LECTURES ABOUT SYMPLECTIC RESOLUTIONS, SYMPLECTIC DUALITY, AND COULOMB BRANCHES

XIAOLONG LIU

Abstract. In this note we will give a survey about symplectic resolutions, symplectic duality and Coulomb branches follows [Kam22].

Contents

1. Introduction

We will mainly follows the survey paper [Kam22]. We work over $\mathbb{C}$.

2. Symplectic resolutions

2.1. Basic definitions.

Here we give some basic definitions.

Definition 2.1. *We consider complex algebraic schemes.*

- *We say a scheme X carries a* Poisson structure *if there is a $\mathbb{C}$-bilinear operation*
$$\{-,-\} : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$$
which is a Lie bracket.
- *Let $f : X \to Y$ be a morphism of Poisson schemes, we say it is a* Poisson morphism *if it induce a homomorphism of Lie algebras.*

Remark 2.2. *Any Poisson structure can be induced by the $\mathscr{O}_X$-linear homomorphism $H : \Omega^1_X \to T_X = \mathrm{Der}(\mathscr{O}_X, \mathscr{O}_X)$ such that $\{f, g\} = H(df)(g)$. In particular, any symplectic variety has a canonical Poisson structure.*

Definition 2.3. *A* symplectic resolution *is a morphism $\pi : Y \to X$ of complex algebraic varieties, where*

- *Y is smooth and carries a symplectic structure.*
- *X is affine, normal, and carries a Poisson structure.*
- *π is projective, birational, and Poisson.*

Definition 2.4. *Let $\pi : Y \to X$ be a symplectic resolution.*

Date: August 31, 2024.
Key words and phrases. symplectic resolutions, symplectic duality, Coulomb branches.

- *π is called* conical *if we are given actions of $\mathbb{C}^\times$ on X and Y such that π is equivariant and $\mathbb{C}^\times$ contracts X to a single point, denoted as 0. We also assume that $\mathbb{C}^\times$ scales the symplectic form with weight 2. The central fiber $F_0 = \pi^{-1}(0)$ is called the* core *of Y.*
- *A conical symplectic resolution is called* Hamiltonian *if we are given Hamiltonian actions of a torus T on X and Y, such that π is T-equivariant. We also assume that the T action commutes under the conical $\mathbb{C}^\times$ action, and that Y^T is finite.*

Here we introduce the most basic and important example:

Example 2.5 (Baby example)**.** *The simplest example of a symplectic resolution is*

$$\pi : Y := T^*\mathbb{P}^1 := \mathrm{Tot}(\Omega^1_{\mathbb{P}^1}) \to X := \mathcal{N}_{\mathfrak{sl}_2}$$

where $\mathcal{N}_{\mathfrak{sl}_2}$ is the variety of nilpotent 2×2 matrices, that is,

$$\mathcal{N}_{\mathfrak{sl}_2} = \left\{ \begin{pmatrix} w & u \\ v & -w \end{pmatrix} \in M_2(\mathbb{C}) : w^2 + uv = 0 \right\} \cong \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$$

which is the singular quadric cone in $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$ where the last isomorphism $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{N}_{\mathfrak{sl}_2}$ given by $(a,b) \mapsto \begin{pmatrix} ab & a^2 \\ b^2 & -ab \end{pmatrix}$. Here we have several descriptions of this morphism.

- *Now via canonical exact sequence of tangent bundle of projective space, we have canonical description $T_{[L]}\mathbb{P}^1 = \{w \in \mathbb{C}^2 : w \cdot L = 0\}$. Hence $T^*_{[L]}\mathbb{P}^1$ is a space of linear functions over $T_{[L]}\mathbb{P}^1$. So this correspond to a matrix $\mathbf{A} \in M_2(\mathbb{C})$ such that $\mathbf{A}L = 0$ and $\mathbf{A}(w) \in L$ for $w \in T_{[L]}\mathbb{P}^1$. So*

  $$T^*\mathbb{P}^1 \cong \left\{ ([L], \mathbf{A}) \in \mathbb{P}^1 \times \mathcal{N}_{\mathfrak{sl}_2} : \mathbf{A}(\mathbb{C}^2) \subset L, \mathbf{A}L = 0 \right\}$$

  with canonical forgetful morphism $\pi : T^*\mathbb{P}^1 \to \mathcal{N}_{\mathfrak{sl}_2}$.
- *Regard $\mathcal{N}_{\mathfrak{sl}_2} \subset \mathbb{C}^3$ as a cone of over a quadric plane curve $C \subset \mathbb{C}^2$. Then consider its projective completion $\overline{C} \subset \mathbb{P}^2$, then $\mathcal{N}_{\mathfrak{sl}_2} = \overline{\mathcal{N}} \backslash H_\infty$ where $\overline{\mathcal{N}} \subset \mathbb{P}^3$ be the cone of $\overline{C}$. Now*

  $$\begin{aligned} Y &\cong \mathsf{Bl}_v \mathcal{N}_{\mathfrak{sl}_2} = (\mathsf{Bl}_v \overline{\mathcal{N}}) \backslash H_\infty = \mathbb{P}_{\overline{C}}(\mathscr{O}_{\overline{C}} \oplus \mathscr{O}_{\overline{C}}(-1)) \backslash \mathbb{P}_{\overline{C}}(\mathscr{O}_{\overline{C}}) \\ &= \mathrm{Tot}(\mathscr{O}_{\overline{C}}(-1)) = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-2)), \end{aligned}$$

  well done. More precisely, we have the following:

  $$\mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-2)) \to \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}), \quad ([z_0 : z_1], \lambda^2(z_0, z_1)^2) \mapsto \lambda(z_0, z_1).$$

Note that $\mathbb{C}^2$ admit a natural Poisson structure, then so is $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{N}_{\mathfrak{sl}_2}$. It is also affine and normal. Moreover $T^*\mathbb{P}^1$ is a smooth and carries a symplectic structure and π is projective, birational, and Poisson. So π is a symplectic resolution.

Actually, we have a conical $\mathbb{C}^\times$-action which scales the matrix and a Hamiltonian action of $T := (\mathbb{C}^\times)^2$, inherited from its action on $\mathbb{C}^2$. So π is a Hamiltonian conical symplectic resolution.

2.2. **Example: cotangent bundles of flag varieties.** The first way to generalize the Example 2.5, we consider any semi-simple group G and its parabolic subgroup P. Then we consider $T^*(G/P)$ which will be seen as a resolution of a nilpotent orbit closure.

Definition 2.6. *Consider a semi-simple Lie algebra $\mathfrak{g}$ with a adjoint algebraic group G.*

- *For a nilpotent element $v \in \mathfrak{g}$, its* nilpotent orbit *$\mathcal{O}_v$ is the orbit of v under the adjoint action of G. Its closure $\overline{\mathcal{O}_v}$ is a* nilpotent orbit closure*.*
- *Consider the standard $\mathfrak{sl}_2$-action of $\mathfrak{g}$ due to Jacobson-Morozov: for a nilpotent element $v \in \mathfrak{g}$, there exist two elements $H, u \in \mathfrak{g}$ such that $[H,v] = 2v, [H,u] = -2u, [v,u] = H$. Thus $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i := \{x \in \mathfrak{g} : [H,x] = ix\}$. Let $\mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$ and P a correspond connected subgroup of G which is a parabolic subgroup. Let $\mathfrak{n} := \bigoplus_{i \geqslant 2} \mathfrak{g}_i$ and $\mathfrak{u} := \bigoplus_{i \geqslant 1} \mathfrak{g}_i$.*
- *The nilpotent orbit $\mathcal{O}_v$ is called* even *if $\mathfrak{g}_1 = 0$ or equivalently if $\mathfrak{g}_{2k+1} = 0$ for all $k \in \mathbb{Z}$.*

Proposition 2.7 (Prop 2.8 and Cor 2.9 in [Fu06])**.** *For a nilpotent orbit $\mathcal{O}_v$, there exists a symplectic form ω (hence Poisson) on $\mathcal{O}_v$ which is called the* Kostant-Kirillov-Souriau form*. Moreover, there exists a G-equivariant proper resolution*

$$\pi : G \times^P \mathfrak{n} \to \overline{\mathcal{O}_v},$$

which is called the Springer resolution *such that π maps the orbit $G \cdot (1, v)$ isomorphically to $\mathcal{O}_v$ and the symplectic form $\pi^*\omega$ on $G \cdot (1, v)$ can be extended to a global 2-form Ω on $G \times^P \mathfrak{n}$. Moreover, Ω is symplectic if and only if $\mathcal{O}_v$ is even.*

As resolution $\pi : G \times^P \mathfrak{n} \to \overline{\mathcal{O}_v}$ factor-through the normalization $\widetilde{\mathcal{O}} \xrightarrow{\nu} \overline{\mathcal{O}_v}$, we have the following directly:

Corollary 2.8. *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure $\overline{\mathcal{O}_v}$ is a symplectic variety. The Springer resolution $G \times^P \mathfrak{n} \to \widetilde{\mathcal{O}}$ is symplectic if and only if $\mathcal{O}_v$ is an even nilpotent orbit.*

But not that for an even nilpotent orbit closure, there can exist some symplectic resolutions not of the above form as we will talking about.

Note that not all nilpotent orbit closure admits a symplectic resolution. But this is true for $\mathfrak{sl}_n$ which we will be particularly interested.

Example 2.9. *Every nilpotent orbit closure in $\mathfrak{sl}_{n+1}$ admits a symplectic resolution. Indeed, note that we have bijction between the set of nilpotent orbit closures and the set of some special partitions, see Proposition 2.2.1 in* [Nam10]*. In the case of $\mathfrak{sl}_{n+1}$, the set of nilpotent orbit closures correspond to the all partitions of $n+1$.*

Let $\mathcal{O}$ be the nilpotent orbit corresponding to the partition $[d_1, ..., d_k]$. The dual partition is defined by $s_j = |\{i | d_i \geqslant j\}|$. The closure is $\overline{\mathcal{O}} = \{A \in \mathfrak{sl}_{n+1} : \dim \ker A^j \geqslant \sum_{i=1}^{j} s_j\}$ which is normal, see [KP79]*. Consider the partial flag variety (which is of course SL_{n+1}/P) $\mathsf{Fl} := \{(V_1, ..., V_l) : \dim V_j = \sum_{i=1}^{j} s_j, V_j \subset V_{j+1}\}$ with the similar description as baby example:*

$$\pi : T^*\mathsf{Fl} \cong \{(\mathbf{A}, V_\bullet) \in \mathfrak{sl}_{n+1} \times \mathsf{Fl} : \mathbf{A} V_i \subset V_{i-1}\} \to \overline{\mathcal{O}}$$

which is of course a symplectic resolution.

But note that not every nilpotent orbit closure admits a symplectic resolution, see Proposition 5.2 in [Fu06]. But conversely we have the following nice result:

Theorem 2.10 (Thm 0.1 in [Fu03])**.** *Suppose that we have a symplectic resolution $\pi : Z \to \widetilde{\mathcal{O}}$ where $\widetilde{\mathcal{O}}$ is a normalization of a nilpotent orbit closure $\overline{\mathcal{O}_v}$, then there exists a parabolic subgroup P of G such that $Z \cong T^*(G/P)$.*

Such orbits are called the Richardson orbits.

Corollary 2.11. *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure in a semisimple Lie algebra admits a symplectic resolution if and only if*

- *$\mathcal{O}$ is a Richardson nilpotent orbit.*
- *There exists a polarization P such that the moment map $T^*(G/P) \to \mathcal{O}$ is birational.*

The conical $\mathbb{C}^\times$ acts by linear scaling on $\overline{\mathcal{O}_v}$ (coming from its embedding in the vector space $\mathfrak{g}$). The maximal torus $T \subset G$ acts Hamiltonianly on $T^*(G/P)$ in the natural way. The fixed point set $(T^*(G/P))^T$ is in bijection with W/W_P.

2.3. More examples.

Example 2.12 (Resolutions of Kleinian singularities)**.** *Generalizing $T^*\mathbb{P}^1$ in a different direction, we take $\Gamma \subset \mathrm{SL}_2$ a finite subgroup. Under the McKay correspondence, such subgroups are in bijection with simply-laced ADE Dynkin diagrams. We only consider*

$$\Gamma \cong \mathbb{Z}/n\mathbb{Z} = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1 \right\} \longleftrightarrow A_{n-1}.$$

The affine GIT quotient $X := \mathbb{C}^2 // \Gamma$ carries a Poisson structure by descending the usual symplectic structure on $\mathbb{C}^2$. Then consider minial resolution. The conical $\mathbb{C}^\times$ action comes from the scaling action on $\mathbb{C}^2$. On the other hand, the Hamiltonian torus T is given by the diagonal matrices in SL_2.

Example 2.13 (Higgs branches of gauge theories)**.** *Consider a reductive group G and a representation V. Then we form $T^*V = V \oplus V^*$ which comes with a moment map $\Phi : T^*V \to \mathfrak{g}^*$. We fix a character $\chi : G \to \mathbb{C}^\times$ and form the GIT quotient*

$$T^*V///_{0,\chi} G := \Phi^{-1}(0)//_\chi G := \mathrm{Proj}\left(\bigoplus_{n \geqslant 0} \mathbb{C}[\Phi^{-1}(0)]^{G, n\chi} \right)$$

where $\mathbb{C}[\Phi^{-1}(0)]^{G,n\chi} = \{f \in \mathbb{C}[\Phi^{-1}(0)] : \sigma(f) = \chi^*(t)^n \otimes f\}$ *where* $\sigma : \mathbb{C}[\Phi^{-1}(0)] \to \Gamma(G, \mathscr{O}_G) \otimes \mathbb{C}[\Phi^{-1}(0)]$ *is the coaction and* χ^* *induced by* χ *with* $\mathbb{C}^\times \cong \operatorname{Spec} \mathbb{C}[t, t^{-1}]$.

We have a natural morphism

$$\pi : Y := T^*V///_{0,\chi}G \to X := T^*V///_{0,0}G$$

since

Example 2.14 (Hypertoric varieties)**.**

Example 2.15 (Quiver varieties)**.**

3. Topologies of symplectic resolutions

[Ach21]

4. Deformations and quantizations

5. Symplectic duality

6. Geometrization and categorification of representations

7. Coulomb branches of 3d gauge theory

8. Affine Grassmannian slices as Coulomb branches

References

[Ach21] Pramod Achar. *Perverse Sheaves and Applications to Representation Theory.* American Mathematical Society, 2021.

[Fu03] Baohua Fu. Symplectic resolutions for nilpotent orbits. *Invent. Math.*, pages 167–186, 2003.

[Fu06] Baohua Fu. A survey on symplectic singularities and symplectic resolutions. *Annales mathématiques Blaise Pascal*, pages 209–236, 2006.

[Kam22] Joel Kamnitzer. Symplectic resolutions, symplectic duality, and coulomb branches. *Bull. London Math. Soc.*, 2022.

[KP79] Hanspeter Kraft and Claudio Procesi. Closures of conjugacy classes of matrices are normal. *Invent. Math.*, pages 227–247, 1979.

[Nam10] Yoshinori Namikawa. Birational geometry for nilpotent orbits yoshinori namikawa. In Ian Morrison Gavril Farkas, editor, *Handbook of Moduli, Volume III*, pages 14 – 51. International Press, 2010.

Institute of Mathematics, AMSS, Chinese Academy of Sciences, 55 Zhongguancun East Road, Beijing, 100190, China

Email address: liuxiaolong@amss.ac.cn