Kuznetsov components, Stability, and Moduli Spaces

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Preface

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Chapter 1

Derived Category and Semi-Orthogonal Decomposition

Here we follows some definitions and results in [8]. Note that when I working in the derived category, we will omit the **R** or **L** of the derived functors. Also, we only consider the schemes over $\mathbb{C}$.

1.1 Basic Definitions

Definition 1.1.1. *A full triangulated subcategory $\mathscr{D}' \subset \mathscr{D}$ is called* admissible *if the inclusion has a right adjoint $\pi : \mathscr{D} \to \mathscr{D}'$*

The orthogonal complement *of a(an admissible) subcategory $\mathscr{D}' \subset \mathscr{D}$ is the full subcategory $\mathscr{D}'^{\perp}$ of all objects $C \in \mathscr{D}$ such that $\mathrm{Hom}(B, C) = 0$ for all $B \in \mathscr{D}'$.*

Definition 1.1.2. *An object $E \in \mathscr{D}$ in a k-linear triangulated category $\mathscr{D}$ is called* exceptional *if*

$$\mathrm{Hom}(E, E[\ell]) = \begin{cases} k, if\ \ell = 0, \\ 0, if\ \ell \neq 0. \end{cases}$$

An exceptional sequence *is a sequence $E_1, ..., E_n$ of exceptional objects such that $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all $i > j$ and all ℓ.*

An exceptional sequence is full *if $\mathscr{D}$ is generated by $\{E_i\}$.*

An exceptional collection $E_1, ..., E_n$ is strong *if in addition $\mathrm{Hom}(E_i, E_j[\ell]) = 0$ for all i, j and all $\ell \neq 0$.*

Definition 1.1.3. *A sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is* semi-orthogonal *if for all $i > j$ we have $\mathscr{D}_j \subset \mathscr{D}_i^{\perp}$. Such a sequence defines a* semi-orthogonal decomposition *(S.O.D.) of $\mathscr{D}$ if $\mathscr{D}$ is generated by the $\mathscr{D}_i$.*

Remark 1.1.4. *Some remarks:*

(a) *If $E \in \mathscr{D}$ is exceptional, then the objects $\bigoplus_i E[i]^{\oplus j_i}$ form an admissible triangulated subcategory $\langle E \rangle \subset \mathscr{D}$.*

(b) *Let $E_1, ..., E_n$ be an exceptional sequence in $\mathscr{D}$. Then the admissible triangulated subcategories $\langle E_1 \rangle, ..., \langle E_n \rangle$ form a semi-orthogonal sequence. If this sequence is a full exceptional sequence, then this forms an S.O.D. of $\mathscr{D}$.*

(c) *Any semi-orthogonal sequence of full admissible triangulated subcategories $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ defines an S.O.D. of $\mathscr{D}$, if and only if any object $A \in \mathscr{D}$ with $A \in \mathscr{D}_i^{\perp}$ for all $i = 1, ..., n$ is trivial. See Lemma 1.61 in [8].*

(d) *If $\mathscr{D}_1, ..., \mathscr{D}_n \subset \mathscr{D}$ is an S.O.D., then $D_1 \subset \langle \mathscr{D}_2, ..., \mathscr{D}_n \rangle^{\perp}$ is an equivalence. See Exercise 1.62 in [8].*

Definition 1.1.5. *Fix an algebraic variety X and a line bundle $\mathscr{L}$ over it.*

(a) *A* right Lefschetz decomposition *of $\mathbf{D}^b(X)$ with respect to $\mathscr{L}$ is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_0, \mathscr{D}_1 \otimes \mathscr{L}, ..., \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(m-1)} \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

(b) *A* left Lefschetz decomposition *of $\mathbf{D}^b(X)$ with respect to $\mathscr{L}$ is a S.O.D of form*

$$\mathbf{D}^b(X) = \left\langle \mathscr{D}_{m-1} \otimes \mathscr{L}^{\otimes(1-m)}, ..., \mathscr{D}_1 \otimes \mathscr{L}^{\otimes(-)}, \mathscr{D}_0 \right\rangle$$

where $0 \subset \mathscr{D}_{m-1} \subset \cdots \subset \mathscr{D}_1 \subset \mathscr{D}_0$.

The subcategories $\mathscr{D}_i$ forming a Lefschetz decomposition will be called blocks*, the largest will be called the* first block*. Usually we will consider right Lefschetz decompositions. So, we will call them simply Lefschetz decompositions. We call a Lefschetz decompositions is* rectangular *if $\mathscr{D}_{m-1} = \cdots = \mathscr{D}_1 = \mathscr{D}_0$.*

If we need to consider the moduli space, we need to consider the family version of S.O.D:

Definition 1.1.6. *A triangulated category $\mathscr{T}$ is S-*linear *if it is equipped with a module structure over the tensor triangulated category $\mathbf{D}^b(S)$. In particular, if X is a scheme over S and $f : X \to S$ is the structure morphism then an S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle$$

*is S-*linear *if each of the subcategories $\mathscr{A}_k$ satisfies that for $A \in \mathscr{A}_k$ and $F \in \mathbf{D}^b(S)$ one has $A \otimes f^*F \in \mathscr{A}_k$.*

Theorem 1.1.7 (Kuznetsov)**.** *If X is an algebraic variety over S with an S-linear S.O.D*

$$\mathbf{D}^b(X) = \langle \mathscr{A}_1, ..., \mathscr{A}_m \rangle \,,$$

then for a change of base morphism $T \to S$ there is, under a certain technical condition, a T-linear S.O.D

$$\mathbf{D}^b(X \times_S T) = \langle \mathscr{A}_{1T}, ..., \mathscr{A}_{mT} \rangle \,,$$

such that $\pi^* A \in \mathscr{A}_{iT}$ for any $A \in \mathscr{A}_i$ and $\pi_*(A') \in \mathscr{A}_i$ for any $A' \in \mathscr{A}_{iT}$ which has proper support over X.

Proof. See [11]. □

1.2 Example I – Projective Bundles

Proposition 1.2.1. *For a smooth projective variety Y we consider the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to Y$ of locally free sheaf $\mathscr{E}$ of rank r on Y, in the sense of Grothendieck. Then for any $a \in \mathbb{Z}$ we claim that $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a+r-1)$ is an S.O.D. of $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.*

Remark 1.2.2. *Hence this is a rectangular Lefschetz decomposition where all $\mathscr{D}_i = \pi^*\mathbf{D}^b(Y)$ and $\mathscr{L} = \mathscr{O}(1)$.*

This combined by the following two things:

Step 1. For any $E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(m), F \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(n)$, we have $\mathrm{Hom}(E, F) = 0$ for any $r - 1 \geq m - n > 0$.

Indeed, we can let $m = 0$ and hence $-r + 1 \leq n < 0$. Let $E = \pi^* E'$ and $F = \pi^* F' \otimes \mathscr{O}(n)$, hence

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(E', \pi_*(\pi^* F' \otimes \mathscr{O}(n))) = \mathrm{Hom}(E', F' \otimes \pi_* \mathscr{O}(n)).$$

It's well-known that $\mathbf{R}^i\pi_*\mathscr{O}(n) = \begin{cases} \mathrm{Sym}^n \mathscr{E}, \text{for } i = 0, \\ 0, \text{for } 0 < i < r-1, \\ \mathrm{Sym}^{-n-r}\mathscr{E}^\vee, \text{for } i = r-1. \end{cases}$ Well done.

Step 2. Categories $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a)$,..., $\pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(a+r-1)$ generates $\mathbf{D}^b(\mathbb{P}(\mathscr{E}))$.

Here we generalize the proof for $\mathbb{P}^n$ in [8] Corollary 8.29. Consider

$$\begin{tikzcd} & \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}) \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, phantom, "\wedge"] & \\ \mathbb{P}(\mathscr{E}) \arrow[dr, "\pi_1"] & & \mathbb{P}(\mathscr{E}) \arrow[dl, "\pi_2"'] \\ & Y & \end{tikzcd}$$

then by the canonical identification

$$\begin{aligned}
&H^0(\mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E}), \mathscr{O}(1) \boxtimes \mathscr{Q}^\vee) \\
&= H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes p_* q^* \mathscr{Q}^\vee) \\
&= H^0(\mathbb{P}(\mathscr{E}), \mathscr{O}(1) \otimes \pi_1^* \pi_{2,*} \mathscr{Q}^\vee) \\
&= H^0(Y, \pi_{1,*} \mathscr{O}(1) \otimes \pi_{2,*} \mathscr{Q}^\vee) \\
&= H^0(Y, \mathscr{E} \otimes \mathscr{E}^\vee)
\end{aligned}$$

where $0 \to \mathscr{Q} \to \pi^* \mathscr{E} \to \mathscr{O}(1) \to 0$ is the universal exact sequence. Let s correspond to the $\mathrm{id}_{\mathscr{E}}$, then $Z(s) = \Delta \subset \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{E})$. By the Koszul resolution of $\mathscr{O}_\Delta$ respect to the s, we have an exact sequence:

$$\begin{aligned}
&0 \to \bigwedge^{r-1} (\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2} (\mathscr{O}(-1) \boxtimes \mathscr{Q}) \\
&\to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.
\end{aligned}$$

(you can also use the Euler exact sequence instead of the universal exact sequence, just as in [8] Corollary 8.29)

Now there is to way to slove this.

The First Way: for any coherent sheaf $\mathscr{F} \in \mathrm{Coh}(\mathbb{P}(\mathscr{E}))$, tensoring $q^* \mathscr{F}$ we have

$$\begin{aligned}
&0 \to \mathscr{O}(-r+1) \boxtimes \bigwedge^{r-1} \mathscr{Q} \otimes \mathscr{F} \to \mathscr{O}(-r+2) \boxtimes \bigwedge^{r-2} \mathscr{Q} \otimes \mathscr{F} \\
&\to \cdots \to \mathscr{O}(-1) \boxtimes (\mathscr{Q} \otimes \mathscr{F}) \to \mathscr{O} \boxtimes \mathscr{F} \to q^* \mathscr{F}|_\Delta \to 0.
\end{aligned}$$

Consider a spectral sequence

$$\begin{aligned}
E_1^{ij} &= \mathbf{R}^i p_* (\mathscr{O}(j) \boxtimes \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F}) = \mathscr{O}(j) \otimes \mathbf{R}^i p_* q^* \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F} \\
&= \mathscr{O}(j) \otimes \pi_1^* \mathbf{R}^i \pi_{2,*} \bigwedge^{-j} \mathscr{Q} \otimes \mathscr{F} \Rightarrow \mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta.
\end{aligned}$$

We know that $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = 0$ if $i+j \neq 0$ and $\mathbf{R}^{i+j} p_* q^* \mathscr{F}|_\Delta = \mathscr{F}$ if $i+j=0$. Since any E_1^{ij} contained in

$$\left\langle \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^* \mathbf{D}^b(Y) \otimes \mathscr{O}(0) \right\rangle,$$

so is $\mathscr{F}$. Hence well done (if you use the Euler exact sequence instead of the universal exact sequence, the similar spectral sequence called the generalized Beilinson spectral sequence as Proposition 8.28 in [8]).

The Second Way: Consider again the Koszul resolution

$$0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q})$$
$$\to \cdots \to \mathscr{O}(-1) \boxtimes \mathscr{Q} \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.$$

Split it into short exact sequences

$$0 \to \bigwedge^{r-1}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to \bigwedge^{r-2}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-2} \to 0,$$
$$0 \to M_{r-2} \to \bigwedge^{r-3}(\mathscr{O}(-1) \boxtimes \mathscr{Q}) \to M_{r-3} \to 0,$$
$$\cdots,$$
$$0 \to M_1 \to \mathscr{O} \boxtimes \mathscr{O} \to \mathscr{O}_\Delta \to 0.$$

Tensor product with q^*F and direct image under the first projection p yields distinguished triangles of Fourier-Mukai transforms:

$$\Phi_{M_{i+1}}(\mathscr{F}) \to \Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \to \Phi_{M_i}(\mathscr{F}) \to \Phi_{M_{i+1}}(\mathscr{F})[1].$$

Easy to see that
$$\Phi_{\bigwedge^i(\mathscr{O}(-1)\boxtimes\mathscr{Q})}(\mathscr{F}) \in \left\langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-i) \right\rangle.$$

By induction we get $F = \Phi_{\mathscr{O}_\Delta} F \in \langle \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}(-r+1), ..., \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O} \rangle$. Well done.

Fully Exceptional Sequence. By the discussed above, we know that pick any fully exceptional sequence $E_1, ..., E_n$ of Y, the set

$$\{\pi^*E_1 \otimes \mathscr{O}(a), ..., \pi^*E_n \otimes \mathscr{O}(a), ..., \pi^*E_1 \otimes \mathscr{O}(a+r-1), ..., \pi^*E_n \otimes \mathscr{O}(a+r-1)\}$$

is a fully exceptional sequence of $\mathbb{P}(\mathscr{E})$ for any $a \in \mathbb{Z}$.

Example 1.2.1. *More general case, such as Grassmann-bundle and even the flag bundle has the similar things. We refer [13].*

We even have the similar about the general Brauer-Severi variety which need the twist derived category. See [2].

1.3 Example II – Blow-Ups

Here we follows section 11.1 in [8]. First we need some results about closed immersions.

Lemma 1.3.1. *Suppose $j : Y \hookrightarrow X$ of codimension C with normal bundle $\mathscr{N}$ is the zero locus of a regular section of a locally free sheaf $\mathscr{E}$ of rank c. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following canonical isomorphisms:*

$$\text{(i)} j^*j_*\mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{N}^\vee[k],$$
$$\text{(ii)} j_*j^*j_*F \simeq j_*\mathscr{O}_Y \otimes j_*F \simeq j_*\left(\bigoplus \bigwedge^k \mathscr{N}^\vee[k] \otimes F\right),$$
$$\text{(iii)} \mathscr{H}om_X(j_*\mathscr{O}_Y, j_*F) \simeq j_*\left(\bigoplus \bigwedge^k \mathscr{N}[-k] \otimes F\right).$$

In particular, we have

$$\mathcal{H}^\ell(j^*j_*F) \simeq \bigoplus_{s-r=\ell} \bigwedge^r \mathscr{N}^\vee \otimes \mathcal{H}^s(F)$$
$$\mathscr{E}xt^\ell_X(j_*\mathscr{O}_Y, j_*F) \simeq j_*\left(\bigoplus_{r+s=\ell} \bigwedge^r \mathscr{N} \otimes \mathcal{H}^s(F)\right).$$

Proof. For (i), by Koszul resolution we get $j^*j_*\mathscr{O}_Y \simeq \bigwedge^* \mathscr{E}^\vee|_Y$. As the differentials in the Koszul complex $\bigwedge^* \mathscr{E}^\vee$ are given by contraction with the defining section, they become trivial on Y. Hence $j^*j_*\mathscr{O}_Y \simeq \bigoplus \bigwedge^k \mathscr{E}^\vee[k]|_Y$. As $\mathscr{E}|_Y \cong \mathscr{N}$, well done.

For (ii), we split the Koszul resolution into the following short exact sequences:

$$\begin{tikzcd}
 & & & M_i \arrow[dr] & & \\
\cdots \arrow[r] & \bigwedge^{i+1} \mathscr{E}^\vee \arrow[rr] \arrow[dr] & & \bigwedge^i \mathscr{E}^\vee \arrow[rr] \arrow[u] & & \bigwedge^{i-1} \mathscr{E}^\vee \arrow[r] & \cdots \\
 & & M_{i+1} \arrow[ur] & & &
\end{tikzcd}$$

Again all these morphisms vanish on Y, we have

$$M_i \otimes j_*F \simeq \left(\bigwedge^i \mathscr{E}^\vee \otimes j_*F\right) \oplus (M_{i+1}[1] \otimes j_*F).$$

Putting these togetherand we get the result.

For (iii), as we have $\mathscr{H}om_X(j_*\mathscr{O}_Y, j_*F) \simeq \left(\bigwedge^i \mathscr{E}^\vee\right)^\vee \otimes j_*F$, then by the similar argument of (ii) we get the result.

The final part follows from (ii)(iii) and the fact that j_* is exact and tensor product with the locally free sheaf commutes with taking cohomology. $\square$

Corollary 1.3.2. *Let $j : Y \hookrightarrow X$ be a smooth hypersurface. Then for any $F \in \mathbf{D}^b(Y)$ there exists the following distinguished triangle*

$$F \otimes \mathscr{O}_Y(-Y)[1] \to j^* j_* F \to F \to F \otimes \mathscr{O}_Y(-Y)[2].$$

Proof. We omit it and refer [8] Corollary 11.4. □

Lemma 1.3.3. *Let $j : Y \hookrightarrow X$ be an arbitrary closed embedding of smooth varieties. Then there exist isomorphisms*

$$\mathcal{H}^i(j^* j_* \mathscr{O}_Y) \simeq \bigwedge^{-i} \mathscr{N}_{Y/X}^{\vee}, \quad \mathscr{E}xt^i_X(j_*\mathscr{O}_Y, j_*\mathscr{O}_Y) \simeq \bigwedge^{i} \mathscr{N}_{Y/X}.$$

Proof. Here we just give an idea, the detail we refer Proposition 11.8 in [8]. Here we first pick a global resolution of locally free sheaves $\mathscr{G}^* \to \mathscr{O}_Y$ and get the free resolution $\mathscr{G}^*_y \to \mathscr{O}_{Y,y}$. Also we can let Y defined by a section of a vector bundle near y, hence we get a local Koszul resolution. Hence at the point y we can get the result from before. Easy to see that this is independent of any choice, we get the result. □

Proposition 1.3.4. *Let $q : \widetilde{X} \to X$ be the blow-up along a smooth subvariety $Y \subset X$. Then for the structure sheaf $\mathscr{O}_Z$ of a subvariety $Z \subset Y$ considered as an object in $\mathbf{D}^b(X)$ one has*

$$\mathcal{H}^k(q^* \mathscr{O}_Z) \simeq (\Omega_\pi^{\otimes -k} \otimes \mathscr{O}_\pi(-k))|_{\pi^{-1}(Z)}$$

where $\pi : \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor.

Proof. We will only show the case that $Y \subset X$ is given as the zero set of a regular section $s \in H^0(X, \mathscr{E})$ of a locally free sheaf $\mathscr{E}$ of rank c. The general case follows from this and the similar argument of Lemma 1.3.3, we refer [8] Proposition 11.12 for details.

Consider $g : \mathbb{P}(\mathscr{E}) \to X$ and consider the Euler sequence

$$0 \to \mathscr{O}_g(-1) \to g^*\mathscr{E} \xrightarrow{\phi} \mathscr{T}_g \otimes \mathscr{O}_g(-1) \to 0.$$

Let $t := \phi(g^*(s)) \in H^0(\mathbb{P}(\mathscr{E}), \mathscr{T}_g \otimes \mathscr{O}_g(-1))$ and consider the zero scheme $Z(t) \subset \mathbb{P}(\mathscr{E})$.

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Hence g induced $Z(t) \to X$ can be identified with the blow-up $q : \widetilde{X} \to X$. Pick the Koszul resolution $\bigwedge^*(\mathscr{O}_g(1) \otimes \Omega_g) \to \mathscr{O}_{\widetilde{X}} \to 0$ of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}$-modules, hence

$$\begin{aligned}\iota_*(\mathcal{H}^k(q^*\mathscr{O}_Z)) &\simeq \iota_*(\mathcal{H}^k(\iota^* g^* \mathscr{O}_Z)) \simeq \mathcal{H}^k(\iota_*\iota^* g^* \mathscr{O}_Z) \\ &\simeq \mathcal{H}^k(g^*\mathscr{O}_Z \otimes \mathscr{O}_{\widetilde{X}}) \simeq \mathcal{H}^k(\bigwedge^{*}(\mathscr{O}_g(1) \otimes \Omega_g)|_{g^{-1}(Z)})\end{aligned}$$

where $\iota : \widetilde{X} = Z(t) \hookrightarrow \mathbb{P}(\mathscr{E})$. If Z is contained in Y , the differentials, which are given by contraction with the section t, vanish and, therefore

$$\mathcal{H}^k(q^*\mathscr{O}_Z) \simeq (\Omega_g^{\otimes -k} \otimes \mathscr{O}_g(-k))|_{g^{-1}(Z)}.$$

Well done. □

Lemma 1.3.5. *Suppose $f : S \to T$ is a projective morphism of smooth projective varieties such that $f_* : \mathbf{D}^b(S) \to \mathbf{D}^b(T)$ sends $\mathscr{O}_S$ to $\mathscr{O}_T$. Then $f^* : \mathbf{D}^b(T) \to \mathbf{D}^b(S)$ is fully faithful and thus describes an equivalence of $\mathbf{D}^b(T)$ with an admissible triangulated subcategory of $\mathbf{D}^b(S)$.*

Proof. Trivial by the projection formula and $f^* \dashv f_*$, which shows directly $\mathrm{id} \simeq f_* f^*$, hence fully faithful. $\square$

Lemma 1.3.6. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Then the functor*

$$\Phi_k = i_*(\mathscr{O}_E(kE) \otimes \pi^*(-)) : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X})$$

is fully faithful for any k. Moreover, Φ_k admits a right adjoint functor.

Proof. The functor Φ_k is a Fourier-Mukai transform with kernel $\mathscr{O}_E(kE)$ considered as on object in $\mathbf{D}^b(Y \times \widetilde{X})$. As such, Φ_k admits in particular right and left adjoint. Now we will use a result due to Bondal-Orlov (Proposition 7.1 in [8]):

- Consider the Fourier-Mukai transform $\Phi_{\mathscr{P}} : \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$ between the derived categories of two smooth projective varieties X and Y given by an object $\mathscr{P} \in \mathbf{D}^b(X \times Y)$. Then the functor $\Phi_{\mathscr{P}}$ is fully faithful if and only if for any two closed points $x, y \in X$ one has

$$\mathrm{Hom}(\Phi_{\mathscr{P}}(\kappa(x)), \Phi_{\mathscr{P}}(\kappa(y))[i]) = \begin{cases} k, \text{if } x = y \text{ and } i = 0; \\ 0, \text{if } x \neq y \text{ or } i < 0 \text{ or } i > \dim(X). \end{cases}$$

For any j and $x \neq y$, this follows from the fact that the result objects have disjoint supports.

Now we let $x = y \in Y$. We need to show that $\mathrm{Ext}^i_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})$ is trivial for $i \notin [0, d = \dim Y]$ and of dimension one for $i = 0$. By Lemma 1.3.3 we get the spectral sequence

$$\begin{aligned} E_2^{p,q} &= H^p(\widetilde{X}, \mathscr{E}xt^q_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x})) = H^p\left(E_x, \bigwedge^q \mathscr{N}_{E_x/\widetilde{X}}\right) \\ &\Rightarrow \mathrm{Ext}^{p+q}_{\widetilde{X}}(\mathscr{O}_{E_x}, \mathscr{O}_{E_x}). \end{aligned}$$

Hence we need to determine $\mathscr{N}_{E_x/\widetilde{X}}$. Consider the exact sequence

$$0 \to \mathscr{N}_{E_x/E} \to \mathscr{N}_{E_x/\widetilde{X}} \to \mathscr{N}_{E/\widetilde{X}}|_{E_x} \to 0,$$

as $\mathscr{N}_{E/\widetilde{X}} = \mathscr{O}_E(E)$ and $\mathscr{N}_{E_x/E} = \mathscr{O}_{E_x}^{\oplus d}$ and since $E_x \cong \mathbb{P}^{c-1}$ one get

$$\mathscr{N}_{E_x/\widetilde{X}} \cong \mathscr{O}_{E_x}(-1) \oplus \mathscr{O}_{E_x}^{\oplus d}$$

by computing the Ext^1. Hence we can directly get the result. □

Proposition 1.3.7. *Let the smooth varieties $Y \subset X$ of codimension $c > 1$, and let $q : \widetilde{X} \to X$ be the blow-up with exceptional divisor $i : E \hookrightarrow \widetilde{X}$ and $\pi : E = \mathbb{P}(\mathscr{N}_{Y/X}) \to Y$ is the contraction of the exceptional divisor. Define*

$$\mathscr{D}_k := \mathrm{Im}(\Phi_{-k} : \mathbf{D}^b(Y) \to \mathbf{D}^b(\widetilde{X}))$$

for $k = -c+1, ..., -1$ and $\mathscr{D}_0 := q^*\mathbf{D}^b(X)$.

Then $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ forms an S.O.D of $\mathbf{D}^b(\widetilde{X})$.

Proof. We divided this into three parts:
Step 1. For $-c+1 \leq \ell < k < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_k^\perp$.

For any $E, F \in \mathbf{D}^b(Y)$ we have

$$\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), i_*(\pi^*E \otimes \mathscr{O}_\pi(\ell))) = \mathrm{Hom}(i^*i_*\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

By Corollary 1.3.2, we get the distinguished triangle:

$$\pi^*F \otimes \mathscr{O}_\pi(1)[1] \to i^*i_*\pi^*F \to \pi_*F \to \pi^*F \otimes \mathscr{O}_\pi(1)[2].$$

Hence we just need to show that

$$\mathrm{Hom}(\pi^*F, \pi^*E \otimes \mathscr{O}_\pi(\ell - k)) = 0 = \mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(1), \pi^*E \otimes \mathscr{O}_\pi(\ell - k)).$$

Both are easily deduced from adjunction $\pi^* \dashv \pi_*$, the projection formula, and $\pi_*\mathscr{O}_\pi(\ell - k) = 0$ for $-c+1 \leq \ell - k < 0$.
Step 2. For $-c+1 \leq \ell < 0$ we have $\mathscr{D}_\ell \subset \mathscr{D}_0^\perp$.

Again use $\pi_*\mathscr{O}_\pi(\ell) = 0$ for $-c+1 \leq \ell < 0$ to conclude this.
Step 3. We have $\mathscr{D}_{-c+1}, ..., \mathscr{D}_{-1}, \mathscr{D}_0$ generates $\mathbf{D}^b(\widetilde{X})$.

For this we let $E \in \mathscr{D}_k^\perp$ for all $-c+1 \leq k < 0$, then we claim that then exists an object $G \in \mathbf{D}^b(Y)$ with $i^*E \otimes \mathscr{O}_\pi(c-1) \simeq \pi^*G$.

By assumption, for any $-c+1 \leq k < 0$ one has $\mathrm{Hom}(i_*(\pi^*F \otimes \mathscr{O}_\pi(k)), E) = 0$ for all $F \in \mathbf{D}^b(Y)$. By Grothendieck duality we get for any $-c+2 \leq k < 1$ one has $\mathrm{Hom}(\pi^*F \otimes \mathscr{O}_\pi(k), i^*E) = 0$. By Proposition 1.2.1 we have $i^*E \in \pi^*\mathbf{D}^b(Y) \otimes \mathscr{O}_\pi(-c+1)$. Hence if we let $E' := E \otimes \mathscr{O}((-c+1)E)$, then $i^*E' \in \pi^*\mathbf{D}^b(Y)$. Pick such $G \in \mathbf{D}^b(Y)$ such that $i^*E' \simeq \pi^*G$.

If $i^*E' \simeq 0$, then $\mathrm{supp}(E') \subset E$ and $E' \in \mathscr{D}_0$.

If not, consider the spectral sequence

$$E_2^{r,s} = \mathrm{Hom}(E', \mathcal{H}^s(q^*\kappa(x))[r]) \Rightarrow \mathrm{Hom}(E', q^*\kappa(x)[r+s]).$$

By Proposition 1.3.4 we have $\mathcal{H}^s(q^*\kappa(x)) \simeq \Omega_{E_x}^{\otimes -s}(-s)$. Hence

$$\begin{aligned} E_2^{r,s} &\simeq \mathrm{Hom}(E', i_*\Omega_{E_x}^{\otimes -s}(-s)[r]) \\ &\simeq \mathrm{Hom}(\pi^*G, \Omega_{E_x}^{\otimes -s}(-s)[r]) \\ &\simeq \mathrm{Hom}(G, \pi_*\Omega_{E_x}^{\otimes -s}(-s)[r]) = 0 \end{aligned}$$

except for $s = 0$. Hence

$$E_2^{m,0} \simeq \mathrm{Hom}(G, \kappa(x)[m]) \simeq \mathrm{Hom}(q^*\kappa(x), E[\dim X - m])^\vee \neq 0$$

for some $m \in \mathbb{Z}$ and some $x \in Y$. Hence if $E \in \mathscr{D}_k^\perp$ for all $-c+1 \leq k < 0$, we cannot have $E \in \mathscr{D}_0^\perp$. Hence well done. □

1.4 Example III – Smooth Quadrics and Grassmannians

Here we follows the results in [10] and just give some results.

Proposition 1.4.1. *Let $\mathsf{Gr}(k, V)$ be the Grassmannian of k-dimensional subspaces in a vector space V of dimension n. Let $\mathscr{U}$ be thebtautological subbundle of rank k. If* $\mathrm{char} k = 0$ *then there is a strong S.O.D*

$$\mathbf{D}^b(\mathsf{Gr}(k, V)) = \left\langle \Sigma^\alpha \mathscr{U}^\vee \right\rangle$$

where α is a Young diagram in the $k \times (n-k)$ rectangle and Σ^α is the associated Schur functor.

Proof. We will not prove this. We refer the original proof in [10]. Note that as in the proof of the projective bundles, if we let $\mathscr{U}^\perp = ((V \otimes \mathscr{O}_{\mathsf{Gr}(k,V)})/\mathscr{U})^\vee$, then we can let a canonical section

$$s \in H^0(\mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V), \mathscr{U}^\vee \boxtimes (\mathscr{U}^\perp)^\vee) = V^\vee \otimes V = \mathrm{End}(V, V)$$

correspond to the id_V. Then s vanishes exactly along the diagonal $\Delta \subset \mathsf{Gr}(k, V) \times \mathsf{Gr}(k, V)$ which induce the Koszul resolution

$$\cdots \to \bigwedge^2 (\mathscr{U} \boxtimes \mathscr{U}^\perp) \to \mathscr{U} \boxtimes \mathscr{U}^\perp \to \mathscr{O}_{\mathsf{Gr}(k,V) \times \mathsf{Gr}(k,V)} \to \mathscr{O}_\Delta \to 0$$

where the i-th term is just the sum $\bigoplus_\alpha \Sigma^\alpha \mathscr{U} \boxtimes \Sigma^{\alpha^*}\mathscr{U}^\perp$ where α runs through Young diagrams with i cells. Hence as before this deduce another generalised Beilinson spectral sequence

$$E_1^{p,q} = \bigoplus_{|\alpha|=-p} \mathbb{H}^q(F \otimes \Sigma^{\alpha^*}\mathscr{U}^\perp) \otimes \Sigma^\alpha \mathscr{U} \Rightarrow \mathcal{H}^{p+q}(F)$$

for any $F \in \mathbf{D}^b(\mathsf{Gr}(k, V))$. □

Remark 1.4.2. *Note that we even have the Lefschetz decomposition on Grassmannians. We refer [6].*

Proposition 1.4.3. *Let $Q \subset \mathbb{P}_k^{n+1}$ be a smooth quadric hypersurface where* $\operatorname{char} k \neq 2$*, then there is a full exceptional collection*

$$\mathbf{D}^b(Q) = \begin{cases} \langle S, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1)\rangle, n \text{ odd}; \\ \left\langle S^-, S^+, \mathscr{O}_Q, \mathscr{O}(Q)(1), ..., \mathscr{O}(Q)(n-1)\right\rangle, n \text{ even}; \end{cases}$$

where $S, S^\pm$ are the spinor bundles.

Remark 1.4.4. *This is also right for the family version, that is, consider a flat fibration in quadrics $f : X \to S$. In other words, assume that $X \subset \mathbb{P}_S(\mathscr{E})$ is a divisor of relative degree 2 where $\mathscr{E}$ is of rank $n+2$ on a scheme S corresponding to a line subbundle $\mathscr{L} \subset \mathrm{Sym}^2\mathscr{E}^\vee$. For each i there is a fully faithful functor $\Phi_i : \mathbf{D}^b(S) \to \mathbf{D}^b(X)$ given by $F \mapsto f^*F \otimes \mathscr{O}_{X/S}(i)$. Then we have a S.O.D*

$$\mathbf{D}^b(X) = \left\langle \mathbf{D}^b(S, \mathscr{C}\ell_0), \Phi_0(\mathbf{D}^b(S)), ..., \Phi_{n-1}(\mathbf{D}^b(S))\right\rangle$$

where $\mathscr{C}\ell_0$ is the sheaf of even parts of Clifford algebras on S associated with the quadric fibration $X \to S$.

1.5 Example IV – Curves

Here we will follows [12]. Let C be a smooth projective curve over $\mathbb{C}$.

Proposition 1.5.1. *When $g(C) = 0$, then $C \cong \mathbb{P}^1$ and we have S.O.D*

$$\mathbf{D}^b(C) = \langle \mathscr{O}_C, \mathscr{O}_C(1)\rangle .$$

Proof. Special case of Proposition 1.2.1. □

Now we consider $g(C) \geq 1$ and show a lemma.

Lemma 1.5.2. *Let $g(C) \geq 1$. Suppose $\mathscr{E} \in \mathrm{Coh}(C)$ is included in a triangle*

$$Y \to \mathscr{E} \to X \to Y[1]$$

with $\mathrm{Hom}^{\leq 0}(Y, X) = 0$*, then $X, Y \in \mathrm{Coh}(C)$.*

Proof. Almost the pure homological algebra, using the fact that $\deg K_C \geq 0$ here. See [7] Lemma 7.2. ☐

Corollary 1.5.3. *Let $g(C) \geq 1$ and $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. Then for any $\mathscr{E} \in \mathrm{Coh}(C)$, there exist coherent sheaves $B \in \mathscr{B} \cap \mathrm{Coh}(C)$ and $A \in \mathscr{A} \cap \mathrm{Coh}(C)$, and an exact sequence of sheaves*

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Proposition 1.5.4. *When $g(C) \geq 1$, then $\mathbf{D}^b(C)$ admits no non-trivial S.O.Ds.*

Proof. Let $\mathbf{D}^b(C) = \langle \mathscr{A}, \mathscr{B} \rangle$ be an S.O.D. By Corollary 1.5.3, for any closed point $x \in C$ there exist $B \in \mathscr{B} \cap \mathrm{Coh}(C)$, $A \in \mathscr{A} \cap \mathrm{Coh}(C)$ such that both of them are sheaves and there exists an exact sequence

$$0 \to B \to \mathscr{O}_x \to A \to 0.$$

Hence $\mathscr{O}_x$ is contained in only one of $\mathscr{A}$ or $\mathscr{B}$. Hence $C(\mathrm{Spec}\,\mathbb{C}) = C_{\mathscr{A}} \sqcup C_{\mathscr{B}}$ by this fact.

By Proposition 3.17 in [8] we know that the set of closed points forms a spanning class, hence if $C_{\mathscr{B}} = \emptyset$ or $C_{\mathscr{A}} = \emptyset$, then $\mathscr{B}$ or $\mathscr{A}$ is trivial. Hence we may let both $C_{\mathscr{B}}$ and $C_{\mathscr{A}}$ are not empty.

We now claim that any coherent sheaf in $\mathscr{B}$ must be torsion. Indeed, otherwise the support of the sheaf coincides with the whole variety C, hence there exists a non-trivial morphism from the sheaf to a closed point which belongs to $\mathscr{A}$. This is a contradiction.

Next we claim that any torsion free sheaf belongs to $\mathscr{A}$. Indeed, let $\mathscr{E}$ be a torsion free sheaf. As before, we have an exact sequence

$$0 \to B \to \mathscr{E} \to A \to 0.$$

Since $\mathscr{E}$ is torsion free, so is B. Combined with the first claim, we see B must be zero, hence $A = \mathscr{E}$.

By Corollary 3.19 in [8] we know that the set of torsion free sheaves forms a spanning class of $\mathbf{D}^b(C)$. Hence $\mathscr{B}$ must be trivial. Well done. ☐

Remark 1.5.5. *Actually the only thing we use the $g(C) \geq 1$ is Corollary 1.5.3. So any smooth projective variety satisfies Corollary 1.5.3 admits no non-trivial S.O.Ds.*

1.6 Example V – Other Examples

Proposition 1.6.1. *Let X be a smooth projective variety with $\omega_X \cong \mathscr{O}_X$, then $\mathbf{D}^b(X)$ admits no non-trivial S.O.Ds.*

Proof. Let there exists an S.O.D $\mathbf{D}^b(X) = \langle \mathscr{A}, \mathscr{B} \rangle$. Hence for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and for any i we have $\mathrm{Hom}(B, A[i]) = 0$. Hence by Serre duality we have

$$\mathrm{Hom}(B, A[i]) = \mathrm{Hom}(A[i], B[n])^\vee = \mathrm{Hom}(A, b[n-i])^\vee = 0.$$

Hence $\mathbf{D}^b(X) = \langle \mathscr{B}, \mathscr{A} \rangle$ is also an S.O.D. Hence $\mathscr{A}, \mathscr{B}$ forms an orthogonal decomposition. Hence by Proposition 3.10 in [8] and the fact that X is connected, this S.O.D must be trivial. □

Lemma 1.6.2. *Let X be a smooth projective variety and $F \in \mathbf{D}^b(X)$ is non-trivial, and $\mathscr{L}$ be a globally generated line bundle. Then*

$$\mathrm{Hom}_X(F, F \otimes \mathscr{L}) \neq 0.$$

Proof. Here we follows [12]. Let $m = \min\{i : \mathscr{H}^i(F) \neq 0\}$ and consider the following standard distinguished triangle

$$\tau_{\leq m} F \to F \to \tau_{\geq m+1} F \to \tau_{\leq m} F[1].$$

Since $\tau_{\leq m} F$ is isomorphic to a shift of a sheaf, we can find $s \in H^0(X, \mathscr{L})$ which induce a non-trivial $\tau_{\leq m} F \to \mathscr{L} \otimes \tau_{\leq m} F$. Consider

$$\begin{array}{ccccccc}
\tau_{\geq m+1}F[-1] & \longrightarrow & \tau_{\leq m}F & \longrightarrow & F & \longrightarrow & \tau_{\geq m+1}F \\
\downarrow{\scriptstyle \sigma_{\geq m+1}[-1]} & & \downarrow{\scriptstyle \sigma_{\leq m}} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_{\geq m+1}} \\
\tau_{\geq m+1}F \otimes \mathscr{L}[-1] & \longrightarrow & \tau_{\leq m}F \otimes \mathscr{L} & \longrightarrow & F \otimes \mathscr{L} & \longrightarrow & \tau_{\geq m+1}F \otimes \mathscr{L}
\end{array}$$

where these four vertical arrows are defined by taking tensor products with the section s. Hence here $\sigma_{\leq m} \neq 0$. Suppose that $\sigma = 0$. Then $\sigma_{\leq m} \neq 0$ factors through a morphism from to $\tau_{\geq m+1} F \otimes \mathscr{L}[-1]$, which is zero since $\tau_{\geq m+1} F \otimes \mathscr{L}[-1]$ has trivial cohomologies up to degree $m+1$. Thus we obtain a contradiction, well done. □

Proposition 1.6.3. *Let X be a smooth projective variety whose canonical line bundle is globally generated. Then $\mathbf{D}^b(X)$ has no exceptional objects.*

Proof. This follows from Lemma 1.6.2 and the duality

$$\mathrm{Hom}(F, F[\dim X]) = \mathrm{Hom}(F, F \otimes \omega_X)^\vee \neq 0.$$

Well done. □

Chapter 2

Examples of Fano Manifolds of Calabi-Yau Type

We just consider the schemes and vector spaces over $\operatorname{Spec}\mathbb{C}$.

2.1 Cubics

2.2 Gushel-Mukai Varieties

2.2.1 Basic Definitions and Properties

Let V_5 be a vector space of dimension 5 and consider the Plücker embedding $\mathsf{Gr}(2, V_5) \hookrightarrow \mathbb{P}\left(\bigwedge^2 V_5\right)$. For any vector space K, consider the cone $\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \subset \mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$ of vertex $\mathbb{P}(K)$. Choose a vector subspace $W \subset \bigwedge^2 V_5 \oplus K$ and a subscheme $Q \subset \mathbb{P}(W)$ defined by defined by one quadratic equation (possibly zero).

Definition 2.2.1. *The scheme*

$$X = \mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W) \cap Q$$

is called a **Gushel-Mukai intersection** *(GM intersection). A GM intersection X is called a* **Gushel-Mukai variety** *(GM variety) if X is a smooth variety of dimension* $\dim W - 5 \geq 1$.

Remark 2.2.2. *Some remarks:*

(a) In the original paper [5] they defined without the smoothness (but always Gorenstein).

(b) Note that all Q and $\mathsf{C}_K(\mathsf{Gr}(2, V_5)) \cap \mathbb{P}(W)$ are Gorenstein, hence all Cohen-Macaulay. So the dimension condition means they are dimensionally transverse, that is, $\operatorname{Tor}_{>0}(\mathscr{O}_Q, \mathscr{O}_{\mathsf{C}_K(\mathsf{Gr}(2,V_5)) \cap \mathbb{P}(W)}) = 0$.

(c) *A GM variety X has a canonical polarization, the restriction H of the hyperplane class on $\mathbb{P}(W)$; we will call (X, H) a* polarized GM variety.

The definition of a GM variety is not intrinsic. We actually have an intrinsic characterization. But before giving these, we will introduce a new definition:

Definition 2.2.3. *Let W be a vector space and let $Y \subset \mathbb{P}(W)$ be a closed subscheme which is an intersection of quadrics, i.e., the twisted ideal sheaf $\mathscr{I}_X(2)$ on $\mathbb{P}(W)$ is globally generated.*

Define $V_X := H^0(\mathbb{P}(W), \mathscr{I}_X(2))$, this yields a surjection $V_X \otimes \mathscr{O}_{\mathbb{P}(W)}(-2) \twoheadrightarrow \mathscr{I}_X$ which induce

$$V_X \otimes \mathscr{O}_X(-2) \twoheadrightarrow \mathscr{I}_X/\mathscr{I}_X^2 = \mathscr{N}^\vee_{X/\mathbb{P}(W)}.$$

We define the excess conormal sheaf *$\mathscr{E}\mathscr{N}^\vee_{X/\mathbb{P}(W)}$ to be the kernel of this map.*

Theorem 2.2.4. *A smooth polarized projective variety (X, H) of dimension $n \geq 1$ is a polarized GM variety if and only if all the following conditions hold:*

(a) *$H^n = 10$ and $K_X = -(n-2)H$.*

(b) *H is very ample and the vector space $W := H^0(X, \mathscr{O}_X(H))^\vee$ has dimension $n+5$.*

(c) *X is an intersection of quadrics in $\mathbb{P}(W)$ and the vector space*

$$V_6 := H^0(\mathbb{P}(W), \mathscr{I}_X(2)) \subset \mathrm{Sym}^2 W^\vee$$

of quadrics through X has dimension 6.

(d) *The twisted excess conormal sheaf $\mathscr{U}_X := \mathscr{E}\mathscr{N}^\vee_{X/\mathbb{P}(W)}(2H)$ of X in $\mathbb{P}(W)$ is simple.*

Proof. We first need to show a smooth polarized GM variety (X, H) satisfies (a)-(d).

For (a), as $\deg(\mathsf{C}_K(\mathsf{Gr}(2, V_5))) = 5$ and they are dimensionally transverse, then $\deg(X) = 10$. Let $\dim K = k$ and hence $K_{\mathsf{C}_K(\mathsf{Gr}(2,V_5))} = -(5+k)H$ by Lemma 2.2.7. Finally we have

$$K_X = (-(5+k) + (10+k) - (n+5) + 2)H = -(n-2)H.$$

For (b), we just need to show $W = H^0(X, \mathscr{O}_X(H))^\vee$. Consider the resolution

$$0 \to \mathscr{O}(-5) \to V_5^\vee \otimes \mathscr{O}(-3) \to V_5 \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_{\mathsf{C}_K \mathsf{Gr}(2,V_5)} \to 0.$$

Restrict it into $\mathbb{P}(W)$ and tensor the resolution of Q as $0 \to \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_Q$, then tensor $\mathscr{O}(1)$ again we get the resolution

$$\begin{aligned} 0 \to \mathscr{O}(-6) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-4) \to (V_5 \otimes \mathscr{O}(-3)) \oplus (V_5^\vee \otimes \mathscr{O}(-2)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-1) \to \mathscr{O}(1) \to \mathscr{O}_X(H) \to 0 \end{aligned}$$

on $\mathbb{P}(W)$. Hence $H^0(X, \mathscr{O}_X(H)) = H^0(\mathbb{P}(W), \mathscr{O}_{\mathbb{P}(W)}(1)) = W^\vee$.

For (c), consider the resolution again:

$$\begin{aligned}0 \to \mathscr{O}(-5) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-3) \to (V_5 \otimes \mathscr{O}(-2)) \oplus (V_5^\vee \otimes \mathscr{O}(-1)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O} \to \mathscr{O}(2) \to \mathscr{O}_X(2H) \to 0\end{aligned}$$

Hence one can show that $H^0(\mathbb{P}(W), \mathscr{I}_X(2)) = V_5 \oplus \mathbb{C}$, hence well done.

For (d), we will use the induction of the dimension. For $n = 1$, this follows from some basic fact of excess normal sheaf and the Mukai's construction about a stable vector bundle of rank 2 on X to show that $\mathscr{U}_X$ is stable, and hence simple. For the detail we refer [5] Theorem 2.3. Hence we now assume $n \geq 2$. Pick a smooth hyperplane section $X' \subset X$ which is also irreducible since $n \geq 2$ by Bertini's theorem. Hence X' is also a GM variety. One can easy to show that in this case $\mathscr{U}_X|_{X'} = \mathscr{U}_{X'}$ (see Lemma A.5 in [5]). Hence we have $0 \to \mathscr{U}_X(-H) \to \mathscr{U}_X \to \mathscr{U}_{X'} \to 0$. Hence

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}).$$

If $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X)) > 1$, then $\dim(\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H))) > 0$. By the similar argument we get

$$0 \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \to \mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-H)) \to \mathrm{Hom}(\mathscr{U}_{X'}, \mathscr{U}_{X'}(-H)) = 0.$$

Hence $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-2H)) \neq 0$. By induction we get $\mathrm{Hom}(\mathscr{U}_X, \mathscr{U}_X(-kH)) \neq 0$ for any $k > 0$. Hence for any $k > 0$ we have $\Gamma(X, \mathscr{U}_X^\vee \otimes \mathscr{U}_X(-kH)) \neq 0$. But these are vector bundles and X is integral of dimension ≥ 2, hence this is impossible.

Now we let a smooth polarized projective variety (X, H) of dimension $n \geq 1$ which satisfies (a)-(d). We need to show that (X, H) is a polarized GM variety.

We know that

$$\det \mathscr{U}_X^\vee = \det(\mathscr{N}_{X/\mathbb{P}(W)}^\vee(2H)) = \mathscr{O}_X(H)$$

and the embedding $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$. Taking wedge product, duality and global sections we get

$$\bigwedge^2 V_6^\vee \to H^0(X, \mathscr{O}_X(H)) = W^\vee.$$

Hence we get $W \to \bigwedge^2 V_6$ which can be factored through an injection $W \to \bigwedge^2 V_6 \oplus K$ for some vector space K. Hence we have

$$\begin{tikzcd}
 & \mathbb{P}(W) \arrow[r, hook] & \mathbb{P}\left(\bigwedge^2 V_6 \oplus K\right) \arrow[d, dashed] \\
X \arrow[ur, hook] \arrow[r] & \mathsf{Gr}(2, V_6) \arrow[r, hook] & \mathbb{P}\left(\bigwedge^2 V_6\right)
\end{tikzcd}$$

where $X \to \mathsf{Gr}(2, V_6)$ induced by $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X$ and is commutative since these are the same linear system. Hence we get $X \subset \mathsf{C}_K^\circ \mathsf{Gr}(2, V_6) = \mathsf{C}_K \mathsf{Gr}(2, V_6) \backslash \mathbb{P}(K)$.

Now by some facts of excess normal sheaves (see Proposition A.3 in [5]), then excess normal sequence induce a functorial diagram:

$$\begin{tikzcd}
0 \arrow[r] & (V_6 \otimes \mathscr{U}_X)/\mathrm{Sym}^2\mathscr{U}_X \arrow[r] \arrow[d] & \bigwedge^2 V_6 \otimes \mathscr{O}_X \arrow[r] \arrow[d] & \det V_6 \otimes \mu^* \mathscr{N}^\vee_{\mathsf{Gr}(2,V_6)/\mathbb{P}(\bigwedge^2 V_6)}(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \det V_6 \otimes \mathscr{U}_X \arrow[r] & \det V_6 \otimes V_6 \otimes \mathscr{O}_X \arrow[r] & \det V_6 \otimes \mathscr{N}_{X/\mathbb{P}(W)}(2) \arrow[r] & 0
\end{tikzcd}$$

which follows from the expression of the excess normal sheaf of $\mathsf{Gr}(2, V_6) \subset \mathbb{P}(\bigwedge^2 V_6)$. The left vertical arrow induces a morphism $\lambda' : V_6 \otimes \mathscr{U}_X \to \det V_6 \otimes \mathscr{U}_X$. As $\mathscr{U}_X$ is simple by (d) we get $\lambda : V_6 \to \det V_6$. Since λ' vanishes on $\mathrm{Sym}^2\mathscr{U}_X$, the image of $\mathscr{U}_X$ in ${}_6 \otimes \mathscr{O}_X$ is contained in $\ker \lambda \otimes \mathscr{O}_X$. Moreover, the middle vertical map in the diagram above is given by $v_1 \wedge v_2 \mapsto \lambda(v_1)v_2 - \lambda(v_2)v_1$.

We claim that $\lambda \neq 0$. If $\lambda = 0$, the middle vertical map in the diagram is zero, which means that all the quadrics cutting out $\mathsf{C}_K \mathsf{Gr}(2, V_6)$ contain $\mathbb{P}(W)$, i.e. $\mathbb{P}(W) \subset \mathsf{C}_K \mathsf{Gr}(2, V_6)$. In other words, $\mathbb{P}(W)$ is a cone over $\mathbb{P}(W') \subset \mathsf{Gr}(2, V_6)$ with vertex a subspace of K. Hence $X \to \mathsf{Gr}(2, V_6)$ factor through $\mathbb{P}(W')$. Hence the vector bundle $\mathscr{U}_X$ is a pullback from $\mathbb{P}(W')$ of the restriction of the tautological bundle of $\mathsf{Gr}(2, V_6)$ to $\mathbb{P}(W')$.

There are two types of linear spaces on $\mathsf{Gr}(2, V_6)$: the first type corresponds to 2-dimensional subspaces containing a given vector and the second type to those contained in a given 3-subspace $V_3 \subset V_6$. If W' is of the first type, the restriction of the tautological bundle to $\mathbb{P}(W')$ is isomorphic to $\mathscr{O} \oplus \mathscr{O}(-1)$, hence $\mathscr{U}_X \cong \mathscr{O} \oplus \mathscr{O}(-H)$ by Lemma 2.2.8. In particular, it is not simple, which is a contradiction. If W' is of the second type, the embedding $\mathscr{U}_X \to V_6 \otimes \mathscr{O}_X$ factors through a subbundle $V_3 \otimes \mathscr{O}_X \subset V_6 \otimes \mathscr{O}_X$. Recall that V_6 is the space of quadrics passing through X in $\mathbb{P}(W)$. Consider the scheme-theoretic intersection M of the quadrics corresponding to the vector subspace V_3. Since the embedding of the excess conormal sheaf factors through $V_3 \otimes \mathscr{O}_X$, the variety X is the complete intersection of M with the 3 quadrics corresponding to the quotient space V_6/V_3. But the degree of X is then divisible by 8, which contradicts the fact that it is 10 by (a). Hence we conclude that $\lambda \neq 0$.

Now let $V_5 := \ker(\lambda)$ is a hyperplane in v_6 which fits in the exact sequence $0 \to V_5 \to V_6 \xrightarrow{\lambda} \det V_6 \to 0$. The composition $\mathscr{U}_X \hookrightarrow V_6 \otimes \mathscr{O}_X \xrightarrow{\lambda} \det V_6 \otimes \mathscr{O}_X$ vanish, hence we get $\mathscr{U}_X \hookrightarrow V_5 \otimes \mathscr{O}_X$.

We now replace V_6 with V_5 and repeat the above argument, then we get a linear map $W \to \bigwedge^2 W_5$ which factor through $\mu : W \hookrightarrow \bigwedge^2 W_5 \oplus K$ which induce again the embedding $X \subset \mathsf{C}_K^\circ \mathsf{Gr}(2, V_5) = \mathsf{C}_K \mathsf{Gr}(2, V_5) \backslash \mathbb{P}(K)$. By the functorial of the excess normal sequence (see Proposition A.3 in [5]) again we get that inside the space V_6 of quadrics cutting out X in $\mathbb{P}(W)$, the hyperplane V_5 is the space of quadratic equations of $\mathsf{Gr}(2, V_5)$, i.e., of Plücker quadrics.

As the Plücker quadrics cut out the cone $\mathsf{C}_K \mathsf{Gr}(2, V_5)$ in $\mathbb{P}\left(\bigwedge^2 V_5 \oplus K\right)$, they cut out $\mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W)$ in $\mathbb{P}(W)$. Since X is the intersection of 6 quadrics by condition (c), we finally obtain

$$X = \mathsf{C}_K \mathsf{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q$$

where Q is some non-Plücker quadric corresponding to a point in $V_6 \backslash V_5$, so X is a GM variety. □

Remark 2.2.5. *This is right for all normal varieties with the similar proof.*

Remark 2.2.6. *The twisted excess conormal sheaf $\mathscr{U}_X$ that was crucial for the proof will be called its* Gushel sheaf. *As we showed in the proof, the projection of X from the vertex $\mathbb{P}(K)$ of the cone $\mathsf{C}_K \mathsf{Gr}(2, V_5)$ defines a rational map $X \dashrightarrow \mathsf{Gr}(2, V_5)$ and the Gushel sheaf $\mathscr{U}_X$ is isomorphic to the pullback under this map of the tautological vector bundle on $\mathsf{Gr}(2, V_5)$. The map $X \dashrightarrow \mathsf{Gr}(2, V_5)$ is thus determined by $\mathscr{U}_X$ and is canonically associated with X. We call this map the* Gushel map *of X.*

Lemma 2.2.7. *Let $X \subset \mathbb{P}^n$ be a subvariety such that $K_X = rH$. Let $\mathsf{C}(X) \subset \mathbb{P}^{n+1}$ be a cone over X, then $K_{\mathsf{C}(X)} = (r-1)H$.*

Proof. We know that the blow-up of of the vertex of $\mathsf{C}(X)$ is

$$\begin{tikzcd} & X' = \mathbb{P}_X(\mathscr{O}_X \oplus \mathscr{O}_X(-H)) \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ \mathsf{C}(X) & & X \end{tikzcd}$$

Let H' be the relative hyperplane class of p. Then

$$K_{X'} = p^*(K_X + H) - 2H' = (r+1)p^*H - 2H'.$$

On the other hand, the morphism π contracts the exceptional section $E \subset X'$ and H' is the pullback of $H_{\mathsf{C}(X)}$. Finally $E \sim_{\text{lin}} H' - p^*H$, hence

$$K_{X'} = (r-1)H' - (r+1)E.$$

Hence $K_{\mathsf{C}(X)} = (r-1)H$. □

Lemma 2.2.8. *Let $Z_p \subset \mathsf{Gr}(k, V)$ be the subscheme parameterizing all k-planes containing the vector p. Then $Z_p \cong \mathsf{Gr}(k-1, n-1)$ and the restriction of the tautological subbundle $\mathscr{S}_{\mathsf{Gr}(k,V)}$ to Z_p splits as the sum of $\mathscr{O}$ and the tautological subbundle $\mathscr{S}_{Z_p}$ of $Z_p \cong \mathsf{Gr}(k-1, n-1)$.*

Proof. This is almost trivial. Indeed, let $V_1 \subset V$ be the 1-dimensional subspace generated by the vector p. Let $V = V_1 \oplus V'$ be a direct sum decomposition. Then for each $k-1$-dimensional subspace $U' \subset V'$ the sum $V_1 \oplus U'$ is a k-dimensional subspace of V. Hence the corresponding subbundle

$$V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{\mathsf{Gr}(k-1,V')} \subset V_1 \otimes \mathscr{O} \oplus V' \otimes \mathscr{O} = V \otimes \mathscr{O}$$

induces a morphism $\mathsf{Gr}(k-1, V') \to \mathsf{Gr}(k, V)$ which is an isomorphism onto Z_p and such that the pullback of the tautological bundle is $V_1 \otimes \mathscr{O} \oplus \mathscr{S}_{Z_p}$. ☐

2.2.2 Some Classifications

Lemma 2.2.9. *Let (X, H) be a polarized variety. If it is projective normal, that is, the canonical map $\mathrm{Sym}^m H^0(X, \mathscr{O}_X(H)) \to H^0(X, \mathscr{O}_X(mH))$ is surjective for any $m \geq 0$, then H must be very ample.*

Proof. By the commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}H^0(X,\mathscr{O}_X(nH)) \arrow[dr, hook] & \\
X \arrow[ur, hook, "|nH|"] \arrow[dr, "|H|"'] & & \mathbb{P}H^0(X,\mathrm{Sym}^n\mathscr{O}_X(H)) \\
 & \mathbb{P}H^0(X,\mathscr{O}_X(H)) \arrow[ur, hook, "n\text{-uple}"] &
\end{tikzcd}$$

we know that $|H|$ also induce an immersion. Hence H is very ample. ☐

Proposition 2.2.10. *Let (X, H) be a smooth polarized variety of dimension $n \geq 2$ such that $K_X = -(n-2)H$ and $H^1(X, \mathscr{O}_X) = 0$. If there is a hypersurface $X' \subset X$ in the linear system $|H|$ such that $(X', H|_{X'})$ is a smooth polarized GM variety, (X, H) is also a smooth polarized GM variety.*

Proof. First we note that for any smooth GM variety (Y, H) the resolution

$$\begin{aligned} 0 \to \mathscr{O}(m-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(m-5) \to (V_5 \otimes \mathscr{O}(m-4)) \oplus (V_5^\vee \otimes \mathscr{O}(m-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(m-2) \to \mathscr{O} \to \mathscr{O}_Y(mH) \to 0 \end{aligned}$$

can imply Y is projective normal, that is, the canonical map $\mathrm{Sym}^m H^0(Y, \mathscr{O}_Y(H)) \to H^0(Y, \mathscr{O}_Y(mH))$ is surjective for any $m \geq 0$.

Back to the result, we need to check the conditions in Theorem 2.2.4. For (a), this follows from $H^n = H \cdot H^{n-1} = H|_{X'}^{n-1} = 10$. Now we know X' is projective normal, so is X by [9] Lemma (2.9). By Lemma 2.2.9 we know H is very ample. By $H^1(X, \mathscr{O}_X) = 0$

we know that $h^0(X, \mathscr{O}_X(H)) = n+5$ by the case of X'. This proves (b), and [9] Lemma (2.10) proves (c). For (d), since $\mathscr{U}_{X'}$ is simple, by the similar proof of (d) in Theorem 2.2.4 we can also show that $\mathscr{U}_X$ is simple. □

Theorem 2.2.11. *Let X be a complex smooth projective variety of dimension $n \geq 1$, together with an ample Cartier divisor H such that $K_X \sim_{\mathrm{lin}} -(n-2)H$ and $H^n = 10$. If we assume that*

- *when $n \geq 3$, we have $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$;*
- *when $n = 2$, the surface X is a Brill-Noether general K3 surface (a K3 surface is called Brill-Noether general if $h^0(S, D)h^0(S, H-D) < h^0(S, H)$ for all divisors D on S not linearly equivalent to 0 or H. When $H^2 = 10$, this is equivalent to the fact that $|H|$ contains a Clifford general smooth curve);*
- *when $n = 1$, the genus-6 curve X is Clifford general (that is, it is neither hyperelliptic, nor trigonal, nor a plane quintic).*

then X is a GM variety.

Before proving this, we need some Lemmas:

Lemma 2.2.12. *Let X be a complex smooth projective variety of dimension $n \geq 3$ with an ample divisor H such that $H^n = 10$ and $K_X \sim_{\mathrm{lin}} -(n-2)H$.*

Then the linear system $|H|$ is very ample and a smooth general $X' \in |H|$ satisfies the same conditions: if $H' := H|_{X'}$, we have $(H')^{n-1} = 10$ and $K_{X'} \sim_{\mathrm{lin}} -(n-3)H'$.

Proof. First we need to show that $h^0(H) > 0$. This follows from the follows result:

- **Lemma 2.2.12.A.** Let X be a smooth Fano variety of dimension $n \geq 3$ such that $-K_X \sim_{\mathrm{lin}} rH$ where H is ample. Then when $r \geq n-2$, then $h^0(H) > 0$.

Proof of Lemma 2.2.12.A. Now we separate it as two cases.

When $r \geq n-1$, use Kodaira vanishing theorem to $(x+r)H + K_X$ we have $h^i(xH) = 0$ for all $i > 0$ and all $x \geq -(n-2)$. Now we let $h^0(H) = 0$ and in these cases we have $\chi(xH) = h^0(xH)$. Hence $\chi(xH)$, as a polynomial, has roots $-1, -2, ..., -(n-2), 1$. As $\chi(0) = 1$ and $\chi(xH)$ as the top coefficient $\frac{H^n}{n!}$, we know that

$$\begin{aligned}\chi(xH) &= \frac{1}{n!}(x+1)(x+2)\cdots(x+n-2)(x-1)(H^n x - n(n-1)) \\ &= \frac{1}{n!}\left(H^n x^n + \left(n(n-3)\frac{H^n}{2} - n(n-1)\right)x^{n-1} + \text{lower terms}\right).\end{aligned}$$

On the other hand, by HRR we get

$$\chi(xH) = \frac{1}{n!}\left(H^n x^n + \frac{1}{2}nrH^n x^{n-1} + \text{lower terms}\right).$$

Hence $\frac{1}{2}nrH^n = n(n-3)\frac{H^n}{2} - n(n-1)$, that is, $r = n - 3 - \frac{2n-2}{H^n}$. But $r \geq n-1$, this is impossible. Hence $h^0(H) > 0$.

When $r = n-2$, we will go through this directly. By Kodaira vanishing theorem again we have $h^i(xH) = 0$ for all $i > 0$ and all $x \geq -(n-3)$. For $x = -(n-2)$, we only have

$$h^i(-(n-2)H) = h^i(K_X) = \begin{cases} 1, & i = n; \\ 0, & 0 \leq i < n. \end{cases}$$

Hence again we have

$$\chi(xH) = \frac{1}{n!}(x+1)(x+2)\cdots(x+n-3)(H^n x^3 + bx^2 + cx + n(n-1)(n-2)).$$

Now as $\chi(-(n-2)H) = (-1)^n$, we can find that $b = \frac{3}{2}H^n(n-2)$ and $c = 2n(n-1) + \frac{1}{2}H^n(n-2)^2$. Hence $h^0(H) > 0$ by taking $x = 1$. $\square$

Hence now $|H|$ is non-empty. $\square$

Lemma 2.2.13. *Let (X, H) be a polarized complex variety of dimension $n \geq 2$ which satisfies the hypotheses of Theorem 2.2.11. A general element of $|H|$ then satisfies the same properties.*

Proof. $\square$

Proof of Theorem 2.2.11. $\square$

Some inverse results:

Proposition 2.2.14. *A smooth projective curve is a GM curve if and only if it is a Clifford general curve of genus* 6.

Proof. Follows from the Theorem 2.2.4 and the Enriques-Babbage theorem in [1] Section III.3. $\square$

Proposition 2.2.15. *A smooth projective surface X is a GM surface if and only if X is a Brill-Noether general polarized K3 surface of degree* 10.

Proof. By Theorem 2.2.11, we just need to show that if X is a GM surface, then X is a Brill-Noether general polarized K3 surface of degree 10. In this case, we have $K_X = 0$ by Theorem 2.2.4(a), and the resolution

$$\begin{aligned} 0 \to \mathscr{O}(-7) \to (V_5^\vee \oplus \mathbb{C}) \otimes \mathscr{O}(-5) \to (V_5 \otimes \mathscr{O}(-4)) \oplus (V_5^\vee \otimes \mathscr{O}(-3)) \\ \to (V_5 \oplus \mathbb{C}) \otimes \mathscr{O}(-2) \to \mathscr{O} \to \mathscr{O}_X \to 0 \end{aligned}$$

implies $H^1(X, \mathscr{O}_X) = 0$, hence X is a K3 surface. Moreover, a general hyperplane section of X is a GM curve, hence a Clifford general curve of genus 6, hence X is Brill-Noether general. $\square$

Proposition 2.2.16. *Let (X, H) be a polarized complex smooth GM variety of dimension $n \geq 3$. Then $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. In particular, the polarization H is the unique GM polarization on X.*

Proof. □

2.3 Debarre-Voisin Varieties

2.4 Iliev-Manivel Varieties

Chapter 3

Kuznetsov Components

Chapter 4

Examples of Derived Equivalences of Kuznetsov Components with K3s

Chapter 5

Stability Conditions on K3 Categories

Chapter 6

Applications: Mukai's program

Chapter 7

Application to Cubic Fourfolds and Gushel-Mukai Manifolds

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