Varieties of Minimal Rational Tangents on the Fano Varieties

Xiaolong Liu

March 11, 2024

Contents

Preface

Here we first develope some history, see also in the introduction in [37].

A Fano manifold is a smooth projective variety whose anti-canonical class $-K_X$ is ample. In dimension 1, the Riemann sphere $\mathbb{P}^1$ is the only example of a Fano manifold. In higher dimensions, two possible sources of complication exist, as usual:

(1) The product of two Fano manifolds is again a Fano manifold, or more generally, many fiber bundles over Fano manifolds with Fano fibers are themselves Fano.

(2) The blow-up of a Fano manifold with a suitable center is again Fano.

To handle complications of these sorts in higher dimensions, the minimal model program (MMP) has been developed since 1980's.

For uniruled varieties, there is another machinery in handling these matters developed in 1990's based on the concept of rationally connected varieties which was nicely surveyed in [57]. In particular, sometimes we focused on the Fano manifolds have Picard number 1. The methods employed in these works include the classical method of double projections, adjunction theory, vector bundle techniques, as well as methods coming from the minimal model program.

In the 1990s, Ngaiming Mok and Jun-Muk Hwang have been trying to develop a geometric theory of Fano manifolds of Picard number 1 from a different perspective, by using rational curves of minimal degree on Fano manifolds Undoubtedly, the importance of rational curves in the study of Fano manifolds is well-known and most works mentioned above also use rational curves extensively as one of the main geometric tools. Before starting discussion, let me roughly describe some motivations and history around this idea.

The story begins with two related conjectures which were outstanding in the 1970's. Both were proposed as a generalization of the uniformization of Riemann surfaces for the genus zero case.

Conjecture 0.1 (Frankel Conjecture, Mori 1979 and Siu-Yau 1980)**.** *If X is a compact Kähler manifold of dimension n with everywhere positive holomorphic bisectional curvature, then $X \cong \mathbb{P}^n_{\mathbb{C}}$.*

Conjecture 0.2 (Hartshorne Conjecture, Mori 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k, if T_X is ample then $X \cong \mathbb{P}^n_k$.*

Note that Hartshorne conjecture implies Frankel conjecture. The Hartshorne conjecture solved by Mori ([69]) and the Frankel conjecture was also solved by Siu-Yau ([88]) using the completely different method which depends heavily on the positive curvature condition and can not be generalized to other Fano manifolds.

But the method of Mori provided a new ground for the study of high-dimensional Fano manifolds. Mori created a method, called bend-and-break, aiming to find the rational curves on Fano manifolds. Finally he shows that any Fano manifold is uniruled. To solve the Hartshorne conjecture, he recover the Fano manifold by the moduli space of rational curves of minimal degree. This is what we concerned here. Finally Mori give a much stronger result than Hartshorne conjecture (see the proof in Theorem 1.78):

Theorem 0.1 (Mori, 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k. If*

(1) $-K_X$ is ample, that is, X is a Fano manifold;

(2) For any non-constant morphism $f : \mathbb{P}^1_k \to X$ the bundle f^*T_X is the sum of line bundles of positive degree.

Then $X \cong \mathbb{P}^n_k$.

After resolving Frankel conjecture, Siu and Yau consider the following generalization:

Conjecture 0.3 (Generalized Frankel Conjecture, Mok 1988)**.** *A Fano manifold with a Kähler metric of non-negative holomorphic bisectional curvature is a Hermitian symmetric space.*

This conjecture was solved by Mok in 1988. Following Mori's proof of Hartshorne conjecture, he consider the space of rational curves of minimal degrees passing a point $\mathcal{K}_x$ and its tangent image $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ as all tangent vectors ofthese curves. Then as Mori's method, he want to use $\mathcal{C}_x$ to recover the Hermitian symmetric space. He use the following result due to Berger:

Theorem 0.2 (Berger's Theorem)**.** *If the holonomy group of a Kähler metric on a Fano manifold at a point x does not act transitively on $\mathbb{P}(\Omega^1_{X,x})$, then X is a Hermitian symmetric space different from the projective space.*

Mok actually show that $\mathcal{C}_x$ is invariant under the action of the holonomy group of a suitable deformation of a given Kähler metric of non-negative holomorphic bisectional curvature. Hence then we can using the Berger's theorem to recover the Hermitian symmetric space.

Now how about the anologue conjecture in algebraic geometry (from now on we just consider the varieties on $\mathbb{C}$)? How do we find the projective manifold with nef tangent bundle? Actually in 1994 we have the following result ([19]):

Theorem 0.3 (Demailly-Peternell-Schneider, 1994)**.** *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $X' \to X$ be a finite étale cover of maximum irregularity $q = q(X') = h^1(X', \mathscr{O}_{X'})$. Then*

(a) $\pi(X') \cong \mathbb{Z}^{\oplus 2q}$.

(b) The Albanese map $\alpha : X' \to \mathsf{Al}(X')$ is a smooth fibration over a q-dimensional abelian variety with nef relative tangent bundle.

(c) The fibres of α are Fano manifolds with nef tangent bundles.

Hence the geometry of projective manifold with nef tangent bundle is determined, up to a finite étale cover, by abelian varieties and Fano manifolds with nef tangent bundles. Actually in 1991, Campana and Peternell consider the following conjecture:

Conjecture 0.4 (Campana-Peternell Conjecture)**.** *Any Fano manifold with nef tangent bundle is a rational homogeneous variety.*

If we solved Campana-Peternell conjecture, then we can classify the projective manifolds with nef tangent bundle up to a finite étale cover.

One of the possible way is to find the algebraic-analogue of Mok's proof of Generalized Frankel conjecture, that is, the Berger's theorem. This seems a difficult way. Maybe we can use minimal rational curves and the splitting model of T_X on these curves to replace the geodesics and parallel translations. But we don't know what is algebraic holonomy groups!

Another is the original possible way:

(1) First prove Campana-Peternell conjecture for smooth varieties with Picard number one.

(2) Then prove that, given any Fano manifold with nef tangent bundle X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can recover the homogeneity of X.

So we need to use our theory to consider the Fano manifolds of Picard number one. Note that we will use the theory of VMRT to consider many results of these manifolds. Such as stability of the tangent bundles, Fano manifolds with non-isomorphic surjective endomorphism, Fano manifolds with big automorphism group, deformation rigidity and Remmert-Van de Ven/Lazarsfeld Problems and so on.

Note that from now on, $\mathbb{P}(-)$ is in the sense of Grothendieck and $\mathbf{P}(-)$ is in the geometric sense and $\mathsf{Grass}(s, V)$ is in the sense of geometry. The $\mathbb{P}^n$-bundle $f : X \to Y$ is that f is smooth and all fibers are $\mathbb{P}^n$. The projective bundle is $\mathbb{P}(\mathscr{E})$. We call the smooth projective varieties are projective manifolds.

Chapter 1

Basic Theory of Rational Curves

The main results here we follows the famous book [57].

1.1 Hilbert Schemes and Chow Schemes

1.1.1 Hilbert Schemes, a Basic Introduction

Definition 1.1. *Let X be an S-scheme, we define the Hilbert functor $\mathscr{H}ilb_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z.*

Fix a Polynomial P and a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{H}ilb^P_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z with Hilbert Polynomial P.

Theorem 1.2 (Grothendieck)**.** *Let S be a noetherian scheme, let $X \to S$ be a projective morphism, and $\mathscr{L}$ a relatively very ample line bundle on X. Then for any polynomial P, the Hilbert functor $\mathscr{H}ilb^P_{X/S}$ is representable by a projective S-scheme $\mathsf{Hilb}^P_{X/S}$. We also have $\mathsf{Hilb}_{X/S} = \coprod_P \mathsf{Hilb}^P_{X/S}$.*

Proof. Note that this notion of projectivity is much general than [31], but is the same when $S = \operatorname{Spec} k$. The proof is to embed it into Grassmannian. The original proof in [29] and we also refer [71], [57] and [23]. $\square$

Remark 1.3. *In [8] we can remove the noetherian hypothesis, by instead assuming strong (quasi-)projectivity of $X \to S$. So also [2].*

Example 1.4. *Some examples and interesting results:*

(a) We have $\mathsf{Hilb}^1_{X/S} = X/S$.

(b) *Let C be a curve over a field k, then*

$$\mathsf{Hilb}^m_{C/k} \cong S^m C := \underbrace{C \times \cdots \times C}_{m} / \mathfrak{S}_m.$$

Hence if C smooth, so is $\mathsf{Hilb}^m_{C/k}$. See also [23] Theorem 7.2.3(1) and Proposition 7.3.3.

(c) *Let S be a smooth surface over a field k, then $\mathsf{Hilb}^m_{S/k}$ is also smooth of dimension $2m$ and hence $\mathsf{Hilb}^m_{S/k} \to S^m X$ (we will see this later for general settings) is a resolution of singularities. Note that $S^m X$ is smooth if and only if X is smooth and $\dim X = 1$ or $m < 2$. See [23] Theorem 7.2.3(2) and Theorem 7.3.4.*

(d) *Let X be a nonsingular variety. Then $\mathsf{Hilb}^m_{X/k}$ is nonsingular for $m \leq 3$. Moreover, for any nonsingular 3-fold the scheme $\mathsf{Hilb}^4_{X/k}$ is singular. See [23] Remark 7.2.5 and 7.2.6.*

(e) *Let $\mathscr{E}$ be a vector bundle of rank $m+1$ over S and let $P_d(n) = \binom{m+n}{m} - \binom{m+n-d}{m}$, then*

$$\mathsf{Hilb}^{P_d}_{\mathbb{P}(\mathscr{E})/S} \cong \mathbb{P}((\mathrm{Sym}^d \mathscr{E})^\vee).$$

(f) *Let $Z \to S$, we have $\mathsf{Hilb}_{X \times_S Z/Z} \cong \mathsf{Hilb}_{X/S} \times_S Z$.*

(g) Hartshorne's Connectedness Theorem: *for every connected noetherian scheme S, $\mathsf{Hilb}^P_{\mathbb{P}^n_S/S}$ is connected.*

(h) *Let X be a connected variety over k, then $\mathsf{Hilb}^n_{X/k}$ is connected for all $n > 0$.*

(i) Murphy's Law: *It has many singularities, that is, for every scheme X finite type over $\mathbb{Z}$ and point $x \in X$, there exists a point $q \in \mathsf{Hilb}^P_{\mathbb{P}^n/k}$ of some Hilbert scheme and an isomorphism*

$$\widehat{\mathscr{O}}_{X,p}[[x_1, ..., x_s]] \cong \widehat{\mathscr{O}}_{\mathsf{Hilb}^P_{\mathbb{P}^n/k},q}[[y_1, ..., y_t]].$$

See [95]. In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ for some n. It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.

(j) *In [89] they gave a full classification of the situation where $\mathsf{Hilb}^P_{\mathbb{P}^n/k}$ smooth.*

Definition 1.5. *Let $X/S, Y/S$ are S-schemes, then we have a functor $\mathscr{H}om_S(X, Y)$ send S-scheme T into a set of T-morphisms $X \times_S T \to Y \times_S T$.*

For a subscheme $B \subset X$ proper over S and $g : B \to Y$, we have a functor $\mathscr{H}om_S(X, Y; g)$ send S-scheme T into a set of T-morphisms $X \times_S T \to Y \times_S T$ such that $f|_{B \times_S T} = g \times_S \mathrm{id}_T$.

Proposition 1.6. *If X/S and Y/S are both projective over S and X is flat over S, then $\mathscr{H}om_S(X,Y)$ represented by an open subscheme $\mathsf{Hom}_S(X,Y) \subset \mathsf{Hilb}_{X\times_S Y/S}$.*

Proof. Any $X \times_S T \to Y \times_S T$ correspond to its graph which is a closed immersion $\Gamma : X \times_S T \to X \times_S Y \times_S T$. As X is flat over S, then $X \times_S T$ is flat over T. Hence we get a morphism $\mathsf{Hom}_S(X,Y) \to \mathsf{Hilb}_{X\times_S Y/S}$. We omit the more details and refer Theorem I.1.10 in [57]. □

Proposition 1.7. *If X/S and Y/S are both projective over S and X, B are both flat over S, then $\mathscr{H}om_S(X,Y;g)$ represented by a subscheme $\mathsf{Hom}_S(X,Y;g) \subset \mathrm{Hom}_S(X,Y)$.*

Proof. Consider the restriction map $R : \mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_S(B,Y)$, then $g : B \to Y$ gives a section $G : S \to \mathrm{Hom}_S(B,Y)$. Hence $\mathsf{Hom}_S(X,Y;g) := R^{-1}(G(S)) \subset \mathsf{Hom}_S(X,Y)$ represents $\mathscr{H}om_S(X,Y;g)$. □

Now we state the deformation theory of Hilbert schemes. We only consider the simpler case that all schemes over a field k. For general case we refer Section 1.2 in [57].

Theorem 1.8. *Let Y be a projective scheme over a field k and $Z \subset Y$ is a subscheme. Then*

(a) We have
$$T_{[Z]}\mathsf{Hilb}_Y \cong \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

(b) The dimension of every irreducible components of Hilb_Y at $[Z]$ is at least
$$\dim \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) - \dim \mathrm{Ext}^1_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

Proof. See Theorem I.2.8 in [57]. For family case we refer Theorem I.2.15 in [57]. □

Corollary 1.9. *Let X, Y are projective varieties over a field k with a morphism $f : X \to Y$. Let Y is smooth over k. Then*

(a) We have
$$T_{[f]}\mathsf{Hom}_k(X,Y) \cong \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

(b) The dimension of every irreducible components of $\mathsf{Hom}_k(X,Y)$ at $[f]$ is at least
$$\dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

Proof. Let $Z \subset X \times_k Y$ be the graph of f, we claim that $\mathscr{I}_Z/\mathscr{I}_Z^2 \cong f^*\Omega^1_Y$. Indeed we have an exact sequence $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega^1_{X\times_k Y}|_Z \to \Omega^1_Z \to 0$. This is split by $\mathscr{O}_Z \cong \mathscr{O}_X \overset{(\mathrm{id}_X,1)}{\to} \mathscr{O}_{X\times_k Y}$. Then we can show the claim. Hence the results follows from Theorem 1.8. The family version we refer Theorem I.2.17 in [57]. □

1.1.2 Chow Schemes, a Basic Introduction

Here we only consider the schemes over a field k such that $\mathrm{char}(k) = 0$. The positive characteristic case is very complicated and we refer Section I.4 in [57].

Definition 1.10. *Let $g_i : U_i \to W$ be a proper morphism of schemes over W. Assume that W is reduced and U_i is irreducible. By generic flatness there is an open subset $W_i \subset g_i(U_i) \subset W$ such that g_i is flat of relative dimension d over W_i. Let $T = \mathrm{Spec}\,\Delta$ be the spectrum of a DVR Δ and $h : T \to W$ a morphism such that $h(T_g) \in W_i$ and $h(T_0) = w \in W$. Let $h^*U_i = U_i \times_h T$ and $\mathscr{J} \subset \mathscr{O}_{h^*U_i}$ the ideal of those sections whose support is contained in the special fiber of $h * U_i \to T$. Let $(U_i)'_T := \underline{\mathrm{Spec}}_T \mathscr{O}_{h^*U_i}/\mathscr{J}$ which is flat over T. Then we let $[Z_0]$ be the fundamental cycle of the central fiber of $(U_i)'_T \to T$, and define*

$$\lim_{h\to w}(U_i/U) := [Z_0] \in Z_d(g_i^{-1}(w) \times_{\kappa(w)} T_0)$$

which is called the cycle theoretic fiber *of g_i at w along h.*

Definition 1.11. *A* well defined family of d-dimensional proper algebraic cycles over W *is a pair $(g : U \to W)$ satisfying the following properties:*

(a) There is a reduced scheme $\mathrm{supp}\,U$ with irreducible components U_i such that $U = \sum_i m_i[U_i]$ is an algebraic cycle.

(b) W is a reduced scheme and $g : \mathrm{supp}\,U \to W$ is a proper morphism.

(c) Let $g_i := g|_{U_i}$. Then every g_i maps onto an irreducible component of W and every fiber of g_i is either empty or has dimension d. In particular there is a dense open subset $W_0 \subset W$ such that every g_i is flat over W_0.

(d) For every $w \in W$ there is a cycle $g^{[-1]}(w) \in Z_d(g^{-1}(w))$ such that for any $h : T \to W$ of spectrum of DVR such that $h(T_0) = w$ and $h(T_g) \in W_0$ we have

$$g^{[-1]}(w) =_{\mathrm{ess}} \sum_i m_i \lim_{h\to w}(U_i/W).$$

That is, both two cycles from a single cycle of $Z_d(g^{-1}(w))$.

Remark 1.12. *If W is normal, then (d) can be implied by (a)-(c). See Theorem I.3.17 in [57].*

Definition 1.13. *Let X be a scheme over S. A* well defined family of proper algebraic cycles of X/S over W/S *is a pair $(g : U/S \to W/S)$ satisfying the following properties:*

(a) $\mathrm{supp}\,U$ is a closed subscheme of $X \times_S W$ and g is the natural projection morphism.

(b) *$(g : U \to W)$ is a well defined family of d-dimensional proper algebraic cycles over W for some d.*

Proposition 1.14. *Assume that $g : U \to W$ is proper and flat of relative dimension d and W is reduced. Let $\sum_i m_i[U_i]$ be the fundamental cycle of U. Then $g : [U] \to W$ is a well defined family of algebraic cycles over W.*

Proof. See Lemma I.3.14 and Corollary I.3.15 in [57]. □

Definition 1.15 (Chow Schemes of Characteristic Zero)**.** *Let X/S and we define a functor $\mathscr{C}how_{X/S}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$.*

Let a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{C}how^{d,d'}_{X/S}$ sends Z/S to the set consists of well defined families of nonnegative proper algebraic cycles of $X \times_S Z/Z$ which is of dimension d and degree d'.

Theorem 1.16. *Let X/S be a scheme, projective over S and $\mathscr{O}(1)$ relatively ample. Then the functor $\mathscr{C}how^{d,d'}_{X/S}$ is representable by a semi-normal and projective S-scheme $\mathsf{Chow}^{d,d'}_{X/S}$. We also have $\mathsf{Chow}_{X/S} = \coprod_{d,d'} \mathsf{Chow}^{d,d'}_{X/S}$.*

Proof. Very complicated, we refer Theorem I.3.21 in [57]. □

Example 1.17. *Let X be a semi-normal variety, then $\mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.*

Proposition 1.18 (Hilbert-Chow)**.** *Let X, Y be S-schemes.*

(a) *We have a natural morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$. This morphism can be factored by dimensions.*

(b) *If X, Y be projective S-schemes and X/S flat, then we have*

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Chow}_{Y/S}.$$

Proof. For (a), consider $[\mathsf{Univ}^{\mathsf{Hilb}} \times_{\mathsf{Hilb}_{X/S}} \mathsf{Hilb}^{\mathrm{sn}}_{X/S}] \to \mathsf{Hilb}^{\mathrm{sn}}_{X/S}$, then by Proposition 1.14 this is a well defined family of algebraic cycles. This gives such morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$.

For (b), by (a) we have

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Hilb}(X \times_S Y/S)^{\mathrm{sn}} \to \mathsf{Chow}_{X \times_S Y/S} \to \mathsf{Chow}_{Y/S}$$

and well done. □

Remark 1.19. *Let X be a semi-normal variety, hence we have $(\mathsf{Hilb}^m_{X/k})^{\mathrm{sn}} \to \mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.*

1.1.3 Small Applications to Curves

For more applications we refer Section II.1 in [57]. Here we only need some easy case. We assume over a field k.

Theorem 1.20. *Let C be a proper curve and $f : C \to Y$ a morphism to a projective variety Y of dimension n such that Y is smooth along $f(C)$. Then*

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + n\chi(\mathscr{O}_C).$$

And equality holds if $H^1(C, f^*T_Y) = 0$, in this case it is smooth at $[f]$.

Proof. By Corollary 1.9(b) we have

$$\begin{aligned}\dim_{[f]} \mathsf{Hom}(C, Y) &\geq \dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X)\\ &= h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y)\\ &= \deg f^*T_Y + n\chi(\mathscr{O}_C)\end{aligned}$$

by Riemann-Roch theorem. The final statement follows from Corollary 1.9(a). □

Proposition 1.21. *Assume that X/S is flat, B/S is flat and finite of degree m and Y/S is smooth of relative dimension n. Then* $\dim \mathsf{Hom}(X, Y; g) \geq \dim \mathsf{Hom}(X, Y) - kn$.

Proof. Let $p : B \to S$ be the projection. By Corollary 1.9 we find that $\mathsf{Hom}(B, Y)$ is smooth over S of relative dimension rank kn. Thus $g(S) \subset \mathrm{Hom}(B, Y)$ is locally defined by kn equations. Pulling back these equations by R we obtain local defining equations. □

Lemma 1.22. *Let $0 \in T$ be the spectrum of a local ring and let U/T be a flat and proper and V/T be a variety. Let $p : U \to V$ as a T-morphism. If $p_0 : U_0 \to V_0$ is a closed immersion (resp. an isomorphism), then so is p.*

Proof. See Lemma I.1.10.1 and Proposition I.7.4.1.2 in [57]. We omit this. □

Theorem 1.23. *Let C be a projective curve over k and Y a smooth variety over k. Let $B \subset C$ be a closed subscheme which is finite over k. Assume that C is smooth along B. Let $g : B \to Y$ be a morphism. Then*

(a) We have
$$T_{[f]}\mathsf{Hom}(C, Y; g) \cong H^0(C, f^*T_Y \otimes \mathscr{I}_B).$$

(b) The dimension of every irreducible component of $\mathsf{Hom}(C, Y; g)$ *at* $[f]$ *is at least*
$$h^0(C, f^*T_Y \otimes \mathscr{I}_B) - h^1(C, f^*T_Y \otimes \mathscr{I}_B).$$

Proof. The original proof we refer [69]. A simple case of family version we refer Theorem II.1.7 in [57]. Here we assume k is algebraically closed. Here $\mathscr{I}_B = \mathscr{O}_C(-s_1 - ... - s_m)$.

Let $X_0 := C \times_k Y$ and let $\gamma_0 : C \cong \Gamma_0 \subset X_0$ be the graph of f. Let $\pi_1 : X_1 := \mathrm{Bl}_{\{s_1\}} X_0 \to X_0$ and Γ_1 be the strict transform of Γ_0. Let $\gamma_1 : C \cong \Gamma_1 \subset X_1$ as C is smooth at s_1. Repeat the process and finally we get $\pi_m : X_m := \mathrm{Bl}_{\{s_m\}} X_{m-1} \to X_{m-1}$ and Γ_m be the strict transform of Γ_{m-1}. Let $\gamma_m : C \cong \Gamma_m \subset X_m$. Then we have $\gamma_0^*(\mathscr{I}_{\Gamma_0}/\mathscr{I}_{\Gamma_0}^2) \cong f^*\Omega_Y^1$ and $\gamma_{i+1}^*(\mathscr{I}_{\Gamma_{i+1}}/\mathscr{I}_{\Gamma_{i+1}}^2) \cong \gamma_i^*(\mathscr{I}_{\Gamma_i}/\mathscr{I}_{\Gamma_i}^2) \otimes \mathscr{O}_C(-s_{i+1})$. Hence we get $\gamma_m^*(\mathscr{I}_{\Gamma_m}/\mathscr{I}_{\Gamma_m}^2) \cong f^*\Omega_Y^1 \otimes \mathscr{I}_B$.

Now we claim that there is an open neighborhood $[\Gamma_m] \in U \subset \mathsf{Hilb}_{X_m}$ such that $\mathsf{Hom}(C, Y; g) \cong U$. Indeed, let $U \subset \mathsf{Hilb}_{X_m}$ be the open set parametrizing those 1-cycles D for which the projection $D \to C$ is an isomorphism. This is open by Lemma 1.22.

First, the universal family of U is contained in $\mathsf{Hom}(C, Y; g)(U)$. Conversely consider $[p_0 : C \times R \to Y \times R] \in \mathsf{Hom}(C, Y; g)(R)$. Let its graph is $G_0 \subset X_0 \times R$. As $\{s_1\} \times R \subset G_0$ and $G_0 \to R$ smooth along $\{s_1\} \times R$, we let $G_1 \subset X_1 \times R$ be the strict transform of G_0. Then $G_1 \cong G_0 \cong C \times R$. Repeat the process and finally we get $X_m \times R \supset C \times R \cong G_m \in \mathsf{Hilb}_{X_m}(R)$. Hence this give the isomorphism $\mathsf{Hom}(C, Y; g) \cong U$. Hence by Theorem 1.8 and we get the result. □

1.2 Families of Rational Curves

We may assume all schemes over a field k of characteristic zero locally of finite type. Note that there are also have the same results by some small modification in the case of positive characteristic, see Section II.2 in [57].

Proposition 1.24. *Let $f : X \to Y$ be a proper morphism of relative dimension one. Assume that if T is the spectrum of a DVR and $h : T \to Y$ a morphism, then every irreducible component of $T \times_Y X$ has dimension two (By Corollary I.3.16 in [57] this is always the case if f is a well defined family of proper algebraic 1-cycles). Then the subset*

$$\{y \in Y : f^{-1}(y) \text{ has geometrically rational components}\} \subset Y$$

is closed in Y.

Proof. See Proposition II.2.2 in [57]. □

Corollary 1.25. *Let $g : U \to V$ be a family of proper algebraic 1-cycles of X/S. Let $U' \subset U$ be the set of points $u \in U$ which are contained in a geometrically rational component of $g^{-1}(g(u))$. The image of the natural morphism $U' \to X$ is called the rational locus of g. It is denoted by $\mathsf{RatLocus}(g : U \to V)$.*

Now let $V \to S$ is proper, then $\mathsf{RatLocus}(g : U \to V)$ is proper over S.

Proof. WLOG we let V is irreducible. Let $U = \sum_i a_i U_i$, then we just need to consider every $g_i : U_i \to V$. Consider the generic fiber D_i of g_i which is a irreducible curve, then if D_i rational, then so is whole g_i by Proposition 1.24. Hence $\mathsf{RatLocus}(g_i : U_i \to V) = \mathrm{Im}(U_i \to X)$ is proper over S. If D_i is not rational, then there is an open subset $\emptyset \neq W \subset V$ such that the fibers of g_i over Ware irreducible and nonrational. Thus

$$\mathsf{RatLocus}(g_i : U_i \to V) = \mathsf{RatLocus}(g_i : g_i^{-1}(V \backslash W) \to V \backslash W).$$

Hence we can apply Noetherian induction. □

Definition 1.26. *Let* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *be a subscheme correspond to the morphisms* $\mathbb{P}^1 \to X$ *birational to its image. By Lemma 1.22 since* $\mathbb{P}^1 \to X$ *birational to its image if and only if it is a immersion at its generic point, then* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *is an open subscheme.*

Definition 1.27. *Let X/S be a scheme, projective over S.*

(a) Let $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} = \bigcup_i W_i$ *be the decomposition into irreducible subschemes of semi-normalization of* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$*. By Proposition 1.18 we have the Hilbert-Chow morphism* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} \to \mathsf{Chow}_{X/S}$*. Let* $V_i' = \overline{\mathrm{Im}(U_i \to \mathsf{Chow}_{X/S})}$*. By Proposition 1.24* V_i' *parametrizes 1-cycles with geometrically rational components, and the generic 1-cycle is irreducible. Let* $V_i \subset V_i'$ *be the open subscheme parametrizing irreducible 1-cycles.*

Let $\eta_i \in V_i$ *be the generic points correspond to curves* C_i*. By generic smoothness* C_i *is a smooth rational curve. Let* V_i^{n} *be the normalization of* V_i*. Then we define the family of rational curves on* X *is*

$$\mathsf{RatCurves}^{\mathrm{n}}(X/S) := \coprod_i V_i^{\mathrm{n}}.$$

with a normalization morphism $\mathsf{RatCurves}^{\mathrm{n}}(X/S) \to \mathsf{Chow}_{X/S}$*.*

If $\mathscr{L}$ *is ample on* X/S*, then we can define* $\mathsf{RatCurves}^{\mathrm{n}}(X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *where* $\mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *is quasi-projective over* S *for any* d*. We define its universal rational curve is*

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) := \left(\mathsf{RatCurves}^{\mathrm{n}}(X/S) \times_{\mathsf{Chow}_{X/S}} \mathsf{Univ}^{\mathsf{Chow}}_{X/S}\right)^{\mathrm{n}}$$

be the normalization.

(b) Fix a section $f : S \to X$*. Similar as (a) we can define* $\mathsf{RatCurves}^{\mathrm{n}}(f, X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(f, X/S)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(f, X/S)$*. This is called family of rational curves passing through* $\mathrm{Im}(f)$*.*

In particular if $S = \mathrm{Spec}\, k$ *where* k *is a field and* $f : (\mathrm{Spec}\, k) = x \in X$*, then we will use the notation* $\mathsf{RatCurves}^{\mathrm{n}}(x, X) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(x, X)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(x, X)$*.*

Theorem 1.28. *(a) Let $f : X \to Y$ be a proper and surjective morphism between irreducible and normal schemes. Assume that the dimension of every fiber is one (hence f is a well defined family of proper 1-cycles by Remark 1.12). Assume that for every $y \in Y$ the cycle theoretic fiber $f^{[-1]}(y)$ is an irreducible and reduced rational curve, then f is a $\mathbb{P}^1$-bundle.*

(b) In the case of the definition, the universal morphisms

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) \to \mathsf{RatCurves}^{\mathrm{n}}(X/S) \textit{ and } \mathsf{Univ}^{\mathsf{rc}}(x,X) \to \mathsf{RatCurves}^{\mathrm{n}}(x,X)$$

are $\mathbb{P}^1$-bundles.

Proof. (b) follows directly from (a), so we just need to prove (a).

One can show that f is smooth at the generic point of every fiber (see Theorem I.6.5 in [57]). For $y \in Y$ pick three different points $x_1, x_2, x_3 \in f^{-1}(y)$ such that f is smooth at x_i. Let $S_i \subset X$ be a Cartier divisor which intersects $f^{[-1]}(y)$ transversally at x_i (there may be other intersection points). Hence $S_i \to Y$ is étale at x_i. Let

$$Z = S_1 \times_Y S_2 \times_Y S_3, \quad z = (x_1, x_2, x_3) \in Z \text{ and } X_Z = X \times_Y Z.$$

So $Z \to Y$ is étale at z, thus X_Z is normal along $f_Z^{-1}(z)$ and f is smooth above y iff f_Z is smooth above z by some commutative algebra. Furthermore, f_Z has three sections $s_i : Z \to X_Z$ corresponding to the S_i. By shrinking Z we may assume that these sections are disjoint.

In $\mathbb{P}^1_Z \to Z$ we have three disjoint sections $p_i : Z \to \mathbb{P}^1_Z$ corresponding to $\{0, 1, \infty\}$. Our aim is the construct an isomorphism $q : \mathbb{P}^1_Z \cong X_Z$ such that $q \circ p_i = s_i$. Let $h : \mathbb{P}^1_Z \times_Z X_Z \to Z$ be the projection. In order to construct the graph of q let $\Gamma \subset \mathsf{Chow}_{\mathbb{P}^1_Z \times_Z X_Z/Z}$ be the closed subvariety parametrizing 1-cycles D with the following properties:

(1) $\deg \mathscr{O}_{\mathbb{P}^1}(1)|_D = 1$;

(2) $\deg \mathscr{O}(s_1(Z))|_D = 1$;

(3) $(p_i(h(D)), s_i(h(D))) \in D$ for $i = 1, 2, 3$.

Let $\mathsf{Univ}^\Gamma \to \Gamma$ be the universal family. We claim that the natural projections $\pi_1 : \mathsf{Univ}^\Gamma \to \mathbb{P}^1_Z$ and $\pi_2 : \mathsf{Univ}^\Gamma \to X_Z$ are isomorphisms.

For any $t \in Z$ consider $h^{-1}(t)$. By construction $(h^{-1}(t))_{\mathrm{red}} \cong \mathbb{P}^1_{\kappa(t)} \times C_t$ where C_t is an irreducible geometrically rational curve, smooth for general t. As D gives a 1-cycle on $(h^{-1}(t))_{\mathrm{red}}$ which has bidegree $(1,1)$, thus D is either the graph of a birational morphism $q_t : \mathbb{P}^1_{\kappa(t)} \to C_t$ or the union of a vertical and of a horizontal section. In the latter case it can not contain all three points $(p_i(t), s_i(t))$. Hence D is the graph of the unique birational morphism q_t such that $q_t(p_i(t)) = s_i(t)$ for $i = 1, 2, 3$. Thus π_1, π_2 are both one-to-one. If C_t is smooth, then q_t is defined over $\kappa(t)$, thus π_1, π_2 are isomorphisms over the generic point of Z. Since X_Z and $\mathbb{P}^1_Z$ are normal, this implies that π_1, π_2 are isomorphisms. Well done. $\square$

Remark 1.29. *In positive characteristic, (a) is right if we assume generic-smoothness.*

Proposition 1.30. *Notation as above definitions, then*

(a) Let $m = \min\{d : \mathsf{RatCurves}^n_d(X/S) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(X/S)$ is proper over S for $k < 2m$.

(b) Let S be a field and let $m(x) = \min\{d : \mathsf{RatCurves}^n_d(x, X) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(x, X)$ is proper for $k < m + m(x)$.

Proof. (b) follows from the same proof of (a). For (a), as $\mathsf{Chow}^{1,k}_{X/S}$ is proper over S, we just need to show that $\bigcup_i V_i \subset \mathsf{Chow}^{1,k}_{X/S}$ is closed where $\mathsf{RatCurves}^n_k(X/S) = \bigcup_i V_i^{\mathrm{n}} \to \bigcup_i V_i$ is finite. Let $\sum_i a_i D_i \in \overline{\mathsf{RatCurves}^n_k(X/S)}$, then every D_i is rational by Proposition 1.24 and $\sum_i a_i \deg D_i = k < 2m$. By assumption $\deg D_i \geq m$, then $\sum_i a_i D_i$ is an irreducible and reduced rational curve. Hence $\mathsf{RatCurves}^n_k(X/S)$ closed. $\square$

Theorem 1.31. *Let $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}$ be the normalization of $\mathsf{Hom}_{\mathsf{bir}}$, then we have the following important results:*

(a) Let X/S projective scheme over S, then there is a natural commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{\quad U \quad} & \mathsf{Univ}^{\mathsf{rc}}(X/S) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{\quad u \quad} & \mathsf{RatCurves}^{\mathrm{n}}(X/S) \end{array}$$

where U and u are smooth of relative dimension 3 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles)

(b) Let X projective scheme over k with a k-point $x \in X(k)$, then there is a natural commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{\quad U \quad} & \mathsf{Univ}^{\mathsf{rc}}(x, X) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{\quad u \quad} & \mathsf{RatCurves}^{\mathrm{n}}(x, X) \end{array}$$

where U and u are smooth of relative dimension 2 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundles)

Proof. These are easy but boring since we consider the characteristic zero. See [57] Theorem II.2.15 and II.2.16. $\square$

Corollary 1.32. *Let X projective scheme over k with a k-point $x \in X(k)$, then*

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong H^0(\mathbb{P}^1, N_C), \quad T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(x, X) \cong H^0(\mathbb{P}^1, N_C \otimes \mathfrak{m}_x)$$

for general point $[C]$ where $f : \mathbb{P}^1 \to C \subset X$ is birational and $N_C = f^*T_X/T_{\mathbb{P}^1}$.

Proof. By Theorem 1.31, canonical morphism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) \to \mathsf{RatCurves}^{\mathrm{n}}(X/k)$ is a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle which is smooth. Hence we have

$$0 \to u^*\Omega^1_{\mathsf{RatCurves}^{\mathrm{n}}(X/k)} \to \Omega^1_{\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)} \to \Omega^1_u \to 0.$$

As $[C]$ general, we have $T_{[f]}\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) = T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)$. Hence

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)/\mathrm{Aut}(\mathbb{P}^1) \cong H^0(\mathbb{P}^1, N_C)$$

by trivial reason. Similar for $\mathsf{RatCurves}^{\mathrm{n}}(x, X)$. □

1.3 Free and Minimal Rational Curves

We will assume all scheme over a algebraically closed field k of characteristic zero.

1.3.1 Free Rational Curves

Definition 1.33. *Let C be a proper curve, X a smooth variety and $f : C \to X$ a morphism. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. We call f is called* free over f *if f is nonconstant and $H^1(C, f^*T_X \otimes \mathscr{I}_B) = 0$ and $f^*T_X \otimes \mathscr{I}_B$ is generated by global sections. Therefore we can define $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ parameterizes the free rational curves.*

Proposition 1.34. *Being free is an open. Hence $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ is open.*

Proof. Trivial by definition. □

Theorem 1.35. *Let C be a proper curve and X a smooth variety. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. Let $F : C \times \mathsf{Hom}(C, X; g) \to X$ be the universal morphism. Then $T_{\kappa(p,[f]), C\times \mathsf{Hom}(C,X;g)} = T_{\kappa(p),C} \oplus H^0(C, f^*T_X \otimes \mathscr{I}_B)$ if $p \notin B$ Consider the differential $df(s) : T_{\kappa(s),C} \to T_{\kappa(f(s)),X}$ and evaluation map*

$$\phi(p, f) : H^0(C, f^*T_X \otimes \mathscr{I}_B) \to f^*T_X \otimes \kappa(p),$$

then $dF(p, [f]) = df(p) + \phi(p, f)$. Furthermore If $\phi(p, f)$ is surjective, then F is smooth at $(p, [f])$. The converse also holds if $H^0(T_C \otimes \mathscr{I}_B) \to T_{\kappa(p),C}$ is surjective.

Proof. Trivial by definitions. □

Corollary 1.36. *If C is smooth and $f : C \to X$ is free over g, then $F : C \times \mathsf{Hom}(C, X; g) \to X$ is smooth along $(C \backslash B) \times [f]$. In particular $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth.*

Proposition 1.37. *Assume that $f : \mathbb{P}^1 \to X$, $g = f|_B$, $\mathrm{length} B \leq 2$ and write $f^*T_X \otimes \mathscr{I}_B = \sum_i \mathscr{O}(a_i)$. Then $\sharp\{i : a_i \geq 0\} = \mathrm{rank} dF(p, [f])$ for all $p \in \mathbb{P}^1 \backslash B$.*

In particular, if

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is smooth at $(p, [f])$ for some $p \in \mathbb{P}^1$, then f is free over g.

Proof. Note that $\mathrm{length} B \leq 2$ implies $H^0(T_{\mathbb{P}^1} \otimes \mathscr{I}_B) \to T_{\kappa(p), \mathbb{P}^1}$ is surjective for all $p \in \mathbb{P}^1 \backslash B$. Then these are trivial by arguments in Theorem 1.35. □

Theorem 1.38 (Kollár-Miyaoka-Mori, 1992)**.** *Let X be a smooth projective variety over k. Let $B \subset \mathbb{P}^1_k$ be a closed subscheme with $\mathrm{length} B \leq 2$ and $g : B \to X$. There are countably many subvarieties $V_i = V_i(B, g) \subset X$ such that if $f : \mathbb{P}^1 \to X$ is a nonconstant morphism such that $f|_B = g$ and $\mathrm{Im}(f) \not\subseteq \bigcup_i V_i$, then f is free over B.*

Proof. Let Z_i be the irreducible components of $\mathsf{Hom}(\mathbb{P}^1, X; g)$ with universal morphisms $F_i : \mathbb{P}^1 \times Z_i \to X$. Let $V_i = \overline{\mathrm{Im}(F_i)}$ if F_i is not dominant, and $V_i = X \backslash U_{F_i}$ if F_i is dominant, where $U_{F_i} \subset X$ is an open and dense subset such that $F_{i,\mathrm{red}} : \mathbb{P}^1 \times Z_{i,\mathrm{red}} \to X$ is smooth over U_{F_i} (this is where we use the char $= 0$ assumption). Then the result is trivial. □

Theorem 1.39. *Let X be a smooth proper variety over k, then the following statements equivalent.*

(1) X is uniruled.

(2) Generic rational curves of X are free.

(3) X has a free rational curve.

Proof. If X is uniruled then since the morphism

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is dominant, it is generic smooth. Hence by Proposition 1.37 the generic rational curves of X are free.

If the generic rational curves of X are free, then X has a free rational curve.

If X has a free rational curve, then the morphism $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth by Corollary 1.36. Hence it has densed image. Hence X is uniruled. □

Remark 1.40. *More properties of uniruled varieties we refer Section IV.1 in [57].*

1.3.2 Minimal Rational Curves

Definition 1.41. *Let X be a smooth projective variety over k of dimension n.*

(a) *A rational curve $f : \mathbb{P}^1 \to X$ is called* standard *(or* unbendable*) if*

$$f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-1-p}$$

where $p + 2 = -\deg f^*K_X$.

(b) *Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a* minimal free rational curve *if it is a free rational curve such that $-\deg f^*K_X$ is minimal.*

(c) *Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a* minimal rational curve *if it is a deformation of the minimal free rational curves. An irreducible component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}(X)$ is called a* minimal rational component *if it contains a rational curve of minimal degree.*

Remark 1.42. *For any non-constant $f : \mathbb{P}^1 \to X$, it can be factored by $f : \mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} X$ where h is birational to its image, then it is a immersion at generic points. Hence $T_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{R}^1}(2) \subset h^*T_X$. Hence $\mathscr{O}_{\mathbb{P}^1}(2\deg g) \subset f^*T_X$. So if we let $f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathscr{O}_{\mathbb{P}^1}(a_n)$ with $a_1 \geq \cdots \geq a_n$, then $a_1 \geq 2$.*

Proposition 1.43. *Let X be a smooth proper variety over k.*

(a) *If X has a free rational curve, then generic free rational curves of X are standard.*

(b) *If X is Fano and $x \in X$ is a general point, let minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}(X)$ and the corresponding component $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(x, X)$ be of minimal degree $p + 2$. Then $\mathcal{K}_x$ is a union of smooth varieties of dimension p and the general points are minimal standard.*

Proof. For (a), let that free rational curve is g, pick an irreducible component $V \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ containing $[g]$. Then by Theorem 1.39 V is dominated to X. Then by Theorem IV.2.4 and Corollary IV.2.9 in [57] there is a $W \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ such that dominated to X and general points in W is standard.

For (b), WLOG we let $\mathcal{K}_x$ irreducible and let $V \subset \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ be the irreducible component correspond to $\mathcal{K}_x$. Now since x is general, by Theorem 1.38 any members of V and hence $\mathcal{K}_x$ are free. Hence for any $[f] \in V$ we have $H^1(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = 0$. Then $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ is smooth at $[f]$ in this case. Hence by Theorem 1.23 V is also smooth at $[f]$ and of dimension $H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = p + 2$. Hence by Theorem 1.31(b) the morephism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) \to \mathsf{RatCurves}^{\mathrm{n}}(x, X)$ is smooth and is an $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundle, hence so is $V \to \mathcal{K}_x$. So $\mathcal{K}_x$ is smooth variety of dimension p. $\square$

Definition 1.44. *Let X be a projective variety and $\mathcal{V} \subset \mathsf{RatCurves}^n(X)$ be a closed irreducible subvariety. Let $u : \mathsf{Hom}^n_{\mathsf{bir}}(\mathbb{P}^1, X) \to \mathsf{RatCurves}^n(X)$ and $V' = u^{-1}(V)$. We say $\mathcal{V}$ (or V') is* unsplit *if V is proper on* $\operatorname{Spec}(k)$.

Corollary 1.45. *Let $\mathcal{V} \subset \mathsf{RatCurves}^n(X)$ is a closed irreducible subvariety of minimal degree, then it is unsplit.*

Proof. Follows from the definition and Corollary 1.30(a). □

1.4 Bend and Break

Bend and Break is a classical method aiming to find the rational curves over the projective varieties which is first observed by S. Mori in [69]. Here we will give the main results proved in [57]. See also the first chapter in [60] for a brief introduction. Here we assume all schemes over a infinity field k.

1.4.1 Main Results of Bend and Break

Definition 1.46. *Let S be a proper surface and $B \subset S$ a proper curve. We say that B is* contractible in S *if there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional.*

Proposition 1.47 (Rigidity Lemma)**.** *Let $f : X \to Y$ be a proper morphism such that $f_*\mathscr{O}_X = \mathscr{O}_Y$. Let $g : X \to Z$ be a morphism. Assume that for some $y \in Y$ there is a factorization*

$$
\begin{tikzcd}
 & & Z \\
X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(y) \arrow[l, hook'] \arrow[ur, dashed, "g|_{f^{-1}(y)}"] \arrow[d, "f_y"'] & \\
Y & \{y\} \arrow[l, hook'] \arrow[uur, "h_y"'] &
\end{tikzcd}
$$

Then there is an open neighborhood $y \in U \subset Y$ and a factorization

$$
\begin{tikzcd}
 & & Z \\
X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(U) \arrow[l, hook'] \arrow[ur, dashed, "g|_{f^{-1}(U)}"] \arrow[d, "f_U"'] & \\
Y & U \arrow[l, hook'] \arrow[uur, "h_U"'] &
\end{tikzcd}
$$

Proof. Let $\Gamma \subset Y \times Z$ be the image of (f,g). Then $p : \Gamma \to Y$ is proper and $p^{-1}(y) = (y, h_y(y))$ is finite over y. Thus there is an open neighborhood $y \in U \subset Y$ such that $p^{-1}(U) \to U$ is finite. Since

$$f_*\mathscr{O}_{f^{-1}(U)} \supset p_*\mathscr{O}_{p^{-1}(U)} \supset \mathscr{O}_U \supset f_*\mathscr{O}_{f^{-1}(U)}$$

which shows that $p^{-1}(U) \to U$ is an isomorphism. □

Corollary 1.48. *Let S be a proper surface and $B \subset S$ a contractible curve. Then $B \cdot B < 0$.*

In particular, let D be an irreducible and proper curve and C an arbitrary curve. Let $B_c = B \times \{c\} \subset B \times C$ where $c \in C$ is arbitrary. Then B_c is not contractible in $B \times C$.

Proof. Since $B \subset S$ a contractible, there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional. We prove this only for S smooth and S' projective. The general case works the same once the definition of intersection numbers is established in general.

Since S' projective, then we can find a finite morphism $f : S' \to \mathbb{P}^2$ since k is infinity. Let $\mathscr{O}(H) = f^*\mathscr{O}(1)$ which is ample and $H \cdot H > 0$ and $H \cdot B = 0$. By Hodge index theorem we have $B \cdot B < 0$.

For the final statement, note that $B_c \cdot B_c = 0$ hence B_c is not contractible. □

Theorem 1.49 (Fundamental Bend and Break, Mori-Miyaoka 1979-1986)**.** *Let B be a smooth proper and irreducible curve over k and S an irreducible, proper and normal surface. Let $p : S \to B$ be a morphism. Assume that there is an open subset $B^0 \subset B$, a smooth projective curve C and an isomorphism*

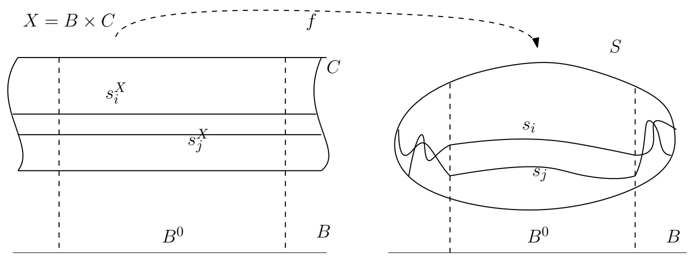
$$f : [C \times B^0 \xrightarrow{\pi} B^0] \cong [p^{-1}(B^0) \xrightarrow{p} B^0].$$

We call a section $s : B \to S$ is called flat if $s(B^0) = \{c\} \times B^0$ under the above isomorphism.

- *(a) If there is a contractible flat section $s_1 : B \to S$, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve intersecting $s_1(B)$.*
- *(b) If k algebraically closed, $g(C) = 0$ and there are two contmctible sections $s_1, s_2 : B \to S$, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ is either reducible or nonreduced.*
- *(c) Let L be a nef $\mathbb{R}$-Cartier divisor on S. If there are $k \geq 1$ contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $L \cdot D \leq \frac{2}{k} L \cdot C$ where C be the general fiber of p.*

(d) *Let L be a nef $\mathbb{R}$-Cartier divisor on S with $L^2 > 0$. If there are k contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $0 < L \cdot D < \frac{2}{k} L \cdot C$ where C be the general fiber of p.*

Proof. Let $X := C \times B$ and $\Gamma \subset X \times_B S$ be the closure of the graph of f. Consider projections p_X, p_S and every flat section s_i induces a flat section $s_i^X : B \to X$:



By Corollary 1.48 the rational map $f : X \dashrightarrow S$ is not defined some where along $s_i^X(B)$ if s_i contractible. Here we only prove (a) and (b). Actually (c) and (d) including the same idea with complicated computation and we refer Theorem II.5.4 in [57].

For (a), since $s_1 : B \to S$ is a contractible flat section, then $f : X \dashrightarrow S$ is not defined some where along $s_1^X(B)$. So we have a exceptional curve $D' \subset \Gamma$ of p_X. One can show that D' is rational, then take $D = p_S(D')$ and we get (a).

For (b), we assume that every fibres of p are integral, then $h^1(\mathscr{O}_{p^{-1}(b)}) = 1 - \chi(\mathscr{O}_{p^{-1}(b)})$ since k is algebraically closed. Then it is independent of $b \in B$ and every fiber of p is isomorphic to $\mathbb{P}^1$. Since p has sections, then S is a minimal ruled surface over B. Now the matrix of intersection form of $s_1(B), s_2(B)$ and $C \times \{b\}$ is $\mathbf{M} = \begin{pmatrix} -a_1 & c & 1 \\ c & -a_2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ where $-a_i = s_i(B)^2 < 0$ by Corollary 1.48 and $c = s_1(B) \cdot s_2(B) \geq 0$. Hence $\det \mathbf{M} = 2c + a_1 a + 2 > 0$ which is impossible since $\dim N_1(S) = 2$ since $N_1(S)$ generated by $s_1(B)$ and $C \times \{b\}$. $\square$

Corollary 1.50. *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $p_1, ..., p_k \in C$ be k distinct points and $g : \{p_1, ..., p_k\} \to X$ a morphism. Assume that there is a smooth, irreducible, proper curve B, an open set $B^0 \subset B$ and a morphism*

$$[h^0 : C \times B^0 \to X \times B^0] \in \mathsf{Hom}(C, X; g)(B^0)$$

such that $h^0(C \times \{b\})$ and $p_X \circ h^0(\{c\} \times B^0)$ are one dimensional for some $b \in B^0$ and $c \in C$.

Then there is a unique normal compactification $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$. Let $p : S \to B$.

(a) *If $k \geq 1$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D which passes through $g(p_1)$.*

(b) *If $C \cong \mathbb{P}^1$, $\dim \operatorname{Im}(p_X \circ h^0) = 2$ and $k \geq 2$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ is either reducible or nonreduced.*

(c) *Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 \leq L \cdot D \leq \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.*

(d) *Let L be a nef $\mathbb{R}$-Cartier divisor on X with $h^* L^2 > 0$ and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 < L \cdot D < \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.*

Proof. If $h^0(C \times \{b\})$ is a point for some $b \in B^0$, then by rigidity lemma $h^0(C \times \{b\})$ is a point for any $b \in B^0$, a contradiction. Thus h^0 is finite on every fiber of $C \times B^0 \to B^0$, hence the natural morphism h^0 is quasifinite. $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$.

If $\operatorname{Im}(p_X \circ h^0)$ is of dimension one, this is not hard to see. If $\operatorname{Im}(p_X \circ h^0)$ is of dimension two, then any p_i determines a contractible flat section of S given by $s_i : B^0 \to \{p_i\} \times B^0$. Then this follows from Theorem 1.49. □

Theorem 1.51 (Bend and Break). *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $f : C \to X$ be a nonconstant morphism.*

(a) *If $\dim_{[f]} \mathsf{Hom}(C, X) \geq \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and*

$$f_*[C] \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i [D_i].$$

(b) *If $g(C) = 0$ and $\dim_{[f]} \mathsf{Hom}(C, X) \geq 2 \dim X + 2$ (holds if $-K_X \cdot C \geq n + 2$), then for every $x_1, x_2 \in f(C)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x_1, x_2 \in \operatorname{supp}(\sum_i a_i D_i)$ and*

$$f_*[C] \sim_{\text{alg}} \sum_i a_i [D_i], \quad \sum_i a_i \geq 2.$$

(c) *Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. If $\dim_{[f]} \mathsf{Hom}(C, X) \geq k \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ ($a_1 > 0$) whose irreducible components are rational curves such that $x \in D_1$ and*

$$f_*[C] \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i [D_i], \quad L \cdot D_1 \leq \frac{2}{k} L \cdot f_* C.$$

Proof. Choose $\{p_1, ..., p_k\} \subset C$ with $g = f|_{\{p_1,...,p_k\}}$, then by Proposition 1.21 we have

$$\dim_{[f]} \mathsf{Hom}(C, X; g) \geq \dim_{[f]} \mathsf{Hom}(C, X) - k \dim X.$$

For (a), we assume $k = 1$ and $f(p_1) = x$ then $\dim_{[f]} \mathsf{Hom}(C, X; g) \geq 1$. Let B^0 be the normalization of an irreducible curve in $\mathsf{Hom}(C, X; g)$ containing $[f]$ and $h^0 : C \times B^0 \to X \times B^0$ the natural cycle morphism. By Corollary 1.50 we have compactifications B and S. Resolve the indeterminacies of $C \times B \dashrightarrow S$ we get

$$\begin{tikzcd} C \times B \arrow[dr, "q"'] & Y \arrow[l, "\rho_X"'] \arrow[r, "\rho_S"] & S \arrow[r, "h"] \arrow[dl, "p"] & X \times B \\ & B & & \end{tikzcd}$$

Pick $b \in B\backslash B^0$ as before we get $(p \circ \rho_S)^{-1}(b) = (q \circ \rho_X)^{-1}(b) = [C_0] + \sum_j e_j[E_j]$ where $C_0 \cong C$ and E_j rational as the exceptional curves of ρ_X. Set $f_x = (h \circ \rho_S)|_{C_0}$ and $\sum_i a_i D_i = (h \circ \rho_S)_*(\sum_j e_j[E_j])$ and well done.

The proof of (b) is similar as (a) using Corollary 1.50(b).

For (c), as before we obtain $D = D_1$ which satisfies all the requirements except that we only know that $D \cap \{f(p_1), ..., f(p_k)\} \neq \emptyset$. By letting the points p_i vary, we conclude that (c) holds except possibly for $k - 1$ points of $f(C)$.

Let $W \subset \mathsf{Chow}^1(X)$ be the connected component of $f_*[C]$. Let $V \subset W$ be the set of those points such that the corresponding cycle Z has the form $Z \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i[D_i]$ where the D_i are rational. By Proposition 1.24 V is closed in W and hence proper. By Corollary 1.25 $\mathsf{RatLocus}(V) \subset X$ is closed. Thus $\mathsf{RatLocus}(V) \cap C$ is a closed subset whose complement has at most $k - 1$ points. Therefore $C \subset \mathsf{RatLocus}(V)$ and this completes the proof. □

Theorem 1.52 (Smooth Bend and Break, Mori 1979-1982)**.** *Let X be a smooth projective variety.*

(a) Let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism. Then for every $x \in f(\mathbb{P}^1)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and

$$f_*[C] \sim_{\text{alg}} \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

(b) Let C be a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Assume that $\deg_C f^*(-K_X) > g(C) \dim X$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and $\deg_C f_x^*(-K_X) \leq g(C) \dim X$ and

$$f_*[C] \sim_{\text{alg}} (f_x)_*[C] + \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

Proof. By using Theorem 1.51(b) to our (a) and 1.51(a) to our (b) and induction on $\deg f^*H$ for some fixed ample divisor H on X, we can get the results. $\square$

1.4.2 Connection of Zero and Positive Characteristics

When we want to find the rational curves on variety X, we need to use the bend and break as Theorem 1.51(c). For any $f : C \to X$ passing $x \in X$ we need to make sure that $\dim_{[f]} \mathsf{Hom}(C, X) \geq k \dim X + 1$ for some k. Now by Theorem 1.20 we have

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + \dim X \chi(\mathscr{O}_C) = -C \cdot K_Y + \dim X - \dim X g(C).$$

If $-K_X \cdot C > 0$, to make sure the latter number larger, we need to find $C' \to C$ such that $-K_X \cdot C'$ larger but $g(C)$ do not change.

For $g(C) = 0$ we can use the large degree map $\mathbb{P}^1 \to \mathbb{P}^1$; for $g(C) = 1$ we use the $\times n$ morphism. But if $g(C) \geq 2$ we do not have such things. Now that in char $= p$ case we have Frobenius map which satisfies this condition. So we need to make char $= 0$ into char $= p$ case and come back to char $= 0$. This is the magic method due to Mori.

Assume that we are given finitely many schemes of finite type X_i, coherent sheaves $\mathscr{F}_i$ and maps g_i defined over a field k. All of these can be described by a finite number of equations (the schemes are given by affine charts and patching functions, the sheaves by finitely presented modules over the affine charts and patchings and the maps are described by their graphs which are schemes themselves). All these equations involve only finitely many elements a_j of the field k.

Let $\mathbb{F} \subset k$ be a subring which denote $\mathbb{F}_p$ if $\mathrm{char}(k) = p$ and $\mathbb{Z}$ if $\mathrm{char}(k) = 0$. Let $R := \mathbb{F}[a_j]$ is a finite type $\mathbb{F}$-algebra.

Lemma 1.53. *Let R be a finitely generated ring over $\mathbb{F}$. Then*

(a) The residue field $R/\mathfrak{m}$ of any maximal ideal $\mathfrak{m} \subset R$ is finite.

(b) The maximal ideals are dense in $\operatorname{Spec} R$.

Proof. (a) is trivial and (b) follows from both cases are Jacobson rings. $\square$

Aftering choose a_j and then R, we may consider X_i, $\mathscr{F}_i$ and g_i defined over $\operatorname{Spec} R$ which we denote them as $X_i^R, \mathscr{F}_i^R$and g_i^R. Hence after base change to $\operatorname{Spec} k$ we again have $X_i, \mathscr{F}_i, g_i$. Hence we constructed data $\{X_i^R, \mathscr{F}_i^R, g_i^R\}$ over $\operatorname{Spec} R$ such that the fibers over $\operatorname{Spec} k$ are the original data $\{X_i, \mathscr{F}_i, g_i\}$. Similarly for maximal ideal $\mathfrak{m} \subset R$ we have data $\{X_i^{\mathfrak{m}}, \mathscr{F}_i^{\mathfrak{m}}, g_i^{\mathfrak{m}}\}$ over $\operatorname{Spec} R/\mathfrak{m}$ which is positive characteristic by the previous Lemma (a).

Definition 1.54. *Let (P) be a property of schemes (morphisms etc.) in algebraic geometry. We say that (P) is of* finite type *if:*

Let K/k be a field extension and X_k a k-scheme. Then (P) holds for X_K iff there is a finitely generated subextension $K/F/k$ such that (P) holds for X_L for every L/F.

Remark 1.55. *A typical property that is not of finite type is: X_K has only finitely many K-points.*

Theorem 1.56 (Meta)**.** *Let $(P_1) \Rightarrow (P_2)$ be a statement in algebraic geometry that we want to prove. Assume the following four conditions:*

(1) (P_1) and (P_2) are of finite type.

(2) If (P_1) holds for the generic fiber of a morphism $X \to Y$, then it holds for every fiber over a nonempty open set.

(3) If (P_2) holds for every fiber of a morphism $X \to Y$ over a (not necessarily open) dense set, then it holds for the generic fiber.

(4) $(P_1) \Rightarrow (P_2)$ holds in positive characteristic.

Then $(P_1) \Rightarrow (P_2)$ always holds.

We may not use this meta-theorem and we will show how to use the propcess before the theorem, that is, a proof of the special (but nice and classical) case of the theorem in the next section.

1.4.3 Applications of General Varieties and Fano Varieties

We assume that all varieties over an algebraically closed field k.

Theorem 1.57 (Kollár-Miyaoka-Mori, 1979-1982-1986-1991)**.** *Let X be a projective variety over k, let C a smooth, projective and irreducible curve, $f : C \to X$ a morphism and M any nef $\mathbb{R}$-divisor. Assume that X is smooth along $f(C)$ and $-K_X \cdot C > 0$.*

Then for every $x \in f(C)$ there is a rational curve $L_x \subset X$ containing x such that

$$M \cdot L_x \leq 2 \dim X \frac{M \cdot C}{-K_X \cdot C}.$$

Proof. Fix the condition in the theorem and consider the following proposition:

(P) M any ample $\mathbb{R}$-divisor and $\varepsilon > 0$ there is a rational curve $L_{x,\varepsilon} \subset X$ containing x such that

$$M \cdot L_{x,\varepsilon} \leq (2 \dim X + \varepsilon) \frac{M \cdot C}{-K_X \cdot C}.$$

Now we prove this theorem with several steps:

► **Step 1.** Prove the proposition (P) for M is ample divisor and char $= p > 0$.

Consider the Frobenius $F^m : C^m \to C$ of degree p^m and consider $f^m : C^m \to X$, then $-K_X \cdot C^m = p^m(-K_X \cdot C)$. Hence by Theorem 1.20 we have

$$\dim_{[f^m]} \mathsf{Hom}(C^m, X) \geq p^m(-K_X \cdot C) + \dim X \chi(\mathscr{O}_C)$$

since X is smooth along $f(C)$. Then for $m \gg 0$ we have $\dim_{[f^m]} \mathsf{Hom}(C^m, X) \geq p^m \frac{-K_X \cdot C}{\dim X + \varepsilon/2} \dim X + 2$. By Theorem 1.51(c) and we get the claim.

▶ **Step 2.** Prove the proposition (P) for char $= 0$.

We just need to show the case when M is ample divisor since $\mathbb{R}$-divisor can be approximated by $\mathbb{Q}$-divisors.

Let $f(p) = x$ and we construct R as before such that $p \subset C \xrightarrow{f} X$ and M over $\operatorname{Spec} R$. Hence we have $p^R, x^R, C^R, f^R, X^R, M^R$. By shrinking $\operatorname{Spec} R$ we may assume $C^R \to \operatorname{Spec} R$ is smooth, $X^R \to \operatorname{Spec} R$ is smooth along $f^R(C^R)$ and M^R is locally free (since $K(R)$ is of char $= 0$).

Let $W_\varepsilon \subset \mathsf{Chow}^1(X_R / \operatorname{Spec} R)$ be the subvariety parametrizing those 1-cycles $Z = \sum_i a_i D_i$ which satisfies that every D_i is rational and $Z \cdot M \leq (2 \dim X + \varepsilon) \frac{M \cdot C}{-K_X \cdot C}$ and $\operatorname{supp}(Z) \cap f^R(X^R) \neq \emptyset$. Consider $\pi : W_\varepsilon \to \operatorname{Spec} R$. We claim that π is surjective.

Indeed, we know that π is proper by Theorem 1.16 and Proposition 1.24. Since the closed points dense in $\operatorname{Spec} R$, we just need to show that $\pi(W_\varepsilon)$ contains all closed points of $\operatorname{Spec} R$. Pick a maximal ideal $\mathfrak{m} \subset R$ and $\{p^{\mathfrak{m}}, x^{\mathfrak{m}}, C^{\mathfrak{m}}, f^{\mathfrak{m}}, X^{\mathfrak{m}}, M^{\mathfrak{m}}\}$ as before over $\operatorname{Spec} R/\mathfrak{m}$ of positive characteristic. Hence by Step 1 we have rational curve $L_{x^{\mathfrak{m}},\varepsilon}$ such that $[L_{x^{\mathfrak{m}},\varepsilon}] \in W_\varepsilon$. Hence we get the claim.

By the claim we find that $W_\varepsilon \times_{\operatorname{Spec} R} \operatorname{Spec} k \neq \emptyset$. Hence we finish this step.

▶ **Step 3.** Prove the theorem.

Now come back to our general theorem. Now M be any nef $\mathbb{R}$-divisor and we fix an ample divisor H. Then $kM + H$ is ample for any $k \geq 0$. By Step 1,2, for any$\varepsilon > 0$ there is a rational curve $L_{x,k,\varepsilon} \subset X$ containing x such that

$$(kM + H) \cdot L_{x,k,\varepsilon} \leq (2 \dim X + \varepsilon) k \frac{M \cdot C}{-K_X \cdot C} + (2 \dim X + \varepsilon) \frac{H \cdot C}{-K_X \cdot C}.$$

Then we have

$$k \left(M \cdot L_{x,k,\varepsilon} - 2 \dim X \frac{M \cdot C}{-K_X \cdot C} \right) + H \cdot L_{x,k,\varepsilon} \leq (2 \dim X + \varepsilon) \frac{H \cdot C}{-K_X \cdot C} + k\varepsilon \frac{M \cdot C}{-K_X \cdot C}.$$

If $M \cdot L_{x,k_0,\varepsilon} - 2 \dim X \frac{M \cdot C}{-K_X \cdot C} \leq 0$ for some k_0, ε, then we take $L_x := L_{x,k_0,\varepsilon}$ and then well done. If not we have

$$H \cdot L_{x,k,\varepsilon} \leq (2 \dim X + \varepsilon) \frac{H \cdot C}{-K_X \cdot C} + k\varepsilon \frac{M \cdot C}{-K_X \cdot C}.$$

for every k, ε. Set $\varepsilon = \frac{1}{k}$ and $k \to \infty$. We obtain a sequence of curves $L_{x,k} := L_{x,k,1/k}$. So $H \cdot L_{x,k}$ is uniformly bounded, thus the $L_{x,k}$ form a bounded family. By Theorem 1.16 $\mathsf{Chow}^1(X)$ has only finitely many components parametrizing 1-cycles of bounded degree. In particular there is a subsequence $k_i \to \infty$ such that $P := P(i) := M \cdot L_{x,k_i} - 2 \dim X \frac{M \cdot C}{-K_X \cdot C}$ is independent of i. Hence

$$k_i P \leq (2 \dim X + 1) \frac{H \cdot C}{-K_X \cdot C} + \varepsilon \frac{M \cdot C}{-K_X \cdot C}, \quad k_i \to \infty.$$

Hence $P \leq 0$ and we take $L_x := L_{x,k_i}$ and well done. □

Theorem 1.58 (Smooth Case)**.** *Let X be a smooth projective variety, C a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Let M be any nef $\mathbb{R}$-divisor. Assume that $-K_X \cdot C > 0$, then for any $x \in f(C)$ there is a rational curve $D_x \subset X$ containing x such that*

$$M \cdot D_x \leq 2 \dim X \frac{M \cdot C}{-K_X \cdot C}, \quad -K_X \cdot D_x \leq \dim X + 1.$$

Proof. Use Theorem 1.52 and Theorem 1.57. This is trivial. □

Remark 1.59. *Both Theorem 1.57 and Theorem 1.58 have generalizations with the same proof, see Theorem II.1.3 and Remark II.5.15 in [57].*

Corollary 1.60 (Fano Case)**.** *Let X be a smooth Fano variety, then for any x there is a rational curve $D_x \subset X$ containing x such that $-K_X \cdot D_x \leq \dim X + 1$. In particular any smooth Fano variety is uniruled.*

1.5 Application I: Basic Theory of Fano Manifolds

Some general theory of Fano varieties we refer [83]. Here we give some important basic theory of Fano manifolds. We consider any schemes over an algebraically closed field k.

1.5.1 Some General Properties

Theorem 1.61. *Let G be a reduced and connected linear algebraic group and X be a proper homogeneous space under the action of G. Pick $x \in X$ and stabilizer $G_x \subset G$. If G_x is reduced (always hold if $\mathrm{char} = 0$), then T_X is generated by global sections and $-K_X$ is very ample.*

Proof. Omitted, we refer Theorem V.1.4 in [57]. □

Proposition 1.62. *Let X be a smooth Fano variety over an algebraically closed field k of characteristic zero.*

(a) We have $\chi(X, \mathscr{O}_X) = 1$ and X is simply connected.

(b) $\mathrm{Pic}(X)$ is finite generated and torsion free.

Proof. For (a), by Kodaira's vanishing theorem we find that $H^m(X, \mathscr{O}_X) = 0$ for all $m > 0$, hence $\chi(X, \mathscr{O}_X) = 1$. If $\pi : X' \to X$ is a connected finite étale cover, then X is also a smooth Fano variety. Hence $\chi(X', \mathscr{O}_{X'}) = 1$. But $\chi(X', \mathscr{O}_{X'}) = \deg \pi \chi(X, \mathscr{O}_X)$. Hence π is an isomorphism.

For (b) we may assume $k = \mathbb{C}$. By exponential sequence one has

$$H^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

By Kodaira's vanishing theorem, we find that $H^m(X, \mathscr{O}_X) = 0$ for all $m > 0$, hence $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. Hence $\mathrm{Pic}(X)$ is finite generated. To show $\mathrm{Pic}(X)$ is torsion free, we just need to show $H^2(X, \mathbb{Z})$ is torsion free. By universal coefficient theorem for cohomology, one has

$$0 \to \mathrm{Ext}^1(H_1(X, \mathbb{Z}), \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

As $\mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$ is torsion free, the only torsion of $H^2(X, \mathbb{Z})$ follows from $H_1(X, \mathbb{Z})$. As $H_1(X, \mathbb{Z}) = \pi_1(X)^{\mathrm{abel}} = 0$ by (a), hence $\mathrm{Pic}(X)$ is torsion free. □

Proposition 1.63. *Let $f : X \to Y$ be a smooth proper morphism of smooth proper varieties over algebraically closed field of characteristic zero.*

(a) Consider the Stein factorization $f : X \xrightarrow{g} Z \xrightarrow{h} Y$, then g is smooth and h is étale.

(b) If Y is simply connected (for example, Y is Fano), then f has connected fibres.

Proof. (b) follows from (a) and Proposition 1.62(a). Here we consider (a).

For (a), now h is finite. By miracle-flatness we just need to show that h is unramified. Indeed, If z is a ramification point and $y = g(z)$, then $g^{-1}(y)$ is non-reduced at z. Hence $f^{-1}(y)$ is also non-reduced, hence not smooth. So h is unramified. □

Theorem 1.64 (Cone Theorem)**.** *Let X be a smooth Fano variety over an algebraically closed field k. On X there are only finitely many families of rational curves C_μ such that $-K_X \cdot C_\mu \leq \dim X + 1$. Let $C_i : 1 \leq i \leq N$ be a set of representatives, then*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_i \mathbb{R}^+[C_i].$$

Proof. A very special case of Theorem 3.7 in [60]. Omitted. □

Proposition 1.65. *Let $f : X \to Y$ be a smooth morphism between smooth projective varieties over an algebraically closed field k.*

(a) If $\dim Y > 0$ then $-K_{X/Y}$ is not (absolutely) ample on X.

(b) If X is Fano, then Y is also Fano.

Proof. For (a), need to add.

For (b), we may assume $\dim Y > 0$. Pick an ample divisor H and $a > 0$ such that $-K_X - af^*H$ is nef. Let $h : C \to Y$ be a non-constant morphism from a smooth projective curve C. Consider $c \xleftarrow{f_C} X_C := X \times_Y C \xrightarrow{g} X$. Now $g^*(-K_X)$ is ample but

$-K_{X_C/C}$ is not by (a). Hence for any $\varepsilon > 0$ there exists an irreducible curve $D \subset X_C$ such that $-K_{X_C/C} \cdot D < \varepsilon(-g^*K_X \cdot D)$. As $-K_{X_C/C} = g^*f^*K_Y - g^*K_X$, we have

$$-g^*f^*K_Y \cdot D > (1-\varepsilon)(-g^*K_X \cdot D) \geq (1-\varepsilon)(ag^*f^*H \cdot D).$$

One can choose $D \to C$ non-constant, so pushforward to C we have

$$\deg h^*(-K_Y) > (1-\varepsilon)a \deg h^*H.$$

Hence since $\varepsilon > 0$ and $h : C \to Y$ are arbitrary, we know that $-K_Y - aH$ is nef. Hence $-K_Y$ is ample and Y is Fano. □

Remark 1.66. *Noe that if f is only flat, this is not true.*

1.5.2 Classifications Via Fano Index

Definition 1.67. *Let X be a smooth Fano variety. The* Fano index *of X is*

$$\mathsf{Index}(X) := \max\{m \in \mathbb{N} : -K_X \sim mH \textit{ for some Cartier divisor } H\}.$$

Theorem 1.68 (Kobayashi-Ochiai, 1970)**.** *Let X be a smooth Fano variety of dimension n over a field of characteristic zero. Then*

(a) $\mathsf{Index}(X) \leq n+1$.

(b) Let $-K_X \sim \mathsf{Index}(X)H$, *then* $\chi(X, \mathscr{O}_X(jH)) = \begin{cases} 1 & j = 0 \\ 0 & -\mathsf{Index}(X) < j < 0 \\ (-1)^n & j = -\mathsf{Index}(X) \end{cases}$.

Moreover we have

$$\chi(X, \mathscr{O}_X(tH)) = \begin{cases} \binom{t+n}{n} & \mathsf{Index} = n+1 \\ \binom{t+n+1}{n+1} - \binom{t+n-1}{n+1} & \mathsf{Index} = n \\ H^n\binom{t+n-1}{n} + \binom{t+n-2}{n-2} & \mathsf{Index} = n-1 \\ H^n\binom{2t+n-2}{2n}\binom{t+n-2}{n-1} + \binom{t+n-2}{n-2} + \binom{t+n-3}{n-2} & \mathsf{Index} = n-2 \end{cases}.$$

Hence $H^n = \begin{cases} 1 & \mathsf{Index} = n+1 \\ 2 & \mathsf{Index} = n \end{cases}$ *and* $h^0(X, \mathscr{O}_X(H)) = \begin{cases} n+1 & \mathsf{Index} = n+1 \\ n+2 & \mathsf{Index} = n \\ H^n + n - 1 & \mathsf{Index} = n-1 \\ \frac{1}{2}H^n + n & \mathsf{Index} = n-2 \end{cases}$.

(c) $\mathsf{Index}(X) = n+1$ *if and only if* $X \cong \mathbb{P}^n$.

(d) $\mathsf{Index}(X) = n$ *if and only if* $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$ *be a smooth quadric.*

Proof. For (a), by Corollary 1.60 we can find a rational curve C such that $-K_X \cdot C \leq n+1$. But $C \cdot H \geq 1$, hence $\mathsf{Index}(X) \leq n+1$.

For (b), $\chi(X, \mathscr{O}_X(jH))$ follows from Kodaira vanishing theorem and Serre duality. Then using this we know some roots of $\chi(X, \mathscr{O}_X(tH))$ correspond to t. Hence others are not hard to find. By Kodaira vanishing theorem again we get $h^0(X, \mathscr{O}_X(H))$ and H^n.

For (c), actually one can show that $\mathscr{O}_X(H)$ is base-point free by Claim V.1.11.7 in [57]. Hence by (b) this induce $p : X \to \mathbb{P}^n$. Let $Y := \mathrm{Im}(p)$, then $1 = H^n = \deg p \deg Y$. Hence $\deg p = \deg Y = 1$. As H is ample, p is finite. Hence p is an isomorphism.

For (d), one can show that $\mathscr{O}_X(H)$ is base-point free by Claim V.1.11.7 in [57]. Hence by (b) this induce $p : X \to \mathbb{P}^{n+1}$. Let $Y := \mathrm{Im}(p)$, then $2 = H^n = \deg p \deg Y$. As $\mathsf{Index}(X) = n$, Y is not linear. Hence $\deg p = 1$ and $\deg Y = 2$. As H is ample, p is finite. Hence p is an isomorphism. □

Remark 1.69. *Some remarks:*

(1) *If one assumes only that $-K_X \sim mH$ is nef and big, then essentially the same proof gives that $X \cong \mathbb{P}^n$ if $m = n+1$. If $m = n$, then either X is a smooth quadric in $X \cong \mathbb{Q}^n \subset \mathbb{P}^{n+1}$ or $p : X \to Y$ is a birational morphism onto a singular quadric of rank 2.*

(2) *Let X be a smooth Fano variety of dimension n (any characteristic) such that $-K_X \sim (n+1)H$, we also have $H^n = 1$.*

Indeed, section of $\mathscr{O}(mH)$ has $\binom{m+n-1}{n}$ conditions vanishing at $x \in X$. So if $H^n > 1$, then $H^0(X, \mathscr{O}_X(mH) \otimes \mathfrak{m}_x^{m+1}) \geq cm^n$ for some $c > 0$ (see also VI.2.15.7 in [57]). Pick a such section D. By Corollary 1.60 we can find a rational curve $x \in C \nsubseteq D$ such that $C \cdot D = m$ since $-K_X \sim (n+1)H$. But $C \cdot D \geq m+1$ which is impossible.

Theorem 1.70 (Fujita, 1990)**.** *Let X be a smooth Fano variety of dimension $n \geq 3$ over a field of characteristic zero such that $\mathsf{Index}(X) = n-1$. Assume $N^1(X) \cong \mathbb{R}$. Let $-K_X = (n-l)H$. Then one of the following holds:*

(a) *$H^n = 1$ and $X \cong X_6 \subset \mathbb{P}(1^{n-1}, 2, 3)$.*

(b) *$H^n = 2$ and $X \cong X_4 \subset \mathbb{P}(1^n, 2)$.*

(c) *$H^n = 3$ and $X \cong X_3 \subset \mathbb{P}(1^{n+1})$.*

(d) *$H^n = 4$ and $X \cong X_{2,2} \subset \mathbb{P}(1^{n+2})$.*

(e) *$H^n = 5$ and X is a linear space section of the Grassmannian $\mathsf{Grass}(2,5) \subset \mathbb{P}^9$ (thus $n \leq 6$).*

Proof. See 8.11 in [26]. □

1.6 Application II: Boundedness of Fano Manifolds

Here we will give a brief introduction about the boundedness of Fano manifolds using rational curves due to Kollár-Miyaoka-Mori (see Section V.2 in [57] or original paper [59] for details). Then we will give a statement of BAB conjecture which has proved by Birkar. We consider schemes over an algebraically closed field k of characteristic zero.

Theorem 1.71 (Kollár-Miyaoka-Mori, 1992)**.** *Let X be a smooth Fano variety of dimension n over k. Then there is a number $d(n)$ (depending only on n) such that any two points of X can be joined by an irreducible rational curve of anticanonical degree at most $d(\dim X)$.*

Proof. This follows from the rational connected varieties, see Section IV.3 and IV.4 and Corollary V.2.14.2 in [57]. □

Proposition 1.72. *Let X be a proper variety of dimension n, $x \in X$ a smooth point and $\mathscr{L}$ an nef and big line bundle on X. Choose $d > 0$ such that a general point $x' \in X$ can be connected to x by an irreducible curve $C_{x'}$ such that $\mathscr{L} \cdot C_{x'} \leq d$. Then $\mathscr{L}^n \leq d^n$.*

Proof. Fix $\varepsilon > 0$ and use a classical result (see Corollary VI.2.15.7 in [57], actually with the similar proof of Remark 1.69(2)) there is a $k > 0$ and a divisor $D_k \in |k\mathscr{L}|$ such that $\mathrm{mult}_x D_k \geq k\sqrt[n]{\mathscr{L}^n} - k\varepsilon$. Pick a general point $x' \notin \mathrm{supp}\, D_k$. Then $C_{x'}$ is not contained in D_k hence

$$kd \geq D_k \cdot C_{x'} \geq \mathrm{mult}_x D_k \geq k\sqrt[n]{\mathscr{L}^n} - k\varepsilon.$$

Hence $d \geq \sqrt[n]{\mathscr{L}^n} - \varepsilon$ and let $\varepsilon \to 0$. □

Theorem 1.73 (Boundedness of Fano Manifolds, Kollár-Miyaoka-Mori 1992)**.** *All n-dimensional Fano Manifolds over k forms a bounded family.*

Proof. By Theorem 1.71 and Proposition 1.72, we know that $(-1)^n K_X^n$ is bounded. Using Matsusaka estimate (see Exercise VI.2.15.8 in [57], proved by Kollár-Matsusaka in [58] in 1983) we know that for any nef and big divisor H, the coefficients of polynomial $\chi(X, \mathscr{O}_X(tH))$ can be bounded by H^m and $K_X \cdot H^{m-1}$. So $\chi(X, \mathscr{O}_X(tK_X))$ has bounded coefficients. In 1970, Matsusaka in [64] shows that there are only finitely many deformation types with fixed Hilbert polynomial. So All n-dimensional Fano Manifolds over k forms a bounded family. □

This finish the story of the smooth Fano varieties. If we have some mild singularities, then this problem is the famous conjecture in birational geometry:

Theorem 1.74 (BAB-Conjecture, Birkar 2016)**.** *Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Then the set of projective varieties X such that (X, B) is ε-lc of dimension d for some boundary B and $-(K_X + B)$ is nef and big, form a bounded family.*

Some History. This is one of the fundamental result of singular Fano varieties and is one of the most important conjectures in birational geometry and it is related to the termination of flips.

As we have seen, Kollár-Miyaoka-Mori in 1992 showed the boundedness of smooth Fano varieties using rational curves. But this can not be used in the BAB-conjecture.

In 1992 Kawamata showed the boundedness of terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one. In 1992 Borisov-Borisov shows this for toric cases. In 1994 V. Alexeev proved the BAB-conjecture for surfaces. In 2000 Kollár-Miyaoka-Mori-Takagi showed the boundedness of canonical $\mathbb{Q}$-Fano threefolds. Then in 2014 C. Jiang proved the weak BAB-conjecture for 3-fold, which is an important step towards the BAB-conjecture.

Finally BAB-Conjecture (along with the Weak BAB Conjecture) in arbitrary dimension was proved by C. Birkar in 2016 by different and much stronger methods, see his papers [10] and [11]. □

Remark 1.75. *The theory of moduli of Fano varieties is an application of J. Alper's theory of good moduli space. Many mathematicians build the whole theory in recent years using K-stability theory.*

In fact, by the theory of Birkar in [10], C. Jiang in 2017 showed that any K-semistable Fano varieties with dimension n and volumn $(-K_X)^n = V$ is bounded. Then there exists $N \gg 0$ such that $|-NK_X|$ gives an embedding to $\mathbb{P}^M$. Fix a Hilbert polynomial and then using the theory of KSBA-moduli space, there is a subspace of that Hilbert space H' correspond what we what. Hence the moduli stack $\mathscr{M}^{\mathrm{Kss}}_{n,V}$ of K-semistable Fano varieties with dimension n and volumn $(-K_X)^n = V$ is $[H'/\mathrm{PGL}]$ which is an algebraic stack of finite type. Then using Alper's theory we construct the separated good moduli space $\mathscr{M}^{\mathrm{Kss}}_{n,V} \to M^{\mathrm{Kps}}_{n,V}$ with ample CM-line bundle.

1.7 Application III: Hartshorne's Conjecture

Hartshorne's Conjecture is first proved by S. Mori in his famous and important paper [69]. This paper is the beginning of the theory of VMRT.

Theorem 1.76 (Hartshorne's Conjecture, Mori 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k, if T_X is ample then $X \cong \mathbb{P}^n_k$.*

Proof. By Theorem 1.78 directly. □

This conjecture motivated by an important conjecture in complex geometry:

Theorem 1.77 (Frankel's Conjecture, Mori 1979 and Siu-Yau 1980)**.** *If X is a compact Kähler manifold of dimension n with everywhere positive holomorphic bisectional curvature, then $X \cong \mathbb{P}^n_{\mathbb{C}}$.*

Proof. By Kodaira embedding theorem to $-K_X$ we know that X is a projective manifold. Then by Theorem 1.76 we get the result. □

Our main result in this section is the following due to Mori which is much stronger than the Hartshorne's Conjecture as we mentioned above.

Theorem 1.78 (Mori, 1979). *Consider n-dimensional smooth projective variety X over an algebraically closed field k. If*

(1) $-K_X$ is ample, that is, X is a Fano manifold;

(2) For any non-constant morphism $f:\mathbb{P}^1_k\to X$ the bundle f^*T_X is the sum of line bundles of positive degree.

Then $X\cong\mathbb{P}^n_k$.

Proof. We will use the following lemmas:

- **Lemma A.** For any $f:\mathbb{P}^1_k\to X$ such that bundle f^*T_X is the sum of line bundles of positive degree, we have $\deg f^*T_X\geq n+1$. If equality holds, then f is an closed embedding and is standard, that is, $f^*T_X\cong\mathscr{O}(2)\oplus\mathscr{O}(1)^{\oplus n-1}$.

  Proof of Lemma A. Let $f^*T_X\cong\mathscr{O}(a_1)\oplus\cdots\oplus\mathscr{O}(a_n)$ where $a_1\geq\cdots\geq a_n$. Then $a_i\geq 1$ and $a_1\geq 2$ by Remark 1.42. Hence $\deg f^*T_X\geq n+1$. If equality holds, then the only possibility is $f^*T_X\cong\mathscr{O}(2)\oplus\mathscr{O}(1)^{\oplus n-1}$. To show f is an embedding, first we now that f is unramified by trivial reason. Others are also easy and we refer to Lemma V.3.7.3.2 in [57]. □

- **Lemma B.** In the case of Theorem, any rational curve can be deformed as a cycle to the sum of rational curves C such that $-K_X\cdot C=n+1$.

  Proof of Lemma B. From bend and break directly. □

Back to the theorem. We let $n\geq 2$. Pick $f:\mathbb{P}^1\to X$ passing a general point $x\in X$ with $0\mapsto x$ and with minimal degree $n+1$ by Lemma B. By Proposition 1.43 the components $V\subset\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1,X;0\mapsto x)=\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1,X;0\mapsto x)$ containing $[f]$ is smooth of dimension $n+1$ and the correspond $\mathcal{K}_x\subset\mathsf{RatCurves}^{\mathrm{n}}_{n+1}(x,X)$ is also smooth of dimension $n-1$. Actually $\gamma:V\to\mathcal{K}_x$ is a principal $G:=\mathrm{Aut}(\mathbb{P}^1;0)$-bundle.
▶ **Step 1.** We claim that $\mathcal{K}_x\cong\mathbb{P}(\Omega^1_{X,x})$.

Consider the tangent $\Phi:V\to\mathbb{V}(\Omega^1_{X,x})$ via $v\mapsto(dv)_0(\frac{d}{dt})$ for uniformizer $t\in\mathscr{O}_{\mathbb{P}^1,0}$ by Lemma A. First we claim that Φ is smooth. Easy to see that Φ is flat and we just need to show $\Phi^{-1}(\Phi(v))$ is smooth. Note that for any finite type k-scheme T and for any morphism $T \to V$ over k, it factors through $\Phi^{-1}(\Phi(v)) \to V$ if and only if the morphism $\mathbb{P}^1_T \to X_T$ coincides on $\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})$ with v_T. Hence

$$\Phi^{-1}(\Phi(v)) \cong V \cap \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; v|_{\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})})$$

which is open and hence smooth with the same proof of Proposition 1.43.

Hence by Lemma A again we get a smooth morphism $\Phi : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$. Hence it is finite étale. Hence $\mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

▶ **Step 2.** Let $F : V \times \mathbb{P}^1 \to \mathcal{K}_x \times X$ defined by $(v,x) \mapsto (\gamma(v), v(x))$, consider $Z := \underline{\mathrm{Spec}}_{\mathcal{K}_x \times X} F_*\mathscr{O}^G$ which is a geometrically quotient by G (can be checked along the principal bundle $V \to \mathcal{K}_x$). As $\psi : Z \to \mathcal{K}_x$ is a $\mathbb{P}^1$-bundle with a section $S \subset Z$ induced by $V \to V \times \mathbb{P}^1$ as $v \mapsto (v, 0)$, then $Z \cong \mathbb{P}(\psi_*\mathscr{O}_Z(S))$ is a projective bundle. Define a universal cycle map $\pi : Z \to X$ induced by G-invariant cycle morphism $V \times \mathbb{P}^1 \to X$. We claim that $\pi : Z \to X$ is étale on $Z\backslash S$ and $\pi(S) = x$.

Actually $\pi(S) = x$ is trivial, to show $\pi|_{Z\backslash S}$ is étale we just need to show $V \times \mathbb{P}^1 \to X$ is smooth. This follows from Corollary 1.36 and Theorem 1.38. Hence we get the claim.

▶ **Step 3.** Consider the Stein factorization we have $\pi : Z \xrightarrow{\phi} U \cong \underline{\mathrm{Spec}}_X \pi_*\mathscr{O}_Z \xrightarrow{\eta} X$. We claim that η is étale, $Z\backslash S \cong U\backslash\{r\}$ where $\phi(S) = r$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

In fact by Stein factorization η is étale outside a codimension ≥ 2 locus, by purity of branced locus we know that η is étale. Now $Z\backslash S \cong U\backslash\{r\}$ where $\phi(S) = r$ follows from Zariski main theorem. Finally we show that $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. Indeed, picka hyperplane $L \subset \mathcal{K}_x$ and a line $C \cong \mathbb{P}^1 \subset S$ such that $\psi(C) \nsubseteq L$. Let $D := \psi^{-1}(L)$, then $C \cdot D = 1$. As $r \in \phi(D)$, we have $\phi^{-1}\phi(D) = D + aS$ for some $a > 0$. So $C \cdot \phi^{-1}\phi(D) = \phi(D) \cdot D = 0$. Hence $C \cdot S = -1$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

▶ **Step 4.** We claim that $U \cong \mathbb{P}^n$.

By Step 3 we have $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$, hence

$$0 \to \mathscr{O}_Z \to \mathscr{O}_Z(S) \to \mathscr{O}_S(-1) \to 0$$

exact. Since $R^1\psi_*\mathscr{O}_Z = 0$, we get

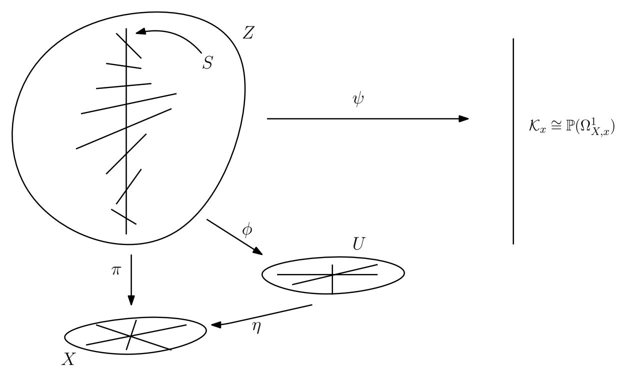
$$0 \to \mathscr{O}_{\mathcal{K}_x} \to \psi_*\mathscr{O}_Z(S) \to \mathscr{O}_{\mathcal{K}_x}(-1) \to 0$$

exact. As $\mathrm{Ext}^1_{\mathbb{P}^{n-1}}(\mathscr{O}(-1), \mathscr{O}) = 0$, we get $\psi_*\mathscr{O}_Z(S) \cong \mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)$. Hence by Step 2 we have $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1))$.

Hence $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)) \cong \mathrm{Bl}_O\mathbb{P}^n$. We can have a contraction map $Z \to \mathrm{Bl}_O\mathbb{P}^n$ makes S to a point $O \in \mathbb{P}^n$ (in fact it is induced by $\psi^*\mathscr{O}(1) \otimes \mathscr{O}(S)$). Hence via $\mathbb{P}^n \leftarrow Z \to U$ we have a birational map $\mathbb{P}^n \dashrightarrow U$. This must be an isomorphism since $Z \cong \mathrm{Bl}_O\mathbb{P}^n$ has only two dimensional Mori cone, hence the only birational contraction is this one (another is that $\mathbb{P}^1$-bundle).

▶ **Step 5.** Finish the proof, that is, we have $X \cong \mathbb{P}^n$.

Since $\mathbb{P}^n$ is simply connected, $U \cong \mathbb{P}^n \to X$ is a Galois covering by Step 3 and 4. Thus $X \cong \mathbb{P}^n$ because any automorphism of $\mathbb{P}^n$ has a fixed point. □



Remark 1.79. *Note that by the proof this is right if we just consider the rational curves containing a sufficient general point.*

Corollary 1.80 (Lazarsfeld, 1984)**.** *Let X be a smooth projective variety over an algebraically closed field k of dimension > 0. Let there is a surjective separable morphism $p : \mathbb{P}^n_k \to X$, then $X \cong \mathbb{P}^n$.*

Proof. By the Chow ring structure of projective space, we know that $\dim X = n$ and p is finite. Hence let R be a ramification divisor of p, we have $p^*(-K_X) = -K_{\mathbb{P}^n} + R$ hence some multiple of $-K_X$ is effective. As p surjective, then $\dim N_1(X) = 1$ Hence $-K_X$ is ample and X is Fano. For a sufficient general point $x \in X$ outside of the ramification divisor, consider $f : \mathbb{P}^1 \to X$ as $0 \mapsto x$. Let C be a normalization of a component in $\mathbb{P} \times_X \mathbb{P}^1$, we have

$$\begin{array}{ccc} C & \xrightarrow{h} & \mathbb{P}^n \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ \mathbb{P}^1 & \xrightarrow{f} & X \end{array}$$

The natural map $r : h^*T_{\mathbb{P}^n} \to h^*p^*T_X = q^*f^*T_X$ is a local isomorphism $q^{-1}(0) \subset C$ since p is étale above x. Write $f^*T_X = \bigoplus_i \mathscr{O}_{\mathbb{P}^1}(a_i)$. For any j we have

$$\bigoplus h^*\mathscr{O}_{\mathbb{P}^1}(1) \to h^*T_{\mathbb{P}^n} \xrightarrow{r} \bigoplus_i q^*\mathscr{O}_{\mathbb{P}^1}(a_i) \to q^*\mathscr{O}_{\mathbb{P}^1}(a_j)$$

which is surjective over an open subspace $U \subset C$. So $q^*\mathscr{O}_{\mathbb{P}^1}(a_j)$ has a section vanishing at somr point. Hence $a_i > 0$ for any i. So by Theorem 1.78 we have $X \cong \mathbb{P}^n$. $\square$

Chapter 2

Several Special Fano Varieties

2.1 More General Facts of Fano Varieties

2.1.1 About Linear Systems

Theorem 2.1 (Fujita 1980-1984)**.** *Let X be a smooth Fano n-fold of index $r \geq n-1$. Then the general element in the fundamental divisor is smooth.*

Proof. See [83] Theorem 2.3.2. □

Theorem 2.2 (Mella 1996)**.** *Let X be a smooth Fano n-fold of index $n-2$. Then the general element in the fundamental divisor is smooth.*

Proof. See [65] Theorem 2.5. □

Corollary 2.3. *Let X be a smooth Fano 3-fold of index 1 and $H^3 \geq 8$ and $\rho(X) = 1$. Then the linear system $|-K_X|$ is very ample and X is projectively normal which is an intersection of quadrics.*

Proof. See [83] Corollary 4.1.13. □

Proposition 2.4. *Let X be a smooth Fano 3-fold of $\mathsf{Index}(X) = r$ with fundamental divisor H.*

(a) If $r \geq 2$, then $\mathsf{Bs}(|-K_X|) = \emptyset$.

(b) If $\dim \mathsf{Bs}(|-K_X|) = 1$, then $\mathsf{Bs}(|-K_X|) \cong \mathbb{P}^1$ as schemes and $\mathsf{Bs}(|-K_X|) \cap (H')^{\mathsf{sing}} = \emptyset$ for general $H' \in |-K_X|$.

(c) If $\dim \mathsf{Bs}(|-K_X|) = 0$, then $\mathsf{Bs}(|-K_X|)$ is a single point, general $H' \in |-K_X|$ at this point has ordinary double singularity and $\mathsf{Bs}(|-K_X|) \in X^{\mathsf{sing}}$.

Moreover, H is base point free except the following cases:

(1) $r = 2$ and $\deg X = 1$ and $|H|$ has simple base point.

(2) $r = 1, g(X) = 3$.

(3) $r = 1, g(X) = 4$.

Proof. See Proposition 2.4.1 and Theorem 2.4.5 in [83]. □

2.1.2 Pseudoindex of Fano Manifolds

Definition 2.5. *For a Fano manifold X, we define the pseudoindex i_X is the minimum of the anticanonical degrees of rational curves on X.*

Theorem 2.6. *Let X be a Fano manifold of dimension $m \geq 2$. Then $i_X \leq n + 1$. Moreover we have the following.*

(a) If $i_X = m + 1$, then $X \cong \mathbb{P}^m$.

(b) If $i_X = m$, then $X \cong \mathbb{Q}^m$.

Proof. This is a generalization of the case in Fano index. See Theorem 4.1 in [72]. □

Proposition 2.7. *Let X be a Fano manifold of dimension n and $i_X \geq 2$ which has only contractions of fiber type. Then $\rho(X) \leq n$. Moreover,*

(a) if $\rho(X) = n$, then $X \cong (\mathbb{P}^1)^n$;

(b) if $\rho(X) = n - 1$, then X is either $(\mathbb{P}^1)^{n-2} \times \mathbb{P}^2$ or $X = (\mathbb{P}^1)^{n-3} \times \mathbb{P}(T_{\mathbb{P}^2})$.

If All its elementary contractions but one are of fiber type. Then $\rho(X) \leq n-1$, equality holding if and only if $X = (\mathbb{P}^1)^{n-3} \times \mathrm{Bl}_p\mathbb{P}^3$.

Proof. We omitted and we refer Proposition 5.1 and 5.2 in [75]. See also the comments of the proof as in Remark 2.4 in [98]. □

2.1.3 More Known Facts of Fano Manifolds

Lemma 2.8. *Let $f : X \to Y$ be a surjective morphism between two smooth projective varieties with connected fibers. Let F be a general fiber of f and $\dim F < (\dim X - 1)$. Then $K_X|_F = K_F$.*

Proof. Note that here F is smooth. Then the differential of f gives an isomorphism of the normal bundle $N_{F/X}$ and the tangent space to Y at the corresponding point. Hence $N_{F/X}$ is trivial. By adjunction formula we get the result. □

Proposition 2.9. *Let X be a Fano manifold with $p : \mathbb{P}(E) \to X$ a projectivisation of a rank r bundle. Suppose that $f : Y \to X$ is a finite morphism. If $\mathbb{P}(f^*(E)) \cong Y \times \mathbb{P}^{r-1}$, then $\mathbb{P}(E) \cong X \times \mathbb{P}^{r-1}$.*

Proof. Let $\eta := \mathscr{O}_{\mathbb{P}(E)}(1)$. We first claim that $r\eta - p^* \det E$ is nef and $(r\eta - p^* \det E)^r = 0$ over $\mathbb{P}(E)$. This follows because the pull-back of $r\eta - p^* \det E$ to $\mathbb{P}(f^*(E))$ has these features. By the same reason $-K_{\mathbb{P}(E)} = r\eta - p^*(\det E + K_X)$ is ample and therefore $\mathbb{P}(E)$ is a Fano manifold and by Kawamata-Shokurov base-point-freeness $r\eta - p^* \det E$ defines a contraction, $g : \mathbb{P}(E) \to Z$, onto a normal projective variety of dimension $r-1$. Any fiber of g is mapped, via p, surjectively onto X, with no positive dimensional fiber. Let F be a general fiber of g. Then, F is smooth and by adjunction and Lemma 2.8 we find out that

$$K_F = (K_{\mathbb{P}(E)})|_F = (p^*K_X + p^*(\det E) - r\eta)|_F = (p^*K_X)|_F.$$

Hence $g|_F$ is finite étale. As X Fano, we have $p|_F : F \to X$ is an isomorphism and hence F is a section of p. Thus we conclude that $\mathbb{P}(E) \cong X \times \mathbb{P}^{r-1}$ since F general. □

For the similar idea, we have:

Proposition 2.10. *Let X be a Fano manifold admitting a projective bundle structure $f : X = \mathbb{P}(\mathscr{E}) \to Y$ of a rank r bundle $\mathscr{E}$ and R the extremal ray corresponding to f. If there exists a proper morphism $g : X \to Z$ onto a variety Z of dimension $r-1$ which does not contract curves of R. Then $X \cong \mathbb{P}^{r-1} \times Y$.*

Proof. Let F a general fiber of g. By dimension reason and condition f does not contract curves in the fibers of g , F dominantes Y (hence surjective). Consider the diagram

$$\begin{tikzcd}
F \arrow[dr, dashed, "s"] \arrow[drr, hook] \arrow[ddr, "\mathrm{id}_F"'] & & & \\
& X_F \arrow[r, dashed, "p_F"] \arrow[d, dashed, "p"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[r, "g"] \arrow[d, "f"] & Z \\
& F \arrow[r, two heads, "f_F"] & Y &
\end{tikzcd}$$

where now s is a section of p such that $p_F \circ s$ is the embedding of F into X. Let $F' \subset X$ be the image of s. Let $\mathscr{E}_F := f_F^*\mathscr{E}$ and hence $X_F = \mathbb{P}(\mathscr{E}_F)$. Hence

$$-K_{X_F} + p^*K_F = r\mathscr{O}_{X_F}(1) - p^* \det \mathscr{E}_F.$$

As $(p^*K_F)|_{F'} = K_{F'} = (K_{X_F})|_{F'}$, hence $(r\mathscr{O}_{X_F}(1) - p^* \det \mathscr{E}_F)|_{F'} = \mathscr{O}_{F'}$. Use the canonical bundle for X again we get $\mathscr{O}_F = (r\mathscr{O}_X(1) - f^* \det \mathscr{E})|_F = (-K_X + f^*K_Y)|_F$. Hence $K_F = (K_X)|_F = f^*K_Y|_F$. Hence $f_F : F \to Y$ is finite unramified (hence étale by dimension reason). Hence as Y Fano by Theorem 1.65(b), $f_F : F \to Y$ is an isomorphism and hence F is a section of f.

As a section, F correspond to $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{L} \to 0$. Hence $N_{F/X} = f_F^*(\mathscr{E}^\vee \otimes \mathscr{L}) = (\mathscr{O}_X(1) \otimes f^*(\mathscr{E}')^\vee)|_F$. But now $N_{F/X} = \mathscr{O}_F^{\oplus r-1}$, we get $(f^*\mathscr{E}')|_F \cong (\mathscr{O}_X(1))|_F^{\oplus r-1}$. Moreover $f_F^*\mathscr{L} = \mathscr{O}_X(1)|_F$, hence pullback the exact sequence, we have $0 \to (\mathscr{O}_X(1))|_F^{\oplus r-1} \to$

$f_F^*\mathscr{E} \to (\mathscr{O}_X(1))|_F \to 0$ on F. As $F \cong Y$, hence Fano and $H^1(F, \mathscr{O}_F) = 0$. Hence this sequence split and $f_F^*\mathscr{E} \cong (\mathscr{O}_X(1))|_F^{\oplus r}$. As f_F is an isomorphism, hence so is $\mathscr{E}$ and $X \cong \mathbb{P}^{r-1} \times Y$. □

2.1.4 Manifolds with Two Bundle Structures

Proposition 2.11. *Let X be a projective manifold of dimension n, endowed with two different $\mathbb{P}$-bundle structures $f : X \to Y$ and $g : X \to Z$ such that $\dim Y + \dim Z = n+1$. Then either $n = 2m-1$, $Y = Z = \mathbb{P}^m$ and $X = \mathbb{P}(T_{\mathbb{P}^m})$ or Y and Z have a $\mathbb{P}$-bundle structure over a smooth curve C and $X = Y \times_C Z$.*

Proof. See Theorem 2 in [76] for the proof. □

Lemma 2.12. *Let X be a Fano manifold of Picard number 2 which admits two different smooth $\mathbb{P}^1$-fibration structures. Then X is isomorphic to G/B with G a semisimple Lie group of type $A_1 \times A_1$, A_2, B_2 or G_2, and B a Borel subgroup of G.*

Very Sketch of the Proof. See Theorem 4.7 in [72] for the detailed proof and here we give a very sketched proof.

Let X be an m-dimensional Fano manifold of Picard number 2 having two smooth $\mathbb{P}^1$-fibrations $\pi_1 : X \to X_1$ and $\pi_2 : X \to X_2$. We assume that $m \geq 3$. Let K_i be the relative canonical divisor of π_i, H_i the pull-back of the ample generator of X_i, $r_{X_i} = \mathsf{Index}(X_i)$. Set $\nu_1 := K_1 \cdot \Gamma_2$, $\mu_1 := H_1 \cdot \Gamma_2$, $\nu_2 := K_2 \cdot \Gamma_1$, $\mu_2 := H_2 \cdot \Gamma_1$. WLOG we may assume that $\nu_2 \geq \nu_1$.

When $m = 2$ this is easy to see and we assume $m \geq 3$. First one can show that $\mu_i, \nu_i > 0$ and $K_j^2 \in \Delta_j H_j^2$ for some $\Delta_j \in \mathbb{Q}$ by Chern-Wu numerical relations and moreover $\Delta_j < 0$. As $-K_j + \frac{\nu_j}{\mu_j} H_j$ is nef but not big, then $(-K_1 + \frac{\nu_1}{\mu_1} H_1)^j H_1^{m-j} \geq 0$ and being 0 when $j = m$. By some argument these inequalities reduce to $(\frac{\nu_1}{\mu_1} + i\sqrt{-\Delta_1})^j \geq (\frac{\nu_1}{\mu_1} - i\sqrt{-\Delta_1})^j$. Hence argument of $\frac{\nu_1}{\mu_1} + i\sqrt{-\Delta_1}$ is π/m and we have $-\Delta_1 = \frac{\nu_1^2}{\mu_1^2} \tan^2(\frac{\pi}{m}) \in \mathbb{Q}$. Hence $m = 3, 4$ or 6.

More calculation we can get $\nu_1\nu_2 = 4\cos^2(\pi/m)$ and hence $(\nu_1, \nu_2) = (1, 1)$ or $(1, 2)$ or $(1, 3)$. More calculation we get $\mu_1 = \mu_2$ and $(m, \mu_1, r_{X_1}, \mu_2, r_{X_2})$ is $(3, 1, 3, 1, 3)$ or $(4, 1, 3, 1, 4)$ or $(6, 1, 3, 1, 5)$. By Theorem 1.68 we can find X_2, then by some more theory on them and well done. □

Theorem 2.13. *Let U be a smooth projective variety which admits two $\mathbb{P}^1$-bundle structures over smooth projective varieties W, S:*

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "\pi"] & W \\ S & \end{tikzcd}$$

such that e dose not contract any π-fiber. Hence the image of π-fibres generate a half-line $R_S \subset \overline{\mathrm{NE}}(W)$. Then R_S is an extremal ray of W and the induced contraction of R_S is a smooth morphism.

Idea of proof. The detailed proof we refer Theorem 2.2 in [49], here we just give some basic ideas.

Give a point $x \in W$, we define $V_0(x) := \{x\}$ and $V_m(x) := e(\pi^{-1}(\pi(e^{-1}(V_{m-1}(x)))))$. Hence $V_m(x)$ is just the set of points that can be connected to x by an S-chain of length m. By the results of rational connected quotient theory, see [18] Proposition 5.7, $V_n(x)$ will stable for some large n. We define $d(x) = \dim V(x)$ and $m(x)$ be the smallest such that $\dim V_{m(x)}(x)$ stable. Let d_S and m_S be those of general fibres. Then after discovering the properties of chain $V_0(x) \subset \cdots \subset V_n(x)$, we can show that $m(y) = m_S$ and $d(y) = d_S$ for any $y \in W$.

Now we can have a morphism $W \to \mathsf{Chow}(W)$ and the union V of above $V(x)$ is a well-defined family of cycles, and hence the pullback of universal family of $\mathsf{Chow}(W)$. Let Z be the normalisation of the image of this map. Hence we have $f : W \to Z$ since W normal which is the S-rational connected quotient. Hence R_S is an extremal ray and f be its contraction.

Now we need to show that the induced contraction of R_S is a smooth morphism. By Rigidity Lemma 1.47 we have

$$\begin{array}{ccc} U & \xrightarrow{e} & W \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & Z \end{array}$$

By symmetry, g is the W-rationally connected quotient morphism of S. By proving the smoothness we just need to show every f-fiber with its reduced structure is a Fano manifold with trivial normal bundle by Lemma 4.13 in [90]. Let $h := f \circ e$, then as the general fiber of h is Fano manifold with Picard number 2 admit two $\mathbb{P}^1$-bundle structures, by Lemma 2.12 we know that this fiber is the complete flag of type $A_1 \times A_1$ or A_2 or B_2 or G_2. By checking case by case and consider dimensions, we can show the result. □

Corollary 2.14. *As the condition of Proposition, we have*

$$\begin{array}{ccc} U & \xrightarrow{e} & W \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & Z \end{array}$$

where f is the contraction of the ray R_S. Then every fiber of $f \circ e$ is a complete flag manifold by Theorem 12.19. In particular, all f-fibers are isomorphic to each other and *they are isomorphic to* $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3, \mathbb{Q}^3, \mathbb{Q}^5$ *or* $K(G_2)$*. Furthermore, if* $-K_e \cdot (\pi\text{-fiber}) = -1$*, then* f*-fibers are isomorphic to* $\mathbb{P}^2, \mathbb{Q}^3$ *or* $K(G_2)$*.*

Theorem 2.15. *Let* X *be a complex projective manifold with Picard number* $\rho(X) = 1$ *and* $\mathscr{E}$ *a rank 2 vector bundle on* X*. Assume that* $Z := \mathbb{P}(\mathscr{E}) \to X$ *admits another smooth morphism* $Z \to Y$ *of relative dimension 1 and* $n := \dim X \geq 2$*. Then,*

(I) X *and* Y *are Fano manifolds with* $\rho = 1$ *and there exists a rank 2 vector bundle* $\mathscr{E}'$ *on* Y *such that* $Z \to Y$ *is given by* $\mathbb{P}_Y(\mathscr{E}')$*.*

(II) If $\mathscr{E}$ *and* $\mathscr{E}'$ *are normalized by twisting with line bundles (i.e.,* $c_1 = 0$ *or* -1*), then* $((X, \mathscr{E}), (Y, \mathscr{E}'))$ *is one of the following, up to exchanging the pairs* $(X, \mathscr{E})$ *and* $(Y, \mathscr{E}')$*:*

- *(a)* $((\mathbb{P}^2, T_{\mathbb{P}^2}), (\mathbb{P}^2, T_{\mathbb{P}^2}))$*.*
- *(b)* $((\mathbb{P}^3, \mathscr{N}), (\mathbb{Q}^3, \mathscr{S}))$*, where* $\mathscr{N}$ *is a null-correlation bundle on* $\mathbb{P}^3$ *(see Definition 12.26) and* $\mathscr{S}$ *is the restriction to the 3-dimensional quadric* $\mathbb{Q}^3$ *of the universal quotient bundle of the Grassmannian* $\mathsf{Grass}(2,4)$*.*
- *(c)* $((\mathbb{Q}^5, \mathscr{C}), (K(G_2), \mathscr{Q}))$*, where* $\mathscr{C}$ *is a Cayley bundle on* $\mathbb{Q}^5$ *and* $\mathscr{Q}$ *is the restriction of universal quotient bundle of Grassmannian.*

Proof. This is the main theorem of [97]. □

2.2 Gushel-Mukai Varieties

2.2.1 Basic Definitions and Properties

Let V_5 be a vector space of dimension 5 and consider the Plücker embedding $\mathsf{Grass}(2, V_5) \hookrightarrow \mathbf{P}\left(\bigwedge^2 V_5\right)$. For any vector space K, consider the cone $\mathsf{C}_K(\mathsf{Grass}(2, V_5)) \subset \mathbf{P}\left(\bigwedge^2 V_5 \oplus K\right)$ of vertex $\mathbf{P}(K)$. Choose a vector subspace $W \subset \bigwedge^2 V_5 \oplus K$ and a subscheme $Q \subset \mathbf{P}(W)$ defined by defined by one quadratic equation (possibly zero).

Definition 2.16. *The scheme*

$$X = \mathsf{C}_K(\mathsf{Grass}(2, V_5)) \cap \mathbf{P}(W) \cap Q$$

is called a ***Gushel-Mukai intersection*** *(GM intersection). A GM intersection* X *is called a* ***Gushel-Mukai variety*** *(GM variety) if* X *is a smooth variety of dimension* $\dim W - 5 \geq 1$*.*

Remark 2.17. *Some remarks:*

(a) In the original paper [17] they defined without the smoothness (but always Gorenstein).

(b) *Note that all Q and $\mathsf{C}_K(\mathsf{Grass}(2, V_5)) \cap \mathbf{P}(W)$ are Gorenstein, hence all Cohen-Macaulay. So the dimension condition means they are dimensionally transverse, that is,* $\mathrm{Tor}_{>0}(\mathscr{O}_Q, \mathscr{O}_{\mathsf{C}_K(\mathsf{Grass}(2,V_5))\cap\mathbf{P}(W)}) = 0$.

(c) *A GM variety X has a canonical polarization, the restriction H of the hyperplane class on $\mathbf{P}(W)$; we will call (X, H) a polarized GM variety.*

The definition of a GM variety is not intrinsic. We actually have an intrinsic characterization. But before giving these, we will introduce a new definition:

Definition 2.18. *Let W be a vector space and let $Y \subset \mathbf{P}(W)$ be a closed subscheme which is an intersection of quadrics, i.e., the twisted ideal sheaf $\mathscr{I}_X(2)$ on $\mathbf{P}(W)$ is globally generated.*

Define $V_X := H^0(\mathbf{P}(W), \mathscr{I}_X(2))$, this yields a surjection $V_X \otimes \mathscr{O}_{\mathbf{P}(W)}(-2) \twoheadrightarrow \mathscr{I}_X$ which induce

$$V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{I}_X/\mathscr{I}_X^2 = \mathscr{N}^\vee_{X/\mathbf{P}(W)}.$$

We define the excess conormal sheaf $\mathscr{E}\mathscr{N}^\vee_{X/\mathbf{P}(W)}$ to be the kernel of this map.

Theorem 2.19. *A smooth polarized projective variety (X, H) of dimension $n \geq 1$ is a polarized GM variety if and only if all the following conditions hold:*

(a) *$H^n = 10$ and $K_X = -(n-2)H$.*

(b) *H is very ample and the vector space $W := H^0(X, \mathscr{O}_X(H))^\vee$ has dimension $n+5$.*

(c) *X is an intersection of quadrics in $\mathbf{P}(W)$ and the vector space*

$$V_6 := H^0(\mathbf{P}(W), \mathscr{I}_X(2)) \subset \mathrm{Sym}^2 W^\vee$$

of quadrics through X has dimension 6.

(d) *The twisted excess conormal sheaf $\mathscr{U}_X := \mathscr{E}\mathscr{N}^\vee_{X/\mathbf{P}(W)}(2H)$ of X in $\mathbf{P}(W)$ is simple.*

Proof. We will show a smooth polarized GM variety (X, H) satisfies (a)-(d) and we will not prove the converse and we refer [17].

For (a), as $\deg(\mathsf{C}_K(\mathsf{Grass}(2, V_5))) = 5$ and they are dimensionally transverse, then $\deg(X) = 10$. Let $\dim K = k$ and hence $K_{\mathsf{C}_K(\mathsf{Grass}(2,V_5))} = -(5+k)H$ by Lemma 2.20. Finally we have

$$K_X = (-(5+k) + (10+k) - (n+5) + 2)H = -(n-2)H.$$

For (b), we just need to show $W = H^0(X, \mathscr{O}_X(H))^\vee$. Consider the resolution

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{C}_K\,\mathsf{Grass}(2,V_5)} \to 0.$$

Restrict it into $\mathbf{P}(W)$ and tensor the resolution of Q as $0 \to \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_Q$, then tensor $\mathscr{O}(1)$ again we get the resolution

$$\begin{aligned} 0 \to \mathscr{O}(-6) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-4) &\to (V_5 \otimes \mathscr{O}(-3)) \oplus (V_5^\vee \otimes \mathscr{O}(-2)) \\ &\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-1) \to \mathscr{O}(1) \to \mathscr{O}_X(H) \to 0 \end{aligned}$$

on $\mathbf{P}(W)$. Hence $H^0(X, \mathscr{O}_X(H)) = H^0(\mathbf{P}(W), \mathscr{O}_{\mathbf{P}(W)}(1)) = W^\vee$.

For (c), consider the resolution again:

$$\begin{aligned} 0 \to \mathscr{O}(-5) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-3) &\to (V_5 \otimes \mathscr{O}(-2)) \oplus (V_5^\vee \otimes \mathscr{O}(-1)) \\ &\to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O} \to \mathscr{O}(2) \to \mathscr{O}_X(2H) \to 0 \end{aligned}$$

Hence one can show that $H^0(\mathbf{P}(W), \mathscr{I}_X(2)) = V_5 \oplus \mathbb{C}$, hence well done.

For (d), we will use the induction of the dimension. For $n = 1$, this follows from some basic fact of excess normal sheaf and the Mukai's construction about a stable vector bundle of rank 2 on X to show that $\mathscr{U}_X$ is stable, and hence simple. For the detail we refer [17] Theorem 2.3. Hence we now assume $n \geq 2$. Pick a smooth hyperplane section $X' \subset X$ which is also irreducible since $n \geq 2$ by Bertini's theorem. Hence X' is also a GM variety. One can easy to show that in this case $\mathscr{U}_X|_{X'} = \mathscr{U}_{X'}$ (see Lemma A.5 in [17]). Hence we have $0 \to \mathscr{U}_X(-H) \to \mathscr{U}_X \to \mathscr{U}_{X'} \to 0$. Hence

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}).$$

If $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X)) > 1$, then $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H))) > 0$. By the similar argument we get

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}(-H)) = 0.$$

Hence $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \neq 0$. By induction we get $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-kH)) \neq 0$ for any $k > 0$. Hence for any $k > 0$ we have $\Gamma(X, \mathscr{U}_X^\vee \otimes \mathscr{U}_X(-kH)) \neq 0$. But these are vector bundles and X is integral of dimension ≥ 2, hence this is impossible. $\square$

Lemma 2.20. *Let $X \subset \mathbb{P}^n$ be a subvariety such that $K_X = rH$. Let $\mathsf{C}(X) \subset \mathbb{P}^{n+1}$ be a cone over X, then $K_{\mathsf{C}(X)} = (r-1)H$.*

Proof. We know that the blow-up of of the vertex of $\mathsf{C}(X)$ is

$$\begin{tikzcd} & X' = \mathbb{P}_X(\mathscr{O}_X \oplus \mathscr{O}_X(-H)) \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathsf{C}(X) & & X \end{tikzcd}$$

Let H' be the relative hyperplane class of p. Then

$$K_{X'} = p^*(K_X + H) - 2H' = (r+1)p^*H - 2H'.$$

On the other hand, the morphism π contracts the exceptional section $E \subset X'$ and H' is the pullback of $H_{\mathsf{C}(X)}$. Finally $E \sim_{\text{lin}} H' - p^*H$, hence

$$K_{X'} = (r-1)H' - (r+1)E.$$

Hence $K_{\mathsf{C}(X)} = (r-1)H$. □

2.2.2 Some Classifications

Lemma 2.21. *Let (X, H) be a polarized variety. If it is projective normal, that is, the canonical map $\operatorname{Sym}^m H^0(X, \mathscr{O}_X(H)) \to H^0(X, \mathscr{O}_X(mH))$ is surjective for any $m \geq 0$, then H must be very ample.*

Proof. By the commutative diagram

$$\begin{tikzcd}
 & \mathbf{P}H^0(X, \mathscr{O}_X(nH)) \arrow[dr, hook] & \\
X \arrow[ur, hook, "|nH|"] \arrow[dr, "|H|"'] & & \mathbf{P}H^0(X, \operatorname{Sym}^n \mathscr{O}_X(H)) \\
 & \mathbf{P}H^0(X, \mathscr{O}_X(H)) \arrow[ur, hook, "n\text{-uple}"] &
\end{tikzcd}$$

we know that $|H|$ also induce an immersion. Hence H is very ample. □

Proposition 2.22. *Let (X, H) be a smooth polarized variety of dimension $n \geq 2$ such that $K_X = -(n-2)H$ and $H^1(X, \mathscr{O}_X) = 0$. If there is a hypersurface $X' \subset X$ in the linear system $|H|$ such that $(X', H|_{X'})$ is a smooth polarized GM variety, (X, H) is also a smooth polarized GM variety.*

Proof. First we note that for any smooth GM variety (Y, H) the resolution

$$\begin{aligned} 0 \to \mathscr{O}(m-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(m-5) \to (V_5 \otimes \mathscr{O}(m-4)) \oplus (V_5^\vee \otimes \mathscr{O}(m-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(m-2) \to \mathscr{O} \to \mathscr{O}_Y(mH) \to 0 \end{aligned}$$

can imply Y is projective normal, that is, the canonical map $\operatorname{Sym}^m H^0(Y, \mathscr{O}_Y(H)) \to H^0(Y, \mathscr{O}_Y(mH))$ is surjective for any $m \geq 0$.

Back to the result, we need to check the conditions in Theorem 2.19. For (a), this follows from $H^n = H \cdot H^{n-1} = H|_{X'}^{n-1} = 10$. Now we know X' is projective normal, so is X by [47] Lemma (2.9). By Lemma 2.21 we know H is very ample. By $H^1(X, \mathscr{O}_X) = 0$ we know that $h^0(X, \mathscr{O}_X(H)) = n+5$ by the case of X'. This proves (b), and [47] Lemma (2.10) proves (c). For (d), since $\mathscr{U}_{X'}$ is simple, by the similar proof of (d) in Theorem 2.19 we can also show that $\mathscr{U}_X$ is simple. □

Theorem 2.23. *Let X be a complex smooth projective variety of dimension $n \geq 1$, together with an ample Cartier divisor H such that $K_X \sim_{\text{lin}} -(n-2)H$ and $H^n = 10$. If we assume that*

- *when $n \geq 3$, we have $\operatorname{Pic}(X) = \mathbb{Z} \cdot H$;*
- *when $n = 2$, the surface X is a* Brill-Noether general K3 surface *(a K3 surface is called* Brill-Noether general *if $h^0(S, D)h^0(S, H - D) < h^0(S, H)$ for all divisors D on S not linearly equivalent to 0 or H. When $H^2 = 10$, this is equivalent to the fact that $|H|$ contains a Clifford general smooth curve);*
- *when $n = 1$, the genus-6 curve X is* Clifford general *(that is, it is neither hyperelliptic, nor trigonal, nor a plane quintic).*

then X is a GM variety.

Before proving this, we need some Lemmas:

Lemma 2.24. *Let X be a complex smooth projective variety of dimension $n \geq 3$ with an ample divisor H such that $H^n = 10$ and $K_X \sim_{\text{lin}} -(n-2)H$.*

Then the linear system $|H|$ is very ample and a smooth general $X' \in |H|$ satisfies the same conditions: if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and $K_{X'} \sim_{\text{lin}} -(n-3)H'$.

Proof. First we need to show that $h^0(H) > 0$. This follows from the follows result:

- **Lemma 2.24.A.** Let X be a smooth Fano variety of dimension $n \geq 3$ such that $-K_X \sim_{\text{lin}} rH$ where H is ample. Then when $r \geq n-2$, then $h^0(H) > 0$.

Proof of Lemma 2.24.A. See Theorem 1.68. □

Hence now $|H|$ is non-empty. Note that in this case H is already the fundamental divisor since $H^n = 10$. Hence by Theorem 2.1 and Theorem 2.2 as in this case the index of X is $\geq n-2$, then the general elements are smooth. Pick such X'. Then if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and by adjunction formula we have $K_{X'} \sim_{\text{lin}} -(n-3)H'$. By Kodaira vanishing theorem we have $H^1(X, \mathscr{O}_X) = 0$. Hence the linear series $|H'|$ is just the restriction of $|H|$ to X' and the base loci of $|H|$ and $|H'|$ are the same. Taking successive linear sections, we arrive at a linear section Y of dimension 3 which is smooth and $K_Y \sim_{\text{lin}} -H_Y$ and $H_Y^3 = 10$.

If $\operatorname{Pic}(Y) = \mathbb{Z} \cdot H_Y$, then by Corollary 2.3 the pair (Y, H_Y) is projectively normal.

If not, then $\rho(X) \geq 2$. By the classification theory (one omitted) of the Fano threefold, Y must be a divisor of bidegree $(3, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ and the pair (Y, H_Y) is again projectively normal.

Hence in both case, we can use the [47] Lemma (2.9) repeatly which imply that (X, H) is projectively normal. Hence by Lemma 2.21 we know H is very ample. □

Lemma 2.25. *Let (X, H) be a polarized complex variety of dimension $n \geq 2$ which satisfies the hypotheses of Theorem 2.23. A general element of $|H|$ then satisfies the same properties.*

Proof. Assume first $n \geq 4$. By Lemma 2.24 we need only to prove that a general smooth $X' \in |H|$ satisfies $\mathrm{Pic}(X') = \mathbb{Z} \cdot H'$ where $H' := H|_{X'}$. By Grothendieck-Lefschetz theorem we have $\mathrm{Cl}(X) \cong \mathrm{Cl}(X')$. Hence $\mathrm{Pic}(X') = \mathbb{Z} \cdot H'$ as $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$.

When $n = 2$, this follows from definitions.

When $n = 3$, X is a smooth Fano 3-fold with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Then by Corollary 2.3 X is an intersection of quadrics. Any smooth hyperplane section S of X is a degree-10 smooth K3 surface which is still an intersection of quadrics. A general hyperplane section of S is still an intersection of quadrics, hence is a Clifford general curve. This proves that S is Brill-Noether general. $\square$

Proof of Theorem 2.23. Induction on n. The case $n = 1$ was proved in Proposition 2.26, so we assume $n \geq 2$. A general hyperplane section X' of X has the same properties by Lemma 2.25, hence is a GM variety by the induction hypothesis. On the other hand, we have $H^1(X, \mathscr{O}_X) = 0$. By Proposition 2.22, we conclude that X is a GM variety. Well done. $\square$

Some inverse results:

Proposition 2.26. *A smooth projective curve is a GM curve if and only if it is a Clifford general curve of genus* 6.

Proof. Follows from the Theorem 2.19 and the Enriques-Babbage theorem in [6] Section III.3. $\square$

Proposition 2.27. *A smooth projective surface X is a GM surface if and only if X is a Brill-Noether general polarized K3 surface of degree* 10.

Proof. By Theorem 2.23, we just need to show that if X is a GM surface, then X is a Brill-Noether general polarized K3 surface of degree 10. In this case, we have $K_X = 0$ by Theorem 2.19(a), and the resolution

$$\begin{aligned} 0 \to \mathscr{O}(-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-5) \to (V_5 \otimes \mathscr{O}(-4)) \oplus (V_5^\vee \otimes \mathscr{O}(-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0 \end{aligned}$$

implies $H^1(X, \mathscr{O}_X) = 0$, hence X is a K3 surface. Moreover, a general hyperplane section of X is a GM curve, hence a Clifford general curve of genus 6, hence X is Brill-Noether general. $\square$

Proposition 2.28. *Let (X, H) be a polarized complex smooth GM variety of dimension $n \geq 3$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. In particular, the polarization H is the unique GM polarization on X.*

Proof. By Theorem 2.23, we just need to show that if (X, H) be a polarized complex smooth GM variety of dimension $n \geq 3$, then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. By Theorem 2.19, X is a Fano variety of degree 10 and is an intersection of quadrics. When $n = 3$, by the proof of Lemma 2.24 we know that $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Now consider $n \geq 4$, a general hyperplane section X' of X satisfies the same properties by Lemma 2.25 and by Grothendieck-Lefschetz theorem again (for general case we refer Theorem 1 in [85]) we have injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X')$. Hence by induction we get the result. $\square$

2.2.3 Grassmannian Hulls

Fix $V_5, V_6, K, W \subset \bigwedge^2 V_5 \oplus K, Q \subset \mathbf{P}(W)$ which defines a smooth GM variety

$$X = \mathsf{C}_K \,\mathsf{Grass}(2, V_5) \cap \mathbf{P}(W) \cap Q.$$

Definition 2.29. *Define $M_X := \mathsf{C}_K \,\mathsf{Grass}(2, V_5) \cap \mathbf{P}(W)$ to be the Grassmannian hull of X. Hence $X = M_X \cap Q$ which is a quadric section of M_X.*

Define $M'_X := \mathsf{Grass}(2, V_5) \cap \mathbf{P}(W')$ to be the projected Grassmannian hull of X where W' defined as the image of the projection $\mu : W \subset \bigwedge^2 V_5 \oplus K \to \bigwedge^2 V_5$.

Remark 2.30. *Note that these two schemes are canonically associated to X via GM datas. See [17] Section 2.*

Now consider the Gushel map $X \to \mathsf{Grass}(2, V_5)$.

Proposition 2.31. *Let X be a such smooth GM variety.*

(i) If $\mu : W \to \bigwedge^2 V_5$ is injective, that is, μ induce $W \cong W'$, then $M_X \cong M'_X$ and Gushel map $X \to \mathsf{Grass}(2, V_5)$ is an embedding which induce

$$X \cong M'_X \cap Q = \mathsf{Grass}(2, V_5) \cap \mathbf{P}(W) \cap Q.$$

In this case we call X a ordinary GM variety. Hence in this case

$$\dim X = \dim W - 5 \leq \dim \bigwedge^2 V_5 - 5 = 5.$$

(ii) If $\ker \mu \neq 0$, then $\dim \ker \mu = 1$, $Q \cap \mathbf{P}(\ker \mu) = \emptyset$ and $M_X = \mathsf{C}_{\mathbf{P}(\ker \mu)} M'_X$ and the Gushel map $X \to \mathsf{Grass}(2, V_5)$ induce $X \to M'_X$ which is a double covering branched at a quadric (which is a ordinary GM variety if $\dim X \geq 2$). In this case we call X a special GM variety. Hence in this case it comes with a canonical involution from the double covering and

$$\dim X = \dim W - 5 \leq \dim \bigwedge^2 V_5 + 1 - 5 = 6.$$

Proof. For (i), this is trivial by the conditions.

For (ii), note that the blow up $\mathrm{Bl}_{\mathbb{P}(\ker\mu)}M_X$ at its vertex is a $\mathbb{P}^{\dim\ker\mu}$-bundle over M'_X. As X is smooth, then $X \cap \mathbf{P}(K) = Q \cap \mathbf{P}(\ker\mu) = \emptyset$. Hence $\dim\ker\mu = 1$ as $\dim Q = \dim\mathbb{P}(W) - 1$. Now as Q is a quadric, then the Gushel map induce $X \to M'_X$ which is a double covering. We have $X \to M'_X$ branched along $\mathsf{Grass}(2, V_5) \cap \mathbf{P}(W') \cap Q$ which is a ordinary GM variety if $\dim X \geq 2$, □

Remark 2.32. *By (ii), we can turn the special GM variety into a ordinary GM variety (as its branched locus). This leads to an important birational operation on the set of all GM varieties which can be described by GM datas. This actually gives a correspondence between special GM n-folds and ordinary GM $(n-1)$-folds. For details we refer [17] Lemma 2.33.*

Remark 2.33. *Hence in this case we know that we only need to assume* $\dim K = 1$ *to construct the whole theory if we just consider the smooth GM varieties.*

2.3 Rational Homogeneous Varieties

2.3.1 Some Lie Algebras and Algebraic Groups

We only consider the objects over $\mathbb{C}$. We will recall some basic things about Cartan decomposition, root system, Weyl groups, Cartan matrix and Dynkin diagrams.

Cartan Decomposition

Definition 2.34. *A Cartan subalgebra of a Lie algebra is a nilpotent subalgebra equal to its own normalizer.*

Remark 2.35. *This shows that Cartan subalgebra is a maximal nilpotent subalgebra:*

Let $\mathfrak{h} \subset \mathfrak{g}$ be a proper subalgebra of a Lie algebra $\mathfrak{g}$. By induction on $\dim\mathfrak{g}$ *we can show that $\mathfrak{h} \neq n_{\mathfrak{g}}(\mathfrak{h})$. Indeed, as $z(\mathfrak{g}) \neq 0$, if $z(\mathfrak{g}) \not\subseteq \mathfrak{h}$ then $\mathfrak{h} \neq n_{\mathfrak{g}}(\mathfrak{h})$ since $z(\mathfrak{g})$ normalizes $\mathfrak{h}$. If $z(\mathfrak{g}) \subset \mathfrak{h}$, apply induction to $\mathfrak{h}/z(\mathfrak{g}) \subset \mathfrak{g}/z(\mathfrak{g})$.*

Definition 2.36. *Let $\mathfrak{g}$ be a Lie algebra. For any $x \in mathfrakg$ consider the characteristic polynomial $P_x(T) = \det(T - \mathsf{ad}(x)|_{\mathfrak{g}})$, let $n(x)$ be the multiplicity of T in $P_x(T)$, or equivalently, the multiplicity of 0 as an eigenvalue of $\mathsf{ad}(x)$. Then we define the rank of $\mathfrak{g}$ is $n = \min\{n(x) : x \in \mathfrak{g}\}$ and $x \in \mathfrak{g}$ is called regular if $n(x) = n$.*

Remark 2.37. *By definition the regular elements forms a Zariski open subset.*

Proposition 2.38. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) Consider the primary decomposition $\mathfrak{g} = \bigoplus_{\lambda\in\mathbb{C}} \mathfrak{g}_x^\lambda$ associated to $\mathsf{ad}(x)$, then $[\mathfrak{g}_x^\lambda, \mathfrak{g}_x^\mu] \subset \mathfrak{g}_x^{\lambda+\mu}$. Hence $\mathfrak{g}_x^0$ is a Lie subalgebra.

(b) *For any regular element $x \in \mathfrak{g}$, the subalgebra $\mathfrak{g}_x^0$ is a Cartan subalgebra of dimension* $\operatorname{rank}\mathfrak{g}$. *In particular, any Lie algebra has a Cartan subalgebra.*

(c) *Any two Cartan subalgebras are conjugate by an elementary automorphism, that is, product of automorphisms of form* $\exp(\mathsf{ad}(x))$.

(d) *For any Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we have* $\dim \mathfrak{h} = \operatorname{rank}\mathfrak{g}$ *and there exists a regular element $x \in \mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}_x^0$.*

Proof. For (a), this is follows from

$$(\mathsf{ad}(x) - \lambda - \mu)^m [y, z] = \sum_{i=1}^{m} \binom{m}{i} [(\mathsf{ad}(x) - \lambda)^i (y), (\mathsf{ad}(x) - \mu)^{m-i}(z)]$$

for $m \gg 0$.

For (b), consider two Zariski open subsets of $\mathfrak{g}_x^0$:

$$U_1 := \{y \in \mathfrak{g}_x^0 : \mathsf{ad}(y)|_{\mathfrak{g}_x^0} \text{ is not nilpotent}\}, \quad U_2 := \{y \in \mathfrak{g}_x^0 : \mathsf{ad}(y)|_{\mathfrak{g}/\mathfrak{g}_x^0} \text{ is invertible}\}.$$

Now $U_2 \neq \emptyset$ since $x \in U_2$. To show $\mathfrak{g}_x^0$ is nilpotent, we just need to show $U_1 = \emptyset$ by Engel's theorem. If not, then $U_1 \cap U_2 \neq \emptyset$. Pick such y in the intersection. Then $n(y) < \dim \mathfrak{g}_x^0 = n(x)$, contradicting the regularity of x. Hence $\mathfrak{g}_x^0$ is nilpotent.

To show $\mathfrak{g}_x^0 = n_{\mathfrak{g}}(\mathfrak{g}_x^0)$, pick $z \in n_{\mathfrak{g}}(\mathfrak{g}_x^0)$, then $[z, x] \in \mathfrak{g}_x^0$, that is, $(\mathsf{ad}(x))^m [z, x] = 0$ for some m. Hence $(\mathsf{ad}(x))^{m+1}(z) = 0$. Hence $z \in \mathfrak{g}_x^0$, well done.

For (c), we omit it and we refer Section III.4 in [86]. Now (d) is a direct corollary of (b) and the proof of (c). See Corollary III.4.2 in [86]. □

Now we consider the decomposition of a Lie algebra.

Theorem 2.39 (Representation of Nilpotent Lie Algebras)**.** *Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}_V$ be a representation. For any linear from $\lambda \in \mathfrak{g}^\vee$ we define the primary space $V^\lambda := \{v \in V : (\rho(g) - \lambda(g))^n v = 0, n \gg 0, \forall g \in \mathfrak{g}\}$. Then if $\mathfrak{g}$ is nilpotent, then each V^λ is stable under $\mathfrak{g}$ and*

$$V = \bigoplus_{\lambda \in \mathfrak{g}^\vee} V^\lambda.$$

Proof. See Bourbaki's Lie algebra VII. □

Definition 2.40. *Let $\mathfrak{g}$ be a Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Consider adjoint action of $\mathfrak{h}$ acting at $\mathfrak{g}$, we get $\mathsf{ad}_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{gl}_{\mathfrak{g}}$. Hence by Theorem 2.39 we have a primary decomposition*

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}^\alpha$$

which is called the Cartan decomposition of $(\mathfrak{g}, \mathfrak{h})$ *where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : (\mathsf{ad}(h) - \alpha(h))^n g = 0, n \gg 0, \forall h \in \mathfrak{h}\}$.*

Now we consider the semisimple Lie algebras which are our main objects.

Theorem 2.41. *Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$.*

(a) *The restricted Killing form $\kappa_{\mathfrak{g}}|_{\mathfrak{h}}$ is nondegenerate.*

(b) *We have $\mathfrak{h}$ is abelian and $c_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

(c) *Every elements of $\mathfrak{h}$ is semisimple.*

(d) *We have Cartan decomposition of $(\mathfrak{g}, \mathfrak{h})$ as*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}^{\alpha}$$

where $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^{\vee} \backslash \{0\}$ be a finite subset such that $\mathfrak{g}^{\alpha} \neq 0$ for that $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}^{\alpha} = \{g \in \mathfrak{g} : \mathsf{ad}(h)g = \alpha(h)g, \forall h \in \mathfrak{h}\}$. Moreover $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \subset \mathfrak{g}^{\alpha+\beta}$.

(e) *The Cartan subalgebra $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$.*

(f) *The Cartan subalgebras of a semisimple Lie algebra are those that are maximal among the subalgebras whose elements are semisimple.*

(g) *Every reguler element is semisimple.*

Proof. For (a), by Proposition 2.38(d) there exists a regular element $x \in \mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}_x^0$. Then we have the primary decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \mathbb{C}^\times} \mathfrak{g}_x^{\lambda}$ associated to $\mathsf{ad}(x)$. Let $x \in \mathfrak{g}_x^a, y \in \mathfrak{g}_x^b$, then we have

$$\kappa_{\mathfrak{g}}(\mathsf{ad}(h)x, y) + \kappa_{\mathfrak{g}}(x, \mathsf{ad}(h)y) = 0.$$

Hence $(a+b)\kappa_{\mathfrak{g}}(x, y) = 0$. Hence $\mathfrak{g}_x^a$ and $\mathfrak{g}_x^b$ are orthogonal with respect to $\kappa_{\mathfrak{g}}$ if $a+b \neq 0$. Hence we have orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \mathbb{C}^\times/\pm} (\mathfrak{g}_x^{\lambda} \oplus \mathfrak{g}_x^{-\lambda}).$$

As $\kappa_{\mathfrak{g}}$ is nondegenerate, then so is $\kappa_{\mathfrak{g}}|_{\mathfrak{h}}$.

For (b), as $z(\mathfrak{g}) = 0$, hence the adjoint representation of $\mathfrak{h}$ make it as a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{\mathfrak{g}}$. By Lie's theorem there exists a base such that $\mathsf{ad}\,\mathfrak{h} \subset \mathfrak{b}_m$. Hence $\mathsf{ad}[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{n}_m$. Hence $\kappa_{\mathfrak{g}}(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$. By (a) we have $[\mathfrak{h}, \mathfrak{h}] = 0$ and hence $\mathfrak{h}$ is abelian. Now $\mathfrak{h} \subset c_{\mathfrak{g}}(\mathfrak{h}) \subset n_{\mathfrak{g}}(\mathfrak{h})$. By definition $\mathfrak{h} = n_{\mathfrak{g}}(\mathfrak{h})$, we have $\mathfrak{h} = c_{\mathfrak{g}}(\mathfrak{h})$.

For (c), by Jordan-Chevalley decomposition we have $x = x_s + x_n$ for any $x \in \mathfrak{h}$. As $\mathsf{ad}(x_s)$ and $\mathsf{ad}(x_n)$ are polynomials of $\mathsf{ad}(x)$, then by (b) $x_s, s_n \in c_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. For any $y \in \mathfrak{h}$ we know that $\mathsf{ad}(y)$ and $\mathsf{ad}(x_n)$ are commute and $\mathsf{ad}(x_n)$ is nilpotent, then $\mathrm{tr}(\mathsf{ad}(y) \circ \mathsf{ad}(x_n)) = 0$. Hence $x_n = 0$ by (a).

For (d), we already have the Cartan decomposition of $(\mathfrak{g}, \mathfrak{h})$:

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : \mathsf{ad}(h)g = \alpha(h)g, \forall h \in \mathfrak{h}\}$ as $\mathfrak{g}$ semisimple and by (b) they can simultaneously diagonalizable. As $\mathfrak{g}^0 = c_{\mathfrak{g}}(\mathfrak{h})$, hence by (b) we have the result. The fact $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ is follows from the direct calculation.

For (e), this is follows by (b) directly.

For (f), this is follows by (b)(e) and the fact that if any $\mathsf{ad}(x)$ is semisimple for $x \in \mathfrak{h}$, then $\mathfrak{h}$ is abelian.

For (g), this is because every regular element is contained in a Cartan subalgebra and use (c). □

Classifications of Semisimple Lie Algebras

We omit the general definitions of root Systems, Weyl Groups, Cartan Matrix, Coxeter Graphs and Dynkin Diagrams. See Chapter V in [86]. Here we state some basic results of the classification theory of semisimple Lie algebras.

Theorem 2.42. *Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We have Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : \mathsf{ad}(h)g = \alpha(h)g, \forall h \in \mathfrak{h}\}$. Fix an $\alpha \in R(\mathfrak{g}, \mathfrak{h})$.

(a) $\mathfrak{g}^\alpha$ and $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{h}$ are both 1-dimensional.

(b) There is a unique element $h_\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(h_\alpha) = 2$.

(c) For each nonzero $x_\alpha \in \mathfrak{g}^\alpha$ there is a unique $y_\alpha \in \mathfrak{g}^{-\alpha}$ such that

$$[x_\alpha, y_\alpha] = h_\alpha, \quad [h_\alpha, x_\alpha] = 2x_\alpha, \quad [h_\alpha, y_\alpha] = 2y_\alpha.$$

Hence $\mathfrak{s}_\alpha := \mathbb{C}x_\alpha \oplus \mathbb{C}h_\alpha \oplus \mathbb{C}y_\alpha = \mathfrak{g}^\alpha \oplus [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \oplus \mathfrak{g}^{-\alpha}$ is a copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Proof. See Chapter VI in [86] or J. Milne's notes [66]. □

Theorem 2.43. *Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. We have Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}^\alpha$$

where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : \mathsf{ad}(h)g = \alpha(h)g, \forall h \in \mathfrak{h}\}$. Then *(a)* *$R(\mathfrak{g},\mathfrak{h})$ is finite, spans $\mathfrak{h}^\vee$ and does not contain 0.*

(b) *For each $\alpha \in R(\mathfrak{g},\mathfrak{h})$, let $h_\alpha \in \mathfrak{h}$ as in Theorem 2.42. Let $\mathfrak{h} \cong \mathfrak{h}^{\vee\vee}$ with $h_\alpha \mapsto \alpha^\vee$, then $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle R(\mathfrak{g},\mathfrak{h}), \alpha^\vee \rangle \in \mathbb{Z}$, and the symmetry $s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ maps $R(\mathfrak{g},\mathfrak{h})$ into $R(\mathfrak{g},\mathfrak{h})$.*

(c) *For no $\alpha \in R(\mathfrak{g},\mathfrak{h})$ does $2\alpha \in R(\mathfrak{g},\mathfrak{h})$.*

Hence $R(\mathfrak{g},\mathfrak{h})$ is a reduced root system in $\mathfrak{h}^\vee$.

Proof. For (a), if $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for any $\alpha \in R(\mathfrak{g},\mathfrak{h})$, then $[h, \mathfrak{g}^\alpha] = 0$. Hence $h \in z(\mathfrak{g}) = 0$ and $h = 0$. Hence $R(\mathfrak{g},\mathfrak{h})$ spans $\mathfrak{h}^\vee$.

For (b), we claim that for any $\alpha, \beta \in R(\mathfrak{g},\mathfrak{h})$ we have $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in R(\mathfrak{g},\mathfrak{h})$. Indeed, regard $\mathfrak{g}$ as an $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$-module via adjoint representation. Let z be a nonzero element in g^β, then $[h_\alpha, z] = \beta(h_\alpha) z$, hence $n := \beta(h_\alpha) \in \mathbb{Z}$ by the representation theory of $\mathfrak{sl}_2$. Moreover we have $y_\alpha^n : \mathfrak{g}^\beta \cong \mathfrak{g}^{\beta - n\alpha}$ if $n \geq 0$ and $x_\alpha^{-n} : \mathfrak{g}^\beta \cong \mathfrak{g}^{\beta - n\alpha}$ if $n \leq 0$. Hence in any case $\beta - n\alpha \in R(\mathfrak{g},\mathfrak{h})$. This finish the claim. Hence the result follows directly from this claim.

For (c), suppose that there is $\alpha \in R(\mathfrak{g},\mathfrak{h})$ such that $2\alpha \in R(\mathfrak{g},\mathfrak{h})$. Hence there exists $y \neq 0$ such that $[h_\alpha, y] = 2\alpha(h_\alpha) y = 4y$. As $h_\alpha = [x_\alpha, y_\alpha]$, we have $[h_\alpha, y] = [x_\alpha, [y_\alpha, y]]$. But $[y_\alpha, y] \in \mathfrak{g}^\alpha = \mathbb{C} x_\alpha$, hence $[h_\alpha, y] = [x_\alpha, [y_\alpha, y]] = 0$. This is impossible. $\square$

The classifications of semisimple Lie algebras as follows:

Theorem 2.44 (Classifications of Semisimple Lie Algebras)**.** *Over $\mathbb{C}$ we have:*

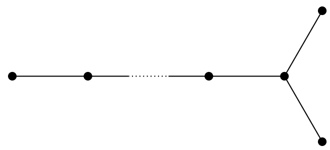
(a) *Every reduced root system arises from a pair of Lie algebras $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$.*

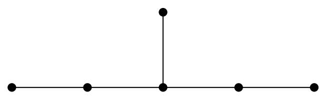
(b) *The root system of a semisimple Lie algebra determines it up to isomorphism. Note that by Proposition 2.38(c) that root system of a semisimple Lie algebra is independent to the Cartan subalgebra up to isomorphism.*

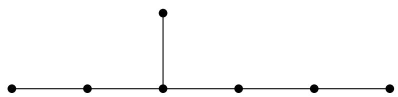
(c) *A decomposition of a pair $(\mathfrak{g},\mathfrak{h})$ as before is equivalent to a decomposition of its root system.*

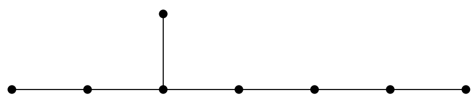
(d) *Any Dynkin diagrams (and equivalently a Cartan matrix) arising from indecomposable root systems are exactly the following type diagrams $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $D_n (n \geq 4)$, E_6, E_7, E_8, F_4 and G_2:*

- *$A_n (n \geq 1)$:*
- *$B_n (n \geq 2)$:*
- *$C_n (n \geq 3)$:*

– *$D_n (n \geq 4)$:* 

– *E_6:* 

– *E_7:* 

– *E_8:* 

– *F_4:*

– *G_2:*

Note that the type E_6, E_7, E_8, F_4, G_2 are called exceptional*.*

(e) *Each type in (d) has a indecomposable root system such that its Dynkin diagram has that type.*

Proof. For (a), we refer Theorem VI.9 in [86].

For (b), we refer Theorem VI.8 and Theorem VI.8' in [86].

For (c), this is trivial.

For (d), we refer Theorem V.4 in [86].

For (e), we consider Section V.16 in [86]. □

Here is an easy but useful criteria for semisimplicity:

Proposition 2.45. *Let $\mathfrak{g}$ be a Lie algebra with a abelian Lie subalgebra $\mathfrak{h}$. For each $\alpha \in \mathfrak{h}^\vee$ we define $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} : \mathsf{ad}(h)g = \alpha(h)g, \forall h \in \mathfrak{h}\}$. Let $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^\vee \backslash \{0\}$ consist of α such that $\mathfrak{g}^\alpha \neq 0$. If*

(a) *We have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathfrak{g}^\alpha$.*

(b) $\dim \mathfrak{g}^\alpha = 1$ *for each $\alpha \in R(\mathfrak{g}, \mathfrak{h})$.*

(c) *For each nonzero $h \in \mathfrak{h}$, there exists an $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ such that $\alpha(h) \neq 0$.*

(d) *If $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, then $-\alpha \in R(\mathfrak{g}, \mathfrak{h})$ and $[[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}], \mathfrak{g}^\alpha] \neq 0$.*

Then $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a Cartan subalgebra.

Proof. Pick a abelian ideal $\mathfrak{a}$. As $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{a}$, we have

$$\mathfrak{a} = \mathfrak{a} \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathfrak{a} \cap \mathfrak{g}^{\alpha}$$

by (a). If $\mathfrak{a} \cap \mathfrak{g}^{\alpha} \neq 0$, then by (b) $\mathfrak{g}^{\alpha} \subset \mathfrak{a}$. As $\mathfrak{a}$ is an ideal, we have $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}] \subset \mathfrak{a}$. As $[\mathfrak{a}, \mathfrak{a}] = 0$, then $[[\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}], \mathfrak{g}^{\alpha}] = 0$ which is contradicting (d). Moreover, if $\mathfrak{a} \cap \mathfrak{h} \neq 0$, let $0 \neq h \in \mathfrak{a} \cap \mathfrak{h}$. By (c) there exists an $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ such that $\alpha(h) \neq 0$. Pick $0 \neq x \in \mathfrak{g}^{\alpha}$, then $[h, x] = \alpha(h)x$. Hence $0 \neq [h, x] \in \mathfrak{g}^{\alpha} \cap \mathfrak{a}$ which is impossible by the previous argument. Hence $\mathfrak{a} = 0$ and $\mathfrak{g}$ is semisimple. Now by (a) directly we find that $\mathfrak{h}$ is a Cartan subalgebra. □

Example 2.46 (Classical Lie Algebras)**.** *We consider several types of subalgebras of $\mathfrak{gl}_{n+1}$. Note that $\mathfrak{gl}_{n+1}$ is not semisimple since $z(\mathfrak{gl}_{n+1})$ are scalar matrixes.*

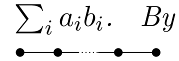
Let $\hat{\mathfrak{h}} \subset \mathfrak{gl}_{n+1}$ be a subalgebra of diagonal objects. Hence $\{E_{ij}\}$ and $\{E_{ii}\}$ are basis of $\mathfrak{gl}_{n+1}$ and $\hat{\mathfrak{h}}$, respectively. Let (ε_i) be a dual basis of $\hat{\mathfrak{h}}$. As for $h \in \hat{\mathfrak{h}}$ we have $[h, E_{ij}] = (\varepsilon_i(h) - \varepsilon_j(h))E_{ij}$, we have $\mathfrak{gl}_{n+1} = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in R} \mathfrak{gl}_{n+1}^{\alpha}$ where $R = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ and $\mathfrak{gl}_{n+1}^{\varepsilon_i - \varepsilon_j} = \mathbb{C} \cdot E_{ij}$.

(a) ***Type** A_n**:** $\mathfrak{sl}_{n+1}$. Let $\mathfrak{h}$ be the subalgebra of diagonal objects. Here $\mathfrak{sl}_{n+1} = \{\mathbf{x} \in \mathfrak{gl}_{n+1} : \mathrm{tr}(\mathbf{x}) = 0\}$.*

Note that $\{E_{i,i} - E_{i+1,i+1}\}_{1 \leq i \leq n}$ is a basis of $\mathfrak{h}$ and $\{E_{i,i} - E_{i+1,i+1}\}_{1 \leq i \leq n} \cup \{E_{i,j} : i \neq j\}$ is a basis of $\mathfrak{sl}_{n+1}$. Note that $\mathfrak{h}^{\vee}$ be a hyperplane of $\hat{\mathfrak{h}}^{\vee}$ consist of $\sum_i a_i \varepsilon_i$ for $\sum_i a_i = 0$. Now we also have

$$\mathfrak{sl}_{n+1} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{sl}_{n+1},\mathfrak{h})} \mathfrak{sl}_{n+1}^{\alpha}$$

where $R(\mathfrak{sl}_{n+1}, \mathfrak{h}) = \{\varepsilon_i - \varepsilon_j : i \neq j\}$. Easy to check the conditions in Proposition 2.45, hence $(\mathfrak{sl}_{n+1}, \mathfrak{h})$ is a semisimple Lie algebra with a Cartan subalgebra.

As $(\varepsilon_i - \varepsilon_{i+1})_i$ be a base of root system $R(\mathfrak{sl}_{n+1}, \mathfrak{h})$, consider the inner product $(\sum_i a_i \varepsilon_i, \sum_i b_i \varepsilon_i) = \sum_i a_i b_i$. By directly calculation we know that the Dynkin diagram is A_n type:  *Moreover $\mathfrak{sl}_{n+1}$ is simple.*

(b) ***Type** B_n**:** $\mathfrak{so}_{2n+1}$. Here the original definition is $\mathfrak{so}_{2n+1} = \{\mathbf{x} \in \mathfrak{gl}_{2n+1} : \mathbf{x} + \mathbf{x}^t = 0\}$. But here we will use an equivalent definition: let $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbf{I} \\ 0 & \mathbf{I} & 0 \end{pmatrix}$ and $\mathfrak{so}_{2n+1} = \{\mathbf{x} \in \mathfrak{gl}_{2n+1} : \mathbf{x}^t S + S\mathbf{x} = 0\}$. Let $\mathfrak{h}$ be the subalgebra of diagonal objects.*

(c) ***Type** C_n**:** $\mathfrak{sp}_{2n}$. Let $\mathfrak{h}$ be the subalgebra of diagonal objects. Here $\mathfrak{sp}_{2n} = \left\{\mathbf{x} \in \mathfrak{gl}_{2n} : \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} \mathbf{x} + \mathbf{x}^t \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} = 0\right\}$.*

(d) ***Type** D_n**:*** $\mathfrak{so}_{2n}$. *Here the original definition is* $\mathfrak{so}_{2n} = \{\mathbf{x} \in \mathfrak{gl}_{2n} : \mathbf{x} + \mathbf{x}^t = 0\}$. *But here we will use an equivalent definition: let* $S = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}$ *and* $\mathfrak{so}_{2n} = \{\mathbf{x} \in \mathfrak{gl}_{2n} : \mathbf{x}^t S + S\mathbf{x} = 0\}$. *Let* $\mathfrak{h}$ *be the subalgebra of diagonal objects.*

Note that the proof of (b)(c)(d) are similar as (a), so we omit it and we refer Milne's notes [66]. Note that $\mathfrak{sl}_n (n \geq 2)$, $\mathfrak{so}_n (n \geq 3)$ *and* $\mathfrak{sp}_n (n \geq 1)$ *are semisimple.*

Remark 2.47. *Note that we use the new but isomorphic definitions for* $\mathfrak{so}_{2n}$ *and* $\mathfrak{so}_{2n+1}$. *Here we will give the reason. We define* $\mathfrak{gl}_n^{\mathbf{T}} := \{\mathbf{M} \in \mathfrak{gl}_n : \mathbf{M}^t\mathbf{T} + \mathbf{T}\mathbf{M} = 0\}$, *then if* $\mathbf{T}$ *and* $\mathbf{S}$ *are congruent, then* $\mathfrak{gl}_n^{\mathbf{S}} \cong \mathfrak{gl}_n^{\mathbf{T}}$. *In our case well done.*

Connections with Semisimple Algebraic Groups

Definition 2.48. *Let* G *be an algebraic group, then we define*

$$\mathsf{Lie}(G) := T_e(G) = \ker(G(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \to G(\mathbb{C})).$$

(a) *Now* Lie *is a functor from algebraic groups to vector spaces (as tangent maps).*

(b) *Consider* $\mathrm{GL}_n := \operatorname{Spec} \mathbb{C}[\{T_{ij}\}_{1 \leq i,j \leq n}, \det(T_{ij})^{-1}]$. *For any* $\mathbf{A} \in \mathfrak{gl}_n$ *we consider* $\mathbf{I}+\varepsilon\mathbf{A}$. *Then* $(\mathbf{I}+\varepsilon\mathbf{A})(\mathbf{I}-\varepsilon\mathbf{A}) = \mathbf{I}$, *hence* $\mathbf{I}+\varepsilon\mathbf{A} \in \ker(\mathrm{GL}_n(\mathbb{C}[\varepsilon]/(\varepsilon^2)) \to \mathrm{GL}_n(\mathbb{C}))$ *and all the elements in it is of this form. Hence* $\mathsf{Lie}(\mathrm{GL}_n) \cong \mathfrak{gl}_n$ *as vector space. Now we define the Lie bracket as* $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$. *Hence* $\mathsf{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ *as Lie algebras.*

(c) *The conjugate action of* G *on itself defines* $\mathsf{Ad} : G \to \mathrm{GL}_{\mathsf{Lie}(G)}$. *Then functor* Lie *induce* $\mathsf{ad} := \mathsf{Lie}(\mathsf{Ad}) : \mathsf{Lie}(G) \to \mathfrak{gl}_{\mathsf{Lie}(G)}$. *So the Lie bracket given by* $[x, y] := \mathsf{ad}(x)(y)$. *Hence* $\mathsf{Lie}(G)$ *is called the* ***Lie algebra of*** G. *Hence* Lie *is a functor from algebraic groups to Lie algebras.*

Example 2.49. *Now we consider the cases in Example 2.46.*

(a) *Now we have defined* $\mathrm{GL}_n := \operatorname{Spec} \mathbb{C}[\{T_{ij}\}_{1 \leq i,j \leq n}, \det(T_{ij})^{-1}]$ *with* $\mathsf{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$ *as Lie algebras.*

(b) *Consider* $\mathrm{SL}_n := \operatorname{Spec} \mathbb{C}[\{T_{ij}\}_{1 \leq i,j \leq n}]/(\det(T_{ij}) - 1)$. *Then* $\mathsf{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n$. *Note that* SL_n *is simply connected almost-simple group.*

  Indeed, as before we have $\mathsf{Lie}(\mathrm{SL}_n) = \{\mathbf{I}+\varepsilon\mathbf{A} \in \mathrm{GL}_n(\mathbb{C}[\varepsilon]/(\varepsilon^2)) : 1 = \det(\mathbf{I}+\varepsilon\mathbf{A}) = 1 + \varepsilon\mathrm{tr}(\mathbf{A})\}$. *Hence* $\mathsf{Lie}(\mathrm{SL}_n) = \{\mathbf{I} + \varepsilon\mathbf{A} \in \mathrm{GL}_n(\mathbb{C}[\varepsilon]/(\varepsilon^2)) : \mathrm{tr}(\mathbf{A}) = 0\}$. *Hence* $\mathsf{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n$.

(c) *Consider* $\mathrm{O}_n := \operatorname{Spec} \dfrac{\mathbb{C}[\{T_{ij}\}_{1 \leq i,j \leq n}, \det(T_{ij})^{-1}]}{((T_{ij})^t(T_{ij}) - \mathbf{I})}$. *Then* $\mathsf{Lie}(\mathrm{O}_n) = \mathfrak{so}_n$.

  Indeed, as before we have $\mathsf{Lie}(\mathrm{O}_n) = \{\mathbf{I}+\varepsilon\mathbf{A} \in \mathrm{GL}_n(\mathbb{C}[\varepsilon]/(\varepsilon^2)) : (\mathbf{I}+\varepsilon\mathbf{A})^t(\mathbf{I}+\varepsilon\mathbf{A}) = \mathbf{I}\}$. *As this is equivalent to* $\mathbf{A}^t + \mathbf{A} = 0$, *we get* $\mathsf{Lie}(\mathrm{O}_n) = \mathfrak{o}_n$.

(d) *We define* $\mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n$*, then* $\mathsf{Lie}(\mathrm{SO}_n) = \mathfrak{so}_n$*. Note that we have*

$$0 \to \mathrm{SO}_n \to \mathrm{O}_n \to \{\pm 1\} \to 0.$$

Note that $\pi_1(\mathrm{O}_n) = \pi_1(\mathrm{SO}_n) = \mathbb{Z}/2\mathbb{Z}$ *for* $\neq 3$*.*

(e) *Consider* $\mathrm{Sp}_n := \mathrm{Spec}\, \dfrac{\mathbb{C}[\{T_{ij}\}_{1\le i,j\le n}, \det(T_{ij})^{-1}]}{\left((T_{ij})^t \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} (T_{ij}) - \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}\right)}$*. Then* $\mathsf{Lie}(\mathrm{Sp}_n) = \mathfrak{sp}_n$ *as (c). Note that* Sp_n *is simply connected almost-simple group.*

(f) *Now we consider the universal covering, the spin groups.*

Fix a $\mathbb{C}$*-vector space* V *(*$\dim V = n$*) with a nonsingular quadratic form on it, that is,* q *is equivalent to* $\sum_{i=1}^{n/2} x_{2i-1}x_{2i}$ *for* n *even and* $x_0^2 + \sum_{i=1}^{(n-1)/2} x_{2i-1}x_{2i}$ *for* n *odd.*

- *Define* $\mathrm{Cl}(V,q) := T^*V/(v\otimes v - q(v))$ *be the* Clifford algebra *associated to* V *and* q*. Then this is a graded algebra with* $\mathrm{Cl}_0(V,q)$ *is of even degree part and* $\mathrm{Cl}_1(V,q)$ *is of odd degree part.*

  Note that $\mathrm{Cl}(V,q) \cong M_{2^{n/2}}(\mathbb{C})$ *if* n *even and* $\mathrm{Cl}(V,q) \cong M_{2^{(n-1)/2}}(\mathbb{C}) \times M_{2^{(n-1)/2}}(\mathbb{C})$ *if* n *odd. Hence all* $\mathbb{C}$*-linear automorphisms of* $\mathrm{Cl}(V,q)$ *are inner associated to the elements in* $\mathrm{Cl}_0(V,q)^\times$*.*
- *Let* $\mathrm{SO}(V,q) := \ker(\mathrm{O}(V,q) \overset{\det}{\to} \mathbb{G}_m)$ *where* $\mathrm{O}(V,q) = \{\mathbf{x} \in \mathrm{GL}_V : q(\mathbf{x}v) = q(v), \forall v \in V\}$ *are algebraic subgroups. In our case* $\mathrm{SO}(V,q) \cong \mathrm{SO}_n$*.*
- *Now for any* $g \in \mathrm{SO}(V,q)$*, the induced* $g : V \cong V$ *induce an isomorphism* $\mathrm{Cl}(V,q) \cong \mathrm{Cl}(V,q)$ *by the universal property. Hence this defines an element* $h \in \mathrm{Cl}_0(V,q)^\times$*.*

  Conversely if $h \in \mathrm{Cl}_0(V,q)^\times$ *is such that* $hVh^{-1} = V$*, then the mapping* $V \to V$ *induced by* $x \mapsto hxh^{-1}$ *is an element of* $\mathrm{SO}(V,q)$*.*

  Hence if we define an algebraic group $\mathrm{GSpin}(V,q) = \{g \in \mathrm{Cl}_0(V,q)^\times : gVg^{-1} = V\}$*, then we have an exact sequence*

  $$1 \to \mathbb{G}_m \to \mathrm{GSpin}(V,q) \to \mathrm{SO}(V,q) \to 1.$$

  Define $\mathrm{Spin}(V,q) = \ker(\mathrm{GSpin}(V,q) \overset{q(-)}{\to} \mathbb{G}_m)$*.*
- *We have the following commutative diagram:*

$$\begin{tikzcd}
 & & 1 \arrow[r] & \mathrm{Spin}(V,q) \arrow[d] & & \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[dr, "x\mapsto x^2"'] & & \mathrm{GSpin}(V,q) \arrow[r] \arrow[d] & \mathrm{SO}(V,q) \arrow[r] & 1 \\
 & & & \mathbb{G}_m \arrow[r] & 1 &
\end{tikzcd}$$

of exact sequences. This induce an exact sequence:

$$1 \to \boldsymbol{\mu}_2 \to \mathrm{Spin}(V, q) \times \mathbb{G}_m \to \mathrm{GSpin}(V, q) \to 1.$$

By some diagram chase we get

$$\mathrm{Spin}(V, q)/\boldsymbol{\mu}_2 \cong \mathrm{GSpin}(V, q)/\mathbb{G}_m \cong \mathrm{SO}(V, q).$$

Hence let $\mathrm{Spin}(V, q) \to \mathrm{SO}_n$ *is a double covering and* $\mathrm{Spin}(V, q)$ *is simply connected when* $n \geq 3$.

Hence we have a double covering $\mathrm{Spin}_n \to \mathrm{SO}_n$ *and hence* $\mathsf{Lie}(\mathrm{Spin}_n) \cong \mathfrak{so}_n$. *Moreover* Spin_n *is simply connected when* $n \geq 3$.

Proposition 2.50. *Now let G be an algebraic group, then we have*

$$\mathsf{Lie}(G) \cong \{\textit{left invariant derivations of } \Gamma(G, \mathscr{O}_G)\} \subset \mathrm{Der}(\Gamma(G, \mathscr{O}_G))$$

as Lie algebras with Lie bracket $[D, D'] = D \circ D' - D' \circ D$. *Note that a left invariant derivation D is defined as satisfies* $\Delta \circ D = (\mathrm{id} \otimes D) \circ \Delta$.

Proof. Well-known that $\mathsf{Lie}(G) \cong \mathrm{Der}(\Gamma(G, \mathscr{O}_G), \mathbb{C})$. So we just need to consider $\mathrm{Der}^{\mathsf{l}}(\Gamma(G, \mathscr{O}_G))$ of left invariant derivations with $\mathrm{Der}(\Gamma(G, \mathscr{O}_G), \mathbb{C})$. Note that let $e : \mathrm{Spec}\,\mathbb{C} \to G$ as $E : \Gamma(G, \mathscr{O}_G) \to \mathbb{C}$. Define $\mathrm{Der}^{\mathsf{l}}(\Gamma(G, \mathscr{O}_G)) \to \mathrm{Der}(\Gamma(G, \mathscr{O}_G), \mathbb{C})$ as $D \mapsto E \circ D$. We omit more details. $\square$

Proposition 2.51. *Consider the functor* Lie.

(a) Lie *is an exact functor.*

(b) Lie *commute with finite inverse limits.*

(c) *Fix an algebraic group, then* Lie *acting on subgroups is injective and preserve order.*

(d) *Let $H \subset G$ be a algebraic subgorup. Then* $\mathsf{Lie}(N_G(H)) = \mathfrak{n}_{\mathsf{Lie}(G)}(\mathsf{Lie}(H))$ *and* $\mathsf{Lie}(C_G(H)) = \mathfrak{c}_{\mathsf{Lie}(G)}(\mathsf{Lie}(H))$.

Proof. Omitted. $\square$

Proposition 2.52. *We have the following useful things.*

(a) *A connected algebraic group G is semisimple if and only if its Lie algebra* $\mathsf{Lie}(G)$ *is semisimple.*

(b) *If $\mathfrak{g}$ be a Lie algebra. We define a functor $G(-)$ such that $G(\mathfrak{g})$ be the Tannaka dual of the neutral tannakian category* $(\mathrm{Rep}(\mathfrak{g}), \mathrm{Forget})$. *Then* $G \dashv \mathsf{Lie}$ *between affine groups and Lie algebras.*

(c) *When* $\mathfrak{g}$ *be a semisimple Lie algebra, then* $G(\mathfrak{g})$ *is also semisimple and* $\mathfrak{g} \cong \mathsf{Lie}(G(\mathfrak{g}))$

Proof. Omitted. □

Here is the classification theory of semisimple algebraic groups:

Definition 2.53. *Let* (V, R) *be a root system. The* **root lattice** $Q = Q(R) = \mathbb{Z} \cdot R$ *is the* $\mathbb{Z}$*-submodule of* V *generated by the roots. The* **weight lattice** $P = P(R)$ *is the lattice dual to* $Q(R^\vee)$*:*

$$P(R) = \{x \in V : \langle x, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

The elements of P *are called the* **weights** *of the root system. We have* $Q(R) \subset P(R)$ *and the quotient* $P(R)/Q(R)$ *is finite (because the lattices generate the same* $\mathbb{Q}$*-vector space).*

Theorem 2.54. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra with Cartan subalgebra* $\mathfrak{h}$*, and let* P *and* Q *be the corresponding weight and root lattices. The action of* $\mathfrak{h}$ *on a* $\mathfrak{g}$*-module* V *decomposes it into a direct sum* $V = \bigoplus_{w \in P} V_w$ *of weight spaces. Let* $D(P)$ *be the diagonalizable group which satisfies* $R \mapsto \mathrm{Hom}_{\mathrm{Groups}}(P, R^\times)$ *is a functor. Thus* $D(P)$ *is a torus such that* $\mathrm{Rep}(D(P))$ *has a natural identification with the category of* P*-graded vector spaces. The functor* $(V, r_V) \mapsto (V, (V_w)_{w \in P})$ *is an exact tensor functor* $\mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(D(P))$ *compatible with the forgetful functors, and hence by dual it defines a homomorphism* $D(P) \to G(\mathfrak{g})$ *with image* $T(\mathfrak{h})$*. Then we have the following.*

(a) $T(\mathfrak{h})$ *is a maximal torus in* $G(\mathfrak{g})$ *and* $\mathfrak{g} \cong \mathsf{Lie}(G(\mathfrak{g}))$ *induce* $\mathfrak{h} \cong \mathsf{Lie}(T(\mathfrak{h}))$*.*

(b) *We have* $D(P) \cong T(\mathfrak{h})$ *and* $X^*(T(\mathfrak{h})) = P$*.*

(c) *We have* $z(G(\mathfrak{g})) = \bigcap_{\alpha \in R} \ker(\alpha : T(\mathfrak{h}) \to \mathbb{G}_m)$*. Hence* $X^*(z(G(\mathfrak{g}))) = P/Q$*.*

Moreover, let $T \subset G$ *be a subtorus of a semisimple algebraic group, then the following are equivalence.*

(1) T *is a maximal torus.*

(2) $T = C_G(T)^0$*.*

(3) $\mathsf{Lie}(T)$ *is a Cartan subalgebra of* $\mathsf{Lie}(G)$*.*

Theorem 2.55 (Classifications of Semisimple Algebraic Groups)**.** *Let* $T \subset G$ *be a maximal subtorus of a semisimple algebraic group. Now the vector space* $\mathsf{Lie}(G)$ *decomposes into eigenspaces under its action*

$$\mathsf{Lie}(G) = \bigoplus_{\alpha \in X^*(T)} \mathsf{Lie}(G)^\alpha$$

(a) *Let* $R(G,T) \subset X^*(T)$ *consist of nonzero* α *such that* $\mathsf{Lie}(G)^\alpha \neq 0$*, then* $(X^*(T) \otimes \mathbb{Q}, R(G,T))$ *is a reduced root system. Moreover* $Q(R(G,T)) \subset X^*(T) \subset P(R(G,T))$.

(b) *Every data consist of a reduced root system* (V,R) *and a lattice* $Q(R) \subset X \subset P(R)$ *arises from a pair* (G,T) *of maximal subtorus of a semisimple algebraic group in (a). Hence they are 1-to-1. Moreover* G *is simply connected if and only if* $X = P(R)$ *and it is centerless if and only if* $X = Q(R)$.

(c) *Let* (G,T) *and* (G',T') *be two pairs of maximal subtori of a semisimple algebraic groups. let* (V,R,X) *and* (V',R',X') *be their associated datas as in (a)(b). Any isomorphism* $V \cong V'$ *sending* R *onto* R' *and* X *into* X' *arises from an isogeny* $G \to G'$ *mapping* T *onto* T'.

2.3.2 Homogeneous Varieties

Definition 2.56. *A smooth projective variety* X *is said to be* homogeneous *if* X *admits a transitive action of an algebraic group* G.

Proposition 2.57. *Let* X *be a projective manifold. Then* X *is homogeneous if and only if* T_X *is globally generated. In particular, the tangent bundle of a homogeneous manifold is nef.*

Proof. Let G be the identity component of group scheme $\underline{\mathrm{Aut}}(X)$. Then G is an algebraic group with Lie algebra $\mathfrak{aut}(X) \cong H^0(X, T_X)$. The evaluation map is denoted by $H^0(X,T_X) \otimes \mathscr{O}_X \to T_X$. On the other hand, for any point $x \in X$, consider the orbit map $\mu_x : G \to X$ as $g \mapsto gx$. Since the differential of μ_x at the identity $e \in G$ coincides with the evaluation at x, then our claim follows. $\square$

2.3.3 Rational Homogeneous Varieties and Dynkin Diagrams

Definition 2.58. *Let* $\mathfrak{g}$ *be a Lie algebra.*

(a) *A maximal solvable Lie subalgebra of* $\mathfrak{g}$ *is called a* Borel subalgebra of $\mathfrak{g}$.

(b) *A Lie subalgebra of* $\mathfrak{g}$ *is called a* parabolic subalgebra of $\mathfrak{g}$ *if it contains a Borel subalgebra of* $\mathfrak{g}$.

Definition 2.59. *Let* G *be a algebraic group.*

(a) *A maximal connected solvable subgroup of* G *is called a* Borel subgroup of G.

(b) *A* parabolic subgroup of G *is a subgroup contains a Borel subgroup of* G.

Remark 2.60 (Quotient by Subgroups)**.** *We refer Section 25.4 in [93] or Section 5.C in [67] for details. Let* $H \subset G$ *be a closed subgroup, then there exists a quotient* G/H *correspond to the orbits (or cosets). In this case* $G \to G/H$ *is universal and faithfully flat. Hence* G/H *is smooth quasi-projective* G*-homogeneous variety.*

Proposition 2.61 (Basic Properties)**.** *We have the following basic properties.*

(a) *Let P be a closed subgroup of G, then P is parabolic if and only if G/P is projective. P is Borel if and only if P is solvable and G/P is projective.*

(b) *Let $\mathfrak{h}$ be the Lie algebra of a connected algebraic group H. Then a Lie subalgebra of $\mathfrak{h}$ is a Borel subalgebra if and only if it is the Lie algebra of a Borel subgroup of H. Similarly a Lie subalgebra of $\mathfrak{h}$ is a parabolic subalgebra if and only if it is the Lie algebra of a parabolic subgroup of H.*

(c) *A parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ contains a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{p} = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{p})$.*

(d) *All Borel subgroup (hence all Borel subalgebra) are closed and conjugate. Any maximal torus contained in a Borel subgroup.*

(e) *Let $B \subset G$ is a Borel subgroup, then $Z(B) = C_G(B) = Z(G)$.*

Proof. See Corollary 28.1.4, 28.1.6, 28.2.3(i) and Proposition 29.4.3 in [93]. □

Lemma 2.62. *Let G be a connected algebraic group.*

(a) *Any finite normal subgroup $H \subset G$, we have $H \subset Z(G)$.*

(b) *We have $\bigcap_{\mathrm{MaxTori} \subset G} T = Z(G)_s$. When G reductive, then $\bigcap_{\mathrm{MaxTori} \subset G} T = Z(G)$.*

Proof. For (a), pick any $h \in H$ and consider $f : G \to G$ as $g \mapsto ghg^{-1}$. Then as H normal we have $f(G) \subset H$. As G connected, then $f(G)$ is a single point. Hence $f(G) = \{h\}$. Hence $h \in Z(G)$ and hence $H \subset Z(G)$.

For (b), see Corollary 28.2.3(ii) in [93] and Proposition 17.61 in [67]. □

Proposition 2.63. *We have the following useful and important properties.*

(a) *Let $q : G \to G'$ be a quotient map of connected algebraic groups and let H be a subgroup variety of G. If H is parabolic (resp. Borel, resp. a maximal unipotent subgroup variety, resp. a maximal torus), then so also is $q(H)$; moreover, every such subgroup of G' arises in this way.*

(b) *For any isogeny $q : G \to G'$ of connected semisimple algebraic groups and parabolic subgroups $H \subset G$ and $q(H) \subset G'$, then*

$$G/H \cong G'/q(H).$$

Proof. For (a), the first statement is easy and the converse we refer Proposition 17.20 in [67].

For (b), by the universal property of quotients, we have the morphism $\bar{q} : G/H \to G'/q(H)$ induced by q. Now $\bar{q}$ is surjective. We claim that it is injective in $\mathbb{C}$-points. Indeed, for any $g_1 H \neq g_2 H$ we have $g_1^{-1} g_2 \notin H$. Hence it is injective if and only if

$g_1^{-1}g_2 \notin (\ker q) \cdot H$ for all such g_1, g_2. As q is an isogeny, then $\ker q$ is normal finite subgroup. By Lemma 2.62(a) we have $\ker q \subset Z(G) \subset H$. Hence $\bar{q}$ is injective in $\mathbb{C}$-points. Finally $\bar{q}$ is bijective in $\mathbb{C}$-points. As G/H and $G'/q(H)$ are proper by (a) and are smooth, then $\bar{q}$ is an isomorphism by Zariski main theorem. □

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Consider the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathfrak{g}^{\alpha}.$$

Let $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ and for any subset $P \subset R(\mathfrak{g},\mathfrak{h})$ we define $\mathfrak{h}_P = \sum_{\alpha \in P} \mathfrak{h}_\alpha$ and $\mathfrak{g}_P = \sum_{\alpha \in P} \mathfrak{g}_\alpha$.

Proposition 2.64. *The subalgebras of $(\mathfrak{g},\mathfrak{h})$, that is, a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ such that $[\mathfrak{a},\mathfrak{h}] \subset \mathfrak{a}$, are exactly subspaces $\mathfrak{a} = \mathfrak{h}' + \mathfrak{g}_P$ where $\mathfrak{h}'$ is a vector subspace of $\mathfrak{h}$ and $P \subset R(\mathfrak{g},\mathfrak{h})$ is a closed subset (that is, if $\alpha, \beta \in P$ and $\alpha+\beta \in R(\mathfrak{g},\mathfrak{h})$ then $\alpha+\beta \in P$). Moreover*

(a) $\mathfrak{a}$ is semisimple if and only if $P = -P$ and $\mathfrak{h}' = \mathfrak{h}_P$.

(b) $\mathfrak{a}$ is solvable if and only if $P \cap (-P) = \emptyset$.

Moreover, let $\mathfrak{b} = \mathfrak{h} + \mathfrak{g}_P$, then $\mathfrak{b}$ is maximal solvable subalgebra if and only if there exists a base S of $R(\mathfrak{g},\mathfrak{h})$ such that $P = R(\mathfrak{g},\mathfrak{h})_+$ if and only if $P \cap (-P) = \emptyset$ and $P \cup (-P) = R(\mathfrak{g},\mathfrak{h})$.

Proof. See Proposition I.8.46, I.8.50 in Milne's notes [66] or Section 20.7 in [93]. □

For a basis $S \subset R(\mathfrak{g},\mathfrak{h})$, we can define a Borel subalgebra:

$$\mathfrak{b}(S) := \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_+} \mathfrak{g}^{\alpha}.$$

For a subset $I \subset S$, let $R(\mathfrak{g},\mathfrak{h})_-(I) = \{\alpha \in R(\mathfrak{g},\mathfrak{h})_- : \alpha = \sum_{\alpha_i \notin I} n_i\alpha_i\}$, then we can define

$$\mathfrak{p}(I) := \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_+} \mathfrak{g}^{\alpha} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_-(I)} \mathfrak{g}^{\alpha}$$

which is a parabolic subalgebra. Let $P(I)$ be the corresponding parabolic subgroup of G with Lie algebra $\mathfrak{p}(I)$ and $\mathfrak{g}$ by Proposition 2.61(b).

Now these things can be describe the rational homogeneous varieties.

Proposition 2.65 (Classification of Parabolic Subgroups)**.** *Let G be semisimple and simply connected. Let P be a parabolic subgroup of G. There exist $g \in G$ and $I \subset S$ such that $g^{-1}Pg = P(I)$.*

Proof. By Proposition 2.61(b)(d) we may choose $g \in G$ such that $P' := g^{-1}Pg \supset B(S)$ where $\mathsf{Lie}(B(S)) = \mathfrak{b}(S)$ as before. Note that $\mathsf{Lie}(P')$ invariant under $\mathsf{ad}|_{\mathfrak{h}}$. Hence $\mathsf{Lie}(P') = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}^{\alpha}$ for some $T \subset R(\mathfrak{g}, \mathfrak{h})$ such that $R(\mathfrak{g}, \mathfrak{h})_+ \subset T$. Let $\alpha \in T$ is negative and $\alpha = \beta + \gamma$ where β, γ are also negative and $-\beta, -\gamma \in T$. Since $[\mathfrak{g}^a, \mathfrak{g}^b] \subset \mathfrak{g}^{a+b}$, we have $\alpha - \beta = \gamma \in T$ and $\alpha - \gamma = \beta \in T$. Hence let $I = S \backslash (-T)$ and well done. $\square$

Corollary 2.66. *Let G is a semisimple algebraic group.*

(a) There is an isogeny

$$G_1 \times \cdots \times G_k \to G, \quad (g_1, ..., g_k) \mapsto g_1 \cdots g_k$$

where G_i are minimal connected normal algebraic subgroups, hence almost-simple.

(b) If morever G is simply connected, then $G = G_1 \times \cdots \times G_k$ as in (a) and let $P \subset G$ be a parabolic subgroup. Then there are parabolic subgroups $P_i \subset G_i$ such that $P = P_1 \times \cdots \times P_k$. In particular

$$G/P \cong G_1/P_1 \times \cdots \times G_k/P_k.$$

Proof. For (a), this follows from the decomposition of semisimple Lie algebra $\mathfrak{g} := \mathsf{Lie}(G) = \bigoplus_{i=1}^{k} \mathfrak{g}_i$ by simple algebras. Let $G_1 := C_G(G(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k))$, then $\mathsf{Lie}(G_1) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k) = \mathfrak{g}_1$ which is also an ideal of $\mathfrak{g}$. Hence G_1 is normal. If G_1 is not almost-simple, then $\mathfrak{g}_1$ will have an ideal other than 0 and $\mathfrak{g}_1$ which is impossible. Now repeat the process. Finally we get $G_1, ..., G_k$. Now as $\mathsf{Lie}(G_1 \times \cdots \times G_k) = \mathsf{Lie}(G_1 \cdots G_k) = \mathfrak{g}$, hence

$$G_1 \times \cdots \times G_k \to G, \quad (g_1, ..., g_k) \mapsto g_1 \cdots g_k$$

is an isogeny.

For (b), since by (a) and G is simply connected, then $G = G_1 \times \cdots \times G_k$. Moreover by Proposition 2.65, let $P \subset G$ be a parabolic subgroup then there are parabolic subgroups $P_i \subset G_i$ such that $P = P_1 \times \cdots \times P_k$. $\square$

Definition 2.67. *A projective quotient G/P of a semisimple algebraic group G and a parabolic subgroup $P \subset G$ is called a rational homogeneous variety.*

Proposition 2.68. *For any rational homogeneous variety G/P where G be a semisimple connected algebraic group with a parabolic subgroup $P \subset G$, we have*

$$G/P \cong G_1/P_1 \times \cdots \times G_k/P_k$$

where G_i are almost-simple group with parabolic subgroups $P_i \subset G_i$.

Proof. From Proposition 2.63 and Corollary 2.66 directly. $\square$

An important result of homogeneous manifold due to Borel-Remmert is the following theorem:

Theorem 2.69 (Borel-Remmert)**.** *For any homogeneous manifold X we have $X \cong A \times G/P$ where A is an abelian variety and G/P be a rational homogeneous variety.*

This theorem tell us that to study the properties of homogeneous manifold is equivalent to study the properties of abelian varieties and rational homogeneous varieties.

2.3.4 Examples of Rational Homogeneous Varieties

We will only give the detailed calculations of A_n and others we omitted. The other roots and lattices we refer Section 21.J in [67] or Section 8 in [66]. We will mainly focus on the Fano varieties of Picard number 1, so we just consider some special cases.

Example 2.70 (Type A_n)**.** *In this type we consider* SL_{n+1} *and it has Dynkin diagram*

And we have

roots	$R = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq n+1, i \neq j\}$
root lattice	$Q(R) = \{\sum_i a_i\varepsilon_i : a_i \in \mathbb{Z}, \sum_i a_i = 0\}$
weight lattice	$P(R) = Q(R) + \langle \varepsilon_1 - (\varepsilon_1 + \cdots + \varepsilon_{n+1})/(n+1)\rangle$
base	$S = \{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_n - \varepsilon_{n+1}\}$

And

$$\mathfrak{sl}_{n+1} = \mathfrak{h} \oplus \bigoplus_{\varepsilon_i - \varepsilon_j \in R} \mathfrak{sl}_{n+1}^{\varepsilon_i - \varepsilon_j} = \{\mathrm{diag}(a_1, ..., a_{n+1}) : a_1 + \cdots a_{n+1} = 0\} \oplus \bigoplus_{\varepsilon_i - \varepsilon_j \in R} \mathbb{C} \cdot E_{ij}.$$

Hence the Borel group $B \subset \mathrm{SL}_{n+1}$ is the group of all upper-triangular matrices in SL_{n+1}, i.e., those automorphisms preserving the standard flag. Moreover, any parabolic subgroup $P \supset B$ can be described as the subgroup that preserves a partial flag in the standard representation. Hence for any $I = \{a_1, ..., a_r\} \subset S \cong \{1, ..., n\}$, we have

$$\mathrm{SL}_{n+1}/P(I) = \mathsf{Flag}(a_1, ..., a_r)$$

the (partial) flag variety*. In particular if $I = \{k\}$, then $\mathrm{SL}_{n+1}/P(I) = \mathsf{Grass}(k, n+1)$.*

A special case of A_n:

Lemma 2.71. *We have $\mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n}) \cong \mathrm{SL}_{n+1}/P(1,n)$, hence it is a rational homogeneous variety. Moreover $\mathbb{P}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}) \cong \mathrm{SL}_{n+1}/P(1,2)$, hence it is a rational homogeneous variety.*

Proof. By the definition of Euler sequence we get a closed embedding $\mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n}) \cong I \subset \mathbb{P}^n \times \mathbb{P}^n$ defined by $\sum_i x_i y_i = 0$. Hence $I \cong I' \subset \mathbb{P}^n \times \mathbb{P}^{n,*}$ as $\{(x, H) : x \in H\}$ which is the partial flag variety $\mathrm{SL}_{n+1}/P(1, n)$. Hence $\mathbb{P}_{\mathbb{P}^n}(T_{\mathbb{P}^n}) \cong \mathrm{SL}_{n+1}/P(1, n)$ which is a rational homogeneous variety.

To consider $\mathbb{P}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n})$, need to add!!! □

Example 2.72 (Type B_n)**.** *In this type we consider* SO_{2n+1} *and it has Dynkin diagram*

Fix a quadratic form q, then the Borel group $B \subset \mathrm{SO}_{2n+1}$ is the subgroup of automorphisms which preserve a fixed complete flag $0 \subset V_1 \subset \cdots \subset V_n$ of isotropic subspaces where $\dim V_r = r$.

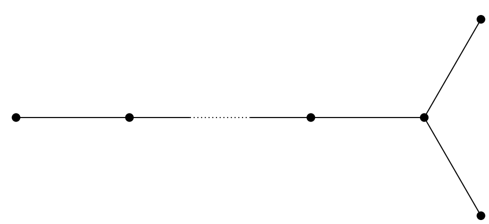
*If $I = \{k\}$, then $\mathrm{SO}_{2n+1}/P(I) = \mathsf{OGrass}(k, 2n+1)$, the **orthogonal Grassmannian**, the space of isotropic k-planes in $\mathbb{C}^{2n+1}$. In particular $\mathbb{S}_n := \mathsf{OGrass}(n, 2n+1)$ which is called the **spinor variety**. Now $B_n/P(1) \cong \mathbb{Q}_{2n-1}$.*

Example 2.73 (Type C_n)**.** *In this type we consider* Sp_{2n} *and it has Dynkin diagram*

The Borel subgroups $B \subset \mathrm{Sp}_{2n}$ are just the subgroups preserving a half-flag of isotropic subspaces, or equivalently a full flag of pairwise complementary subspaces.

*If $I = \{k\}$, then $\mathrm{Sp}_{2n}/P(I) = \mathsf{SGrass}(k, 2n)$, the **symplectic Grassmannian**, the space of isotropic k-planes in the symplectic space $\mathbb{C}^{2n}$. In particular $\mathsf{Lag}(2n) := \mathsf{SGrass}(n, 2n)$ which is called the **Lagrangian Grassmannian**.*

Example 2.74 (Type D_n)**.** *In this type we consider* SO_{2n} *and it has Dynkin diagram*



Fix a quadratic form q, then the Borel group $B \subset \mathrm{SO}_{2n}$ is the subgroup of automorphisms which preserve a fixed complete flag $0 \subset V_1 \subset \cdots \subset V_{n-1}$ of isotropic subspaces where $\dim V_r = r$.

If $I = \{k\}$ for $k \le n-2$, then $\mathrm{SO}_{2n}/P(I) = \mathsf{OGrass}(k, 2n)$ as type B_n. When $k = n-1, n$, we have $\mathrm{SO}_{2n}/P(I) = \mathbb{S}_{n-1}$. Now $D_n/P(1) \cong \mathbb{Q}_{2n-2}$.

Example 2.75 (Exceptional Types)**.** *These are E_6, E_7, E_8, F_4, G_2 types.*

For E, F, we denote them E_k/P_l for $I = \{l\}$. In particular we have $\mathbb{OP}^2 = E_6/P_1$ which is called the Cayley plane*. For G, we have $G_2/P_1 = \mathbb{Q}_5$ and let $K(G_2) := G_2/P_2$. Moreover $K(G_2) \cong \mathbb{P}^{13} \cap \mathsf{Grass}(2,7) \subset \mathbb{P}^{20}$.*

Example 2.76 (Special Cases of Short Roots)**.** *Consider the following rational homogeneous space of short root which can be constructed from long roots:*

(a) We have $B_n/P(n) \cong \mathbb{S}_n \cong D_{n+1}/P(n+1)$.

(b) We have $C_n/P(1) \cong \mathsf{SGrass}(1,2n) \cong \mathbb{P}^{2n-1} \cong A_{2n-1}/P(1)$.

(c) We have $G_2/P(1) \cong \mathbb{Q}_5 \cong B_3/P(1)$.

2.3.5 Basic Properties of Rational Homogeneous Varieties

Theorem 2.77. *For a rational homogeneous manifold X, we have $-K_X$ is ample (hence X Fano) and globally generated.*

Proof. See Theorem V.1.4 in [57]. □

Theorem 2.78. *Fix a rational homogeneous manifold $G/P(I)$ where $I \subset S \subset R$ of root system. From the classification theory of parabolic subgroups it immediately follows that given two subsets $J \subset I$, the inclusion $P(I) \subset P(J)$ provides a proper surjective morphism $p^{I,J} : G/P(I) \to G/P(J)$. Moreover, the fibers of this morphism are rational homogeneous manifolds, determined by the marked Dynkin diagram obtained from S by removing the nodes in J and marking the nodes in $I \backslash J$.*

Conversely all contractions are all of the form $p^{I,J}$ for $J \subset I \subset S$. In particular, $\mathrm{Pic}(G/P(I)) \cong \mathbb{Z}^{\sharp(I)}$ *and the Mori cone* $\mathrm{NE}(G/P(I)) \subset N^1(G/P(I))$ *is simplicial.*

Linearizable Bundles on Rational Homogeneous Spaces

Now fix a rational homogeneous space G/P.

Definition 2.79. *As $\pi : G \to G/P$ is a principal P-bundle, for any representation $\rho : P \to \mathrm{GL}_r$ we let E_ρ be the associated vector bundle, that is, $E_\rho = G \times \mathbb{A}^r_{\mathbb{C}}/P$ where $P \curvearrowright G \times \mathbb{A}^r_{\mathbb{C}}$ as $p(g,v) = (gp^{-1}, pv)$.*

Remark 2.80. *We have $E_{\rho_1} \oplus E_{\rho_1} = E_{\rho_1 \oplus \rho_2}$ and $\bigwedge^k E_\rho = E_{\bigwedge^k \rho}$ and $E_{\rho_1} \otimes E_{\rho_1} = E_{\rho_1 \otimes \rho_2}$.*

Remark 2.81. *Using this we can describe $H^0(G/P, E_\rho)$.*

Remark 2.82. *We can define the G-linearization of a bundle here without cocycle condition since X is a variety.*

Theorem 2.83 (Matsushima)**.** *A vector bundle E of rank r over G/P is G-linearizable if and only if there exists a representation $\rho : P \to \mathrm{GL}_r$ such that $E \cong E_\rho$.*

Proof. Let a vector bundle E of rank r over G/P is G-linearizable, then $\pi : E \to G/P$ then the action of G restricted to P takes $\pi^{-1}(P)$ to $\pi^{-1}(P)$ and this is a $\rho : P \to \mathrm{GL}_r$. Now $E \cong E_\rho$ defined by taking $e \in \pi^{-1}(gP)$ to $(g, g^{-1}e)$. Conversely, G act on E_ρ as $g'(g, v) = (g'g, v)$. $\square$

Theorem 2.84. *Let G semisimple and simply connected. A vector bundle E over $X = G/P$ is G-linearizable if and only if $\theta_g^* E \cong E$ for any $g \in G$ where the action of G on G/P gives $G \to \mathrm{Aut}(G/P)$ as $g \mapsto \theta_g$.*

Proof. Consider an algebraic group $\underline{\mathrm{Aut}}_X(E) \subset \underline{\mathrm{Aut}}(E)^0$ consist of automorphisms preserving each fiber and acting linearly on them. Define another algebraic group $H \subset \underline{\mathrm{Aut}}(E)^0$ consist of automorphisms acting linearly on any fibers and induces on X by elements in G. Hence we have $0 \to \underline{\mathrm{Aut}}_X(E) \to H \to G \to 0$.

Now this induce surjection $\phi : \mathsf{Lie}(H) \twoheadrightarrow \mathsf{Lie}(G)$. As the image of a solvable algebra is still a solvable algebra, we have $\phi(\mathrm{rad}(\mathsf{Lie}(H))) = 0$. By Levi-Malcev theorem to $\mathsf{Lie}(H)$ there exists a semisimple Lie subalgebra $\mathfrak{s} \subset \mathsf{Lie}(H)$ such that $\psi : \mathfrak{s} \to \mathsf{Lie}(G)$ is surjective. Now $\mathfrak{s} = \bigoplus_{i=1}^k \mathfrak{s}_i$ of simple algebras, then there exists $j < k$ such that $\ker\psi \cong \bigoplus_{i=1}^j \mathfrak{s}_i$ and $\mathsf{Lie}(G) \cong \bigoplus_{i=j+1}^k \mathfrak{s}_i$.

Then there exists Lie subgroup (not necessary algebraic) $G' \subset H$ such that $\mathsf{Lie}(G') \cong \mathsf{Lie}(G)$. Hence G' acts over E and now G is a covering of G' and also G acts over E as we wanted. $\square$

Proposition 2.85 (Ise)**.** *Let G be semisimple and simply connected. Every line bundle on G/P is G-linearizable.*

Proof. As it is Fano, we have $H^i(G/P, \mathscr{O}) = 0$ for all $i > 0$. Hence by exponential sequence we have $\mathrm{Pic}(G/P) = H^1(G/P, \mathscr{O}^*) \cong H^2(G/P, \mathbb{Z})$. so that $\mathrm{Pic}(G/P)$ is discrete and the G-action on it given by $L \mapsto g^*L$ is trivial. $\square$

More Properties of Rational Homogeneous Spaces

Here are some results we will use.

Theorem 2.86. *(a) (Blanchard) Let G be a connected algebraic group acting over a projective variety X with $H^1(X, \mathscr{O}) = 0$. Then there exists a representation $\rho : G \to \mathrm{PGL}(V)$ and an embedding $X \subset \mathbf{P}(V)$ such that the original action is induced by ρ. In particular the action is given by projective linear transformations. Hence if X is a projective variety with $H^1(X, \mathscr{O}) = 0$ then $\underline{\mathrm{Aut}}^0(X)$ is linear algebraic.*

(b) *(Borel Fixed Point) Let G be a solvable linear algebraic group. Then any action of G on a projective variety X has a fixed point.*

(c) *Let G be a linear group acting transitively and effectively (that is, $G \to \underline{\mathrm{Aut}}(X)$ is injective) over a variety X. Then G is semisimple. By (c) if $H^1(X, \mathscr{O}) = 0$ then the assumption that G is linear can be dropped.*

Theorem 2.87. *We have the following properties.*

(a) *A homogeneous projective variety X with $b_1(X) = 0$ is rational.*

(b) *Every rational homogeneous variety $G/P(I)$ is rational.*

Proof. By Theorem 2.86(c) we just need to show (b). Now as before we have

$$\mathsf{Lie}(P(I)) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_+} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_-(I)} \mathfrak{g}^\alpha$$

where $\mathfrak{g} = \mathsf{Lie}(G)$ and $R(\mathfrak{g}, \mathfrak{h})_-(I) = \{\alpha \in R(\mathfrak{g}, \mathfrak{h})_- : \alpha = \sum_{\alpha_i \notin I} n_i \alpha_i\}$.

Let $\mathfrak{u}_- := \bigoplus_{\alpha \notin R(\mathfrak{g},\mathfrak{h})_-(I), \alpha < 0} \mathfrak{g}^\alpha$, then it is a solvable (even nilpotent) subalgebra. Consider $\mathsf{ad} : \mathfrak{g} \hookrightarrow \mathfrak{gl}_\mathfrak{g}$ since $\mathfrak{g}$ semisimple. Now $\mathsf{ad}(\mathfrak{u}_-)$ consists of nilpotent endomorphisms. By Engel theorem, in a convenient basis of $\mathfrak{g}$ we have that $\mathfrak{u}_-$ is in the subalgebra of strictly lower-triangular matrices. Then at level of Lie groups we have $U_- \subset G \subset \mathrm{GL}_N$ such that U_- is in the subgroup of unipotent matrices.

In particular $\exp : \mathfrak{u}_- \to U_-$ has inverse given by $\log : U_- \to \mathfrak{u}_-$ as $\mathbf{A} \mapsto \sum_{n=0}^{\infty} (-1)^n \frac{(\mathbf{A}-\mathbf{I})^n}{n}$. From the matrix description we have $U_- \cap P(I) = \{e\}$, so that the morphism $U_- \to G/P(I)$ is injective and it is dominant too by dimensional reasons. Hence $G/P(I)$ is rational. □

Proposition 2.88. *Consider the rational homogeneous variety $G/P(I)$, then*

$$H^2(G/P(I), \mathbb{Z}) \cong \mathrm{Pic}(G/P(I)) \cong \mathbb{Z}^{\sharp(I)}.$$

Proof. WLOG we let G is simply connected. First we have

$$\begin{aligned}
\mathsf{Lie}(P(I)) &= \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_+} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})_-(I)} \mathfrak{g}^\alpha \\
&= \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})(I)} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \notin R(\mathfrak{g},\mathfrak{h})_+(I), \alpha > 0} \mathfrak{g}^\alpha \\
&= \mathfrak{s}_P \oplus \bigoplus_{i=1}^{\sharp(I)} [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}] \oplus \bigoplus_{\alpha \notin R(\mathfrak{g},\mathfrak{h})_+(I), \alpha > 0} \mathfrak{g}^\alpha
\end{aligned}$$

where $\mathfrak{s}_P = \mathsf{Lie}(S_P)$ is semisimple by Proposition 2.64 and $I = \{\alpha_1, ..., \alpha_{\sharp(I)}\}$. Now S_P is called the $\mathsf{semisimple\ part}$ of $P(I)$. By definition $\mathfrak{s}_P$ is covered by the copies of $\mathfrak{sl}_2$.

By Theorem 2.83 and Proposition 2.85, $\mathrm{Pic}(G/P(I))$ is just the groups of 1-dim representation of $P(I)$. Let $V_\lambda = \{v : \rho(h)(v) = \lambda(h)v, \forall h \in \mathfrak{h}\}$ be a one-dimensional $\mathsf{Lie}(P(I))$-module, where λ is the corresponding weight in $\mathfrak{h}^\vee$ where $\rho : \mathsf{Lie}(P(I)) \to \mathfrak{gl}_V$. Then V_λ restricted to $\mathfrak{s}_P$ is trivial since $\mathfrak{s}_P$ semisimple. Easy to see that $\mathfrak{g}^\alpha V_\lambda \subset V_{\alpha+\lambda}$, then the representation is trivial when restricted to $\bigoplus_{\alpha\notin R(\mathfrak{g},\mathfrak{h})_+(I),\alpha>0} \mathfrak{g}^\alpha$. Hence this representation is obtained from the abelian $\sharp(I)$-dimensional piece $\bigoplus_{i=1}^{\sharp(I)}[\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$ which is the Lie algebra of a torus $\mathbb{G}_m^{\sharp(I)}$. Hence well done. □

Proposition 2.89 (Ise)**.** *Fix a rational homogeneous variety $G/P(I)$. A representation $\rho : P(I) \to \mathrm{GL}_V$ is completely reducible if and only if $\rho|_U$ is trivial where $U \subset P(I)$ correspond to $\bigoplus_{\alpha\notin R(\mathfrak{g},\mathfrak{h})_+(I),\alpha>0} \mathfrak{g}^\alpha$. Now U is called the unipotent part of $P(I)$.*

Proof. Let ρ is completely reducible. Hence WLOG we let ρ is irreducible. Let $Y \subset P(I)$ correspond to $\bigoplus_{i=1}^{\sharp(I)}[\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}] \oplus \bigoplus_{\alpha\notin R(\mathfrak{g},\mathfrak{h})_+(I),\alpha>0} \mathfrak{g}^\alpha$. From the theorem of Lie there exists a basis in V such that $\rho(y)$ is upper triangular for every $y \in Y$. Since $\mathsf{Lie}(U) = [\mathsf{Lie}(Y), \mathsf{Lie}(Y)]$ we get that $d\rho(u)$ is strictly upper triangular for every $u \in \mathsf{Lie}(U)$. It follows that there exists a nonzero $v \in V$ such that $\rho(u)v = v$ for any $u \in U$.

Let $F := \{v \in V : \rho(u)v = v, \forall u \in U\}$, then $F \neq 0$ as before. As U is normal, it is easy to check that F is $P(I)$-invariant so that by the assumption $F = V$. This means that $\rho|_U$ is trivial.

Conversely we let $\rho|_U$ is trivial. Then at the level of Lie algebras ρ comes from a representation of $\mathsf{Lie}\, S_P \oplus \mathfrak{z}$ where $\mathfrak{z} = \bigoplus_{i=1}^{\sharp(I)}[\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$. Any such representation is the tensor product of a representation of $\mathfrak{z}$ (abelian Lie algebra of $\mathbb{G}_m^{\sharp(I)}$) and a representation of $\mathsf{Lie}(S_P)$ which are both completely reducible. □

2.3.6 Borel-Weil Theory

Lemma 2.90. *Let G be a semisimple group, we have $\pi_2(G) = 0$.*

Proof. Omit it. □

Lemma 2.91. *Let G be semisimple and simply connected. Let $P(I) \subset G$ be a parabolic subgroup. Then $\pi_1(P(I)) = \pi_2(G/P(I)) = \mathbb{Z}^{\sharp(I)}$ and $\pi_1(S_P) = 0$ where S_P be the semisimple part of $P(I)$.*

Proof. See the Proposition 10.8 in the survey [81]. □

Proposition 2.92 (Classification of Irreducible Bundles)**.** *Let G be semisimple and simply connected. Let $P(I) \subset G$ be a parabolic subgroup. Let $I := \{\alpha_1, ..., \alpha_k\}$ be a subset of simple roots. Let $\lambda_1, ..., \lambda_k$ be the corresponding set of fundamental weights, that is, $\{\lambda_i\}$ is dual basis of $\{h_{\alpha_i}\}$ as in Theorem 2.42. Then all the irreducible representations of $P(I)$ are*

$$V \otimes L_{\lambda_1}^{n_1} \otimes \cdots \otimes L_{\lambda_k}^{n_k}$$

where V is a representation of S_P and $n_i \in \mathbb{Z}$ (by the Lemma 2.91, λ_i define representations of S_P).

Proof. Now we have $\mathsf{Lie}(P(I))$ Follows from Proposition 2.89 and Lemma 2.91. □

Theorem 2.93 (Borel-Weil). *Let G be semisimple and simply connected. Let $P(I) \subset G$ be a parabolic subgroup. Let $I := \{\alpha_1, ..., \alpha_k\}$ be a subset of simple roots. Let $\lambda_1, ..., \lambda_k$ be the corresponding set of fundamental weights. Then $L_{\lambda_1}^{n_1} \otimes \cdots \otimes L_{\lambda_k}^{n_k}$ is very ample if and only if it is ample if and only if all $n_i > 0$.*

Proof. See Theorem 10.16 in [81]. □

Corollary 2.94. *The ample generator $\mathscr{L}$ of the Picard group of the rational homogeneous varieties G/P of Picard number 1 is very ample. In particular, it gives a G-equivariantly embedding $G/P \hookrightarrow \mathbb{P}(H^0(G/P), \mathscr{L})$.*

Theorem 2.95 (Bott, 1957). *We have $H^i(G/P, T_{G/P}) = 0$ for all $i > 0$ for any rational homogeneous variety G/P. In particular it is locally rigid.*

2.4 Special Rational Homogeneous Spaces

2.4.1 Hermitian Symmetric Spaces

2.4.2 Homogeneous Contact Manifolds

2.5 Del-Pezzo Manifolds

Chapter 3

Varieties of Minimal Rational Tangents

We will assume the base field is $\mathbb{C}$.

3.1 Basic Properties

In this section we will discover some fundamental and important properties of tangent map $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x})$ with VMRT $\mathcal{C}_x$ for any smooth Fano variety X. First we need to find some properties of singular rational curves.

Definition 3.1. *Let X be a smooth uniruled variety over $\mathbb{C}$ and $x \in X$ is a point. Choose a (dominanted) minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ and the corresponding component $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(x, X)$ be of minimal degree $p+2$. Consider the rational map*

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x}), \quad [i : C \subset X] \mapsto \left.\frac{di}{dt}\right|_{t=0}$$

where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{C,0}$, defined on curves smooth at x. We define the variety of minimal rational tangents or VMRT $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ at x is the closure of the image of τ_x. Moreover, we define

$$\mathcal{C} := \overline{\bigcup_{x \text{ general}} \mathcal{C}_x}^{\mathrm{zar}} \subset \mathbb{P}(\Omega^1_X)$$

the total variety of minimal rational tangents or total VMRT.

Remark 3.2. *Note that there are only finitely many choice of minimal rational component $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$, hence there are only finitely many choice of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$, at least for general point $x \in X$.*

Theorem 3.3 (Kebekus [52], 2002)**.** *Let X be a smooth uniruled variety and $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ a (dominanted) minimal rational component. Let $\mathcal{K}'_x \subset \mathcal{K}$ be the locus of curves passing through x where $x \in X$ be a general point (hence $\mathcal{K}_x \to \mathcal{K}'_x$ is a normalization). consider the closed subvarieties*

$$\mathcal{K}_x^{\mathsf{sing}} := \{[C] \in \mathcal{K}'_x : C \text{ singular}\}, \quad \mathcal{K}_x^{\mathsf{sing},x} := \{[C] \in \mathcal{K}'_x : C \text{ singular at } x\}.$$

Then the following holds.

(a) The space $\mathcal{K}_x^{\mathsf{sing}}$ has dimension at most one, and the subspace $\mathcal{K}_x^{\mathsf{sing},x}$ is at most finite. Moreover, if $\mathcal{K}_x^{\mathsf{sing},x}$ is not empty, the associated curves are unramified .

(b) If there exists a line bundle $\mathscr{L} \in \mathrm{Pic}(X)$ that intersects the curves with multiplicity 2, then $\mathcal{K}_x^{\mathsf{sing}}$ is at most finite and $\mathcal{K}_x^{\mathsf{sing},x}$ is empty.

Proof. See the original paper [52] or the sketch in Theorem 2.12 in the survey [53]. □

Remark 3.4. *There is another thing about the singular rational curves: if there is a curve parametrized by $\mathcal{K}_x$ singular at x, then there is also a curve parametrized by $\mathcal{K}_x$ with a cuspidal singularity. See V.3.6 in [57].*

Corollary 3.5. *By Theorem 3.3(a), very curve parametrized by $\mathcal{K}_x$ is unramified at x (i.e., its normalization is unramified at $0 \mapsto x$).*

Theorem 3.6 (Kebekus-2002, Hwang-Mok-2004)**.** *Let X be a smooth uniruled variety and $\mathcal{K} \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(X)$ a (dominanted) minimal rational component. Let $x \in X$ be a general point, consider the tangent map*

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x}), \quad [f : \mathbb{P}^1 \to X] \mapsto \left.\frac{df}{dt}\right|_{t=0}.$$

(a) τ_x is actually a finite morphism, we can called it tangent morphism*.*

(b) $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is a birational morphism, hence

(c) $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is the normalization.

Proof. (a) and (b) implies (c) in this case.

For (a) (proved in [52]), we will first show that $\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(\Omega^1_{X,x})$ actually can be a morphism. We have two arguments with the same result:

(M1) By Theorem 1.31(b) we have q as follows

$$\begin{tikzcd} \mathcal{K}_x \arrow[r, "q"] \arrow[dr, "\tau_x"'] & \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \arrow[d, "t_x"] \\ & \mathbb{P}(\Omega^1_{X,x}) \end{tikzcd}$$

where $t_x : \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \to \mathbb{P}(\Omega^1_{X,x})$ sends f to $(df)_0(\frac{d}{dt})$ for uniformizer $t \in \mathscr{O}_{\mathbb{P}^1,0}$ since it is unramified by Corollary 3.5.

(M2) Consider the universal morphism and cycle morphism

$$\begin{tikzcd} \mathsf{Univ}^{\mathrm{n}}(x,X) & \mathcal{U}_x^n \arrow[l, hook'] \arrow[r, "\iota_x"] \arrow[d, "\pi_x"] & X \\ \mathsf{RatCurves}^{\mathrm{n}}(x,X) & \mathcal{K}_x \arrow[l, hook'] & \end{tikzcd}$$

We have a section $\mathcal{K}_x \cong \sigma_\infty \subset \mathcal{U}_x^n$ contracted to $x \in X$ via ι_x which is canonical by Theorem 3.3(a). By Corollary 3.5 again we can consider a nowhere vanishing morphism of vector bundles

$$T_{\mathcal{U}_x^n/\mathcal{K}_x}|_{\sigma_\infty} \to \iota_x^*(T_{X,x})$$

and yields $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathbb{P}(\Omega^1_{X,x})$.

Now we need to show τ_x is finite. If not, we have a curve $C \subset \mathcal{K}_x$ contracted by τ_x. Let the normalization of universal family $U \to C$ is again a $\mathbb{P}^1$-bundle. Let the corresponding section is $s_\infty \subset U$. Consider $N_{s_\infty/U}$. Since s_∞ contracted into a point, its normal bundle is negative. But this is the tangent morphism, the normal bundle nedd to be trivial. This is impossible. Hence τ_x is finite.

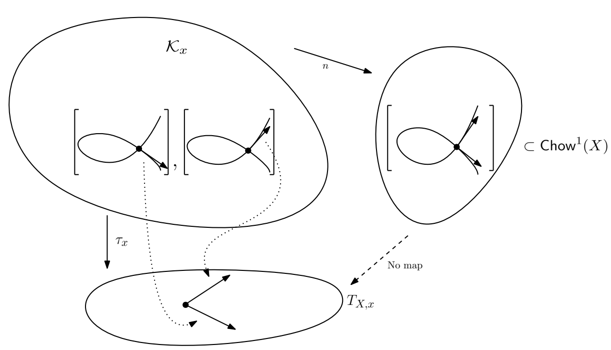
For (b), proved in [46] Theorem 1 and we will omit it. $\square$

Remark 3.7. *Note that by the proof of (a) we have* $\tau_x^*(\mathscr{O}(1)) \cong \mathscr{O}_{\sigma_\infty}(K_{\mathcal{U}_x^n/\mathcal{K}_x})$.

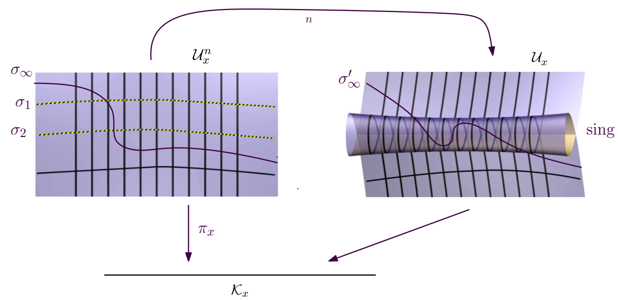
Remark 3.8. *Note also that we need to think (M1) and (M2) deeply as follows:*

The fundamental question is that if the minimal rational curve C not smooth at x (however it is unramified at x by Corollary 3.5), how to choose the different tangent vectors?

(M1) In this method, since $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1; 0) \cong \mathcal{K}_x$ *we know that there are several curves in $\mathcal{K}_x$ maps to $[C]$ and their tangent vectors separated by the tangent vectors of C at x since C is not smooth at x. The diagram as follows:*



(M2) *In this method, the section $\sigma_\infty \cong \mathcal{K}_x$ will meet the sections of singular points at finite points. For example in local case where $\sigma_\infty \subset \iota_x^{-1}(x)$ be that section and σ_1, σ_2 are preimage of singular locus* sing $\subset \mathcal{U}_x$*:*



Hence the choice of tangent vectors are canonical. Another interesting method is that we can use the universal property of the blow-up:

$$\begin{tikzcd} & & \mathrm{Bl}_x X \arrow[d, "b"] \\ \mathcal{K}_x & \mathcal{U}_x^n \arrow[l] \arrow[r, "\iota_x"'] \arrow[ur, dashed, "\hat{\iota}_x"] & X \end{tikzcd}$$

Then we have $\tau_x = \hat{\iota}_x|_{\sigma_\infty} : \mathcal{K}_x \cong \sigma_\infty \to E = \mathbb{P}(\Omega^1_{X,x})$.

Remark 3.9. *In fact in [52] they show that $\iota_x^{-1}(x) = \sigma_\infty \cup \{$finite points$\}$. Moreover the tangent morphism $d\iota_x$ has rank one along σ_∞.*

Proposition 3.10. *Let X be a smooth uniruled variety and $x \in X$ be a general point, then the morphism $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is unramified at $[f] \in \mathcal{K}_x$ if and only if $[f]$ is standard.*

Proof. We follows Proposition 1.4 in the survey [37] or Proposition 2.7 in [5]. Consider

$$\begin{tikzcd} \mathsf{Hom}^{\mathsf{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & V_x \arrow[l] \arrow[r, "\phi_x"] \arrow[dr, "\psi_x"'] & \mathcal{K}_x \arrow[d, "\tau_x"] \\ & & \mathbb{P}(\Omega^1_{X,x}) \end{tikzcd}$$

Pick any $[C] \in \mathcal{K}_x$ and its normalization $[f] \in V_x$, then we need to consider $(d\psi_x)_{[f]} : T_{[f]}V_x \to T_{\psi_x[f]}\mathbb{P}(\Omega^1_{X,x})$. Now $T_{[f]}V_x \cong H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$ and $T_{\psi_x[f]}\mathbb{P}(\Omega^1_{X,x}) \cong T_xX/\hat{\psi}_x[f]$ where $\hat{\psi}_x[f]$ denotes the 1-dimensional subspace of T_xX corresponding to the point $\psi_x[f]$. If $v \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$, then we let a deformation f_s with $f_0 = f$ such that

$\frac{df_s}{ds}|_{t=0} = v$. Then

$$(d\psi_x)_{[f]}(v) = \frac{d}{ds}\bigg|_{s=0} \frac{df_s}{dt}\bigg|_{t=0} = \frac{d}{dt}\bigg|_{t=0} \frac{df_s}{ds}\bigg|_{s=0} = \frac{dv}{dt}\bigg|_{t=0} \in T_xX/\hat{\psi}_x[f] = f^*T_X|_0/T_o\mathbb{P}^1$$

where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{\mathbb{P}^1,0}$. For a $v \neq 0$ such that v not be zero after quotient by $T_o\mathbb{P}^1$, we find that $(d\psi_x)_{[f]}(v) = 0$ if and only if $\mathscr{O}(2) \subset f^*T_X|_0/T_o\mathbb{P}^1$ if and only if $[f]$ is standard. □

Remark 3.11. *Hence we give another proof of that τ_x is generically finite.*

Corollary 3.12. *Let X be a smooth uniruled variety and $x \in X$ be a general point. If every irreducible component of $\mathcal{C}_x$ is smooth, then all curves parametrized by $\mathcal{K}_x$ are smooth at x.*

Proof. Since every irreducible component of $\mathcal{C}_x$ is smooth, τ_x is unramified by Theorem 3.6 (in fact, the restriction of τ_x to each irreducible component of $\mathcal{K}_x$ is an isomorphism). Thus, by Proposition 3.10, f is standard for every member $[f] \in \mathcal{K}_x$. Hence there is no curve parametrized by $\mathcal{K}_x$ has a cuspidal singularity. Then the result follows from Remark 3.4. □

Corollary 3.13. *Let X be a smooth uniruled variety and $x \in X$ be a general point. We assume that under the embedding $X \subset \mathbb{P}^N$, any point in X lies in a line on X. Then $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is an embedding, hence $\mathcal{C}_x$ is smooth.*

Proof. Note that the map τ_x is injective, because any line through x is uniquely determined by its tangent direction. Hence we just need to show that τ_x is unramified. By Proposition 3.10 we just need to show that any minimal rational curve, that is, these lines C containing x is standard. Indeed, let $T_X|_C \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_1)$ with $a_1 \geq \cdots \geq a_n \geq 0$. Hence $a_i \geq 2$. As $T_X|_C \subset T_{\mathbb{P}^n}|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus N-1}$, we get $a_1 = 2$ and $1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and C is standard. □

Corollary 3.14. *If X be a smooth prime Fano variety of Fano index $\mathsf{Index}(X) > \frac{n+1}{2}$ with dimension n, then X satisfies the conditions in Corollary 3.13. Hence $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is an embedding for a general point $x \in X$, hence $\mathcal{C}_x$ is smooth.*

Proof. For any minimal rational curve C (let the anticanonical degree is $p+2$), we have

$$n + 1 \geq p + 2 = -K_X \cdot C = \mathsf{Index}(X) C \cdot \mathscr{L}$$

where $\mathscr{L}$ generates $\mathrm{Pic}(X)$. As $\mathsf{Index}(X) > \frac{n+1}{2}$, then C must be a line under the embedding given by $\mathscr{L}$. □

Proposition 3.15. *Let X be a smooth uniruled variety and $x \in X$ be a general point. For general $[C] \in \mathcal{K}_x$ with normalization $f : \mathbb{P}^1 \to C \subset X$ with minimal degree $p+2$. Define $T_xX_C^+ \subset T_xX$ be the subspace correspond to the positive part, that is, the stalk of*

$$\mathrm{Im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes \mathscr{O} \to f^*T_X(-1)] \otimes \mathscr{O}(1) \subset f^*T_X$$

at x. Then $\mathbb{P}((T_xX_C^+)^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ is the projective tangent space of $\mathcal{C}_x$ at $\tau_x([f])$.

Proof. As general curve, we just consider the standard one. By proposition 3.10, if $v \in H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0)$, then the differential sends v to $\frac{dv}{dt}\big|_{t=0}$ where t be the uniformizer of $\mathfrak{m}_0 \subset \mathscr{O}_{\mathbb{P}^1,0}$. Since v lies in the positive part, then so is $\frac{dv}{dt}$. As $\dim \mathcal{C}_x = p = \dim \mathbb{P}(\mathscr{O}(2) \oplus \mathscr{O}(1)^p)$, then well done. □

3.2 Basic Examples of VMRT

3.2.1 Projective Spaces

Proposition 3.16. *If $X = \mathbb{P}^n$, then $\tau_x : \mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.*

Proof. By the proof of Theorem 1.78 or Corollary 3.14. □

Conversely we introduce some characterizations of projective spaces. Some of them we have proved and some of them are easy to prove. We also will to prove some of them using VMRT theory.

Theorem 3.17 (Cho-Miyaoka-Barron, 2002)**.** *Let X be a smooth projective variety of dimension n and $x_0 \in X$ be a general point . Then the following fourteen conditions are equivalent:*

- *(a)* $X \cong \mathbb{P}^n$.
- *(b)* Hirzebruch-Kodaira-Yau condition: *X homotopic to $\mathbb{P}^n$.*
- *(c)* Kobayashi-Ochiai condition: *X is Fano and $c_1(X)$ is divisible by $n+1$ in $H_2(X, \mathbb{Z})$.*
- *(d)* Frankel-Siu-Yau condition: *X carries a Kähler metric of positive holomorphic bisectional curvature.*
- *(e)* Hartshorne-Mori condition: *T_X is ample.*
- *(f)* Mori condition: *X is Fano and $T_X|_C$ is ample for any rational curves C.*
- *(g)* Doubly transitive group action: *The action of* $\mathrm{Aut}(X)$ *on X is doubly transitive.*
- *(h)* Remmert-Vande Ven-Lazarsfeld condition: *There exists a surjective morphism from a suitable projective space onto X.*
- *(i)* Length condition: *X is uniruled and $-K_X \cdot C \geq n+1$ for any curve $C \subset X$.*

(j) *Length condition on rational curves:* X *is uniruled and* $-K_X \cdot C \geq n+1$ *for any rational curve* $C \subset X$.

(k) *Length condition on rational curves with base point:* X *is uniruled and* $-K_X \cdot C \geq n+1$ *for any rational curve* $C \subset X$ *passing through a general point* $x_0 \in X$.

(l) *VMRT condition:* X *is uniruled and* $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

First Comments. Actually there is a much general condition in the original paper [16] implies all of these, but we will omit it. Note that we also omit the proof of $(k) \Rightarrow (a)$ since it use that general condition. But we finally will prove $(i) \Rightarrow (l) \Rightarrow (a)$ by using VMRT theory as in

Here are some trivial implications. We have (a) implies everything. We have $(i) \Rightarrow (j) \Rightarrow (k)$ and $(d) \Rightarrow (e) \Rightarrow (f)$. Morever $(c) \Rightarrow (i)$ and $(f) \Rightarrow (j)$ are also trivial. Note also that $(a) \Leftrightarrow (d) \Leftrightarrow (e) \Leftrightarrow (f)$ are proved in Theorem 1.76, Theorem 1.77 and Theorem 1.78. Note also that $(h) \Rightarrow (k)$ and $(h) \Rightarrow (a)$ is proved also in Corollary 1.80. For $(g) \Rightarrow (f)$ we refer Page 45 in [16]. $\square$

Proof of $(b) \Rightarrow (c)$. As X homotopic to $\mathbb{P}^n$, then X is simply connected. By the proof of Proposition 1.62(b) we have $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z}) = H^2(\mathbb{P}^n, \mathbb{Z}) \cong \mathbb{Z}$. Pick an ample generator h and let $c_1(X) = mh$. As $c_1^n(X)$ is homotopic invariant up the sign (see [32]), we have $m = \pm(n+1)$. If $m = n+1$ then well done.

If $m = -(n+1)$ and we will show that this is impossible. In this case K_X is ample, then X has KE-metric by several works [7][102][103]. The Chern number $c_1^{n-2}(2(n+1)c_2 - nc_1^2)$ is again homotopic invariant up the sign. By Chen-Ogiue-Yau's result ([15][102][103]) this would imply that the universal cover of X is the open unit ball, contradicting the assumption that the compact manifold X is simply connected. $\square$

Finally we will prove $(i) \Rightarrow (l) \Rightarrow (a)$ **using VMRT**.

Proof of $(i) \Rightarrow (l)$. By Theorem 3.6(a), we have $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathbb{P}(\Omega^1_{X,x})$ is finite. Since $\dim \mathcal{K}_x = n-1 = \dim \mathbb{P}(\Omega^1_{X,x})$, we know that τ_x is surjective. By Theorem 3.6(b) we find that τ_x is birational (**Note that the proof of 3.6(b) in [46] is to reduce the general case to our case. So we can not use this at all. But for convenience we will use this directly**). Hence by Zariski main theorem we know that $\tau_x : \mathcal{K}_x \cong \sigma_\infty \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$ are isomorphisms. $\square$

Proof of $(l) \Rightarrow (a)$. This is the same proof of the final step of Hartshorne's conjecture 1.78. As $\tau_x : \mathcal{K}_x \to \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x})$ where by Theorem 3.6 τ_x is a normalization, hence $\mathcal{K}_x \cong \mathcal{C}_x \cong \mathbb{P}(\Omega^1_{X,x}) \cong \mathbb{P}^{n-1}$.

By Stein factorization we have $\iota_x : \mathcal{U}^n_x \xrightarrow{A} Y \xrightarrow{B} X$ where $A(\sigma_\infty) = \{\mathrm{pt}\}$ and B finite. Similarly pushforward $0 \to \mathscr{O}_{\mathcal{U}^n_x} \to \mathscr{O}_{\mathcal{U}^n_x}(\sigma_\infty) \to \mathscr{O}_{\sigma_\infty}(\sigma_\infty) \to 0$ to $\mathcal{K}_x$ and consider Ext^1 we have $\mathcal{U}^n_x \cong \mathbb{P}_{\mathcal{K}_x}(\mathscr{O} \oplus \mathscr{O}(-1))$ and get $Y \cong \mathbb{P}^n$. Finally by Corollary 1.80 we get $X \cong \mathbb{P}^n$. $\square$

Remark 3.18. *Note that the history about the characterizations of projective space is very long and we refer Remark 5.2 in [16]. Noe also that there is an anologue of quadric hypersurfaces, see Remark 5.3 in [16].*

Theorem 3.19 (Wahl, 1983)**.** *Let X be a complex projective non-singular variety, let $\mathscr{L}$ be an ample line bundle. If $H^0(X, T_X \otimes \mathscr{L}^{-1}) \neq 0$, then $(X, \mathscr{L})$ is $(\mathbb{P}^n, \mathscr{O}(1))$ or $(\mathbb{P}^n, \mathscr{O}(2))$.*

Proof. See the main theorem in the paper [96]. □

Theorem 3.20 (Andreatta-Wisniewski, 2001)**.** *If $\mathscr{E}$ is an ample locally free subsheaf of T_X, then $X \cong \mathbb{P}^n$ and $E \cong \mathscr{O}(1)^{\oplus r}$ or $\mathscr{E} \cong T_{\mathbb{P}^n}$.*

Proof. This is the main theorem in the paper [4]. □

3.2.2 Fano Hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be a smooth Fano hypersurface of degree d where $n \geq 3$. Hence now $d \leq n+1$. We first consider the following general result which will be useful later:

Proposition 3.21. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d* ***over any field*** *k. If $n \geq 3$ then*

$$\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathscr{O}_X(1).$$

Proof. For the proof over any field we refer XII. Cor 3.6 in [30]. We only prove the case where $k = \mathbb{C}$. By exponential sequence one has

$$H^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

By the Lefschetz hyperplane theorem we have $H^1(X, \mathscr{O}_X) = H^2(X, \mathscr{O}_X) = 0$ since $n \geq 3$. Hence $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. By the Lefschetz hyperplane theorem again we have $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathscr{O}_X(1)$. Well done. □

To consider $\mathcal{C}_x$ for $x \in X$, we first consider when does the lines lie over the $X \subset \mathbb{P}^{n+1}$. Let $F(t_0, ..., t_{n+1})$ be the homogeneous polynomial of degree d defining X and let $x = [x_0 : ... : x_{n+1}] \in X$ be a general point.

Proposition 3.22. *If $d \leq n$, then $\mathcal{C}_x$ is the smooth complete intersection of multi-degree $(2, 3, ..., d)$.*

Proof. A line through x given by $l = [x_0 + \lambda y_0 : ... : x_{n+1} + \lambda y_{n+1}]$ where $[y_0 : ... : y_{n+1}] \in \mathbb{P}^{n+1}$ be some point. Hence $l \subset X$ if and only if $F(x_0 + \lambda y_0, ..., x_{n+1} + \lambda y_{n+1}) = 0$ for any λ. So this if and only if $\sum_{i=0}^d \lambda^i \frac{1}{i!}(\Delta_x(y))^i F(x) = 0$ where $\Delta_x(y) = \sum_i y_i \frac{\partial}{\partial t_i}$. Hence this if and only if

$$\Delta_x(y)F(x) = 0, (\Delta_x(y))^2 F(x) = 0, \cdots, (\Delta_x(y))^d F(x) = 0.$$

Note that the first one is just the defining equation of $\mathbb{P}(\Omega^1_{X,x})$, hence well done. □

Remark 3.23. *Some situations:*

(a) *When $d = 2$ then X is the hyperquadric $\mathbb{Q}_n$ which is homogeneous. Hence VMRT $\mathcal{C}_x \cong \mathbb{Q}_{n-2} \subset \mathbb{P}(\Omega^1_{X,x})$.*

(b) *When d is high and $d < n$, then VMRT os Calabi-Yau or of general type.*

(c) *When $d = n$ then VMRT is finite and of cardinality $n!$.*

(d) *When $d = n + 1$ there exists no line but has finite conics (see V.4.4.4 in [57]).*

3.2.3 Grassmannians

Let $X = \mathsf{Grass}(s, V)$ is Grassmannian of $s > 0$-dimensional subspaces where $\dim V = r + s$. Pick a general point $x = [W] \in X$.

Proposition 3.24. *In this case $\tau_x : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$ is just the Segre embedding*

$$\tau_x : \mathbb{P}(W) \times \mathbb{P}((V/W)^*) \hookrightarrow \mathbb{P}(W \otimes (V/W)^*).$$

Proof. Via Plücker embedding X covered by lines, hence by Corollary 3.13 τ_x is an embedding. Note that a line on $\mathsf{Grass}(s, V)$ through a point $x = [W] \in X = \mathsf{Grass}(s, V)$ is determined by a choice of subspace W' of dimension $s - 1$ contained in W and a subspace W'' of dimension $s + 1$ containing W. Then that line consist of subspaces of dimension s which are containing W' and contained in W''. So $\mathcal{K}_x \cong \mathbb{P}(W) \times \mathbb{P}((V/W)^*)$. Hence easy to see the tangent morphism is just Segre embedding:

$$\tau_x : \mathcal{K}_x \cong \mathbb{P}(W) \times \mathbb{P}((V/W)^*) \hookrightarrow \mathbb{P}(W \otimes (V/W)^*) \cong \mathbb{P}(\Omega^1_{X,x}).$$

Well done. □

3.2.4 Moduli Space of Stable Bundles over Curves

Consider a smooth projective curve C of genus $g \geq 2$.

Proposition 3.25. *Consider the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of degree d. If d is odd, then $M_{2;\mathscr{D},d}(C)$ is a $(3g-3)$-dimensional Fano manifold of Picard number 1 (it is prime). Moreover $M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},1}(C)$ in this case. In particular, when $g = 2$ the space $M_{2;\mathscr{D},1}(C)$ is a intersection of two quadrics in $\mathbb{P}^5$.*

Proof. We refer [70], omit it. □

Corollary 3.26. *When $g = 2$, the VMRT is just four points in $\mathbb{P}(\Omega^1_{X,x})$ given by the intersection of two conics.*

Proof. See the proof of Proposition 3.22. □

For $g \geq 3$ we will construct some kind of rational curves on $X = M_{2;\mathscr{D},1}(C)$ which is called the Hecke curves. There are two equivalent constructions:

(M1) Pick $[W] \in X$ which is $(1,1)$-stable, that is, any sub-line-bundle has degree < 0, is dense in X by [70]. Consider $\pi : \mathbf{P}(W) \to C$ and $\eta \in \mathbf{P}(W)$ with $y = \pi(\eta) \in C$.

First we get a new bundle W^η of rank 2:

$$0 \to W^\eta \to W \to \mathscr{O}_y \otimes (W_y/\eta) \to 0.$$

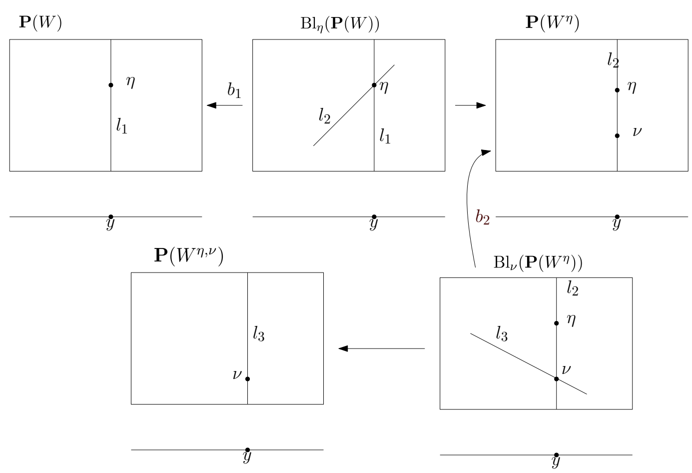
Hence $\deg((W^\eta)^\vee) = \det(W)^{-1} \otimes \mathscr{O}(y)$. Now for any $\nu \in \mathbf{P}((W^\eta)^\vee_y)$ we have another new bundle V^ν of rank 2:

$$0 \to V^\nu \to (W^\eta)^\vee \to \mathscr{O}_y \otimes ((W^\eta)^\vee_y/\nu) \to 0.$$

So $\det(V^\nu)^\vee = \det W$ and V^ν is stable. Then $\{(V^\nu)^\vee : \nu \in \mathbf{P}((W^\eta)^\vee_y)\}$ is a rational curve on X.

Since by dual we have $0 \to W^\vee \xrightarrow{f} (W^\eta)^\vee \to \mathscr{O}_y = \mathrm{Ext}^1 \to 0$. Let $\nu' = \mathrm{coker} f$ and then $W \cong (V^{\nu'})^\vee$. Hence $\{(V^\nu)^\vee : \nu \in \mathbf{P}((W^\eta)^\vee_y)\}$ is a rational curve on X passing through W which is the Hecke curve.

(M2) This is more geometric. Pick $[W] \in X$ which is $(1,1)$-stable and the process as follows:



Consider the blow-up $b_1 : \mathrm{Bl}_\eta(\mathbf{P}(W)) \to \mathbf{P}(W)$ over $\eta \in \mathbf{P}(W)$ over $y \in C$ with fiber $l_1 = \mathbf{P}(W_y)$. The exceptional divisor is $l_2 \cong \mathbf{P}(T_\eta \mathbf{P}(W))$. The strict transform of l_1 is a (-1)-curve since $0 = (l_1 + l_2)^2$. Hence blow-down the l_1 we get a new ruled surface $\mathbf{P}(W^\eta)$. For the choose of tangent direction $\nu \in l_2 = \mathbf{P}(T_\eta \mathbf{P}(W)) = \mathbf{P}(W^\eta_y)$, we blow-up ν again and we get $b_2 : \mathrm{Bl}_\nu(\mathbf{P}(W^\eta)) \to \mathbf{P}(W^\eta)$

and blow-down via (-1)-curve l_2 and we get a new ruled surface $\mathbf{P}(W^{\eta,\nu})$. When ν is tangent to l_1, then we have $W^{\eta,\nu} = W$. Hence $\{W^{\eta,\nu} : \nu \in \mathbf{P}(T_\eta \mathbf{P}(W))\}$ is a rational curve on X passing $[W]$.

Proposition 3.27. *Consider a smooth projective curve C of genus $g \geq 3$. and the moduli space $X = M_{2;\mathscr{D},1}(C)$ of stable bundles of rank 2 and degree 1. Let $\mathscr{L}$ be the ample generator of Picard group, then $-K_X = 2\mathscr{L}$ and Hecke curves have degree 2 with respect to $\mathscr{L}$. Hecke curves are smooth in the smooth lucus of X. Moreover, Hecke curves are minimal rational curves on X.*

Proof. We refer [70] for the proof of that fact that $-K_X = 2\mathscr{L}$, Hecke curves have degree 2 with respect to $\mathscr{L}$ and Hecke curves are smooth in the smooth lucus of X.

For the last statement, the original proof is Proposition 8 in [36]. The basic idea is follows. We just need to show that there are no rational curves of degree 1. By Kodaira's stability, if a rational curve of degree 1 exists at a generic point of X for some C, such a curve exists at a generic point of X for any C of the same genus. So if a rational curve of degree 1 exists at a generic point of X for our C, then pick a hyperelliptic curve C' and its X' is also in this case. But in the hyperelliptic case X' is the set of $(g-2)$-dimensional linear subspaces in the intersection of two quadrics in $\mathbb{P}^{2g+1}$ determined by the hyperelliptic curve, see Theorem 1 in [21]. If lines exist through generic points of X', we have at least a $(3g-3)-(g-1) = (2g-2)$-dimensional family of $(g-1)$-dimensional linear subspaces in the intersection of the two quadrics. By Theorem 2 in [21] the set of $(g-1)$-dimensional linear subspaces of the intersection of the two quadrics is equivalent to the Jacobian of C' which has dimension g. Hence this is impossible since $g \geq 3$. $\square$

Proposition 3.28. *Consider a smooth projective curve C of genus $g \geq 3$. and the moduli space $X = M_{2;\mathscr{D},1}(C)$ of stable bundles of rank 2 and degree 1.*

(a) Then for any $(1,1)$-stable $[W] \in X$, the Hecke curves associated to two distinct η_1, η_2 are distinct rational curves on X.

(b) We have $\mathcal{K}_{[W]} \cong \mathbf{P}_C(W) = \mathbb{P}(W^\vee)$ and the tangent morphism $\tau_{[W]} : \mathcal{K}_{[W]} \to \mathbb{P}(\Omega^1_{X,[W]})$ is given by the linear system $\pi^* K_C \otimes T_{\mathbf{P}_C(W)/C} = 2\pi^* K_C - K_{\mathbf{P}_C(W)}$. Moreover $\mathcal{C}_{[W]}$ is nondegenerate in $\mathbb{P}(\Omega^1_{X,[W]})$.

Proof. For (a) this is 5.13 in [70] and we omit it.

For (b), we give the main idea and the details we refer Proposition 11 in [36]. By (a) we know that the set of Hecke curves is just $\mathbf{P}_C(W) \subset \mathcal{K}_{[W]}$. As $\dim \mathcal{K}_{[W]} = \dim \mathbf{P}_C(W) = 2$ we have $\mathcal{K}_{[W]} \cong \mathbf{P}_C(W) = \mathbb{P}(W^\vee)$. Moreover, by Euler sequence we have $\pi_* T_{\mathbf{P}_C(W)/C} = \mathrm{ad}(W)^\vee$, then traceless endomorphism bundle of W, and $R^1\pi_* T_{\mathbf{P}_C(W)/C} = 0$ where $\pi : \mathbf{P}_C(W) \to C$. As the tangent space of X is just $H^1(C, \mathrm{ad}(W))$, we have the tangent morphism $\tau_{[W]} : \mathbf{P}_C(W) \to \mathbf{P}H^1(C, \mathrm{ad}(W))$. As

$$H^0(\mathbf{P}_C(W), \pi^* K_C \otimes T_{\mathbf{P}_C(W)/C}) = H^0(C, K_C \otimes \mathrm{ad}(W)^\vee) \cong H^1(C, \mathrm{ad}(W))^\vee,$$

it is not different to see that $\tau_{[W]}$ is given by the linear system $\pi^* K_C \otimes T_{\mathbf{P}_C(W)/C}$. □

3.3 Distribution and Its Basic Properties

Definition 3.29. *Let X be a smooth uniruled variety with fixed minimal rational component $\mathcal{K}$. For general $x \in X$ we have VMRT $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$. Consider its linear span $W'_x \subset T_xX$. As x varies over an zariski open subset (which is our meaning of general) U we have a subbundle $W' \subset T_U$. Define its annihilator $(W')^\perp \subset \Omega^1_X$ and the annihilator $W \subset T_V$ (saturation of W') of $(W')^\perp \subset \Omega^1_X$ where V is a open subset of codimension ≥ 2.*

Lemma 3.30. *Given any subset $E \subset X$ of codimension ≥ 2, we can find a standard minimal rational curve disjoint from E.*

Proof. Choose a standard minimal rational curve C through a general point $x \notin E$. Let $N_C \cong \mathscr{O}(1)^{\oplus p} \oplus \mathscr{O}^{\oplus n-1-p}$ and choose sections $\sigma_1, ..., \sigma_p$ of N_C correspond to the independent sections of $\mathscr{O}(1)^{\oplus p}$ vanishing at x, and sections $\sigma_{p+1}, ..., \sigma_{n-1}$ generates $\mathscr{O}^{\oplus n-1-p}$. Since no obstruction, we have a $(n-1)$-dimensional deformation of C whose initial velocities are contained in the linear span of $\sigma_1, ..., \sigma_{n-1}$. If all members meets E, this means we have a 1-dimensional subfamily passing through a given point $y \in E$ since $\mathrm{codim} E \geq 2$. Hence in the linear span of $\sigma_1, ..., \sigma_{n-1}$ there exists a non-zero section vanishing at y. But this is impossible since $\sigma_1, ..., \sigma_{n-1}$ are pairwisely independent outside x. Hence well done. □

3.3.1 Levi Tensor of the Distribution

Definition 3.31. *Fix a distribution $\mathcal{D} \subset T_M$ for a complex manifold. For any $x \in M$ and any two vectors $u, v \in \mathcal{D}_x$, let their local sections $\tilde{u}, \tilde{v}$. Then we define the* Levi tensor *of $\mathcal{D}$, which is a section of $\mathscr{H}om\left(\bigwedge^2 \mathcal{D}, T_M/\mathcal{D}\right)$, as*

$$\mathsf{Levi}^{\mathcal{D}}_x(u, v) := [\tilde{u}, \tilde{v}]_x \pmod{} D_x.$$

Remark 3.32. *In the old survey [37], this is called the Frobenius bracket tensor.*

Proposition 3.33. *Let X be a smooth uniruled variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ associated to a distribution W. If W is a proper distribution, then it is not integrable at general points.*

Proof. For the whole proof we refer Proposition 2.2 in [37]. Here we give some idea. If W is integrable, then by Frobenius theorem it defines a non-trivial foliation on $X \backslash E$ for some $\operatorname{codim} E \geq 2$. By some argument one can compactify the leaves of this foliations into algebraic subvarieties.

Pick a Chow schemes Chow_W of compactifications of these leaves. Choosing a hypersurface H in Chow_W generically, we get a hypersurface L in X which is the closure of the codimension 1 part of the union of compactificated leaves corresponding to H. A generic member of $\mathcal{K}$ lies in a leaf of $\mathcal{D}$ but is disjoint from H, hence disjoint from L, a contradiction to the Picard number condition on X. $\square$

Proposition 3.34. *Let X be a smooth uniruled variety with fixed minimal rational component $\mathcal{K}$ associated to a distribution W. Let $\mathcal{T}_x \subset \mathsf{Grass}(1, \mathbf{P}(W_x)) \subset \mathbf{P}\left(\bigwedge^2 W_x\right)$ be lines tangent to the smooth locus of $\mathcal{C}_x$. Then $\mathcal{T}_x$ is contained in the projectivization of the kernel of the Levi tensor $\mathsf{Levi}_x^W(-,-) : \bigwedge^2 W_x \to T_xX/W_x$.*

Proof. By Proposition 3.15 we just need to show that $\mathsf{Levi}_x^W(\alpha, \beta) = 0$ for any $\alpha \in W_x$ correspond to the general point of $\mathcal{C}_x$ and $\beta \in T_xX_\alpha^+$. So WLOG we let both of them are non-zero. Hence we just need to show that there is a local complex analytic surface through x tangent to W in the neighborhood of x whose tangent space at x containing α, β.

Choose a standard rational curve C through x whose tangent vector is α (as α general) and fix $y \in C$ with $x \neq y$. Now β correspond to the positivr part of $T_X|_C$, thus there exists a non-zero section σ of the normal bundle so that $\sigma(y) = 0$ and $\sigma(x) = \beta$. As $H^1(C, N_C \otimes \mathfrak{m}_y) = 0$, we can find a deformation C_t of C fix y with initial velocity β. This fomrs a local complex analytic surface through x whose tangent space at x spanned by α, β. Moreover its tangent space at z near x spanned by T_zC_t and $\sigma_t(z)$ where $\sigma_t \in H^0(C_t, N_{C_t} \otimes \mathfrak{m}_y)$. By Proposition 3.15 again we know that σ_t in the tangent space of $\mathcal{C}_z$, hence in W_z. Hence this surface tangent to W. Well done. $\square$

Corollary 3.35. *Let X be a smooth uniruled variety of Picard number 1 with fixed minimal rational component. For a general point $x \in X$, the VMRT $\mathcal{C}_x$ cannot be an irreducible linear proper subspace.*

Proof. Follows directly from Proposition 3.33 and Proposition 3.34. $\square$

3.3.2 Nondegeneracy of the Distribution

In this small section we will consider when the VMRT $\mathcal{C}_x$ is nondegenerate.

Proposition 3.36. *Let W be a vector space with a non-linear cone $J \subset W$ such that $\dim J > \frac{1}{2} \dim W$ and $\mathbf{P}(J)$ is a smooth subvariety of $\mathbf{P}(W)$. Let $\mathcal{T} \subset \mathbf{P}\left(\bigwedge^2 W\right)$ be the variety of tangent lines of $\mathbf{P}(J)$, then $\mathcal{T}$ is nondegenerate in $\mathbf{P}\left(\bigwedge^2 W\right)$.*

Proof. This is a boring result deduced by dimension-counting and Zak's theorem in the projective geometry about tangencies. We refer the proof of Proposition 2.6 in [37]. □

Theorem 3.37. *Let X be a smooth uniruled variety of Picard number 1 and dimension n with $\dim \mathcal{C}_x = p > \frac{n-3}{2}$, then if $\mathcal{C}_x$ is smooth for some general point, then it is nondegenerate.*

Proof. If it is degenerate, defining the non-trivial distribution W of rank $m < n$. Since $\mathcal{C}_x$ is smooth and $\dim \mathcal{C}_x = p > \frac{n-3}{2}$, the Levi tensor of W vanish identically by Proposition 3.34 and 3.36. But by Proposition 3.33 this is impossible! □

Corollary 3.38. *Let X be a prime smooth Fano variety of dimension n with $\mathsf{Index}(X) > \frac{n+1}{2}$, then the VMRT is nondegenerate.*

Proof. This follows directly from this Theorem and Corollary 3.14. □

3.3.3 Cauchy Characteristic of the Distribution

Definition 3.39. *Let a distribution $\mathcal{D}$ on a complex manifold X regarded as a subsheaf of T_X. The* Cauchy characteristic of $\mathcal{D}$ *is a subsheaf defined as*

$$\mathsf{Ch}(\mathcal{D})(U) := \{f \in \mathcal{D}(U) : \mathsf{Levi}^{\mathcal{D}}(f, g) = 0, \forall g \in \mathcal{D}(U)\}.$$

Remark 3.40. *Actually $\mathsf{Ch}(\mathcal{D})$ is a integrable distribution over the open subset where it is locally free.*

Lemma 3.41. *Let $g : M \to N$ be a submersion of complex manifolds so that the fibers of g define a distribution $\ker(dg)$ on M. Let $\mathcal{D}$ be a distribution on N, define the pull-back distribution is $(g^*\mathcal{D})_m = (dg)^{-1}(\mathcal{D}_{g(m)})$. Then we have*

$$\ker(dg) \subset \mathsf{Ch}(g^*\mathcal{D}).$$

Proof. Almost trivial. Omitted. □

Proposition 3.42. *Let X be a smooth Fano variety of Picard number 1. Consider the total VMRT*

$$\mathcal{C} := \overline{\bigcup_{x \text{ general}} \mathcal{C}_x} \subset \mathbb{P}(\Omega^1_X)$$

and consider the universal cycle morphisms $\mathcal{K} \xleftarrow{\rho} \mathcal{U} \xrightarrow{\mu} X$. Note that the normalization $(\mu^{-1}(x))^n = \mathcal{K}_x$ and the tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ induce a rational map $\tau : \mathcal{U} \dashrightarrow \mathcal{C}$ which is generically finite. The image of τ of fibers of ρ induce a multi-valued foliation $\mathscr{F}$ and the leaf of it is the lift of the minimal rational curve to its tangent vectors.

Define a distribution $\mathcal{P}$ of rank $2p+1$ on generic part of $\mathcal{C}$ as

$$\mathcal{P}_\alpha := (d\pi)^{-1}(\mathbb{P}((T_x X_\alpha^+)^\vee))$$

where $\pi : \mathcal{C} \to X$ sends $\alpha \mapsto x$.

Now choose an analytic open subspace $O \subset \mathcal{U}$ such that $\tau|_O$ is biholomorphic, we can regard O as an open subset of $\mathcal{C}$ and $\mathscr{F}$ be a univalent foliation on O. If $\mathcal{C}_x$ has generically finite Gauss map for general $x \in O$, then $\mathscr{F} = \mathsf{Ch}(\mathcal{P})$ on O.

Remark 3.43. *Let us examine what the condition on Gauss map means in this remark.*

It is perhaps easier to look at the affine case. So let $Z \subset \mathbb{A}^n_{\mathbb{C}}$ be an affine variety of dimension m and let $z \in Z$ be a generic smooth point. Let $z_1, ..., z_m$ be a local coordinate system of Z at z and $w_1, ..., w_n$ be an affine coordinate system. The Gauss map of Z is just associating to z its tangent space $T_z(Z)$. If the Gauss map is not generically finite, its differential has kernel in a neighborhood of z. Let $v \in T_z(Z)$ be in the kernel of the differential of the Gauss map. This means that in the direction of v, the tangent spaces $T_z(Z)$ remain constant to the first order as x varies in a neighborhood of z.

In particular, for any local vector field ω on Z as $\omega = \sum_i a_i(z_1, ..., z_m)\frac{\partial}{\partial w_i}$ and its derivative in the direction of v is $D_v\omega = \sum_i v(a_i(z_1, ..., z_m))\frac{\partial}{\partial w_i}$ also tangent to Z at z.

Conversely, one can see that if v is a tangent vector to Z at z so that $D_v\omega(z) \in T_z Z$ for any local vector field ω on Z, then v is in the kernel of the differential of the Gauss map. This can be applied to a projective subvariety of $\mathbb{P}^{n-1}$ by taking its affine cone.

Sketched proof of Proposition 3.42. Now assume all we work are on O.

One one side (without assuming the Gauss map), if we definr the distribution $\mathcal{Q}$ generically on $\mathcal{K}$ as $\mathcal{Q}_{[C]} = H^0(C, \mathscr{O}(1)^{\oplus p}) \subset T_{[C]}\mathcal{K} = H^0(C, N_C)$. then by Proposition 3.15 we have $\mathcal{P} = \rho^*\mathcal{Q}$. By Lemma 3.41, we have $\mathscr{F} \subset \mathsf{Ch}(\rho^*\mathcal{Q}) = \mathsf{Ch}(\mathcal{P})$.

Conversely, if there exists a vector in $\mathsf{Ch}(\mathcal{P})_\alpha$ not in $\mathscr{F}_\alpha$, then there must a vector v tangent to the fibers of $\pi : \mathcal{C} \to X$, that is, $v \in T_\alpha\mathcal{C}_x$ where $x = \pi(\alpha)$ by Jacobi indentity. The condition $v \in \mathsf{Ch}(\mathcal{P})_\alpha$ as $\mathsf{Levi}^{\mathcal{P}}_\alpha(v, \mathcal{P}) \subset \mathcal{P}$. Hence

$$\mathsf{Levi}^{\mathcal{P}}_\alpha(v, \mathcal{P} \cap T_{\mathbb{P}(\Omega^1_{X,x})}) \subset \mathcal{P} \cap T_{\mathbb{P}(\Omega^1_{X,x})}.$$

As $\mathcal{P}_\alpha \cap T_\alpha\mathbb{P}(\Omega^1_{X,x}) = T_\alpha(\mathcal{C}_x)$, we have $\mathsf{Levi}^{\mathcal{P}}_\alpha(v, T_{\mathcal{C}_x}) \subset T_{\mathcal{C}_x}$. Hence v is must in this kernel of the Gauss map since $v \in T_\alpha\mathcal{C}_x$. Hence well done. □

3.4 Cartan-Fubini Type Extension Theorem

3.4.1 Some History

In this small section we will follows the introduction survey [40]. The beginning of these problems is the following theorem due to Liouville:

Theorem 3.44 (Liouville)**.** *For any conformal map $f : U_1 \to U_2$ between two domains in sphere S^n for $n \geq 2$, there is a Möbius transformation $f : S^n \to S^n$ satisfying $f = F|_{U_1}$.*

As a natural extension in the projective geometry, we may ask:

Theorem 3.45. *For any holomorphic conformal map $f : U_1 \to U_2$ between two domains in $\mathbb{Q}^n, n \geq 3$, there is a biholomorphic automorphism $F \in \mathrm{Aut}(\mathbb{Q}^n)$ satisfying $f = F|_{U_1}$.*

As a generalization, we consider the following theorems:

Theorem 3.46 (Fubini-Cartan-Jensen-Musso)**.** *Let $X_1, X_2 \subset \mathbb{P}^{n+1}$ be two smooth hypersurfaces of degree $d \geq 2$. If a biholomorphic map $f : U_1 \to U_2$ between two domains $U_1 \subset X_1$ and $U_2 \subset X_2$ preserves the structures given by both the second fundamental form and the Fubini cubic form, then there is a biholomorphic morphism $F : X_1 \to X_2$ satisfying $f = F|_{U_1}$.*

In our sense of VMRT, we may consider the following questions:

Problem 3.1. *Let X be a smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}$ so that the VMRT $\mathcal{C}_x$ at a general point $x \in X$. Does $\mathcal{C}_x$ determine X in the following sense:*

Let X' be any smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}'$ for which we denote the VMRT $\mathcal{C}'_{x'}$ for general $x' \in X'$. Suppose there exists connected analytic open subsets $U \subset X, U' \subset X'$ and a biholomorphic map $\phi : U \to U'$ with isomorphism $\psi : \mathbf{P}T_U \to \mathbf{P}T_{U'}$ compactible with ϕ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}'_{\phi(x)}$ for general $x \in U$. Do we have a biholomorphic map $X \to X'$?

This question is not right for the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of odd degree d over a smooth projective curve C of genus $g = 2$.

This question is right for $\mathbb{P}^n$ by Cho-Miyaoka and right for any irreducible Hermitian symmetric space by Hwang-Mok.

3.4.2 The Main Result

We will follows the survey [37] and paper [44]. We consider the following theorem due to Hwang-Mok:

Theorem 3.47 (Cartan-Fubini Type Extension Theorem)**.** *Let X be a smooth Fano variety of Picard number 1 with the choise of minimal rational component $\mathcal{K}$ so that the VMRT $\mathcal{C}_x$ at a general point $x \in X$ is of positive dimension $p > 0$ and the Gauss map of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ is generically finite.*

Let X' be any smooth Fano variety of Picard number 1 *with the choise of minimal rational component $\mathcal{K}'$ for which we denote the VMRT $\mathcal{C}'_{x'}$ for general $x' \in X'$.*

Suppose there exists connected analytic open subsets $U \subset X, U' \subset X'$ and a biholomorphic map $\phi : U \to U'$ so that the differential $\phi_* : \mathbf{P}T_U \to \mathbf{P}T_{U'}$ sends $\mathcal{C}_x$ isomorphically to $\mathcal{C}'_{\phi(x)}$ for general $x \in U$, then ϕ can extended to a biholomorphic map $X \to X'$.

Remark 3.48. *Several remarks:*

(a) *Althrough this theorem is not true for projective space (note that the Gauss map is not generically finite), the Problem 3.1 is true for projective space.*

(b) *Actually the Gauss map of $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ is generically finite (actually finite by Zak's results) for any non-linear smooth projective variety, see [28]. Hence the theorem is right for any examples we want to see, except projective space, with $p > 0$.*

Sketched proof of Theorem 3.47. We will follows the sketch in [37] Theorem 3.2 and we refer the detailed proof in [44]. We will follows the several steps.

▶ **Step 1. Show the map ϕ sends local pieces of $\mathcal{K}$ in U to local pieces of $\mathcal{K}'$ in U'.**

Consider the Proposition 3.42, then since ϕ_* sends $\mathcal{C}|_U$ to $\mathcal{C}'|_{U'}$, then it sends $\mathcal{P}$ to $\mathcal{P}'$. Hence it sends $\mathscr{F}$ to $\mathscr{F}'$. Well done.

▶ **Step 2. To extend the domain of ϕ from the analytic open set to an étale open set.**

Suppose C the standard minimal rational curve intersecting U. ϕ is defined on $C \cap U$ and we want to extend it to other points on C. To define the extension at a point $y \in C$, consider a deformation C_t of C fixing the point y since $p > 0$. Now consider the local pieces $U \cap C_t$. By Step 1, $\phi(U \cap C_t)$ is a local piece of some minimal rational curve C'_t belonging to $\mathcal{K}'$. We claim that these curves C'_t have a unique common point y'.

Indeed the common point y' exists because it exists when y is chosen to be inside U. It is unique because C'_t do not have deformations fixing two or more points. In fact, if such a deformation exists, then its initial velocity is a section of the normal bundle of a standard minimal rational curve vanishing at two or more points, a contradiction to the splitting type. Hence we proved the claim. Hence we can define y' as the image of y and then we can extend ϕ along standard minimal rational curves intersecting U (this has some problems, but we have shown in bold font below). This enlarges the domain of definition of ϕ to a bigger open set $\widehat{U}$. Applying the same argument to $\widehat{U}$, we can analytically continue along standard minimal rational curves intersecting $\widehat{U}$.

We can repeat this procedure until the domain of definition covers a Zariski open subset in X. But **there is a gap in this extension argument. A point outside U may belong to different standard minimal rational curves intersecting U. So**

when we carry out the analytic continuation, we end up with a multi-valued extension of ϕ. So what we get at the end is a multi-valued extension of ϕ over an étale open subset $\widetilde{U}$ of X, namely a quasi-projective variety $\widetilde{U}$ with an étale morphism $\widetilde{U} \to X$ covering a Zariski open subset of X and a morphism $\widetilde{\phi} : \widetilde{U} \to X'$ extending ϕ. We skipped many technique things and we refer [44].

▶ **Step 3. To extend the domain from the étale open set to a Zariski open set.**

To extend $\widetilde{\phi}$ to a morphism Φ_0, defined on a Zariski open subset X_0 of X, we have to reduce the multi-valuedness of $\widetilde{\phi}$. First of all, we can reduce the multi-valuedness of $\widetilde{\phi}$ by identifying two points $u_1, u_2 \in \widetilde{U}$ if $\nu(u_1) = \nu(u_2)$ and $\widetilde{\phi}(u_1) = \widetilde{\phi}(u_2)$ where $\nu : \widetilde{U} \to X$ be that étale morphism. So let us assume that there is no such two distinct points. Then we claim that ν must be 1-to-1.

Indeed, if not then by Lemma 3.50 we can choose a standard minimal rational curve C generically and pick a generic point $x \in C$. Then there exists an irreducible component C' of $\nu^{-1}(C)$ containing a pair of points $u_1, u_2 \in \widetilde{U}$ with $\nu(u_1) = \nu(u_2) = x$ and $\widetilde{\phi}(u_1) \neq \widetilde{\phi}(u_2)$. Now let C_t be a deformation of C with x fixed, which exists by $p > 0$, then their inverse images under ν contains components C'_t which are deformations of C' fixing u_1 and u_2. Then their images under $\widetilde{\phi}$ define a family of standard rational curves in X' fixing two distinct points $\widetilde{\phi}(u_1) \neq \widetilde{\phi}(u_2)$, a contradiction. This finishes Step 3.

▶ **Step 4. To extend the domain from the Zariski open set to the whole Fano manifold X.**

By applying the same extension to $\phi^{-1} : U' \to U$, we see that the rational map Φ_0 in Step 3 is birational. For Step 4, if Φ_0 has exceptional set $E \subset X$ of codimension 1 which is contracted to a set $Z \subset X'$ of codimension 2. From the Picard number condition, all members of $\mathcal{K}$ intersect E. It follows that generic members of $\mathcal{K}$ must intersect Z, a contradiction to Lemma 3.30. Hence Φ_0 is a birational map with no exceptional set.

Hence Φ_0 induce the isomorphisms $H^0(X, -mK_X) \cong H^0(X', -mK_{X'})$. Hence Φ_0 induce

$$\Phi : X \cong \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, -mK_X) \cong \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X', -mK_{X'}) \cong X'.$$

Well done. □

Remark 3.49. *In the proof, the hypothesis of Gauss map is used only in step 1 and the hypothesis of $p > 0$ is used only in step 2,3.*

Lemma 3.50. *Let $\pi : Y \to X$ be a generically finite morphism from a normal variety Y onto a Fano manifold X with Picard number 1. Suppose for a generic standard rational curve $C \subset X$ belonging to a chosen minimal rational component, each component of the inverse image $\pi^{-1}(C)$ is birational to C by π. Then $\pi : Y \to X$ itself is birational.*

Proof. Let π is not birational. By Stein factorization π can be factored into $Y \xrightarrow{g} Y' \xrightarrow{h} X$ where g is birational and h is finite. By Proposition 1.62(a) we know that h is not étale. Hence we can choose a ramification divisor $R \subset Y$ such that $\pi(R) \subset X$ is also a divisor.

By genericity of C, we may assume that $\pi^{-1}(C)$ lies on the smooth part of the normal variety Y. Let C_1 be any irreducible component of $\pi^{-1}(C)$. Then C_1 is also a rational curve and deformations of C_1 give deformations of C since $\pi|_{C_1}$ is birational. It follows that the space of deformations of C and the space of deformations of C_1 have equal dimensions. So we have $K_X \cdot C = K_Y \cdot C_1$. This implies C is disjoint from the ramication divisor R. Since this holds for any components of $\pi^{-1}(C)$, we know that C is disjoint from $\pi(R)$. But this is impossible by the assumption that X is of Picard number 1. □

3.4.3 More Comments

We may ask what is the difference between Problem 3.1 and Theorem 3.47. We will follows [37] and consider the case $X = \mathbb{Q}^n \subset \mathbb{P}^{n+1}$. For more general setting and more detailed computations about conformal differential geometry we refer paper [42].

Actually the VMRT is a hyperquadric in $\mathbb{P}(\Omega^1_{X,x})$. Hence they generate a subbundle of $\mathbb{P}\Omega_X$ with fibers isomorphic to hyperquadrics. This gives a conformal structure on X.

Definition 3.51. *A* conformal structure *on a complex manifold M is vector bundle morphism $\sigma : \mathrm{sym}^2 T_M \to \mathscr{L}$ for some line bundle $\mathscr{L}$ which gives a nondegenerate symmetric bilinear form at each fiber T_xM.*

The null-cone *$\mathcal{C} \subset \mathbf{P}T_M$ is the zero locus of bilinear form σ whose fibers are $\mathcal{C}_x \subset \mathbf{P}T_xM$.*

After choose a local trivialization of $\mathscr{L}$, we have locally

$$\sigma = \sum_{ij} g_{ij}(z) dz^i \otimes dz^j$$

for local coordinates $z_1, ..., z_n$ and (g_{ij}) are nondegenerate symmetric matrix. Consider the curvature tensor

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial z^k} - \frac{\partial \Gamma^i_{jk}}{\partial z^l} + \sum_{\mu} (\Gamma^\mu_{jl}\Gamma^i_{\mu k} - \Gamma^\mu_{jk}\Gamma^i_{\mu j}) = \mathsf{Weyl} + m\mathsf{Ric} + n\mathsf{Sca}.$$

Also, the geodesic defined by $\frac{d^2\gamma^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$. Althrough R^i_{jkl} depends on the choice of the trivialization, the Weyl tensor Weyl is not and so is the geodesics which tangent to the null-cone ($\mathsf{null\text{-}geodesics}$). If $\mathsf{Weyl} = 0$ we say the conformal structure is flat.

For our X, the conformal structure given by the VMRT is flat which can be seen by the choise of a flattening coordinate system! This is an example of Harish-Chandra coordinate on Hermitian symmetric spaces. In this case the minimal rational curves are null-geodesics.

In the sense of Theorem 3.47, $\phi_*\mathcal{C}_x = \mathcal{C}'_{\phi(x)}$ means the conformal structure defined at generic points on X' is flat. Hence the difference between Problem 3.1 and Theorem 3.47 is just the Weyl tensor Weyl.

Now we give an very special example which shows how to use VMRT to handle the curvature:

Example 3.52. *Let X be a Fano manifold of Picard number 1 with VMRT are hyperquadric. Hence we have a conformal structure given on a Zariski open set of X. No we assume that the conformal structure can be extends to the whole X. Then the Weyl tensor $\mathsf{Weyl} \in H^0(X, \bigwedge^2 \Omega_X \otimes \mathscr{E}nd(T_X))$ vanish.*

Proof. Consider a standard minimal rational curve C and $T_X|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-2} \oplus \mathscr{O}$ in this case. We need to show $\mathsf{Weyl}(u \wedge v) \in \mathrm{End} T_x X$ vanish at all $u, v \in T_x X$. By Proposition 3.36 we just need to consider $u \wedge v$ for $u \in \mathcal{C}_x$ and $v \in T_u\mathcal{C}_x$. Let u in the $\mathscr{O}(2)$-part vanish at two points and v vanish at one point. Hence $\mathsf{Weyl}(u \wedge v)$ has three zeros. If it is not zero, then since $\mathsf{Weyl}(u \wedge v)$ be a section of $\mathscr{E}nd(T_X|_C) = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus 2n-4} \oplus \cdots$. Hence if it is not zero, then it can not have three zeros! $\square$

Chapter 4

Some Basic Applications of VMRT

4.1 Stability of the Tangent Bundles

4.1.1 Basic Facts about Stability of the Tangent Bundles

Proposition 4.1 (Simpleness)**.** *Let X be a smooth uniruled variety. If the VMRT $\mathcal{C}_x$ is irreducible and nondegenerate for some choise of minimal rational component, then T_X is simple.*

Proof. Let $\xi \in \mathrm{End}(T_X)$. Let x general and $v \in T_xX$ be a tangent vector to standard minimal rational curve C through x. Consider the extended vector field $\tilde{v}$ on C having two distinct zeroes. Then $\xi(\tilde{v}) \in \Gamma(T_X|_C)$ vanishing at two distinct points. As C is standard, then either $\xi(\tilde{v}) = 0$ or $\xi(\tilde{v})$ is proportional to $\tilde{v}$. Hence v is the eigenvector of ξ in T_xX. As this is true for any choice of v tangent to some standard minimal rational curve C through x and since $\mathcal{C}_x$ is nondegenerate, then ξ act as scalar multiplication in T_xX. Since $\xi(\tilde{v})$ is the constant multiple of $\tilde{v}$, the eigenvalues must be constant on C. Hence ξ must be a scalar multiplication and T_X is simple. $\square$

Now we consider the stability of tangent bundles. We will follows Section 2.4 in the survey [37] and the paper [35]. This is a standard method developed in [35]. Note that the results in this small section hold for any rational component $\mathcal{K}'$ of Chow schemes but we don not care.

Now we will assume X be an n-dimensional smooth Fano variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ of degree $p+2$. Then to show the stability of T_X we just need to show that for any subsheaf $\mathscr{F} \subset T_X$ of rank $1 \leq k \leq n-1$ we have $\frac{c_1(\mathscr{F})\cdot(-K_X)^{n-1}}{k} < \frac{c_1(T_X)\cdot(-K_X)^{n-1}}{n}$. As Picard number is 1, we can check this over a generic standard minimal rational curve C. Hence for a sheaf $\mathscr{F}$ of rank r, which can be assumed to be locally free over C by Lemma 3.30, we can define $\mu(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot C}{r}$. Note that $\mu(\mathscr{F})$ depends only on $\mathscr{F}$ and $\mathcal{K}$ and does not depend on the choice of C. For example $\mu(T_X) = \frac{p+2}{n}$.

Example 4.2 (Baby version for $\mathbb{P}^n$)**.** *We will show that $T_{\mathbb{P}^n}$ is stable. For any subsheaf $\mathscr{F} \subset T_{\mathbb{P}^n}$ Choose a generic line C, so that $\mathscr{F}|_C$ is a vector bundle and splits as $\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ where $a_1 \geq \cdots \geq a_r$. Since $T_{\mathbb{P}^n}|_C \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$, if $\mu(\mathscr{F}) \geq \mu(T_X) = \frac{n+1}{n}$, then $a_1 = 2$. This implies that the line C is tangent to the distribution $\mathscr{F}$. But this is true for any generic choice of C. Hence $\mathscr{F}$ must have rank n, and we are done.*

Proposition 4.3. *Suppose that T_X is not stable (resp. not semi-stable). Then we can find a subsheaf $\mathscr{F} \subset T_X$ of rank $r, 1 < r < n$ with torsion free quotient $T_X/\mathscr{F}$, satisfying $\mu(\mathscr{F}) \geq \mu(T_X)$ (resp. $\mu(\mathscr{F}) > \mu(T_X)$), whose Levi tensor $\mathsf{Levi}^{\mathscr{F}}$ vanishs for general x.*

Proof. Consider a subsheaf $\mathscr{F} \subset T_X$ of rank r smaller than n with maximal values of $\mu(\mathscr{F}) \geq \mu(T_X) > 0$. Moreover, we can choose such $\mathscr{F}$ so that $T_X/\mathscr{F}$ is torsion free. In fact, if $T_X/\mathscr{F}$ has torsion $(T_X/\mathscr{F})_{\mathrm{Tor}}$ for a such choice of $\mathscr{F} \subset T_X$, the inverse image $\mathscr{F}'$ of $(T_X/\mathscr{F})_{\mathrm{Tor}}$ in T_X under the quotient map is a subsheaf of rank r with $\mu(\mathscr{F}') \geq \mu(\mathscr{F})$, and we may choose $\mathscr{F}'$ as our $\mathscr{F}$.

First we have $r > 1$. Indeed, if $r = 1$ then $\mathscr{F}^{\vee\vee}$ is an ample line subbundle of T_X (since Picard number is 1) and hence X is a projective space by Theorem 3.19, a contradiction to the assumption that T_X is not stable.

By the choice, $\mathscr{F}$ is semi-stable and $\bigwedge^2 \mathscr{F}$ is also semi-stable. Let the image of the Levi tensor $\mathsf{Levi}^{\mathscr{F}} : \bigwedge^2 \mathscr{F} \to T_X/\mathscr{F}$ is $\mathscr{G}$. If it has positive rank, by semi-stability, we have $\mu(\mu(G)) \geq \mu\left(\bigwedge^2 \mathscr{F}\right) = 2\mu(\mathscr{F}) > \mu(\mathscr{F})$.

Suppose the rank of $\mathscr{G}$ is equal to the rank of $T_X/\mathscr{F}$. Then $\mu(\mathscr{G}) \leq \mu(T_X/\mathscr{F}) \leq \mu(T_X) \leq \mu(\mathscr{F})$. A contradiction to $\mu(\mu(G)) > \mu(\mathscr{F})$.

Suppose if $\mathscr{G}$ has positive, but strictly smaller rank than that of $T_X/\mathscr{F}$. let $\mathscr{G}' \subset T_X$ be the kernel sheaf of $T_X \to (T_X/\mathscr{F})/\mathscr{G}$. Let m be the rank of $\mathscr{G}'$ with $r < m < n$. Then

$$\mu(\mathscr{G}') = \frac{r}{m}\mu(\mathscr{F}) + \frac{m-r}{m}\mu(\mathscr{G}) \geq \mu(\mathscr{F})$$

which is a contradiction to the choice of $\mathscr{F}$. □

Proposition 4.4. *Let $\mathscr{F}$ be any subsheaf of T_X with rank $< n$. If generic curves in $\mathcal{K}$ are tangent to $\mathscr{F}$, then $\mathscr{F}$ cannot be integrable at generic points.*

Proof. Assume that $\mathscr{F}$ is integrable. Let $Z \subset X$ be the singular loci of the foliation defined by $\mathscr{F}$. The codimension of Z is ≥ 2. Thus a generic member of $\mathcal{K}$ is disjoint from Z (Lemma 3.30) and lies in a single leaf of $\mathscr{F}$.

For a given point $x \in X \backslash Z$, let D_x be the set of points which can be joined to x by a connected curve each component of which is a member of $\mathcal{K}$ disjoint from Z. Then

D_x is a constructible set (see Section IV.4 in [57]) and the collection of D_x's for generic $x \in X$ defines a meromorphic foliation $\mathcal{D}$ on X. Clearly, D_x is contained in the leaf of $\mathscr{F}$ containing x. It follows that $\mathcal{D}$ is a nontrivial foliation of X. Let $\mathsf{Chow}_{\mathcal{D}}$ be the Chow variety whose generic points corresponds to leaves of $\mathcal{D}$. Choosing a hypersurface H in $\mathsf{Chow}_{\mathcal{D}}$ generically, we get a hypersurface L in X which is the closure of the codimension 1 part of the union of $\mathcal{D}$-leaves corresponding to H. A generic member of $\mathcal{K}$ lies in a leaf of $\mathcal{D}$ but is disjoint from H, hence disjoint from L, a contradiction to the Picard number condition on X. $\square$

Corollary 4.5. *For the choice of Proposition 4.3, we have $\mu(\mathscr{F}) \leq 1$.*

Proof. Let $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ for $a_1 \geq \cdots \geq a_r$. If $\mu(\mathscr{F}) = \sum_{i=1}^r a_i/r > 1$, then $a_1 = 2$ and implying that C is tangent to $\mathscr{F}$. By Proposition 4.4 this is impossible. $\square$

Theorem 4.6. *If $p = n-1$ or 0, then T_X is stable. If $p = n-2$, then T_X is semi-stable.*

Proof. For $p = n-1, n-2$, this is immediate from $\mu(T_X) = \frac{p+2}{n} \geq 1$ and Corollary 4.5. For $p = 0$ assuming that T_X is not stable, choose $\mathscr{F}$ as in Proposition 4.3 and choose a generic C from $\mathcal{K}$ so that both $\mathscr{F}$ and $T_X/\mathscr{F}$ are locally free on C. Let $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$ for $a_1 \geq \cdots \geq a_r$. As $T_X|_C = \mathscr{O}(2) \oplus \mathscr{O}^{\oplus n-1}$, then $a_1 = 2$ and implying that C is tangent to $\mathscr{F}$. By Proposition 4.4 this is impossible. $\square$

Theorem 4.7. *Let X be a smooth Fano variety of Picard number 1. Assume that for a general point of the VMRT $\alpha \in \mathcal{C}_x$ and for any $k-1$-dimensional $\mathbb{P}(F_x^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ we have $\dim(\mathbb{P}(F_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) < \frac{k}{n}(p+2) - 1$ where $p = \dim \mathcal{C}_x$. Then T_X is stable.*

Proof. If T_X is not stable, choose $\mathscr{F}$ as in Proposition 4.3. For general $C \in \mathcal{K}$ we have $\mathscr{F}|_C = \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_k)$ for $a_1 \geq \cdots \geq a_k$. As $\mathscr{F}|_C \subset T_X|_C$ we have $a_1 \leq 2$. If $a_1 = 2$, then C tangent to $\mathscr{F}$ and this is impossible by Proposition 4.4. Hence $1 = a_1 = \cdots = a_q > a_{q+1} \geq \cdots$ for some $q \leq k$. As $\mu(\mathscr{F}) = \sum_{i=1}^k \frac{a_i}{k} \geq \mu(T_X) = \frac{p+2}{n}$ and hence $q \geq \frac{k}{n}(p+2)$. Let $x \in C$ general with tangents correspond to $\alpha \in \mathcal{C}_x$, then by definition we have $\dim(\mathbb{P}(\mathscr{F}_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) \geq q-1 \geq \frac{k}{n}(p+2)-1$ which is impossible by the hypothesis. $\square$

Proposition 4.8. *Let X be a prime smooth Fano variety of dimension n with $\mathsf{Index}(X) > \frac{n+1}{2}$, then T_X is stable.*

Proof. If not, by Theorem 4.7 we have a $k-1$-dimensional $\mathbb{P}(F_x^\vee) \subset \mathbb{P}(\Omega^1_{X,x})$ we have $\dim(\mathbb{P}(F_x^\vee) \cap \mathbb{P}((T_x X_\alpha^+)^\vee)) \geq \frac{k}{n}(p+2) - 1$ where $p = \dim \mathcal{C}_x$.

Consider the projection $\psi : \mathbb{P}(\Omega^1_{X,x}) \backslash \mathbb{P}(F_x^\vee) \to \mathbb{P}^{n-k-1}$ and let q be the dimension of the generic fiber of $\psi|_{\mathcal{C}_x}$. Then $q \geq \frac{k}{n}(p+2)$. Let T be the projective tangent space of $\psi(\mathcal{C}_x)$ at general point $\alpha \in \psi(\mathcal{C}_x)$, then $\dim \psi^{-1}(T) = \dim T + k = p - q + k$. This $\psi^{-1}(T)$ tangent to Y along $(\psi|_{\mathcal{C}_x})^{-1}(\alpha)$. By Corollary 3.14 $\mathcal{C}_x$ is smooth, hence by Zak's theorem on tangencies we can find that $q \leq \frac{k}{2}$. As $q \geq \frac{k}{n}(p+2)$ we get $\mathsf{Index}(X) = p+2 \leq \frac{n}{2}$ which is impossible by the hypothesis. $\square$

4.1.2 For Low Dimensional Fano manifolds

We will follows the paper [35]. As in the previous section, we fix X be an n-dimensional smooth Fano variety of Picard number 1 with fixed minimal rational component $\mathcal{K}$ of degree $p+2$.

Recall that as Picard number is 1, we can check this over a generic standard minimal rational curve C. Hence for a sheaf $\mathscr{F}$ of rank r, which can be assumed to be locally free over C by Lemma 3.30, we can define $\mu(\mathscr{F}) := \frac{c_1(\mathscr{F})\cdot C}{r}$. Note that $\mu(\mathscr{F})$ depends only on $\mathscr{F}$ and $\mathcal{K}$ and does not depend on the choice of C. For example $\mu(T_X) = \frac{p+2}{n}$.

Proposition 4.9 (For $p = 1$)**.** *If $p = 1$ and $n \leq 6$, then T_X is semi-stable, and stable except possibly when $n = 6$.*

Proof. If T_X is not semi-stable, choose $\mathscr{F}$ as in Proposition 4.3. From $\mathscr{F}|_C \subset \mathscr{O}(2) \oplus \mathscr{O}(1) \oplus \mathscr{O}^{\oplus n-2}$ with $T_X/\mathscr{F}|_C$ being locally free and $\mu(F) > 0$, we see $\mathscr{F}|_C = \mathscr{O}(1) \oplus \mathscr{O}^{\oplus r-1}$ by Proposition 4.4. From $\frac{1}{r} = \mu(\mathscr{F}) > \mu(T_X) = \frac{3}{n}$ and $r > 1$, we get $n > 6$. If T_X is semi-stable but not stable, we have $\mu(\mathscr{F}) = \mu(T_X)$ and $n = 3r$. $\square$

Proposition 4.10 (For $p = 2$)**.** *Suppose $p = 2$ and $n > 4$. If T_X is not stable, the for any $\mathscr{F}$ as in Proposition 4.3 we have $\mu(F) < 1$.*

Proof. We need several conclusions on surfaces:

- **Lemma A.** Let $W \subset \mathbb{P}^{n-1}$ be an irreducible surface with $n > 4$, which is not necessarily smooth. Suppose there exists a line l in P^{n-1} so that the tangent spaces to W at all generic points of W contain l. Then W is a plane.
- **Lemma B.** Let S be a normal projective surface. Suppose for a generic point $s \in S$, there exists a family $\mathscr{D}_s$ of rational curves through s, parametrized by a complete curve Λ_s, so that each member of the family is irreducible and reduced as a cycle. Then $S \cong \mathbb{P}^2$.

For the proof see also Lemma 1,2 in [35].

By Corollary 4.5 for $\mathscr{F} \subset T_X$ in Proposition 4.3 we have $\mu(\mathscr{F}) \leq 1$. If $\mu(\mathscr{F}) = 1 > \frac{4}{n} = \mu(T_X)$, we see that the only possible splitting type of $\mathscr{F}$ on a generic member C is $\mathscr{O}(1) \oplus \mathscr{O}(1)$ because the splitting type of $T_X|_C$ and $T_X/\mathscr{F}$ is locally free on C. By Lemma A and Theorem 4.7, $\mathcal{C}_x$ for generic x is a finite union of planes intersecting along the line $\mathbf{P}\mathscr{F}_x$.

By this observation, consider

$$\begin{tikzcd} \mathbb{P}(\Omega_X) & \mathcal{U} \arrow[l, dashed, "\Phi"'] \arrow[r, "\phi"] \arrow[d, "\psi"] & X \\ & \mathcal{K} & \end{tikzcd}$$

where ψ is the universal family with cycle map ϕ and tangent map Φ. One can show that $\psi' : \Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)) \to \mathcal{K}' := \psi(\Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)))$ is a 1-dimensional fibration and $\mathcal{K}' \subset \mathcal{K}$ is codimension 1.

Let $C \subset X$ be the image of a generic fiber of ψ' under ϕ. For a smooth point $y \in C$, let $z \in \Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee))$ be its inverse image under ϕ. Then by the definition of the tangent map, the fibers of ψ' correspond to curves in X tangent to the meromorphic foliation $\mathscr{F}$.

From the minimality of $\mathcal{K}$ and the fact that Φ_x is generically finite on each component of $\mathcal{U}_x$ for a generic x, while $\mathbb{P}(\mathscr{F}_x^\vee)$ is ample on each component of $\mathcal{C}_x$ for a generic x, we can choose a generic point x so that each curve corresponding to a point of $\mathcal{K}_x = \psi(\mathcal{U}_x) = \psi(\phi^{-1}(x))$ is reduced and irreducible and $\mathcal{K}' := \mathcal{K}_x \cap \psi(\Phi^{-1}(\mathbb{P}(\mathscr{F}^\vee)))$ consists of 1-dimensional components, and there exists at least one component of $\mathcal{K}'_x$ for each component of $\mathcal{K}_x$.

Let S' be the closure of the $\mathscr{F}$-leaf through x. The 1-dimensional families of curves corresponding to $\mathcal{K}'_x$ lie on the $\mathscr{F}$-leaf through x and their tangents span $\mathscr{F}$ at x. Thus S' is the closure of the union of curves corresponding to $\mathcal{K}'_x$ and is an algebraic surface. For each generic point $s \in S'$, S' is the closure of the $\mathscr{F}$-leaf through s. The families of curves corresponding to $\mathcal{K}'_s$ consist of irreducible and reduced cycles. By Lemma B, the normalization S of S' is $\mathbb{P}^2$. Thus $\mathcal{K}'_x$ is just the set of lines through a generic point on $\mathbb{P}^2$, and is irreducible for a generic choice of x. Hence $\mathcal{K}_x$ and hence $\mathcal{U}_x$ and $\mathcal{C}_x$ are irreducible.

Since $\mathcal{C}_x$ is irreducible, the collection of $\mathcal{C}_x$ in $\mathbb{P}(\Omega^1_{X,x})$ at generic x, defines a meromorphic distribution $\mathscr{F}'$ of rank 3. For a generic member C we have $\mathscr{F}'|_C = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus 2}$ and $T_X/\mathscr{F}'|_C = \mathscr{O}^{\oplus n-3}$. This implies that $\mathscr{F}'$ is integrable. A contradiction to Proposition 4.4. $\square$

Lemma 4.11 (Reid, 1977)**.** *Let X be a Fano manifold of dimension n. Let $\mathscr{G} \subset T_X$ be a proper reflexive subsheaf. Then $c_1(\mathscr{G}) < c_1(X)$. In particular, T_X is stable if* $\mathsf{Index}(X) = 1$.

Proof. Pick such $\mathscr{G} \subset T_X$ of rank $p < n$. If $c_1(\mathscr{G}) \geq c_1(X)$, then we have a nonzero $\det \mathscr{G} \to \bigwedge^p T_X$. Hence

$$0 \neq H^0(X, \bigwedge^p T_X \otimes \det G^\vee) = H^0(X, \Omega_X^{n-p} \otimes \det T_X \otimes \det G^\vee).$$

If $c_1(\mathscr{G}) > c_1(X)$, then this is impossible by Kodaira-Nakano vanishing theorem. If $c_1(\mathscr{G}) = c_1(X)$, then by Hodge symmetry this is impossible by Kodaira vanishing theorem as $H^{n-p}(X, \mathscr{O}_X) = 0$. □

Theorem 4.12. *Fano* 5*-folds with Picard number* 1 *have stable tangent bundles.*

Proof. For $p = 0, 1, 4$, the result follows from Theorem 4.6 and Proposition 4.9. If $p = 3$, the index of X is either 5 or 1. If the index is 5, X is a hyperquadric by Theorem 1.68(d). If the index is 1, done by Lemma 4.11. If $p = 2$, and T_X is not stable, choose $\mathscr{F}$ with $1 \geq \mu(\mathscr{F}) \geq \frac{4}{5} = \mu(T_X)$. Since $\mu(\mathscr{F})$ is a rational number with denominator $2, 3$, or 4, we get $\mu(\mathscr{F}) = 1$, a contradiction by Proposition 4.10. □

Theorem 4.13. *Fano* 6*-folds with Picard number* 1 *have semi-stable tangent bundles.*

Proof. If $p = 0, 1, 4, 5$, the result follows from Theorem 4.6 and Proposition 4.9. If $p = 3$, X has index 5 or 1. If the index is 1, done by Lemma 4.11. If the index is 5, done by [84] Theorem 3(1).

If $p = 2$ and T_X is not semi-stable, choose $\mathscr{F}$ as in Proposition 4.3 and $1 \geq \mu(\mathscr{F}) > \frac{4}{6} = \mu(T_X)$. Hence we have $\mu(\mathscr{F}) = 1, \frac{4}{5}, \frac{3}{4}$. But $\mu(\mathscr{F}) = 1$ is not possible by Proposition 4.10. The case $\mu(\mathscr{F}) = \frac{4}{5}$ implies that $\mathscr{F}|_C = \mathscr{O}(1)^{\oplus 4} \oplus \mathscr{O}$, violating the locally freeness of $T_X/\mathscr{F}|_C$. The same contradiction for $\mu(\mathscr{F}) = \frac{3}{4}$. □

4.1.3 For Hecke Curves on Moduli Space of Bundles on Curves

We will follows the paper [36]. For a smooth projective curve C of genus g. Consider the moduli space $M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of degree d. If d is odd (we will assume d odd in whole section), then $M_{2;\mathscr{D},d}(C)$ is a $(3g-3)$-dimensional Fano manifold of Picard number 1 (it is prime). Moreover $M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},1}(C)$ in this case. In particular, when $g = 2$ the space $M_{2;\mathscr{D},1}(C)$ is a intersection of two quadrics in $\mathbb{P}^5$.

Proposition 4.14. *Let $g \geq 3$. For a general $[W] \in X$ and tangent morphism $\tau_{[W]} : \mathcal{K}_{[W]} \to \mathbb{P}(\Omega^1_{X,[W]})$ which is given by the linear system $2\pi^* K_C - K_{\mathbf{P}_C(W)}$ (by Proposition 3.28). Given any linear subspace $\mathbb{P}(F^\vee) \subset \mathbb{P}(\Omega^1_{X,[W]})$ of dimension $r-1$, its intersection with the projective tangent space at a generic point of $\mathcal{C}_{[W]}$ is either empty or has dimension smaller than $(4r/(3g-3)) - 1$.*

Proof. Let such $\mathbb{P}(F^\vee) \subset \mathbb{P}(\Omega^1_{X,[W]})$ of dimension $r - 1$ we have

$$\dim(\mathbb{P}(F^\vee) \cap \mathbb{P}((T_{[W]}X^+_\alpha)^\vee)) \geq \frac{4r}{3g-3} - 1$$

for generic $\alpha \in \mathcal{C}_{[W]}$.

Since the surface $\mathcal{C}_{[W]}$ is nondegenerate in $\mathbb{P}(\Omega^1_{X,[W]})$ (see Proposition 3.28(b)), the intersection can have dimension 0 or 1. If the intersection has dimension 1, then the projection from $\mathbb{P}(F^\vee)$ sends the tangent space at a generic point of $\mathcal{C}_{[W]}$ to zero. Thus the projection sends $\mathcal{C}_{[W]}$ to a point. This implies that $\mathcal{C}_{[W]}$ is contained in some linear subspace containing $\mathbb{P}(F^\vee)$, a contradiction to the nondegeneracy of $\mathcal{C}_{[W]}$. It follows that the intersection has dimension 0 and $r \leq \frac{3}{4}(g-1)$. Moreover, the projection from $\mathbb{P}(F^\vee)$ projects $\mathcal{C}_{[W]}$ to a curve $\ell \subset \mathbb{P}^{3g-4-r}$.

Suppose the $\tau_{[W]}$-image of a generic fiber of $\pi : \mathbf{P}_C(W) \to C$ is dominant over ℓ. Since the image of this fiber under $\tau_{[W]}$ is of degree less than or equal to 2, ℓ must be contained in a plane. This implies that $\mathcal{C}_{[W]}$ is contained in some $\mathbb{P}^{r+2}$ containing $\mathbb{P}(F^\vee)$, a contradiction to the nondegeneracy of $\mathcal{C}_{[W]}$ again. Thus the projection to $\mathbb{P}^{3g-4-r}$ contracts generic fibers of π to a point. It follows that the $\tau_{[W]}$-image of a generic fiber of π is contained in some linear subspace $\mathbb{P}^r$ containing $\mathbb{P}(F^\vee)$ as a hyperplane, and it intersects $\mathbb{P}(F^\vee)$.

Let $\Xi \subset |2\pi^*K_C - K_{\mathbf{P}_C(W)}|$ be the subsystem of dimension $3g-4-r$ defining the projection of $\mathbf{P}_C(W)$ to $\mathbb{P}^{3g-4-r}$ from $\mathbb{P}(F^\vee)$. Let $D \subset \mathbf{P}_C(W)$ be the base locus of Ξ. Hence D corresponds to the intersection of $\mathcal{C}_{[W]}$ with $\mathbb{P}(F^\vee)$. Hence generic fibers of $\pi : \mathbf{P}_C(W) \to C$ intersect D twice, counting multiplicity. Using the notation of [31] for ruled surface, we have $D \sim_{\text{num}} 2C_0 + df$ and $2\pi^*K_C - K_{\mathbf{P}_C(W)} \sim_{\text{num}} 2C_0 + (2g-2+e)f$. Thus the moving part of the system Ξ is just the pullback of a linear system on X of degree $2gg-2+e-d$. By Nagata's result about the intersection number of ruled surface in [74], we have $0 < C_0^2 = -e \leq g$. Since C_0 is ample by [31] Proposition 2.21, we have $D \cdot C_0 > 0$ and $-2e + d > 0$. So Ξ is the pullback of a linear system of degree less than or equal to $3g-3$. By the Riemann-Roch theorem and Clifford's theorem (see [31] page 343), we have $\dim \Xi \leq \max((3/2)(g-1), 2g-3) = 2g-3$. Combined with $\dim \Xi = 3g-4-r$, we get $g \leq r+1$, a contradiction to $r \leq (3/4)(g-1)$. □

Theorem 4.15. *Let the moduli space $X := M_{2;\mathscr{D},d}(C) \cong M_{2;\mathscr{D},d}(C)$ of stable bundles of rank 2 with fixed determinant $\mathscr{D}$ of odd degree d over a smooth projective curve C of genus g. If $g \geq 2$, then T_X is stable.*

Proof. For $g = 2$, this can be directly deduced by Proposition 3.25 and Corollary 3.26. For $g \geq 3$, this follows directly from Theorem 4.7 and Proposition 4.14. □

4.1.4 Need to add

4.2 Rigidity of Generically Finite Morphisms

4.2.1 Varieties of Distinguished Tangents

Here we will the inverse image of minimal rational curves and hence need to construct the non-rational things. Here we will follows [37], see also Section 1 in [43].

Definition 4.16 (h-stratification)**.** *For a morphism $h : M \to Z$ of quasi-projective varieties, the h-stratification of M is a decomposition $M = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_k$ induced by h such that*

- ***(h1)*** *Each M_i is smooth and $h(M_i)$ is also smooth.*
- ***(h2)*** *For any tangent vector v to $h(M_i)$, we can find a local holomorphic arc in M_i whose image under h tangent to v.*
- ***(h3)*** *When a connected Lie group acts on M and Z and h equivariant, then each M_i is invariant under the group action.*

Proposition 4.17. *This h-stratification can always be constructed.*

Proof. Repeatedly using the usual stratification of a variety into smooth and singular locus, we can get **(h1)**. We can stratify each stratum further by the rank of the restriction of h to the stratum to achieve the condition **(h2)**. But after this new stratification, **(h1)** may be violated. Then we apply the singular loci stratification to each stratum again. After finitely many steps of applying these two stratifying procedures, we end up with the stratification satisfying both **(h1)** and **(h2)**. Since this procedure is canonical, **(h3)** is automatic. □

Definition 4.18 (Varieties of Distinguished Tangents)**.** *Given a smooth projective variety Y and a point $y \in Y$. Consider $\mathcal{N} \subset \mathsf{Chow}^1(y, Y)$ be an irreducible component of Chow schemes of curves passing through $y \in Y$. $\mathcal{N}' \subset \mathcal{N}$ be the open subscheme consist of curves smooth at y.*

Consider $\mathcal{N}'_{\mathrm{red}} = \coprod_i N^i$ correspond to geometric genus. Pick a N^j and tangent morphism $\Phi : N^j \to \mathbb{P}(\Omega^1_{Y,y})$. Using Φ-stratification as above we have $N^j = M^j_1 \sqcup M^j_2 \sqcup \cdots \sqcup M^j_k$. We define $\overline{\Phi(M^j_i)}$ be a variety of distinguished tangents for the choice of $\mathcal{N}, N^j$ and M^j_i.

Given a curve $l \subset Y$ smooth at $y \in Y$, there exists a unique variety of distinguished tangents determined by some choice of $\mathcal{N}, N^j$ and M^j_i. We denote it $\mathcal{D}_y(l) \subset \mathbb{P}(\Omega^1_{Y,y})$.

Proposition 4.19. *Given a smooth projective variety Y and a point $y \in Y$. We havethe following properties:*

- ***(d1)*** *There only countably many varieties of distinguished tangents in $\mathbb{P}(\Omega^1_{Y,y})$.*
- ***(d2)*** *Let $\mathcal{D}_y \subset \mathbb{P}(\Omega^1_{Y,y})$ be a variety of distinguished tangents. Then for any tangent vector v to $\mathcal{D}_y$, we can find a family of curves l_t belonging to $\mathcal{N}$ smooth at y so that the derivative of the tangent directions $\mathbb{P}((T_y l_t)^\vee)$ at $t = 0$ is v.*
- ***(d3)*** *Suppose a connected Lie group G acts on Y fixing y. Then any variety of distinguished tangents in $\mathbb{P}(\Omega^1_{Y,y})$ is G-invariant under the isotropy action of G on $\mathbb{P}(\Omega^1_{Y,y})$.*

Proof. We can easy to see that **(d1)** follows from the fact that there are only countably many irreducible components of the Chow scheme. **(d2)** follows from the property **(h2)** of h-stratification. **(d3)** follows from the property **(h3)** of h-stratification. □

Remark 4.20. *The property **(d1)** is the key to the rigidity result we will discuss. **(d2)** is one of the key points of the definition of varieties of distinguished tangents. Unlike the standard minimal rational curves, it is very rare that we have good information on the normal bundle of high genus curves. As a result, their deformation theory can be very tricky. But **(d2)** automatically takes care of obstructions to deformations. **(d3)** is useful in the study of homogeneous spaces.*

Proposition 4.21. *Given a smooth projective variety Y and a point $y \in Y$.*

(a) Let l_z $(z \in Z)$ be a family of curves passing through y parameterized by an irreducible variety Z such that l_z is smooth at y for general z. Let

$$\mathcal{Z} = \overline{\bigcup_{z \in Z\, general} \mathbb{P}((T_y l_z)^\vee)} \subset \mathbb{P}(\Omega^1_{Y,y}).$$

Then $\mathcal{Z} \subset \mathcal{D}_y(l_z)$ for general $z \in Z$.

(b) Let $y \in Y$ be a sufficiently general point. Then

$$\dim \mathcal{D}_y(l) \leq \dim Y - 1 - h^0(\tilde{l}, \mathscr{H}om(\nu^* T_Y / T_{\tilde{l}}, \mathscr{O}_{\tilde{l}}))$$

where $\nu : \tilde{l} \to l$ be the normalization.

Proof. For (a), by definition $\mathcal{Z} \subset \bigcup_{z \in Z} \mathcal{D}_y(l_z)$. Since by **(d1)** this union is countable, then $\mathcal{Z} \subset \mathcal{D}_y(l_z)$ for general $z \in Z$.

For (b), given a tangent vector v to $\mathcal{D}_y(l)$, we can find a deformation $F \to \Delta$ whose fibers are l_t and $l_0 := l$. Let $\widetilde{F} \to F$ be the normalization. Then generic fibers of $\widetilde{F} \to \Delta$ are smooth. Since fibers of it are of constant geometric genus by assumption and of constant arithmetic genus by flatness, all fibers of it are smooth and the normalization map $\widetilde{F} \to F$ gives a family of normalizations $\nu_t : \tilde{l}_t \to l_t$. Then the Kodaira-Spencer class κ of the deformation can be regarded as an element of $H^0(\tilde{l}, \nu^* T_Y / T_{\tilde{l}})$ with $\kappa_y = 0$.

For any $w \in H^0(\tilde{l}, \mathscr{H}om(\nu^* T_Y / T_{\tilde{l}}, \mathscr{O}_{\tilde{l}}))$ the pairing $\langle w, \kappa \rangle$ should be a constant function on $\tilde{l}$ and $d\langle w, \kappa \rangle = 0$. Hence

$$0 = d\langle w, \kappa \rangle (T_y \tilde{l}) = \left\langle dw(T_y \tilde{l}), \kappa_y \right\rangle + \left\langle w_y, d\kappa((T_y \tilde{l})) \right\rangle = \langle w, \kappa \rangle_y .$$

Hence we need have

$$\dim \mathcal{D}_y(l) \leq \dim Y - 1 - h^0(\tilde{l}, \mathscr{H}om(\nu^* T_Y / T_{\tilde{l}}, \mathscr{O}_{\tilde{l}}))$$

where $\nu : \tilde{l} \to l$ be the normalization. □

Remark 4.22. *Note that VMRT is a special case of varieties of distinguished tangents. Here $\dim \mathcal{C}_x = n - 1 - h^0(C, N_C^*) = p$.*

4.2.2 Pull-back of VMRT under Generically Finite Morphisms

Proposition 4.23. *Let $f : Y \to X$ be a generically finite morphism from a projective manifold Y to a Fano manifold X of Picard number 1, different from $\mathbb{P}^n$. For a general $x \in X$ out side the branced locus and let $\mathcal{C}_x$ be VMRT. Then for $y \in f^{-1}(x)$, each irreducible component of $df_y^{-1}(\mathcal{C}_x) \subset \mathbb{P}(\Omega^1_{Y,y})$ is a variety of distinguished tangents where $df_y : T_yY \to T_xX$.*

Proof. For general proof we refer Proposition 3 in [43]. Here we assume all curves are smooth. Pick an irreducible component $A \subset \mathcal{C}_x$, then by Proposition 4.21(a) we have $df_y^{-1}(A) \subset \mathcal{D}_y(l)$ for some curve $l \subset Y$ where $f(l)$ is a general member of A. As $x \in X$ out side the branced locus and $\dim A = p$, then $\dim df_y^{-1}(A) = p$. But by Proposition 4.21(b) we have $\dim \mathcal{D}_y(l) \leq \dim Y - 1 - h^0(l, N_l^*)$. Since $h^0(f(l), N^*_{f(l)}) = n - 1 - p \leq h^0(l, N_l^*)$, we get $\dim \mathcal{D}_y(l) \leq p$. Hence $\dim \mathcal{D}_y(l) = df_y^{-1}(A)$ and well done. □

4.2.3 Rigidity of Generically Finite Morphisms-I

Theorem 4.24. *Let Y be a projective manifold and $\{X_t\}_{t\in\Delta}$ be a family of Fano manifold of Picard number 1 with minimal rational components $\mathcal{K}_t$ such that the Cartan-Fubini type extension theorem 3.47 holds where Δ be a unit disc.*

Then for any family of generically finite morphisms $f_t : Y \to X_t$ there exists a family of biholomorphic morphisms $g_t : X_0 \to X_t$ with $g_0 = \mathrm{id}$ and the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "f_0"] \arrow[dr, "f_t"'] & X_0 \arrow[d, dashed, "\exists g_t"] \\ & X_t \end{tikzcd}$$

Proof. Let U be an analytic open subset of Y such that $f_t|_U$ is biholomorphic for all $t \in \Delta$ and let $U_t := f_t(U)$. For a generic $y \in U$ and let $x_t := f_t(y)$, the components of $df_t^{-1}(\mathcal{C}_{x_t})$ form a family of varieties of distinguished tangents by Proposition 4.23. But by **(d1)** we find that $df_t^{-1}(\mathcal{C}_{x_t}) = df_0^{-1}(\mathcal{C}_{x_0})$ for any $t \in \Delta$. Hence if we define $\phi_t := f_t \circ (f_0|_U)^{-1} : U_0 \to U_t$, then it preserves VMRTs. By Cartan-Fubini type extension theorem 3.47 we find that ϕ_t can extends to a biholomorphic morphism $g_t : X_0 \to X_t$. Well done. □

Remark 4.25. *This theorem is not right for projective spaces.*

A direct corollary:

Corollary 4.26. *For a given projective manifold Y, there are only countably many smooth hypersurface of degree $\leq n-1$ in $\mathbb{P}^{n+1}$ which can be the image of a generically finite morphism from Y.*

Remark 4.27. *By the work of Kobayashi-Ochiai on the varieties of general type, there are finitly many when degree $\geq n+3$. See [55].*

By using semi-positivity of direct images of powers of dualizing sheaves, we can show there are countably many when degree $= n+2$.

4.2.4 Webs, Discriminantal divisors and Their Inverse

Now we need to consider the case $p = 0$. In this case $\dim \mathcal{K}_x = 0$ and the normal bundle of standard minimal rational curves are trivial. Hence we need to discuss the case when the normal bundle are trivial. We will follows [37]. Our definition here is different from that of the original paper [45], but suffices for our purpose here.

Definition 4.28. *Let Y be a smooth projective variety. Let a projective variety $\mathcal{M}$ with finitely many components in the reduction of the Chow scheme of Y is called a* web*, if*

(a) Generic members of each component of $\mathcal{M}$ are curves with only nodal singularities and with trivial normal bundles.

(b) Members of each component of $\mathcal{M}$ cover a Zariski open subset in Y.

Consider the universal family $\mathcal{M} \xleftarrow{\rho} \mathcal{U} \xrightarrow{\mu} Y$. Note that μ is generically finite.

The $\deg \mu$ *is called the* degree of the web *$\mathcal{M}$. As before, we can define the tangent map $\tau : \mathcal{U} \to \mathbb{P}(\Omega^1_Y)$. Let $\mathcal{C} \subset \mathbb{P}(\Omega^1_Y)$ be the closure of the image $\tau(\mathcal{U})$ and $\pi : \mathcal{C} \to Y$ be the natural projection, which is generically finite. An irreducible hypersurface $M \subset Y$ is called a* discriminantal divisor *of the web $\mathcal{M}$ if π is not étale over a generic point of M.*

Proposition 4.29. *For a Fano manifold X of Picard number 1 which has a minimal rational component $\mathcal{K}$ with $p = 0$, the set $\mathbf{H}$ of discriminantal divisors of the web $\mathcal{K}$ is non-empty. Moreover a member of $\mathcal{K}$ intersects $\mathbf{H}$ at least at two distinct points on the normalization $\mathbb{P}^1$.*

Proof. Suppose $\mathbf{H}$ is empty. Then μ (since π) étale outside a set of codimension ≥ 2. By Lemma 3.30 a generic minimal rational curve is disjoint from that set, so its inverse image in $\mathcal{U}$ must have d distinct components from the simply-connectedness of $\mathbb{P}^1$ where d is the degree of $\mathcal{M}$. Thus $\mu : \mathcal{U} \to X$ is a birational morphism by Lemma 3.50. Since μ is unramified in a neighborhood of a generic fiber of $\rho : \mathcal{U} \to \mathcal{K}$, this is a contradiction to the Picard number of X since $\mu(\rho^{-1}(v))$ is disjoint to $\mu(\rho^{-1}(H))$. Now for the last statement, apply the same argument to $\mathbb{A}^1_{\mathbb{C}}$ and then well done. $\square$

Lemma 4.30. *Given a web $\mathcal{M}$ on Y and an irreducible hypersurface $H \subset Y$, a component C of a member of $\mathcal{M}$ passing through a generic point $h \in H$ is either transversal to H at every point of $H \cap C$ or contained in H.*

Proof. Trivial since μ is unramified in a neighborhood of a generic fiber of $\rho : \mathcal{U} \to \mathcal{K}$. □

The following Proposition provides many examples of webs whose members are not necessarily rational curves:

Proposition 4.31. *Let $f : Y' \to Y$ be a generically finite morphism between projective manifolds. Suppose Y has a web $\mathcal{M}$. Then for a generic member C of $\mathcal{M}$ each component of $f^{-1}(C)$ is a curve with nodal singularity whose normal bundle is trivial.*

Proof. A generic member of each component of the web $\mathcal{M}$ intersects the branch locus of f transversally from Lemma 4.30. From this we see that each component of $f^{-1}(C)$ has only nodal singularities. Now the $n-1$ independent sections of the conormal bundle of C can be pulled back to those of components of $f^{-1}(C)$, which gives the triviality of the normal bundle of each component of $f^{-1}(C)$. □

Definition 4.32. *By Proposition 4.31 the components of $f^{-1}(C)$ form a web which is called the* inverse image web *and denote $f^{-1}(\mathcal{M})$.*

Proposition 4.33. *Let $f : Y' \to Y$ be a generically finite morphism between projective manifolds. For a discriminantal divisor $M \subset Y$ of the web $\mathcal{M}$, each component of $f^{-1}(M)$ on which f is generically finite, is a discriminantal divisor of $f^{-1}(\mathcal{M})$.*

Proof. It suffices to show that if a hypersurface H of Y' is not a discriminantal divisor of $f^{-1}(\mathcal{M})$, then $f(H)$ is not a discriminantal divisor of $\mathcal{M}$. We may assume that H is a ramification divisor of f. Let d be the degree of $\mathcal{M}$. Through a general point $h \in H$, there are d distinct curves $C_1, ..., C_d$, belonging to $f^{-1}(\mathcal{M})$ which has d distinct tangent vectors. We claim that at most one of C_i is not contained in H. In fact, if C_1, C_2 are not contained in H, then $f(C_1)$ and $f(C_2)$ are transversal to $f(H)$ by Lemma 4.30. This implies that C_1 and C_2 are tangent to the kernel of df_h. so they are tangent to each other at h, a contradiction. Since $f|_H$ is unramified at h (since it is general), d or $d-1$ members among $C_1, ..., C_d$, which are contained in H, are sent to curves in $f(H)$ with distinct tangents at $f(h)$. Thus $f(C_1), ..., f(C_d)$ have d distinct tangents at $f(h)$. Thus $f(H)$ is not a discriminantal divisor. □

4.2.5 Rigidity of Generically Finite Morphisms-II

Now we will consider the case $p = 0$ proved in [45], using the webs and discriminantal divisors as we discussed above.

Theorem 4.34. *Let Y be a projective manifold and $\{X_t\}_{t\in\Delta}$ be a family of Fano manifold of Picard number 1 with minimal rational components $\mathcal{K}_t$ with $p = 0$ where Δ be a unit disc.*

Then for any family of generically finite morphisms $f_t : Y \to X_t$ there exists a family of biholomorphic morphisms $g_t : X_0 \to X_t$ with $g_0 = \mathrm{id}$ and the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "f_0"] \arrow[dr, "f_t"'] & X_0 \arrow[d, dashed, "\exists g_t"] \\ & X_t \end{tikzcd}$$

Proof. The key point is that the inverse image web $f_t^{-1}(\mathcal{K}_t)$ is independent of t. This is because there are only countably many webs on Y from the countability of the number of components of the Chow scheme. Let M_t be the union of all discriminantal divisors of $\mathcal{K}_t$. Then $f^{-1}(M_t)$ is also independent of t from Proposition 4.33. Fix a general member C of any component of $f^{-1}(\mathcal{K}_t)$. By a general argument, which we will skip, we can reduce the proof to showing that any two points on C which have the same image under f_0 have the same image under f_t for any t. Since $f^{-1}(M_t)$ is independent of t, we know that any two points which are sent to the same point in M_0 are sent to the same point in M_t. But by Proposition 4.29, at least two points of $f_t(C)$ are in M_t. Thus $f_t|_C$ can be regarded as meromorphic functions on the curve C with the same zeroes and poles, and so they are constant multiples of each other, which implies that any two points with the same value of f_0 must have the same value of f_t. $\square$

4.3 Special Remmert-Van de Ven/Lazarsfeld Problem

In this section we will show a special case in [43]. We will discuss the general case for homogeneous Fano manifold of Picard number 1 in further chapters.

Theorem 4.35. *Let X be a smooth projective variety and $\mathsf{Grass}(s, V)$ be a Grassmannian. If $f : \mathsf{Grass}(s, V) \to X$ be a surjective morphism, then either $X \cong \mathbb{P}^n$ or f is an isomorphism.*

Sketch of the proof. WLOG we let $\dim V \geq 2s$ and $s > 1$ since if $s = 1$, then this follows from Corollary 1.80. The tangent space at $[W]$ is naturally isomorphic to $\mathrm{Hom}(W, V/W)$. The isotropy subgroup at $[W]$ is the group of linear automorphisms of V preserving W. Under the action of this group, $\mathbf{P}\,\mathrm{Hom}(W, V/W)$ has orbits $S^1, ..., S^s$ where $S^k \subset \mathrm{Hom}(W, V/W)$ consist of lucs of rank $= k$. The VMRT $\mathcal{C}_{[W]} \subset \mathbf{P}\,\mathrm{Hom}(W, V/W)$ corresponds to S^1 by Proposition 3.24. It is well-known that the closure of S^k is an irreducible subvariety of $\mathbf{P}\,\mathrm{Hom}(W, V/W)$ whose singular locus is precisely the closure of S^{k-1}, for $1 < k < s$, with $S^0 = \emptyset$. Consider the fiber subbundle $\mathscr{S}^k \subset \mathbf{P}T_{\mathsf{Grass}(s,V)}$ whose fiber at $[W]$ is the closure of S^k.

Given a surjective morphism $f : \mathsf{Grass}(s, V) \to X$ with X different from the projective space, let $U \subset X$ be a small connected open set disjoint from the branch locus.

Hence f is finite and let U_1, U_2 be two components of $f^{-1}(U)$ and $\phi : U_1 \to U_2$ be the biholomorphism induced by f. Since X is different from $\mathbb{P}^n$, the VMRT is a proper subvariety of $\mathbb{P}(\Omega^1_{X,x})$ for $x \in U$ by Theorem 3.17(l). Thus $df_y^{-1}(\mathcal{C}_x) = \mathscr{S}^l_y$ for some $l < s$ by Proposition 4.23 because a variety of distinguished tangents must be $\mathscr{S}^k_y$ for some k by **(d3)**. It means that preserves $\mathscr{S}^l$ and hence $\mathscr{S}^1$ because $\mathscr{S}^{k-1}$ is precisely the singular locus of $\mathscr{S}^k$. From the Cartan-Fubini type extension applied to the $\mathsf{Grass}(s, V)$ and ϕ, then ϕ can be extended to an automorphism of $\mathsf{Grass}(s, V)$.

Since U_1, U_2 can be chosen as any components of $f^{-1}(U)$, we see that f is a Galois covering outside the ramification locus. Moreover one can show that an automorphism extending must fix the ramification locus of f pointwise. Thus there exists a finite group G acting on $\mathsf{Grass}(s, V)$ fixing an effective divisor H pointwise. But one can show that if a homogeneous Fano manifold of Picard number 1 has a finite group action fixing a hypersurface pointwise, the Fano manifold must be either the projective space or the hyperquadric and the quotient by the group must be the projective space, a contradiction to the assumption that X is not the projective space! $\square$

Chapter 5

VMRT of Rational Homogeneous Varieties

Now we mainly consider the rational homogeneous varieties of Picard number 1, that is, by Proposition 2.88 the $G/P(I)$ for $\sharp(I) = 1$.

5.1 More Properties of Rational Homogeneous Varieties

Proposition 5.1. *Let $G/P(I)$ be a rational homogeneous variety of Picard number 1, consider the minimal G-equivalence embedding $G/P \hookrightarrow \mathbb{P}(H^0(G/P), \mathscr{L})$ as Corollary 2.94, then it is covered by lines in $\mathbb{P}(H^0(G/P), \mathscr{L})$.*

Proof. For this we refer Theorem V.1.15 in [57]. Note that via the transitive action of G, the line can be cover the whole variety. □

More precisely we have the following:

Let $G/P(I)$ for $I = \{\alpha_p\} \subset S$ the p-th nodes be a rational homogeneous varieties of Picard number 1. Let I' be the set of nodes connected to α_p in the Dynkin diagram.

Theorem 5.2 (Landsberg-Manivel, 2003)**.** *As the previous assumption.*

(a) If α_p is a long node, then $G/P(I')$ is the variety of lines on $G/P(I)$.

(b) If α_p is a short node, then the variety of lines on $G/P(I)$ is irreducible with two G-orbits, and $G/P(I')$ is the closed one.

Proof. We refer [61]. □

5.2 Basic Results of VMRT of Rational Homogeneous Varieties

Theorem 5.3. *Let $X = G/P(r)$ be the rational homogeneous manifold of Picard number one determined by the connected Dynkin diagram D marked at the node r. Assume moreover that r is a node associated to a long root. Then the VMRT of X at every point is a rational homogeneous manifold associated to the Dynkin diagram obtained from D by removing the node r, and marking the nodes connected with r. In particular, it is a product of homogeneous manifolds of Picard number one.*

Proof. See [61] for the proof. □

If the node r is a short root, the VMRT may still be computed, but it is homogeneous only in certain cases:

Proposition 5.4. *With the same notation as in the Theorem, assume that the homogeneous space is one of the following:*

$$B_n/P(n), \quad C_n/P(1), \quad G_2/P(1),$$

then the VMRT of X at every point is isomorphic, respectively, to

$$\mathsf{Grass}(n-1, n+1), \quad \mathbb{P}^{2n-2}, \quad \mathbb{Q}_3.$$

Proof. This follows from Example 2.76 and Theorem 5.3. □

Finally, the remaining cases are not homogeneous, but they have been described in the following way:

Proposition 5.5. *With the same notation as in the Theorem, we have:*

(a) For $C_n/P(r), r = 2, ..., n-1$, then the VMRT is $\mathbb{P}(\mathscr{O}_{\mathbb{P}^{r-1}}(2) \oplus \mathscr{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r})$ embedded by the complete linear system of the tautological bundle $\mathscr{O}(1)$.

(b) For $F_4/P(3)$, then the VMRT is non-trivial smooth $\mathbb{Q}_4$-fibration over $\mathbb{P}^1$.

(c) For $F_4/P(4)$, then the VMRT is smooth hyperplane section of $\mathbb{S}_4$.

Proof. See [61] for the proof. □

Here we will give a complete table of VMRT of rational homogeneous spaces.

Dynkin	node r	$X = G/P(r)$	VMRT	Embeddings
A_n	$\leq n$	$\mathsf{Grass}(r, n+1)$	$\mathbb{P}^{r-1} \times \mathbb{P}^{n-r}$	$\mathscr{O}(1,1)$
B_n	$\leq n-2$	$\mathsf{OGrass}(r, 2n+1)$	$\mathbb{P}^{r-1} \times \mathbb{Q}_{2(n-r)-1}$	$\mathscr{O}(1,1)$
	$n-1$	$\mathsf{OGrass}(n-1, 2n+1)$	$\mathbb{P}^{n-2} \times \mathbb{P}^1$	$\mathscr{O}(1,2)$
	n	$\mathbb{S}_n$	$\mathsf{Grass}(n_1, n+1)$	$\mathscr{O}(1)$
C_n	1	$\mathbb{P}^{2n-1}$	$\mathbb{P}^{2n-2}$	$\mathscr{O}(1)$
	$\leq n-1$	$\mathsf{SGrass}(r, 2n)$	$\mathbb{P}(\mathscr{O}_{\mathbb{P}^{r-1}}(2) \oplus \mathscr{O}_{\mathbb{P}^{r-1}}(1)^{2n-2r})$	
	n	$\mathsf{Lag}(2n)$	$\mathbb{P}^{n-1}$	$\mathscr{O}(2)$
D_n	$\leq n-3$	$\mathsf{OGrass}(r, 2n)$	$\mathbb{P}^{r-1} \times \mathbb{Q}_{2(n-r-1)}$	$\mathscr{O}(1,1)$
	$n-2$	$\mathsf{OGrass}(n-2, 2n)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-3}$	$\mathscr{O}(1,1,1)$
	$n-1, n$	$\mathbb{S}_{n-1}$	$\mathsf{Grass}(n-2, n)$	$\mathscr{O}(1)$
E_k	1	$E_k/P(1)$	$\mathbb{S}_{k-2}$	$\mathscr{O}(1)$
	2	$E_k/P(2)$	$\mathsf{Grass}(3, k)$	$\mathscr{O}(1)$
	3	$E_k/P(3)$	$\mathbb{P}^1 \times \mathsf{Grass}(2, k-1)$	$\mathscr{O}(1,1)$
	4	$E_k/P(4)$	$\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^{k-4}$	$\mathscr{O}(1,1,1)$
	5	$E_k/P(5)$	$\mathsf{Grass}(3,5) \times \mathbb{P}^{k-5}$	$\mathscr{O}(1,1)$
	6	$E_k/P(6)$	$\mathbb{S}_4 \times \mathbb{P}^{k-6}$	$\mathscr{O}(1,1)$
	7	$E_k/P(7)$	$E_6/P(6) \times \mathbb{P}^{k-7}$	$\mathscr{O}(1,1)$
	8	$E_k/P(8)$	$E_7/P(7)$	$\mathscr{O}(1)$
F_4	1	$F_4/P(1)$	SGrass$(3,6)$	$\mathscr{O}(1)$
	2	$F_4/P(2)$	$\mathbb{P}^1 \times \mathbb{P}^2$	$\mathscr{O}(1,2)$
	3	$F_4/P(3)$	Proposition 5.5	
	4	$F_4/P(4)$	Proposition 5.5	
G_2	1	$\mathbb{Q}_5$	$\mathbb{Q}_3$	$\mathscr{O}(1)$
	2	$K(G_2)$	$\mathbb{P}^1$	$\mathscr{O}(3)$

5.3 Determined by VMRT

Theorem 5.6. *Let X be a Fano manifold of Picard number 1. If for some general $x \in X$ the VMRT of it is projectively isomorphic to a VMRT of some rational homogeneous space G/P, then $X \cong G/P$.*

Proof. Need to add.

Note that the case of irreducible Hermitian symmetric spaces and homogeneous contact spaces proved by Mok in [68]. The case of long roots was proved by Hong-Hwang in [33]. The case of short roots of type C_n was proved by Hwang-Li in [41]. The case of short roots of type F_4 was proved by Hwang and others. $\square$

5.4 VMRT of Hermitian Symmetric Spaces

5.5 VMRT of Homogeneous Contact Spaces

Chapter 6

Minimal Sections and Dual VMRT

6.1 The Contact Structure and Symplectic Resolution

Definition 6.1. *A smooth variety M is called a* contact manifold *if it supports a surjective morphism from T_M to a line bundle $\mathscr{L}$, whose kernel is maximally non integrable, and it is called* symplectic *if there exists an everywhere nondegenerate closed to form $\sigma \in H^0(M, \Omega_M^2)$.*

Given a contact form $\theta \in H^0(M, \Omega_M \otimes \mathscr{L})$ on a smooth variety M, the total space $\widehat{M}$ of the line bundle $\mathscr{L}$ is a symplectic manifold. A projective birational morphism $\hat{f} : \widehat{M} \to \widehat{N}$ from a symplectic manifold $\widehat{M}$ to a normal variety $\widehat{N}$ is called a symplectic resolution of $\widehat{N}$. This type of resolutions have been extensively studied by Fu, Kaledin, Verbitsky, Wierzba, and others. We refer the interested reader to [24] and the references there for a survey on this topic.

Come back to our case, let X be a projective manifold (without assumption about T_X), then $\mathcal{X} := \mathbb{P}T_X \overset{\phi}{\to} X$ supports a contact structure $\mathcal{F}$ defined as the kernel of

$$\theta : T_{\mathcal{X}} \overset{d\phi}{\to} \phi^* T_X = \phi^* \phi_* \mathscr{O}(1) \to \mathscr{O}(1),$$

that is, the Levi tensor on the distribution $\mathcal{F}$ defines a symplectic form on $\mathcal{F}_x$ for each x. Globally we have a non-degenerate form induced by the Levi tensor $\mathcal{F} \otimes \mathcal{F} \to T_{\mathcal{X}}/\mathcal{F} \cong \mathscr{O}_{\mathcal{X}}(1)$. In particular we have $\mathcal{F} \cong \mathcal{F}^\vee(1)$. Note that it fits in the following commutative diagram, with exact sequences:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\mathcal{X}/X} \arrow[r] \arrow[d, "="] & \mathcal{F} \arrow[r] \arrow[d] & \Omega_{\mathcal{X}/X}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\mathcal{X}/X} \arrow[r] & T_{\mathcal{X}} \arrow[r] \arrow[d, "\theta"] & \phi^* T_X \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{O}(1) \arrow[r, "="] \arrow[d] & \mathscr{O}(1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

For the verification by the local and we refer Proposition 2.14 in [54] for details. Briefly, around every point with local coordinates $(x_1, ..., x_m)$ and vector fields $(\zeta_1, ..., \zeta_m)$, satisfying $\zeta_i(x_j) = \delta_{ij}$. Then the contact structure is determined by the 1-form $\sum_i \zeta_i dx_i$. (so T_X is big and semiample as before, then we get a symplectic resolution $\widehat{\mathrm{ev}} : \widehat{\mathcal{X}} \to \widehat{\mathcal{Y}}$)

Following Beauville's work [9], the existence of a contact form on $\mathcal{X}$ implies the existence of a symplectic form on $\widehat{\mathcal{X}}$. Locally analytically, the symplectic form induced by θ is the standard symplectic form on the cotangent bundle, given by $\sum_i d\zeta_i \wedge dx_i$.

Theorem 6.2. *Let M be a projective contact manifold such that K_M is not nef. Then, either M is a Fano manifold of Picard number 1 or $M = \mathbb{P}(T_Z)$ for some smooth projective variety Z.*

Proof. We refer [54]. $\square$

We remark that it is conjectured that the only Fano contact manifolds of Picard number one are rational homogeneous: more concretely, minimal nilpotent orbits of the adjoint action of a simple Lie group G on $\mathbb{P}(\mathfrak{g})$.

6.2 Minimal Sections

Definition 6.3. *Let X be a Fano manifold which is not a projective space. For a general minimal standard rational curve $[l]$, a minimal section $\bar{l}$ of $\mathbb{P}(T_X)$ over the curve l is a section which is given by a surjection $f^*T_X \twoheadrightarrow \mathscr{O}_{\mathbb{P}^1}$ (exist since X is not a projective space).*

Proposition 6.4. *Let X be an n-dimensional uniruled projective manifold (not be the projective space) equipped with a dominating component $\mathcal{K}$ of minimal rational curves* *(degree c+2 we assume) and a general standard minimal rational curve $f : \mathbb{P}^1 \to C \subset X$. Let $\mathcal{X} := \mathbb{P}T_X \xrightarrow{\phi} X$. We may consider the irreducible component $\overline{\mathcal{K}} \subset \mathsf{RatCurves}^n(\mathcal{X})$ containing a minimal section $\overline{C}$ of $\mathcal{X}$ over $[C]$ and the corresponding universal family, fitting in a commutative diagram:*

$$\begin{tikzcd} \overline{\mathcal{K}} \arrow[d, "\bar\phi"] & \overline{\mathcal{U}} \arrow[l, "\bar p"'] \arrow[r, "\bar q"] \arrow[d] & \mathcal{X} \arrow[d, "\phi"] \\ \mathcal{K} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd}$$

Now let $\bar f : \mathbb{P}^1 \to \mathcal{X}$ be the normalization of the minimal section $\overline{C}$, then $\overline{\mathcal{K}}$ is smooth at $[\overline{C}]$, of dimension $2n-3$, and for some $e \le c$ we have

$$\bar f^* T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(-1)^{\oplus e} \oplus \mathscr{O}(1)^{\oplus e} \oplus \mathscr{O}^{\oplus 2n-3-2e}.$$

Proof. First, the fibers of $\bar\phi$ over every standard deformation of C are isomorphic to $\mathbb{P}^{n-c-2}$, so $\dim \overline{\mathcal{K}} = 2n-3$.

Next, we have $f^*T_X \cong \mathscr{O}(2)\oplus\mathscr{O}(1)^{\oplus c}\oplus\mathscr{O}^{\oplus n-c-1}$ and by definition of minimal section we have $\bar f^*\mathscr{O}(1) = \mathscr{O}$. By the relative Euler sequence we have

$$\bar f^* T_{\mathcal{X}/X} \cong \mathscr{O}(-2) \oplus \mathscr{O}(-1)^{\oplus c} \oplus \mathscr{O}^{\oplus n-c-2}.$$

By the contact structure $\mathcal{F}$ as we defined in the previous subsection, we have exact $0 \to T_{\mathcal{X}/X} \to \mathcal{F} \to \Omega_{\mathcal{X}/X}(1) \to 0$ which deduce

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-1)^{\oplus c} \oplus \mathscr{O}^{\oplus n-c-2} \to \bar f^*\mathcal{F} \to \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus c} \oplus \mathscr{O}^{\oplus n-c-2} \to 0$$

since $\bar f^*\mathscr{O}(1) = \mathscr{O}$. On the other hand, $\bar f^*\mathscr{O}(1) = \mathscr{O}$ also implies that $d\bar f : T_{\mathbb{P}^1} \to f^*T_X$ factors via $\bar f^*\mathcal{F}$, hence this bundle has a direct summand of the form $\mathscr{O}(2)$. Being $\mathcal{F}$ a contact structure, it follows that $\bar f^*\mathcal{F} \cong \bar f^*\mathcal{F}^\vee$, so this bundle has a direct summand $\mathscr{O}(-2)$, as well. Hence for some $e \le c$ we have

$$\bar f^*\mathcal{F} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(-1)^{\oplus e} \oplus \mathscr{O}(1)^{\oplus e} \oplus \mathscr{O}^{\oplus 2n-4-2e}.$$

Hence $\bar f^*T_{\mathcal{X}}$ has two possible cases: one is $\mathscr{O}(-2)\oplus\mathscr{O}(2)\oplus\mathscr{O}(-1)^{\oplus e}\oplus\mathscr{O}(1)^{\oplus e}\oplus\mathscr{O}^{\oplus 2n-3-2e}$ and another is $\mathscr{O}(2)\oplus\mathscr{O}(-1)^{\oplus e+2}\oplus\mathscr{O}(1)^{\oplus e}\oplus\mathscr{O}^{\oplus 2n-4-2e}$. But the fact that $\dim\overline{\mathcal{K}} = 2n-3$ implies that $h^0(\mathbb{P}^1, \bar f^*T_{\mathcal{X}}) \ge 2n$, which allows us to discard the second option. Finally we have $\overline{\mathcal{K}}$ is smooth at $[\overline{C}]$ since $h^0(\mathbb{P}^1, \bar f^*T_{\mathcal{X}}) = 2n$. Well done. □

6.3 Basic Facts About Dual Varieties

Definition 6.5. *Let $X \subset \mathbb{P}^N$ be a subvariety of dimension n. We define the conormal variety of $X \subset \mathbb{P}^N$ is*

$$\mathsf{Conormal}(X) := \overline{\{(p,H) \in \mathbb{P}^N \times \mathbb{P}^{N,*} : p \in X_{\text{smooth}}, \mathbb{P}(\Omega^1_{X,p}) \subset H\}} \subset \mathbb{P}^N \times \mathbb{P}^{N,*}.$$

The dual variety X^* of X is the image of $\mathsf{Conormal}(X)$ in $\mathbb{P}^{N,*}$.

Proposition 6.6. *Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a subvariety of dimension n. Then* $\mathsf{Conormal}(X)|_{X_{\text{smooth}}} = \mathbb{P}(N_{X_{\text{smooth}}}(-1)) \subset \mathbb{P}V \times \mathbb{P}V^\vee$ *and hence*

$$\mathsf{Conormal}(X) = \overline{\mathbb{P}(N_{X_{\text{smooth}}}(-1))}^{\text{zar}} \subset \mathbb{P}V \times \mathbb{P}V^\vee.$$

In particular, we have also $\mathsf{Conormal}(X)|_{X_{\text{smooth}}} = \mathbb{P}(N_{X_{\text{smooth}}})$ *which is not very canonical to our original definition.*

Proof. Consider the Euler sequence we have

$$0 \to \Omega^1_{\mathbb{P}(V)} \to \mathscr{O} \otimes V(-1) \to \mathscr{O} \to 0.$$

Hence we have a surjection $\mathscr{O} \otimes V^\vee \twoheadrightarrow T_{\mathbb{P}(V)}(-1) \twoheadrightarrow N_{X_{\text{smooth}}}(-1)$ which induce the inclution

$$\mathbb{P}(N_{X_{\text{smooth}}}(-1)) \subset \mathbb{P}(\mathscr{O} \otimes V^\vee) = \mathbb{P}V \times \mathbb{P}V^\vee.$$

By the meaning of the Euler sequence we get the results. □

Definition 6.7. *As above, we define* $\operatorname{def}(X) := N - 1 - \dim X^*$ *is the* **dual defect of** X *and if* $\operatorname{def}(X) > 0$ *we call* X *is* **dual defective**.

For more properties of dual varieties we refer the book [1]. See also Example 3.2.21 in [27] and Section 10.6 in [22].

Theorem 6.8 (Reflexivity)**.** *If $X \subset \mathbb{P}^N$ is any variety and $X^* \subset \mathbb{P}^{N,*}$ its dual, then the conormal variety* $\mathsf{Conormal}(X) \subset \mathbb{P}^N \times \mathbb{P}^{N,*}$ *is equal to* $\mathsf{Conormal}(X^*) \subset \mathbb{P}^{N,*} \times \mathbb{P}^N$ *with the factors reversed. It follows that $X^{**} = X$.*

Proof. See the proof of Theorem 10.20 in [22]. □

6.4 Dual VMRT

Now we consider our main definition and main result. For more things we refer Section 3.A in [25].

Definition 6.9. *Let X be an n-dimensional uniruled projective manifold (not be the projective space) equipped with a dominating component $\mathcal{K}$ of minimal rational curves. Let $\mathcal{X} := \mathbb{P}T_X \xrightarrow{\phi} X$. We may consider the irreducible component $\overline{\mathcal{K}} \subset \mathsf{RatCurves}^n(\mathcal{X})$ containing a minimal section $\overline{C}$ of $\mathcal{X}$ over $[C] \in \mathcal{K}$ and the corresponding universal family, fitting in a commutative diagram:*

$$\begin{tikzcd}
\overline{\mathcal{K}} \arrow[d, "\bar{\phi}"] & \overline{\mathcal{U}} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] \arrow[d] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{K} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X
\end{tikzcd}$$

Then we define the **total dual VMRT of $\mathcal{K}$** *is*

$$\check{\mathcal{C}} := \overline{\bar{q}(\overline{\mathcal{U}})}^{\mathrm{zar}} = \overline{\bigcup_{[l]\in\mathcal{K}\ general} \bar{l}}^{\mathrm{zar}} \subset \mathbb{P}T_X = \mathcal{X}$$

of minimal sections. Define the **dual VMRT $\check{\mathcal{C}}_x$ at general point** *x is the fibre of $\check{\mathcal{C}} \to X$ at $x \in X$.*

Theorem 6.10. *Let X be an n-dimensional uniruled projective manifold equipped with a dominating component $\mathcal{K}$ of minimal rational curves. Let $x \in X$ be a general point. Then $\check{\mathcal{C}}_x$ is the dual variety of $\mathcal{C}_x$, that is, $\check{\mathcal{C}}_x = \mathcal{C}_x^* \in \mathbb{P}(T_xX)$.*

Moreover, let $e = \mathrm{def}(\mathcal{C}_x)$. Then for a minimal section $\overline{C}$ over a general standard rational curve $[C] \in \mathcal{K}_x$ with normalization $\bar{f} : \mathbb{P}^1 \to \overline{C} \subset \mathbb{P}T_X = \mathcal{X}$, we have

$$\bar{f}^*T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(-1)^{\oplus e} \oplus \mathscr{O}(1)^{\oplus e} \oplus \mathscr{O}^{\oplus 2n-3-2e}.$$

Proof. Here we follows the proofs in [72] which is the similar idea as the proof of Proposition 3.10. Let the normalization of C is $f : \mathbb{P}^1 \to C \subset X$. Actually as x and C general, the tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is unramified at $[C]$ by Proposition 3.10. Hence we may use it to identify the tangent space of $\mathcal{C}_x$ at $P := \tau_x([C])$.

Now consider the blow-up $\beta : \mathrm{Bl}_xX \to X$ with exceptional divisor $E = \mathbb{P}(\Omega^1_{X,x})$. Note that we have a filtration $T_xX \supset V_1(f) \supset V_2(f)$ correspond to $f^*T_X \supset \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus c} \supset \mathscr{O}(2)$. Then by previous argument we have $T_P\mathcal{C}_x = V_1(f)/V_2(f)$. By the universal property of blow-up, we have the following evaluation morphisms:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; 0 \mapsto x) \arrow[d, "\mathrm{ev}"] \arrow[dr, "\mathrm{ev}'"] & \\
X & \mathrm{Bl}_xX \arrow[l, "\beta"']
\end{tikzcd}$$

Hence $T_P\mathcal{C}_x = \mathrm{dev}'_{(0,[f])}(\{0\} \times H^0(\mathbb{P}^1, f^*T_X(-1)))/V_2(f)$ and we may identify the space $H^0(\mathbb{P}^1, f^*T_X(-1))$ with the global sections of f^*T_X vanishing at 0. Choosing now a set of local coordinates $(t, t_2, ..., t_m)$ of X around x such that $f(\mathbb{P}^1)$ is given by $t_2 = \cdots = t_m = 0$ and t is a local parameter of $f(\mathbb{P}^1)$, and writing Bl_xX in terms of these coordinates, it's easy to check that, modulo $V_2(f)$, $\mathrm{dev}'_{(0,[f])}$ sends every section s vanishing at 0 to $\frac{ds}{dt}|_{s=0}$ as in 3.10, hence it follows that its image is $V_1(f)$. Hence by the description in Proposition 6.6 we get the result.

For the last statement, by Proposition 6.4 we have

$$\bar{f}^*T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(-1)^{\oplus E} \oplus \mathscr{O}(1)^{\oplus E} \oplus \mathscr{O}^{\oplus 2n-3-2E}$$

for some $E \leq c$. We need to show that $E = e = \mathrm{def}(\mathcal{C}_x)$. Equivalently, we need to show that $\dim \bar{q}(\overline{\mathcal{U}}) = 2n - 2 - f$. Let $H \subset \mathsf{Hom}(\mathbb{P}^1, \mathcal{X})$ be a component containing

$[\bar{f}]$. Consider the rank of the differential of $H \times \mathbb{P}^1 \to \mathcal{X}$ by Theorem 1.35, the result follows then by noting that E equals the dimension of the kernel of the evaluation of global sections $H^0(\mathbb{P}^1, \bar{f}^*T_{\mathcal{X}}) \otimes \mathscr{O} \to \bar{f}^*T_{\mathcal{X}}$. Well done. □

6.5 Positivity and Dual VMRT

Here we follows the paper [25]. First we give some notations and some basic results we will use about the divisorial Zariski decomposition in 2.B in [25].

Definition 6.11. *Let D be a pseudoeffective $\mathbb{R}$-divisor on a projective manifold X. Recall that for a prime divisor Γ on X we can define*

$$\sigma_\Gamma(D) = \lim_{\varepsilon \to 0^+} \inf\{\mathrm{mult}_\Gamma D' : D' \geq 0, D' \sim_{\mathbb{R}} D + \varepsilon A\}$$

where A is any fixed ample divisor. One can show that there are only finitely many prime divisors Γ on X such that $\sigma_\Gamma(D) > 0$. Hence we can define

$$N_\sigma(D) := \sum_\Gamma \sigma_\Gamma(D)\Gamma, \quad P_\sigma(D) := D - N_\sigma(D).$$

The decomposition $D = N_\sigma(D) + P_\sigma(D)$ is called the divisorial Zariski decomposition of D.

Note that $N_\sigma(D)$ is an effective $\mathbb{R}$-Weil divisor and $P_\sigma(D)$ is a movable $\mathbb{R}$-divisor. In particular, for any prime divisor Γ the restriction $P_\sigma(D)|_\Gamma$ is pseudoeffective.

Lemma 6.12. *Let D be a pseudoeffective $\mathbb{R}$-Weil divisor on a projective manifold X. Then*

(a) $\mathrm{supp}(N_\sigma(D))$ *is precisely the divisor $\mathbb{B}^1_-(D)$ which is the union of codimension 1 components of $\mathbb{B}_-(D) = \bigcup_A \mathrm{BaseLocus}(D + A)$ for ample A such that $D + A$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor.*

(b) If D is not movable and $[D]$ generates an extremal ray of $\overline{\mathrm{Eff}}(X)$, then there exists a unique prime divisor $\Gamma \subset X$ such that $[\Gamma] \in \mathbb{R}_{>0}[D]$. Moreover, we have $\Gamma = \mathrm{supp}(N_\sigma(D)) = \mathbb{B}^1_-(D)$.

Proof. See the Lemma 2.4 in [25]. □

Lemma 6.13. *Let X be a projective variety. Let $\mathscr{E}$ be a vector bundle over X and let $\delta \in N^1(X)$ be a $\mathbb{Q}$-Cartier divisor class. Let $\pi : \mathbb{P}\mathscr{E} \to X$.*

(a) The divisor $\mathscr{O}(1) + \pi^*\delta$ is pseudoeffective if and only if for an arbitrary big $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor D on X and an arbitrary $\mathbb{Q}$-Cartier $\mathbb{Q}$-Weil divisor Δ on X such that $[\Delta] = \delta$, there exists an effective $\mathbb{Q}$-Weil divisor N satisfying

$$N \sim_{\mathbb{Q}} \mathscr{O}(1) + \pi^*(\Delta + D).$$

(b) *The divisor $\mathscr{O}(1)+\pi^*\delta$ is big if and only if divisor $\mathscr{O}(1)+\pi^*\delta-\pi^*\gamma$ is pseudoeffective for some big $\mathbb{Q}$-Cartier class γ.*

Proof. In the [25] they called the $\mathbb{Q}$-twisted vector bundle but we will not use these. We refer Proposition 2.7 in [25] for the proof. $\square$

Here is our main result, see Theorem 3.4 in [25]:

Theorem 6.14. *Let X be a Fano manifold of Picard number 1 equipped with a minimal rational component $\mathcal{K}$. Let H be the ample generator and let Λ be the tautological divisor of $\pi : \mathbb{P}\mathscr{E} \to X$. Assume that the VMRT $\mathcal{C}_x \subset \mathbb{P}(\Omega^1_{X,x})$ at a general point $x \in X$ is not dual defective. Denote by a and b the unique integers such that*

$$[\check{\mathcal{C}}] \sim_{\text{num}} a\Lambda - b\pi^*H$$

Then $a = \deg \check{\mathcal{C}}_x$ and the following statements hold.

(a) *T_X is big if and only if $b > 0$.*
(b) *If T_X is big, then $bH \cdot \mathcal{K} \leq 2$ with equality if and only if there exists a minimal section $\overline{C}$ over a general standard rational curve $[C] \in \mathcal{K}$ such that $\check{\mathcal{C}}$ is smooth along $\overline{C}$.*
(c) *If T_X is big, then $[\check{\mathcal{C}}]$ generates an extremal ray of $\overline{\text{Eff}}(\mathbb{P}T_X)$; that is, we have*

$$\overline{\text{Eff}}(\mathbb{P}T_X) = \left\langle [\check{\mathcal{C}}], \pi^*H \right\rangle.$$

Proof. Note that $a = \deg \check{\mathcal{C}}_x$ is trivial by $[\check{\mathcal{C}}] \sim_{\text{num}} a\Lambda - b\pi^*H$ which restrict to the fiber.

For (a), if $b > 0$ then T_X is big by Lemma 6.13(b). Conversely, if T_X is big, consider the pseudoeffective threshold of X is

$$\alpha_X = \alpha(x, H) := \max\{a \in \mathbb{R}_{>0} : \Lambda - a\pi^*H \text{ is pseudoeffective}\}.$$

Note that $\check{\mathcal{C}}$ is dominated by minimal sections $\overline{C}$ over standard rational curves in $\mathcal{K}$ and we have $(\Lambda - \alpha_X\pi^*H)\cdot\overline{C} = -\alpha_X\pi^*H \cdot C < 0$. Hence $(\Lambda - \alpha_X\pi^*H)|_{\check{\mathcal{C}}}$ is not pseudoeffective. In particular, the R-divisor $\Lambda - \alpha_X\pi^*H$ is not movable and the total dual VMRT $\check{\mathcal{C}}$ is contained in the effective Weil divisor $\Gamma := \text{supp}(N_\sigma(\Lambda - \alpha_X\pi^*H)) = \mathbb{B}^1_-(\Lambda - \alpha_X\pi^*H)$ by Lemma 6.12(a). As X has Picard number 1, it follows that $\rho(\mathbb{P}(T_X)) = 2$ and $R := \mathbb{R}_{\geq 0}[\Lambda - \alpha_X\pi^*H]$ is an extremal ray of $\overline{\text{Eff}}(\mathbb{P}T_X)$. Then it follows from Lemma 6.12(b) that Γ is a prime divisor generating the extremal ray R. This yields that $\Gamma = \check{\mathcal{C}}$ and hence $b > 0$.

For (b), consider minimal section $\overline{C}$ of a general standard minimal rational curve $[C] \in \mathcal{K}$ with normalization $\bar{f} : \mathbb{P}^1 \to \mathbb{P}T_X$. We have the following exact sequence

$$\mathscr{I}_{\check{\mathcal{C}}}/\mathscr{I}^2_{\check{\mathcal{C}}} \cong \mathscr{O}_{\check{\mathcal{C}}}(-a\Lambda + b\pi^*H) \to \Omega_{\mathbb{P}T_X}|_{\check{\mathcal{C}}} \to \Omega_{\check{\mathcal{C}}} \to 0.$$

Pull back it via $\bar{f}$ we have

$$\mathscr{O}_{\mathbb{P}^1}(bH \cdot C) \xrightarrow{\iota} \bar{f}^*\Omega_{\mathbb{P}T_X} \to \bar{f}^*\Omega_{\check{\mathcal{C}}} \to 0.$$

As we may assume that $\overline{C}$ is not contained in the singular locus of $\check{\mathcal{C}}$ since it is general, then ι is generically finite. By (a) as $b > 0$, since by Theorem 6.10 we have

$$\bar{f}^*T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}^{\oplus 2n-3},$$

then $bH \cdot \mathcal{K} \leq 2$ with equality if and only if ι is an injection of vector bundles. By Nakayama's lemma, the latter one is equivalent to the smoothness of $\check{\mathcal{C}}$ along $\overline{C}$.

For (c), by the argument in the proof of (a) we have $\mathrm{supp}(N_\sigma(\Lambda - \alpha_X \pi^* H)) = \check{\mathcal{C}}$ and $\check{\mathcal{C}} \in \mathbb{R}_{\geq 0}[\Lambda - \alpha_X \pi^* H]$. Since T_X is big and X has Picard number 1, we have

$$\overline{\mathrm{Eff}}(\mathbb{P}T_X) = \langle \Lambda - \alpha_X \pi^* H, \pi^* H \rangle = \left\langle [\check{\mathcal{C}}], \pi^* H \right\rangle.$$

Well done. $\square$

Chapter 7

About Campana-Peternell Conjecture-I

Here we will follows the survey [72]. An important motivation of the theory of VMRT is the conjecture generalize the Hartshorne conjecture in [12]:

Conjecture 1 (Campana-Peternell Conjecture)**.** *Any Fano manifold whose tangent bundle is nef is rational homogeneous.*

Another version of the same problem is the following.

Conjecture 2. *Let X be a Fano manifold, and assume that T_X is nef. Then T_X is globally generated.*

Note that we have discussed the VMRT in $\mathbb{P}(\Omega_X)$. On the other hand, we may consider the projectivization of the dual bundle, $\mathbb{P}(T_X)$, which we have already introduced to define the nefness of T_X associated to the Campana-Peternell conjecture.

7.1 Basic Facts about Fano Varieties with Nef Tangent Bundle

Here we state some basic facts about the Fano manifolds with nef tangent bundle.

Lemma 7.1. *Let X be a projective manifold with T_X nef. Then any effective divisor D is nef.*

Proof. Not hard. See Proposition 2.12 in [12]. □

Lemma 7.2. *Let X be a Fano manifold with nef tangent bundle and let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism. Then $\mathsf{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$ and, being H the irreducible component of $\mathsf{Hom}(\mathbb{P}^1, X)$ containing $[f]$, the restriction of the evaluation morphism $H \times \mathbb{P}^1 \to X$ is dominant.*

Proof. Since T_X is nef, for any nonconstant morphism $f : \mathbb{P}^1 \to X$ it holds that f^*T_X is globally generated, and in particular $H^1(\mathbb{P}^1, f^*T_X) = 0$. Then $\mathsf{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$. By Theorem 1.35 and then $H \times \mathbb{P}^1 \to X$ is dominant. □

Lemma 7.3. *Let X be a Fano manifold with nef tangent bundle of dimension n. Consider a minimal rational components $\mathcal{K} \subset \mathsf{RatCurves}^n(X)$ of degree d and universal family $\mathcal{U}$, then $\dim \mathcal{K} = n+d-3$ and the cycle map $e : \mathcal{U} \to X$ is smooth with connected fibres of dimension $d-2$.*

Proof. From the fact that T_X is nef, Theorem 1.31 and Corollary 1.36, we know that the cycle map $e : \mathcal{U} \to X$ is smooth with fibres of dimension $d-2$. By Proposition 1.63(b) we know that $e : \mathcal{U} \to X$ has connected fibres. □

Theorem 7.4. *Let X be a Fano manifold with nef tangent bundle, then $\rho(X) \leq \dim X$.*

(a) Any Mori-contraction $\pi : X \to Y$ is a Mori fiber space. Moreover π is smooth, and Y and the fibers of π are also Fano manifolds with nef tangent bundle.

(b) The Mori cone $\mathrm{NE}(X)$ is simplicial, that is, it is generated by linearly independent elements.

(c) For any Mori-contraction $\pi : X \to Y$ and every $y \in Y$ the following properties hold:

(c1) $\rho(\pi^{-1}(y)) = \rho(X) - \rho(Y)$.

(c2) $j_*(\mathrm{NE}(\pi^{-1}(y))) = \mathrm{NE}(X) \cap N_1(\pi^{-1}(y))$ where $j : \pi^{-1}(y) \subset X$.

(d) More generally, let $f : X \to Y$ be a Mori contraction determined by an extremal ray. Let $F \subset Y$ be a projective manifold such that $N_{F/Y} \cong \mathcal{O}_F^{\oplus l}$, then $W := f^{-1}(F)$ is also a Fano manifold with nef tangent bundle.

Proof. We will omit the proof of the smoothness of π in (a) and whole (b)(c). We refer Corollary 3.2, Theorem 3.3 and Proposition 3.7 in [72]. See also the appendix in [90].

For (a), by cone-theorem and Lemma 7.2 one can easy to see that the Mori-contraction $\pi : X \to Y$ is a Mori fiber space. Finally, since π is smooth, via the exact sequences defining the relative tangent bundle, Proposition 1.65 and the normal bundles to the fibers we know that Y and the fibers of π are also Fano manifolds with nef tangent bundle.

For (d), in this case we have $T_{W/F} \cong T_{X/Y}|_W$. So we have

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & 0 \arrow[r] & N_{W/X} \arrow[r] \arrow[u] & f_W^* N_{F/Y} \arrow[u] & \\
0 \arrow[r] & T_{X/Y}|_W \arrow[r] \arrow[u] & T_X|_W \arrow[r] \arrow[u] & f^*(T_Y)|_W = f_W^*(T_Y|_F) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & T_W|_F \arrow[r] \arrow[u] & T_W \arrow[r] \arrow[u] & f_W^* T_F \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

By the snake lemma we have $N_{W/X} \cong f_W^*(N_{F/Y}) \cong \mathscr{O}_W^{\oplus l}$. See the middle column, as $T_X|_W$ nef and $c_1(N_{W/X}) = 0$, then T_W is nef. As $-K_W = (-K_X)|_W$, then W is also Fano. □

Here we introduce some notations we will use.

Situation 1. *Now X is a Fano manifold with nef tangent bundle T_X.*

We will denote by $\phi : \mathbb{P}(T_X) \to X$ the canonical projection, by $\mathscr{O}_{\mathbb{P}(T_X)}(1)$ the corresponding tautological line bundle. In particular we have $\mathscr{O}(-K_{\mathbb{P}(T_X)}) = \mathscr{O}_{\mathbb{P}(T_X)}(\dim X)$. Throughout this ***chapter*** *we will always assume that T_X is not ample, i.e. that X is not a projective space by Hartshorne's conjecture 1.76. This hypothesis allows us to consider the following:*

Let $\rho(X) = n$, by Theorem 7.4(b) we will denote by $R_1, ..., R_n$ the extremal rays of $\mathrm{NE}(X)$*. For every $i = 1, ..., n$ the corresponding elementary contraction will be denoted by $\pi_i : X \to X_i$, and its relative canonical divisor by $K_i := K_{\pi_i}$. We will denote by Γ_i a rational curve of minimal degree such that $[\Gamma_i] \in R_i$, general in the corresponding unsplit family of rational curves $\mathcal{K}_i$ by Corollary 1.45, by $\mathcal{K}_i \overset{p_i}{\leftarrow} \mathcal{U}_i \overset{q_i}{\to} X$ the universal morphisms. Let $\overline{\Gamma}_i$ be the minimal sections of Γ_i and $f_i : \mathbb{P}^1 \to \Gamma_i$ and $\overline{f}_i$ be the normalizations os Γ_i and $\overline{\Gamma}_i$.*

Remark 7.5. *Note that in this situation the function $(-) \cdot \mathscr{O}_{\mathbb{P}(T_X)}(1)$ is a supporting function in sense of Definition II.4.9.3 in [57].*

7.2 Semiampleness of Tangent Bundles

As a general philosophy, if the Campana-Peternell conjecture is true, one should be able to recognize the homogeneous structure of X by looking at the loci of $\mathbb{P}(T_X)$ in which

$\mathscr{O}(1)$ is not ample. The expectancy is that $\mathscr{O}(1)$ is semiample and that those loci appear as the exceptional loci of the associated contraction.

Example 7.6 (For Rational Homogeneous Varieties)**.** *Need to add.*

7.2.1 Basic Facts

Theorem 7.7. *Consider the Situation 1, the Mori cone* $\mathrm{NE}(\mathbb{P}(T_X))$ *is generated by the class of a line in a fiber of* $\phi : \mathbb{P}(T_X) \to X$ *and by the classes of minimal sections* $\overline{\Gamma}_i$*. Moreover, the following are equivalent:*

(a) T_X *is big.*

(b) T_X *is semiample and big.*

(c) There exist an effective $\mathbb{Q}$*-divisor* Δ *satisfying* $\Delta \cdot \overline{\Gamma}_i < 0$ *for all* i*.*

Proof. Consider $\phi_* : N_1(\mathbb{P}(T_X)) \to N_1(X)$. Let $N_0 \subset \mathrm{NE}(\mathbb{P}(T_X))$ be a subcone generated by $\overline{\Gamma}_i$, then ϕ_* induce an isomorphism of N_0 with $\mathrm{NE}(X)$. By the definition of minimal section, elements of N_0 will be killed by $\mathscr{O}_{\mathbb{P}(T_X)}(1)$. As this intersection function is actually a supporting function, then N_0 is the face of $\mathrm{NE}(\mathbb{P}(T_X))$ (see Lemma II.4.10.1 in [57]). Since $\mathrm{NE}(\mathbb{P}(T_X)) \subset (\phi_*)^{-1}\mathrm{NE}(X) \cap \{Z \in N_1(\mathbb{P}(T_X)) : Z \cdot \mathscr{O}_{\mathbb{P}(T_X)}(1) \geq 0\}$, the first claim follows.

The equivalence of (a) and (b) follows from the Basepoint-free theorem. Now consider the equivalence of (a) and (c). Note that $\mathscr{O}_{\mathbb{P}(T_X)}(1)$ is big if and only if $\mathscr{O}_{\mathbb{P}(T_X)}(1)$ lies in the interior of the pseudo-effective cone of $\mathbb{P}(T_X)$ or, equivalently, if and only if for every ample divisor H and sufficiently small $\varepsilon \in \mathbb{Q}_{>0}$, $\Delta = L - \varepsilon H$ is effective. Well done. □

7.2.2 A Birational Contraction

Definition 7.8. *A contraction* $\varepsilon : X \to Y$ *is called*

(a) elementary *if all curves contracted by* ε *are numerically proportional, or equivalently, the relative Picard number is* 1*;*

(b) Mori *if* $-K_X$ *is* ε*-ample;*

(c) crepant *if* $K_X = \varepsilon^* K_Y$ *and* ε *is birational.*

Let X be a smooth projective variety with semiample and big tangent bundle T_X, then consider the evaluation (contraction) morphism

$$\mathrm{ev} : \mathcal{X} := \mathbb{P}T_X \to \mathcal{Y} := \mathrm{Proj} \bigoplus_{r \geq 0} H^0(\mathbb{P}T_X, \mathscr{O}(r)) = \mathrm{Proj} \bigoplus_{r \geq 0} H^0(X, \mathrm{sym}^r T_X)$$

with connected fiber (as a Stein factorization of the linear system) which is birational (since big) and finite type (since semiample).

Alternatively one may consider the total spaces $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$ of the tautological line bundles $\mathscr{O}(1)$ on the Proj-schemes $\mathcal{X}$ and $\mathcal{Y}$, and the natural map:

$$\begin{tikzcd} \widehat{\mathcal{X}} = \underline{\mathrm{Spec}}_{\mathcal{X}} \bigoplus_{r\in\mathbb{Z}} \mathscr{O}_{\mathcal{X}}(r) \arrow[r, "\widehat{\mathrm{ev}}"] \arrow[d, "\mathbb{G}_m"] & \widehat{\mathcal{Y}} = \underline{\mathrm{Spec}}_{\mathcal{Y}} \bigoplus_{r\in\mathbb{Z}} \mathscr{O}_{\mathcal{Y}}(r) \arrow[d, "\mathbb{G}_m"] \\ \mathcal{X} = \mathbb{P}T_X \arrow[r, "\mathrm{ev}"] & \mathcal{Y} = \mathrm{Proj} \bigoplus_{r\geq 0} H^0(X, \mathrm{sym}^r T_X) \end{tikzcd}$$

Lemma 7.9. *In this case,* ev *and* $\widehat{\mathrm{ev}}$ *are crepant contractions, that is, pull back of the canonical divisor is also just the canonical divisor. In particular, their positive dimensional fibers are uniruled.*

Proof. The proof in both cases is analogous and we just consider ev. For instance, we have $R^i\mathrm{ev}_*\mathscr{O}_{\mathcal{X}} = R^i\mathrm{ev}_*(\omega_{\mathcal{X}}\otimes\mathscr{O}(\dim(X))) = 0$ for $i > 0$ by GR vanishing theorem. Then ev is a rational resolution and $\omega_{\mathcal{Y}}$ is a line bundle, isomorphic to $\mathrm{ev}_*\omega_{\mathcal{X}}$ (cf. Section 5.1 in [60]). But then $\omega_{\mathcal{X}} \otimes \mathrm{ev}^*\omega_{\mathcal{Y}}^{-1}$ is effective and vanishes on the $\overline{\Gamma}_i$'s, hence it is numerically proportional to $\mathscr{O}(1)$. Since it is also exceptional, it is trivial.

For the uniruledness of the fibers, we take (by Theorem 7.7) an effective $\mathbb{Q}$-divisor Δ satisfying that (X, Δ) is klt and that $-\Delta$ is ε-ample, and use Theorem 1 in [51]. □

Proposition 7.10. *With the same notation as above, if moreover* ev *is an elementary divisorial contraction, then its exceptional locus is an irreducible divisor D, and any one dimensional fiber consists of either a smooth $\mathbb{P}^1$ or the union of two $\mathbb{P}^1$'s meeting in a point.*

Proof. Omitted and this is a special case of Theorem 1.3 in [100]. □

7.3 The 1-ample Case of Campana-Peternell Conjecture

Let X be a smooth projective variety with tangent bundle T_X, then consider the evaluation (contraction) morphism

$$\mathrm{ev} : \mathcal{X} := \mathbb{P}T_X \to \mathcal{Y} := \mathrm{Proj} \bigoplus_{r\geq 0} H^0(\mathbb{P}T_X, \mathscr{O}(r)) = \mathrm{Proj} \bigoplus_{r\geq 0} H^0(X, \mathrm{sym}^r T_X).$$

Definition 7.11. *We say T_X is k-ample if the dimension of every component of a fiber of* ev *is at most k-dimensional.*

Lemma 7.12. *Let X be a Fano manifold such that T_X is nef, big and k-ample, and let $\pi : X \to X'$ be a Mori contraction. Then $T_{X'}$ is $(k - \dim X + \dim X')$-ample.*

Proof. By Theorem 7.4(a) π is smooth and we have surjection $T_X \to \pi^* T_{X'}$ which induce the inclusion $\mathbb{P}(\pi^* T_{X'}) = \mathbb{P}(T_{X'}) \times_{X'} X \hookrightarrow \mathbb{P}(T_X)$. Hence we have the following factorization:

$$\begin{tikzcd}
\mathbb{P}(T_{X'}) \arrow[r, "\mathrm{ev}'"] & \mathcal{Y}' \arrow[drr] & & \\
\mathbb{P}(\pi^* T_{X'}) \arrow[u] \arrow[r, hook] & \mathbb{P}(T_X) = \mathcal{X} \arrow[rr, "\mathrm{ev}"] & & \mathcal{Y}
\end{tikzcd}$$

As $\mathbb{P}(\pi^* T_{X'}) \to \mathbb{P}(T_{X'})$ has fibers of dimension $\dim X - \dim X'$, then well done. $\square$

In this section we will give a sketch of the proof of the special case of Campana-Peternell conjecture follows the survey [72] and the original paper we refer [90].

Theorem 7.13. *Let X be a Fano manifold such that T_X is nef, big and 1-ample. Then X is rational homogeneous.*

Proof of Theorem 7.13 for Picard number > 1. By Lemma 7.12, any Mori contraction $\pi : X \to X'$ must have one-dimensional fibers and its image must have ample tangent bundle, which is $\mathbb{P}^{\dim X - 1}$ by Hartshorne's conjecture. Therefore, in our situation, X has at least two $\mathbb{P}^1$-fibrations over $\mathbb{P}^{\dim X-1}$. By applying Lemma 2.12 and well done. $\square$

Now we consider the case of Picard number 1. We consider the notations as before: consider the irreducible component $\overline{\mathcal{K}} \subset \mathsf{RatCurves}^n(\mathcal{X})$ containing a minimal section $\overline{C}$ of $\mathcal{X}$ over $[C]$ and the corresponding universal family, fitting in a commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{K}} \arrow[d, "\bar{\phi}"] & \overline{\mathcal{U}} \arrow[l, "\bar{p}"'] \arrow[r, "\bar{q}"] \arrow[d] & \mathcal{X} \arrow[d, "\phi"] \\
\mathcal{K} & \mathcal{U} \arrow[l, "p"'] \arrow[r, "q"] & X
\end{tikzcd}$$

Lemma 7.14. *Let X be a Fano manifold of dimension m such that T_X is nef, big and 1-ample but not ample (hence $X \neq \mathbb{P}^m$).*

(a) Let $\bar{f} : \mathbb{P}^1 \to \mathcal{X}$ be a general minimal section of $\mathcal{X}$ over a general minimal rational curve $f : \mathbb{P}^1 \to X$. Then

$$\bar{f}^* T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}^{\oplus 2m-3} \oplus \mathscr{O}(2)$$

and the exceptional locus of ev *is equal to $\mathcal{D} := \bar{q}(\overline{\mathcal{U}})$.*

(b) Being $\bar{f} : \mathbb{P}^1 \to \mathcal{X}$ be the normalization of any curve $[\overline{\Gamma}]$ of $\overline{\mathcal{K}}$, we have:

$$\bar{f}^* T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(-1)^{\oplus e} \oplus \mathscr{O}(1)^{\oplus e} \oplus \mathscr{O}^{\oplus 2m-2e-3}$$

for some $e \geq 0$. Moreover, the variety $\overline{\mathcal{K}}$ is a smooth projective contact variety with dimension $2m-3$.

(c) *The natural map $\bar{\phi} : \overline{\mathcal{K}} \to \mathcal{K}$ is an isomorphism and $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus m-2} \oplus \mathscr{O}$ for every $[f] \in \mathcal{K}$.*

Proof. For (a), by some arguments as Theorem 4.1 in the original paper [90] we can show that $e = \mathrm{def} = 0$. Hence by Theorem 6.10 we have $\bar{f}^*T_{\mathcal{X}} \cong \mathscr{O}(-2) \oplus \mathscr{O}^{\oplus 2m-3} \oplus \mathscr{O}(2)$. In particular, this implies that $\bar{q}(\overline{\mathcal{U}})$ is an irreducible divisor. By the definition of mimnimal sections, we have $\bar{q}(\overline{\mathcal{U}}) \subset \mathrm{Exc}(\mathrm{ev})$. Since the hypotheses also imply that ev is elementary, then by Proposition 7.10 we have $\bar{q}(\overline{\mathcal{U}}) = \mathrm{Exc}(\mathrm{ev})$.

For (b), let $\mathscr{J}$ be the ideal of $\overline{\Gamma} \subset \mathcal{X}$. By (a) and Proposition 7.10 the curve $\overline{\Gamma}$ is smooth. Moreover, by GR vanishing theorem we have $R^i\mathrm{ev}_*\mathscr{O}_{\mathcal{X}} = 0$ for $i > 0$. Hence push-forward the sequence $0 \to \mathscr{J}^2 \to \mathscr{O}_{\mathcal{X}} \to \mathscr{O}_{\mathcal{X}}/\mathscr{J}^2 \to 0$ we have $R^1\mathrm{ev}_*\mathscr{O}_{\mathcal{X}}/\mathscr{J}^2 = R^2\mathrm{ev}_*\mathscr{J}^2 = 0$ since T_X is 1-ample. Hence we have (Why???) $H^1(\overline{\Gamma}, \mathscr{O}_{\mathcal{X}}/\mathscr{J}^2) = (R^1\mathrm{ev}_*\mathscr{O}_{\mathcal{X}}/\mathscr{J}^2) \otimes \mathscr{O}_p = 0$ where $\{p\} = \mathrm{ev}(\overline{\Gamma})$. Hence by $0 \to N_{\overline{\Gamma}/\mathcal{X}} \to \mathscr{O}_{\mathcal{X}}/\mathscr{J}^2 \to \mathscr{O}_{\overline{\Gamma}} \to 0$ we have $H^1(N_{\overline{\Gamma}/\mathcal{X}}) = 0$ and hence the splitting type of the normal bundle $N_{\overline{\Gamma}/\mathcal{X}}$ does not contain any integer bigger than 1. Consider the contact sheaf $\mathcal{F}$ as before we have the following commutative diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\overline{\Gamma}} \arrow[r] \arrow[d, "="] & \bar{f}^*\mathcal{F} \arrow[r] \arrow[d] & \bar{f}^*\mathcal{F}/T_{\overline{\Gamma}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\overline{\Gamma}} \arrow[r] & \bar{f}^*T_{\mathcal{X}} \arrow[r] \arrow[d] & N_{\overline{\Gamma}/\mathcal{X}} \arrow[r] \arrow[d] & 0 \\
 & & \mathscr{O}_{\overline{\Gamma}} \arrow[r, "="] \arrow[d] & \mathscr{O}_{\overline{\Gamma}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Then well done by the contact structure as $\bar{f}^*\mathcal{F} \cong \bar{f}^*\mathcal{F}^\vee$ as it is contained in the same components with minimal sections.

Now by the same arguments in Proposition 6.4 we know that $\overline{\mathcal{K}}$ is a smooth projective variety with dimension $2m-3$. We omit the proof of the fact that $\overline{\mathcal{K}}$ is contact and we refer Corollary 5.27 in [72] or the original Lemma 3.3 in [90].

For (c), now $\overline{\mathcal{K}}$ and $\mathcal{K}$ are both smooth by (b). The general fiber of $\bar{\phi}$ is a projective space of dimension $m-c-2$, with $c := -K_X \cdot \Gamma - 2$ for $[\Gamma] \in \mathcal{K}$. If $m-c-2 > 0$, then $\bar{\phi}$ would be a Mori contraction of the contact manifold $\overline{\mathcal{K}}$ by (b). By Theorem 6.2 it would follow that $\overline{\mathcal{K}} \cong \mathbb{P}(T_{\mathcal{K}})$ and, in particular, $\dim(\mathcal{K}) = m-1$. Together with the nefness of T_X, this implies that $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}^{\oplus m-1}$ for all $[f] \in \mathcal{K}$. Consider

$$\begin{tikzcd} \mathbb{P}^1 \times \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) & \mathbb{P}^1 \times \mathsf{Hom}^n_{\mathsf{bir}}(\mathbb{P}^1, X) \arrow[l, "="'] \arrow[r] \arrow[dr, "f"'] & \mathcal{U} \arrow[d, "q"] \\ & & X \end{tikzcd}$$

Hence by Theorem 1.35 we find that $\mathcal{U} \to X$ is étale. As X is simply connected, then $\mathcal{U} \cong X$, contradicting that X has Picard number one since we also have $\mathcal{U} \to \mathcal{K}$. Hence $m - c - 2 = 0$ and $\bar{\phi} : \overline{\mathcal{K}} \to \mathcal{K}$ is birational. If $\bar{\phi}$ is not an isomorphism, by birational geometry it would factor via a Mori contraction since $\mathcal{K}$ smooth. This is possible by Theorem 6.2. Hence $\bar{\phi} : \overline{\mathcal{K}} \to \mathcal{K}$ is an isomorphism.

Finally note that, being $\bar{\phi}$ an isomorphism, the number of zeroes appearing in the splitting type of f^*T_X for any $[f] \in \mathcal{K}$ is equal to one. Looking at the general element, which is standard, we obtain that $-K_X \cdot \mathbb{P}^1 = m$ for every $[f] \in \mathcal{K}$. Hence for any $[f] \in \mathcal{K}$ the splitting type of f^*T_X contains no negative elements (since nef), an integer ≥ 2 (since smooth), and at most one zero. Hence the only possibility is $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus m-2} \oplus \mathscr{O}$. Well done. □

Proof of Theorem 7.13 for Picard number 1. In our case we will show that X is a smooth quadric hypersurface. Let $\dim X = m$.

Let $\mathcal{D} := \bar{q}(\overline{\mathcal{U}}) \subset \mathcal{X}$ be the exceptional divisor of ev by Lemma 7.14(a), and L be a divisor associated to the tautological line bundle $\mathscr{O}(1)$ on $\mathcal{X}$, and write $\mathcal{D} = aL - \phi^*B$ for some divisor B on X where $\phi : \mathcal{X} \to X$ as before. For general $x \in X$ the space $\mathcal{D}_x := \phi^{-1}(x) \cap \mathcal{D}$ is the dual VMRT. Since the VMRT $\mathcal{C}_x$ is a hypersurface by Lemma 7.14(c), then its dual cannot be a hyperplane in $\mathbb{P}(T_xX)$ (otherwise $\mathcal{C}_x$ would be a point), and we may write $a > 1$.

By Proposition 7.10 every positive dimensional fiber of ev is either $\mathbb{P}^1$ or a union of two $\mathbb{P}^1$'s meeting at a point. For any irreducible component $\overline{\Gamma}$ in the fiber, we have $\mathcal{D} \cdot \overline{\Gamma} = -2$ in the first case and $\mathcal{D} \cdot \overline{\Gamma} = -1$ in the second case (Why???). Both of them we have $L \cdot \overline{\Gamma} = 0$.

If there is an exceptional fiber in the second case, we have $B \cdot \overline{\Gamma} = 1$. It follows that B is the ample generator of $\mathrm{Pic}(X)$ and $-K_X = mB$, so that X is necessarily a smooth quadric by Theorem 1.68(d).

If all the exceptional fibres are in the first case, then $\mathcal{D} \cdot \overline{\Gamma} = -2$. In this case $\bar{q} : \overline{\mathcal{U}} \to \mathcal{D}$ is a bijective immersion, hence an isomorphism. Since moreover the family $\mathcal{U} \to \mathcal{K}$ is isomorphic to $\overline{\mathcal{U}} \to \overline{\mathcal{K}}$, by Lemma 7.14(c), it allows to identify the restriction $\phi|_{\mathcal{D}} : \mathcal{D} \to X$ with the evaluation morphism $q : \mathcal{U} \to \mathcal{X}$. Hence it is smooth by Lemma 7.3. Hence we have $0 \to \mathscr{O}_{\mathcal{D}}(-\mathcal{D}) \to (\Omega^1_{\mathcal{X}/X})|_{\mathcal{D}} \to T_{\mathcal{D}/X} \to 0$. Hence $c_{m-1}(\Omega^1_{\mathcal{X}/X} \otimes \mathscr{O}_{\mathcal{D}}(\mathcal{D})) = 0$. Now using the relative Euler sequence one can easy to calculate it. Then we have $a = 1$, which is impossible as we argued, or $\mathcal{D} \sim_{\text{num}} aL + \frac{a}{m}\phi^*K_X$.

Since $\mathcal{D} \cdot \overline{\Gamma} = 2$ and $\Gamma \cdot K_X = -m$ we get $a = 2$. Hence $\mathcal{D}$ defines a section in $H^0(\mathcal{X}, 2L + \frac{2}{m}\phi^* K_X) = H^0(X, \mathrm{sym}^2 T_X \otimes \mathscr{O}(\frac{2}{m}K_X))$ which is a nowhere degenerate symmetric form. Now by Lemma 7.16 we find that $X \cong \mathbb{Q}_m$. Well done. $\square$

Lemma 7.15 (Twisted Trivial Bundles)**.** *Let X be Fano manifold of Picard number 1. Let $\mathscr{E} \subset T_X$ is a subbundle such that there exists an integer a such that for any $[f] \in \mathcal{K}$, a unsplit family, we have $f^*\mathscr{E} = \mathscr{O}(a)^{\oplus r}$. Then there exists a (uniquely defined) line bundle L over X such that $\deg f^*L = a$ and $\mathscr{E} \cong L^{\oplus r}$.*

Proof. We omit the proof and we refer Proposition 1.2 in [4]. $\square$

Lemma 7.16 (Ye, 1994)**.** *Let X be Fano manifold of dimension $n \geq 3$ of Picard number 1, with $\mathcal{K}$ an unsplit and dominating family of rational curves. Suppose that T_X is $\mathcal{K}$-uniform, that is, all f^*T_X are the same splitting type for any $[f] \in \mathcal{K}$. If $T_X \cong \Omega_X \otimes L$ for an ample line bundle , then $X \cong \mathbb{Q}_n$.*

Proof. For any standard $f : \mathbb{P}^1 \to C \subset X$, we have $f^*T_X = \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus a} \oplus \mathscr{O}^{\oplus b}$. By $T_X \cong \Omega_X \otimes L$ we get $f^*L = \mathscr{O}(2)$ and $a = n-2$ and $b = 1$.

Let H be the ample generator. Then the class $c_1(H) \in H^1(X, \Omega_X) = \mathrm{Ext}^1(\mathscr{O}_X, \Omega_X)$ correspond to $0 \to \Omega_X \to V' \to \mathscr{O}_X \to 0$ and we get $0 \to L^{-1} \to (V')^\vee \otimes L^{-1} \to \Omega_X \to 0$. By the Kodaira vanishing theorem to L we get $H^1(X, \Omega_X) \cong H^1(X, (V')^\vee \otimes L^{-1}) \cong \mathrm{Ext}^1(L, (V')^\vee)$. Hence $c_1(H)$ can be again associated to a sequence $0 \to (V')^\vee \to V \to L \to 0$. Repeat this process for f^*V for any $[f] \in \mathcal{K}$, we can easy to see that $f^*V \cong \mathscr{O}(1)^{\oplus n+2}$. By Lemma 7.15 we have $V \cong \mathscr{O}(H)^{\oplus n+2}$ where the divisor H has degree 1 on $\mathcal{K}$ and $L = \mathscr{O}(2H)$. So we get $-K_X = nH$. By Theorem 1.68(d) we have $X \cong \mathbb{Q}_n$. $\square$

Remark 7.17. *There is a similar result and method for $\mathbb{P}^n$ and Grassmannians. We refer [101] for details.*

Chapter 8

Fano Manifolds with Non-isomorphic Surjective Endomorphism

There is a interesting conjecture:

Conjecture 3. *Let X be a Fano manifold of Picard number 1 of dimension n. Suppose that X admits a non-isomorphic surjective endomorphism. Then $X \cong \mathbb{P}^n$.*

Recently the Conjecture 3 holds for

(a) Almost homogeneous spaces.

(b) Smooth hypersurfaces of a projective space.

(c) Fano threefolds.

(d) Fano manifolds containing a rational curve with trivial normal bundle.

(e) Fano fourfolds with Fano index $\mathsf{Index} \geq 2$.

(f) Del Pezzo manifolds, i.e., the Fano index $\mathsf{Index} = \dim X - 1$.

See introduction in [87] for the references. See also in [50] for the new method for some cases in arbitrary characteristic.

In this chapter we follows the paper [87] to consider the case when T_X is big.

8.1 The Case with Big Tangent Bundle and VMRTs not Dual Defective

First we recall some situations and notations we will use.

Situation 2. *Let $f : X \to Y$ be a non-isomorphic surjective morphism between Fano manifolds of Picard number 1. Assume that Y is not isomorphic to a projective space. Then X is not isomorphic to a projective space, either by Corollary 1.80.*

(1) We consider the commutative diagram associated with the injection (since $\Omega_{X/Y}$ is torsion and then $\Omega_{X/Y}^{\vee} = T_{X/Y} = 0$) $0 \to T_X \to f^*T_Y$ where Γ is the graph of the induced rational map $\mathbb{P}(f^*T_Y) \dashrightarrow \mathbb{P}(T_X)$:

$$\begin{tikzcd}
 & & \Gamma \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\
\mathbb{P}(T_Y) \arrow[d, "\phi"] & \mathbb{P}(f^*T_Y) \arrow[l, "\tilde{f}"'] \arrow[rr, dashed] \arrow[dr, "\varphi"] & & \mathbb{P}(T_X) \arrow[dl, "\tau"] \\
Y & & X \arrow[ll, "f"'] &
\end{tikzcd}$$

(2) Denote by ξ (resp. η) the tautological line bundle of $\mathbb{P}(T_X)$ (resp. $\mathbb{P}(T_Y)$). Let $\tilde{\eta} := \tilde{f}^*\eta$ which is the tautological line bundle of $\mathbb{P}(f^*T_Y)$.

(3) Let $\mathcal{K}$ (resp. $\mathcal{G}$) be a dominating family of minimal rational curves on X (resp. on Y). Assume that both VMRTs along a general point are not dual defective.

(4) Denote by $\mathcal{D}_X \subset \mathbb{P}(T_X)$ and $\mathcal{D}_Y \subset \mathbb{P}(T_Y)$ the total dual VMRTs of $\mathcal{K}$ and $\mathcal{G}$ respectively. By our assumption, both $\mathcal{D}_X$ and $\mathcal{D}_Y$ are irreducible hypersurfaces.

(5) Let H_X be the ample generator of the Picard group $\mathrm{Pic}(X)$.

First we will collect some basic definitions and facts from symplectic geometry.

Definition 8.1. *Let M be a complex manifold equipped with a closed holomorphic 2-form ω. For a point $z \in M$, let*

$$\mathsf{Null}_z(M) := \{u \in T_zM : \omega(u, v) = 0, \forall v \in T_zM\}.$$

*This defines a distribution, called the **null distribution** on a Zariski open subset of M.*

*Now let (M, ω) be a symplectic manifold equipped with a non-degenerate symplectic 2-form ω. Given an irreducible subvariety $Z \subset M$, we consider the restriction $\omega|_{Z_{\text{smooth}}}$. The rank of the null distribution of $\omega|_{Z_{\text{smooth}}}$ is no more than the codimension $\mathrm{codim}_M Z$ and if the equality holds, then we say that Z is **coisotropic**. The null distribution on Z defines a foliation on a Zariski open subset of Z which we call the **null foliation of ω on Z**.*

Theorem 8.2 (Shao-Zhong, 2023)**.** *Let $f : X \to Y$ be a finite morphism between Fano manifolds of Picard number 1. Let $\mathcal{K}$ and $\mathcal{G}$ be the dominating families of minimal rational curves on X and Y whose VMRTs along a general point are not dual defective.*

Suppose that Y is not isomorphic to a projective space and the induced rational map $\mathbb{P}(T_X) \dashrightarrow \mathbb{P}(T_Y)$ sends the total dual VMRT $\mathcal{D}_X$ to the total dual VMRT of $\mathcal{D}_Y$. Then f is an isomorphism.

Proof. Let $T^*X = \underline{\mathrm{Spec}}_X \mathrm{Sym} T_X$ be the affine cone of $\mathbb{P}T_X$ and consider the rational map

$$\Phi : T^*X \dashrightarrow T^*Y, \quad (s,t) \mapsto (f(s), (df_s^*)^{-1}(t))$$

defined outside the ramification divisor.

Let ω be a natural symplectic form on T^*Y. Now Φ induce a 2-form on T^*X defined by

$$\Phi^*\omega(u,v) = \omega(d\Phi_z u, d\Phi_z v), \quad u, v \in T_z(T^*X).$$

As $d\Phi_z$ is isomorphism, the form $\Phi^*\omega$ is a symplectic form on sub open subset of T^*X. Suppose that C is a general leaf of the null foliation of $\Phi^*\omega$ on $\mathsf{AffineCone}(\mathcal{D}_X) \subset T^*X$. For a general $z \in C$, we have $\Phi^*\omega(u,v) = 0$ for arbitrary $v \in T_zC$ and $u \in T_z\mathsf{AffineCone}(\mathcal{D}_X)$. Consider the image $\Phi(C)$. By our assumption $\mathsf{AffineCone}(\mathcal{D}_X)$ is mapped onto $\mathsf{AffineCone}(\mathcal{D}_Y)$ along the rational map Φ; hence $\Phi(C)$ is contained in $\mathsf{AffineCone}(\mathcal{D}_Y)$. Given any $u' \in T_{\Phi(z)}\mathsf{AffineCone}(\mathcal{D}_Y)$ and $v' \in T_{\Phi(z)}\Phi(C)$ we have

$$\omega(u', v') = \Phi^*\omega(d\Phi_z^{-1}(u'), d\Phi_z^{-1}(v')) = 0.$$

Therefore, $\Phi(C)$ is a leaf of the null foliation of ω on $\mathsf{AffineCone}(\mathcal{D}_Y)$. Hence by [39] Proposition 2.4, both $\mathsf{AffineCone}(\mathcal{D}_X)$ and $\mathsf{AffineCone}(\mathcal{D}_Y)$ are coisotropic (hence $\mathcal{D}_X$ and $\mathcal{D}_Y$) and the closure of C and the closure of $\Phi(C)$ (as in $\mathbb{P}T_*$) are minimal sections over minimal rational curves; moreover, a general minimal section of τ (resp. ϕ) can be realized as the closure of a leaf of the null foliation of $\Phi^*\omega$ (resp. ω) on $\mathcal{D}_X$ (resp. $\mathcal{D}_Y$).

Let $\mathcal{M}_X \subset \mathsf{Chow}^1(\mathbb{P}(T_X))$ and $\mathcal{M}_Y \subset \mathsf{Chow}^1(\mathbb{P}(T_Y))$ be the families of minimal sections of τ and ϕ, respectively. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_X \arrow[r, hook] & \mathsf{Chow}^1(\mathbb{P}(T_X)) \arrow[r, dashed] \arrow[d, "\tau_*"] & \mathsf{Chow}^1(\mathbb{P}(T_Y)) \arrow[d, "\phi_*"] & \mathcal{M}_Y \arrow[l, hook'] \\
\mathcal{K} \arrow[r, hook] & \mathsf{Chow}^1(X) \arrow[r, dashed] & \mathsf{Chow}^1(Y) & \mathcal{G} \arrow[l, hook']
\end{tikzcd}$$

As $\mathcal{M}_X$ is sent to $\mathcal{M}_Y$ via the first horizontal map, we obtain the induced map $\mathcal{K} \dashrightarrow \mathcal{G}$ via the second horizontal map which is also dominant. In particular, f maps a general minimal rational curve $[l] \in \mathcal{K}$ to a general minimal rational curve $[l'] \in \mathcal{G}$. Then for a general point $x \in X$ away from the ramification divisor, there exists a general standard element $[l] \in \mathcal{K}_x$ which is birational to its image $l' := f(l)$, noting that the normal bundle cannot have sections vanishing along two distinct points. Therefore, from the normal bundle sequence, we obtain that

$$K_X \cdot l = K_Y \cdot l' = K_Y \cdot f_*(l) = K_X \cdot l - R \cdot l.$$

Hence $R = 0$ since otherwise R will be ample. Hence f is finite unramified since X is of Picard number 1. By miracle flatness f is finite étale. But by Proposition 1.62 Y is simply connected, hence f is an isomorphism. $\square$

Here is our main theorem:

Theorem 8.3 (Shao-Zhong, 2023). *Let X and Y be the Fano manifolds of Picard number 1. Suppose that the VMRT $\mathcal{C}_x \subset \mathbb{P}(\Omega_{X,x})$ (resp. $\mathcal{C}'_y \subset \mathbb{P}(\Omega_{Y,y})$) at a general point $x \in X$ (resp. $y \in Y$) is not dual defective. Suppose further that the tangent bundle T_X is big. Then any surjective morphism $X \to Y$ has to be an isomorphism unless Y is a projective space; in particular, X admits no non-isomorphic surjective endomorphism unless it is a projective space.*

Proof. We are in Situation 2. As ξ is big, it follows from Theorem 6.14 that $[\mathcal{D}_X] \sim_{\mathrm{num}} a\xi - b\tau^*H_X$ where $a = \deg \check{\mathcal{C}}_x > 0$ and $b > 0$ is an integer; moreover, the total dual VMRT $\mathcal{D}_X$ is extremal in the pseudo-effective cone $\overline{\mathrm{Eff}}(\mathbb{P}T_X) = \langle [\mathcal{D}_X], \tau^*H_X \rangle$. Here, as $\mathbb{P}T_X$ is simply connected, the numerical equivalence of integral Cartier divisors is indeed a linear equivalence by Lemma 8.4(b). Since $\mathcal{D}_X$ is covered by minimal sections $\bar{l} \in \mathbb{P}T_X$ of $\mathcal{K}$ such that $\xi \cdot l = 0$, we have $\mathcal{D}_X \cdot l < 0$. Let $\mathcal{D}'_X := \beta_*(\alpha_*^{-1}\mathcal{D}_X)$ be the proper transform along the birational map $\mathbb{P}(f^*T_X) \dashrightarrow \mathbb{P}T_X$.

Recall the diagram in Situation 2:

$$\begin{tikzcd}
& & \Gamma \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\
\mathbb{P}(T_Y) \arrow[d, "\phi"] & \mathbb{P}(f^*T_Y) \arrow[l, "\tilde{f}"'] \arrow[rr, dashed] \arrow[dr, "\varphi"] & & \mathbb{P}(T_X) \arrow[dl, "\tau"] \\
Y & & X \arrow[ll, "f"'] &
\end{tikzcd}$$

By injection $T_X \hookrightarrow f^*T_X$ we have the injection

$$H^0(X, \mathrm{Sym}^a T_X \otimes \mathscr{O}_X(-bH_X)) \hookrightarrow H^0(X, \mathrm{Sym}^a(f^*T_Y) \otimes \mathscr{O}_X(-bH_X)).$$

Hence since α and β are birational and hence with connected fibres by Zariski main theorem, we have

$$\begin{aligned} &H^0(X, \mathrm{Sym}^a T_X \otimes \mathscr{O}_X(-bH_X)) = H^0(X, \tau_*(a\xi) \otimes \mathscr{O}_X(-bH_X)) \\ &= H^0(\mathbb{P}T_X, a\xi - b\tau^*H_X) = H^0(\Gamma, a\alpha^*\xi - b\alpha^*\tau^*H_X) \end{aligned}$$

and similarly $H^0(X, \mathrm{Sym}^a(f^*T_Y) \otimes \mathscr{O}_X(-bH_X)) = H^0(\Gamma, a\beta^*\tilde{\eta} - b\beta^*\varphi^*H_X)$. Hence we have the injection

$$H^0(\Gamma, a\alpha^*\xi - b\alpha^*\tau^*H_X) \hookrightarrow H^0(\Gamma, a\beta^*\tilde{\eta} - b\beta^*\varphi^*H_X).$$

Hence by linear equivalence $[\mathcal{D}_X] \sim a\xi - b\tau^* H_X$ there exists $m \geq 0$ such that $a\tilde{\eta} - b\varphi^* H_X \sim \mathcal{D}'_X + m\varphi^* H_X$. Hence $\mathcal{D}'_X \sim a\tilde{\eta} - (m+b)\varphi^* H_X$. Since both α and β are birational and $\mathcal{D}_X$ is dominant over X, it follows that $\mathcal{D}'_X$ is a prime divisor.

As the total dual VMRT $\mathcal{D}_Y$ is covered by minimal sections $\bar{c}$ such that $\eta \cdot \bar{c} = 0$, by the projection formula, its pullback $\tilde{f}^* \mathcal{D}_Y$ is covered by curves $\bar{c}'$ such that $\tilde{\eta} \cdot \bar{c}' = 0$. In particular,

$$\bar{c}' \cdot \mathcal{D}'_X = -(m+b)\varphi^* H_X \cdot \bar{c}' < 0.$$

Hence $\bar{c}' \subset \mathcal{D}'_X$ and hence $\tilde{f}^* \mathcal{D}_Y \subset \mathcal{D}'_X$. As $\mathcal{D}'_X$ is a prime divisor, we have $\tilde{f}^* \mathcal{D}_Y = \mathcal{D}'_X$. Hence by Theorem 8.2 and well done. □

Lemma 8.4. *Let X be a smooth projective variety over $k = \bar{k}$.*

(a) (Matsusaka) We have $Z^1_{\mathrm{hom}}(X) = Z^1_{\mathrm{num}}(X)$.

(b) When $k = \mathbb{C}$ and X is simply connected, then

$$Z^1_{\mathrm{hom}}(X) = Z^1_{\mathrm{num}}(X) = Z^1_{\mathrm{rat}}(X).$$

Proof. For (a), this is a famous theorem due to Matsusaka which is a special case of one of the standard conjecture. We refer Appendix A in [82].

For (b), by (a) we just need to prove that $Z^1_{\mathrm{hom}}(X) = Z^1_{\mathrm{rat}}(X)$. Consider the exponential sequence we have

$$H^1(X, \mathscr{O}_X) \to \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X).$$

As X is simply connected, we have $H_1(X, \mathbb{Z}) = 0$. By universal coefficient theorem for cohomology we have $H^1(X, \mathbb{C}) = 0$. By Hodge decomposition we have $H^1(X, \mathscr{O}_X) = 0$. Hence $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is injective. Hence $Z^1_{\mathrm{hom}}(X) = Z^1_{\mathrm{rat}}(X)$. □

8.2 Examples for Rational Homogeneous Varieties

8.3 Applications of Bigness of Tangent Bundle

Lemma 8.5. *Let $f : X \to Y$ be a generically finite surjective morphism between smooth projective varieties. If the tangent bundle T_Y is not big (resp. not pseudo-effective), then T_X is not big (resp. not pseudo-effective) either.*

Proof. We need the following lemma as Theorem 5.13 in [94]:

- **Lemme A.** et $f : V \to W$ be a surjective morphism of complex varieties and D be a Cartier divisor in W, then

$$\kappa(V, f^* D) = \kappa(W, D).$$

Consider the surjective morphism $\mathbb{P}(f^*T_Y) \to \mathbb{P}(T_Y)$ induced by f. Then the tautological line bundles of $\mathbb{P}(f^*T_Y)$ and $\mathbb{P}(T_Y)$ have same Kodaira dimension by Lemma A. Hence, T_Y is big if and only if f^*T_Y is big. Since f is generically surjective and T_X is locally free, there is a natural injection $0 \to T_X \to f^*T_Y$, and thus the non-bigness of T_Y implies the non-bigness of T_X.

Now suppose that T_Y is not pseudo-effective. Let η (resp. $\tilde{\eta}$) be the tautological line bundle of $\mathbb{P}(T_Y)$ (resp. $\mathbb{P}(f^*T_Y)$) and let $\pi : \mathbb{P}(T_Y) \to Y$ and $\tilde{\pi} : \mathbb{P}(f^*T_Y) \to X$ be the natural projections. Let A be any ample divisor on Y. Then $\eta + \frac{1}{n}\pi^*A$ is not $\mathbb{Q}$-effective for any sufficiently large integer n. Hence $\tilde{\eta} + \frac{1}{n}\tilde{\pi}^*f^*A$ is not $\mathbb{Q}$-effective for any sufficiently large integer n by Lemma A. Applying Lemma 2.2 in [34] (a result about pseudo-effective) and the injection $0 \to T_X \to f^*T_Y$, we see that T_X is not pseudo-effective. □

8.3.1 Smooth Complete Intersections

Lemma 8.6. *Let X be a smooth non-linear Fano complete intersection of dimension ≥ 3. Then the VMRT is not dual defective along a general point.*

Proof. □

Theorem 8.7. *Let X be a non-linear smooth complete intersection of multi-degree $\mathbf{d} = (d_1, \cdots, d_k)$ in a projective space. Then the tangent bundle T_X is big if and only if X is a quadric hypersurface. Moreover, suppose that X is very general in its deformation amily. Then T_X is pseudo-effective if and only if $\mathbf{d} = (2)$ or $\mathbf{d} = (2, 2)$.*

Proof. □

8.3.2 Del-Pezzo Manifolds

Need to add.

8.3.3 Gushel-Mukai Manifolds

Need to add.

8.4 More Applications for the Conjecture

We will show the following which is a special case of the Conjecture 3:

Theorem 8.8. *Let X be a Gushel-Mukai manifold of Picard number 1 or a non-linear smooth complete intersection of dimension ≥ 3. Then X admits no non-isomorphic surjective endomorphism.*

Proof. □

8.5 Connections with Bott Vanishing

Here we follows the special case in [50] for smooth projective varieties over $\mathbb{C}$.

Definition 8.9. *An endomorphism $f : X \to X$ is said to be int-amplified if there is an ample Cartier divisor H on X such that $f^*H - H$ is ample.*

Definition 8.10. *Let X be a smooth projective variety over a field. We say that X satisfies Bott vanishing if we have $H^i(X, \Omega_X^j(A)) = 0$ for every $i > 0, j \geq 0$, and A an ample Cartier divisor.*

Here is our main theorem in this section:

Theorem 8.11 (Kawakami-Totaro, 2023)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Suppose that X admits an int-amplified endomorphism, then X satisfies Bott vanishing.*

Proof. Let f be that int-amplified endomorphism, hence f is finite. Let $d = \deg f$. By Tag 0FLB we have a trace map $\mathrm{tr}_f : f_*\Omega_X^j \to \Omega_X^j$ such that $\mathrm{tr}_f \circ f^* = d\mathrm{id}$. Hence $\frac{\mathrm{tr}_f}{d}$ gives a splitting of the pullback $f^* : \Omega_X^j \hookrightarrow f_*\Omega_X^j$. Taking the pushforward by f, we obtain a split injective map $f_*\Omega_X^j \hookrightarrow (f^2)_*\Omega_X^j$. Hence we get a split injective map $\Omega_X^j \hookrightarrow (f^2)_*\Omega_X^j$. Repeat this process we find that for every positive integer e we have the split injective map

$$\Omega_X^j(A) \hookrightarrow ((f^e)_*\Omega_X^j)(A) = (f^e)_*(\Omega_X^j((f^e)^*(A)))$$

where A ample. As f finite and then we have the split injective map

$$H^i(X, \Omega_X^j(A)) \hookrightarrow H^i(X, \Omega_X^j((f^e)^*(A)))$$

for any positive integer e.

Pick an ample divisor H such that $f^*H - H$ is ample. Then there exists $c \in \mathbb{Q}_{>1}$ such that $f^*H - cH$ ample. Hence $(f^e)^*H - c^eH$ nef for all $e \in \mathbb{Z}_{>0}$. Now there exists also rational $u > 0$ such that $A - uH$ ample. Hence $(f^e)^*A - u(f^e)^*H$ nef for all $e \in \mathbb{Z}_{>0}$. Hence $(f^e)^*A - uc^eH$ nef for all $e \in \mathbb{Z}_{>0}$. Now by Fujita vanishing theorem (see Theorem 1.4.35 in [63]), there is $m \in \mathbb{Z}_{>0}$ such that $H^i(X, \Omega_X^j(mH + D)) = 0$ for any nef divisor D. As $c > 1$, pick some e such that $uc^e \geq m$, we find that $(f^e)^*A - mH$ is nef. Hence $H^i(X, \Omega_X^j((f^e)^*A)) = 0$. Finally we get $H^i(X, \Omega_X^j(A)) = 0$ and well done. $\square$

Proposition 8.12. *Let X be a smooth Fano variety over $\mathbb{C}$ satisfies Bott vanishing, then X is locally rigid, that is, $H^1(X, T_X) = 0$. Hence there are only finitely many smooth complex Fano varieties in each dimension admit an int-amplified endomorphism.*

Proof. When $\dim X = 1$, then $X = \mathbb{P}^1$ and well done. When $\dim X > 1$, by Serre duality we have $H^1(X, T_X) = H^{\dim X - 1}(X, \Omega_X \otimes K_X)^\vee = 0$ since X satisfies Bott vanishing. □

Corollary 8.13. *Let X be a smooth projective variety over $\mathbb{C}$ with $\operatorname{Pic}(X) = \mathbb{Z}$.*

(a) If X admits a non-isomorphic surjective endomorphism, then X satisfies Bott vanishing.

(b) If X is Fano and admits a non-isomorphic surjective endomorphism, then X satisfies Bott vanishing. In particular X is locally rigid.

Proof. For (a), if X admits a non-isomorphic surjective endomorphism f, then by Zariski main theorem $\deg f > 1$. Hence $f^*H - H$ is ample since $\operatorname{Pic}(X) = \mathbb{Z}$. Hence f is an int-amplified endomorphism, then X satisfies Bott vanishing by Theorem 8.11.

For (b), by (a) X satisfies Bott vanishing. Hence by Proposition 8.12 that X is locally rigid. □

Chapter 9

Fano Manifolds with Big Automorphism Group

There is a interesting conjecture:

Conjecture 4. *Let X be a Fano manifold of Picard number 1 of dimension n. Then $\dim \mathfrak{aut}(X) \leq n^2+2n$ and with equality if and only if $X \cong \mathbb{P}^n$ where $\mathfrak{aut}(X) = H^0(X, T_X)$ is the Lie algebra of $\underline{\mathrm{Aut}}(X)$.*

Chapter 10

Deformation Rigidity

Chapter 11

Remmert-Van de Ven/Lazarsfeld Problem

We refer [43], [46] and [62].

Theorem 11.1.

Chapter 12

About Campana-Peternell Conjecture-II

Recall the conjecture in [12].

Conjecture 5 (Campana-Peternell Conjecture)**.** *Any Fano manifold whose tangent bundle is nef is rational homogeneous.*

12.1 One of Possible Way to the CP-Conjecture

Here we will follows the survey [72]. To prove the Conjecture 5, in the original paper [12] they find a way:

(1) First prove CP Conjecture for smooth varieties with Picard number one.

(2) Then prove that, given any Fano manifold with nef tangent bundle X and a contraction $f : X \to Y$, from the homogeneity of Y and of the fibers of f one can recover the homogeneity of X.

Unfortunately proving homogeneity in the Picard number one case turned out to be a very hard problem. In [73] they have have the following converse steps:

(A) Prove Conjecture 5 for maximal Picard number.

(B) Prove that any Fano manifold with nef tangent bundle is dominated by one of such manifolds.

Then by the main theorem of [62] we get the result!

12.1.1 Sketch Proof of the First Step

The original paper is [77] and we will follows the proof in Section 6 in the survey [72]. First we define the Fano manifold with nef tangent bundle with maximal Picard number:

Definition 12.1. *We define a variety X is a Flag-Type manifold (FT manifold) if it is a Fano manifold of nef tangent bundle such that elementary contractions are $\mathbb{P}^1$-bundles.*

Furthermore, we introduce the width of a Fano manifold X as a measure of how far X is from being an FT-manifold.

Definition 12.2. *Given a Fano manifold of nef tangent bundle X with Picard number n, consider the Situation 1. We define*

$$\tau(X) := \sum_{i=1}^{n}(-K_X \cdot \Gamma_i - 2) \in \mathbb{Z}_{\geq 0}.$$

Lemma 12.3. *Let X be a Fano manifold of nef tangent bundle such that elementary contractions are $\mathbb{P}^1$-bundles, then $-K_X \cdot \Gamma = 2$ for minimal rational curves in the extremal ray $\Gamma \in R$.*

Proof. Consider the contraction $f : X \to X'$ induced by R, then since $K_X = K_f + f^*K_{X'}$ we get the result as $K_f \cdot \Gamma = -2$. □

Proposition 12.4. *Let X be a Fano manifold of nef tangent bundle with Picard number n and dimension m. Consider the Situation 1.*

(a) If $\tau(X) = 0$, then $\mathcal{K}_i \xleftarrow{p_i} \mathcal{U}_i \xrightarrow{q_i} X$ correspond to Γ_i satisfies that $q_i : \mathcal{U}_i \to X$ is an isomorphism.

(b) If $\tau(X) = 0$, then the elementary contraction correspond to Γ_i is $p_i : X \cong \mathcal{U}_i \to \mathcal{K}_i$ which is a $\mathbb{P}^1$-bundle.

(c) We have X is an FT-manifold if and only if $\tau(X) = 0$.

(d) Let M be a Fano manifold of nef tangent bundle admit a contraction $f : M \to X$ where X is an FT-manifold. Then there exists a smooth variety Y such that $M \cong X \times Y$.

Proof. For (a), if $\tau(X) = 0$, then for any i we have $-K_X \cdot \Gamma_i = 2$. By Lemma 7.3 we know that q_i is smooth. Now $\dim \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X) = m + 2$ and hence by Theorem 1.31 we have $\dim \mathcal{K}_i = \dim \mathsf{RatCurves}^{\mathrm{n}}(X) = m - 1$. Hence $\dim \mathcal{U}_i = m = \dim X$. Hence q_i is étale. As X is Fano we have $q_i : \mathcal{U}_i \to X$ is an isomorphism.

For (b), this is trivial by (a).

For (c), if $\tau(X) = 0$ then X is an FT-manifold by (b). Conversely if X is a FT-manifold, then this follows from Lemma 12.3.

For (d) we refer Proposition 5 in [73]. □

Remark 12.5. *Note that (d) shows that FT-manifold is the "maximal" Fano manifolds of nef tangent bundles.*

Here we fix some more notations:

Situation 3. *See the Situation 1 in the case of FT-manifolds. By Proposition 12.4, the families $p_i : \mathcal{U}_i \to \mathcal{K}_i$ coincide with contractions $\pi_i : X \to X_i$.*

Moreover given any subset $I \subset D := \{1, ..., n\}$, the rays $R_i, i \in I$, span an extremal face by Theorem 7.4(b), that we will denote by R_I. We will denote by $\pi_I : X \to X_I$ the corresponding extremal contraction, by $T_I := T_{\pi_I}$ the relative tangent bundle, and by $K_I := -\det T_I$ the relative canonical divisor. Alternatively, we will denote by $\pi^I : X \to X^I$ the contraction of the face R^I spanned by the rays R_i such that $i \in D \backslash I$. For $I \subset J \subset D$ we will denote the contraction of the extremal face $\pi_{I,*}(R_J) \subset N_1(X_I)$ by $\pi_{I,J} : X_I \to X_J$ or by $\pi^{D\backslash I, D\backslash J} : X^{D\backslash I} \to X^{D\backslash J}$.

Bott-Samelson Varieties

We consider the Situation 3 without assume T_X is nef. Let $D = \{1, ..., n\}$ and a sequence $\ell = (l_1, ..., l_r)$ where $l_i \in D$. We define $\ell[s] := (l_1, ..., l_{r-s})$ and $\ell[r] = \emptyset$.

Definition 12.6. *For any sequence $\ell = (l_1, ..., l_r)$ of D we will define smooth varieties $\mathcal{Z}_{\ell[s]}$ for $s = 0, ..., r$ which is called the* Bott-Samelson varieties of X associated with ℓ, *together with morphisms*

$$\mathcal{Z}_{\ell[s+1]} \underset{p_{\ell[s+1]}}{\overset{\sigma_{\ell[s+1]}}{\rightleftarrows}} \mathcal{Z}_{\ell[s]} \xrightarrow{f_{\ell[s]}} X$$

in the following way:

For $s = r$ we let $\mathcal{Z}_{\ell[r]} := X$ and $f_{\ell[r]} = \mathrm{id}$. For $s < r$, if we have defined $\mathcal{Z}_{\ell[s+1]}$ and $f_{\ell[s+1]}$, then consider the cartesian diagram:

$$\begin{tikzcd}
\mathcal{Z}_{\ell[s]} \arrow[rr, "f_{\ell[s]}"] \arrow[d, shift left, "p_{\ell[s+1]}"] & & X \arrow[d, "\pi_{l_{r-s}}"] \\
\mathcal{Z}_{\ell[s+1]} \arrow[u, shift left, "\sigma_{\ell[s+1]}"] \arrow[urr, "f_{\ell[s+1]}"] \arrow[rr, "g_{\ell[s+1]}"'] & & X_{l_{r-s}}
\end{tikzcd}$$

Note that $p_{\ell[s+1]}$ is a $\mathbb{P}^1$-bundle with a section $\sigma_{\ell[s+1]}$ follows from $f_{\ell[s+1]}$; more precisely it is a projectivization of an extension of $\mathscr{O}_{\mathcal{Z}_{\ell[s+1]}}$ by $f^*_{\ell[s+1]}K_{l_{r-s}}$ which is the pullback of relative Euler sequence of $\mathcal{Z}_{\ell[s]}$ via $\sigma_{\ell[s+1]}$.

For $\phi : Y \to X$, we may define $\mathcal{Z}_{\ell[s]}(Y) := \mathcal{Z}_{\ell[s]} \times_X Y$ which are called the Bott-Samelson varieties of X associated with $\phi : Y \to X$ and ℓ. *Note that this is functorial. Note also that $\mathcal{Z}_{\ell[s]}(Y)$ can be constructed as $\mathcal{Z}_{\ell[r]} := Y$ and $f_{\ell[r]} = \phi$ and do the same thing as before. When $Y = \{x\}$ via the inclusion ϕ, we let $Z_{\ell[s]} := \mathcal{Z}_{\ell[s]}(Y)$.*

Here we mainly consider $Z_{\ell[s]}$. We define $\beta_{i(r-i)}$ be the class of the fibers of $p_{\ell[r-i+1]}$ in $N_1(Z_{\ell[r-i]})$. For $s \leq r-i$ we define $\beta_{i(s)}$ be the composition pushforward via $\sigma_{\ell[r-j]}$, $j = i, ..., r-s-1$. By definition we have $f_{\ell[s],*}\beta_{i(s)} = [\Gamma_{l_i}]$. We define $\beta_i := \beta_{i(0)}$.

Hence $\{\beta_i, i = 1, ..., r\}$ be a basis of $N_1(Z_\ell)$ with a dual basis $\{H_i\}$ in $N^1(Z_\ell)$.

Proposition 12.7. *Consider the same situation as before.*

(a) Define

$$N_a := \sum_{i \leq a, l_i = l_a} H_i \in N^1(Z_\ell)$$

with dual elements $\gamma_b \in N_1(Z_\ell)$. Then the Mori cone (resp. the nef cone) of Z_ℓ is a simplicial cone generated by γ_b (resp. N_a).

(b) Let $J := \{i : l_i = l_k$ for some $k > i\}$, then the Stein factorization of $f_\ell : Z_\ell \to X$ is the contraction associated to the extremal face of $\mathrm{NE}(Z_\ell)$ generated by $\{\gamma_i : i \in J\}$.

Proof. See Corollary 3.9, 3.10 in [77]. □

Definition 12.8. *We call ℓ is a* maximal reduced sequence *if it satisfying that $r = m := \dim X$ and $\dim f_{\ell[s]}(Z_{\ell[s]}) = m - s$ for every s (since X is rationally chain connected by curves Γ_j, it is always possible to find a sequence of this kind).*

Cartan Matrix of FT-Manifolds

Consider again the Situation 3.

Definition 12.9. *Let X be an FT-manifold of Picard number n. We define the* Cartan matrix *is $M(X) = (-K_i \cdot \Gamma_j)_{ij}$.*

Theorem 12.10. *Let X be an FT-manifold of Picard number n.*

(a) Let $I \subset D$ which is not empty. Then every fiber of $\pi_I : X \to X_I$ is an FT-manifold whose Cartan matrix is $M_I(X)$, the $I \times I$-principal submatrix of $M(X)$.

(b) We will write $M(X) = M_I(X) \times M_J(X)$ if $M(X)_{ij} = 0$ for any $(i,j) \in (I \times J) \cup (J \times I)$ where $I, J \subset D$ are complementary. Then in this case we have $X \cong X(I) \times X(J)$ where $X(I), X(J)$ are FT-manifolds with Cartan matrices $M_I(X), M_J(X)$, respectively.

(c) The Cartan matrix of X is the Cartan matrix of a semisimple Lie algebra.

Sketch of the proof. For (a) and (b) we refer Proposition 6,7 in [73].

For (c), we use induction on Picard number n. For $n = 2$ this is directly from Lemma 2.12. Hence we may assume that $n \geq 3$ and that the statement holds for FT-manifolds of Picard number $\leq n-1$.

By some Lie theory, see the proof in [72] Proposition 6.13, $M(X)$ is either of finite or of affine type. We let here is in the latter case. In this case, by some Lie theory again there exists a linear combination $\sum_1^n m_i\Gamma_i$ such that $m_i \in \mathbb{R}_{>0}$ and $K_i \cdot \sum_1^n m_i\Gamma_i = 0$ for all i. As $M(X)$ is of integer coefficients, we may assume $m_i \in \mathbb{Z}_{>0}$. By some more theory of rational curves (see [57] Section II.7), $\sum_1^n m_i\Gamma_i$ is smoothable and therefore it is numerically equivalent to an irreducible rational curve Γ.

Let $f : \mathbb{P}^1 \to \Gamma \subset X$ be the normalization morphism, and $p \in \mathbb{P}^1$ be any point. Consider a maximal reduced sequence $\ell = (l_1, ..., l_m)$ of X for the point $f(p)$, and the corresponding Bott-Samelson varieties $Z_{\ell[s]}$ associated to ℓ and $f(p)$. At the same time, consider the associated Bott-Samelson varieties $Z'_{\ell[s]} := \mathscr{Z}_{\ell[s]}(\mathbb{P}^1)$ associated to f. We will denote by $f'_{\ell[s]} : Z'_{\ell[s]} \to X$ their evaluations.

We first claim that $Z'_{\ell[s]} \cong \mathbb{P}^1 \times Z_{\ell[s]}$. Indeed, $Z'_{\ell[s]}$ is the projectivization of a rank two bundle on $Z'_{\ell[s+1]}$, which, by induction, is isomorphic to $\mathbb{P}^1 \times Z_{\ell[s+1]}$, appearing as an extension

$$0 \to \mathscr{O}((f'_{\ell[s]})^* K_{m-s}) \to \mathscr{F}'_{\ell[s]} \to \mathscr{O}_{Z'_{\ell[s]}} \to 0.$$

By the construction of the curve Γ, $(f'_{\ell[s]})^* K_{m-s}$ has intersection zero with the fibers of the projection $p_2 : \mathbb{P}^1 \times Z_{\ell[s+1]} \to Z_{\ell[s+1]}$, then $\mathscr{F}'_{\ell[s]}$ is trivial on these fibers. Hence the sequence is the pullback via p_2 of

$$0 \to \mathscr{O}(f^*_{\ell[s]} K_{m-s}) \to \mathscr{F}_{\ell[s]} \to \mathscr{O}_{Z_{\ell[s]}} \to 0.$$

Hence $Z'_{\ell[s]} \cong \mathbb{P}^1 \times Z_{\ell[s]}$ and we get the claim.

Let $j_{\ell[s]} : Z_{\ell[s]} \to Z'_{\ell[s]} \cong \mathbb{P}^1 \times Z_{\ell[s]}$ as $z \mapsto (p, z)$, then by functorial we have $f'_{\ell[s]} \circ j_{\ell[s]} = f_{\ell[s]}$. Consider

$$\begin{tikzcd}
Z_{\ell[1]} \arrow[d, shift left, "j_{\ell[1]}"] \arrow[drr, "f_{\ell[1]}"] & & \\
Z'_{\ell[1]} \cong \mathbb{P}^1 \times Z_{\ell[1]} \arrow[u, shift left, "p_2"] \arrow[d, "p_1"] \arrow[rr, "f'_{\ell[1]}"'] & & X \arrow[d, "\pi_{l_{m-1}}"] \\
\mathbb{P}^1 & & X_{l_{m-1}}
\end{tikzcd}$$

Since the image of $\pi_{l_{m-1}} \circ f'_{\ell[1]}$ in X is numerically equivalent to Γ, then it does not contract fibers of p_2. By the choice of ℓ it is generically finite when restricted to fibers of p_1. Hence this implies that $\pi_{l_{m-1}} \circ f'_{\ell[1]}$ is generically finite (for details we refer Lemma 7 in [73]). This is impossible since $\dim Z'_{\ell[1]} = \dim X_{l_{m-1}} + 1$. $\square$

Relative Duality and Reflection Groups for General Spaces

We will now present a generalization of the previous result holds for Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles; in other words, it avoids the assumption of the nefness of the tangent bundle of X.

Lemma 12.11. *Let $\pi : M \to Y$ be a $\mathbb{P}^1$-bundle over a smooth manifold Y, denote by Γ one of its fibers and by K its relative canonical divisor. Then for every Cartier divisor D on M, setting $l := D \cdot \Gamma$ and $\operatorname{sgn}(\alpha) := \alpha/|\alpha|$ for $\alpha \neq 0$, $\operatorname{sgn}(0) := 1$.*

(a) *We have:*
$$H^i(M, \mathscr{O}_M(D)) \cong H^{i+\operatorname{sgn}(l+1)}(M, \mathscr{O}_X(D + (l+1)K)).$$
In particular $\chi(M, \mathscr{O}_M(D)) = -\chi(M, \mathscr{O}_M(D + (l+1)K))$.

(b) *We have*
$$(\pi_*\mathscr{O}_M(K - D))^\vee \cong \pi_*(\mathscr{O}_M(D + (l+1)K)).$$

Proof. We refer Lemma 2.3 and Lemma 2.4 in [77]. □

Now let X be a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles. We use the similar notations as before. For every elementary contraction $\pi_i : X \to X_i$ we will consider the linear map $r_i : N^1(X) \to N^1(X)$ given by $r_i(D) = D + (D\cdot\Gamma_i)K_i$, which is a reflection, i.e. it is an involution that fixes the hyperplane $\Gamma_i^\perp := \{D : D \cdot \Gamma_i = 0\}$. Moreover $r_i(K_i) = -K_i$ by Lemma 12.3.

Theorem 12.12. *The group $W \subset \mathrm{GL}(N^1(X))$ generated by the reflections $\{r_i : i = 1, ..., n\}$ is a finite group.*

Proof. Consider the dual action of W on the vector space $N_1(X) = N^1(X)^\vee$, defined by $w^\vee(C) \cdot D = C \cdot w(D)$, for all $D \in N^1(X), C \in N_1(X)$. This action is clearly faithful. Moreover the matrix of every element $r_i^\vee \in \mathrm{GL}(N_1(X))$ with respect to the basis $\{\Gamma_1, ..., \Gamma_n\}$ has integral coefficients and determinant ± 1, hence the same properties hold for the matrices of any $w^\vee \in \mathrm{GL}(N_1(X))$.

Consider the Euler characteristic which is a polynomial of degree $\leq \dim X$ on $N^1(X)_\mathbb{Z}$ which can be extended to $N^1(X)$. Define $T : D \mapsto D + \frac{K_X}{2}$. Let $\chi^T := \chi_X \circ T$, then by Lemma 12.3 and Lemma 12.11 we have $\chi^T(D) = -\chi^T(r_i(D))$ for any r_i and D. Hence $\chi^T(D) = -\chi^T(w(D))$ for all D and $w \in W$. Hence $\chi^T(w(\Gamma_i^\perp)) = -\chi^T(w \circ r_i \circ w^{-1} w(\Gamma_i^\perp)) = -\chi^T(w(\Gamma_i^\perp))$. Hence $\chi^T(w(\Gamma_i^\perp)) = 0$ and then χ^T vanishes on any hyperplane of the form $w(\Gamma_i^\perp)$.

Hence in particular, it follows that the cardinality of this set $Z \subset \mathbb{P}(N^1(X))$ of hyperplanes is smaller than or equal to the degree of χ^T, i.e. the dimension of $\dim X$. Hence finite. Therefore, to show that W is finite, it is enough to consider the induced action of W on Z^n, and show that the isotropy subgroup $W_0 \subset W$ of elements of W

fixing the point $([\Gamma_1], ..., [\Gamma_n])$ is finite. If $w \in W_0$, then the matrix of $w^\vee$ with respect to the basis $\{\Gamma_1, ..., \Gamma_n\}$ is diagonal, hence all its diagonal coefficients are equal to ± 1. In particular the image of W_0 in $\mathrm{GL}(N_1(X))$ is finite and, since the action of W on $N_1(X)$ is faithful, W_0 is finite as well. □

Corollary 12.13. *With the same notation as above, $\{-K_i : i = 1, ..., n\}$ is a basis of $N^1(X)$ and $R := \{w(-K_i) : w \in W, i = 1, ..., n\}$ (which is a finite set by Theorem 12.12) is a root system with Weyl group W. In particular the Cartan matrix $M(X)$, which coincide as above, of a Fano manifold X whose elementary contractions are smooth $\mathbb{P}^1$-fibrations is the Cartan matrix of a semisimple Lie algebra.*

Proof. By Theorem 12.12 directly check, we refer Corollary 2.10 and Proposition 2.13 in [77]. Note that we consider any inner product $(-,-)$ in $N_1(X)$. A new inner product $\langle -,- \rangle$ defined by $\langle x, y \rangle := \sum_{w \in W}(w(x), w(y))$, for all $x, y \in N_1(X)$, is $W^\vee$-invariant. The reflections $r_i^\vee$ are orthogonal with respect to the scalar product $\langle -,- \rangle$. □

Dynkin Diagrams and Homogeneous Models

Now consider an FT-manifold, or more general, a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles X. By the previous two small sections we have the Dynkin diagram associated to X:

Definition 12.14. *The Dynkin diagram $\mathcal{D}(X)$ of X is the graph having $n := \rho(X)$ nodes, such that the nodes in the i-th and j-th position are joined by $(-K_i \cdot \Gamma_j)(-K_j \cdot \Gamma_i)$, which is equal to $= 0, 1, 2$ or 3 edges. When two nodes are joined by multiple edges we write an arrow on them pointing to the node j if $-K_i \cdot \Gamma_j < -K_j \cdot \Gamma_i$. The set of nodes of $\mathcal{D}(X)$ will be identified with $D = 1, ..., n$.*

Let G/B be the complete flag manifold associated to the Dynkin diagram $\mathcal{D}(X)$ where B is the Borel subgroup. We call G/B be the rational homogeneous model *of X. We will use for G/B a similar notation as for X, adding an overline to distinguish the two cases (so we will use $\bar{\pi}_i, \overline{\Gamma}_i, -\overline{K}_i, ...$).*

We will also consider the isomorphism of vector spaces $\psi : N^1(X) \to N^1(G/B)$ defined by $\psi(-Ki) = -\overline{K}_i$. This isomorphism sends $-K_X$ to $-K_{G/B}$.

Proposition 12.15. *Let X be a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles. With the same notation as above, for every line bundle $\mathscr{L}$ we have $h^i(X, \mathscr{L}) = h^i(G/B, \psi(\mathscr{L}))$ for any i. In particular the dimension of X equals the dimension of its homogeneous model G/B.*

Proof. See Corollary 2.25 in [77]. □

Proposition 12.16. *Let X be a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles. Let $\ell = (l_1, ..., l_m)$ is a sequence such that $w(\ell) := r_{l_1} \circ \cdots \circ r_{l_m}$ is a* *reduced expression of the longest element in W, that is, $w(\ell)$ cannot be written as a composition of a smaller number of reflections r_i and for every Weyl group W there is a unique element of maximal length (see [77] section 3 for details). Then the morphism $f_\ell : Z_\ell \to X$ is surjective and birational.*

Sketch of the proof. By the Schubert variety of G/B, we can find that $\dim f_\ell(Z_\ell) = \dim Z_\ell$, see Corollary 3.18 in [77]. As ℓ is of the longest, via the case of G/B and Proposition 12.15 that $\dim X = \dim G/B$, we know that f_ℓ is surjective and of same dimension.

Let $\mathscr{L}$ be an ample line bundle on X and let $\overline{\mathscr{L}} := \psi(\mathscr{L})$. Hence by Lemma 12.18 to show that f_ℓ is birational is enough to show that, for all $s \gg 0$, the restriction map $H^0(X, \mathscr{L}^{\otimes s}) \to H^0(Z_\ell, f_\ell^* \mathscr{L}^{\otimes s})$ which is an injection by the surjectivity of f_ℓ, is an isomorphism. Note that for G/B we have $\bar{f}_\ell$ is birational and $H^0(G/B, \overline{\mathscr{L}}^{\otimes s}) \cong H^0(\overline{Z}_\ell, \bar{f}_\ell^* \overline{\mathscr{L}}^{\otimes s})$ which are the well-known results, see paper [20]. Hence by Proposition 12.15 we just need to show that

$$H^0(Z_\ell, f_\ell^* \mathscr{L}^{\otimes s}) \cong H^0(\overline{Z}_\ell, \bar{f}_\ell^* \overline{\mathscr{L}}^{\otimes s})$$

for all $s \gg 0$. This is an application of Euler characteristic and Kawamata-Viehweg vanishing Theorem. We omit it and we refer Proposition 3.17 in [77]. □

Note that Theorem 12.10(b) holds more generally as we considered:

Corollary 12.17. *Let X be a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles. Assume that $\mathcal{D}(X) = \mathcal{D}_1 \sqcup \mathcal{D}_2$. Then $X \cong X_1 \times X_2$, where X_1 and X_2 are Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles and whose Dynkin diagrams are $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.*

Proof. Actually if we consider the reduced sequence of maximal length ℓ_1, ℓ_2 of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, we have $Z_{\ell_1} \times Z_{\ell_2} \cong Z_{\ell_1 \ell_2}$ and $\ell_1 \ell_2$ is also a reduced sequence of maximal length, see Lemma 3.2 and in [77]. Hence $f_{\ell_1 \ell_2}$ is birational.

By Proposition 12.7(b) $f_{\ell_1 \ell_2}$ is the contraction determined by the extremal face of $\overline{\mathrm{NE}}(Z_{\ell_1 \ell_2})$ generated by $R := \{\gamma_i : l_i = l_k \text{ for some } k > i\}$. Hence this face is the convex hull of the two extremal faces generated by the subsets in ℓ_1 and ℓ_2. This can give us $X \cong X_1 \times X_2$ such that both of them are Fano manifolds whose elementary contractions are $\mathbb{P}^1$-bundles. □

Lemma 12.18. *Let $f : X \to Y$ be a surjective morphism between reduced projective schemes and let $\mathscr{L}$ be an ample invertible sheaf on Y. Assume that $H^0(Y, \mathscr{L}^{\otimes n}) \to H^0(X, f^* \mathscr{L}^{\otimes n})$ is an isomorphism for all $n \gg 0$, then $f_* \mathscr{O}_X = \mathscr{O}_Y$.*

Proof. Consider the sheaf exact sequence $0 \to \mathscr{O}_Y \to f_* \mathscr{O}_X \to \mathscr{Q} \to 0$ on Y since f surjective and X, Y reduced. Then we have

$$0 \to H^0(Y, \mathscr{L}^{\otimes n}) \to H^0(X, f^* \mathscr{L}^{\otimes n}) \to H^0(Y, \mathscr{Q} \otimes \mathscr{L}^{\otimes n}) \to H^1(Y, \mathscr{L}^{\otimes n}) \to \cdots .$$

Hence for all $n \gg 0$ we have $H^0(Y, \mathscr{Q} \otimes \mathscr{L}^{\otimes n}) = 0$. As $\mathscr{L}$ ample we get $\mathscr{Q} = 0$. Well done. $\square$

Homogeneity of FT-Manifolds and More General

Finally we will prove the following:

Theorem 12.19. *Let X be a Fano manifold whose elementary contractions are smooth $\mathbb{P}^1$-fibrations. Then $X \cong G/B$ where G is a semisimple algebraic group and B is a Borel subgroup.*

Remark 12.20. *For FT-manifolds X, we may consider the Situation 3 and use the smoothness of contractions $\pi^I : X \to X^I$ (Theorem 7.4(a)) by the following steps: let $\overline{X}$ be the rational homogeneous model.*

(1) Find an increasing sequence $I_1 \subset I_2 \subset \cdots \subset D$ such that $\sharp(I_{k+1}) = \sharp(I_k) + 1$ and $\overline{X}^{I_{k+1}}$ is a complete family of lines in $\overline{X}^{I_k}$.

(2) Show that $X^{I_1} \cong \overline{X}^{I_1}$.

(3) Show that $X^{I_k} \cong \overline{X}^{I_k}$ implies $X^{I_{k+1}} \cong \overline{X}^{I_{k+1}}$.

Moreover, it has been shown in [73] that Step (3) works if we further assume that FT-manifolds whose Dynkin diagrams are proper subdiagrams of D are homogeneous. Thus, in principle one could use this strategy to prove Theorem 12.19 for FT-manifolds by induction on the number of nodes of D, choosing appropriately the sequence I_k at every step so that it assures suitable initial isomorphism $X^{I_1} \cong \overline{X}^{I_1}$.

Actually in the step (2), we need to find $X^{I_1} \cong \overline{X}^{I_1}$ case by case. But here in general case we will use the different method in [77] which compare X and $\overline{X}$ via Bott-Samelson varieties. By Corollary 12.17 we just need to consider the case with connected Dynkin diagrams.

Proposition 12.21. *Let X be a Fano manifold whose elementary contractions are $\mathbb{P}^1$-bundles. Assume that its Dynkin diagram D is connected, different from F_4 and G_2. Then there exists a reduced sequence $\ell = (l_1, ..., l_m)$ associated to the longest element of W such that*

$$Z_{\ell[s]} \cong \overline{Z}_{\ell[s]}, \text{ for any } s = 0, ..., m-1.$$

Basic ideas of the proof. The arguments leading to the result are rather technical. We refer the reader to [77] for details, and include here a few words about the ideas behind them.

Now looking at the construction of the Bott-Samelson varieties, we know that given $Z_{\ell[s+1]}$, the next $Z_{\ell[s]}$ come from the extension

$$0 \to f^*_{\ell[s+1]} K_{l_{m-s}} \to \mathscr{F}_{\ell[s]} \to \mathscr{O}_{\mathcal{Z}_{\ell[s+1]}} \to 0$$

which correspond to an element $\zeta_{\ell[s]} \in H^1(Z_{\ell[s+1]}, f^*_{\ell[s+1]}K_{l_{m-s}})$. Hence we just need to find a reduced sequence of maximal length m such that for any s the cocycle $\zeta_{\ell[s]}$ is uniquely determined up to homotheties.

By looking at the restriction of the sequence to curves $\beta_i(s+1)$ one sees that the extension cannot be trivial if $J := \{i < r - s : l_i = l_{m-s}\}$ is not empty. So we just need to show that for any $s = 0, ..., m-1$ we have

$$H^1(Z_{\ell[s+1]}, f^*_{\ell[s+1]}K_{l_{m-s}}) = \begin{cases} 0, & J = \emptyset; \\ 1, & J \neq \emptyset \end{cases}$$

Now in [77] we check that if $D \neq F_4, G_2$ has no multiple edges any reduced sequence of maximal length satisfies the required property. This is not the case when D is of type B or C, but we may still choose carefully the sequence ℓ so that the whole process works. □

Proof of Theorem 12.19 different from F_4 and G_2. By Proposition 12.16 we know that f_ℓ and $\overline{f}_\ell$ are birational, since Proposition 12.21 it is enough to compare the extremal faces defining them, which are the same by Proposition 12.7(b). □

Proof of Theorem 12.19 for G_2. This is follows from Lemma 2.12. □

Ideas of Theorem 12.19 for F_4. For F_4 we can not use the similar way as Proposition 12.21, not that via the computer, authors has checked that for none of the 2144892 possible sequences providing a reduced expression of the longest element of W. Hence we need to find another way. Actually we will consider the way we considered in Remark 12.20. Here we only give the basic ideas and thee details we refer Section 5 in [77].

So first we need to show that $\pi^I : X \to X^I$ are smooth. To do this, we prove first that the fibers of these contractions are birational images of Bott-Samelson varieties, and these images may be proved to be homogeneous because we know that Theorem 12.19 holds in the case in which the Dynkin diagram is a proper subdiagram of F_4. Hence Remark 12.20 reduce the problem to showing that X^1 is isomorphic to its homogeneous model $\overline{X}^1$. Now by Theorem 5.6 we just need to show that the VMRT's at a general point of both varieties are projectively isomorphic.

For the rational homogeneous model, this is the rational homogeneous space corresponding to the Dynkin diagram C_3 marked on the third node. For our manifold X^1 we may consider the family of lines passing through one point, which is the image via π^1 of a Bott-Samelson variety Z_ℓ. We may then prove that Z_ℓ is isomorphic to the corresponding Bott-Samelson variety of $\overline{X}$, and hence the proof boils down to studying the morphism from Z_ℓ into X^1. □

Corollary 12.22. *Any Fano manifold X of nef tangent bundle with $\tau(X) = 0$ is isomorphic to the quotient of a semisimple group G by its Borel subgroup B.*

Proof. Follows directly from Proposition 12.4(c) and Theorem 12.19. □

12.1.2 The Second Step

Now by Corollary 12.22 we know that the CP-Conjecture 5 follows from the following conjecture:

Conjecture 6. *Given a Fano manifold X of nef tangent bundle satisfying $\tau(X) > 0$ which is not the product of positive dimensional varieties. Then there exists a surjective morphism $f : X' \to X$ from a Fano manifold X' of nef tangent bundle, which is not a product of positive-dimensional varieties, such that $\tau(X') < \tau(X)$.*

Need to add.

12.2 For Lower Dimensions

Here is our main result:

Theorem 12.23. *Let X be a Fano manifold with nef tangent bundle of $\dim X := m \leq 5$, then X is rational homogeneous. More precisely:*

(1) For $m = 1$, we have $X \cong \mathbb{P}^1$.

(2) For $m = 2$, we have $X \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$.

(3) For $m = 3$, we have $X \cong \mathbb{P}^3$ or $\mathbb{Q}^3$ or $\mathbb{P}^1 \times \mathbb{P}^2$ or $\mathbb{P}(T_{\mathbb{P}^2})$ or $(\mathbb{P}^1)^3$.

(4) For $m = 4$, we have $X \cong \mathbb{P}^4$ or $\mathbb{Q}^4$ or $\mathbb{P}^1 \times \mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{Q}^3$ or $(\mathbb{P}^2)^2$ or $\mathbb{P}(T_{\mathbb{P}^2})$ or $\mathbb{P}(\mathscr{N})$ for the null-correction bundle over $\mathbb{P}^3$ (see Definition 12.26), or $(\mathbb{P}^1)^2 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}(T_{\mathbb{P}^2})$ or $(\mathbb{P}^1)^4$.

(5) For $m = 5$, we omitted, see [98] and [49].

For $m \leq 3$, this is right in [12]. For $m = 4$ is completed by Hwang (as [38]). The case $n = 5$ with $\rho(X) > 1$ solved by Watanabe in [98]. And for $n = 5$ with $\rho(X) = 1$, this was solved by Kanemitsu in [49].

For dimension ≤ 2. For $m = 1$, this is trivial. For $m = 2$, note that X is minimal. Let $X \neq \mathbb{P}^2$, then X is a ruled surface over a curve C. By Theorem 7.4(a) we know that $C = \mathbb{P}^1$. Hence $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(a) \oplus \mathscr{O}(b))$. If $a \neq b$, then X contains an exceptional rational curve induce the birational contractions. This is impossible by Theorem 7.4(a). Hence $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}) \cong \mathbb{P}^1 \times \mathbb{P}^1$. □

12.2.1 Fundamental Lemmas

We always use the following lemmas:

Lemma 12.24. *Let $f : X \to Y$ be a $\mathbb{P}^d$-bundle. If Y is a curve or is rational, then there exists a vector bundle of rank $d + 1$ on Y such that $X \cong \mathbb{P}_Y(\mathscr{E})$.*

Proof. Consider group schemes over Y as $1 \to \mathbb{G}_m \to \mathrm{GL}_{d+1} \to \mathrm{PGL}_d \to 1$ which induce a long exact sequence

$$H^1_{\text{ét}}(Y, \mathrm{GL}_{d+1}) \to H^1_{\text{ét}}(Y, \mathrm{PGL}_d) \to H^2_{\text{ét}}(Y, \mathbb{G}_m) = \mathrm{Br}(Y).$$

It's well-known that if Y is a curve or rational, then $\mathrm{Br}(Y) = 0$. Hence $H^1_{\text{ét}}(Y, \mathrm{GL}_{d+1}) \to H^1_{\text{ét}}(Y, \mathrm{PGL}_d)$ is surjectve.

On the other hand, a $\mathbb{P}^d$-bundle f defines a cocycle $[f] \in H^1_{\text{ét}}(Y, \mathrm{PGL}_d)$. Then f is given by the projectivization of a vector bundle if and only if there exists a preimage of $[f]$ in $H^1_{\text{ét}}(Y, \mathrm{GL}_{d+1})$. Hence well done. $\square$

Lemma 12.25 (Sato-Hirschowitz-Schneider, 1976-1980)**.** *The uniform bundle on $\mathbb{P}^n$ is a vector bundle $\mathscr{E}$ such that the splitting type of $\mathscr{E}|_\ell$ are the same for all lines in $\mathbb{P}^n$. Then any rank n uniform bundle on $\mathbb{P}^n$ is of the following types:*

$$\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n), \quad T_{\mathbb{P}^n}(a), \quad \Omega_{\mathbb{P}^n}(a).$$

Proof. See the history in the Section 1.3 in [79]. $\square$

Definition 12.26. *Let n is odd and consider $n+1$ matrix:*

$$\mathbf{A} := \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$$

which is non-singular and $\langle \mathbf{A}\boldsymbol{x}, \boldsymbol{x}\rangle = 0$ for all $\boldsymbol{x} \in \mathbb{C}^{n+1}$. Choose a homogeneous basis of $\mathbb{P}^n$ and its dual basis in $\mathbb{P}^{n,*}$, the matrix $\mathbf{A}$ induce $\Phi : \mathbb{P}^n \cong \mathbb{P}^{n,*}$. Since $\langle \mathbf{A}\boldsymbol{x}, \boldsymbol{x}\rangle = 0$, we have $(x, \Phi(x))\mathbb{P}(T_{\mathbb{P}^n}) \subset \mathbb{P}^n \times \mathbb{P}^{n,*}$. This defines a section $g : \mathbb{P}^n \to \mathbb{P}(T_{\mathbb{P}^n})$ of $p : \mathbb{P}(T_{\mathbb{P}^n}) \to \mathbb{P}^n$. Hence induce a surjection $T_{\mathbb{P}^n} \twoheadrightarrow \mathscr{O}(a)$.

Now we determine a. Now in this case we have $\int c_n(T_{\mathbb{P}^n}(-a)) = 0$. Moreover we have

$$\int c_n(T_{\mathbb{P}^n}(-a)) = \sum_{i=0}^{n} (-a)^{n-i} \int c_i(T_{\mathbb{P}^n}) = -\sum_{i=0}^{n} (-a)^{n-i+1} \binom{n+1}{i} = 1 - (1-a)^{n+1}.$$

Hence $a = 2$ or 0. But $a \neq 0$ since $\int c_n(T_{\mathbb{P}^n}) \neq 0$. Hence $a = 2$ and we have $T_{\mathbb{P}^n} \twoheadrightarrow \mathscr{O}(2)$. Now thsi forms a short exact sequence

$$0 \to \mathscr{N} \to T_{\mathbb{P}^n}(-1) \twoheadrightarrow \mathscr{O}(1) \to 0$$

and we call such $\mathscr{N}$ is the null correlation bundle over $\mathbb{P}^n$ (n odd). Note that we call show that such $\mathscr{N}$ is simple bundle and $c(\mathscr{N}) = 1 + h^2 + h^4 + \cdots + h^{n-1}$.

Lemma 12.27. *Let $f : Z \to S$ be a smooth contraction of Fano varieties, then $\rho(F) = \rho(Z) - \rho(S)$ for every fiber F.*

Proof. We know that $\mathrm{Pic}(Z) \otimes \mathbb{Q} \cong H^2(Z, \mathbb{Q})$ and $\mathrm{Pic}(F) \otimes \mathbb{Q} \cong H^2(F, \mathbb{Q})$. As S simply connected, the monodromy action is trivial. Hence $H^2(Z, \mathbb{Q}) \to H^2(F, \mathbb{Q})$ is surjective by Deligne's invariant cycle theorem. Hence $\rho(F) = \dim N_1(F, Z)$. Note that $\rho(Z) - \rho(S) = \dim N_1(F, Z)$ by Lemma 3.3 in [14] $\square$

12.2.2 Lower Dimensions for Picard Number Bigger Than One

Here we give the basic idea for the case $\rho(X) > 1$ where X be a Fano manifold with nef tangent bundle. Now X admits at least two contractions of extremal rays. By Theorem 7.4(a), these contractions are smooth morphisms, and their fibers and targets are again Fano manifolds with nef tangent bundles. Hence, induction applies!

For dimension $= 3$ *of* $\rho(X) > 1$. This can be checked by the classification theory of Fano 3-folds. But here we follows the arguments in [12].

Now $\rho(X) > 1$. In this case we will use the induction as above. Consider a contraction $f : X \to Y$. Then Y is a Fano manifold with nef tangent bundle.

If $\dim Y = 1$, then $Y \cong \mathbb{P}^1$. So the fibres are $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. If in the first case, by Lemma 12.24 we have $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(a) \oplus \mathscr{O}(b) \oplus \mathscr{O}(c))$ on Y. Hence $a = b = c$ by Theorem 7.4(a). So $X \cong \mathbb{P}^1 \times \mathbb{P}^2$. We claim that the second case is impossible. Indeed, now X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle on $\mathbb{P}^1$. View as analytic topology, we have $R^2 f_* \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^{\oplus 2}$ since it is locally constant. By the second page of Leray spectral sequence we can find that $H^2(X, \mathbb{Z}) = \mathbb{Z}^{\oplus 3}$. By the proof in Proposition 1.62, we have $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$. Hence $\rho(X) = 3$. This is impossible since $\rho(X) = \rho(Y) + 1$. See also Theorem 7.4(c1).

If $\dim Y = 2$, then $f : X \to Y$ is a $\mathbb{P}^1$-bundle. By Lemma 12.24 we have $X = \mathbb{P}_Y(\mathscr{E}) \to Y$. Here $Y = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We just consider $Y = \mathbb{P}^2$ and the case $\mathbb{P}^1 \times \mathbb{P}^1$ can be induced by the similar method and are much easier and hence $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Now $f : X = \mathbb{P}_Y(\mathscr{E}) \to Y = \mathbb{P}^2$. For any line $\ell \subset Y$, we let $\mathscr{E}|_\ell \cong \mathscr{O}(a) \oplus \mathscr{O}$ for $a \geq 0$, after some twist. If $a > 0$, we can find a section $\bar{\ell} \subset \mathbb{P}(\mathscr{E}|_\ell)$ such that $\ell^2 = -a$. Consider the exact sequence

$$0 \to N_{\bar{\ell}/\mathbb{P}(\mathscr{E}|_\ell)} = \mathscr{O}(-a) \to N_{\bar{\ell}/X} \to N_{\mathbb{P}(\mathscr{E}|_\ell)/X}|_{\bar{\ell}} = \mathscr{O}(-a) \to 0.$$

Hence we have $\int c_1(N_{\bar{\ell}/X}) = 1 - a$. As $T_X|_{\bar{\ell}}$ nef, we have $a = 1$. Hence $\mathscr{E}|_\ell \cong \mathscr{O}(1) \oplus \mathscr{O}$. In this case we can easy to see that actually $\mathscr{E}|_\ell \cong \mathscr{O}(1) \oplus \mathscr{O}$ for any line $\ell \subset Y$ since $\int c_1(\mathscr{E}) \cap \ell = 1$. Hence by the theory of uniform bundle as in Lemma 12.25, we have $\mathscr{E}$ is $\mathscr{O}(1) \oplus \mathscr{O}$ or $T_{\mathbb{P}^n}(-1)$ or $\Omega_{\mathbb{P}^n}(2)$. When $\mathscr{E}$ is $\mathscr{O}(1) \oplus \mathscr{O}$ this is impossible. By Lemma 2.71 we have $X \cong \mathbb{P}(T_{\mathbb{P}^2}) \cong \mathbb{P}(\Omega_{\mathbb{P}^2})$, well done. $\square$

Idea for dimension $= 4$ *for* $\rho(X) > 1$. For $\rho(X) > 1$, this was solved in [13]. The main method is similar as above. But we need more detailed analysis. Here we give an example and idea of these. Consider a contraction $f : X \to Y$ induced by one ray.

If $\dim Y = 1$, then $Y \cong \mathbb{P}^1$ and its fibres F are either $\mathbb{P}^3$ or $\mathbb{Q}^3$ since $\rho(X) = 2$ now and by the dimension 3 case as above. For $F = \mathbb{P}^3$ case $X = \mathbb{P}^1 \times \mathbb{P}^3$ as before. For $F = \mathbb{Q}^3$, we need to consider another contraction $g : X \to Y'$ induced by another ray. As no curve in F is contracted by g, we know that $g|_F$ is finite! Hence $\dim Y' = 3$, fibres of g are $\mathbb{P}^1$ and $\rho(Y') = 1$. Hence $Y' \cong \mathbb{P}^3$ or $\mathbb{Q}^3$. So we need to consider the Fano bundles of rank 2 on $\mathbb{P}^3$ and $\mathbb{Q}^3$. Fortunately we have the whole classifications of these objects, see the paper [91] for details. Except the obvious one, we can use the follows principle to exclude the possible cases: (i) the number of contractions is more than 2; (ii) another contraction is not f as above.

If $\dim Y = 2$, then $Y \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. For $Y \cong \mathbb{P}^2$, by the reason of Picard number, we know that X be the projective bundle on it. So we need to find the Fano bundle of rank 3 on $\mathbb{P}^2$. Fortunately we have the whole classifications of these objects, see the paper [92] for details. Except the obvious one, we can show many cases have tangent bundles which are not nef via the second contraction. For example in this case we need to exclude $X = \mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathscr{O})$. The Euler sequence give use $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}(-1) \oplus \mathscr{O}) \subset \mathbb{P}^3 \times \mathbb{P}^2$. The second contraction is the projection onto $\mathbb{P}^3$. If its tangent bundle is nef, then this contraction is a $\mathbb{P}^1$-bundle. Now using the results in [91] again and we can get the result.

For $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ this is much simpler. By the reason of Picard number again, we know that X be the projective bundle $\mathbb{P}(\mathscr{E})$ on it. So $\mathscr{E}|_{\mathbb{P}^1 \times \{s\}} = \mathscr{O}(a)^{\oplus 3}$ and $\mathscr{E}|_{\{t\} \times \mathbb{P}^1} = \mathscr{O}(b)^{\oplus 3}$. So by normalisation we can get this for any s, t and hence $\mathscr{E}$ is trivial. Hence $X \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

For $\dim Y = 3$, then there are many cases as above and below of case of $\rho(Y) = 1$. We omitted and it is similar as above. But note that $X = \mathbb{P}(T_{\mathbb{P}^2}) \times_{T_{\mathbb{P}^2}} \mathbb{P}(T_{\mathbb{P}^2})$ in the original paper [13] is not have nef tangent bundle! See the arguments in Lemma 3.3 in [98] which find a line ℓ in it such that $-K_X \cdot \ell = 1$ which is impossible if it has nef tangent bundle (note also that $\mathbb{P}(T_{\mathbb{P}^2}) \times_{T_{\mathbb{P}^2}} \mathbb{P}(T_{\mathbb{P}^2}) \cong \mathrm{Bl}_\Delta(\mathbb{P}^2 \times \mathbb{P}^2)$). $\square$

Idea for dimension $= 5$ *with* $\rho(X) > 1$. First we have the following general properties:

As before we have at least two contractions $f : X \to Y$ and $g : X \to Z$ by two extremal rays.

(1) Let X_y and X_z are some fibres over Y, Z, then it is either $\mathbb{P}^d (1 \leq d \leq 4)$ or $\mathbb{Q}^d (3 \leq d \leq 4)$.

(2) We have $\dim X_z \leq \dim Y < 5$ and $\dim X_y \leq \dim Z < 5$.

(3) If $\dim Z = \dim X_y$ and $X_y \cong \mathbb{P}^d$ (resp. $\dim Y = \dim X_z$ and $X_z \cong \mathbb{P}^d$), then we have $X \cong \mathbb{P}^d \times Y$ (resp. $X \cong \mathbb{P}^d \times Z$).

(4) If $\dim Z = \dim X_y$ and $X_y \cong \mathbb{Q}^d$ (resp. $\dim Y = \dim X_z$ and $X_z \cong \mathbb{Q}^d$) for $3 \leq d \leq 4$, then Z is a $\mathbb{P}^d$ or $\mathbb{Q}^d$ and X is a $\mathbb{P}^{5-d}$-(projective)bundle over Z (resp. Y).

Note that (1)(2) are easy and (3) follows directly from Lemma 12.24 and Proposition 2.10. (4) follows directly from Theorem 11.1 and (2).

Next as before we need to consider the case when the base variety Y varies: let $f : X \to Y$ be a $\mathbb{P}^1$-(projective) bundle, the the following holds.

(a) If $Y \cong \mathbb{P}^4$, then $X \cong \mathbb{P}^1 \times \mathbb{P}^4$.

(b) If $Y \cong \mathbb{Q}^4$, then $X \cong \mathbb{P}^1 \times \mathbb{Q}^4$ or $\mathbb{P}(\mathscr{S}_i)$ for the two spinor bundles $\mathscr{S}_i$ on $\mathbb{Q}^4$.

(c) If $Y \cong \mathbb{P}^1 \times \mathbb{P}^3$ (resp. $\mathbb{P}^1 \times \mathbb{Q}^3$), then $X \cong (\mathbb{P}^1)^2 \times \mathbb{P}^3$ or $\mathbb{P}^1 \times \mathbb{P}(\mathscr{N})$ (resp. $X \cong (\mathbb{P}^1)^2 \times \mathbb{Q}^3$ or $\mathbb{P}^1 \times \mathbb{P}(\mathscr{S})$) where $\mathscr{N}$ be the null-correlation bundle on $\mathbb{P}^3$, $\mathscr{S}$ the spinor bundle on $\mathbb{Q}^3$.

(d) If $Y \cong \mathbb{P}(\mathscr{N})$, then $X \cong \mathbb{P}^1 \times \mathbb{P}(\mathscr{N})$.

(e) If $Y \cong (\mathbb{P}^2)^2$, then $X \cong \mathbb{P}^1 \times (\mathbb{P}^2)^2$ or $\mathbb{P}^2 \times \mathbb{P}(T_{\mathbb{P}^2})$.

For the definition of spinor bundle we refer [80]. This follows from directly check as before using the lower situations. See Proposition 3.9 in [98] for the details and we omitted them here.

Note that our case satisfies the conditions in Proposition 2.7, so if X be a Fano 5-fold with nef tangent bundle of $\rho(X) \geq 4$, then $X \cong (\mathbb{P}^1)^5$ or $(\mathbb{P}^1)^3 \times \mathbb{P}^2$ or $(\mathbb{P}^1)^2 \times \mathbb{P}(T_{\mathbb{P}^2})$.

Come back to the main theorem. Let X be such variety with at least two contractions $f : X \to Y$ and $g : X \to Z$ by two extremal rays. We need to consider the $\dim Y = 1, ..., 4$ and using (1)-(4) to find all posibilities and then using (a)-(e) to find the answers. During the process, we can also use the above situation in $\rho(X) > 3$. In some cases f, g are two different projective bundles, we can also use Proposition 2.11. For details we refer the Theorem 4.1 in [98]. $\square$

12.2.3 Lower Dimensions for Picard Number One

First we recall the following fundamental result:

Theorem 12.28. *Let X be a Fano manifold of dimension $m \geq 2$ with Picard number 1 and has nef tangent bundle, then $3 \leq i_X \leq n+1$. Moreover we have the following.*

(a) If $i_X = 3$, then $X \cong \mathbb{P}^2$ or $\mathbb{Q}^3$ or $K(G_2)$.

(b) If $m = 5$, then X is $\mathbb{P}^5$ or $\mathbb{Q}^5$ or $K(G_2)$ or $i_X = 4$.

Proof. See Theorem 4.2 in [72] and [98] Corollary 5.2. $\square$

For dimension $= 3$ *with* $\rho(X) = 1$. This can be checked by the classification theory of Fano 3-folds. But here we follows the arguments in [12].

Now $\rho(X) = 1$. We will show that in this case we have $X \cong \mathbb{P}^3$ or $\mathbb{Q}^3$. By Theorem 2.6(a)(b), we just need to show $i_X \leq 2$ are impossible. As T_X nef, we let $i_X = 2$ (and $\mathsf{Index}(X) = 2$ in this case). Let C be such rational curve. We first claim that C is smooth. By Proposition 2.4, if H is the fundamental divisor, then H is base-point free or has just one base point. If H is base-point free, as $H \cdot C = 1$ then well done. If H has just one base point x_0. Let $s \in C^{\mathsf{sing}}$, then we must have $s \in \mathsf{Bs}(|H|)$. Hence $s = x_0$. By $H \cdot C = 1$ again we have $C \subset \mathsf{Bs}(|H|)$. This is impossible. Hence C is smooth. This proves the claim.

Now by Lemma 7.3, we find that for this minimal rational components we have $u : \mathcal{U} \to X$ is finite étale. As X Fano we have $\mathcal{U} \cong X$ and has Picard number 1. This is impossible since we also have $\mathcal{U} \to \mathcal{K}$ with $\dim \mathcal{K} = 2 > 0$. Well done. See also Theorem 12.28. ☐

Idea for dimension $= 4$ *for* $\rho(X) = 1$. By Theorem 12.28 we get that $X \cong \mathbb{P}^4$ or $\mathbb{Q}^4$. ☐

For dimension $= 5$ *for* $\rho(X) = 1$. By Theorem 12.28(b) we get that $X \cong \mathbb{P}^5$ or $\mathbb{Q}^5$ or $K(G_2)$ or $i_X = 4$. In the next section we will show that $i_X = 4$ is impossible (Theorem 12.35). ☐

12.3 For Five Dimension and for Special Picard Number One and Pseudoindex Four

Here we will finish the proof of CP-conjecture of dimension five. Moreover, we will also show the case of with $\rho(X) = 1$ and $i_X = 4$ such that the evaluation morphism $e : U \to X$ as before is a $\mathbb{P}^2$-bundle. Note that in this case the universal family of minimal rational components U has dimension $\dim X + 2$. Here we mainly follows the paper [49].

12.3.1 Some Preparations

We consider two lemmas.

Lemma 12.29. *Let X be a Fano manifold of nef tangent bundle with Picard number* 1 *and $i_X = 4$, $e : U \to X$ the evaluation morphism of minimal rational curves and F an arbitrary e-fiber. Then the following hold.*

(a) $H_2(F, \mathbb{Q}) \to H_2(U, \mathbb{Q})$ *is injective.*

(b) $H^{1,1}(U, \mathbb{Q}) \to H^{1,1}(F, \mathbb{Q})$ *is surjective.*

(c) For distinct (-1)*-curves* $C_1, C_2 \subset F$ *we have* $[C_1] \not\sim_{\mathrm{num}} [C_2]$ *in* $N_1(U)$*.*

Proof. (a)(b) follows from Deligne's invariant cycle theorem.

For (c), now we have $[C_1] \nsim_{\text{num}} [C_2]$ in $N_1(F)$, hence there exists $\mathscr{L} \in \operatorname{Pic}(F)$ such that $\deg_{C_1}(\mathscr{L}) \neq \deg_{C_2}(\mathscr{L})$. Consider $[\mathscr{L}] \in H^{1,1}(F, \mathbb{Q})$, by (b) we can get $D \in H^{1,1}(U, \mathbb{Q})$ whose restriction to F is $[\mathscr{L}]$. By Lefschetz (1,1)-theorem we can get such $\overline{\mathscr{L}} \in \operatorname{Pic}(U) \otimes \mathbb{Q}$ with $\overline{\mathscr{L}}|_F \sim_{\text{num}} \mathscr{L}$. Well done. $\square$

Lemma 12.30. *Let X be a Fano manifold of nef tangent bundle with Picard number one and $i_X = 4$. Consider again $e : U \to X$ of families of minimal rational curves. Then there exists a K_U-negative curve contained in an e-fiber. In particular, there exists a K_U-negative extremal ray R of $\overline{\mathrm{NE}}(U)$ which is contracted by e.*

Proof. By cone theorem we just need to show K_U is not e-nef. If not, K_U is e-nef, then every e-fiber is a minimal surface of non-negative Kodaira dimension. By the result about families of minimal surface of non-negative Kodaira dimension, for any rational curve $C = \mathbb{P}^1 \to X$ this family $U \times_X C \to C$ is isotrivial (see [78] Theorem 0.1). As C is simply connected, this family is trivial. Hence there exists a minimal surface of non-negative Kodaira dimension S such that $(U \times_X C \to C) \cong (S \times C \to C)$, see [78] Lemma 1.6.

Consider a section C_1 of $S \times C \to C$ correspond to $[C] \in V$ and is contracted by $\pi \circ f_U : S \times C \to V$. As the diagram below, taking base-change and taking another section C_2 correspond to $C_1 \subset S \times C$:

$$\begin{tikzcd}
 & & C \arrow[r, "f"] & X & \\
C_2 \arrow[r] & C_1 \times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \times C \arrow[u] \arrow[r, "f_U"] \arrow[d] \arrow[ur, phantom, "\llcorner"] & U \arrow[u, "e"'] \arrow[r, "\pi"] & V \\
 & C_1 \arrow[r] & S & &
\end{tikzcd}$$

Then easy to see that $\dim \pi(S \times C) = 2$.

Finally we have two methods to finish the proof.

- This implies that all rational curves parametrized by $\pi(S \times C) \subset V$ have the same image $f(C)$. This contradicts the fact that V is a (normalization of) parameter space of rational curves.
- Since $\dim \pi(S \times C) = 2$, we know that $\pi(e^{-1}(x))$ is independent of $x \in C$. Moreover, since $\rho(X) = 1$, X is V-rationally connected by Lemma 3 in [56]. Hence $\dim V = 2$ as C varies. This contradicts the dimension of V.

Well done. $\square$

Then the result is follows:

Theorem 12.31. *Let X be a Fano manifold of nef tangent bundle with Picard number one and $i_X = 4$, let $\dim X = n$ then one of the following holds:*

(1) The evaluation morphism as before $e : U \to X$ is a $\mathbb{P}^2$-bundle.

(2) There exists the commutative diagram:

$$\begin{tikzcd}
 & X \\
U \arrow[ur, "e"] \arrow[r, "f"'] \arrow[d, "\pi"'] & W \arrow[u, "g"'] \arrow[d, "q"'] \\
V \arrow[r, "p"'] & Y
\end{tikzcd}$$

with the following properties:

(a) f and g are $\mathbb{P}^1$-bundles.

(b) p and q are smooth elementary Mori contractions.

Hence the evaluation morphism $e : U \to X$ is a composition of two $\mathbb{P}^1$-bundles.

Proof. By Lemma 12.30, we have a contraction ϕ and the factorization $e : U \xrightarrow{\phi} W' \xrightarrow{\psi} X$.

If $W' \cong X$, then e is an elementary Mori contraction and every fiber of e is isomorphic to $\mathbb{P}^2$ by Lemma 12.27. Hence the case (1) occurs.

Now we assume that $W' \ncong X$. Then every fiber of ϕ has dimension at most 1. Hence we need the classification theory of Mori contractions of relative dimension 1, which is the following:

- **Theorem A.** Let $f : A \to B$ be a Mori contraction of a smooth variety A of dimension n around a fixed fiber $F = f^{-1}(b)$ such that $\dim F = 1$.

  (1) If f is birational then F is irreducible, $F \cong \mathbb{P}^1$, $-K_A \cdot F = 1$ and its normal bundle is $N_{F/A} = \mathscr{O}(-1) \oplus \mathscr{O}^{\oplus n-2}$. The target B is smooth and f is a blow-up of a smooth codimension 2 subvariety of B.

  (2) If f is of fiber type then B is smooth and f is a flat conic bundle. In particular one of the following is true:

  (2.a) F is a smooth $\mathbb{P}^1$ and $-K_A \cdot F = 2$, $N_{F/A} = \mathscr{O}^{\oplus n-1}$.

  (2.b) $F = C_1 \cup C_2$ is a union of two smooth rational curves meeting at one point and $-K_A \cdot C_i = 1$, $N_{F/A}|_i = \mathscr{O}^{\oplus n-1}$ and $N_{C_i/A} = \mathscr{O}(-1) \oplus \mathscr{O}^{\oplus n-2}$ for $i = 1, 2$.

(2.c) F is a smooth $\mathbb{P}^1$, $-K_A \cdot F = 1$ and the fiber structure F' on F is of multiplicity 2 (a non reduced conic); the normal bundle of F' is trivial while $N_{F/A}$ is either $\mathscr{O}(1) \oplus \mathscr{O}(-1)^{\oplus 2} \oplus \mathscr{O}^{\oplus n-4}$ or $\mathscr{O}(1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}^{\oplus n-3}$ depending on whether the discriminant locus of the conic bundle is smooth at b or not.

For the proof we refer [3].

So we need to consider two cases.

(I) **Case I.** ϕ is birational.

In this case, by Theorem A, W' is a smooth projective variety and ϕ is a blow-up of a smooth codimension two subvariety $Z \subset W'$. We will denote by E the exceptional divisor. We prove that every e-fiber is isomorphic to $\mathbb{F}_1$ and the case (2) occurs. Consider the diagram:

$$\begin{tikzcd}
E \arrow[r] \arrow[dr, hook] \arrow[dd, "\pi_E"'] & Z \arrow[dr, hook] \arrow[drr, "\psi_Z"] & & \\
& U \arrow[r, "\phi"'] \arrow[dl, "\pi"] & W' \arrow[r, "\psi"'] \arrow[dl, dashed, "\beta"] & X \\
V \arrow[r, dashed, "\alpha"] & M & &
\end{tikzcd}$$

Note that α, β, M will appear later. We have the following steps:

(I.1) **Step I.1. $\psi_Z : Z \to X$ is finite and hence surjective.**

Indeed, otherwise there would exist a curve $D \subset Z$ contracted by ψ. Then $e^{-1}(\psi(D))\phi^{-1}(D)$ by the dimensional reason. Hence $\psi^{-1}(\psi(D)) = D$, contradicting the fact that ψ is of relative dimension two.

(I.2) **Step I.2. $\psi_Z : Z \to X$ is an isomorphism.**

Indeed, $\psi_Z : Z \to X$ is generically one-to-one by Lemma 12.29(c). Hence this follows from Zariski main theorem.

(I.3) **Step I.3. $\pi_E : E \to V$ is surjective.**

Otherwise, $\pi_E : E \to V$ is a contraction of fiber type. Since π is a $\mathbb{P}^1$-bundle, $\pi_E : E \to \pi(E)$ is a $\mathbb{P}^1$-bundle and $\pi(E)$ is smooth by descent theory. by Theorem A we have $Z \cong \mathbb{P}^3$ since $i_X = i_Z = 4$. However, the universal family of lines on $\mathbb{P}^3$ is a $\mathbb{P}^2$-bundle. This contradicts the fact that ϕ is a birational morphism.

(I.4) **Step I.4. $N_1(E) \cong N_1(V)$ and $\overline{\mathrm{NE}}(E) \cong \overline{\mathrm{NE}}(V)$ by $(\pi_E)_*$.**

Indeed, this since $\phi(V) = 2$ since $\pi_E : E \to V$ is surjective and $\rho(V) \geq 2$.

(I.5) **Step I.5. Contraction M of V.**
By Step I.4 and rigidity lemma, we have the α, β and M induced by same ray of ϕ_E.

(I.6) **Step I.6. $\dim M = n$.**
Indeed, $\beta|_Z : Z \to M$ is a finite surjective morphism since it is surjective and $\rho(Z) = \rho(M) = 1$.

(I.7) **Step I.7. Every fiber of e is isomorphic to $\mathbb{F}_1$ and ψ is a $\mathbb{P}^2$-bundle.**
If $K_{W'}$ is ψ-nef, then ψ is isotrivial on every rational curve on X as in Lemma 12.30. Hence $\dim M = 2$ by the same arguments in Lemma 12.30. This contradicts Step I.6. Therefore ψ is an elementary Mori contraction, and hence every fiber of ψ is isomorphic to $\mathbb{P}^2$ by Lemma 12.27.

(I.8) **Step I.8. Conclusions.**
By Step I.7, e is a Mori contraction. Hence we get another factorization $e : U \xrightarrow{f} W \xrightarrow{g} X$. The same argument as in Case II below shows that f and g are $\mathbb{P}^1$-bundles.

(II) **Case II.** ϕ is of fiber type. Hence by Theorem A we have thw following cases of fibres F of ϕ:

(a) $F \cong \mathbb{P}^1$ and $-K_U \cdot F = 2$ and $N_{F/U} = \mathscr{O}^{\oplus n}$.

(b) $F = C_1 \cup C_2$ is a union of two smooth rational curves meeting at one point and $-K_U \cdot C_i = 1$ for $i = 1, 2$.

(c) $F_{\text{red}} \cong \mathbb{P}^1$ and F is of multiplicity 2 (a non reduced conic), $N_{F_{\text{red}}/U}$ is either $\mathscr{O}(1) \oplus \mathscr{O}(-1)^{\oplus 2} \oplus \mathscr{O}^{\oplus n-2}$ or $\mathscr{O}(1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}^{\oplus n-1}$.

Now let $f := \phi$ and $g := \psi$ and $W := W'$. We will show that case (2) occurs.

(II.1) **Step II.1. f is of type (a) as above.**
Indeed, if f is of type (b), then as $C_1 \not\approx_{\text{num}} C_1$ in $N_1(U)$ by Lemma 12.29(c), this is impossible since f is elementary contraction.
Furthermore, if a (-1)-curve C in the fiber, then $N_{C/U} = \mathscr{O}(-1) \oplus \mathscr{O}^{\oplus n}$. Hence (c) is impossible. Hence the only possible case is (a).

(II.2) **Step II.2. g is also a $\mathbb{P}^1$-bundle.**
By Step II.1, g is a smooth fibration of relative dimension one. By Theorem 2.13, there exists the V (resp. W)-rationally connected quotient morphism $q : W \to Y$ (resp. $p : V \to Y$).
Assume that the genus of g-fibers are positive, then g is isotrivial on any rational curve on X (see the introduction and Remark 1.5 in [78], for example). Hence $\dim Y = 1$ and the relative dimension of q is n by the similar

arguments as the second method in Lemma 12.30. Therefore, the restriction $g|_{q\text{-fiber}}$ is finite and surjective onto X. Note that any q-fiber is rational homogeneous manifold as in Corollary 2.14 and that $i_X = 4$. Hence, by Theorem 11.1, we have $X \cong \mathbb{P}^3$. This gives a contradiction. Hence g is a $\mathbb{P}^1$-bundle, completing the proof.

Well done! $\square$

12.3.2 Main Results

Finally comes to our two main results. Here is our first main result in this section:

Theorem 12.32. *Let X be a Fano manifold of nef tangent bundle with Picard number one and $i_X = 4$ such that the evaluation morphism $e : U \to X$ as before is a $\mathbb{P}^2$-bundle, then X is rational homogeneous. In particular $X \cong \mathbb{P}^3$ or $\mathsf{Lag}(6) = \mathsf{SGrass}(3, 6)$.*

Proof. Fix any π-fiber C. We first claim that $T_e|_C = \mathscr{O}(-1)^{\oplus 2}$.

Indeed, consider base change:

$$\begin{tikzcd}
 & U_C \arrow[r, "e_C"] \arrow[dl, "\pi_C"'] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d] \\
V & U \arrow[l, "\pi"'] \arrow[r, "e"] & X
\end{tikzcd}$$

We have a natural section $s : C \to U_C$ of e_C. As e is a $\mathbb{P}^2$-bundle, we have $U_C = \mathbb{P}\mathscr{F} \to C$ by Lemma 12.24 where $\mathscr{F} = \mathscr{O}(a) \oplus \mathscr{O}(b) \oplus \mathscr{O}(c)$ for $a \geq b \geq c$ and $c_1(\mathscr{F}) = 0, 1, 2$ after some twist. Let $\xi = \mathscr{O}_{U_C}(1)$ now. Since π_C contracts $s(C)$, the section s correspond to $\mathscr{F} \to \mathscr{O}(c)$. As by construction π_C contracts only finite curves, we have $a \geq b > c$ now. Moreover, we have $K_U \cdot C = -2$ and $K_X \cdot C = -4$, we have $K_e \cdot C = 2$. As $-K_e = 3\xi + e_C^*(-c_1(\mathscr{F}))$, we have $c_1(\mathscr{F}) = 3c + 2$. Hence $c_1(\mathscr{F}) = 2$ and $c = 0$. Hence $\mathscr{F} = \mathscr{O}(1)^{\oplus 2} \oplus \mathscr{O}$ and we get the claim.

Now consider $M := \mathbb{P}(T_e) \xrightarrow{p} U$, then the fiber of $e \circ p$ are $\mathbb{P}(T_{\mathbb{P}^2})$ which is a rational homogeneous space. As $\rho(M/X) = 2$, consider another Mori-contraction of it, then by Theorem 2.78 and Theorem 4.4 in [90] we have another $\mathbb{P}^1$-bundle $q : M \to N$ with $N \to X$ factor through $M \to X$. Similarly, since by the claim we have $T_e|_C = \mathscr{O}(-1)^{\oplus 2}$, any fiber of $\pi \circ p$ is $\mathbb{P}^1 \times \mathbb{P}^1$, hence we have third $\mathbb{P}^1$-bundle which fix the following diagram:

$$\begin{tikzcd}
L \arrow[d] & M \arrow[l, "r"'] \arrow[r, "q"] \arrow[d, "p"'] & N \arrow[d] \\
V & U \arrow[l, "\pi"'] \arrow[r, "e"] & X
\end{tikzcd}$$

with different p, q, r. By Theorem 2.13 for q, r, we have

$$\begin{tikzcd} M \arrow[r, "q"] \arrow[d, "r"'] & N \arrow[d] \\ L \arrow[r] & Q \end{tikzcd}$$

and $M \to Q$ is a contraction of a 2-dimensional extremal face. Then M is a Fano manifold (Why???) whose elementary contractions are $\mathbb{P}^1$-bundles since $\rho(M) = 3$. Hence M is a rational homogeneous manifold by Theorem 12.19, and so is X by Theorem 11.1. $\square$

The we consider the second main result. But before that, we need two lemmas.

Lemma 12.33. *Let X be a Fano manifold of nef tangent bundle with Picard number one and $i_X = 4$ and consider the situation (2) in Theorem 12.31 with:*

$$\begin{tikzcd} & X \\ U \arrow[ur, "e"] \arrow[r, "f"'] \arrow[d, "\pi"'] & W \arrow[u, "g"'] \arrow[d, "q"'] \\ V \arrow[r, "p"'] & Y \end{tikzcd}$$

Then

$$-K_f \cdot C = -K_g \cdot f(C) = -1$$

for any π-fiber C.

Proof. By assumption we have $K_U \cdot C = -2$ and $K_X \cdot e(C) = -4$, we have $-K_e \cdot C = -2$. So we just need to show that $-K_f \cdot C < 0$ and $-K_g \cdot f(C) < 0$.

(1) Now $U \times_W C \cong \mathbb{F}_m$ and we have

$$\begin{tikzcd} \mathbb{F}_m \cong U \times_W C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C \cong \mathbb{P}^1 \arrow[d] & \\ U \arrow[r, "f"] \arrow[d, "\pi"'] & W \arrow[r, "g"] \arrow[d, "q"'] & X \\ V \arrow[r, "p"] & Y & \end{tikzcd}$$

There exists a section C' over C corresponding to $C \subset U$. Since V is a family of rational curves, $\pi : U \times_W C \to \pi(U \times_W C)$ is generically finite. Hence we have $m \neq 0$ and C' is the negative section of this Hirzebruch surface. Hence $K_f \cdot C' = K_{U \times_W C} \cdot C' - f^* K_C \cdot C' = m - 2 + 2 = m > 0$. Well done.

(2) Consider $W \times_X C$ again be a Hirzebruch surface, consider:

$$\begin{tikzcd}
 & W \times_X C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C \arrow[d] \\
U \arrow[r, "f"] \arrow[d, "\pi"'] & W \arrow[r, "g"] \arrow[d, "q"'] & X \\
V \arrow[r, "p"] & Y &
\end{tikzcd}$$

First we claim that the morphism $q : W \times_X C \to q(W \times_X C)$ is generically finite for some C. Otherwise, Hence $\dim Y = 1$ and the relative dimension of q is n by the similar arguments as the second method in Lemma 12.30. Therefore, $g|_{q\text{-fiber}}$ is finite and surjective onto X. Note that any q-fiber is rational homogeneous manifold as in Corollary 2.14 and that $i_X = 4$. Hence, by Theorem 11.1, we have $X \cong \mathbb{P}^3$. This gives a contradiction. However, the evaluation morphism of lines on $\mathbb{P}^3$ is a smooth $\mathbb{P}^2$-fibration. This contradicts our assumption. Hence we proved the claim. Now as the section of $W \times_X C \to C$ contracted by q, we find that $K_g \cdot C > 0$ as in (1).

Well done! $\square$

Lemma 12.34. *Let X be a smooth Fano manifold of dimension $2n-1$ whose Picard number is one and $g : W \to X$ a $\mathbb{P}^1$-bundle over X. Assume that there exists another nontrivial contraction $W \to Y$ onto a variety Y of dimension $\leq n$. Then $-K_g$ is nef.*

Proof. As g is a $\mathbb{P}^1$-bundle over X, we first claim that there exists a vector bundle $\mathscr{G}$ such that $W \subset \mathbb{P}(\mathscr{G})$ and $W \in |2\eta|$ where η is the tautological divisor on $\mathbb{P}(\mathscr{G})$ and $\mathscr{G} \cong \mathscr{G}^\vee$.

Indeed, let $\mathscr{E} := (g_*(\omega_{W/X}^{-1}))^\vee$ (this is not so good since we use Grothendieck's $\mathbb{P}$ instead of geometric $\mathbf{P}$) which is a vector bundle of rank 3. Now $W \subset \mathbb{P}(\mathscr{E})$ is a conic in each fiber, so it is a divisor of relative degree 2, hence it is given by a global section of a line bundle that can be written as $\mathscr{O}_{\mathbb{P}(\mathscr{E})/X}(2) \otimes g^*\mathscr{L}$ for some line bundle $\mathscr{L}$ on X. Its global section, therefore, is a section of $\mathrm{Sym}^2\mathscr{E} \otimes \mathscr{L}$, i.e., a morphism $\mathscr{E}^\vee \otimes \mathscr{L}^\vee \to \mathscr{E}$. Since all the conics are nondegenerate, this morphism is an isomorphism. Hence $\mathscr{E}^\vee \cong \mathscr{E} \otimes \mathscr{L}$. Taking the determinants of $\mathscr{E}^\vee \cong \mathscr{E} \otimes \mathscr{L}$, one gets $(\det \mathscr{E}^\vee)^{\otimes 2} \cong \mathscr{L}^{\otimes 3}$. As $\mathrm{Pic}(X)$ is cyclic as the hypothesis and Proposition 1.62(b), there exists a line bundle $\mathscr{M}$ such that $\mathscr{L} \cong \mathscr{M}^{\otimes 2}$. Hence let $\mathscr{G} := \mathscr{E} \otimes \mathscr{M}$ and we get the claim.

By adjunction we have $-K_g = \eta_W$ and $\eta^3 \sim_{\text{num}} -g^*(c_2(\mathscr{G}))\eta$. Let H be the pullback of the ample generator of $\mathrm{Pic}(X)$ via g and let $\tau := \sup\{\frac{K_g \cdot C}{H \cdot C} : C \text{ are irreducible curves}\}$ and if C is in the fiber of g, then define this value is $-\infty$. In this case $-K_g + \tau H$ is nef but not ample. By definition $\eta_W + \tau H = -K_g + \tau H$ is a pullback of a $\mathbb{R}$-divisor on Y (why???). Hence $(\eta_W + \tau H)^i = 0$ for $i = n+1, ..., 2n$. Hence on $\mathbb{P}(\mathscr{G})$ we have $(\eta+\tau H)^i \cdot \eta \cdot H^{2n-i} = 0$ for any $i = n+1, ..., 2n$. Using the relation $\eta^3 \sim_{\text{num}} -g^*(c_2(\mathscr{G}))\eta$ and $H^{2n} = 0$, we have

$$\sum_{i=1}^{n} \binom{2n-j+1}{2i-1} M_{2i-1} \tau^{2n-2i-j+2} = 0$$

for any $j = 1, ..., n$ where $M_{2i-1} = (-g^*(c_2(\mathscr{E})))^{i-1} \cdot H^{2n-i}$. Since $M_1 \neq 0$, the matrix $\left(\binom{2n-j+1}{2i-1}\tau^{2n-2i-j+2}\right)_{n\times n}$ maps the nonzero vector (M_{2i-1}) to zero. Hence $\det\left(\binom{2n-j+1}{2i-1}\tau^{2n-2i-j+2}\right) = 0$. But $\det\left(\binom{2n-j+1}{2i-1}\right) \neq 0$, by some argument of linear algebra we get $\tau = 0$. Hence $-K_g$ is nef. □

Now this is our second and the final main result in this section:

Theorem 12.35. *The CP-conjecture holds for dimension five.*

Proof. We have proved the case $\rho(X) > 1$. For $\rho(X) = 1$, by Theorem 12.28(b) we get that $X \cong \mathbb{P}^5$ or $\mathbb{Q}^5$ or $K(G_2)$ or $i_X = 4$. Hence we just need to show that $i_X = 4$ is impossible.

Now by Theorem 12.31 and Theorem 12.32, we just need to consider the diagram

$$\begin{array}{ccccc} U & \xrightarrow{f} & W & \xrightarrow{g} & X \\ \pi\downarrow & & \downarrow q & & \\ V & \xrightarrow[p]{} & Y & & \end{array}$$

such that f and g are $\mathbb{P}^1$-bundles and p and q are smooth elementary Mori contractions. By Lemma 12.33 and Corollary 2.14 we know that q is a smooth morphism of relative dimension $2, 3, 5$ and fibres $\mathbb{P}^2$, $\mathbb{Q}^3$ and $K(G_2)$, respectively.

If q is a $\mathbb{P}^2$-bundle, then W has two $\mathbb{P}$-bundles and we have $b_4(W) = 1 + b_2(Y) + b_4(Y) = b_2(X) + b_4(X)$ and $b_6(W) = b_2(Y) + b_4(Y) + b_6(Y) = b_4(X) + b_6(X)$. As $b_4(X) = b_6(X)$ and $b_2(Y) = b_6(Y)$, we have $b_4(X) = 1$ and $b_4(Y) = 0$. But $b_4(Y) \geq 1$ hence this is impossible.

If q is a $\mathbb{Q}^3$-bundle or a $K(G_2)$-bundle, then $-K_g$ is nef by Lemma 12.34. This is impossible by Lemma 12.33. This finish the proof. □

12.4 Large Picard Number

See [99] and [48].

12.5 Need to add

Index

Bibliography

[1] Evgueni A. Tevelev. *Projective Duality and Homogeneous Spaces*. Springer Berlin, Heidelberg, 2005.

[2] Jared Alper. *Stacks and Moduli*. Working draft, November 15, 2023.

[3] Tetsuya Ando. On extremal rays of the higher dimensional varieties. *Invent. math.*, 81:347–357, 1985.

[4] Marco Andreatta and Jarosław A. Wiśniewski. On manifolds whose tangent bundle contains an ample subbundle. *Invent. Math.*, 146:209–217, 2001.

[5] Carolina Araujo. Rational curves of minimal degree and characterizations of projective spaces. *Math. Ann.*, 335:937–951, 2006.

[6] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*. Springer-Verlag, New York, 1985.

[7] Thierry Aubin. Equations de type mange-ampère sur les variétés kähleriennes compactes. *Bull. Math.*, 102:63–95, 1978.

[8] Allen B. Altman and Steven L. Kleiman. Compactifying the picard scheme. *Adv. in Math.*, 35(1):50–112, 1980.

[9] Arnaud Beauville. Riemannian holonomy and algebraic geometry. *Enseign. Math.*, 53:97–126, 2007.

[10] Caucher Birkar. Anti-pluricanonical systems on fano varieties. *Ann. of Math.*, 190:345–463, 2019.

[11] Caucher Birkar. Singularities of linear systems and boundedness of fano varieties. *Ann. of Math.*, 193:347–405, 2021.

[12] Frédéric Campana and Thomas Peternell. Projective manifolds whose tangent bundles are numerically effective. *Math. Ann.*, 289:169–187, 1991.

[13] Frédéric Campana and Thomas Peternell. 4-folds with numerically effective tangent bundles and second betti numbers greater than one. *Manuscripta Math.*, 79:225–238, 1994.

[14] Cinzia Casagrande. Quasi-elementary contractions of fano manifolds. *Compositio Math.*, 144:1429–1460, 2008.

[15] Bang-Yen Chen and Koichi Ogiue. Some characterizations of complex space forms in terms of chern classes. *Quat. J. Math.*, 26:459–464, 1975.

[16] Koji Cho, Yoichi Miyaoka, and N. I. Shepherd-Barron. Characterizations of projective space and applications to complex symplectic manifolds. *Adv. Stud. Pure Math.*, 35:1–88, 2002.

[17] O. Debarre and A. Kuznetsov. Gushel-mukai varieties: classification and birationalities. *Algebr. Geom.*, 5:15–76, 2018.

[18] Olivier Debarre. *Higher-Dimensional Algebraic Geometry*. Springer New York, 2001.

[19] Jean-Pierre Demailly, Thomas Peternell, and Michael Schneider. Compact complex manifolds with numerically effective tangent bundles. *J. Algebraic Geom.*, 1994.

[20] Michel Demazure. Désingularisation des variétés de schubert généralisées. *Annales scientifiques de l'École Normale Supérieure*, pages 53–88, 1974.

[21] U. V. Desale and S. Ramanan. Classification of vector bundles of rank 2 on hyperelliptic curves. *Invent. Math.*, 38:161–185, 1976/77.

[22] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[23] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angele Vistoli. *Fundamental algebraic geometry – Grothendieck's FGA explained*. American Mathematical Society, 2005.

[24] Baohua Fu. A survey on symplectic singularities and symplectic resolutions. *Ann. Math. Blaise Pascal*, 13:209–236, 2006.

[25] Baohua Fu and Jie Liu. Normalised tangent bundle, varieties with small codegree and pseudoeffective threshold. *J. Inst. Math. Jussieu*, pages 1–58, 2022.

[26] T. Fujita. *Classification theories of polarized varieties*. Cambridge University Press, 1990.

[27] William Fulton. *Intersection Theory.* Springer New York, 1998.

[28] Phillip Griffiths and Joseph Harris. Algebraic geometry and local differential geometry. *Ann. scient. Éc. Norm. Sup.*, 12:355–452, 1979.

[29] Alexander Grothendieck. *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert.* Soc. Math. France, Paris, 1960-61.

[30] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2).* Advanced Studies in Pure Math., 1962.

[31] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977.

[32] Friedrich Hirzebruch. *Einleitung*, pages 1–10. Springer Berlin Heidelberg, Berlin, Heidelberg, 1956.

[33] Jaehyun Hong and Jun-Muk Hwang. Characterization of the rational homogeneous space associated to a long simple root by its variety of minimal rational tangents. *Algebraic Geometry in East Asia—Hanoi 2005. Tokyo: Math Soc Japan*, pages 217–236, 2008.

[34] Andreas Höring, Jie Liu, and Feng Shao. Examples of fano manifolds with non-pseudoeffective tangent bundle. *Journal of the London Mathematical Society*, pages 27–59, 2022.

[35] Jun-Muk Hwang. Stability of tangent bundles of low dimensional fano manifolds with picard number 1. *Math. Ann.*, 312:599–606, 1998.

[36] Jun-Muk Hwang. Tangent vectors to hecke curves on the moduli space of rank 2 bundles over an algebraic curve. *Duke Math. J.*, 101:179–187, 2000.

[37] Jun-Muk Hwang. *Geometry of minimal rational curves on Fano manifolds*, pages 335–393. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[38] Jun-Muk Hwang. Rigidity of rational homogeneous spaces. *Proceedings of ICM. 2006 Madrid*, vol. II:613–626, 2006.

[39] Jun-Muk Hwang. Dual cones of varieties of minimal rational tangents. *Adv. Stud. Pure Math.*, 65:123–141, 2015.

[40] Jun-Muk Hwang. Cartan-fubini type extension theorems. *Acta. Math. Vietnam*, 41:369–377, 2016.

[41] Jun-Muk Hwang and Qifeng Li. Characterizing symplectic grassmannians by varieties of minimal rational tangents. *J. Differential Geom.*, 119:309–381, 2021.

[42] Jun-Muk Hwang and Ngaiming Mok. Uniruled projective manifolds with irreducible reductive g-structures. *Journal für die reine und angewandte Mathematik*, pages 5–64, 1997.

[43] Jun-Muk Hwang and Ngaiming Mok. Holomorphic maps from rational homogeneous spaces of picard number 1 onto projective manifolds. *Invent. Math.*, 136:209–231, 1999.

[44] Jun-Muk Hwang and Ngaiming Mok. Cartan-fubini type extension of holomorphic maps for fano manifolds of picard number 1. *J. Math. Pures Appl.*, 80:563–575, 2001.

[45] Jun-Muk Hwang and Ngaiming Mok. Finite morphisms onto fano manifolds of picard number 1 which have rational curves with trivial normal bundles. *J. Alg. Geom.*, 12:627–651, 2003.

[46] Jun-Muk Hwang and Ngaiming Mok. Birationality of the tangent map for minimal rational curves. *Asian J. Math.*, 9:51–64, 2004.

[47] V A Iskovskih. Fano 3-folds. i. *Mathematics of the USSR-Izvestiya*, 11(3):485, 1977.

[48] Akihiro Kanemitsu. Fano n-folds with nef tangent bundle and picard number greater than $n-5$. *Math. Z.*, 284:195–208, 2016.

[49] Akihiro Kanemitsu. Fano 5-folds with nef tangent bundles. *Math. Res. Lett.*, 24:1453–1475, 2017.

[50] Tatsuro Kawakami and Burt Totaro. Endomorphisms of varieties and bott vanishing. *arXiv: 2302.11921*, pages 1–18, 2023.

[51] Yujiro Kawamata. On the length of an extremal rational curve. *Invent. Math.*, 105:609–611, 1991.

[52] Stefan Kebekus. Families of singular rational curves. *J. Algebraic Geom.*, 11:245–256, 2002.

[53] Stefan Kebekus and Luis Solá Conde. *Existence of Rational Curves on Algebraic Varieties, Minimal Rational Tangents, and Applications*, pages 359–416. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[54] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142:1–15, 2000.

[55] Shoshichi Kobayashi and Takushiro Ochiai. Meromorphic mappings onto compact complex spaces of general type. *Invent. Math.*, 31:7–16, 1975.

[56] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational curves on fano varieties. In *Classification of Irregular Varieties*, pages 100–105, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.

[57] János Kollár. *Rational Curves on Algebraic Varieties*. Springer Berlin, Heidelberg, 1996.

[58] János Kollár and Teruhisa Matsusaka. Riemann-roch type inequalities. *Amer. J. Math.*, 105:229–252, 1983.

[59] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of fano manifolds. *J. Diff. Geom.*, 36:765–769, 1992.

[60] János Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge University Press, 1998.

[61] J. Landsberg and L. Manivel. On the projective geometry of rational homogeneous varieties. *Comment. Math. Helv.*, 78:65–100, 2003.

[62] Chi-Hin Lau. Holomorphic maps from rational homogeneous spaces onto projective manifolds. *J. Algebraic Geom.*, pages 223–256, 2009.

[63] Robert Lazarsfeld. *Positivity in Algebraic Geometry I. Classical Setting: Line Bundles and Linear Series*. Springer-Verlag Berlin Heidelberg, 2004.

[64] Teruhisa Matsusaka. On canonically polarised varieties (ii). *Amer. J. Math.*, 92:283–292, 1970.

[65] Massimiliano Mella. Existence of good divisors on mukai varieties. *J. Algebraic Geom.*, 8:197–206, 1999.

[66] J.S. Milne. *Lie Algebras, Algebraic Groups, and Lie Groups*. `https://www.jmilne.org/math/CourseNotes/LAG.pdf`, 2013.

[67] J.S. Milne. *Algebraic Groups: the Theory of Group Schemes of Finite Type over a Field*. Cambridge University Press, 2017.

[68] Ngaiming Mok. Recognizing certain rational homogeneous manifolds of picard number 1 from their varieties of minimal rational tangents. *AMS/IP Stud Adv Math*, 42:41–61, 2008.

[69] Shigefumi Mori. Projective manifolds with ample tangent bundles. *Ann. of Math.*, 110:593–606, 1979.

[70] Narasimhan M.S. and Ramanan S. Geometry of hecke cycles i, in ramanujan–a tribute. *Springer*, pages 291–345, 1978.

[71] David Mumford. *Lectures on curves on an algebraic surface.* Princeton University Press, 1966.

[72] Roberto Muñoz, Gianluca Occhetta, Luis E. Solá Conde, Kiwamu Watanabe, and Jarosław A. Wiśniewski. A survey on the campana-peternell conjecture. *Rend. Istit. Mat. Univ. Trieste*, 47:127–185, 2015.

[73] Roberto Muñoz, Gianluca Occhetta, Luis Solá Conde, and Kiwamu Watanabe. Rational curves, dynkin diagrams and fano manifolds with nef tangent bundle. *Math. Ann.*, 361:583–609, 2015.

[74] Masayoshi Nagata. On self-intersection number of a section on a ruled surface. *Nagoya Math. J.*, 37:191–196, 1970.

[75] Carla Novelli and Gianluca Occhetta. Ruled fano fivefolds of index two. *Indiana University Mathematics Journal*, 56:207–241, 2007.

[76] Gianluca Occhetta and Jarosław A. Wiśniewski. On euler-jaczewski sequence and remmert-van de ven problem for toric varieties. *Math. Z.*, 241:35–44, 2002.

[77] Gianluca Occhetta, Luis Solá Conde, Kiwamu Watanabe, and Jarosław A. Wiśniewski. Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations: a geometric characterization of flag varieties. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, pages 573–607, 2017.

[78] Keiji Oguiso and Eckart Viehweg. On the isotriviality of families of elliptic surfaces. *J. Algebraic Geom.*, 10:569–598, 2001.

[79] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector Bundles on Complex Projective Spaces, With an Appendix by S. I. Gelfand.* Birkhäuser Basel, 1980.

[80] Giorgio Ottaviani. Spinor bundles on quadrics. *Trans. Amer. Math. Soc.*, 307:301–316, 1988.

[81] Giorgio Ottaviani. Rational homogeneous varieties. *https://people.dimai.unifi.it/ottaviani/rathomo/rathomo.pdf*, pages 1–70, 1995.

[82] Jacob P. Murre, Jan Nagel, and Chris A. M. Peters. *Lectures on the Theory of Pure Motives.* AMS, 2013.

[83] A.N. Parshin and I.R. Shafarevich. *Algebraic Geometry V: Fano Varieties.* Springer Berlin, Heidelberg, 1999.

[84] T. Peternell and A. Wiśniewski. On stability of tangent bundles of fano manifolds with $b_2 = 1$. *J. Alg. Geom.*, 4:363–384, 1995.

[85] G.V. Ravindra and V. Srinivas. The grothendieck-lefschetz theorem for normal projective varieties. *J. Algebraic Geo.*, 15(3):563–590, 2006.

[86] Jean-Pierre Serre. *Complex Semisimple Lie Algebras*. Benjamin, New York, 1966.

[87] Feng Shao and Guolei Zhong. Bigness of tangent bundles and dynamical rigidity of fano manifolds of picard number 1 (with an appendix by jie liu). *arXiv: 2311.15559*, pages 1–21, 2023.

[88] Yum-Tong Siu and Shing-Tung Yau. Compact kähler manifolds of positive bisectional curvature. *Invent. Math.*, pages 189–204, 1980.

[89] Roy Skjelnes and Gregory G. Smith. Smooth hilbert schemes: their classification and geometry. *arXiv:2008.08938.*, 2020.

[90] Luis Solá Conde and Jarosław A. Wiśniewski. On manifolds whose tangent bundle is big and 1-ample. *Proc. London Math. Soc.*, 89:273–290, 2004.

[91] Michał Szurek and Jarosław A. Wiśniewski. Fano bundles over $\mathbb{P}^3$ and $\mathbb{Q}_3$. *Pacific J. Math.*, 141:197–208, 1990.

[92] Michał Szurek and Jarosław A. Wiśniewski. On fano manifolds, which are $\mathbb{P}^k$-bundles over $\mathbb{P}^2$. *Nagoya Math. J.*, 120:89–101, 1990.

[93] Patrice Tauvel and Rupert W. T. Yu. *Lie Algebras and Algebraic Groups*. Springer Berlin, Heidelberg, 2005.

[94] Kenji Ueno. *Classification theory of algebraic varieties and compact complex spaces*. Springer-Verlag, Berlin-New York, 1975.

[95] Ravi Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*, 164(3):569–590, 2006.

[96] J. Wahl. A cohomological characterization of $\mathbb{P}^n$. *Invent. Math.*, 72:315–322, 1983.

[97] Kiwamu Watanabe. $\mathbb{P}^1$-bundles admitting another smooth morphism of relative dimension one. *Journal of Algebra*, 414:105–119, 2014.

[98] Kiwamu Watanabe. Fano 5-folds with nef tangent bundles and picard numbers greater than one. *Math. Z.*, 276:39–49, 2014.

[99] Kiwamu Watanabe. Fano manifolds with nef tangent bundle and large picard number. *Proc. Japan Acad. Ser. A Math. Sci.*, 91:89–94, 2015.

[100] J. Wierzba. Contractions of symplectic varieties. *J. Algebraic Geom.*, 12:507–534, 2003.

[101] Jarosław A. Wiśniewski. *Uniform Vector Bundles on Fano Manifolds and an Algebraic Proof of Hwang-Mok Characterization of Grassmannians*, pages 329–340. Springer Berlin Heidelberg, 2002.

[102] Shing-Tung Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci.*, 74:1798–1799, 1977.

[103] Shing-Tung Yau. On the ricci curvature of a compact kiihler manifold and the complex monge-ampere equation i. *Comm. Pure Appl. Math.*, 31:339–411, 1978.