Algebraic Cycles and Hodge Theory

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Contents

1 Introduction

The reader of course need to be familiar with the book [3] including the basic theory and schemes, cohomology, curves and surfaces. We will also use the intersection theory frequently such as the main contents of [2] or [1] and the reader should familiar with these. Finally we will omit the most basic theory of complex Hodge theory, such as the first seven chapters in [5].

We will focus on the final part of the book [6]. There are three topics of Hodge theory in this book but we just discuss the final part of them. We will also use the Serre's GAGA-principle without explanation.

2 Some Background of Mixed Hodge Theory

2.1 Basic Definition and Properties

Definition 2.1. *A rational (real) mixed Hodge structure of weight n is given by a $\mathbb{Q}$-vector space ($\mathbb{R}$-vector space) H equipped with an increasing filtration W_iH called the weight filtration , and a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$, called the Hodge filtration $F^kH_{\mathbb{C}}$. Such that the induced Hodge filtration on each $\mathrm{Gr}_i^W H$ make $\mathrm{Gr}_i^W H$ to be a Hodge structure of weight $n+i$.*

These filtrations are required to be bounded. Recall that a morphism $\alpha : (U, F) \to (V, G)$ is said to be strict if $\mathrm{Im}\alpha \cap G^pV = \alpha(F^pU)$. It's easy to show that the morphism of rational pure Hodge structures are strict for Hodge filtration (even in type (r, r), see [5] Lemma 7.23).

This is an analogue theory of Hodge decomposition of pure Hodge structures:

Lemma 2.2. *Let (H, W, F) be a mixed Hodge structure. Then there exists a decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

with $H^{p,q} \subset F^pH_{\mathbb{C}} \cap W_{p+q-n}H_{\mathbb{C}}$, such that via the projection $W_{p+q-n}H_{\mathbb{C}} \to \mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}$, the space $H^{p,q}$ can be identified with

$$H^{p,q}(\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}) := F^p\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}} \cap \overline{F^q\mathrm{Gr}_{p+q-n}^W H_{\mathbb{C}}}.$$

More generally, we have

$$W_iH_{\mathbb{C}} = \bigoplus_{p+q \le n+i} H^{p,q}, F^iH_{\mathbb{C}} = \bigoplus_{p \ge i} H^{p,q}.$$

This decomposition is preserved by the morphisms of mixed Hodge structures.

Proof. This is pure linear algebra, we omit it and refer [6] Lemma 4.21. □

Remark 2.3. *Unlike the pure case, the decomposition above may satisfies $H^{p,q} \neq \overline{H^{p,q}}$, although this does become true after projection to $\mathrm{Gr}^W_{p+q} H_{\mathbb{C}}$.*

Theorem 2.4 (P. Deligne, 1971)**.** *The morphisms*

$$\alpha : (H, W, F) \to (H', W', F')$$

of (rational or real) mixed Hodge structures are strict for the filtrations W and F.

Proof. We will only show the statement for W since the statement for H is similar.

Pick $l' \in \alpha(H_{\mathbb{C}}) \cap W_i H'$ and we write $l' = \alpha(l)$ with $l = \sum_{p,q} l^{p,q}$ by Lemma 2.2. As $l' \in W'_i H'_{\mathbb{C}}$, then $\alpha(l^{p,q}) = 0$ for $p+q > n+i$ by Lemma 2.2 again. Hence $l' \in \alpha(W_i H_{\mathbb{C}})$ and well done. □

2.2 A Classical Example of Mixed Hodge Structure

We consider a smooth complex variety U with a compactification X such that $X \backslash U = D$, a effective normal crossing divisor.

Definition 2.5. *Define a subsheaf $\Omega^k_X(\log D) \subset \Omega^k_X(*D)$ such that $\alpha \in \Gamma(V, \Omega^k_X(\log D))$ if α is a meromorphic differential form on V, holomorphic on $V \backslash D$ and admits a pole of order at most 1 along (each component of) D, and the same holds for $d\alpha$. Hence $d = \partial$ in it and we call the complex $(\Omega^*_X(\log D), \partial)$ the* logarithmic de Rham complex *.*

Lemma 2.6. *Let $z_1, ..., z_n$ be local coordinates on an open set $V \subset X$, in which $D \cap V$ is defined by the equation $z_1 \cdots z_r = 0$. Then $\Omega^k_X(\log D)|_V$ is a sheaf of free $\mathscr{O}|_U$-modules with basis*

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$$

where $i_s \leq r$, $j_s > r$ and $l + m = k$. In particular, $\Omega^k_X(\log D)$ is locally free.

Proof. Almost trivial, see [5] Lemma 8.16. □

Proposition 2.7. *Let inclusion $j : U \hookrightarrow X$, then we have a canonical inclusion $\Omega^k_X(\log D) \subset j_* \Omega^k_U \subset j_* \mathscr{A}^k_U$ which give us a morphism of complex*

$$\Omega^*_X(\log D) \to j_* \mathscr{A}^*_U.$$

Then this is a quasi-isomorphism. In particular we have

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_X(\log D)).$$

Proof. This is not hard to see and we refer [5] Proposition 8.18. From this we have $\mathbb{H}^k(X, \Omega_X^*(\log D)) \cong \mathbb{H}^k(X, j_*\mathscr{A}_U^*)$. As $\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_U^\infty$-modules which is a resolution of $\mathbb{C}_U$, then $j_*\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_X^\infty$-modules, so it is acyclic and

$$\mathbb{H}^k(X, j_*\mathscr{A}_U^*) \cong H^k\Gamma(X, j_*\mathscr{A}_U^*) = H^k\Gamma(U, \mathscr{A}_U^*) = H^k(U, \mathbb{C}).$$

Hence we get the result. □

For now we will give $H^k(U, \mathbb{Q})$ (or $H^k(U, \mathbb{R})$) a mixed Hodge structure. First we will give two filtrations over $\Omega_X^*(\log D)$.

We define the Hodge filtration over $\Omega_X^*(\log D)$ to be

$$F^p\Omega_X^*(\log D) = \Omega_X^{\geq p}(\log D).$$

For weight filtration, we define $W_l\Omega_X^*(\log D)$ to be

$$W_l\Omega_X^*(\log D) = \begin{cases} \bigwedge^l \Omega_X^1(\log D) \wedge \Omega_X^{*-l}, & 0 \leq l \leq r, \\ 0, & l > r. \end{cases}$$

(We often let $W^k := W_{-k}$)

Now for simplicity, we let the divisor D is simply normal crossing with $D = \bigcup_i D_i$ where each $D_i \subset X$ is a smooth hypersurface, and the intersection of any l hypersurfaces $D_{i_1}, ..., D_{i_l}$ is transverse. We equip I with a total order. We let

$$D^{(k)} := \coprod_{K \subset I, |K|=k} D_K = \coprod_{K \subset I, |K|=k} \bigcap_{i \in K} D_i$$

with inclusions $j_k : D^{(k)} \to X$ and $j_M : D_M \to X$.

Proposition 2.8. *There exists a natural isomorphism*

$$W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

Proof. This morphism defined by Poincaré residue map . Give a local coordinates in $V \subset X$ we define $\mathrm{Res}^V : \Gamma(V, W_k\Omega_X^*(\log D)) \to \Gamma(V, j_{k,*}\Omega_{D^{(k)}}^{*-k})$ as

$$\begin{aligned} \alpha &= \sum_{K \subset \{1,\dots,r\} \subset I, |K| \leq k} \alpha_{K,L} dz_L \wedge \frac{dz_K}{z_K} \\ &\mapsto (\mathrm{Res}^V \alpha)_M = \left((2\pi\sqrt{-1})^k \sum_L \alpha_{M,L} dz_L|_{D_M \cap V} \right)_M. \end{aligned}$$

Note that this annihilates the sections of $W_{k-1}\Omega_X^*(\log D)$ and change coordinates only change the elements in $W_{k-1}\Omega_X^*(\log D)$, Hence we get a well-defined residue map:

$$\alpha : W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

This is an isomorphism is easy to see. We refer [5] Proposition 8.32. □

Now these two filtrations induce two filtrations over $R\Gamma(X, \Omega_X^*(\log D))$, and hence over $H^k(U, \mathbb{C})$ by Proposition 2.7. So the arguments in [5] is far from complete and we need some derived-version filtration of these, such as mixed Hodge complex. We omitted this and we refer section 3.3 in [4].

Theorem 2.9 (P. Deligne, 1971)**.** *The discussion above equip $H^k(U, \mathbb{C})$ a mixed Hodge structure which is independent with X, D.*

Proof. This follows from some analysis of the weight spectral sequence (induced by $W^* = W_{-*}$), here we give a sketch.

By the general theory of spectral sequence, we have

$$_WE_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)).$$

By Proposition 2.8 we have $\mathrm{Gr}_W^p \Omega_X^*(\log D) \cong j_{-p,*}\Omega_{D^{(-p)}}^{*+p}$, hence

$$\begin{aligned}\mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)) &= \mathbb{H}^{2p+q}(X, j_{-p,*}\Omega_{D^{(-p)}}^*) \\ &= \mathbb{H}^{2p+q}(D^{(-p)}, \Omega_{D^{(-p)}}^*) = H^{2p+q}(D^{(-p)}, \mathbb{C}).\end{aligned}$$

We can also get that the differential

$$d_1 : \quad \begin{array}{ccc} H^{2p+q}(D^{(-p)}, \mathbb{C}) & \longrightarrow & H^{2p+q+2}(D^{(-p-1)}, \mathbb{C}) \\ \downarrow \cong & & \downarrow \cong \\ \bigoplus_{|K|=-p} H^{2p+q}(D_K, \mathbb{C}) & \longrightarrow & \bigoplus_{|L|=-p-1} H^{2p+q+2}(D_L, \mathbb{C}) \end{array}$$

has component $d_{1,K}^L$ equal to zero for $L \not\subseteq K$, and equal to $(-1)^{q+s} j_{K,*}^L$ when $K = \{i_1 < \cdots < i_p\}$ and $L = K \backslash \{i_s\}$ where $j_K^L : D_K \to D_L$ (see Proposition 8.34 in [5]). Hence we can deduce any pages of weight spectral sequence! By some analysis we can get the result which omitted, we refer Theorem 3.4.7 and section 3.4.1.5 in [4]. □

3 Cycle Classes and Abel–Jacobi Map

3.1 Cycle Classes and Cycle Map

The case of general complex manifolds with closed analytic subsets

Let X be a $n + r$-dimensional complex manifold with a codimension r closed analytic subset Z, we will associated Z to be a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$.

Lemma 3.1. *If $Y \subset X$ be a closed complex submanifold of codimension k, then the natural map $H^l(X, \mathbb{Z}) \to H^l(X \backslash Y, \mathbb{Z})$ is an isomorphism for $l \leq 2k - 2$.*

Proof. Trivial, just need to look at the long exact sequence induced by the good pair $(X, X\backslash Y)$ and using Thom's isomorphism and the excision theorem. □

Come back to our case, as in algebraic geometry, we can have a filtration

$$\emptyset = Z_{n+1} \subset \cdots \subset Z_0 = Z$$

where $\dim Z_i = n - i$ and $Z_k \backslash Z_{k-1}$ is a closed complex submanifold of dimension $n - k$ in $X \backslash Z_{k-1}$ (see [5] Theorem 11.11 for the proof).

We apply this Lemma to each $X\backslash Z_k \subset X\backslash Z_{k+1}$, we have

$$H^{2r}(X, \mathbb{Z}) \cong H^{2r}(X\backslash Z_1, \mathbb{Z}).$$

Here $Z\backslash Z_1$ is smooth. So we just need to consider the case when Z is a smooth complex submanifold in X!

If Z is a smooth complex submanifold of codimension r in X, then by Thom's isomorphism and the excision theorem, we have the following diagram

$$\begin{tikzcd}
H^{2r}(X, X\backslash Z; \mathbb{Z}) \arrow[r, "j_Z"] \arrow[d, "="] & H^{2r}(X, \mathbb{Z}) \\
H^{2r}(X, X\backslash Z; \mathbb{Z}) \arrow[r, "{\cong, T}"] & H^0(Z, \mathbb{Z})
\end{tikzcd}$$

Then we define $[Z] = j_Z(T^{-1}(1)) \in H^{2r}(X, \mathbb{Z})$.

Remark 3.2. *We an also using the most natural way: if $Z = \sum_i n_i Z_i$, we can define $[Z] = \sum_i n_i [Z_i]$ where $[Z_i] = \mathrm{PD}(j_{i,*}([Z_i']_{\text{fund}}))$ and $j_i : Z_i' \to X$ is a resolution of singularity of Z_i, $[Z_i']_{\text{fund}}$ is the fundamental homology class and PD denotes Poincaré duality!*

Here we give some discription using the de Rham cohomology without proof:

Proposition 3.3. *Let $U \subset X$ be a neighbourhood of Z isomorphic to a neighbourhood V of the section in the normal bundle $N_{Z/X}$. Let ω be a closed form of degree k with support in V, satisfying*

$$\int_{\pi^{-1}(z)} \omega = 1$$

where $\pi : V \to Z$ is the projection. Then the form ω is a representative in de Rham cohomology of the class $[Z]$.

Proof. See Lemma 11.14 in [5]. □

The case of compact Kähler manifolds

Let X be a $n+r$-dimensional compact Kähler manifold with a codimension r closed analytic subset Z. We have associated Z to be a cohomology class $[Z] \in H^{2r}(X, \mathbb{Z})$. Now using Hodge decomposition, we will discuss the type of $[Z]$ in $H^{2r}(X, \mathbb{C}) = \bigoplus_{p+q=2r} H^{p,q}(X)$.

Theorem 3.4. *The image of $[Z]$ in $H^{2r}(X, \mathbb{C})$ lies in $H^{r,r}(X)$.*

Proof. Here we need to use the following two results (for the proof, see [5] Lemma 7.30 and Theorem 11.21, using the de Rham discription we discussed above this is easy to prove):

(i) If Y be a compact Kähler manifold of dimension m, then

$$H^{p,q}(Y) = \left(\bigoplus_{k+l=2m-p-q, (k,l) \neq (m-p, m-q)} H^{k,l}(Y) \right)^{\perp}$$

where the orthogonality is relative to the Poincaré duality on Y.

(ii) (Lelong, 1957) The current $\omega \mapsto \int_{Z_{\text{smooth}}} \omega$ maps to zero on the exact forms. Hence it is an element in $H^{2n}(X, \mathbb{C})^*$. Then this element is equal to the image of $[Z]$under the morphism

$$H^{2r}(X, \mathbb{Z}) \to H^{2r}(X, \mathbb{C}) \to H^{2n}(X, \mathbb{C})^*.$$

By (i), we just need to show that $\int_X [Z] \wedge \alpha = 0$ for any α of type (p, q), $p+q = 2n$, $(p, q) \neq (n, n)$. Then this is trivial by (ii). □

The case of complex smooth (quasi-)projective varieties

Let X be a complex smooth quasi-projective variety of dimension n and $Z \in \mathcal{Z}_k(X)$, then we give $[Z] \in H^{2n-2k}(X, \mathbb{Z})$ as above.

Proposition 3.5. *If $Z \sim_{\text{rat}} 0$, then $[Z] = 0 \in H^{2n-2k}(X, \mathbb{Z})$, hence we give the class map*

$$\text{cl} : \text{CH}_l(X) \to H^{2n-2l}(X, \mathbb{Z}), Z \mapsto [Z].$$

We denote its kernal $\text{CH}_l(X)_{\text{hom}}$.

Proof. WLOG we can assume X is projective. Let $W \subset X$ is of dimension $k+1$ and $\phi \in K(W)^*$, we just need to show $[\tau_* \text{div}(\phi)] = 0$ where $\tau : W' \to W \to X$ be a resolution of singularity of W.

We can easy to see $[\tau_* \text{div}(\phi)] = \tau_* [\text{div}(\phi)]$ where $\tau_* : H^2(W', \mathbb{Z}) \to H^{2n-2k}(X, \mathbb{Z})$ defined by Poincaré duality by Remark 3.2. Hence we just need to show $[\text{div}(\phi)] = 0 \in H^2(W', \mathbb{Z})$. This follows from Lelong's fundamental theorem that $[D] = c_1(\mathscr{O}(D)) \in H^2(W', \mathbb{Z})$. □

Proposition 3.6. *Let $f : X \to Y$ be morphism of smooth quasi-projective varieties.*

(i) If $Z \in \mathrm{CH}^l(X), Z' \in \mathrm{CH}^k(X)$, then

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(Z) \cup \mathrm{cl}(Z') \in H^{2k+2l}(X, \mathbb{Z});$$

(ii) if $Z \in \mathrm{CH}^k(Y)$, then $f^* \mathrm{cl}(Z) = \mathrm{cl}(f^*Z) \in H^{2k}(X, \mathbb{Z})$;

(iii) if f proper and $Z \in \mathrm{CH}^k(X)$, then $f_* \mathrm{cl}(Z) = \mathrm{cl}(f_*Z) \in H^{2k-2\dim X+2\dim Y}(Y, \mathbb{Z})$.

Proof. We have showed (iii) in the proof of Proposition 3.5. We first show the case of closed immersion of (ii). Indeed, by moving lemma we may assume Z and X intersect generically transverse. We may assume Z is irreducible. By Lemma 3.1 we may assume Z and X intersect transversely. Then compare the normal bundle and well done.

Then we prove (i). We know by definition that

$$[Z \times Z'] = p_1^*[Z] \cup p_2^*[Z'].$$

Let diagonal $\delta : \Delta_X = X \subset X \times X$, then by the case of closed immersion of (ii) we have

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(\delta^*(Z \times Z')) = \delta^* \mathrm{cl}(Z \times Z') = \delta^*(p_1^*[Z] \cup p_2^*[Z']) = \mathrm{cl}(Z) \cup \mathrm{cl}(Z'),$$

well done.

For the general case of (ii), this follows from (i) and decomposition

$$f : X \xrightarrow{\text{graph}} X \times Y \to Y,$$

well done. □

3.2 Hodge Classes and Hodge Conjecture

Definition 3.7. *If $(V_{\mathbb{Z}}, F^*V_{\mathbb{C}})$ is a pure Hodge structure of weight $2p$, then we denote the set of* Hodge classes

$$\mathrm{Hdg}(V) := V_{\mathbb{Z}} \cap V^{p,p}.$$

Now consider a compact Kähler manifold X with its standard Hodge structure, then we have:

Theorem 3.8. *The cohomology class of analytic subsets of X and the Chern classes of the holomorphic vector bundles over X are all Hodge classes.*

Proof. The first way is Theorem 3.4. For the second one, we know that if X is algebraic, this is trivial since the Chern classes in the cohomology group can come from the Chern classes in the Chow group! Indeed, we can tensoring some higher times ample bundle and get a map from X to a Grassmannian such that the Chern class of this bundle comes from the Schubert classes in the Grassmannian (see [1] Proposition 10.2). In general, we have a classical result that

$$H^*(P(\mathscr{E})) \cong \frac{H^*(X)[\zeta]}{\zeta^r + c_1(\mathscr{E})\zeta^{r-1} + \cdots + c_r(\mathscr{E})}$$

and $\zeta = c_1(\mathscr{O}_{P\mathscr{E}}(1))$ is of type $(1,1)$. Hence well done. $\square$

Corollary 3.9. *If X is algebraic, these two subgroups of $\mathrm{Hdg}(X)$ coincide.*

Proof. Omitted. $\square$

Converse is true in codimension 1 which is classical Lefschetz $(1,1)$-Theorem:

Theorem 3.10. *In this case, the group $\mathrm{Hdg}^2(X, \mathbb{Z})$ is equal to the image of $c_1 : \mathrm{Pic}X \to H^2(X, \mathbb{Z})$.*

But this is false in general. Actually, J. Kollár in 1992 gave a counterexample. His counterexample is given by hypersurfaces X of degree d in the projective space $\mathbb{P}^4$. Such a hypersurface satisfies $H^2(X, \mathbb{Z}) = \mathbb{Z}$, and the class of a plane curve $\mathbb{P}^2 \cap X$ is equal to d times the generator of $H^2(X, \mathbb{Z})$. He shows that this generator is not, however, in general, the class of an algebraic cycle.

In $\mathbb{Q}$-coefficient, we have the following famous conjectures:

Conjecture 1 (Hodge Conjecture)**.** *Let X be a projective manifold, and $\alpha \in \mathrm{Hdg}^{2k}(X)$. Then a multiple $N\alpha$ with $N \neq 0$ is the class of an algebraic cycle.*

Conjecture 2 (Generalized Hodge Conjecture, Grothendieck)**.** *Let X be a smooth algebraic variety, and $L \subset H^{2k+l}(X, \mathbb{Q})$ a rational sub-Hodge structure contained in $F^k H^{2k+l}(X)$. Then there exist (not necessarily smooth) algebraic subvarieties $j_i : Y_i \to X$ of codimension k such that L is contained in $\sum_i j_{i,*} H_{2n-l}(Y_i, \mathbb{Q})$ where $\dim X = n+k$.*

3.3 The Abel–Jacobi Map

Let X be a compact Kähler manifold.

Definition 3.11. *We know $H^{2k-1}(X, \mathbb{R}) \cong H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)$ and consider the lattice $H^{2k-1}(X, \mathbb{Z}) \subset H^{2k-1}(X, \mathbb{R})$, we get the k-th intermediate Jacobian $J^{2k-1}(X)$ as a complex torus*

$$J^{2k-1}(X) := H^{2k-1}(X, \mathbb{C})/(F^k H^{2k-1}(X) \oplus H^{2k-1}(X, \mathbb{Z})).$$

Remark 3.12. *Note that in general $J^{2k-1}(X)$ is not an abelian variety! But $J^1(X) = \mathrm{Pic}^0(X)$ is.*

Definition 3.13. *Pick $Z \in \mathcal{Z}^k(X)_{\text{hom}}$, then we can find a differentiable chain $\Gamma \subset X$ of real dimension $2n-2k+1$ such that $\partial\Gamma = Z$. Consider $\int_\Gamma \in A^{2n-2k+1}(X)^*$, When we restrict it into $F^{n-k-1}H^{2n-2k+1}(X)^*$, we find that by the reason of type and Stokes's formula, $\int_\Gamma \in F^{n-k-1}H^{2n-2k+1}(X)^*$ is independent of the choice of the representative of cohomology class. One can also easy to see that if we descend $\int_\Gamma$ in*

$$F^{n-k-1}H^{2n-2k+1}(X)^*/H_{2n-2+1}(X,\mathbb{Z}) \cong J^{2k-1}(X),$$

it is independent of the choice of Γ such that $\partial\Gamma = Z$! Hence we give a morphism called the Abel-Jacobi map

$$\Phi_X^k : Z \in \mathcal{Z}^k(X)_{\text{hom}} \to J^{2k-1}(X), \quad Z \mapsto \int_\Gamma .$$

Theorem 3.14 (Griffiths, 1968)**.** *Let Y be a connected complex manifold, $y_0 \in Y$ a reference point, and $Z \subset Y \times X$ a cycle of codimension k. We will assume that $Z = \sum_i n_i Z_i$ where each Zi is flat over Y, then the map*

$$\phi : Y \to J^{2k-1}(X), \quad y \mapsto \Phi_X^k(Z_y - Z_{y_0})$$

is holomorphic.

Proof. See Theorem 12.4 in [5]. □

Now we consider two special cases when $k = 1$ and $k = \dim X$.

When $k = 1$, by Proposition 12.7 in [5] we can show that $c_1(\mathcal{O}(Z)) = \Phi_X^1(Z) \in J^1(X) = \text{Pic}^0(X)$. Then we get:

Corollary 3.15 (Abel's Theorem)**.** *If D be a divisor homologous to 0 in X, then $\Phi_X^1(D) = 0$ if and only if $\mathcal{O}_X(D)$ is trivial.*

When $k = n = \dim X$, we find that if X is connected, such a cycle is homologous to 0 if and only if it is of degree 0.

Definition 3.16. *We define the Albanese variety $\text{Alb}(X) := J^{2n-1}(X)$ and the holomorphic map*

$$\text{alb}_X : X \to \text{Alb}(X), \quad x \mapsto \Phi_X^{2n-1}(x - x_0)$$

is called the Albanese map .

Remark 3.17. *(a) The Albanese map satisfies the following universal property: For every holomorphic map $f : X \to T$ with values in a complex torus and satisfying $f(x_0) = 0$, there exists a unique morphism of complex tori such that:*

$$\begin{tikzcd} X \arrow[r, "\text{alb}"] \arrow[d, "f"'] & \text{Alb}(X) \arrow[dl, dashed, "\exists! g"] \\ T & \end{tikzcd}$$

Hence this give us a purely algebraic construction. For details see Theorem 12.15 in [5];

(b) *actually for sufficiently large k, the morphism*

$$\mathrm{alb}_X^k : X^k \to \mathrm{Alb}(X), \quad (x_1, ..., x_k) \mapsto \sum_i \mathrm{alb}_X(x_i)$$

is surjective. Hence we can use this to show that $\mathrm{Alb}(X)$ is an abelian variety. For details see Lemma 12.11 and Corollary 12.12 in [5].

Theorem 3.18. *Let X, Y are two compact Kähler manifolds with Y connected and $Z \subset Y \times X$ be a cycle of codimension k which is flat over Y. Hence we have the holomorphic map $\phi : Y \to J^{2k-1}(X)$ given by $y \mapsto \Phi_X^k(Z_y - Z_{y_0})$. Hence by the universal property, we get a morphsim of complex tori $\psi : \mathrm{Alb}(Y) \to J^{2k-1}(X)$ factor through ϕ. Then ψ induced by $[Z]^{1,2k-1} : H^{2\dim Y-1}(Y, \mathbb{Z}) \to H^{2k-1}(X, \mathbb{Z})$, the $(1, 2k-1)$-Künneth component. In particular, the image of ψ is a complex subtorus of $J^{2k-1}(X)$ having the property that its tangent space at 0 is contained in $H^{k-1,k}(X)$.*

Proof. The main result is not hard to see (Theorem 12.17 in [5]). Let $m = \dim Y$, then the morphism of Hodge structures $[Z]$ is of bidegree $(k-m, k-m)$, and as the Hodge structure on $H^{2m-1}(Y)$ is of type $(m, m-1) + (m-1, m)$, the image of $[Z]$ is thus contained in $H^{k,k-1}(X) \oplus H^{k-1,k}(X)$. Well done. □

When X be a smooth projective variety, there is a compactible relation with the intersection product.

Proposition 3.19. *Now for any $Z \in \mathrm{CH}^k(X)$, we have the map $\mathrm{cl}(Z)\cup : H^{2l-1}(X, \mathbb{Z}) \to H^{2l+2k-1}(X, \mathbb{Z})$ which can descend to $\mathrm{cl}(Z)\cup : J^{2l-1}(X) \to J^{2l+2k-1}(X)$. If let $Z' \in \mathrm{CH}^l(X)_{\mathrm{hom}}$, then by Proposition 3.6(i) we have $Z \cdot Z' \in \mathrm{CH}^{k+l}(X)_{\mathrm{hom}}$. Then we have*

$$\Phi_X^{k+l}(Z \cdot Z') = \mathrm{cl}(Z) \cup \Phi_X^l(Z').$$

Proof. □

Finally in this case with $\dim X = n$, we introduce two more maps induced by the Abel-Jacobi map.

We know that if $Z \sim_{\mathrm{alg}} 0$, then $Z \sim_{\mathrm{hom}} 0$ for any Weil cohomology theory (see [2]). Then we define $\mathrm{Griff}^k(X) := \mathcal{Z}^k(X)_{\mathrm{hom}} / \mathcal{Z}^k(X)_{\mathrm{alg}}$.

Remark 3.20. *When $k = 1, n$ and X connected, we have $\mathrm{Griff}^k(X) = 0$. Here we need a classical result:*

- *Let X by any variety and $x_i \in X$. Then there is an irreducible curve C on X containing x_i. (Sketch of the proof, which is so classical: Given any two points on a projective variety, blow them up and re embed the blownup variety in $\mathbb{P}^N$. Then by Bertini, any general linear section of the right codimension will meet the variety in an irreducible curve which also meets both exceptional divisors. Then blowing back down gives an irreducible curve connecting the original points.)*

For $k = 1$, we know that any two homologous trivial divisor lies in the connected variety $\mathrm{Pic}^0(X)$, then by this results with normalizing that curve gives a map from just one smooth connected curve that connects your two points. Hence $\mathrm{Griff}^1(X) = 0$.

For $k = n$, as any 0-cycle lies over an irreducible curve in X by this result, normalizing that curve and we get $\mathrm{Griff}^n(X) = 0$.

But when $2 \leq k < n$ in general $\mathrm{Griff}^k(X) \neq 0$. So we define $\mathrm{Griff}^k(X)$ to find their difference and to deduce some new Abel-Jacobi maps as follows.

We define $J^{2k-1}(X)_{\mathrm{alg}} \subset J^{2k-1}(X)$ be the largest subtorus with tangent space contained in $H^{k-1,k}(X)$. Hence by Theorem 3.18 we know that $\Phi_X^k(\mathcal{Z}^k(X)_{\mathrm{alg}}) \subset J^{2k-1}(X)_{\mathrm{alg}}$. We can also define $J^{2k-1}(X)_{\mathrm{trans}} := J^{2k-1}(X)/J^{2k-1}(X)_{\mathrm{alg}}$ as the transcendental part of the intermediate Jacobian.

Remark 3.21. *This is because in general $J^{2k-1}(X)$ is not algebraic but $J^{2k-1}(X)_{\mathrm{alg}}$ is always algebraic, and hence $J^{2k-1}(X)_{\mathrm{alg}}$ is an abelian variety! Indeed, by the classical Hodge–Riemann bilinear relation: (see [5] Theorem 6.32)*

- *Consider a Hermitian form H_k on $H^k(X, \mathbb{C})$ defined by*

$$H_k(\alpha, \beta) = (\sqrt{-1})^k \int_X \omega^{n-k} \wedge \alpha \wedge \bar{\beta},$$

  then the form $(-1)^{\frac{k(k-1)}{2}}(\sqrt{-1})^{p-q-k} H_k$ is positive definite on the complex subspace $H^{p,q}(X) \cap H^k(X, \mathbb{C})_{\mathrm{prim}}$.

we can get the result directly.

Hence we can define the continuous (algebraic) part of the Abel-Jacobi map $\Phi^k_{X,\mathrm{alg}} : \mathcal{Z}^k(X)_{\mathrm{alg}} \to J^{2k-1}(X)_{\mathrm{alg}}$ and the transcendental part of the Abel-Jacobi map:

$$\begin{tikzcd}
\mathcal{Z}^k(X)_{\mathrm{hom}} \arrow[r, "\Phi_X^k"] \arrow[d, two heads] & J^{2k-1}(X) \arrow[d, two heads] \\
\mathrm{Griff}^k(X) \arrow[r, "\Phi^k_{X,\mathrm{trans}}"] & J^{2k-1}(X)_{\mathrm{trans}}
\end{tikzcd}$$

3.4 Deligne Cohomology

Definition 3.22. *Let X be a complex manifold and $p \geq 1$, we define the* Deligne complex *$\mathbb{Z}_D(p)$ is*

$$0 \to \mathbb{Z} \xrightarrow{(2\pi\sqrt{-1})^p} \mathscr{O}_X \xrightarrow{d} \Omega_X \to \cdots \to \Omega_X^{p-1} \to 0.$$

We define the Deligne cohomology *$H_D^k(X, \mathbb{Z}(p)) := \mathbb{H}^k(X, \mathbb{Z}(p))$.*

Remark 3.23. *We have $\mathbb{Z}_D(1) \simeq_{\text{qis}} \mathscr{O}_X^*[-1]$ and $H_D^2(X, \mathbb{Z}(1)) = H^1(X, \mathscr{O}_X^*)$.*

Proposition 3.24. *If X is a compact Kähler manifold, then there exists a long exact sequence*

$$\begin{aligned}\cdots \to H_D^k(X, \mathbb{Z}(p)) \to H^k(X, \mathbb{Z})& \\ \to H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C}) &\to H^{k+1}D(X, \mathbb{Z}(p)) \to \cdots .\end{aligned}$$

Proof. First we consider

$$0 \to \Omega_X^{\leq p-1}[-1] \to \mathbb{Z}_D(p) \to \mathbb{Z} \to 0$$

which induce a long exact sequence and we see that we just need to show $\mathbb{H}^k(X, \Omega_X^{\leq p-1}) = H^k(X, \mathbb{C})/F^p H^k(X, \mathbb{C})$. From the basic fact of Hodge structure (e.g. Proposition 7.5 in [5]) $\mathbb{H}^k(X, \Omega_X^{\geq p}) = F^p H^k(X, \mathbb{C})$ and the exact sequence

$$0 \to \Omega_X^{\geq p} \to \Omega_X^* \to \Omega_X^{\leq p-1} \to 0$$

we can get the result directly. □

Corollary 3.25. *In this case, we have exact sequence*

$$0 \to J^{2p-1}(X) \to H_D^{2p}(X, \mathbb{Z}(p)) \to \mathrm{Hdg}^{2p}(X, \mathbb{Z}) \to 0.$$

Proof. This follows directly from the $k = 2p$ in theorem and the fact

$$\mathrm{Hdg}^{2p}(X, \mathbb{Z}) = \ker(H^{2p}(X, \mathbb{Z}) \to H^k(X, \mathbb{C})/F^p H^{2p}(X)).$$

Well done. □

Here we give another method to compute the Deligne cohomology $H_D^{2p}(X, \mathbb{Z}(p))$.

Definition 3.26. *Let $\Xi^l_{\mathrm{diff}}(X)$ be a subgroup of $\mathrm{Hom}(Z_l^{\mathrm{diff}}, \mathbb{R}/\mathbb{Z})$ consist of*

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