NOTES ON THE GEOMETRY OF HYPERTORIC VARIETIES

XIAOLONG LIU

Abstract. In this note we will introduce the basic theory of hypertoric varieties.

Contents

1. Introduction

1.1. **Background/Motivation.**

1.2. **Related works and some future direction.** Need to add.

1.3. **Notations and remarks.** We work over $\mathbb{C}$.

2. Recollection of the basic theory of toric varieties

We will follows [Ful93], [CLS11] and [Tel22] to recollect something we need.

3. Basic definitions and resolutions of hypertoric varieties

3.1. **About Poisson and symplectic structures and symplectic resolutions.** Here we give an introduction of these and we refer [Bea00] and [Fu06] for more details. See also [Fu03] for more examples and results.

Definition 3.1. *We consider complex algebraic schemes.*

Date: October 27, 2024.
2020 *Mathematics Subject Classification.* 14M25, 14J42 .
Key words and phrases. hypertoric varieties, GIT quotient.

- *We say a scheme X carries a* Poisson structure *if there is a $\mathbb{C}$-bilinear operation*
$$\{-,-\}:\mathscr{O}_X\times\mathscr{O}_X\to\mathscr{O}_X$$
which is a Lie bracket.
- *Let $f:X\to Y$ be a morphism of Poisson schemes, we say it is a* Poisson morphism *if it induce a homomorphism of Lie algebras.*

Remark 3.2. *Any Poisson structure can be induced by the $\mathscr{O}_X$-linear homomorphism $H:\Omega^1_X\to T_X=\mathrm{Der}(\mathscr{O}_X,\mathscr{O}_X)$ such that $\{f,g\}=H(df)(g)$. In particular, any symplectic variety has a canonical Poisson structure.*

We also have the relative version of Poisson schemes and we omit them here.

Definition 3.3. *Let Y_0 be a normal variety.*

- *A pair (Y_0,ω_0) of the normal algebraic variety Y_0 and a 2-form ω_0 on the smooth locus $(Y_0)_{\mathsf{sm}}$ is called a* symplectic variety *if ω_0 is symplectic and there exists (or equivalently, for any) a resolution $\pi:Y\to Y_0$ such that the pull-back of ω_0 by π extends to a holomorphic 2-form ω on Y.*
- *The resolution $\pi:Y\to Y_0$ is called* symplectic *if ω is also symplectic.*

Some basic properties:

Proposition 3.4 (Prop.1.6 in [Fu06])**.** *Let W be a symplectic variety with a resolution $\pi:Z\to W$, then the following statements are equivalent:*

(1) π is crepant;
(2) π is symplectic;
(3) K_Z is trivial.

Next, we now care about the following special case:

Definition 3.5. *An affine symplectic variety $(Y_0=\mathrm{Spec}\,R,\omega_0)$ with $\mathbb{C}^*$-action (called* conical *$\mathbb{C}^*$-*action) is called a* conical symplectic variety *if it satisfies:*

- *The grading induced from the $\mathbb{C}^*$-action to the coordinate ring R is positive, i.e., $R=\bigoplus_{i\geqslant 0}R_i$ and $R_0=\mathbb{C}$.*
- *ω_0 is homogeneous with respect to the $\mathbb{C}^*$-action, i.e., there exists $\ell\in\mathbb{Z}$ (the weight of ω_0) such that $t^*\omega_0=t^\ell\omega_0$ ($t\in\mathbb{C}^*$).*

Remark 3.6. *We can show that the weight ℓ is always positive.*

3.2. **Algebraic symplectic quotients and hypertoric varieties.** Note that hypertoric varieties are examples of symplectic varieties.

Consider the exact sequence
$$0\to\mathbb{Z}^{n-d}\xrightarrow{B}\mathbb{Z}^n\xrightarrow{A}\mathbb{Z}^d\to 0$$
where $A=[\boldsymbol{a}_1,...,\boldsymbol{a}_n]\in M_{d\times n}(\mathbb{Z})$ and $B^T=[\boldsymbol{b}_1,...,\boldsymbol{b}_n]\in M_{(n-d)\times n}(\mathbb{Z})$ (the Gale duality of $\{\boldsymbol{a}_1,...,\boldsymbol{a}_n\}$). Acting $\mathrm{Hom}(-,\mathbb{C}^*)$ we get
$$1\to\mathbb{T}^d\xrightarrow{A^T}\mathbb{T}^n\xrightarrow{B^T}\mathbb{T}^{n-d}\to 1$$
an exact sequence of algebraic tori.

Via the natural action of $\mathbb{T}^n$ on $T^*\mathbb{C}^n\cong\mathbb{C}^{2n}$, we have the action of $\mathbb{T}^d$ on $T^*\mathbb{C}^n\cong\mathbb{C}^{2n}$ as
$$\boldsymbol{t}\cdot(z_1,...,z_n,w_1,...,w_n)=(\boldsymbol{t}^{\boldsymbol{a}_1}z_1,...,\boldsymbol{t}^{\boldsymbol{a}_n}z_n,\boldsymbol{t}^{-\boldsymbol{a}_1}w_1,...,\boldsymbol{t}^{-\boldsymbol{a}_n}w_n)$$
where $\boldsymbol{t}^{\boldsymbol{a}_i}:=t_1^{a_{1,i}}\cdots t_d^{a_{d,i}}$. The moment map of this given by
$$\mu:T^*C^n\to\mathfrak{t}_d^*=\mathbb{C}^d,\quad(z_1,...,z_n,w_1,...,w_n)\mapsto\sum_{i=1}^n\boldsymbol{a}_iz_iw_i.$$

Definition 3.7. *Fix a character $\alpha\in\mathbb{Z}^d=\mathrm{Hom}(\mathbb{T}^d,\mathbb{C}^*)$ and a point $\xi\in\mathbb{C}^d$.*

- *We define the* Lawrence toric variety *as*
$$X(A,\alpha):=(\mathbb{C}^{2n})^{\alpha\text{-}ss}/\!\!/\,\mathbb{T}^d=\mathrm{Proj}\left(\bigoplus_{k\geqslant 0}\mathbb{C}[z_i,w_j]^{\mathbb{T}^d,k\alpha}\right)$$

where $(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \{u \in \mathbb{C}^{2n} :$ *there exists* $f \in \mathbb{C}[z_i, w_j]$ *such that* $f(u) \neq 0$ *and* $\sigma(f) = \alpha^*(t)^k \otimes f$ *for* $k > 0\}$ *where* $\mathbb{C}^* = \operatorname{Spec}\mathbb{C}[t, 1/t]$ *and coaction morphism* $\sigma : \mathbb{C}[z_i, w_j] \to \Gamma(\mathscr{O}_{\mathbb{T}^d}) \otimes \mathbb{C}[z_i, w_j]$. *Note that* $\mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha} = \{f \in \mathbb{C}[z_i, w_j] : \sigma(f) = \alpha^*(t)^k \otimes f\}$.

- *We define the* hypertoric variety *(or* toric hyperkähler variety*) as*

$$Y(A, \alpha, \xi) := \mu^{-1}(\xi)^{\alpha\text{-}ss} /\!\!/ \mathbb{T}^d = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}\right)$$

similar as above.

Remark 3.8. *We can write the semistable locus as follows:*

$$(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \left\{(z_i, w_j) \in \mathbb{C}^{2n} : \alpha \in \sum_{i:z_i\neq 0} \mathbb{Q}_{\geqslant 0}\boldsymbol{a}_i + \sum_{j:w_j\neq 0} \mathbb{Q}_{\geqslant 0}(-\boldsymbol{a}_j)\right\}$$

and $\mu^{-1}(\xi)^{\alpha\text{-}ss} = \mu^{-1}(\xi) \cap (\mathbb{C}^{2n})^{\alpha\text{-}ss}$.

Remark 3.9. *Note that we have a natural morphism* $\Pi : X(A, \alpha) \to X(A, 0)$ *and* $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$ *with the same reason. Indeed, we consider the case of hypertoric varieties. Note that*

$$Y(A, 0, \xi) = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\cdot 0}\right) = \operatorname{Spec}\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}.$$

Then inclusion $\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d} \subset \bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}$ *induce* $\operatorname{Spec}\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha} \to \operatorname{Spec}\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}$. *Since the grade induced by* $\mathbb{C}^*$*-action and this morphism is* $\mathbb{C}^*$*-invariant, then we get* $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$. *Note moreover that* $\mu^{-1}(\xi)^{\alpha\text{-}ss} \subset \mu^{-1}(\xi) = \mu^{-1}(\xi)^{0\text{-}ss}$.

Remark 3.10. *The hypertoric varieties are the special case of the following general contruction.*

Consider a reductive group G *and a representation* V. *Then we form* $T^*V = V \oplus V^*$ *which comes with a moment map* $\Phi : T^*V \to \mathfrak{g}^*$ *given by cup of* $T_xV^* \to \mathfrak{g}^*$ *as* $T_eG \to T_x(Gx) \subset T_xV$. *We fix a character* $\chi : G \to \mathbb{C}^\times$ *and form the GIT quotient*

$$\Phi^{-1}(\xi) /\!\!/_\chi G := \Phi^{-1}(\xi)^{\chi\text{-}ss} /\!\!/ G = \operatorname{Proj}\left(\bigoplus_{n\geqslant 0} \mathbb{C}[\Phi^{-1}(\xi)]^{G, n\chi}\right).$$

We have a natural projective morphism as before

$$\pi : Y := \Phi^{-1}(\xi) /\!\!/_\chi G \to X := \Phi^{-1}(\xi) /\!\!/_0 G = \operatorname{Spec}\mathbb{C}[\Phi^{-1}(0)]^G$$

carry Poisson structures coming from the usual symplectic structure on T^*V. *This construction will not usually give a symplectic resolution; for example,* Y *may not be smooth and* $Y \to X$ *might not be birational. Here in the physics literature,* Y *is called the* Higgs branch of the 3d supersymmetric gauge theory defined by G, V. G *is called the* gauge group *and* N *is called the* matter*.*

There is a conical $\mathbb{C}^\times$ *action on* Y *coming from its scaling action of* T^*V. *In order to define a Hamiltonian torus action, we need one piece of data. We choose an extension* $1 \to G \to \widetilde{G} \to T \to 1$, *where* T *is the flavor torus, and an action of* $\widetilde{G}$ *on* V, *extending the action of* G. *Then we obtain a residual Hamiltonian action of* T *on* Y *and* X. *In general, this action does not have finitely many fixed points.*

Example 3.11. *Another special case, we introduce the Nakajima quiver varieties, first introduced by Nakajima. We fix a finite directed graph* $Q = (I, E)$, *with head and tail maps* $h, t : E \to I$. *Also, we fix two dimension vectors* $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$. *For* $i \in I$,*let* $V_i = \mathbb{C}^{u_i}, W_i = \mathbb{C}^{w_i}$ *and consider the space of representations of the quiver* Q *on the vector space* $\oplus V_i$ *framed by* $\oplus W_i$.

$$N = \bigoplus_{e\in E} \operatorname{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i\in I} \operatorname{Hom}(V_i, W_i).$$

This big vector space N *has a natural action of* $G = \prod_i \mathrm{GL}(V_i)$. *We form the cotangent bundle* T^*N *and take the Hamiltonian reduction by the action of* G. *The resulting space* $Y = \Phi^{-1}(0)/\!/_\chi G$ *is called a* Nakajima quiver variety*. Here we choose* $\chi : G \to \mathbb{C}^\times$ *to be given by the product of the determinants. On* Y, *we have a Hamiltonian action of* $T = \prod_i (\mathbb{C}^\times)^{w_i}$ *inherited from its action on* $\oplus W_i$. *(In other words, we take* $\widetilde{G} = G \times T$.*)*

Note that the space Y *is always smooth but* $\pi : Y \to X$ *is not always birational. Also, the Hamiltonian torus action does not always have finitely many fixed points.*

Here we give two examples of Nakajima quiver varieties.

- *Consider a linearly oriented type A_{n-1}-quiver with $\mathbf{v} = (1, ..., n-1), \mathbf{w} = (0, ..., 0, n)$:*

$$\bullet(V_1) \longrightarrow \bullet(V_2) \longrightarrow \cdots \longrightarrow \bullet(V_{n-1}) \longrightarrow \blacksquare(\mathbb{C}^n)$$

  Then $N = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$ with $G = \prod_{i=1}^{n-1} \mathrm{GL}_i$. Then $Y \cong T^*\mathsf{Fl}_n$ with $X = \mathcal{N}_{\mathfrak{sl}_n}$.
- *Another important example is a quiver with one vertex and one self-loop with $V = \mathbb{C}^n$ and $W = \mathbb{C}^r$.*

$$\begin{tikzcd} \bullet(\mathbb{C}^n) \arrow[loop above] \arrow[r] & \blacksquare(\mathbb{C}^r) \end{tikzcd}$$

  In this case, Y is the moduli space of rank r, torsion-free sheaves on $\mathbb{P}^2$, framed at ∞ with second Chern class n.

3.3. **Symplectic resolutions of hypertoric varieties.** We will consider when $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$ will be a symplectic resolution. So we need to consider the condition that $\mu^{-1}(\xi)^{\alpha\text{-ss}} = \mu^{-1}(\xi)^{\alpha\text{-st}}$. First we will compute their stabilizer group.

Let $(\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{C}^{2n}$ and set $J_{\boldsymbol{z},\boldsymbol{w}} := \{j \in \{1, ..., n\} : z_j \neq 0 \text{ or } w_j \neq 0\}$, then we have

$$\mathrm{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d = \ker(\mathbb{T}^d \xrightarrow{A^T_{J_{\boldsymbol{z},\boldsymbol{w}}}} \mathbb{T}^{|J_{\boldsymbol{z},\boldsymbol{w}}|}).$$

Hence by some linear algebra we have

Corollary 3.12 (Coro.2.7 in [Nag21])**.** *We have:*

(1) $\mathrm{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d$ *is finite if and only if* $\sum_{j \in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Q}\boldsymbol{a}_j = \mathbb{Q}^d$*;*

(2) $\mathrm{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d = 1$ *if and only if* $\sum_{j \in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Z}\boldsymbol{a}_j = \mathbb{Z}^d$*.*

Definition 3.13. *In this setting, we call A is* unimodular *if all $d \times d$-minors of A are 0 or ± 1.*

Remark 3.14. *Note that A is unimodular if and only if B is.*

Hence for a unimodular A, we have $\sum_{j \in J} \mathbb{Q}\boldsymbol{a}_j = \mathbb{Q}^d$ iff $\sum_{j \in J} \mathbb{Z}\boldsymbol{a}_j = \mathbb{Z}^d$ for $J \subset \{1, ..., n\}$.

Let A is a unimodular matrix and we define

$$\mathcal{H}_A := \{H \subset \mathbb{R}^d : H \text{ is generated by some of the } \boldsymbol{a}_j \text{ and of codimension} = 1\}.$$

We say α generic if $\alpha \notin \bigcup_{H \in \mathcal{H}_A} H$.

Lemma 3.15 (Lem.2.10 and Coro.2.11 in [Nag21])**.** *In the case, for any $\alpha \in \mathbb{Z}^d$ and $\xi \in \mathbb{C}^d$, we have $(\mu^{-1}(\xi))^{\alpha\text{-ss}} \neq \varnothing$. If α generic, then $(\mu^{-1}(\xi))^{\alpha\text{-ss}} = (\mu^{-1}(\xi))^{\alpha\text{-st}}$ with free action by $\mathbb{T}^d$. In particular, if α generic then $X(A, \alpha)$ is $2n - d$-dimensional smooth Poisson variety and for any ξ, $Y(A, \alpha, \xi)$ is a $2n - 2d$-dimensional smooth symplectic variety.*

Theorem 3.16 (Thm.2.16 in [Nag21])**.** *For a unimodular A and generic α and any $\xi \in \mathbb{C}^d$, the morphism*

$$\pi_\xi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$$

is a projective symplectic resolution and if $\xi = 0$, then it is conical.

Sketch. First, by $\mu : \mathbb{C}^{2n} \xrightarrow{\Psi} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^d$ with $\Psi : (\boldsymbol{z}, \boldsymbol{w}) \mapsto \sum_j z_j w_j \boldsymbol{e}_j$ is flat. Then from dimension counting we get $\mu^{-1}(\xi)$ is of equidimension $2n - d$. As it define by d polynomials, we know that $\mu^{-1}(\xi) \in \mathbb{C}^{2d}$ is a complete intersection and hence Cohen-Macaulay. After showing that the codimension of singular locus $\geqslant 2$, then $\mu^{-1}(\xi)$ is normal by Serre's condition. Finally we can construct an open subset and show that π_ξ is identity over it which force it is birational. Moreover, the result follows from Lemma 3.15 and the following easy fact (see Proposition 2.15 in [Nag21]):

- If $\pi : Y \to Y_0$ is projective birational morphism with Y is a nonsingular sympecltic variety, then π is a symplectic resolution.

Well done. □

Remark 3.17. *Note that we have the more general results. In [Bel23] Lemma 2.4 and Proposition 2.5, without assuming A is unimodular, shows that if we choose α, α' such that $\mu^{-1}(\xi)^{\alpha'\text{-ss}} \subset \mu^{-1}(\xi)^{\alpha\text{-ss}}$, then there exists a projective birational Poisson morphism $Y(A, \alpha', \xi) \to Y(A, \alpha, \xi)$. Moreover, any hypertoric variety $Y(A, \alpha, \xi)$ has symplectic singularities.*

4. Basic geometry of hypertoric varieties

4.1. **Hypertoric varieties with hyperplane arrangements.** Here we consider the case $\xi = 0$. Then we define $Y(A, \alpha) := Y(A, \alpha, 0)$. It defined by

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$

where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$.

Then we can define $H_i := \{x \in \mathbb{R}^{n-d} : x \cdot \boldsymbol{b}_i + r_i = 0\}$ for $i = 1, ..., n$ where $\boldsymbol{r} = (r_1, ..., r_n) \in \mathbb{Z}^n$ be a lifting of α along A. This defines a hyperplane arrangement $\mathcal{A} := \{H_1, ..., H_n\}$. Here we can denote $Y(\mathcal{A}) := Y(A, \alpha)$.

Definition 4.1. *In this setting, for such hyperplane arrangement $\mathcal{A}$:*

- *we call $\mathcal{A}$ is* simple *if for any subset of m hyperplanes with nonempty intersections, they intersect of codimension m.*
- *we call $\mathcal{A}$ is* unimodular *if for any $n - d$ linear independent $\{\boldsymbol{b}_{i_1}, ..., \boldsymbol{b}_{i_{n-d}}\}$ spans $\mathbb{C}^{n-d}$ over $\mathbb{Z}$.*
- *we call $\mathcal{A}$ is* smooth *if it is simple and unimodular.*

Remark 4.2. *Note that $\mathcal{A}$ is unimodular if and only if B is unimodular if and only if A is unimodular.*

Proposition 4.3 (3.2/3.3 in [BD00])**.** *The hypertoric variety $Y(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth.*

Note that $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement, meaning that $r_i = 0$ for all i, so that all of the hyperplanes pass through the origin. Then we have the following result:

Corollary 4.4. *For any central arrangement $\mathcal{A}$, there exists a* simplification *$\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$ of $\mathcal{A}$ by which we mean an arrangement defined by the same vectors $\{\boldsymbol{b}_i\}$, but with a different choice of $\alpha, \boldsymbol{r}$ such that $\widetilde{\mathcal{A}}$ is simple. This will give us an* equivariant orbifold resolution *$Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$. When A is unimodular, this will give us a resolution of singularities which recover the special case of Theorem 3.16.*

4.2. **The cores and homotopy models.** Consider again $\xi = 0$. Then we have an equivariant orbifold resolution

$$\pi : Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$$

where $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement with simplification $\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$.

Definition 4.5. *In this case, we call $\mathfrak{c}(\widetilde{\mathcal{A}}) := \pi^{-1}(0)$ the* core *of $Y(\widetilde{\mathcal{A}})$.*

Now we will give a toric interpretation of the core $\mathfrak{c}(\widetilde{\mathcal{A}})$. For any $J \subset \{1, ..., n\}$, define the polyhedron

$$P_J := \{x \in \mathbb{R}^{n-d} : x \cdot \boldsymbol{b}_i + r_i \geqslant 0 \text{ if } i \in J \text{ and } x \cdot \boldsymbol{b}_i + r_i \leqslant 0 \text{ if } i \notin J\}.$$

Define

$$\mathfrak{E}_J := \{(\boldsymbol{z}, \boldsymbol{w}) \in T^*\mathbb{C}^n : w_i = 0 \text{ if } i \in J \text{ and } z_i = 0 \text{ if } i \notin J\}$$

and define $\mathfrak{X}_J := \mathfrak{E}_J /\!\!/_\alpha \mathbb{T}^d$, which induce the inclusion

$$\mathfrak{X}_J \hookrightarrow \mu^{-1}(0) /\!\!/_\alpha \mathbb{T}^d = Y(\widetilde{\mathcal{A}}).$$

Theorem 4.6 (Section 6 in [BD00]/ section 3.2 in [Pro04])**.** *In this setting, we have:*

1. *the scheme $\mathfrak{X}_J$ is isomorphic to the toric variety correspond to the weighted polytope P_J;*
2. *we have $\mathfrak{c}(\widetilde{\mathcal{A}}) = \bigcup_{J:P_J \text{ bounded}} \mathfrak{X}_J$, hence $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a union of compact toric varieties glued together along toric subvarieties as prescribed by the combinatorics of the polytopes P_J and their intersections in $\mathbb{R}^{n-d}$.*

Sketch. Note that (1) follows from the surjectivity real moment maps and some classification theorems, see Lemma 3.8 in [Pro04]. For (2), see Proposition 3.11 in [Pro04]. □

Remark 4.7. *This is right even for $\widetilde{\mathcal{A}}$ is not simple.*

Finally we consider some homotopy results.

Theorem 4.8 (6.5 in [BD00] and section 6 in [HS02]). *In this setting, we have:*

(1) the core $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a deformation retract of $Y(\widetilde{\mathcal{A}})$;

(2) the inclusion

$$Y(\widetilde{\mathcal{A}}) = \mu^{-1}(0) /\!\!/_{\alpha} \mathbb{T}^d \hookrightarrow T^*\mathbb{C}^n /\!\!/_{\alpha} \mathbb{T}^d = X(\widetilde{\mathcal{A}})$$

is a homotopy equivalence where $X(\widetilde{\mathcal{A}})$ is the corresponding Lawrence toric variety.

4.3. **Universal Poisson structure of hypertoric varieties.** In this section we will give a concrete description of universal Poisson structure of hypertoric varieties. At the begining, we consider some general results. Here we will follows [Nag21].

Definition 4.9. *For a Poisson variety $(Y, \{-,-\}_0)$ and an affine scheme $(B, 0)$ with fixed point 0, we call a Poisson B-scheme $(\mathcal{Y}, \{-,-\})$ a* Poisson deformation *of Y if $\mathcal{Y} \to B$ is flat, each fiber is a Poisson scheme, and the central fiber is isomorphic to $(Y, \{-,-\}_0)$ as a Poisson variety.*

A Poisson deformation $(\mathcal{Y}, \{-,-\}) \to B$ is called infinitesimal *if $B = \operatorname{Spec} A$ where A is an Artinian algebra with residue field $\mathbb{C}$.*

Definition 4.10. *A Poisson deformation $(\mathcal{Y}, \{-,-\}) \to B$ of a Poisson variety $(Y, \{-,-\}_0)$ is called* universal at 0 *if for each infinitesimal Poisson deformation $(\mathcal{X}, \{-,-\}') \to (\operatorname{Spec} A, \mathfrak{m}_A)$ there exists a unique morphism $f : \operatorname{Spec} A \to B$ such that $f(\mathfrak{m}_A) = 0$ and the diagram*

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Y} \arrow[d] \\
\operatorname{Spec} A \arrow[r, "f"] & B
\end{tikzcd}$$

which is cartesian.

In general we have the following:

Theorem 4.11 ([Nam15]). *Let Y_0 be a conical symplectic variety with a projective symplectic resolution $\pi : Y \to Y_0$. Then there exists the universal Poisson deformation spaces $\mathcal{Y} \to H^2(Y, \mathbb{C})$ and $\mathcal{Y}_0 \to H^2(Y, \mathbb{C})/W$ of Y and Y_0, respectively, and they satisfy the following $\mathbb{C}^*$-commutative diagram:*

$$\begin{tikzcd}
& Y \arrow[rr, "\pi"] \arrow[dl, hook'] \arrow[dd] & & Y_0 \arrow[dl, hook'] \arrow[dd] \\
\mathcal{Y} \arrow[rr, "\Pi", crossing over] \arrow[dd, "\bar{\mu}"] & & \mathcal{Y}_0 & \\
& 0 \arrow[rr, mapsto] \arrow[dl, hook'] & & \bar{0} \arrow[dl, hook'] \\
H^2(Y, \mathbb{C}) \arrow[rr, "\psi"] & & H^2(Y, \mathbb{C})/W \arrow[from=uu, crossing over, "\bar{\mu}_W"] &
\end{tikzcd}$$

where ψ is a Galois cover with finite Galois group W acts linearly on $H^2(Y, \mathbb{C})$ which is called the Namikawa–Weyl group *of Y_0.*

Some comments. First, the singular locus $(Y_0)_{\mathrm{sing}}$ is stratified by smooth symplectic varieties. Let $\Sigma_{\mathrm{codim} \geqslant 4}$ denote the union of strata of codimension 4 or higher, and define $\Sigma_{\mathrm{codim} 2} := (Y_0)_{\mathrm{Sing}} \backslash \Sigma_{\mathrm{codim} \geqslant 4}$. Then, for each component Z_k of the connected component decomposition $\Sigma_{\mathrm{codim}\ 2} = \bigsqcup_{k=1}^{s} Z_k$, one can consider a transversal slice S_{ℓ_k} through a point $x \in Z_k$. Since $S_{\ell_k} = S_{\Delta_{\ell_k}}$ is a symplectic surface, i.e., the ADE type surface singularity with the corresponding Dynkin diagram Δ_{ℓ_k}, so $\pi : Y \to Y_0$ is locally (at x) isomorphic to $p \times \mathrm{id} : \tilde{S}_{\ell_k} \times \mathbb{C}^{2m-2} \to S_{\ell_k} \times \mathbb{C}^{2m-2}$, where $2m = \dim Y_0$ and p is the minimal resolution of S_{ℓ_k}. We consider all (-2)-curves C_i $(1 \leqslant i \leqslant \ell_k)$ in $\tilde{S}_{\ell_k}$ and set

$$\Phi_{\ell_k} := \left\{ \sum_{i=1}^{\ell_k} d_i [C_i] \,\middle|\, d_i \in \mathbb{Z} \text{ s.t. } \left(\sum_{i=1}^{\ell_k} d_i [C_i] \right)^2 = -2 \right\} \subset H^2(\tilde{S}_{\ell_k}, \mathbb{R}).$$

Then, Φ_{ℓ_k} defines the corresponding ADE type root system in $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$, and the associated usual Weyl group $W_{S_{\ell_k}}$ acts on $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$. However this description is local at each point on Z_k, and the number of irreducible components of $\pi^{-1}(Z_k)$ may be less than ℓ_k globally. In fact, the following homomorphism is defined by the monodromy:

$$\rho_k : \pi_1(Z_k) \to \mathrm{Aut}(\Delta_{\ell_k}),$$

where Δ_{ℓ_k} is the associated Dynkin diagram and $\mathrm{Aut}(\Delta_{\ell_k})$ is its graph automorphism group. Then, we can define the subgroup of $W_{S_{\ell_k}}$ as

$$W_{Z_k} := W_{S_{\ell_k}}^{\mathrm{Im}\,\rho_k} := \{\sigma \in W_{S_{\ell_k}} \mid \sigma\iota = \iota\sigma^2 (\iota \in \mathrm{Im}\,\rho_k)\}.$$

Finally, taking the direct product of them, we get the Namikawa-Weyl group:

$$W := \prod_k W_{Z_k}.$$

Well done. $\square$

In our case of hypertoric varieties, we have the following results:

Theorem 4.12 (Thm 3.11 in [Nag21])**.** *Let A be a unimodular matrix and $\alpha \in \mathbb{Z}^d$ be a generic element. If for B, $\boldsymbol{b}_i \neq 0 (1 \leqslant i \leqslant n)$ and we take B as*

$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(s)} \end{pmatrix}, \quad B^{(k)} = \left.\begin{pmatrix} \boldsymbol{b}^{(k)} \\ \boldsymbol{b}^{(k)} \\ \vdots \\ \boldsymbol{b}^{(k)} \end{pmatrix}\right\} \ell_k$$

where if $k_1 \neq k_2$, then $\boldsymbol{b}^{(k_1)} \neq \pm \boldsymbol{b}^{(k_2)}$. Then the diagram of Theorem 4.11 for the affine hypertoric variety $Y(A,0)$ is obtained as

$$\begin{tikzcd}
 & Y(A,\alpha) \arrow[rr, "\pi"] \arrow[dl, hook] \arrow[dd] & & Y(A,0) \arrow[dl, hook] \arrow[dd] \\
X(A,\alpha) \arrow[rr, "\Pi_{W_B}"] \arrow[dd, "\bar{\mu}_\alpha"] & & X(A,0)/W_B \arrow[dd, "\bar{\mu}_{W_B}"] & \\
 & 0 \arrow[rr, mapsto] \arrow[dl, hook] & & \bar{0} \arrow[dl, hook] \\
\mathbb{C}^d \arrow[rr, "\psi"] & & \mathbb{C}^d/W_B &
\end{tikzcd}$$

where Π_{W_B} is the composition of $X(A,\alpha) \to X(A,0)$ and the quotient map of $X(A,0)$ by $W_B := \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_s}$.

Sketch. First we need to show that $\bar{\mu}_\alpha : X(A,\alpha) \to \mathbb{C}^d$ and $\bar{\mu}_0 : X(A,0) \to \mathbb{C}^d$ are Poisson deformations of $Y(A,\alpha)$ and $Y(A,0)$, respectively. Note that $X(A,\alpha)$ is smooth and $X(A,0)$ is Cohen-Macaulay by a result due to Hochster, then by miracle-flatness $\bar{\mu}_\alpha$ and $\bar{\mu}_0$ are flat. Then these are right by definition.

Next we need to analyze the structure of $\Sigma_{\mathrm{codim}2}$ in order to describe the Namikawa–Weyl group. Note that in this case we already have the following diagram:

$$\begin{tikzcd}
 & Y(A,\alpha) \arrow[rr, "\pi"] \arrow[dl, hook] \arrow[dd] & & Y(A,0) \arrow[dl, hook] \arrow[dd] \\
X(A,\alpha) \arrow[rr, "\Pi"] \arrow[dd, "\bar{\mu}_\alpha"] & & X(A,0) \arrow[dd, "\bar{\mu}_0"] & \\
 & 0 \arrow[rr, mapsto] \arrow[dl, hook] & & \bar{0} \arrow[dl, hook] \\
\mathbb{C}^d \arrow[rr, "="] & & \mathbb{C}^d/W_B &
\end{tikzcd}$$

If one can construct a good W_B-action on $X(A,0)$ and $\mathbb{C}^d$, then one can show $W = W_B$ and construct the universal Poisson deformation of $Y(A,0)$ (Lemma 3.8 in [Nag21]).

Note that we have already take B as

$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(s)} \end{pmatrix}, \quad B^{(k)} = \left.\begin{pmatrix} \boldsymbol{b}^{(k)} \\ \boldsymbol{b}^{(k)} \\ \vdots \\ \boldsymbol{b}^{(k)} \end{pmatrix}\right\} \ell_k$$

where if $k_1 \neq k_2$, then $\boldsymbol{b}^{(k_1)} \neq \pm \boldsymbol{b}^{(k_2)}$. Then we let $W_B := \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_s}$ act $\mathbb{C}^{2n}$ as $z_i \mapsto z_{\sigma(i)}, w_i \mapsto w_{\sigma(i)}$ and act on $\mathbb{C}^n$ as $u_i \mapsto u_{\sigma(i)}$. Now one can show that W_B-action on $\mathbb{C}^{2n}$ induce an action on $X(A,0)$ and W_B-action on $\mathbb{C}^n$ induce an action on $\mathbb{C}^d$ via $A : \mathbb{C}^n \to \mathbb{C}^d$. Then we get the result. $\square$

Remark 4.13. *By definition, the W_B-action on $\mathbb{C}^{2n}$ does not commute with the $\mathbb{T}^d$-action on it in general.*

5. Wall-crossing structures, Mukai flops and counting crepent resolutions

Here we will follows [HD14]. We always assume our case is unimodular.

5.1. **Wall-chamber structure of semistable conditions.** We review our setting of hypertoric varieties:

Consider the exact sequence

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$

where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$ (the Gale duality of $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_n\}$). Acting $\mathrm{Hom}(-, \mathbb{C}^*)$ we get

$$1 \to \mathbb{T}^d \xrightarrow{A^T} \mathbb{T}^n \xrightarrow{B^T} \mathbb{T}^{n-d} \to 1$$

an exact sequence of algebraic tori.

Now we also have $0 \to \mathbb{C}^d \xrightarrow{A^T} \mathbb{C}^n \xrightarrow{B^T} \mathbb{C}^{n-d} \to 0$. Now we let A is a unimodular matrix. We have defined the hyperplane arrangement

$$\mathcal{H}_A := \{H \subset \mathbb{R}^d : H \text{ is generated by some of the } \boldsymbol{a}_j \text{ and of codimension} = 1\}.$$

Here we will give another description and more precise analysis of this.

Definition 5.1. *For each subset $C \subset \{1, ..., n\}$, let $\mathfrak{k}_C := \mathbb{C}^d \cap \mathrm{span}\{e_i : i \in C\}$ where e_i are standard vectors. Then we call C is a circuit if $\dim \mathfrak{k}_C = 1$.*

Note that $\boldsymbol{v} \in \mathfrak{k}_C := \mathbb{C}^d \cap \mathrm{span}\{e_i : i \in C\}$ if and only if $A^T \cdot \boldsymbol{v} \in \mathrm{span}\{e_i : i \in C\}$ if and only if $\boldsymbol{v} \in (\mathrm{span}\{\boldsymbol{a}_i : i \notin C\})^{\perp}$. Hence

$$H_C := (\mathfrak{k}_C)^{\perp}_{\mathbb{R}} = \mathrm{span}\{\boldsymbol{a}_i : i \notin C\} \subset \mathbb{R}^d.$$

This gives another description of the hyperplane arrangement $\mathcal{H}_A$ and we will give it a name:

Definition 5.2. *For each circuit C, the associated discriminantal hyperplane is $H_C \subset \mathbb{R}^d$ as above. The discriminantal arrangement is the collection of all discriminantal hyperplanes which is our $\mathcal{H}_A$ as above.*

We have the following statement which is strengthen our results as before:

Proposition 5.3 ([Kon00]). *A character $\alpha \in \mathbb{Z}^d$ such that $Y(A, \alpha)$ is smooth if and only if it does not lie on any discriminantal hyperplane.*

Here using circuit we can define orientation of wall more easier.

Definition 5.4. *Let C be a circuit and $\alpha \in \mathbb{Z}^d \subset \mathbb{R}^d$ a character with $\alpha \notin H_C$. Let β^{α}_C be a generator of $(\mathfrak{k}_C)_{\mathbb{Z}} \cong \mathbb{Z}$ such that $\alpha \cdot \beta^{\alpha}_C > 0$. We then define*

$$C^{\alpha}_{+} := \{i \in C : \boldsymbol{a}_i \cdot \beta^{\alpha}_C > 0\}, \quad C^{\alpha}_{-} := \{i \in C : \boldsymbol{a}_i \cdot \beta^{\alpha}_C < 0\}.$$

We refer to the partition $C = C^{\alpha}_{+} \sqcup C^{\alpha}_{-}$ as an orientation of C.

That is, $i \in C_+^\alpha$ if $\boldsymbol{a}_i$ and α are in the same connected component of $\mathbb{R}^d \backslash H_C$, and $i \in C_-^\alpha$ if they are in different components. As A is unimodular, then we have

$$\beta_C^\alpha = \sum_{i \in C_+^\alpha} e_i - \sum_{i \in C_-^\alpha} e_i$$

as we consider it in $\mathbb{C}^n$ (we always doing this).

Proposition 5.5. *Let α be a smooth character, then*

$$\mu^{-1}(0) = \left\{ (\boldsymbol{z}, \boldsymbol{w}) \in T^*\mathbb{C}^n : \sum_{i \in C_+^\alpha} z_i w_i = \sum_{i \in C_-^\alpha} z_i w_i \text{ for all circuits } C \right\}.$$

Proof. View $\mathbb{C}^d$ in $\mathbb{C}^n$ via A^T, the moment map given by

$$\mu(\boldsymbol{z}, \boldsymbol{w}) : \mathbb{C}^d \to \mathbb{C}, \quad (x_i) \mapsto \sum_i z_i w_i x_i.$$

Hence $\mu(\boldsymbol{z}, \boldsymbol{w}) = 0$ if and only if $\sum_i z_i w_i e_i^*(\boldsymbol{x}) = 0$ for any $\boldsymbol{x} \in \mathbb{C}^d$. From the definition of circuit, $\mathbb{C}^d$ is generated over $\mathbb{Z}$ by the subtori $\mathfrak{k}_C$. It follows that $\mu(\boldsymbol{z}, \boldsymbol{w}) = 0$ if and only if $\sum_i z_i w_i e_i^*(\beta_C^\alpha) = 0$ for all circuits C, which gives the claim immediately. □

Definition 5.6. *Let C be a circuit for the action of $\mathbb{T}^d$ on $T^*\mathbb{C}^n$. As $\dim \mathfrak{k}_C = 1$, let $\mathbb{T}_C$ be the rank-1 subtorus of $\mathbb{T}^d$ whose Lie algebra is $\mathfrak{k}_C$. We further denote by $\overline{\mathbb{T}}_C$ the quotient torus $\mathbb{T}^d/\mathbb{T}_C$, and by $\bar{\mathfrak{k}}_C$ its Lie algebra $\mathbb{C}^d/\mathfrak{k}_C$.*

Hence in this case $(\bar{\mathfrak{k}}_C)^*_{\mathbb{R}} = H_C$. The torus $\overline{\mathbb{T}}_C$ does not naturally act on $\mathbb{C}^n$, but since $\mathbb{T}_C$ acts trivially on the coordinates z_i and w_i for $i \notin C$, we do have an action of $\overline{\mathbb{T}}_C$ on $E_C := \operatorname{span}\{e_i : i \notin C\}$. Then we have an action of $\overline{\mathbb{T}}_C$ on $T^*E_C \subset T^*\mathbb{C}^n$.

Definition 5.7. *A character of $\mathbb{T}^d$ is said to be* **subsmooth** *if it lies on exactly one discriminantal hyperplane.*

5.2. **Mukai flops and its family-version.**

5.3. **Wall-crossing of hypertoric varieties as Mukai flops in a family.**

5.4. **An application: counting their projective crepent resolutions.**

6. Cohomology of hypertoric varieties

References

[BD00] Roger Bielawski and Andrew Dancer. The geometry and topology of toric hyperkähler manifolds. *Comm. Anal. Geom.*, pages 727–760, 2000.

[Bea00] Arnaud Beauville. Symplectic singularities. *Invent. Math.*, pages 541–549, 2000.

[Bel23] Gwyn Bellamy. Coulomb branches have symplectic singularities. *https://arxiv.org/abs/2304.09213*, 2023.

[CLS11] David Cox, John Little, and Hal Schenck. *Toric Varieties*. American Mathematical Society, 2011.

[Fu03] Baohua Fu. Symplectic resolutions for nilpotent orbits. *Invent. Math.*, pages 167–186, 2003.

[Fu06] Baohua Fu. A survey on symplectic singularities and symplectic resolutions. *Ann. Math. Blaise Pascal*, pages 209–236, 2006.

[Ful93] William Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[HD14] Brad Hannigan-Daley. Hypertoric varieties and wall-crossing. *PH.D. Thesis*, 2014.

[HS02] Tamás Hausel and Bernd Sturmfels. Toric hyperkähler varieties. *Doc. Math.*, pages 495–534, 2002.

[Kon00] Hiroshi Konno. Cohomology rings of toric hyperk¨ahler manifolds. *Internat. J. Math.*, pages 1001–1026, 2000.

[Nag21] Takahiro Nagaoka. The universal poisson deformation of hypertoric varieties and some classification results. *Pacific Journal of Mathematics*, pages 459–508, 2021.

[Nam15] Yoshinori Namikawa. Poisson deformations and birational geometry. *J. Math. Sci. Univ. Tokyo*, pages 339–359, 2015.

[Pro04] Nicholas Proudfoot. Hyperkähler analogues of kähler quotients. *Ph.D. Thesis, https://arxiv.org/abs/math/0405233v1*, 2004.

[Tel22] Simon Telen. Introduction to toric geometry. *https://arxiv.org/abs/2203.01690*, 2022.

Institute of Mathematics, AMSS, Chinese Academy of Sciences, 55 Zhongguancun East Road, Beijing, 100190, China

Email address: liuxiaolong@amss.ac.cn