PERVERSE SHEAVES

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Abstract. In this note we will introduce the basic theory of perverse sheaves, including constructible sheaves, perverse sheaves, nearby and vanishing cycles. Moreover we will also give a glimpse of $\mathscr{D}$-modules, the Riemann-Hilbert correspondence and mixed Hodge modules. Finally we will consider some applications of the theory, such as enumerative geometry and representation theory.

Contents

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1. Introduction

1.1. **Background/Motivation.** Perverse sheaves were discovered in the fall of 1980 by Beilinson-Bernstein- Deligne-Gabber in [BBDG18], sitting at the confluence of two major developments of the 1970s: the intersection homology theory of Goresky-MacPherson, and the Riemann-Hilbert correspondence, due to Kashiwara and Mebkhout.

We will first follows the book [Ach21] to learn the basic theory of perverse sheaves. We will focus the theory of algebraic varieties over $\mathbb{C}$ and using analytic topology. We will also give a quike discussion about pure-algebraic theory using étale topology and étale cohomology. See also original [BBDG18].

We will also discuss some applications of this theory. Such as representation theory and enumerative geometry, especially the relative DT conjecture.

The prerequisites are: familiarity with the language of derived and triangulated categories; familiarity with introductory algebraic topology and some topology of complex algebraic varieties; familiarity with basic algebraic geometry.

1.2. **Related works and some future direction.** Need to add.

Acknowledgments. Need to add.

2. Recollection of the basic theory of sheaves

2.1. **Six functors.** Here we recollect some definitions of sheaves. Including six functors.

Definition 2.1. *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

- *Let $\mathscr{F} \in \mathsf{Sh}(Y, R)$, then the* pullback *$f^{-1}(\mathscr{F})$ of $\mathscr{F}$ is the sheafification of*
$$f^{-1}_{\mathrm{pre}}(\mathscr{F}) : U \mapsto \varinjlim_{V\subset Y \ open, V\supset f(U)} \mathscr{F}(V).$$
This is an exact functor.
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* pushforward *$f_*(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_*(\mathscr{F})(U) := \mathscr{F}(f^{-1}(U))$.*
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* proper pushforward *$f_!(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_!(\mathscr{F})(U) := \{s \in \mathscr{F}(f^{-1}(U)) : f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U$ is proper$\}$.*
- *Let $\mathscr{F} \in \mathsf{D}^-(X, R)$ and $\mathscr{G} \in \mathsf{D}^-(Y, R)$, we can define* external tensor product *as*
$$\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G} := p_1^{-1}\mathscr{F} \otimes^{\mathbf{L}} p_2^{-1}\mathscr{G}$$
where p_i are projections.
- *We can define*
$$\mathbb{R}\mathscr{H}om(-,-) : \mathsf{D}^-(X,R)^{\mathrm{op}} \times \mathsf{D}^+(X,R) \to \mathsf{D}^+(X,R),$$
$$- \otimes^{\mathbf{L}} - : \mathsf{D}^{\pm}(X,R) \times \mathsf{D}^{\pm}(X,R) \to \mathsf{D}^{\pm}(X,R).$$

Here we recollect some useful and basic results about these functors.

Proposition 2.2 ([Ach21])**.** *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

(1) f^{-1} is exact and $f_*, f_!$ are left exact functor. So we can define
$$\mathbf{R}f_*, \mathbf{R}f_! : \mathsf{D}^+(X,R) \to \mathsf{D}^+(Y,R), \quad f^{-1} : \mathsf{D}(Y,R) \to \mathsf{D}(X,R).$$
Moreover, consider $f : X \to Y$ and $g : Y \to Z$, then we have $(g \circ f)^{-1} = f^{-1}g^{-1}$ and $(g \circ f)_* = g_* \circ f_*$. If X, Y, Z are Hausdorff and locally compact, then $(g \circ f)_! = g_! \circ f_!$.

(2) If $h : Y \hookrightarrow X$ is a locally closed embedding, then for any $\mathscr{F} \in \mathsf{Sh}(Y, R)$ the sheaf $h_!(\mathscr{F})$ is the sheafification of $h_{!,\mathrm{pre}}\mathscr{F}$ which maps U to $\Gamma(U \cap Y, \mathscr{F})$ if $U \cap \overline{Y} \subset Y$ and 0 otherwise. Moreover in this case $h_!$ is exact. Note that $h_!(\mathscr{F})_x \cong \begin{cases} \mathscr{F}_x & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$

(3) We have
$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}f_*\mathscr{G})$$
for any $\mathscr{F} \in \mathsf{D}^-(Y,R)$ and $\mathscr{G} \in \mathsf{D}^+(X,R)$.

(4) *We have*
$$\mathbf{R}\mathscr{H}om(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G}, \mathscr{H}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}\mathscr{H}om(\mathscr{G}, \mathscr{H}))$$
for any $\mathscr{F}, \mathscr{G} \in \mathsf{D}^-(X,R)$ *and* $\mathscr{H} \in \mathsf{D}^+(X,R)$.

Proposition 2.3 ([Ach21], Prop. 1.4.21)**.** *Let* $f : X \to X'$ *and* $g : Y \to Y'$ *be continuous maps.*

(1) *For* $\mathscr{F} \in \mathsf{D}^-(X', R)$ *and* $\mathscr{G} \in \mathsf{D}^-(Y', R)$, *there is a natural isomorphism*
$$f^{-1}\mathscr{F} \boxtimes^{\mathbf{L}} g^{-1}\mathscr{G} \cong (f \times g)^{-1}(\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}).$$
(2) *For* $\mathscr{F}, \mathscr{F}' \in \mathsf{D}^-(X,R)$ *and* $\mathscr{G}, \mathscr{G}' \in \mathsf{D}^-(Y,R)$, *there is a natural isomorphism*
$$(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{F}') \boxtimes^{\mathbf{L}} (\mathscr{G} \otimes^{\mathbf{L}} \mathscr{G}') \cong (\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}) \otimes^{\mathbf{L}} (\mathscr{F}' \boxtimes^{\mathbf{L}} \mathscr{G}').$$
(3) *Assume that* $\mathrm{gl.dim}(R) < \infty$ *has finite global dimension and that our spaces are Hausdorff and locally compact. For* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *and* $G \in \mathsf{D}^+(Y,R)$, *there is a natural isomorphism*
$$\mathbf{R}f_!\mathscr{F} \boxtimes^{\mathbf{L}} \mathbf{R}f_!\mathscr{G} \cong \mathbf{R}(f \times g)_!(\mathscr{F} \boxtimes^{\mathbf{L}} \mathscr{G}).$$

Remark 2.4. *If* $\mathrm{gl.dim}(R) < \infty$, *(1)(2) also hold for* D^+.

Theorem 2.5 (Proper base change, [Ach21] Thm. 1.2.13)**.** *Consider a cartesian square*
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

(1) *If all the spaces are Hausdorff and locally compact, then for any* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *we have isomorphism*
$$g^{-1}f_!\mathscr{F} \cong f'_!(g')^{-1}\mathscr{F}.$$
(2) *If* f *is proper, then for any* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *we have isomorphism*
$$g^{-1}f_*\mathscr{F} \cong f'_*(g')^{-1}\mathscr{F}.$$

Theorem 2.6 (Projection formula, [Ach21] Thm. 1.4.9)**.** *Let* $f : X \to Y$ *be a continuous map of Hausdorff and locally compact spaces, and assume that* $\mathrm{gl.dim}(R) < \infty$. *For* $\mathscr{F} \in \mathsf{D}^+(X,R)$ *and* $\mathscr{G} \in \mathsf{D}^+(Y,R)$, *there is a natural isomorphism*
$$\mathbf{R}f_!\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G} \cong \mathbf{R}f_!(\mathscr{F} \otimes^{\mathbf{L}} f^{-1}\mathscr{G}).$$

Remark 2.7 (Change of scalars)**.** *We can also consider change of scalars. For ring map* $\phi : R \to R'$, *we can define* $\mathrm{Res}_{R,R'} : \mathsf{Sh}(X,R') \to \mathsf{Sh}(X,R)$ *and* $R' \otimes - : \mathsf{Sh}(X,R) \to \mathsf{Sh}(X,R')$ *as usual. Note that* $\mathrm{Res}_{R,R'}$ *is exact and* $R' \otimes -$ *is right exact. So we have* $\mathrm{Res}_{R,R'} : \mathsf{D}(X,R') \to \mathsf{D}(X,R)$ *and* $R' \otimes^{\mathbf{L}} - : \mathsf{D}^-(X,R) \to \mathsf{D}^-(X,R')$. *We have the following results about these. Let* $f : X \to Y$ *be a continuous map, and let* $\phi : R \to R'$ *be a ring homomorphism.*

- *For* $\mathscr{F} \in \mathsf{D}^-(X,R)$, *there is a natural isomorphism*
$$f^{-1}(R' \otimes^{\mathbf{L}} \mathscr{F}) \cong R' \otimes^{\mathbf{L}} f^{-1}(\mathscr{F}).$$
If $\mathrm{gl.dim}(R) < \infty$, *this holds for* D^+.
- *If* X, Y *are Hausdorff and locally compact and* R *is Noetherian with* $\mathrm{gl.dim}(R) < \infty$, *then*
$$R' \otimes^{\mathbf{L}} \mathbf{R}f_!\mathscr{F} \cong \mathbf{R}f_!(R' \otimes^{\mathbf{L}} \mathscr{F}).$$

Finally, we will consider the functor $f^!$.

Theorem 2.8 (Verdier)**.** *Let* $f : X \to Y$ *be a continuous map of Hausdorff and locally compact spaces with a Noetherian ring* R *such that* $\mathrm{gl.dim}(R) < \infty$. *Assume that* $f_!$ *has finite cohomological dimension. Then there exists a triangulated functor*
$$f^! : \mathsf{D}^+(Y,R) \to \mathsf{D}^+(X,R)$$
such that we have
$$\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{F}, \mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G})$$
for any $\mathscr{F} \in \mathsf{D}^-(X,R)$ *and* $\mathscr{G} \in \mathsf{D}^+(Y,R)$.

Sketch. Here we give a sketch and the details we refer [KS94] Theorem 3.1.5 and Proposition 3.1.10. Now first using the assumption that R is Noetherian with $\mathrm{gl.dim}(R) < \infty$, we can show that $\underline{R}_X$ has a flat and soft resolution $\mathscr{K}$ of finite terms. For $\mathscr{F} \in \mathsf{Ch}^-(\mathsf{Sh}(X,R))$ and $\mathscr{G} \in \mathsf{Ch}^+(\mathsf{Sh}(Y,R))$, we can define a chain complex of presheaves

$$\mathsf{E}(\mathscr{F},\mathscr{G}) : U \mapsto \mathrm{Hom}_{\mathsf{Ch}}(f_!(\mathscr{F} \otimes j_{U!}(\mathscr{K}|_U)), \mathscr{G})$$

with inclusion $j_U : U \subset X$. It is actually a complex of sheaves and hence we have a functor

$$\mathsf{E} : \mathsf{K}^-(\mathsf{Sh}(X,R))^{\mathrm{op}} \times \mathsf{K}^+(\mathsf{Sh}(Y,R)) \to \mathsf{K}^+(\mathsf{Sh}(X,R)).$$

We can find that $\mathsf{E}(\mathscr{F},\mathscr{G}) \cong \mathscr{H}om_{\mathsf{Ch}}(\mathscr{F}, \mathsf{E}(\underline{R}_X,\mathscr{G}))$ with $\mathscr{H}om_{\mathsf{Ch}}(f_!(\mathscr{F} \otimes \mathscr{K}),\mathscr{G}) \cong f_*\mathsf{E}(\mathscr{F},\mathscr{G})$.

Moreover, we find that $\mathsf{E}(\mathscr{F}, -)$ maps injective complex into injective complex. Hence we have

$$\mathbf{R}\mathsf{E} : \mathsf{D}^-(\mathsf{Sh}(X,R))^{\mathrm{op}} \times \mathsf{D}^+(\mathsf{Sh}(Y,R)) \to \mathsf{D}^+(\mathsf{Sh}(X,R))$$

with $\mathbf{R}\mathsf{E}(\mathscr{F},\mathscr{G}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}\mathsf{E}(\underline{R}_X,\mathscr{G}))$ and $\mathbf{R}\mathscr{H}om_{\mathsf{Ch}}(\mathbf{R}f_!(\mathscr{F} \otimes \mathscr{K}),\mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathsf{E}(\mathscr{F},\mathscr{G})$. Finally, we define $f^!$ as:

$$f^! : \mathsf{D}^+(Y,R) \to \mathsf{D}^+(X,R), \quad \mathscr{G} \mapsto \mathbf{R}\mathsf{E}(\underline{R}_X,\mathscr{G}).$$

Hence $\mathbf{R}\mathscr{H}om(\mathbf{R}f_!\mathscr{F},\mathscr{G}) \cong \mathbf{R}f_*\mathbf{R}\mathscr{H}om(\mathscr{F}, f^!\mathscr{G})$ for any $\mathscr{F} \in \mathsf{D}^-(X,R)$ and $\mathscr{G} \in \mathsf{D}^+(Y,R)$. □

Remark 2.9. *Why we define as this? Note that if we have proved this theorem and $f^!$ maps sheaves to sheaves, then we have:*

$$\mathscr{H}om(\mathscr{F}, f^!\mathscr{G})(U) = \mathrm{Hom}(j_{U!}j_U^{-1}\mathscr{F}, f^!\mathscr{G}) = \mathrm{Hom}(\mathbf{R}f_!(j_{U!}j_U^{-1}\mathscr{F}),\mathscr{G})$$

with $j_{U!}j_U^{-1}\mathscr{F} = \mathscr{F} \otimes j_{U!}\underline{R}_U$.

Proposition 2.10 ([Ach21], Prop. 1.5.6-1.5.9)**.** *Assume R is Noetherian ring with $\mathrm{gl.dim}(R) < \infty$.*

1. *Let $f : X \to Y$ and $g : Y \to Z$ are continuous maps of Hausdorff and locally compact spaces such that $f_!, g_!$ have finite cohomological dimension, then we have $f^!g^! \cong (g \circ f)^!$.*
2. *Consider cartesian*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

   of continuous maps of Hausdorff and locally compact spaces with $f_!$ has finite cohomological dimension, then $g'_*(f')^! \cong f^!g_*$.

3. *Let $f : X \to Y$ is a continuous map of Hausdorff and locally compact spaces such that $f_!$ have finite cohomological dimension, then*

$$f^!\mathbf{R}\mathscr{H}om(\mathscr{F},\mathscr{G}) \cong \mathbf{R}\mathscr{H}om(f^{-1}\mathscr{F}, f^!\mathscr{G})$$

   for any $\mathscr{F} \in \mathsf{D}^b(Y,R)$ and $\mathscr{G} \in \mathsf{D}^+(Y,R)$.

4. *Let $f : X \to Y$ is a continuous map of Hausdorff and locally compact spaces of finite soft dimension, then there is a natural map*

$$f^!\mathscr{F} \otimes^{\mathbf{L}} f^{-1}\mathscr{G} \to f^!(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G})$$

   and when $\mathscr{G}$ is a local system of finite type, this map is an isomorphism.

Definition 2.11. *Let X be a Hausdorff and locally compact space of finite soft dimension. The* dualizing complex *of X, denoted by $\omega_X \in \mathsf{D}^+(X,R)$ given by*

$$\omega_X := f^!\underline{R}_{pt}$$

for $f : X \to \{pt\}$. The Borel-Moore-Verdier duality functor *is the functor*

$$\mathbb{D} : \mathsf{D}^-(X,R)^{op} \to \mathsf{D}^+(X,R), quad \mathscr{F} \mapsto \mathbf{R}\mathscr{H}om(\mathscr{F},\omega_X).$$

Corollary 2.12. *Assume R is Noetherian ring with $\mathrm{gl.dim}(R) < \infty$ and let $f : X \to Y$ be a continuous map of locally compact spaces of finite soft dimension.*

1. *There is a canonical isomorphism $f^!\omega_Y \cong \omega_X$.*
2. *For any $\mathscr{F} \in \mathsf{D}^-(X,R)$, there is a natural isomorphism $\mathbf{R}f_*\mathbb{D}(F) \cong \mathbb{D}(\mathbf{R}f_!F)$. For any $\mathscr{G} \in \mathsf{D}^-(Y,R)$, there is a natural isomorphism $f^!\mathbb{D}(\mathscr{F}) \cong \mathbb{D}(f^{-1}\mathscr{F})$.*

2.2. **Open and closed embeddings.**

2.3. Local Systems.

3. Constructible sheaves

3.1. Preliminaries from algebraic geometry.

3.2. Stratifications and constructible sheaves.

3.3. Artin's vanishing theorem.

3.4. Constructibility theorem.

3.5. Verdier duality theorem.

3.6. More compatibilities.

3.7. Borel-Moore homology and fundamental classes.

4. Perverse sheaves

4.1. Perverse sheaves.

4.2. Intersection cohomology complexes.

4.3. Affine pushforward.

4.4. Smooth pullback and smooth descent.

4.5. Semismall maps.

4.6. The decomposition theorem and the hard Lefschetz theorem.

5. Nearby and vanishing cycles

5.1. Basic things.

5.2. Properties.

5.3. Beilinson's theorem.

6. A glimpse of the algebraic theory

7. About $\mathscr{D}$-modules and mixed Hodge modules

7.1. $\mathscr{D}$-modules and Riemann-Hilbert correspondence.

7.2. Mixed Hodge modules.

8. More Applications

8.1. Relative Donaldson-Thomas Theory for 4-folds.

8.2. For geometric representation theory.

References

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