞ -categories based on \mathbf{Top} and \mathbf{sSets} are essentially equivalent. But it seems no one uses categories enriched in \mathbf{Top} except as motivation.

3.1 Categories Enriched in Topological Spaces

Our first model for an $(\infty, 1)$ -category is the categories enriched in topological spaces:

Definition 3.1. A category enriched in topological spaces is a category \mathcal{C} such that for all objects X, Y in \mathcal{C} , the set $\text{Hom}_{\mathcal{C}}(X, Y)$ of morphisms $f : X \rightarrow Y$ is given the structure of a topological space (generally a nice topological space, e.g. Hausdorff, ..., and homotopy equivalent to a CW complex), and for objects X, Y, Z the composition $\mu_{X,Y,Z} : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$ is a continuous map. Moreover, there is a homotopy between $\mu_{W,X,Y} \circ (\mu_{W,X,Y} \times \text{id}) \rightarrow \mu_{W,X,Y} \circ (\text{id} \times \mu_{X,Y,Z})$.

The higher-morphism of \mathcal{C} defined as follows:

- A 1-morphism $f : X \rightarrow Y$ is a point of $\text{Hom}_{\mathcal{C}}(X, Y)$.
- If $f, g : X \rightarrow Y$ are 1-morphisms, a 2-morphism $\eta : f \Rightarrow g$ is a continuous path $\eta : [0, 1] \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ with $\eta(0) = f$ and $\eta(1) = g$. Note that η is invertible with $\eta^{-1}(s) = \eta(1 - s)$.
- If $\eta, \zeta : f \Rightarrow g$ are 2-morphisms, a 3-morphism $\aleph : \eta \Rightarrow \zeta$ is a continuous map $\aleph : [0, 1]^2 \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ such that for $s, t \in [0, 1]$ we have

$$\aleph(0, t) = f, \aleph(1, t) = g, \aleph(s, 0) = \eta, \aleph(s, 1) = \zeta.$$

- n -morphisms are continuous maps $[0, 1]^{n-1} \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ with prescribed boundary conditions on $\partial([0, 1]^{n-1})$.
- Moreover, if $\eta : f \Rightarrow g, \zeta : g \Rightarrow h$ are 2-morphisms, the vertical composition $\zeta \odot \eta : f \Rightarrow h$ is $(\zeta \odot \eta)(s) = \eta(2s)$ if $s \in [0, 1/2]$ and $(\zeta \odot \eta)(s) = \zeta(2s - 1)$ if $s \in [1/2, 1]$. This is not associative, but is associative up to homotopy, i.e. up to 3-isomorphism. Other kinds of composition can be defined in a similar way.

Definition 3.2. For any higher category \mathcal{C} (as we will used), the homotopy category $\text{Ho}(\mathcal{C})$ which is an ordinary category, where objects X of $\text{Ho}(\mathcal{C})$ are objects of \mathcal{C} , and morphisms $[f] : X \rightarrow Y$ in $\text{Ho}(\mathcal{C})$ are 2-isomorphism classes of 1-morphisms $f : X \rightarrow Y$ in \mathcal{C} .

Now for a category enriched in topological spaces \mathcal{C} . we have $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) = \pi_0(\text{Hom}_{\mathcal{C}}(X, Y))$.

3.2 Model Categories

Now we introduce some model categories invented by Quillen to abstract methods of homotopy theory into category theory.

Definition 3.3. A model category is a complete and cocomplete category \mathcal{M} equipped with three distinguished classes of morphisms: the weak equivalences \mathcal{W} , the fibrations \mathcal{F} , and the cofibrations \mathcal{C} . These must satisfy:

- $\mathcal{W}, \mathcal{F}, \mathcal{C}$ are closed under composition and include identities.
- $\mathcal{W}, \mathcal{F}, \mathcal{C}$ are closed under retracts. Here f is a retract of g if there exist i, j, r, s such that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{\quad i \quad} & Y & \xrightarrow{\quad r \quad} & X \\ \downarrow f & & \downarrow g & & \downarrow f \\ X' & \xrightarrow{\quad j \quad} & Y' & \xrightarrow{\quad s \quad} & X' \end{array} \quad \begin{array}{c} \text{id}_X \\ \text{id}_{X'} \end{array}$$

- For $f : X \rightarrow Y, g : Y \rightarrow Z$ in \mathcal{M} , if two of $f, g, g \circ f$ are in \mathcal{W} then so is the third.
- A (co)fibration which is also a weak equivalence is called *acyclic*. Acyclic cofibrations have the left lifting property with respect to fibrations, and cofibrations have the left lifting property with

respect to acyclic fibrations. Explicitly, if the square below commutes, where i is a cofibration, p is a fibration, and i or p is acyclic, then there exists h as shown:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow i & \nearrow h & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

- (e) Every morphism f in \mathcal{M} can be written as $f = p \circ i$ for a fibration p and an acyclic cofibration i .
- (f) Every morphism f in \mathcal{M} can be written as $f = p \circ i$ for an acyclic fibration p and a cofibration i .

Some basic elements in a model category:

Definition 3.4. Let $(\mathcal{M}, \mathcal{W}, \mathcal{F}, \mathcal{C})$ be a model category with initial object \emptyset and final object $*$.

- (a) An object $X \in \mathcal{M}$ is called **fibrant** if $[X \rightarrow *] \in \mathcal{F}$, and **cofibrant** if $[\emptyset \rightarrow X] \in \mathcal{C}$.
- (b) If $X \in \mathcal{M}$ and there is a weak equivalence $w : C \rightarrow X$ with C cofibrant then C is a **cofibrant replacement** for X . If there is a weak equivalence $w : X \rightarrow F$ with F fibrant then F is a **fibrant replacement** for X . Such replacements always exist.
- (c) If $X \in \mathcal{M}$, a **cylinder object** is an object $X \times [0, 1]$ in \mathcal{M} with a factorization $X \sqcup X \xrightarrow{c} X \times [0, 1] \xrightarrow{w} X$ of the codiagonal $X \sqcup X \rightarrow X$, with c a cofibration and w a weak equivalence. Cylinder objects exist by Definition 3.3(d).
- (d) If $X \in \mathcal{M}$, a **path object** is an object $\text{Map}([0, 1], X)$ in \mathcal{M} with a factorization $X \xrightarrow{w} \text{Map}([0, 1], X) \xrightarrow{f} X \times X$ of the diagonal $X \rightarrow X \times X$, with w a weak equivalence and f a fibration. Path objects exist by Definition 3.3(e).
- (e) Morphisms $f, g : X \rightarrow Y$ are called **(left) homotopy equivalent** if there exists $h : X \times [0, 1] \rightarrow Y$ with $h \circ c = f \sqcup g$ where c as in (c).
- (f) The **homotopy category** is $\text{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}]$, the category obtained by formally inverting all weak equivalences. Note that this is independent of \mathcal{C}, \mathcal{F} .

Note that two of $\mathcal{W}, \mathcal{F}, \mathcal{C}$ determine the third. Now we introduce an important theorem:

Theorem 3.5 (Fundamental Theorem of Model Categories). *Let \mathcal{M} be a model category. Then $\text{Ho}(\mathcal{M})$ is equivalent to the category whose objects are fibrant-cofibrant objects in \mathcal{M} , and whose morphisms are homotopy classes of morphisms in \mathcal{M} .*

Example 3.6. (a) The category **Top** of topological spaces has a model structure with \mathcal{W} the weak homotopy equivalences, and \mathcal{F} the Serre fibrations (maps with the homotopy lifting property for CW complexes). In this case by Theorem 3.5, $\text{Ho}(\mathbf{Top})$, which is the homotopy category of homotopy types, can be described as the category whose objects are CW complexes and morphisms are homotopy classes of continuous maps.

- (b) If R is a commutative ring then $\text{Com}(\text{Mod}_R)$ has two canonical model structures with weak equivalences quasi-isomorphisms and
 - cofibrations morphisms $\phi : E^\bullet \rightarrow F^\bullet$ with $\phi^k : E^k \rightarrow F^k$ injective for all k ;
 - fibrations morphisms $\phi : E^\bullet \rightarrow F^\bullet$ with $\phi^k : E^k \rightarrow F^k$ surjective for all k .

The first one is the injective model category and the second one is the projective model category. In this case by Theorem 3.5, $\text{Ho}(\text{Com}(\text{Mod}_R)) = \mathbf{D}(R)$, can be described as the category whose objects are either K -injective, or K -projective, complexes, and morphisms are homotopy classes of maps between these complexes.

- (c) Let \mathcal{A} be a Grothendieck abelian category (such as $\text{Qcoh}(X)$ for any scheme X). Then $\text{Com}(\mathcal{A})$ has a model structure with weak equivalences quasi-isomorphisms and cofibrations morphisms $\phi : E^\bullet \rightarrow F^\bullet$ with $\phi^k : E^k \rightarrow F^k$ injective for all k . In this case by Theorem 3.5, $\text{Ho}(\text{Com}(\mathcal{A})) = \mathbf{D}(\mathcal{A})$, can be described as the category whose objects are either K -injective complexes, and morphisms.

Next we consider some another constructions about model categories.

Definition 3.7 (Derived Functors). *Let $(\mathcal{C}, \mathcal{W})$ and $(\mathcal{D}, \mathcal{V})$ be a relative category (that is, a category with a subcategory of weak equivalences). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and we say $\mathbf{R}F : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$ be a right-derived functor of F we have*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \lambda_{\mathcal{C}} \downarrow & \eta \swarrow \cong & \downarrow \lambda_{\mathcal{D}} \\ \mathrm{Ho}(\mathcal{C}) & \xrightarrow{\mathbf{R}F} & \mathrm{Ho}(\mathcal{D}) \end{array}$$

then any natural transformation $\lambda_{\mathcal{D}} \circ F \rightarrow \mathbf{R}F \circ \lambda_{\mathcal{C}}$ factors through η , and this factorisation is unique up to natural isomorphism in $\mathrm{Ho}(\mathcal{D})$ — this condition ensures that $\mathbf{R}F$ is unique up to weak equivalence. Similarly for left-derived functor.

Definition 3.8. A functor $G : \mathcal{C} \rightarrow \mathcal{D}$ of model categories is *right Quillen* if it has a left adjoint F and preserves fibrations and trivial fibrations. Dually, F is *left Quillen* if it has a right-adjoint and F preserves cofibrations and trivial cofibrations. $F \dashv G$ is in that case called a *Quillen adjunction*. A Quillen adjunction $F \dashv G$ is said to be a *Quillen equivalence* if $\mathbf{R}G : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$ is an equivalence of categories, in which case it has quasi-inverse $\mathbf{L}F$.

Lemma 3.9. Let $F \dashv G$ be a adjunction of functors of model categories, then F is left Quillen is and only if G is right Quillen.

Proof. Easy from lifting properties. □

Theorem 3.10 (Quillen). *If G is right Quillen, then the right-derived functor $\mathbf{R}G$ exists and is given on objects by $A \mapsto G\hat{A}$, for $A \rightarrow \hat{A}$ a fibrant replacement. Dually, left Quillen functors give left-derived functors by cofibrant replacement.*

Remark 3.11. To get a functor, we can take fibrant replacements functorially, but on objects the choice of fibrant replacement doesn't matter (and in particular need not be functorial), because it turns out that right Quillen functors preserve weak equivalences between fibrant objects. The proof is an exercise with path objects.

Next we consider another construction.

Definition 3.12. Let \mathcal{C} be a model category with a small category I . Then we have constant functor $\Delta : \mathcal{C} \rightarrow \mathcal{C}^I$. This descends to $\Delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C}^I)$ where now \mathcal{C}^I has two model structures, projective one and injective one (see Section A.2.8 in [Lur09]).

Now $\Delta : \mathcal{C} \rightarrow \mathcal{C}^I$ is left Quillen for the injective model structure on \mathcal{C}^I , and right Quillen for the projective model structure. So $\Delta : \mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{C}^I)$ possesses both a right and a left adjoint, called the *homotopy limit* and *homotopy colimit* $\mathrm{Holim}_I, \mathrm{Hocolim}_I : \mathrm{Ho}(\mathcal{C}^I) \rightarrow \mathrm{Ho}(\mathcal{C})$.

Here are some motivation:

Example 3.13. In **Top**, fibre products $X \times_{f,Z,g} Y$ and pushouts $X \sqcup_{f,Z,g} Y$ exist, but if we change f, g by homotopies, then $X \times_{f,Z,g} Y, X \sqcup_{f,Z,g} Y$ need not stay homotopy equivalent. So fibre products and pushouts in **Top** are the wrong idea for homotopy theory. Instead one defines the homotopy fibre product and homotopy pushout as before. In this case, we have

$$X \times_{f,Z,g}^h Y := X \times_{f,Z,\mathrm{ev}_0} \mathrm{Map}([0,1], Z) \times_{\mathrm{ev}_1,Z,g} Y$$

and

$$X \sqcup_{f,Z,g}^h Y := X \sqcup_{f,Z,t_0} ([0,1] \times Z) \sqcup_{t_1,Z,g} Y.$$

For example $\{y\} \times_Y^h \{y\} = \Omega(Y; y)$ be the loop space.

What do these mean? Let \mathcal{C} be a model category, in general homotopy limits and colimits are not limits and colimits (do not satisfy universal properties) in either the ordinary categories \mathcal{C} , or the ordinary categories $\mathbf{Ho}(\mathcal{C})$. Instead, the correct interpretation is that there are secretly $(\infty, 1)$ -categories \mathcal{C}^∞ with homotopy categories $\mathbf{Ho}(\mathcal{C})$, and homotopy limits and colimits are actually ∞ -category limits and colimits in \mathcal{C}^∞ . Thus model category techniques effectively give ways to do constructions in an ∞ -category, without defining ∞ -categories.

- Model categories are strict forms of $(\infty, 1)$ -categories.

That is, composition of morphisms in a model category is strictly associative (other notions of $(\infty, 1)$ -category have composition non-associative, or even not uniquely defined). There are ‘strictification theorems’ which allow you to pass from looser forms of $(\infty, 1)$ -categories (e.g. Segal categories) to model categories. Typically, general constructions are done in the looser kinds of $(\infty, 1)$ -category, and explicit computations are done in model categories.

3.3 Simplicial Sets and Simplicial Categories

Definition 3.14. (a) The simplex category Δ has objects set $[n] = \{0, 1, \dots, n\}$ for $n \geq 0$ and morphisms $f : [n] \rightarrow [m]$ the order preserving functions, that is, if $0 \leq i \leq j \leq n$ then $f(i) \leq f(j)$.

A simplicial set is a functor $F : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$. A morphism of simplicial sets $\eta : F \rightarrow G$ is a natural transformation of functors $\eta : F \Rightarrow G$. This makes simplicial sets into a category \mathbf{sSets} .

- (b) The topological n -simplex Δ_{top}^n is

$$\Delta_{\text{top}}^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

If $f : [n] \rightarrow [m]$ is order-preserving we define $f_{\text{top}} : \Delta_{\text{top}}^m \rightarrow \Delta_{\text{top}}^n$ by

$$f_{\text{top}}(x_0, \dots, x_m) = (y_0, \dots, y_n), \quad y_i = \sum_{j \in \{0, \dots, m\} : f(j)=i} x_j.$$

This defines a functor $G : \Delta^{\text{op}} \rightarrow \mathbf{Top}$ mapping $[n] \mapsto \Delta_{\text{top}}^n$ and $f \mapsto f_{\text{top}}$.

- (c) Let $F : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$ be a simplicial set. We define a topological space X_F with a triangulation, the topological realization of F , by

$$X_F := \left(\coprod_{n \geq 0} F([n]) \times \Delta_{\text{top}}^n \right) / \sim$$

where \sim generated by $(s, f_{\text{top}}(x_0, \dots, x_m)) \sim (F(f)s, (x_0, \dots, x_m))$. In this way we can define a topological realization functor $\mathbf{TR} : \mathbf{sSets} \rightarrow \mathbf{CGHaus} \subset \mathbf{Top}$, for \mathbf{CGHaus} the subcategory of compactly generated Hausdorff spaces. It is a right Kan extension

$$\begin{array}{ccc} \Delta^{\text{op}} & \xrightarrow{\text{Yoneda}} & \mathbf{sSets} \\ & \searrow G & \downarrow \mathbf{TR} \\ & & \mathbf{CGHaus} \hookrightarrow \mathbf{Top} \end{array}$$

- (d) There is also a functor $\mathbf{Sing} : \mathbf{Top} \rightarrow \mathbf{sSets}$ which maps a space X to the functor $F : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$ with $F([n]) := \text{Map}(\Delta_{\text{top}}^n, X)$ for objects $[n]$ and $F(f) := - \circ f_{\text{top}} : \text{Map}(\Delta_{\text{top}}^n, X) \rightarrow \text{Map}(\Delta_{\text{top}}^m, X)$ for morphisms $f : [n] \rightarrow [m]$, so that $f_{\text{top}} : \Delta_{\text{top}}^m \rightarrow \Delta_{\text{top}}^n$.

Note that there are model category structures on \mathbf{Top} and \mathbf{sSets} such that $\mathbf{TR} : \mathbf{sSets} \rightarrow \mathbf{Top}$ and $\mathbf{Sing} : \mathbf{Top} \rightarrow \mathbf{sSets}$ are homotopy inverses (\mathbf{Top} and \mathbf{sSets} are Quillen equivalent model categories), and the homotopy categories $\mathbf{Ho}(\mathbf{Top})$ and $\mathbf{Ho}(\mathbf{sSets})$ are equivalent categories.

The weak equivalences in the model category \mathbf{sSets} are morphisms η for which $\mathbf{TR}(\eta)$ is a weak homotopy equivalence of topological spaces. The fibrations are ‘Kan fibrations’, and the cofibrations morphisms $\eta : F \Rightarrow G$ such that $\eta([n]) : F([n]) \Rightarrow G([n])$ is injective for all n . All simplicial sets are cofibrant. The fibrant objects are called ‘Kan complexes’. The category \mathbf{sSets} is used as a model for ∞ -groupoids.

Definition 3.15. A simplicial category \mathcal{S} is a category enriched in simplicial sets. That is, \mathcal{S} is a ‘category’ in which for all objects X, Y in \mathcal{S} , the morphisms $\text{Hom}(X, Y)$ is a simplicial set, and composition $\mu_{X, Y, Z} : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$ is a morphism of simplicial sets. Composition is strictly associative, that is,

$$\mu_{W, Y, Z} \circ (\text{id}_{\text{Hom}(Y, Z)} \times \mu_{W, X, Y}) = \mu_{W, Y, Z} \circ (\mu_{X, Y, Z} \times \text{id}_{\text{Hom}(W, X)}),$$

rather than ‘associative up to homotopy’.

Note that unlike additive categories, in which $\text{Hom}(X, Y)$ is an abelian group, is still an ordinary category since it is a set. For simplicial categories, it is probably best not to think of $\text{Hom}(X, Y)$ as a set with extra structure (though you could think of the underlying set as $\mathbf{TR}(\text{Hom}(X, Y))$), so a simplicial category is not an ordinary category with extra structure.

Remark 3.16. Any ordinary category \mathcal{C} can be made into a simplicial category \mathcal{S} by defining $\text{Hom}_{\mathcal{S}}(X, Y)$ to map $[n] \mapsto \text{Hom}_{\mathcal{C}}(X, Y)$ and $f \mapsto \text{id}_{\mathcal{C}}(X, Y)$ and for all $[n]$ and $f : [m] \rightarrow [n]$.

Here is a related notion:

Definition 3.17. Given a category \mathcal{C} , a simplicial object in \mathcal{C} is a functor $F : \Delta^{\text{op}} \rightarrow \mathcal{C}$. In particular, a simplicial object in \mathbf{Cat} is a functor $F : \Delta^{\text{op}} \rightarrow \mathbf{Cat}$.

Given a simplicial category \mathcal{S} , we can define $F_{\mathcal{S}} : \Delta^{\text{op}} \rightarrow \mathbf{Cat}$, a simplicial object in \mathbf{Cat} , by taking $F_{\mathcal{S}}([n])$ to be the category with the same objects as \mathcal{S} and with morphisms $X \rightarrow Y$ to be the set $\text{Hom}_{\mathcal{S}}(X, Y)([n])$. Then simplicial categories correspond to special simplicial objects in \mathbf{Cat} .

For example, a simplicial commutative \mathbb{K} -algebra is a functor $A : \Delta^{\text{op}} \rightarrow \mathbf{Alg}_{\mathbb{K}}$. This will be important for Derived Algebraic Geometry as one model for ‘derived commutative \mathbb{K} -algebras’ are simplicial commutative \mathbb{K} -algebras. (If $\text{char} \mathbb{K} = 0$, another model is cdgas as before.) So roughly speaking, a derived scheme should be a topological space with a homotopy sheaf (∞ -sheaf) of simplicial commutative \mathbb{K} -algebras. A simplicial commutative \mathbb{K} -algebra is a cosimplicial affine \mathbb{K} -scheme.

Return to the simplex category Δ .

Definition 3.18. Define morphisms in Δ , the face maps $\delta^{n,i} : [n-1] \rightarrow [n]$ and degeneracy maps $\sigma^{n,i} : [n+1] \rightarrow [n]$ for $i = 0, \dots, n$ by

$$\delta^{n,i}(j) = \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases}, \quad \sigma^{n,i}(j) = \begin{cases} j, & j \leq i \\ j-1, & j > i \end{cases}$$

That is, $\delta^{n,i}$ misses i , and $\sigma^{n,i}$ repeats i .

Now let $S : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$ be a simplicial set. Write $S_n = S([n])$ and define face maps $d_{n,i} = S(\delta^{n,i}) : S_n \rightarrow S_{n-1}$ and degeneracy maps $s_{n,i} = S(\sigma^{n,i}) : S_n \rightarrow S_{n+1}$.

These satisfy the identities

$$\begin{aligned} \delta^{n,j} \circ \delta^{n-1,i} &= \delta^{n,i} \circ \delta^{n-1,j-1}, & 0 \leq i < j \leq n, \\ \sigma^{n,j} \circ \sigma^{n+1,i} &= \sigma^{n,i} \circ \sigma^{n+1,j+1}, & 0 \leq i \leq j \leq n, \\ \sigma^{n,j} \circ \delta^{n+1,i} &= \delta^{n,i} \circ \sigma^{n-1,j-1}, & 0 \leq i < j \leq n, \\ \sigma^{n-1,j} \circ \delta^{n,i} &= \delta^{n-1,i-1} \circ \sigma^{n-2,j}, & 0 \leq j < j+1 < i \leq n, \\ \sigma^{n,j} \circ \delta^{n+1,i} &= \text{id}, & i = j \text{ or } j+1. \end{aligned}$$

The category Δ is generated by the $\delta^{n,i}$, $\sigma^{n,i}$ subject only to these relations.

For $S : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$, eversing directions of morphisms, these satisfy

$$\begin{aligned} d_{n-1,i} \circ d_{n,j} &= d_{n-1,j-1} \circ d_{n,i}, & 0 \leq i < j \leq n, \\ s_{n+1,i} \circ s_{n,j} &= s_{n+1,j+1} \circ s_{n,i}, & 0 \leq i \leq j \leq n, \\ d_{n+1,i} \circ s_{n,j} &= s_{n-1,j-1} \circ d_{n,i}, & 0 \leq i < j \leq n, \\ d_{n,i} \circ s_{n-1,j} &= s_{n-2,j} \circ d_{n-1,i-1}, & 0 \leq j < j+1 < i \leq n, \\ d_{n+1,i} \circ s_{n,j} &= \text{id}, & i = j \text{ or } j+1. \end{aligned}$$

As Δ is generated by the $\delta^{n,i}, \sigma^{n,i}$ subject with those relations, to define a simplicial set S it is enough to give sets S_n for $n \geq 0$ and maps $d_{n,i} = S(\delta^{n,i}) : S_n \rightarrow S_{n-1}$ and $s_{n,i} = S(\sigma^{n,i}) : S_n \rightarrow S_{n+1}$ for $0 \leq i \leq n$ satisfying these relations, and all the other morphisms $S(f)$ in S for $f : [k] \rightarrow [l]$ can be written as compositions of the $d_{n,i}$ and $s_{n,i}$. This gives us a way to draw a picture of a simplicial set:

$$\begin{array}{ccccccc}
 & & & \longleftarrow d_{3,0} \longrightarrow & & & \\
 & & \longleftarrow d_{2,0} \longrightarrow & \longrightarrow s_{2,0} \longrightarrow & & & \\
 \longleftarrow d_{1,0} \longrightarrow & \longrightarrow s_{1,1} \longrightarrow & \longleftarrow d_{3,1} \longrightarrow & & & & \\
 S_0 \longrightarrow s_{0,0} \longrightarrow S_1 & \longleftarrow d_{2,1} \longrightarrow S_2 & \longrightarrow s_{2,1} \longrightarrow S_3 & \cdots & & & \\
 \longleftarrow d_{1,1} \longrightarrow & \longrightarrow s_{1,0} \longrightarrow & \longleftarrow d_{3,2} \longrightarrow & & & & \\
 & \longleftarrow d_{2,2} \longrightarrow & \longrightarrow s_{2,2} \longrightarrow & & & & \\
 & & \longleftarrow d_{3,3} \longrightarrow & & & &
 \end{array}$$

3.4 Kan Complexes and Weak Kan Complexes

Definition 3.19. (a) For $n \geq 0$, the *standard n -simplex* Δ^n is the simplicial set $\text{Hom}(-, [n]) : \Delta^{\text{op}} \rightarrow \mathbf{Sets}$.

(b) Define morphisms of simplicial sets $\delta^{n,i} : \Delta^{n-1} \rightarrow \Delta^n$ for $i = 0, 1, \dots, n$ by $\delta^{n,i} = - \circ \delta^{n,i}$, where $\delta^{n,i} : [n-1] \rightarrow [n]$ is the face map. Then $\delta^{n,i}$ is an injective morphism in \mathbf{sSets} , and $\delta^{n,i}(\Delta^{n-1})$ is a simplicial subset of Δ^n , that is, $\delta^{n,i}(\Delta^{n-1})([k]) \subset \Delta^n([k])$ in \mathbf{Sets} for each $k \geq 0$.

(c) Define the $n-1$ -sphere $\partial\Delta^n$, as a simplicial subset of Δ^n , by

$$\partial\Delta^n = \bigcup_{i=0, \dots, n} \delta^{n,i}(\Delta^{n-1}),$$

where for each $k \geq 0$ we take the union in subsets of $\Delta^n([k])$. It is a simplicial set with an inclusion $\partial\Delta^n \hookrightarrow \Delta^n$.

(d) For $k = 0, \dots, n$, define the k -horn \mathbb{A}_k^n , as a simplicial subset of Δ^n , by

$$\mathbb{A}_k^n = \bigcup_{i=0, \dots, n; i \neq k} \delta^{n,i}(\Delta^{n-1}),$$

It is a simplicial set with an inclusion $\mathbb{A}_k^n \hookrightarrow \Delta^n$. It is an inner horn if $0 < k < n$.

Definition 3.20. A simplicial set S is a *Kan complex* if for all $0 \leq k \leq n$ and all $f : \mathbb{A}_k^n \rightarrow S$ in \mathbf{sSets} , there exists a (not necessarily unique) morphism $g : \Delta^n \rightarrow S$ making the following diagram commute:

$$\begin{array}{ccc}
 \mathbb{A}_k^n & \xrightarrow{f} & S \\
 \searrow & & \nearrow g \\
 & \Delta^n &
 \end{array}$$

Then we say that all horns in S have fillers.

We call S a *weak Kan complex* if the above holds for all $0 < k < n$. Then we say that all inner horns in S have fillers.

Example 3.21. If X is a topological space then $\mathbf{Sing}(X)$ is a Kan complex.

Indeed, we must fill in the diagram in **Top**:

$$\begin{array}{ccc}
 \mathbf{TR}(\mathbb{A}_k^n) & \xrightarrow{f} & X \\
 \searrow & & \nearrow g \\
 & \mathbf{TR}(\Delta^n) &
 \end{array}$$

This is possible as $\mathbf{TR}(\Delta^n)$ retracts onto $\mathbf{TR}(\mathbb{A}_k^n)$.

3.5 Quasicategories

Quasicategories are a model (arguably the best) for $(\infty, 1)$ -categories, developed by Joyal and Lurie. Lurie went on to use them as the foundation for his theory of Derived Algebraic Geometry.

Definition 3.22. *A quasicategory is a weak Kan complex.*

Next we first define the nerve $N(\mathcal{C})$ of a category \mathcal{C} , then we will use them to explain how to treat a quasicategory like an $(\infty, 1)$ -category.

Definition 3.23. *Let \mathcal{C} be a small category. Define a simplicial set $N(\mathcal{C})$ called the nerve of \mathcal{C} as follows:*

- 0-simplices (elements of $N(\mathcal{C})([0])$) are objects $X \in \mathcal{C}$;
- 1-simplices (in $N(\mathcal{C})([1])$) are morphisms $X_0 \xrightarrow{f_1} X_1$ in \mathcal{C} .
- n -simplices are sequences $X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{n-1} \xrightarrow{f_n} X_n$ in \mathcal{C} .
- Face maps $d_{n,i} : N(\mathcal{C})([n]) \rightarrow N(\mathcal{C})([n-1])$ omit X_0, f_1 for $i = 0$, omit X_n, f_n for $i = n$, and compose f_i, f_{i+1} for $0 < i < n$.
- Degeneracy maps $s_{n,i} : N(\mathcal{C})([n]) \rightarrow N(\mathcal{C})([n+1])$ insert $\text{id}_{X_i} : X_i \rightarrow X_i$ into the sequence.

Functors $F : \mathcal{C} \rightarrow \mathcal{D}$ induce morphisms $N(\mathcal{C}) \rightarrow N(\mathcal{D})$.

We can characterize when a simplicial set is a nerve via horn-filling.

Proposition 3.24. (a) *A simplicial set S is isomorphic to the nerve $N(\mathcal{C})$ of a small category \mathcal{C} iff all inner horns $f : \Delta_k^n \rightarrow S$ have unique fillers.*

This implies that $N(\mathcal{C})$ is a weak Kan complex for all \mathcal{C} .

(b) *A simplicial set S is isomorphic to the nerve $N(\mathcal{C})$ of a small groupoid \mathcal{C} iff all horns $f : \Delta_k^n \rightarrow S$ for $n > 1$ have unique fillers.*

(c) *The nerve $N(\mathcal{C})$ of a small category \mathcal{C} is a Kan complex iff \mathcal{C} is a groupoid.*

Note that Proposition suggests that we can see weak Kan complexes as generalizations of categories, and Kan complexes as generalizations of groupoids. We already know that Kan complexes are the fibrant-cofibrant objects in **sSets**, so by the fundamental theorem of model theory, the category with objects Kan complexes and homotopy classes of morphisms between them is equivalent to $\text{Ho}(\mathbf{sSets})$, so a model for ∞ -groupoids.

3.6 Stable Infinity Categories

We will give a reason why we need stable ∞ -categories:

Example 3.25. *I would argue that triangulated categories are not quite the ‘right’ theory. However, they are a very good approximation - you can work with them for years and not notice the problems.*

As a signal that there should be something more, recall that if \mathcal{T} is a triangulated category, and $u : X \rightarrow Y$ a morphism in \mathcal{T} , there is $\text{cone}(u) \in \mathcal{T}$, in a distinguished triangle $X \rightarrow Y \rightarrow \text{cone}(u) \rightarrow X[1]$ in \mathcal{T} . This is begging to be turned into a cone functor: we would like a category $\text{Mor}(\mathcal{T})$ of morphisms in \mathcal{T} , and a functor $\text{cone} : \text{Mor}(\mathcal{T}) \rightarrow \mathcal{T}$ mapping $u \mapsto \text{cone}(u)$ on objects. To try to define cone on morphisms in $\text{Mor}(\mathcal{T})$, consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & \text{cone}(u) & \longrightarrow & X[1] \\ \downarrow f & & \downarrow g & & \downarrow \text{cone}(f,g) & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \longrightarrow & \text{cone}(u') & \longrightarrow & X'[1] \end{array}$$

and extension via the definition of triangulated categories. But it is not unique, so we cannot define cone.

So the explanation is \mathcal{T} should be a higher category (an ∞ -category)! In this case \mathfrak{n} -morphisms in $\text{Mor}(\mathcal{T})$ correspond to $(\mathfrak{n} + 1)$ -morphisms in \mathcal{T} . So to define cone on 1-morphisms in $\text{Mor}(\mathcal{T})$, we should be using 2-morphisms in \mathcal{T} . We consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \longrightarrow & \text{cone}(u) & \longrightarrow & X[1] \\ \downarrow f & \nearrow \eta & \downarrow g & & \downarrow \text{cone}(f,g;\eta) & & \downarrow f[1] \\ X' & \xrightarrow{u'} & Y' & \longrightarrow & \text{cone}(u') & \longrightarrow & X'[1] \end{array}$$

where $\eta : u' \circ f \Rightarrow g \circ u$ is a 2-morphism. Then $\text{cone}(f,g;\eta)$ should exist and be unique up to 2-isomorphism. When we pass to the homotopy category $\text{Ho}(\mathcal{T})$, this choice of η is forgotten, which is why we lose uniqueness of $\text{cone}(f,g)$. Note moreover that if we want \mathcal{T} and $\text{Mor}(\mathcal{T})$ to be objects of the same type we cannot truncate to \mathfrak{n} -categories for any finite \mathfrak{n} — we need $\mathfrak{n} = \infty$.

3.7 Dg-categories and Segal Categories

4 Derived Schemes and Derived Stacks

4.1 Higher Stacks

4.2 Derived Schemes and Derived Stacks

5 Geometry of Derived Stacks

6 Need to add

References

- [EP23] Jon Eugster and Jon P Pridham. An introduction to derived (algebraic) geometry. *Rendiconti di Matematica*, September 2023.
- [Lur09] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009.