

Note for the Virtual Fundamental Class

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1 Introduction

We will follow [BF97][AB84][GP99] and we will also use [Ric22].
We need [Har77][Ful98][EH16].

Here we will consider $\mathbb{P}(-) = \mathbf{Proj} \mathrm{Sym}(-)^\vee$ for bundles and the vector bundle is both space and sheaf via $\mathbf{Spec} \mathrm{Sym}(-)^\vee$. For a cone $C = \mathbf{Spec}_X \mathcal{S}^*$, we define $\mathbb{P}(C) := \mathbf{Proj}_X \mathcal{S}^*$ and $\mathbb{P}(C \oplus \mathcal{O}) := \mathbf{Proj}_X \mathcal{S}^*[z]$ which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

2 Review of Basic Intersection Theory

We will follow [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field k .

2.1 Basic Facts of Refined Gysin Pullback

Here we will follow Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

Definition 2.1 (Intersection Product). *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k . Consider the cartesian diagram*

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let \mathcal{I} be the ideal of i and \mathcal{J} be the ideal of j , then we have surjection

$$\bigoplus_n f^*(\mathcal{I}^n / \mathcal{I}^{n+1}) \rightarrow \bigoplus_n \mathcal{J}^n / \mathcal{J}^{n+1} \rightarrow 0$$

*which induce embedding $C_{W/V} \hookrightarrow g^*N_{X/Y}$. Note that $C_{W/V}$ is also a scheme of pure dimension k since $\mathbb{P}(C_{W/V} \oplus \mathcal{O})$ is the exceptional divisor of $\mathrm{Bl}_W(Y \times \mathbb{A}^1)$. Let $0 : W \rightarrow g^*N_{X/Y}$ be the zero-section of $\pi : g^*N_{X/Y} \rightarrow W$, then we define*

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \mathrm{CH}_{k-d}(W)$$

as the intersection class.

Proposition 2.2. *Consider the situation of Definition 2.1.*

(a) *We have $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$.*

(b) Let \mathcal{Q} be the universal quotient bundle of $q : \mathbb{P}(g^*N_{X/Y} \oplus \mathcal{O}) \rightarrow W$, then

$$X \cdot V = q_*(c_d(\mathcal{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathcal{O})]).$$

(c) If $j : W \hookrightarrow V$ is a regular embedding of codimension d' , then $X \cdot V = c_{d-d'}(g^*N_{X/Y}/N_{W/V}) \cap [W]$.

Proof. Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98]. \square

Definition 2.3 (Refined Gysin Pullback). *Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick $f : Y' \rightarrow Y$ be a morphism. Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xhookrightarrow{j} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Then we define $i^! : Z_k Y' \rightarrow \mathrm{CH}_{k-d} X'$ as $\sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i$. Now $i^!$ can be decomposed as:

$$i^! : Z_k Y' \xrightarrow{\sigma} Z_k C_{X'/Y'} \rightarrow \mathrm{CH}_k(g^*N_{X/Y}) \xrightarrow{0^*} \mathrm{CH}_{k-d} X'$$

where $\sigma : Z_k Y' \rightarrow Z_k C_{X'/Y'}$ given by $[V] \mapsto [C_{V \cap X'/V}]$. By the technique of deformation to the normal cone, this can be descend to the Chow-group level as $\sigma : \mathrm{CH}_k Y' \rightarrow \mathrm{CH}_k C_{X'/Y'}$ (see Proposition 5.2 in [Ful98]) which is called the *specialization to the normal cone*. Hence this induce the refined Gysin pullback

$$i^! : \mathrm{CH}_k Y' \rightarrow \mathrm{CH}_{k-d} X', \quad \sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i.$$

Proposition 2.4. *Consider the situation of Definition 2.3. Consider*

$$\begin{array}{ccc} X'' & \xhookrightarrow{i''} & Y'' \\ q \downarrow & \lrcorner & p \downarrow \\ X' & \xhookrightarrow{i'} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

(a) If p proper and $\alpha \in \mathrm{CH}_k(Y'')$, then $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathrm{CH}_{k-d}(X')$.

- (b) If p is flat of relative dimension n and $\alpha \in \mathbf{CH}_k(Y'')$, then $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathbf{CH}_{k+n-d}(X'')$.
- (c) If i' is also a regular embedding of codimension d and $\alpha \in \mathbf{CH}_k(Y'')$, then $i^! \alpha = (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X'')$.
- (d) If i' is a regular embedding of codimension d' , then for $\alpha \in \mathbf{CH}_k(Y'')$ we have

$$i^!(\alpha) = c_{d-d'}(q^*(g^* N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X'').$$

We call $g^* N_{X/Y}/N_{X'/Y'}$ the *excess normal bundle*.

- (e) Let F be any vector bundle on Y' , then for $\alpha \in \mathbf{CH}_k(Y'')$ we have

$$i^!(c_m(F) \cap \alpha) = c_m((i')^* F) \cap i^!(\alpha) \in \mathbf{CH}_{k-d-m}(X').$$

Proof. See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98]. \square

Corollary 2.5. Let $i : X \hookrightarrow Y$ be a regular embedding of codimension d , then

$$i^* i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathbf{CH}_*(X).$$

Proof. By Proposition 2.4(d) directly. \square

Proposition 2.6. The refined Gysin pullback have the following properties.

- (a) Let $i : X \hookrightarrow Y$ and $j : S \hookrightarrow T$ are regular embeddings of codimension d, e , respectively. Consider cartesian:

$$\begin{array}{ccccc} X'' & \hookrightarrow & Y'' & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow j' & \lrcorner & \downarrow j \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{g} & T \\ \downarrow & \lrcorner & \downarrow f & & \\ X & \xrightarrow{i} & Y & & \end{array}$$

Then for any $\alpha \in \mathbf{CH}_k(Y'')$, we have

$$j^! i^!(\alpha) = i^! j^!(\alpha) \in \mathbf{CH}_{k-d-e}(X'').$$

- (b) Let $i : X \hookrightarrow Y$ and $j : Y \hookrightarrow Z$ are regular embeddings of codimension d, e , respectively. Consider cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & Z \end{array}$$

Then ji is a regular embedding of codimension $d + e$ and for all $\alpha \in \text{CH}_k(Z')$ we have

$$(ji)^!(\alpha) = i^! j^!(\alpha) \in \text{CH}_{k-d-e}(X').$$

Proof. See Theorem 6.4 and Theorem 6.5 in [Ful98]. \square

Proposition 2.7. *Consider cartesian:*

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \end{array}$$

- (a) *If i is a regular embedding of codimension d and p and pi are flat of relative dimension $n, n-d$, respectively. Then i' is a regular embedding of codimension d and $p', p'i'$ are flat, and for $\alpha \in \text{CH}_k(Z')$ we have*

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i^!((p')^*\alpha).$$

- (b) *If i is a regular embedding of codimension d and p is smooth of relative dimension n , and pi is a regular embedding of codimension $d-n$. Then for $\alpha \in \text{CH}_k(Z')$ we have*

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

Proof. See Proposition 6.5 in [Ful98]. \square

Remark 2.8. *Some remarks.*

- (a) *For local complete intersection morphism $f : X \rightarrow Y$, we can decompose it into $f : X \xrightarrow{i} P \xrightarrow{p} Y$ where i is a closed regular embedding of constant codimension and p is smooth of constant relative dimension. Then we can define $f^! := i^!(p)^*$. See Section 6.6 in [Ful98] for more properties.*
- (b) *If Y is nonsingular of dimension n , then we can define the following intersection product: Let $f : X \rightarrow Y$ and $p : X' \rightarrow X$ and $q : Y' \rightarrow Y$. Let $x \in \text{CH}_k(X')$ and $y \in \text{CH}_l(Y')$, consider the cartesian*

$$\begin{array}{ccc} X' \times_Y Y' & \longrightarrow & X' \times Y' \\ \downarrow & \lrcorner & \downarrow p \times q \\ X & \xrightarrow{\gamma_f} & X \times Y \end{array}$$

and define $x \cdot_f y := \gamma_f^!(x \times y) \in \text{CH}_{k+l-n}(X' \times_Y Y')$.

So when $x, y \in \text{CH}_*(Y)$, then let $X = Y$ and $X' = |x|, Y' = |y|$, then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case $\text{CH}_*(Y)$ is a ring which is called *Chow ring*.

Finally we will discuss something about equivalence and supportness.

Definition 2.9. Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d with normal bundle $N_{X/Y}$. Pick V be a scheme of pure dimension k . Consider the cartesian diagram

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let C_1, \dots, C_r be the irreducible components of $C_{W/V}$, then $[C_{W/V}] = \sum_{i=1}^r m_i [C_i]$. Let $Z_i = \pi(C_i)$ where $\pi : g^*N_{X/Y} \rightarrow W$ and we call them the *distinguished varieties* of the intersection of V by X . Let $N_i := (g^*N_{X/Y})|_{Z_i}$ and let $0_i : Z_i \rightarrow N_i$ be the zero-sections. Let $\alpha_i := 0_i^*[C_i] \in \text{CH}_{k-d}(Z_i)$ and hence we have $X \cdot V = \sum_{i=1}^r m_i \alpha_i \in \text{CH}_{k-d}(W)$.

Pick any closed set $S \subset W$, we define

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i \alpha_i \in \text{CH}_{k-d}(S)$$

as the part of $X \cdot V$ supported on S .

Definition 2.10. Let $X_i \hookrightarrow Y$ be closed regular embeddings of codimension d_i . Let $V \subset Y$ be a k -dimensional subvariety. Consider

$$\begin{array}{ccc} \bigcap_i X_i \cap V & \hookrightarrow & V \\ \downarrow & \lrcorner & \downarrow \delta \\ X_1 \times \dots \times X_r & \hookrightarrow & Y \times \dots \times Y \end{array}$$

Then we can get $X_1 \cdot \dots \cdot X_r \cdot V \in \text{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$.

Let Z be a connected component of $\bigcap_i X_i \cap V$, we will consider

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z \in \text{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

Proposition 2.11. *As in the previous situation, we have*

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cap s(Z, V) \right\}_{\dim V - \sum_i d_i}.$$

If $Z \hookrightarrow V$ is a regular embedding, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If V, Z are both non-singular, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

Proof. See Proposition 9.1.1 in [Ful98]. □

2.2 Localized Chern Class

Here we will follow Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

Definition 2.12. *Let $E \rightarrow X$ be a vector bundle of rank e over a purely n -dimensional scheme X . Let $s : X \rightarrow E$ be a section, consider the cartesian*

$$\begin{array}{ccc} Z(s) & \longrightarrow & X \\ i \downarrow & \lrcorner & \downarrow s \\ X & \xrightarrow{0} & E \end{array}$$

with zero-section $0 : X \rightarrow E$ which is a regular section by trivial reason. We define

$$c_{\text{loc}}(E, s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \text{CH}_{n-e}(Z(s))$$

be the localized (top) Chern class of E with respect to s .

Proposition 2.13. *Consider the situation of Definition 2.12.*

- (a) *We have $i_*(c_{\text{loc}}(E, s)) = c_e(E) \cap [X]$.*
- (b) *Each irreducible component of $Z(s)$ has codimension at most e in X . If $\text{codim}_{Z(s)} X = e$, then $c_{\text{loc}}(E, s)$ is a positive cycle whose support is $Z(s)$.*
- (c) *If s is a regular section, then $c_{\text{loc}}(E, s) = [Z(s)]$.*

- (d) Let $f : X' \rightarrow X$ be a morphism, $s' = f^*s$ be a induced section of f^*E .
 Let $g : Z(s') \rightarrow Z(s)$ be the induced morphism.
- (d1) If f flat, then $g^*c_{\text{loc}}(E, s) = c_{\text{loc}}(f^*E, s')$.
- (d2) If f is proper of varieties, then $g_*c_{\text{loc}}(f^*E, s') = \deg(X'/X)c_{\text{loc}}(E, s)$.

Proof. For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_*0^![X] = 0^*s_*[X] = s^*s_*[X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{array}{ccc} Z(s') & \longrightarrow & X' \\ \downarrow & \lrcorner & \downarrow s' \\ X' & \xrightarrow{0_{f^*E}} & f^*E \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{0_E} & E \end{array}$$

and Proposition 2.4. □

3 A Brief of Cotangent Complexes

Here we will give a quike introduction of cotangent complexes. We will consider Deligne-Mumford stacks locally of finite type over k . Morphisms are quasicompact and quasiseparated. We work over étale site.

Theorem 3.1. *For every morphism $f : X \rightarrow Y$ of DM-stacks (resp. finite type morphism of noetherian DM-stacks), there exists a complex*

$$\mathbb{L}_{X/Y} : \cdots \rightarrow \mathbb{L}_{X/Y}^{-1} \rightarrow \mathbb{L}_{X/Y}^0 \rightarrow 0$$

of flat \mathcal{O}_X -modules with quasi-coherent (resp., coherent) cohomology, whose image $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$ (resp. $\mathbf{D}_{\text{Coh}}^-(X_{\text{ét}})$) is also denoted by $\mathbb{L}_{X/Y}$. This is called the cotangent complex of f . It satisfies the following properties.

- (a) $H^0(X, \mathbb{L}_{X/Y}) = \Omega_{X/Y}^1$.
- (b) *The morphism f is smooth if and only if f is locally of finite presentation and $\mathbb{L}_{X/Y}$ is a perfect complex supported in degree 0. In this case, there is a quasi-isomorphism $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}^1[0]$.*

(c) If f factors as $X \hookrightarrow Z$ defined by a sheaf of ideals \mathcal{I} and a smooth morphism $Z \rightarrow Y$, then

$$\mathbb{L}_{X/Y} \cong [0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{Z/Y}^1|_X \rightarrow 0]$$

in $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$ with $\Omega_{X/Y}^1$ in degree 0. If in addition f is generically smooth, then $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}^1[0]$. Moreover, if f is lci, then $\mathbb{L}_{X/Y}$ is perfect of perfect amplitude contained in $[-1, 0]$.

(d) If we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

then there is a morphism $(\mathbf{L}g')^*\mathbb{L}_{X/Y} \rightarrow \mathbb{L}_{X'/Y'}$. When f or g is flat, then it is a quasi-isomorphism.

(e) If $X \xrightarrow{f} Y \rightarrow Z$ is a composition of morphisms of DM-stacks, then there is an exact triangle

$$\mathbf{L}f^*\mathbb{L}_{Y/Z} \rightarrow \mathbb{L}_{X/Z} \rightarrow \mathbb{L}_{X/Y} \rightarrow \mathbf{L}f^*\mathbb{L}_{Y/Z}[1]$$

in $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$. This induces a long exact sequence on cohomology

$$\cdots \rightarrow H^{-1}(\mathbb{L}_{X/Z}) \rightarrow H^{-1}(\mathbb{L}_{X/Y}) \rightarrow f^*\Omega_{Y/Z}^1 \rightarrow \Omega_{X/Z}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0.$$

Proof. In the level of ring maps $A \rightarrow B$, this is constructed by standard simplicial free A -resolution $B \rightarrow P(B)_*$ where $P(B)_n = A[\cdots [A[B]] \cdots]$ as

$$\mathbb{L}_{B/A} := \Omega_{P(B)_*/A} \otimes_{P(B)_*} B.$$

See Tag 08UV Tag 0D0N Tag 0FK3 Tag 08QQ Tag 08T4. \square

Remark 3.2. For the general algebraic stacks, any quasicompact and quasiseparated 1-morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ there exists a relative cotangent complex

$$\mathbb{L}_f \in \mathbf{D}_{\text{Coh}}^{\leq 1}(\mathcal{X}_{\text{lis-ét}})$$

over lisse-étale site of \mathcal{X} . Existence is good, but the fact that the cotangent complex trespasses to positive degree forces one to pay more attention when performing the cutoff. If the diagonal of f is unramified (as we consider now), then this problem goes away, in the sense that $\mathbb{L}_f \in \mathbf{D}_{\text{Coh}}^{\leq 0}(\mathcal{X}_{\text{lis-ét}})$. We refer section C.3 in [Ric22] for more comments about this and the generalization of the properties as above.

4 Foundations of Virtual Fundamental Class

We will follow [BF97]. Here an algebraic stack (or Artin stack) over a field k is assumed to be quasi-separated and locally of finite type over k .

4.1 About Cones

We will let X be a Deligne-Mumford stack now.

Definition 4.1. *Let X be a DM-stack.*

- (a) *We call an affine X -scheme $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ is a cone over X if the quasi-coherent algebra \mathcal{S} is graded as $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$ with $\mathcal{S}^0 = \mathcal{O}_X$ and \mathcal{S}^1 is coherent and \mathcal{S} is generated by \mathcal{S}^1 .*
- (b) *A morphism of cones over X is an X -morphism induced by a graded morphism of graded sheaves of \mathcal{O}_X -algebras. A closed subcone is the image of a closed immersion of cones.*

Remark 4.2. (a) *The fiber product of cones over X is still a cone over X .*

- (b) *For every cone $C \rightarrow X$, it has a zero section $0 : X \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}^0$.*
- (c) *For every cone $C \rightarrow X$, the grade induce a \mathbb{G}_m -action $\mathbb{G}_m \times C = \underline{\mathrm{Spec}}_X \mathcal{S}[t, t^{-1}] \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}[t, t^{-1}]$ via $s_0 + \dots + s_d \mapsto \sum_i a_i t^i$ where $s_i \in \mathcal{S}^i$. Since no negative power of t occurs, we can in fact replace \mathbb{G}_m by \mathbb{A}^1 . So we have the \mathbb{A}^1 -action $\gamma : \mathbb{A}^1 \times C \rightarrow C$ induced by $\mathcal{S} \rightarrow \mathcal{S}[x]$ via $\mathcal{S}^i \ni s \mapsto sx^i$. Note that here \mathbb{A}^1 is not a group scheme and the action here, as expected, to be the commutativity of the following diagrams:*

$$\begin{array}{ccc}
 C & \xrightarrow{(1, \mathrm{id})/(0, \mathrm{id})} & \mathbb{A}^1 \times C \\
 & \searrow \mathrm{id}/0 & \downarrow \gamma \\
 & & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{A}^1 \times \mathbb{A}^1 \times C & \xrightarrow{\mathrm{id} \times \gamma} & \mathbb{A}^1 \times \mathbb{A}^1 \times C \\
 m \times \mathrm{id} \downarrow & & \downarrow \gamma \\
 \mathbb{A}^1 \times C & \xrightarrow{\gamma} & C
 \end{array}$$

where $m(x, y) = xy$.

- (d) *So a morphism of cones $f : C \rightarrow D$ over X is just the \mathbb{A}^1 -equivariant X -morphism respecting the zero section, that is, the following commutativity of the diagram:*

$$\begin{array}{ccccc}
 \mathbb{A}^1 \times C & \longrightarrow & C & \xleftarrow{0_C} & X \\
 \mathrm{id} \times f \downarrow & & f \downarrow & \nearrow 0_D & \\
 \mathbb{A}^1 \times D & \longrightarrow & D & &
 \end{array}$$

Definition 4.3. Let \mathcal{F} be a coherent sheaf of X , then we can define $C(\mathcal{F}) := \underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathcal{F})$ which is a group scheme over X since it can be represented as $C(\mathcal{F})(T) = \mathrm{Hom}(\mathcal{F}_T, \mathcal{O}_T)$. We call a cone of this form is an **abelian cone** over X .

Remark 4.4. (a) A fibered product of abelian cones is an abelian cone.

(b) A vector bundle $E = \underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathcal{E}^\vee)$ is a special case.

(c) Any cone $C = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$ is canonically a closed subcone of an abelian cone $A(C) = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{S}^1$, called the **abelian hull** of C . The abelian hull is a vector bundle if and only if \mathcal{S}^1 is locally free.

(d) The **abelianization** $C \mapsto A(C)$ is a functor has the forgetful functor as a right adjoint. So we have

$$\mathrm{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \mathrm{Hom}_{\mathbf{Cone}_X}(C, A).$$

(e) Let \mathbf{Alg}_X^o as the category of quasicoherent graded \mathcal{O}_X -algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{Alg}_X^o & \xrightarrow{\underline{\mathrm{Spec}}_X} & \mathbf{Cone}_X^{\mathrm{op}} \\ \mathrm{Sym} \uparrow & & \uparrow \\ \mathbf{LocFree}_X & \xrightarrow{\underline{\mathrm{Spec}}_X \mathrm{Sym}(-)^\vee} & \mathbf{Vect}_X^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{Coh}_X & \xrightarrow{\underline{\mathrm{Spec}}_X \mathrm{Sym}} & \mathbf{AbCone}_X^{\mathrm{op}} \end{array}$$

Example 4.5. Two important examples. Let $X \hookrightarrow Y$ be a closed immersion of ideal \mathcal{I} . Then $C_{X/Y} := \underline{\mathrm{Spec}}_X \bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$ is called the **normal cone** of X in Y . The associated abelian cone $N_{X/Y} = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{I} / \mathcal{I}^2$ is called the **normal sheaf** of X in Y .

Lemma 4.6. About smoothness:

(a) Let $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ be a cone over X . Then $C_{X/C} \cong \mathcal{S}^1 \cong 0^* \Omega_{C/X}$.

(b) A cone C over X is a vector bundle if and only if it is smooth over X .

(c) Let $C \rightarrow D$ be a smooth morphism of cones of relative dimension n over X . Then the induced morphism $A(C) \rightarrow A(D)$ is also smooth of relative dimension n .

Proof. For (a), note that $C_{X/C} \cong \mathcal{S}^1$ is trivial by definition. Moreover, $0 : X \rightarrow C$ is the zero section and we have $0 \rightarrow C_{X/C} \rightarrow 0^*\Omega_{C/X} \rightarrow \Omega_{X/X} = 0$ exact (see Tag 0474). Well done.

For (b), let $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$ and assume that $C \rightarrow X$ has constant relative dimension r . Then $\mathcal{S}^1 = 0^*\Omega_{C/X}$ is locally free of rank r . As $C \hookrightarrow A(C)$ where $A(C)$ is a vector bundle and $\dim C = \dim A(C)$, we know that C is a vector bundle.

For (c), apply the exact triangle of cotangent complex to $X \rightarrow C \rightarrow D$ and (a), we have an exact sequence

$$0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow 0^*\Omega_{C/D} \rightarrow 0$$

where $C = \underline{\text{Spec}}_X \mathcal{S}$ and $D = \underline{\text{Spec}}_X \mathcal{T}$. So locally we have $A(C) = A(D) \times_X \underline{\text{Spec}}_X \text{Sym}(0_C^*\Omega_{C/D})$. Well done. \square

Definition 4.7. A sequence of cone morphisms

$$0 \rightarrow E \xrightarrow{i} C \rightarrow D \rightarrow 0$$

is called **exact** if E is a vector bundle and locally over X there is a morphism of cones $C \rightarrow E$ splitting i and inducing an isomorphism $C \cong E \times_X D$.

Remark 4.8. As $E \rightarrow X$ is smooth and surjective by Lemma 4.6, if $0 \rightarrow E \xrightarrow{i} C \rightarrow D \rightarrow 0$ then locally we have $C \cong E \times_X D$ which force that $C \rightarrow D$ is smooth and surjective! Similarly $i : E \rightarrow C$ is a closed embedding.

Lemma 4.9. We have the following useful results.

- (a) Given a short exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0$ of coherent sheaves on X , with \mathcal{E} locally free, then $0 \rightarrow C(\mathcal{E}) \rightarrow C(\mathcal{F}) \rightarrow C(\mathcal{F}') \rightarrow 0$ is exact, and conversely is also true.
- (b) Let $0 \rightarrow E \rightarrow F \xrightarrow{f} G \rightarrow 0$ be an exact sequence of abelian cones over X with E a vector bundle. Assume that $D \subset G$ is a closed subcone, then the induced sequence $0 \rightarrow E \rightarrow f^{-1}(D) =: C \rightarrow D \rightarrow 0$ is exact.
- (c) Let $f : C \rightarrow D$ be a morphisms of cones over X which is smooth surjective, then the induced diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ A(C) & \xrightarrow{A(f)} & A(D) \end{array}$$

is cartesian. Moreover, we have $D = [C/E]$ (see Lemma 4.12(a)) and $A(D) = [A(C)/E]$, where $E := C \times_{D,0} X = A(C) \times_{A(D),0} X$.

(d) Let E be a vector bundle over X and then the sequence $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$ is exact if and only if the abelian hulls $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$ is exact and $C \rightarrow D$ is smooth and surjective.

Proof. For (a), we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. As exactness is local, we may assume \mathcal{E} is free. Then the first sequence is exact if and only if $\mathcal{F}' \oplus \mathcal{E} = \mathcal{F}$ if and only if the second sequence is exact as cones, since $\text{Sym}(\mathcal{F}' \oplus \mathcal{E}) = \text{Sym}(\mathcal{F}') \otimes \text{Sym}(\mathcal{E}) = \text{Sym}(\mathcal{F})$.

For (b), note that this can be checked locally, so we can let we can assume that $\mathcal{F} = \mathcal{G} \oplus \mathcal{E}^\vee$ where $E = \text{Spec}_X \text{Sym} \mathcal{E}^\vee$ and $F = \text{Spec}_X \text{Sym} \mathcal{F}$ and $G = \text{Spec}_X \text{Sym} \mathcal{G}$. Let $D = \text{Spec}_X \mathcal{T}$, then we have surjection $\text{Sym}(\mathcal{G}) \rightarrow \mathcal{T}$. By definition, we have

$$\begin{aligned} C &= F \times_G D = \text{Spec}_X (\text{Sym}(\mathcal{F}) \otimes_{\text{Sym}(\mathcal{G})} \mathcal{T}) \\ &= \text{Spec}_X ((\text{Sym}(\mathcal{G}) \otimes \text{Sym} \mathcal{E}^\vee) \otimes_{\text{Sym}(\mathcal{G})} \mathcal{T}) \\ &= \text{Spec}_X (\text{Sym} \mathcal{E}^\vee \otimes \mathcal{T}). \end{aligned}$$

This means locally $C = E \oplus D$ and the splitting $C \rightarrow E$ is induced by $F \rightarrow E$. Well done.

For (c), let $E := C \times_{D,0} X$ and $E' := A(C) \times_{A(D)} D$ with embedding $E \hookrightarrow E'$, then both of them are vector bundles by Lemma 4.6(b)(c) and hence $E = E'$. We have cartesians

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow \wr & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \begin{array}{ccc} E & \longrightarrow & X \\ \downarrow \wr & & \downarrow \\ A(C) & \longrightarrow & A(D) \end{array}$$

By the properties of commutative affine group schemes, we have $A(D) = [A(C)/E]$. But how about $[C/E]$? Now we have

$$\begin{array}{ccccc} & & & & D \\ & & & \nearrow & \\ C & \xrightarrow{\quad \wr \quad} & [C/E] & \nearrow & \\ \downarrow & & \downarrow & & \\ A(C) & \longrightarrow & A(D) & & \end{array}$$

Since $C \rightarrow [C/E]$ and $C \rightarrow D$ are both smooth and surjective, we know that $[C/E] \rightarrow D$ is flat and surjective. But by closed embeddings $[C/E] \rightarrow A(D)$ and $D \rightarrow A(D)$, we know that $[C/E] \rightarrow D$ is also a closed embedding. Thus $D = [C/E]$, well done.

For (d), note that all the question is locally on X . First we assume $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D \rightarrow 0$ is exact. Then by (a), to show that $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$ is exact, we only need to show that $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$ is exact where $E = \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$ and $C = \underline{\text{Spec}}_X \mathcal{S}$ and $D = \underline{\text{Spec}}_X \mathcal{T}$. First since f is faithfully flat and quasi-compact, we know that $\mathcal{T}^1 \rightarrow \mathcal{S}^1$ is injective. And since i is a closed embedding, $\mathcal{S}^1 \rightarrow \mathcal{E}^\vee$ is surjective. Now by local splitting, we know that locally we have $\text{Sym}(\mathcal{E}^\vee) \otimes \mathcal{T} = \mathcal{S}$. In particular, we have $\mathcal{T}^1 \oplus \mathcal{E}^\vee = \mathcal{S}^1$. Thus the exactness of $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$ is obtained. Conversely we assume that after taking abelian hull, the sequence is exact. Now the result follows from (a) and (c). \square

Proposition 4.10. *Let $C \rightarrow D$ be a smooth, surjective morphism of cones. If we let $E = C \times_{D,0} X$, then the sequence*

$$0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$$

is exact. Conversely if $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$ is exact, then $E \cong C \times_{D,0} X$.

Proof. Let $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$ and $D = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{T}^i$.

Let $E = C \times_{D,0} X = \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$, by Lemma 4.9(d) we just need to show that $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$ is exact, that is, $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$ is exact by Lemma 4.9(a). Note that $\text{Sym } \mathcal{E}^\vee = \mathcal{S} \otimes_{\mathcal{T}} (\mathcal{T} / \mathcal{T}^{\geq 1})$ which force $\mathcal{E}^\vee \cong \mathcal{S}^1 / \mathcal{T}^1$. Well done.

Conversely, assume that the sequence $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$ is exact and $F = C \times_{D,0} X$. Then by the universal property of fibre product, we get a morphism $E \rightarrow F$. From the construction, it is easy to see that $\mathcal{F}^\vee \rightarrow \mathcal{E}^\vee$ is surjective. Since they are both bundles of the same rank over X , we know that $E = F$. \square

Definition 4.11. (a) *If E is a vector bundle and $f : E \rightarrow C(\mathcal{F})$ a morphism of abelian cones. Then there is an E -action as $E \times_X C(\mathcal{F}) \rightarrow C(\mathcal{F})$ as $(\nu, \gamma) \mapsto f\nu + \gamma$.*

(b) *If E is a vector bundle and $d : E \rightarrow C$ a morphism of cones, we say that C is an E -cone, if C is invariant under the action of E on $A(C)$.*

(c) *A morphism ϕ from an E -cone C to an F -cone D is a commutative diagram of cones*

$$\begin{array}{ccc} E & \xrightarrow{d_E} & C \\ \downarrow \phi & & \downarrow \phi \\ F & \xrightarrow{d_F} & D \end{array}$$

(d) *If $\phi : (E, d_E, C) \rightarrow (F, d_F, D)$ and $\psi : (E, d_E, C) \rightarrow (F, d_F, D)$ are morphisms, we call them homotopic, if there exists a morphism of cones $k : C \rightarrow F$, such that $kd_E = \psi - \phi = d_F k$.*

Lemma 4.12. *Some useful lemmas:*

- (a) *Let $f : C \rightarrow D$ be a smooth surjective cone morphism with $E = C \times_{D,0} X$, then C is an E -cone.*
- (b) *Let $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D = [C/E] \rightarrow 0$ be a sequence of algebraic X -spaces with E a bundle, C is a E -cone, i a closed embedding and $f : C \rightarrow D = [C/E]$ is the universal family. Then locally on X , there is a $j : C \rightarrow E$ split i and induces an isomorphism $(f, j) : C \rightarrow D \times_X E$.*
- (c) *Let $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D \rightarrow 0$ be a sequence of algebraic X -spaces with sections and \mathbb{A}^1 -actions such that E a bundle, C is a E -cone, i is a closed embedding and f is \mathbb{A}^1 -equivariant. Then D is a cone with the sequence exact if and only if $D \cong [C/E]$.*

Proof. For (a), this follows from directly check. We omit it.

For (b), since the question is local we can assume that E is a trivial bundle and X is a scheme. Let $i' : E \rightarrow A(C)$ and $C = \text{Spec}_X \mathcal{S}$ and $E = \text{Spec}_X \text{Sym } \mathcal{E}^\vee$. Then the surjection $\mathcal{S}^1 \rightarrow \mathcal{E}^\vee$ has a splitting $\mathcal{E}^\vee \hookrightarrow \mathcal{S}^1$, which gives $j' : A(C) \rightarrow E$ such that $j' \circ i' = \text{id}_E$. Then we just define $j : C \rightarrow E$ as composition with $C \rightarrow A(C)$ and j' . Hence $j \circ i = \text{id}_E$.

Now since $C \rightarrow D$ is also a principal E -bundle, and we have a E -equivariant D -morphism $(f, j) : C \rightarrow D \oplus E$ from C to the trivial principal bundle. Since they are both E -principal bundle, we know that (f, j) is an isomorphism.

For (c), let $D = [C/E]$. We know that $D \rightarrow X$ is affine since locally on X we have $C \cong D \times_X E \rightarrow E$ is affine and (b) and faithfully flat descent. By construction we have $E = C \times_{D,0} X$, hence by Proposition 4.10 we just need to show D is a cone. Now as $D \rightarrow X$ affine we have $D = \text{Spec}_X \mathcal{T}$. If $C = \text{Spec}_X \mathcal{S}$, then $\mathcal{T} \subset \mathcal{S}$ as $C \rightarrow D$ is faithfully flat. Hence it has graded structure $\mathcal{T} = \bigoplus_{i \geq 0} \mathcal{T} \cap \mathcal{S}^i$ as f is \mathbb{A}^1 -equivariant. As it have zero section, we have $\mathcal{T}^0 = \mathcal{O}_X$. Finally we have \mathbb{A}^1 -equivariant embedding $D \hookrightarrow [A(C)/E]$ and $[A(C)/E]$ is a cone by Lemma 4.9(c). Hence \mathcal{T} generated by the coherent sheaf \mathcal{T}^1 .

Conversely, we assume D is a cone and that sequence is exact. Let $D' = [C/E]$. By the universal property of quotient, we have a natural map $g : D' \rightarrow D$. Since $0 \rightarrow E \rightarrow C \rightarrow D' \rightarrow 0$ is also exact by the first case, by exactness we have locally $C \cong E \times_X D \cong E \times_X D'$. Note that these isomorphisms compatible with $g : D' \rightarrow D$, hence by faithfully flat descent we have g is an isomorphism. \square

Proposition 4.13. *Let X be a DM-stack.*

- (a) Let E be a vector bundle. Consider the sequence of cone morphisms $0 \rightarrow E \xrightarrow{i} C \xrightarrow{\phi} D \rightarrow 0$ with i a closed embedding. Then it is exact if and only if C is a E -cone, $\phi : C \rightarrow D$ is faithfully flat and the diagram

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ \downarrow p & \lrcorner & \downarrow \phi \\ C & \xrightarrow{\phi} & D \end{array}$$

is cartesian with projection p and action σ .

- (b) Let $(C, 0, \gamma)$ and $(D, 0, \gamma)$ be algebraic X -spaces with sections and \mathbb{A}^1 -actions and let $\phi : C \rightarrow D$ be an \mathbb{A}^1 -equivariant X -morphism, which is smooth and surjective. Let $E = C \times_{D,0} X$. Assume that E is a vector bundle. Then C is an E -cone (resp. abelian cone, vector bundle) over X if and only if D is a cone (resp. abelian cone, vector bundle) over X and C is affine over X .

Proof. For (a), if it is exact, locally we have $C \cong E \times_X D$. So E act on C locally as $E \times E \times_X D \rightarrow E \times_X D$ given by $(f, (e, d)) \mapsto (i(f) + e, d)$. So C is a E -cone. Now $\phi : C \rightarrow D$ is trivially faithfully flat. The cartesian diagram follows from Lemma 4.12(c).

Conversely, since ϕ is fppf, this diagram is also cocartesian by Proposition V.1.3.1 in [Li18] which force $D = [C/E]$. Hence the results follows from Lemma 4.12(c).

For (b), let C is an E -cone over X . Then we have $g : [C/E] \rightarrow D$. We claim that g is an isomorphism. Indeed, by the diagram in (a), we know that g induces an isomorphism $g' : E \times_X C = C \times_{[C/E]} C \rightarrow C \times_D C$. Note that we have a cartesian diagram:

$$\begin{array}{ccc} C \times_{[C/E]} C & \longrightarrow & C \times_D C \\ \downarrow & \lrcorner & \downarrow \\ [C/E] & \hookrightarrow & [C/E] \times_D [C/E] \end{array}$$

where $C \times_D C \rightarrow [C/E] \times_D [C/E]$ is faithfully flat, hence $[C/E] \hookrightarrow [C/E] \times_D [C/E]$ is an isomorphism. So g is a monomorphism. But since $C \rightarrow [C/E]$ and $C \rightarrow D$ are faithfully flat, hence epimorphism. Thus g is also an epimorphism, hence an isomorphism. Thus $D \cong [C/E]$ and the result follows from Lemma 4.12(c).

Now assume that $C = A(C)$ is an abelian cone, then taking hull to $0 \rightarrow E \rightarrow C \rightarrow D = [C/E] \rightarrow 0$. By Lemma 4.9(c)(d) we have $A(D) = [A(C)/E] = [C/E] = D$. Hence D is also an abelian cone.

Finally assume that C is a bundle. Then by the previous case we know that D is an abelian cone. The $\mathcal{T}^1 = \ker(\mathcal{S}^1 \twoheadrightarrow \mathcal{E}^\vee)$ is clearly locally

free since \mathcal{C}^1 and \mathcal{E} are where $C = \underline{\text{Spec}}_X \mathcal{S}$, $D = \underline{\text{Spec}}_X \mathcal{T}$ and $E = \underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$.

Conversely we let D is a cone and C is affine over X . Hence we have $C = \underline{\text{Spec}}_X \mathcal{S}$ where $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$ and $\mathcal{S}^1 = \mathcal{O}_X$. By the same reason E is affine over X . Hence we have $C = \underline{\text{Spec}}_X \mathcal{F}$ where $\mathcal{F} = \bigoplus_{i \geq 0} \mathcal{F}^i$ and $\mathcal{F}^1 = \mathcal{O}_X$. If we let $D = \underline{\text{Spec}}_X \mathcal{T}$, then $\mathcal{F} = \mathcal{S}/(\mathcal{T}^{\geq 1} \mathcal{S})$.

Apply the exact triangle of cotangent complex to $X \xrightarrow{0_C} C \rightarrow D$, we have an exact sequence

$$0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 = C_{X/C} \rightarrow \mathcal{E}^\vee := 0_C^* \Omega_{C/D} \rightarrow 0.$$

As $\mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 = \mathcal{S}^1 \oplus \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2$, we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{S}^1 & \longrightarrow & \mathcal{F}^1 \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 & \longrightarrow & \mathcal{E}^\vee \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2 & \xrightarrow{=} & \mathcal{E}^\vee/\mathcal{F} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Locally on X we can assume that \mathcal{E} is free and $\mathcal{T}^1 \oplus \mathcal{E}^\vee = \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2$. Then as $\mathcal{F}^1 \subset \mathcal{E}^\vee$, we know that \mathcal{F}^1 . Since \mathcal{T}^1 is also coherent, we know that so is \mathcal{S}^1 . Finally we just need to show \mathcal{S} generated by \mathcal{S}^1 as by Lemma 4.12(a) here C will be an E -cone.

Then locally on X we can choose generators of $\mathcal{T}^1, \mathcal{F}^1, \mathcal{E}^\vee/\mathcal{F}^1 = \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2$ such that gives a surjective \mathcal{O}_X -algebra morphism $\phi : \mathcal{T} \oplus \text{Sym} \mathcal{E}^\vee \twoheadrightarrow \mathcal{S}$ which induce $\mathcal{T} \oplus \text{Sym} \mathcal{F}^1 \rightarrow \mathcal{T} \oplus \text{Sym} \mathcal{E}^\vee \twoheadrightarrow \mathcal{S}$ is graded. Tensoring $(-) \otimes_{\mathcal{T}} \mathcal{O}_X$ with ϕ we get surjection $\phi' : \text{Sym} \mathcal{E}^\vee \twoheadrightarrow \mathcal{F}$. This induce the closed immersion $E \hookrightarrow \underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$. Since they are both smooth of a same relative dimension over X and $\underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$ is a vector bundle, hence $E \cong \underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$ and ϕ' is an isomorphism. Hence $\mathcal{F} = \text{Sym}(\mathcal{F}^1)$ and \mathcal{F}^1 is locally free. As $\text{Sym}(\mathcal{F}^1) \subset \text{Sym} \mathcal{E}^\vee \xrightarrow{\phi'} \mathcal{F} = \text{Sym}(\mathcal{F}^1)$ is identity, this force $\mathcal{E}^\vee = \mathcal{F}^1$. As this can be check locally, we have $\mathcal{E}^\vee = \mathcal{F}^1$ in whole X . By the diagram above, we have $\mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2 = \mathcal{E}^\vee/\mathcal{F}^1 = 0$. This means \mathcal{S} generated by \mathcal{S}^1 . Well done. \square

Remark 4.14. In the original paper [BF97] they claim (a) is enough for the surjectivity of f .

4.2 Cone Stack

Let X be a Deligne-Mumford stack.

Definition 4.15. Let \mathfrak{C} be an algebraic stack over X , together with a section $0 : X \rightarrow \mathfrak{C}$. An \mathbb{A}^1 -action on $(\mathfrak{C}, 0)$ is given by a morphism of X -stacks $\gamma : \mathbb{A}^1 \times \mathfrak{C} \rightarrow \mathfrak{C}$ and three 2-isomorphisms θ_1, θ_0 and θ_γ between the 1-morphisms in the following diagrams.

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{(1, \text{id})/(0, \text{id})} & \mathbb{A}^1 \times \mathfrak{C} \\ \text{id}/0 \searrow & \xRightarrow{\theta_1/\theta_0} & \swarrow \gamma \\ & \mathfrak{C} & \end{array}$$

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\text{id} \times \gamma} & \mathbb{A}^1 \times \mathfrak{C} \\ \downarrow m \times \text{id} & \xRightarrow{\theta_\gamma} & \downarrow \gamma \\ \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\gamma} & \mathfrak{C} \end{array}$$

The 2-isomorphisms θ_1, θ_0 and θ_γ are required to satisfy certain compatibilities.

Definition 4.16. Let $(\mathfrak{C}, 0, \gamma)$ and $(\mathfrak{D}, 0, \gamma)$ be X -stacks with sections and \mathbb{A}^1 -actions. Then an \mathbb{A}^1 -equivariant morphism $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ is a triple $(\phi, \eta_0, \eta_\gamma)$, where $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ is a morphism of algebraic X -stacks and η_0 and η_γ are 2-isomorphisms between the morphisms in the following diagrams.

$$\begin{array}{ccc} X & \xrightarrow{0} & \mathfrak{C} \\ & \searrow \eta_0 & \downarrow \phi \\ & 0 & \mathfrak{D} \end{array}$$

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\text{id} \times \phi} & \mathbb{A}^1 \times \mathfrak{D} \\ \downarrow \gamma & \xRightarrow{\eta_\gamma} & \downarrow \gamma \\ \mathfrak{C} & \xrightarrow{\phi} & \mathfrak{D} \end{array}$$

Again, the 2-isomorphisms have to satisfy certain compatibilities.

Definition 4.17. Let $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \rightarrow \mathfrak{D}$ and $(\psi, \eta'_0, \eta'_\gamma) : \mathfrak{C} \rightarrow \mathfrak{D}$ be two \mathbb{A}^1 -equivariant morphisms. An \mathbb{A}^1 -equivariant isomorphism $\zeta : \phi \rightarrow \psi$ is a

2-isomorphism $\zeta : \phi \rightarrow \psi$ such that the diagrams

$$\begin{array}{ccc}
0 & \xrightarrow{\eta_0} & \phi \circ 0 \\
& \searrow \eta'_0 & \downarrow \zeta \circ 0 \\
& & \psi \circ 0
\end{array}
\qquad
\begin{array}{ccc}
\phi \circ \gamma & \longrightarrow & \gamma \circ (\text{id} \times \phi) \\
\downarrow \zeta \circ \gamma & & \downarrow \gamma \circ (\text{id} \times \zeta) \\
\psi \circ \gamma & \longrightarrow & \gamma \circ (\text{id} \times \psi)
\end{array}$$

commute.

Example 4.18. Let C be a E -cone, then consider the quotient stack $[C/E]$. We claim that $[C/E]$ a zero section and an \mathbb{A}^1 -action.

Indeed, the zero section $0 : X \rightarrow [C/E]$ given by $X \leftarrow E \rightarrow C$. The \mathbb{A}^1 -action of $\alpha \in \mathbb{A}^1(T)$ on $(P, f) \in [C/E](T)$ defined by $(\alpha P, \alpha f)$ where $\alpha P = P \times^{E, \alpha} E$ and $\alpha f : P \times^{E, \alpha} E \rightarrow C$ given by $[p, v] \mapsto \alpha f(p) + d(v)$ where $d : E \rightarrow C$.

Moreover, if $\phi : (E, C) \rightarrow (F, D)$ is a morphism of vector bundle cones we get an induced \mathbb{A}^1 -equivariant morphism $\tilde{\phi} : [C/E] \rightarrow [D/F]$.

Lemma 4.19. Some useful results.

- (a) A homotopy $k : \phi \rightarrow \psi$ of two morphisms of vector bundle cones $\phi, \psi : (E, C) \rightarrow (F, D)$ gives rise to an \mathbb{A}^1 -equivariant 2-isomorphism $\tilde{k} : \tilde{\phi} \rightarrow \tilde{\psi}$ of \mathbb{A}^1 -equivariant morphisms of stacks with \mathbb{A}^1 -action.
- (b) Conversely, let two morphisms of vector bundle cones $\phi, \psi : (E, C) \rightarrow (F, D)$ with an \mathbb{A}^1 -equivariant 2-isomorphism $\zeta : \tilde{\phi} \rightarrow \tilde{\psi}$ of \mathbb{A}^1 -equivariant morphisms of stacks with \mathbb{A}^1 -action. Then $\zeta = \tilde{k}$ for unique homotopy $k : \phi \rightarrow \psi$.

Proof. For (a), similar to Proposition 4.29. For (b) TBC... \square

Proposition 4.20. Let C be an E -cone and D an F -cone and let $\phi : (E, C) \rightarrow (F, D)$ be a morphism. If the diagram

$$\begin{array}{ccc}
E & \longrightarrow & C \\
\downarrow & \ulcorner & \downarrow \phi \\
F & \xrightarrow{d} & D
\end{array}$$

is cartesian and $F \times_X C \rightarrow D$ by $(\mu, \gamma) \mapsto d\mu + \phi(\gamma)$ is surjective, then $[C/E] \rightarrow [D/F]$ is an isomorphism of algebraic X -stacks with \mathbb{A}^1 -action.

Proof. For the same proof of Proposition 4.30. \square

Definition 4.21. (a) We call an algebraic stack $(\mathfrak{C}, 0, \gamma)$ over X with section and \mathbb{A}^1 -action a **cone stack**, if, étale locally on X , there exists a cone C over X and an \mathbb{A}^1 -equivariant morphism $C \rightarrow \mathfrak{C}$ that is smooth and surjective and such that $E = C \times_{\mathfrak{C}, 0} X$ is a vector bundle over X .

- (b) The morphism $C \rightarrow \mathfrak{C}$ is called a *local presentation of \mathfrak{C}* . The section $0 : X \rightarrow \mathfrak{C}$ is called the *vertex of \mathfrak{C}* .
- (c) Let \mathfrak{C} and \mathfrak{D} be cone stacks over X . A *morphism of cone stacks* $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ is an \mathbb{A}^1 -equivariant morphism of algebraic X -stacks. A *2-isomorphism of cone stacks* is just an \mathbb{A}^1 -equivariant 2-isomorphism.
- (d) A cone stack \mathfrak{C} over X is called *abelian cone stack* (resp. *vector bundle stack*), if, locally in X , one can find presentations $C \rightarrow \mathfrak{C}$, where C is an abelian cone (resp. vector bundle).

Remark 4.22. Some basic properties of cone stacks.

- (a) If $C \rightarrow \mathfrak{C}$ is a global presentation with $E = C \times_{\mathfrak{C},0} X$, then C is an E -cone with $\mathfrak{C} \cong [C/E]$ as stacks with \mathbb{A}^1 -action. This follows from Proposition 4.10 and 4.13 and Lemma 4.12.
- (b) If $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ is a morphism of cone stacks, then, étale locally on X , ϕ is \mathbb{A}^1 -equivariantly isomorphic to $[C/E] \rightarrow [D/F]$, where $E \rightarrow F$ is a morphism of vector bundles over X and $C \rightarrow D$ is a morphism from the E -cone C to the F -cone D .
- (c) A 2-isomorphism of cone stacks $\zeta : \phi \rightarrow \psi$, where $\phi, \psi : \mathfrak{C} \rightarrow \mathfrak{D}$, is étale locally over X given by a homotopy of morphisms of vector bundle cones. This follows from Lemma 4.19(b).
- (d) Let $C \rightarrow \mathfrak{C}$ and $D \rightarrow \mathfrak{D}$ be two local presentation of a cone stack \mathfrak{C} over X , then so is $C \times_{\mathfrak{C}} D \rightarrow \mathfrak{C}$.

Indeed, we only need to show that $C \times_{\mathfrak{C}} D$ is a cone. Since $C \rightarrow \mathfrak{C}$ and $D \rightarrow \mathfrak{C}$ are affine, we know that $C \times_{\mathfrak{C}} D \rightarrow \mathfrak{C} \rightarrow X$ is also affine. Then $C \times_{\mathfrak{C}} D$ is a cone by Proposition 4.13(b) and the result follows.

- (e) Every fibered product of cone stacks is a cone stack.
- (f) If \mathfrak{C} is a representable cone stack over X , then it is a cone.

Indeed, locally on X , $\mathfrak{C} \rightarrow X$ is \mathbb{A}^1 -isomorphic to a cone. In particular, as $\mathfrak{C} \rightarrow X$ is representable, it is affine. Then we assume that $C = \underline{\text{Spec}}_X \mathcal{S}$. Since there is a non-trivial \mathbb{A}^1 -action on C and has a section, we know that \mathcal{S} is a graded algebra with $\mathcal{S}^0 = \mathcal{O}_X$. To show C is a cone, we only need to show that \mathcal{S}^1 is coherent and \mathcal{S} is locally generated by \mathcal{S}^1 . These are both local property, then they hold since locally $\mathfrak{C} \rightarrow X$ is \mathbb{A}^1 -isomorphic to a cone.

- (g) If \mathfrak{C} is abelian (a vector bundle stack), then for every local presentation $C \rightarrow \mathfrak{C}$ the cone C will be abelian (a vector bundle).

Example 4.23. Note that all cones are cone stacks and all morphisms of cones are morphisms of cone stacks. For a vector bundle E on X , the

classifying stack $\mathbf{B}_X E$ is a cone stack. Every homomorphism of vector bundles $\phi: E \rightarrow F$ gives rise to a morphism of cone stacks.

Proposition 4.24. *Every cone stack is a closed subcone stack of an abelian cone stack. There exists a universal such abelian cone stack. It is called the abelian hull.*

Proof. Just glue the stacks obtained from the abelian hulls of local presentations. \square

Definition 4.25. (a) Let \mathfrak{E} be a vector bundle stack and $\mathfrak{E} \rightarrow \mathfrak{C}$ a morphism of cone stacks. We say that \mathfrak{C} is an \mathfrak{E} -cone stack, if $\mathfrak{E} \rightarrow \mathfrak{C}$ is locally isomorphic (as a morphism of cone stacks) to the morphism $[E_1/E_0] \rightarrow [C/F]$ coming from a commutative diagram

$$\begin{array}{ccc} E_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & C \end{array}$$

where C is both E_1 - and F -cone. The natural action $\mathfrak{E} \times_X \mathfrak{C} \rightarrow \mathfrak{C}$ induced by $E_1 \times C \rightarrow C$.

(b) Let $\mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D}$ be a sequence of morphisms of cone stacks where \mathfrak{C} is an \mathfrak{E} -cone stack. If

(b1) $\mathfrak{C} \rightarrow \mathfrak{D}$ is a smooth epimorphism.

(b2) The diagram

$$\begin{array}{ccc} \mathfrak{E} \times_X \mathfrak{C} & \xrightarrow{\sigma} & \mathfrak{C} \\ p \downarrow & \lrcorner & \downarrow \\ \mathfrak{C} & \longrightarrow & \mathfrak{D} \end{array}$$

is cartesian where σ is action and p is projection.

Then we call $0 \rightarrow \mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ is a **short exact sequence of cone stacks**. As before, this is equivalent to \mathfrak{C} being locally isomorphic to $\mathfrak{E} \times_X \mathfrak{D}$.

Proposition 4.26. *The sequence $0 \rightarrow \mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$ of morphisms of cone stacks is exact if and only if locally in X there exist commutative diagrams*

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & F & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & D \longrightarrow 0 \end{array}$$

where the top row is a short exact sequence of vector bundles and the bottom row is a short exact sequence of cones, such that $\mathfrak{E} \rightarrow \mathfrak{E} \rightarrow \mathfrak{D}$ is isomorphic to $[E_1/E_0] \rightarrow [C/F] \rightarrow [D/G]$.

Proof. The statement is local on X . To prove the only if part we can assume $\mathfrak{C} = \mathfrak{E} \times_X \mathfrak{D}$, and then it is trivial. To prove the if part, note that both short exact sequences are locally split. \square

4.3 A Picard Stack of Special Type

General Theory

First we will consider the case of complex of two terms.

Definition 4.27. Let X be a topos.

- (a) Let $d : E^0 \rightarrow E^1$ a homomorphism of abelian sheaves on X , which we shall consider as a complex of abelian sheaves on X . Via d , the abelian sheaf E^0 acts on E^1 and we may consider the quotient stack of this action, denoted

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := [E^1/E^0]$$

which is a Picard stack over X .

- (b) Now if $d : F^0 \rightarrow F^1$ is another homomorphism of abelian sheaves on X and $\phi : E^\bullet \rightarrow F^\bullet$ is a homomorphism of complexes, then we get an induced morphism of Picard stacks

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \rightarrow \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

given by $(P, f) \mapsto (P \times^{E^0, \phi^0} F^0, \phi^1(f))$ where $\phi^1(f)$ is the map

$$\phi^1(f) : P \times^{E^0, \phi^0} F^0 \rightarrow F^1, \quad [p, \nu] \mapsto \phi^1(f(p) + d(\nu)).$$

- (c) Now, if $\psi : E^\bullet \rightarrow F^\bullet$ is another homomorphism of complexes, then the homotopy $k : \phi \rightarrow \psi$ is a homomorphism of abelian sheaves $k : E^1 \rightarrow F^0$, such that $kd = \psi^0 - \phi^0$ and $dk = \psi^1 - \phi^1$.

Remark 4.28. Note that roughly speaking, a Picard stack is a stack together with an ‘addition’ operation, that is both associative and commutative. For the precise definition of Picard stack see Sect. 1.4 of Exposé XVIII in [AGV73].

Here the quotient stack is similar as before: the groupoid $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$ is the category of pairs (P, f) , where P is an E^0 -torsor over U and $f : P \rightarrow E^1|_U$ is an E^0 -equivariant morphism of sheaves on U .

Proposition 4.29. *As in the considtion of definition, if we have a homotopy $k : \phi \rightarrow \psi$, then this can induce isomorphism $\theta : \mathcal{H}^1/\mathcal{H}^0(\phi) \rightarrow \mathcal{H}^1/\mathcal{H}^0(\psi)$ of morphisms of Picard stacks from $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ to $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)$.*

Proof. Pick object $U \in \text{ob}(X)$ and $(P, f) \in \mathcal{H}^1/\mathcal{H}^0(E^\bullet)(U)$, then $\theta(U)(P, f) : \mathcal{H}^1/\mathcal{H}^0(\phi)(U)(P, f) \rightarrow \mathcal{H}^1/\mathcal{H}^0(\psi)(U)(P, f)$ in $\mathcal{H}^1/\mathcal{H}^0(F^\bullet)(U)$ is the isomorphism of $F^0|_U$ -torsors

$$\theta(U)(P, f) : P \times^{E^0, \phi^0} F^0 \rightarrow P \times^{E^0, \psi^0} F^0$$

given by $[p, \nu] \mapsto [p, kf(p) + \nu]$ such that the diagram of $F^0|_U$ -sheaves

$$\begin{array}{ccc} P \times^{E^0, \phi^0} F^0 & & \\ \theta(U)(P, f) \downarrow & \searrow \phi^1(f) & \\ P \times^{E^0, \psi^0} F^0 & \xrightarrow{\psi^1(f)} & F^1 \end{array}$$

commutes. □

Proposition 4.30. *Let $\phi : E^\bullet \rightarrow F^\bullet$ is a homomorphism of complexes of abelian sheaves in the topos X . If ϕ induces isomorphisms on kernels and cokernels (i.e. if ϕ is a quasi-isomorphism), then*

$$\mathcal{H}^1/\mathcal{H}^0(\phi) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \rightarrow \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$$

is an isomorphism of Picard stacks over X .

Proof. First let us treat the case that ϕ is a homotopy equivalence, that is, there is a homotopy inverse of ϕ such that compositions are homotopic to id_{E^\bullet} and id_{F^\bullet} , respectively. By Proposition 4.29 well done.

Next we assume ϕ is an epimorphism. In this case $E^1 \rightarrow [F^1/F^0]$ is an epimorphism, so we just need to prove the diagram

$$\begin{array}{ccc} E^0 \times E^1 & \xrightarrow{d+\text{id}} & E^1 \\ \downarrow p & & \downarrow \\ E^1 & \longrightarrow & [F^1/F^0] \end{array}$$

is cartesian as in this case this will be a cocartesian diagram! This quickly reduces to proving that

$$\begin{array}{ccc} E^1 \times E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ E^1 \times F^0 & \longrightarrow & F^1 \end{array}$$

is cartesian, which, in turn, is equivalent to

$$\begin{array}{ccc} E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ F^0 & \longrightarrow & F^1 \end{array}$$

being cartesian, which is a consequence of the assumptions.

Finally in general case, let us note that a general ϕ factors as a homotopy equivalence followed by an epimorphism, then well done. Indeed, consider $E^\bullet \oplus F^0$, which is homotopy equivalent to E^\bullet . Define a homomorphism $\psi : E^\bullet \oplus F^0 \rightarrow F^\bullet$ by $\psi^0(\nu, \mu) = \phi^0(\nu) + \mu$ and $\psi^1(\xi, \mu) = \phi^1(\xi) + \mu$. Then ψ is surjective and $\phi = \psi \circ i$ where $i : E^\bullet \hookrightarrow E^\bullet \oplus F^0$ is the canonical embedding. \square

Now we consider the general case.

Definition 4.31. *Let X be a topos and E^\bullet be a complex of abelian sheaves on X , then we define*

$$\mathcal{H}^1/\mathcal{H}^0(E^\bullet) := \mathcal{H}^1/\mathcal{H}^0(\tau^{[0,1]}E^\bullet).$$

Lemma 4.32. *Let X be a ringed topos with structure sheaf of rings \mathcal{O}_X .*

- (a) *We can define $\mathcal{H}^1/\mathcal{H}^0(E^\bullet)$ and homomorphisms can defined over $\mathbf{D}(\mathcal{O}_X)$.*
- (b) *Let $\phi, \psi : E^\bullet \rightarrow F^\bullet$ be two morphisms in $\mathbf{D}(\mathcal{O}_X)$. Then, if for some choice of $\mathcal{H}^1/\mathcal{H}^0(\phi)$ and $\mathcal{H}^1/\mathcal{H}^0(\psi)$ we have $\mathcal{H}^1/\mathcal{H}^0(\phi) \cong \mathcal{H}^1/\mathcal{H}^0(\psi)$ as morphisms of Picard stacks, then $\phi = \psi$.*
- (c) *Consider the zero morphism $0(E, F) : \mathcal{H}^1/\mathcal{H}^0(E^\bullet) \rightarrow \mathcal{H}^1/\mathcal{H}^0(F^\bullet)$. Then $\text{Aut}(0(E, F)) = \text{Hom}_{\mathbf{D}(\mathcal{O}_X)}^{-1}(E^\bullet, F^\bullet)$.*

Proof. For (b)(c), see Sect. 1.4 of Exposé XVIII in [AGV73]. For (a), the quasi-isomorphism induce an isomorphism of Picard stacks, see Proposition 4.30. \square

Example 4.33. *Consider E^\bullet and we focus on $d^0 : E^0 \rightarrow E^1$.*

- (1) *If d^0 is a monomorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \text{coker}(d^0)$ is a sheaf.*
- (2) *If d^0 is an epimorphism, then $\mathcal{H}^1/\mathcal{H}^0(E^\bullet) = \mathbf{B}_X \ker(d^0)$ is a gerbe.*

Application

Come back to our case, let X be a DM-stack over a field k , then consider the big fppf topos X_{fppf} and small étale topos $X_{\text{ét}}$. Then we have the morphism of topoi

$$v : X_{\text{fppf}} \rightarrow X_{\text{ét}}.$$

- (a) Then we can get $\mathbf{L}v^* : \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}}) \rightarrow \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}})$. We may let $M_{\text{fppf}}^\bullet := \mathbf{L}v^* M^\bullet$ for any $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}})$.
- (b) We also have $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_{X_{\text{fppf}}}) : \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}}) \rightarrow \mathbf{D}^+(\mathcal{O}_{X_{\text{fppf}}})$. We may let $M^{\bullet, \vee} := \mathbf{R}\mathcal{H}om(M^\bullet, \mathcal{O}_{X_{\text{fppf}}})$ for any $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}})$.

Remark 4.34. We will consider the stack $\mathcal{H}^1/\mathcal{H}^0(M_{\text{fppf}}^{\bullet, \vee})$ for $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}})$. Note that in this case

$$\mathcal{H}^1/\mathcal{H}^0(M_{\text{fppf}}^{\bullet, \vee}) = \mathcal{H}^1/\mathcal{H}^0((\tau^{\geq -1} M_{\text{fppf}}^\bullet)^\vee).$$

Remark 4.35. For a complex E^\bullet , we define $Z^i(E^\bullet) = \ker(E^i \rightarrow E^{i+1})$ and $C^i(E^\bullet) = \text{coker}(E^{i-1} \rightarrow E^i)$.

Definition 4.36. We call an object $L^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ satisfies Condition (*) if

- (1) $H^i(L^\bullet) = 0$ for all $i > 0$.
- (2) $H^i(L^\bullet)$ is coherent for all $i = 0, -1$.

Here are some fundamental results:

Proposition 4.37. Let X be a DM-stack.

- (a) Let $L^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ satisfy Condition (*). Then the X -stack $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee})$ is an abelian cone stack over X . Moreover, if L^\bullet is of perfect amplitude contained in $[-1, 0]$, then $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee})$ is a vector bundle stack.
- (b) If $\phi : E^\bullet \rightarrow L^\bullet$ is a homomorphism in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$, where E^\bullet and L^\bullet satisfy (*), then we get an induced morphism of algebraic stacks

$$\phi^\vee : \mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee}) \rightarrow \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee}).$$

Then ϕ^\vee is a morphism of abelian cone stacks. Moreover, $H^0(\phi)$ is surjective if and only if ϕ^\vee is representable.

- (c) The morphism ϕ^\vee is a closed immersion if and only if $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. Moreover, ϕ^\vee is an isomorphism if and only if $H^0(\phi)$ and $H^{-1}(\phi)$ are isomorphisms.
- (d) Let $E^\bullet \rightarrow F^\bullet \rightarrow G^\bullet \rightarrow E^\bullet[1]$ be a distinguished triangle in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$, where E^\bullet and F^\bullet satisfy (*) and G^\bullet is of perfect amplitude contained in $[-1, 0]$. Then the induced sequence

$$\mathcal{H}^1/\mathcal{H}^0(G_{\text{fppf}}^{\bullet, \vee}) \rightarrow \mathcal{H}^1/\mathcal{H}^0(F_{\text{fppf}}^{\bullet, \vee}) \rightarrow \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee})$$

is a short exact sequence of abelian cone stacks over X .

Proof. For (a), as the claim is étale local, we may assume L^\bullet consists of free \mathcal{O}_X -modules with $L^i = 0$ for $i > 0$ and L^0, L^{-1} have finite rank. Then $L_{\text{fppf}}^\bullet = v^* L^\bullet$ and $L_{\text{fppf}}^{\bullet, \vee}$ is taking dual of L_{fppf}^\bullet component-wise. Hence we have

$$\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee}) = [Z^1(L^{\vee, \bullet})/L^{\vee, 0}]$$

which is an abelian cone stack given by $L^{\vee, 0} \rightarrow Z^1(L^{\vee, \bullet}) = C(C^{-1}L^\bullet)$.

When L^\bullet is of perfect amplitude contained in $[-1, 0]$, then $\mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee})$ is a vector bundle stack since étale locally as above we have $Z^1(L^{\vee, \bullet}) = L^{\vee, 1}$.

For (b), the fact that ϕ^\vee is a morphism of abelian cone stacks is immediate from the definition. The second question is étale local in X , so we may assume that E^\bullet and L^\bullet are complexes of free \mathcal{O}_X -modules and that $E^i = L^i = 0$, for $i > 0$, and that L^0, L^{-1}, E^0 and E^{-1} are of finite rank. Consider the commutative diagram

$$\begin{array}{ccc} C^{-1}(E^\bullet) & & \\ & \searrow & \\ & F & \xrightarrow{\quad} E^0 \\ & \downarrow & \downarrow \\ & B^{-1}(L^\bullet) & \longrightarrow L^0 \end{array}$$

of coherent sheaves with fiber product F . This force $0 \rightarrow F \rightarrow E^0 \oplus C^{-1}(L^\bullet) \rightarrow L^0$ exact. Then its easy to see that $H^0(\phi)$ is surjective if and only if $0 \rightarrow F \rightarrow E^0 \oplus C^{-1}(L^\bullet) \rightarrow L^0 \rightarrow 0$ exact. Hence taking duality we get $0 \rightarrow L^{\vee, 0} \rightarrow E^{\vee, 0} \times_X Z^1(L^{\vee, \bullet}) \rightarrow C(F) \rightarrow 0$ exact. Then by Proposition 4.20 we get

$$[Z^1(L^{\vee, \bullet})/L^{\vee, 0}] \cong [C(F)/E^{\vee, 0}].$$

This force the following cartesians

$$\begin{array}{ccc} C(F) & \xrightarrow{\quad} & Z^1(E^{\vee, \bullet}) \\ \downarrow & \searrow & \downarrow \\ \mathcal{H}^1/\mathcal{H}^0(L_{\text{fppf}}^{\bullet, \vee}) & \xrightarrow{\phi^\vee} & \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee}) \end{array}$$

hence ϕ^\vee is representable.

For the converse, note that $\phi^\vee : [Z^1(L^{\vee, \bullet})/L^{\vee, 0}] \rightarrow [Z^1(E^{\vee, \bullet})/E^{\vee, 0}]$ representable implies that $[Z^1(L^{\vee, \bullet})/L^{\vee, 0}] = [W/E^{\vee, 0}]$. Then we have the

commutative diagram:

$$\begin{array}{ccc}
Z^1(L^{\vee, \bullet}) \times_X L^{\vee, 0} & \longrightarrow & Z^1(L^{\vee, \bullet}) \\
\downarrow & & \downarrow \\
Z^1(L^{\vee, \bullet}) \times_X E^{\vee, 0} & \longrightarrow & W \\
\downarrow & & \downarrow \\
Z^1(L^{\vee, \bullet}) & \longrightarrow & [W/E^{\vee, 0}]
\end{array}$$

such that the the whole diagram and the lower diagram are cartesian, then this force the upper square is cartesian. So we get cartesian

$$\begin{array}{ccccc}
L^{\vee, 0} & \longrightarrow & Z^1(L^{\vee, \bullet}) \times_X L^{\vee, 0} & \longrightarrow & Z^1(L^{\vee, \bullet}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
E^{\vee, 0} & \longrightarrow & Z^1(L^{\vee, \bullet}) \times_X E^{\vee, 0} & \longrightarrow & W
\end{array}$$

Hence $L^{\vee, 0} \cong E^{\vee, 0} \times_W Z^1(L^{\vee, \bullet}) \rightarrow E^{\vee, 0} \times_X Z^1(L^{\vee, \bullet})$ is a closed immersion. This implies that $E^0 \oplus C^{-1}(L^{\bullet}) \rightarrow L^0$ is an epimorphism.

For (c), following the previous argument in (b), ϕ^{\vee} is a closed immersion if and only if $C(F) \rightarrow Z^1(E^{\vee, \bullet})$ is. This is equivalent to $C^{-1}(E^{\bullet}) \rightarrow F$ being surjective. A simple diagram chase shows that this is equivalent to $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective. The ‘moreover’ follows similarly.

For (d), the question is étale local, so assume that E^i and F^i are 0 for $i > 0$ and vector bundles for $i = 0, -1$, and that $G^i = E^{i+1} \oplus F^i$, that is, $G^{\bullet} = \text{cone}(E^{\bullet} \rightarrow F^{\bullet})$. If we consider the small enough étale locally, we may let $G^i = 0$ for $i \leq -2$ as G^{\bullet} is of perfect amplitude contained in $[-1, 0]$. Now we have to prove that

$$0 \rightarrow [Z^1(G^{\vee, \bullet})/G^{\vee, 0}] \rightarrow [Z^1(F^{\vee, \bullet})/F^{\vee, 0}] \rightarrow [Z^1(E^{\vee, \bullet})/E^{\vee, 0}] \rightarrow 0$$

is a short exact sequence of cone stacks. Now by directly check, we have the exact sequence of sheaves

$$0 \rightarrow C^{-1}(E^{\bullet}) \rightarrow C^{-1}(F^{\bullet}) \oplus E^0 \rightarrow C^{-1}(G^{\bullet}) \rightarrow 0.$$

Hence consider

$$\begin{array}{ccccccc}
0 & \longrightarrow & C^{-1}(E^{\bullet}) & \longrightarrow & C^{-1}(F^{\bullet}) \oplus E^0 & \longrightarrow & C^{-1}(G^{\bullet}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E^0 & \longrightarrow & F^0 \oplus E^0 & \longrightarrow & G^0 = F^0 \longrightarrow 0
\end{array}$$

with exact rows. Finally by Proposition 4.26 we get the result. \square

4.4 About Normal Cones

Here we will consider some useful results about normal cones of DM-stacks.

Consider the commutative diagram of algebraic stacks

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ \downarrow u & & \downarrow v \\ X & \xrightarrow{i} & Y \end{array}$$

with where i and j are local immersions. Then there is a natural morphism of cones over X'

$$\alpha : C_{X'/Y'} \rightarrow C_{X/Y}.$$

If the diagram is cartesian, then α is a closed immersion. If, moreover, v is flat, then α is an isomorphism.

Proposition 4.38. *Consider a commutative diagram of DM-stacks*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ & \searrow i & \downarrow f \\ & & Y \end{array}$$

where i and i' are local immersions and f is smooth. Then the sequence of morphisms of cones over X

$$(i')^* T_{Y'/Y} \rightarrow C_{X/Y'} \rightarrow C_{X/Y}$$

is exact.

Proof. The question is local, so we can assume them are schemes and that i' and i are immersions. This is then Example 4.2.6 in [Ful98]. \square

Lemma 4.39. *Let $f : U \rightarrow M$ be a local immersion of affine k -schemes of finite type, where M is smooth over k . Then the normal cone $C_{U/M} \hookrightarrow N_{U/M}$ is invariant under the action of $f^* T_M$ on $N_{U/M}$. In other words, $C_{U/M}$ is an $f^* T_M$ -cone.*

Proof. Consider projections $p_i : M \times M \rightarrow M$, we consider two diagrams:

$$\begin{array}{ccc} U & \xrightarrow{\Delta f} & M \times M \\ & \searrow f & \downarrow p_i \\ & & M \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{f} & M \\ & \searrow \Delta f & \downarrow \Delta \\ & & M \times M \end{array}$$

The first one give us exact sequence of abelian cones on U :

$$0 \rightarrow f^*T_M N_{U/M \times M} \xrightarrow{j_i} N_{U/M} \xrightarrow{p_{i,*}} 0$$

and the second one give us a homomorphism of abelian cones $s : N_{U/M} \rightarrow N_{U/M \times M}$ which is a section of both $p_{i,*}$.

Using $(j_1, p_{1,*})$ we make the identification $N_{U/M \times M} = f^*T_M \times N_{U/M}$ and $p_{2,*}$ is identified with the action of f^*T_M on $N_{U/M}$. Since the same functorialities of normal sheaves used so far are enjoyed by normal cones, we get that under the identification above the subcone $C_{U/M \times M} \subset N_{U/M \times M}$ corresponds to $f^*T_M \times C_{U/M}$ and the action $p_{2,*} : f^*T_M \times N_{U/M} \rightarrow N_{U/M}$ restricts to $p_{2,*} : f^*T_M \times C_{U/M} \rightarrow C_{U/M}$. \square

4.5 Intrinsic Normal Cone

Now let X be a Deligne-Mumford stack, locally of finite type over k . Now we will construct the intrinsic normal cone and intrinsic normal sheaf of X and their basic properties.

Definition 4.40. *We denote the abelian cone stack*

$$\mathfrak{N}_X := \mathcal{H}^1 / \mathcal{H}^0((\mathbb{L}_{X, \text{fppf}}^\bullet)^\vee)$$

and call it the *intrinsic normal sheaf of X* where $\mathbb{L}_X^\bullet \in \mathbf{D}^{\leq 0}(\mathcal{O}_{X_{\text{ét}}})$ is the cotangent complex which satisfies the condition $(*)$.

Definition 4.41. (a) *A local embedding of X is a pair (U, M) with a diagram $X \xleftarrow{i} U \xrightarrow{f} M$ where*

- (a1) U is an affine k -scheme of finite type;
- (a2) $i : U \rightarrow X$ is an étale morphism;
- (a3) M is a smooth affine k -scheme of finite type;
- (a4) $f : U \rightarrow M$ is a local immersion.

(b) *A morphism of local embeddings $\phi : (U', M') \rightarrow (U, M)$ is a pair of morphisms $\phi_U : U' \rightarrow U$ and $\phi_M : M' \rightarrow M$ such that*

- (b1) ϕ_U is an étale X -morphism;
- (b2) ϕ_M is smooth morphism such that

$$\begin{array}{ccc} U' & \xrightarrow{f'} & M' \\ \downarrow \phi_U & & \downarrow \phi_M \\ U & \xrightarrow{f} & M \end{array}$$

commutes.

Remark 4.42. If (U', M') and (U, M) are local embeddings of X , then $(U' \times_X U, M' \times M)$ is naturally a local embedding of X which we call the product of local embeddings, even though it may not be the direct product in the category of local embeddings of X .

Now we consider the local presentation of intrinsic normal sheaf \mathfrak{N}_X . Indeed, consider a local embedding $X \xleftarrow{i} U \xrightarrow{f} M$ of X , then we have a natural homomorphism

$$\phi : \mathbb{L}_X^\bullet|_U \rightarrow [\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1]$$

in $\mathbf{D}^{\leq 0}(\mathcal{O}_{U_{\text{ét}}})$ where \mathcal{I} be the ideal correspond to f and $[\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1] \in \mathbf{D}^{[-1,0]}(\mathcal{O}_{U_{\text{ét}}})$. Moreover, by Theorem 3.1(c) we know that ϕ induces an isomorphism on H^{-1} and H^0 . By Proposition 4.30 we get an induced isomorphism of cone stacks

$$\phi^\vee : [N_{U/M}/f^*T_M] \cong i^*\mathfrak{N}_X.$$

In other words, $N_{U/M}$ is a local presentation of the abelian cone stack \mathfrak{N}_X .

Theorem 4.43. *There exists a unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ such that for every local embedding (U, M) of X we have $\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]$, that is, the diagram*

$$\begin{array}{ccc} C_{U/M} & \hookrightarrow & N_{U/M} \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{C}_X|_U & \hookrightarrow & \mathfrak{N}_X|_U \end{array}$$

Proof. If $\chi : (U', M') \rightarrow (U, M)$ is a morphism of local embeddings, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{L}_X^\bullet|_{U'} & & \\ \downarrow \phi|_{U'} & \searrow \phi' & \\ [\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1]|_{U'} & \xrightarrow{\tilde{\chi}} & [\mathcal{I}'/(\mathcal{I}')^2 \rightarrow (f')^*\Omega_{M'}^1] \end{array}$$

in $\mathbf{D}^{\leq 0}(\mathcal{O}_{U'_{\text{ét}}})$ because of the naturality of ϕ and thus induce the commutative diagram

$$\begin{array}{ccc} [N_{U'/M'}/(f')^*T_{M'}] & \xrightarrow{\tilde{\chi}^\vee} & [N_{U/M}/f^*T_M]|_{U'} \\ (\phi')^\vee, \cong \downarrow & \swarrow \phi^\vee|_{U'}, \cong & \\ \mathfrak{N}_X|_{U'} & & \end{array}$$

in $\mathbf{D}^{\leq 0}(\mathcal{O}_{U'_{\text{ét}}})$. In particular, $\tilde{\chi}^\vee$ is an isomorphism of cone stacks over U' .

Now by Lemma 4.39 χ induce a morphism from the $(f')^*T_{M'}$ -cone $C_{U'/M'}$ to the $f^*T_M|_{U'}$ -cone $C_{U/M}|_{U'}$. By Proposition 4.26 the pair $(C_{U/M} \hookrightarrow N_{U/M})|_{U'}$ is the quotient of $(C_{U'/M'} \hookrightarrow N_{U'/M'})$ by the action of $(f')^*T_{M'/M}$ since the kernel of $(f')^*T_{M'} \rightarrow f^*T_M|_{U'}$ is $(f')^*T_{M'/M}$. This implies that the isomorphism above

$$\tilde{\chi}^\vee : [N_{U'/M'}/(f')^*T_{M'}] \cong [N_{U/M}/f^*T_M]|_{U'}$$

identifies the closed subcone stack $[C_{U'/M'}/(f')^*T_{M'}]$ with the closed subcone stack $[C_{U/M}/f^*T_M]|_{U'}$. This give us the unique closed subcone stack $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$ with the properties above. \square

Definition 4.44. *This unique closed subcone stack \mathfrak{C}_X is called the intrinsic normal cone of X .*

Theorem 4.45. *The intrinsic normal cone \mathfrak{C}_X is of pure dimension zero which abelian hull is the intrinsic normal sheaf \mathfrak{N}_X .*

Proof. The second claim follows because the normal sheaf is the abelian hull of the normal cone, for any local embedding.

To prove the claim about the dimension of \mathfrak{C}_X , consider a local embedding (U, M) of X , giving rise to the local presentation $C_{U/M}$ of \mathfrak{C}_X . Assume that M is of pure dimension. We then have a cartesian diagram of U -stacks

$$\begin{array}{ccc} C_{U/M} \times f^*T_M & \longrightarrow & C_{U/M} \\ \downarrow & \lrcorner & \downarrow \\ C_{U/M} & \longrightarrow & [C_{U/M}/f^*T_M] \end{array}$$

Thus $C_{U/M} \rightarrow [C_{U/M}/f^*T_M]$ is a smooth epimorphism of relative dimension $\dim M$. So since $C_{U/M}$ is of pure dimension $\dim M$ (see the comments on the Definition 2.1), the stack $[C_{U/M}/f^*T_M]$ has pure dimension $\dim M - \dim M = 0$. Well done. \square

Finally, we discuss some basic properties of them.

Proposition 4.46. *Let X be a DM-stack.*

(a) *The following are equivalent.*

(a1) *X is a local complete intersection.*

(a2) *\mathfrak{C}_X is a vector bundle stack.*

(a3) *$\mathfrak{C}_X = \mathfrak{N}_X$.*

If X is smooth, we have $\mathfrak{C}_X = \mathfrak{N}_X = \mathbf{B}_X(T_X)$.

(b) We have $\mathfrak{N}_{X \times Y} = \mathfrak{N}_X \times \mathfrak{N}_Y$ and $\mathfrak{C}_{X \times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$.

(c) Let $f : X \rightarrow Y$ be a local complete intersection morphism. Then we have a natural short exact sequence of cone stacks

$$\mathfrak{N}_{X/Y} := \mathcal{H}^1/\mathcal{H}^0(\mathbb{T}_{X/Y}^\bullet) \rightarrow \mathfrak{C}_X \rightarrow f^*\mathfrak{C}_Y.$$

Proof. (a) is trivial. (b) follows from the fact that if C is an E -cone and D is an F -cone, then $C \times D$ is an $E \times F$ -cone and there is a canonical isomorphism of cone stacks $[C/E] \times [D/F] \rightarrow [C \times D/E \times F]$.

For (c), by Theorem 3.1(c)(e) we have an exact triangle

$$\mathbf{L}f^*\mathbb{L}_Y \rightarrow \mathbb{L}_X \rightarrow \mathbb{L}_{X/Y} \rightarrow \mathbf{L}f^*\mathbb{L}_Y[1]$$

in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ and $\mathbb{L}_{X/Y}$ is of perfect amplitude contained in $[-1, 0]$. By Proposition 4.37(d) we have a short exact sequence of abelian cone stacks

$$\mathfrak{N}_{X/Y} \rightarrow \mathfrak{N}_X \rightarrow f^*\mathfrak{N}_Y.$$

So the claim is local in X and we may assume that we have a diagram

$$\begin{array}{ccccc} X & \xhookrightarrow{i} & M'' & \longrightarrow & M' \\ & \searrow f & \downarrow & \lrcorner & \downarrow \\ & & Y & \longrightarrow & M \end{array}$$

where the square is cartesian, the vertical maps are smooth, the horizontal maps are local immersions, i is regular and M is smooth. Then we have a morphism of short exact sequences of cones on X

$$\begin{array}{ccccc} i^*T_{M''/Y} & \longrightarrow & T_{M'}|_X & \longrightarrow & T_M|_X \\ \downarrow & & \downarrow & & \downarrow \\ N_{X/M''} & \longrightarrow & C_{X/M'} & \longrightarrow & C_{Y/M}|_X \end{array}$$

Hence by Proposition 4.26 we get the result. \square

4.6 About Obstruction Theories

Intrinsic Normal Sheaf as Obstruction

Let X be a DM-stack with intrinsic normal sheaf \mathfrak{N}_X . Let $T \hookrightarrow \bar{T}$ be a closed immersion with ideal \mathcal{J} such that $\mathcal{J}^2 = 0$. If we have $g : T \rightarrow X$, may we have the extension $\bar{g} : \bar{T} \rightarrow X$ of g ? What is the obstruction of this deformation?

First, by Theorem 3.1(e) we have a composition of canonical morphisms

$$\mathbf{L}g^*\mathbb{L}_X^\bullet \rightarrow \mathbb{L}_T^\bullet \rightarrow \mathbb{L}_{T/\overline{T}}^\bullet.$$

Since $\tau^{\geq -1}\mathbb{L}_{T/\overline{T}}^\bullet = \mathcal{J}[1]$, this homomorphism may be considered as an element $\omega(g) \in \text{Ext}^1(g^*\mathbb{L}_X^\bullet, \mathcal{J})$. Then the basic deformation theory find that an extension $\overline{g} : \overline{T} \rightarrow X$ of g exists if and only if $\omega(g) = 0$ and if $\omega(g) = 0$ the extensions form a torsor under $\text{Ext}^0(g^*\mathbb{L}_X^\bullet, \mathcal{J}) = \text{Hom}(g^*\Omega_X, \mathcal{J})$.

Here we will use the intrinsic normal sheaf \mathfrak{N}_X to interpret this. Recall the morphism as above

$$\mathbf{L}g^*\mathbb{L}_X^\bullet \rightarrow \mathbb{L}_T^\bullet \rightarrow \mathbb{L}_{T/\overline{T}}^\bullet.$$

This induce a morphism

$$\mathbf{ob}(g) : C(\mathcal{J}) = \mathcal{H}^1/\mathcal{H}^0(\mathbb{L}_{T/\overline{T}, \text{fppf}}^{\bullet, \vee}) \rightarrow \mathcal{H}^1/\mathcal{H}^0(\mathbf{L}g^*\mathbb{L}_{X, \text{fppf}}^{\bullet, \vee}) = g^*\mathfrak{N}_X$$

since $\tau^{\geq -1}\mathbb{L}_{T/\overline{T}}^\bullet = \mathcal{J}[1]$ by Theorem 3.1(c). Consider another morphism

$$\mathbf{0}(g) : C(\mathcal{J}) \rightarrow T \xrightarrow{0} g^*\mathfrak{N}_X.$$

- Consider a sheaf $\mathcal{J} \text{som}(\mathbf{ob}(g), \mathbf{0}(g))$ of 2-isomorphisms of cone stacks from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$, restricted to $T_{\text{ét}}$.
- Denote the sheaf of extensions $\overline{T} \rightarrow X$ of g by $\mathcal{E}xt(g, T)$ on $T_{\text{ét}}$.

Proposition 4.47. *There is a canonical isomorphism*

$$\mathcal{E}xt(g, T) \xrightarrow{\cong} \mathcal{J} \text{som}(\mathbf{ob}(g), \mathbf{0}(g))$$

of sheaves on $T_{\text{ét}}$. Hence in particular, extensions of g to \overline{T} exist if and only if $\mathbf{ob}(g)$ is \mathbb{A}^1 -equivariantly isomorphic to $\mathbf{0}(g)$.

Proof. Locally we can have an embedding $i : X \hookrightarrow M$ where M is smooth of ideal \mathcal{J} . Then by the formally-smoothness of M we have the lifting:

$$\begin{array}{ccc} X & \xhookrightarrow{i} & M \\ \uparrow g & & \uparrow h \\ T & \hookrightarrow & \overline{T} \longrightarrow \text{Spec}(k) \end{array}$$

and such extensions is a $\text{Hom}(g^*i^*\Omega_M, \mathcal{J})$ -torsor. Now, any such h induce $h^\# : g^*\mathcal{J}/\mathcal{J}^2 \rightarrow \mathcal{J}$. By the local description before the Theorem 4.43, $\mathbf{ob}(g)$ induced by

$$h^\# : \mathbf{L}g^*[\mathcal{J}/\mathcal{J}^2 \rightarrow i^*\Omega_M] \rightarrow [\mathcal{J} \rightarrow 0].$$

Now the torsor structure induce the following homotopy

$$\begin{array}{ccccccc}
0 & \longrightarrow & g^* \mathcal{I} / \mathcal{I}^2 & \longrightarrow & g^* i^* \Omega_M & \longrightarrow & 0 \\
& & \downarrow h^\sharp, (\tilde{h})^\sharp & \swarrow \text{dashed} & \downarrow h^\sharp, (\tilde{h})^\sharp & & \\
0 & \longrightarrow & \mathcal{I} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

of extensions $h^\sharp, (\tilde{h})^\sharp$.

Now let $\bar{g} : \bar{T} \rightarrow X$ be an extension of g . Then easy to see that $(i \circ g)^\sharp = 0$, so that we get a homotopy from any local h^\sharp as above to 0, or in other words a local \mathbb{A}^1 -equivariant isomorphism from $\mathbf{ob}(g)$ to $\mathbf{0}(g)$ by Proposition 4.29. Since these local isomorphisms glue, we get the required map

$$\mathcal{E}xt(g, T) \rightarrow \mathcal{I} \text{som}(\mathbf{ob}(g), \mathbf{0}(g)).$$

Now we consider the inverse. Let $\theta : \mathbf{ob}(g) \rightarrow \mathbf{0}(g)$ be a 2-isomorphism of cone stacks. By Lemma 4.19(a), θ defines for every local h as above an extension of h^\sharp to $\bar{h}^\sharp : g^* i^* \Omega_M \rightarrow \mathcal{I}$. So we can get $h' : \bar{T} \rightarrow M$ such that $(h')^\sharp = 0$ by the changing via homotopy \bar{h}^\sharp . So h' factor through X and we get $h' : \bar{T} \rightarrow X$. Gluing them we get the inverse. \square

Proposition 4.48. *There is a canonical isomorphism*

$$\mathcal{A}ut(\mathbf{0}(g)) \xrightarrow{\cong} \mathcal{H}om(g^* \Omega_X, \mathcal{I})$$

of sheaves on $T_{\text{ét}}$.

Proof. Again similar as above, Lemma 4.19(a) shows that the automorphisms of $\mathbf{0}(g)$ are (locally) the homomorphisms from $g^* i^* \Omega_M$ to \mathcal{I} vanishing on $g^* \mathcal{I} / \mathcal{I}^2$. The exact sequence

$$\mathcal{I} / \mathcal{I}^2 \rightarrow i^* \Omega_M \rightarrow \Omega_X \rightarrow 0$$

give the result. \square

Remark 4.49. *This shows that the sheaf $\mathcal{E}xt(g, T) \cong \mathcal{I} \text{som}(\mathbf{ob}(g), \mathbf{0}(g))$ is a formal $\mathcal{H}om(g^* \Omega_X, \mathcal{I})$ -torsor. So if $\mathbf{ob}(g) \cong \mathbf{0}(g)$, the set $\text{Hom}(\mathbf{ob}(g), \mathbf{0}(g))$ is a torsor under the group $\text{Hom}(g^* \Omega_X, \mathcal{I})$.*

Obstruction Theories

Here we consider more general setting.

Definition 4.50. *Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ satisfies condition $(*)$. Then a homomorphism $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ is called an **obstruction theory** for X if $H^0(\phi)$ is an isomorphism and $H^{-1}(\phi)$ is surjective.*

Now we will give some equivalent conditions of obstruction theory which is connected to the obstruction of extensions as above.

Situation 1. Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ satisfies condition $(*)$ with a homomorphism $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$. Let $\mathfrak{E} := \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee})$ with induced morphism of cone stacks $\phi^\vee : \mathfrak{N}_X \rightarrow \mathfrak{E}$.

Let $T \hookrightarrow \bar{T}$ be a closed immersion with ideal \mathcal{J} such that $\mathcal{J}^2 = 0$ with a morphism $g : T \rightarrow X$. Then we can consider the obstruction class $\omega(g) \in \text{Ext}^1(g^*\mathbb{L}_X^\bullet, \mathcal{J})$. Define $\phi^*\omega(g) \in \text{Ext}^1(g^*E^\bullet, \mathcal{J})$ be the pullback and $\phi^\vee(\text{ob}(g))$ be the composition

$$C(\mathcal{J}) \xrightarrow{\text{ob}(g)} g^*\mathfrak{N}_X \xrightarrow{g^*\phi^\vee} g^*\mathfrak{E}$$

of cone stacks over T . Moreover $\mathbf{0} : C(\mathcal{J}) \rightarrow T \rightarrow g^*\mathfrak{E}$ is the vertex.

Theorem 4.51. Let X be a DM-stack and $E^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ satisfies condition $(*)$ with a homomorphism $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$. Let $\mathfrak{E} := \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee})$ with induced morphism of cone stacks $\phi^\vee : \mathfrak{N}_X \rightarrow \mathfrak{E}$. Then the following are equivalent.

- (a) $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ is an obstruction theory for X .
- (b) The induced $\phi^\vee : \mathfrak{N}_X \rightarrow \mathfrak{E}$ is a closed immersion of cone stacks over X . In this case we call $\phi^\vee(\mathfrak{C}_X) \subset \mathfrak{E}$ the **obstruction cone** of the obstruction theory for X where $\mathfrak{C}_X \subset \mathfrak{N}_X$ is the intrinsic normal cone.
- (c) Consider any choice of Situation 1, then the obstruction $\phi^*\omega(g) \in \text{Ext}^1(g^*E^\bullet, \mathcal{J})$ vanishes if and only if an extension \bar{g} of g to \bar{T} exists; and if $\phi^*\omega(g) = 0$, then the extensions form a torsor under $\text{Ext}^0(g^*E^\bullet, \mathcal{J}) = \text{Hom}(g^*H^0E^\bullet, \mathcal{J})$.
- (d) Consider any choice of Situation 1, then we have isomorphism

$$\mathcal{E}xt(g, T) \xrightarrow{\cong} \mathcal{J} \text{som}(\phi^\vee \text{ob}(g), \mathbf{0})$$

of sheaves on $T_{\text{ét}}$.

Proof. Note that (1) \Leftrightarrow (2) follows from Proposition 4.37(c). By the similar proof as Proposition 4.47 we know that (2) \Rightarrow (4). By Lemma 4.32(b) we know that (4) \Rightarrow (3). So we just need to consider (3) \Rightarrow (1).

We show $H^0(\phi)$ is an isomorphism. Let $X = \text{Spec } R$ as this is local. For any R -algebra A and R -module M , let $T := \text{Spec } A$ and $\bar{T} := \text{Spec}(A \oplus M)$ for the nilpotent extension. Easy to see that there is extension $\bar{g} : \bar{T} \rightarrow X$. So we have a bijection

$$\text{Hom}(H^0(\mathbb{L}_X^\bullet) \otimes A, M) \rightarrow \text{Hom}(H^0(E^\bullet) \otimes A, M).$$

This implies easily that $H^0(\phi)$ is an isomorphism.

We show $H^{-1}(\phi)$ is surjective. As this is étale local and only depends on $\tau^{\geq -1}E^\bullet$, we may assume X is an affine scheme, $i : X \rightarrow W$ a closed embedding in a smooth affine scheme W , and let \mathcal{I} be the ideal of X in W . Also $E^\bullet = [E^{-1} \rightarrow f^*\Omega_W]$ as a complex of coherent sheaves (see the proof of Proposition 4.37(b)). As in this case $\mathbb{L}_X^\bullet = [\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_W]$, we claim that $E^{-1} \rightarrow \mathcal{I}/\mathcal{I}^2$ is surjective.

Indeed, let $M := \text{Im}(E^{-1} \rightarrow \mathcal{I}/\mathcal{I}^2)$. Let $T = X$ and $M' \subset \mathcal{I}$ be the primage of M and let $\bar{T} \subset W$ defined by M' . So we can extend $g = \text{id}_X$ to the inclusion $\bar{g} : \bar{T} \rightarrow W$. Let $\pi : \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{I}/M'$ be the natural projection. By assumption π factors via E^0 if and only if g extends to a map $\bar{T} \rightarrow X$, if and only if $\pi \circ \phi^{-1}$ factors via E^0 . As $\pi \circ \phi^{-1}$ is the zero map, it certainly factors. Therefore π also factors via E^0 . Consider now the commutative diagram with exact rows

$$\begin{array}{ccccccc} E^{-1} & \longrightarrow & E^0 & \longrightarrow & H^0(E^\bullet) & \longrightarrow & 0 \\ \downarrow \phi^{-1} & & \downarrow = & & \downarrow = & & \\ \mathcal{I}/\mathcal{I}^2 & \longrightarrow & E^0 & \longrightarrow & H^0(E^\bullet) & \longrightarrow & 0 \end{array}$$

By an easy diagram chasing argument, the fact that π factors via E^0 together with $\pi \circ \phi^{-1} = 0$ implies $\pi = 0$, hence well done. \square

Remark 4.52. See more things about the obstruction of small extensions we refer the final part of Section 4 in [BF97].

4.7 Vistoli's Rational Equivalence

Before starting the theory of virtual class, we need some results of Vistoli. We will follow something in [Vis89].

Definition 4.53. Let X be a stack.

- (a) The group $Z_k(X)$ of cycles of dimension k is generated by all integral closed substacks of dimension k . And $Z_*(X) := \bigoplus_k Z_k(X)$.
- (b) The group of rational equivalences on X is

$$W_k(X) := \bigoplus_G K(G)^*, \quad W_*(X) := \bigoplus_k W_k(X)$$

where the direct sum is taken over all integral substacks G of X of dimension $k + 1$.

(c) If X is a scheme, there is a canonical homomorphism

$$\partial_X : W_*(X) \rightarrow Z_*(X).$$

This is commute with proper pushforward and flat pullback.

Remark 4.54. Note that when X be a DM-stack, we can restricting Z_* and W_* to the étale site of X , we get two sheaves \mathcal{Z}_* and \mathcal{W}_* on X . As Z_* and W_* commute with proper pushforward and flat pullback, $\partial : \mathcal{W}_* \rightarrow \mathcal{Z}_*$ is a morphism of sheaves, so we get a homomorphism $\partial_X : W_*(X) \rightarrow Z_*(X)$.

Recall that we consider again the cartesian diagram of algebraic stacks

$$\begin{array}{ccc} X' & \xhookrightarrow{i} & Y' \\ u \downarrow & \lrcorner & \downarrow v \\ X & \xhookrightarrow{j} & Y \end{array}$$

with i and j are local immersions and v is a regular local immersion and Y is smooth of constant dimension. Then this induce the cartesians

$$\begin{array}{ccccc} N_{Y'/Y} \times_Y C_{X/Y} & \longrightarrow & u^* C_{X/Y} & \longrightarrow & C_{X/Y} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ j^* N_{Y'/Y} & \longrightarrow & X' & \xrightarrow{u} & X \\ \downarrow & \lrcorner & \downarrow j & \lrcorner & \downarrow i \\ N_{Y'/Y} & \xrightarrow{\rho} & Y' & \xhookrightarrow{v} & Y \end{array}$$

Theorem 4.55 (Vistoli). Consider the above situation, if Y is a scheme, then there is a canonical rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ such that

$$\partial\beta(Y', X) = [C_{u^* C_{X/Y}/C_{X/Y}}] - [\rho^* C_{X'/Y'}].$$

Proof. See Lemma 4.6 in [Vis89]. □

Corollary 4.56. In this case we have $v^! [C_{X/Y}] = [C_{X'/Y'}] \in \text{CH}_*(u^* C_{X/Y})$.

Proof. Let $0 : u^* C \rightarrow N \times_Y C$ be the zero section, then by definition of refined Gysin pullback

$$0^* [C_{u^* C_{X/Y}/C_{X/Y}}] = v^! [C] \in \text{CH}_*(u^* C_{X/Y}).$$

Moreover

$$0^* [\rho^* C_{X'/Y'}] = 0^* \rho^! [C_{X'/Y'}] = C_{X'/Y'}.$$

By Theorem 4.55 we get $v^! [C] = [C_{X'/Y'}] \in \text{CH}_*(u^* C_{X/Y})$. □

But now we need to consider the Vistoli rational equivalence at the level of stacks. So we need some base-change result about this:

Proposition 4.57. *Vistoli's rational equivalence commutes with any smooth base change $\phi : Y_1 \rightarrow Y$.*

Proof. If ϕ is étale, this is Lemma 4.6(ii) in [Vis89]. Vistoli's proof is based on the fact that the following commute with étale base change: blowing up a scheme along a closed subscheme; normalization; order of a Cartier divisor along an irreducible Weil divisor on a reduced, equidimensional scheme. But all these operations do in fact commute with smooth base change. Hence well done. \square

Corollary 4.58. *We have Vistoli's rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ for any algebraic stacks. Moreover, if Y is a DM-stack, then $v^![C_{X/Y}] = [C_{X'/Y'}] \in \text{CH}_*(u^*C_{X/Y})$ holds.*

Proof. Follows directly from the previous Proposition. \square

Now we again consider the general case. We assume $i : X \rightarrow Y$ can factor as $X \xrightarrow{\tilde{i}} \tilde{Y} \xrightarrow{\pi} Y$ where \tilde{i} is another local immersion and π is of relative Deligne-Mumford type (i.e. has unramified diagonal) and is smooth of constant fiber dimension.

Then the previous diagram can be fused into a large diagram of cartesianians:

$$\begin{array}{ccccc}
N_{Y'/Y} \times_Y C_{X/\tilde{Y}} & \longrightarrow & u^*C_{X/\tilde{Y}} & \longrightarrow & C_{X/\tilde{Y}} \\
\downarrow & & \downarrow & & \downarrow \alpha \\
N_{Y'/Y} \times_Y C_{X/Y} & \longrightarrow & u^*C_{X/Y} & \longrightarrow & C_{X/Y} \\
\downarrow & & \downarrow & & \downarrow \\
j^*N_{Y'/Y} & \longrightarrow & X' & \xrightarrow{u} & X \\
\downarrow & & \downarrow \tilde{j} & & \downarrow \tilde{i} \\
\pi^*N_{Y'/Y} & \xrightarrow{\tilde{\rho}} & \tilde{Y}' & \xrightarrow{\tilde{v}} & \tilde{Y} \\
\downarrow & & \downarrow & & \downarrow \pi \\
N_{Y'/Y} & \xrightarrow{\rho} & Y' & \xrightarrow{v} & Y
\end{array}$$

Hence by Proposition 4.38 $\alpha : C_{X/\tilde{Y}} \rightarrow C_{X/Y}$ is a $T_{\tilde{Y}/Y} \times_{\tilde{Y}} C_{X/Y}$ -bundle.

Proposition 4.59. *We have $\alpha^*(\beta(Y', X)) = \beta(\tilde{Y}', X) \in W_*(N_{Y'/Y} \times_Y C_{X/\tilde{Y}})$.*

Proof. In the compatibilities of β proved in [Vis89] we reduce to the case that $\tilde{Y} = \mathbb{A}_Y^n$. Then one checks that Vistoli's construction in the case directly. \square

Proposition 4.60. *Back to the original diagram, assume that Y is of Deligne-Mumford type. Vistoli's rational equivalence $\beta(Y', X) \in W_*(N_{Y'/Y} \times_Y C_{X/Y})$ is invariant under the natural action of $j^* N_{Y'/Y} \times_Y T_Y$ on $N_{Y'/Y} \times_Y C_{X/Y}$.*

Proof. The vector bundle $i^* T_Y$ acts on the X -cone $C_{X/Y}$ by Lemma 4.39. Pulling back from X to $j^* N_{Y'/Y}$ gives the natural action of $j^* N_{Y'/Y} \times_Y T_Y$ on $N_{Y'/Y} \times_Y C_{X/Y}$. Using the construction of the proof of Lemma 4.39 the claim follows from Proposition 4.59 applied to $\tilde{Y} = Y \times Y$ and $\tilde{i} = \Delta \circ i : X \rightarrow Y \times Y$. \square

4.8 Virtual Fundamental Classes

Definition 4.61. *Let X be a DM-stack and an obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ for X . We say ϕ is a **perfect obstruction theory** if E^\bullet is of perfect amplitude contained in $[-1, 0]$.*

We will construct the virtual fundamental class associated to a perfect obstruction theory. But before that, we will discuss a toy version which is a local model of the general theory.

Local Model, an Intuition

Consider Y be a smooth variety of dimension d with a vector bundle $E = \text{Spec}_Y \text{Sym } \mathcal{E}^\vee$ over it of rank r . Let $s : Y \rightarrow E$ be a section and $0 : Y \rightarrow E$ be the zero section, then we consider the zero locus of s as:

$$\begin{array}{ccc} X = Z(s) & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow s \\ Y & \xrightarrow{0} & E \end{array}$$

Note that $X = Z(s)$ defined by the ideal $\mathcal{I} := \text{Im}(s^\vee : \mathcal{E}^\vee \rightarrow \mathcal{O}_Y)$.

Now the most nice condition is that when s is a regular section, then $\dim Z(s) = d - r$. But in general this might not be true! So we define $d^{\text{vir}}(X) = d - r$ to be the **virtual dimension** of X .

Moreover, we define the **virtual fundamental class** of X is

$$[X]^{\text{vir}} := c_{\text{loc}}(E, s) := 0^!([Y]) = 0^*(C_{Z(s)/Y}) \in \text{CH}_{d^{\text{vir}}(X)}(X)$$

which is the localized (top) Chern class of E with respect to s .

In this case the perfect obstruction theory is

$$\begin{array}{ccc} E^\bullet : & \mathcal{E}^\vee|_X & \longrightarrow \Omega_Y|_X \\ & \downarrow s^\vee|_X & \downarrow \text{id} \\ \mathbb{L}_X^\bullet : & \mathcal{I}/\mathcal{I}^2 & \xrightarrow{d_X} \Omega_Y|_X \end{array}$$

Remark 4.62. Any perfect obstruction theory on a DM-stack is locally of this form. See the Remark 1.7 in [Tod21].

Consider a perfect obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ for X . Locally, we embed X into a smooth scheme M with defining ideal \mathcal{I} .

$$\begin{array}{ccccccc} & & \text{cone}(\psi) = \mathcal{E}^\vee[1] & & & & \\ & & \uparrow & & & & \\ E^\bullet & \xrightarrow{\phi} & \tau^{\geq -1}\mathbb{L}_X^\bullet & \longrightarrow & \text{cone}(\phi) = P[2] & \longrightarrow & E^\bullet[1] \\ & \nearrow \psi & & & & & \\ & \Omega_Y|_X & & & & & \\ & \uparrow & & & & & \\ & \mathcal{E}^\vee & & & & & \end{array}$$

Here \mathcal{E} locally free. Moreover we can represent the morphism ϕ as a morphism of complexes from $[\mathcal{E}^\vee \rightarrow \Omega_Y|_X]$ to $[\mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_Y|_X]$. Then extend \mathcal{E} to a vector bundle \mathcal{F} on Y and use the surjection $\mathcal{E}^\vee \rightarrow \mathcal{I}/\mathcal{I}^2$ and projectivity of \mathcal{E}^\vee to lift it to $\mathcal{F}^\vee \rightarrow \mathcal{I}$. This defines a section of \mathcal{F} cutting out $X \subset Y$.

Remark 4.63. This also shows why we consider intrinsic normal cone and the obstruction theory. We need a global version of the previous case.

General Construction

In the global version of a DM-stack X , fix a perfect obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$. We want to intersect the intrinsic normal cone \mathfrak{C}_X with the vertex of $\mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee})$ to get the virtual fundamental class $[X]_\phi^{\text{vir}}$. Here we begin our story.

Remark 4.64. Actually by the general theory of the intersection theory of general algebraic stacks had been developed in [Kre99] after our [BF97], so we can define $[X]_\phi^{\text{vir}} = 0^*[\mathfrak{C}_X]$ as the local case. But there we will follow the original method in [BF97] instead of this by assume we have global resolution.

Definition 4.65. Let a DM-stack X and a perfect obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$. We define the virtual dimension of X with respect to the ϕ is

$$d_\phi^{\text{vir}}(X) := \text{rank}(E^\bullet) = \dim E^0 - \dim E^{-1}$$

if locally E^\bullet is written as a complex of vector bundles $[E^{-1} \rightarrow E^0]$.

Definition 4.66. Let $F^\bullet = [F^{-1} \rightarrow F^0]$ be a homomorphism of vector bundles on X considered as a complex of \mathcal{O}_X -modules concentrated in degrees -1 and 0 . An isomorphism $F^\bullet \rightarrow E^\bullet$ in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ is called a *global resolution* of E^\bullet .

Note also that the intersection theory in [Vis89] holds when we consider the rational coefficient Chow group with there is a scheme F and a proper surjective morphism $F \rightarrow X$. In our case, if $X \rightarrow \text{Spec } k$ is separated, then this condition holds, see Theorem 4.12 in [DM69].

So in this whole section we will assume X to be a separated DM-stack locally of finite type over a field k .

Definition 4.67. In this condition, consider a perfect obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ and admits a global resolution $F^\bullet \rightarrow E^\bullet$ with $F^\bullet = [F^{-1} \rightarrow F^0]$. Then $\mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee}) = [F^{-1, \vee}/F^{0, \vee}]$. Consider the cartesian

$$\begin{array}{ccc} C(F^\bullet) & \xhookrightarrow{\quad} & F^{-1, \vee} \\ \downarrow & \searrow & \downarrow \\ \mathfrak{C}_X & \xhookrightarrow{\quad} & \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet, \vee}) = [F^{-1, \vee}/F^{0, \vee}] \end{array}$$

Then define virtual fundamental class of X associated to the perfect obstruction theory ϕ is:

$$[X]_\phi^{\text{vir}} := 0^!(C(F^\bullet)) \in \text{CH}_{d_\phi^{\text{vir}}(X)}(X)_{\mathbb{Q}}$$

where $0 : X \rightarrow F^{-1, \vee}$ be the zero section with refined Gysin pullback $0^!$.

Proposition 4.68. The virtual fundamental class $[X]_\phi^{\text{vir}}$ of X associated to the perfect obstruction theory ϕ is independent of the global resolution F^\bullet used to construct it.

Proof. Give another global resolution H^\bullet . WLOG assume that $H^\bullet \rightarrow E^\bullet$ and $F^\bullet \rightarrow E^\bullet$ are given by morphisms of complexes. Then we get an induced homomorphism $H^0 \oplus F^0 \rightarrow E^0$. Consider cartesian diagram

$$\begin{array}{ccc} K^{-1} & \longrightarrow & H^0 \oplus F^0 \\ \downarrow & \lrcorner & \downarrow \\ E^{-1} & \longrightarrow & E^0 \end{array}$$

Let $K^0 = H^0 \oplus F^0$ and hence we get another global resolution K^\bullet . So we just need to consider F^\bullet and K^\bullet .

Now $K^{-1,\vee} \rightarrow E^{-1,\vee}$ is an epimorphism. Consider cartesians

$$\begin{array}{ccccccc} X & \xrightarrow{0} & C(K^\bullet) & \longrightarrow & C(F^\bullet) & \longrightarrow & \mathfrak{C}_X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{0} & K^{-1,\vee} & \xrightarrow{\alpha} & F^{-1,\vee} & \longrightarrow & \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee}) \end{array}$$

Now α is smooth surjective, the virtual fundamental class using F^\bullet is equal to

$$(\alpha \circ 0)^! [C(F^\bullet)] = 0^! \alpha^! [C(F^\bullet)] = 0^! [C(K^\bullet)]$$

which is also the virtual fundamental class using K^\bullet . \square

Remark 4.69 (Virtual Structure Sheaf). *Let a perfect obstruction theory $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$. Let $\mathfrak{C} := \mathcal{H}^1/\mathcal{H}^0(E_{\text{fppf}}^{\bullet,\vee})$ be the abelian cone stack.*

Consider the presentation $\mathfrak{C} = [E^{-1,\vee}/E^{0,\vee}]$, then $\mathfrak{C}_X \hookrightarrow \mathfrak{C}$ induce a subcone $C \subset E^{-1,\vee}$. We then set

$$[\mathcal{O}_{X,\phi}^{\text{vir}}] := [\mathcal{O}_C \otimes_{\mathcal{O}_{E^{-1,\vee}}}^{\mathbb{L}} \mathcal{O}_X] = \sum_{i \geq 0} (-1)^i [\mathcal{T}or_i^{\mathcal{O}_{E^{-1,\vee}}}(\mathcal{O}_C, \mathcal{O}_X)] \in K_0(X)$$

which is called the virtual structure sheaf of X relative to the perfect obstruction theory ϕ .

Now assume that there is a Chern character $\mathfrak{c} : K_0(X) \rightarrow \text{CH}_(X)_{\mathbb{Q}}$ and E^\bullet admits a global resolution, then we can define the virtual fundamental class as:*

$$[X]_{\phi}^{\text{vir}} = \text{td}(E^\bullet) \cap \mathfrak{c}([\mathcal{O}_{X,\phi}^{\text{vir}}]) \in \text{CH}_{d_{\phi}^{\text{vir}}(X)}(X)_{\mathbb{Q}}.$$

These two constructions agree when they are both possible.

Some Examples

Example 4.70 (Trivial Obstructions). *Let \mathbb{L}_X^\bullet is of perfect amplitude contained in $[-1, 0]$ (such as X is a complete intersection) then \mathbb{L}_X^\bullet itself is a perfect obstruction theory. Any embedding of X into a smooth DM-stack gives rise to a global resolution of \mathbb{L}_X^\bullet . The virtual fundamental class $[X]^{\text{vir}}$ thus obtained is equal to $[X]$, the ‘usual’ fundamental class.*

Example 4.71 (No Obstructions). *If E^\bullet is perfect, $H^0(E^\bullet)$ is locally free and $H^1(E^\bullet) = 0$, then X is smooth and $d_{E^\bullet}^{\text{vir}}(X) = \dim X$ and the virtual fundamental class $[X]_{E^\bullet}^{\text{vir}} = [X]$, the usual fundamental class.*

Example 4.72 (Locally Free Obstructions). *If X is smooth, E^\bullet is perfect and $H^1(E^{\bullet,\vee})$ is locally free, then $d_{E^\bullet}^{\text{vir}}(X) = \text{rank}(H^1(E^{\bullet,\vee}))$ and the virtual fundamental class*

$$[X]_{E^\bullet}^{\text{vir}} = c_{\text{rank}(H^1(E^{\bullet,\vee}))}(H^1(E^{\bullet,\vee})) \cap [X].$$

Example 4.73 (Products). *Consider two perfect obstruction theories $\phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ and $\psi : F^\bullet \rightarrow \mathbb{L}_Y^\bullet$. Then $\mathbb{L}_{X \times Y}^\bullet = \mathbb{L}_X^\bullet \boxplus \mathbb{L}_Y^\bullet$, then we have also a perfect obstruction theory $\phi \boxplus \psi : E^\bullet \boxplus F^\bullet \rightarrow \mathbb{L}_{X \times Y}^\bullet$. If both E^\bullet, F^\bullet have global resolutions, then so is $E^\bullet \boxplus F^\bullet$. By Proposition 4.46(b) we have*

$$[X \times Y]_{\phi \boxplus \psi}^{\text{vir}} = [X]_\phi^{\text{vir}} \times [Y]_\psi^{\text{vir}} \in \text{CH}_{d_\phi^{\text{vir}}(X) + d_\psi^{\text{vir}}(Y)}(X \times Y).$$

Pullback of Virtual Fundamental Class

We will show the pullback formula via local complete intersection morphism for now as in [BF97]. For the general case, we refer [Man12].

Consider a cartesian diagram of DM-stacks

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow g & \lrcorner & \downarrow f \\ Y' & \xrightarrow{v} & Y \end{array}$$

where v is a local complete intersection morphism. Let $\Phi : E^\bullet \rightarrow \mathbb{L}_X^\bullet$ and $\Psi : F^\bullet \rightarrow \mathbb{L}_{X'}^\bullet$ be two perfect obstruction theories.

Definition 4.74. *A compatibility datum (relative to v) for E^\bullet and F^\bullet is a triple (ϕ, ψ, χ) of morphisms in $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ giving rise to a morphism of distinguished triangles*

$$\begin{array}{ccccccc} u^*E^\bullet & \xrightarrow{\phi} & F^\bullet & \xrightarrow{\psi} & g^*\mathbb{L}_{Y'/Y} & \xrightarrow{\chi} & u^*E^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u^*\mathbb{L}_X^\bullet & \longrightarrow & \mathbb{L}_{X'}^\bullet & \longrightarrow & \mathbb{L}_{X'/X}^\bullet & \longrightarrow & u^*\mathbb{L}_X^\bullet[1] \end{array}$$

Given a compatibility datum, we call E^\bullet and F^\bullet compatible (over v).

Assume that E^\bullet and F^\bullet are endowed with such a compatibility datum. Then we get (Proposition 4.37(d)) a short exact sequence of vector bundle stacks

$$g^*\mathfrak{N}_{Y'/Y} = g^*\mathcal{H}^1/\mathcal{H}^0(\mathbb{T}_{Y'/Y, \text{fppf}}^\bullet) \rightarrow \mathfrak{F} = \mathcal{H}^1/\mathcal{H}^0(\mathbb{E}_{\text{fppf}}^{\bullet,\vee}) \xrightarrow{\phi} u^*\mathfrak{E} = u^*\mathcal{H}^1/\mathcal{H}^0(\mathbb{E}_{\text{fppf}}^{\bullet,\vee}).$$

Lemma 4.75. *If Y and Y' are smooth and v a regular local immersion, then $\mathfrak{N}_{Y'/Y} = N_{Y'/Y}$ is the normal bundle and we denote $N := g^*N_{Y'/Y}$. Then there is a (canonical) rational equivalence $\beta(Y', X) \in W_*(N \times \mathfrak{F})$ such that*

$$\partial\beta(Y', X) = [\phi^*C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}] - [N \times \mathfrak{C}_{X'}].$$

Proof. Let $X \rightarrow M$ be a local embedding, where M is smooth. We get an induced cartesian diagrams

$$\begin{array}{ccccc} N \times_X C_{X/Y \times M} & \longrightarrow & u^*C_{X/Y \times M} & \longrightarrow & C_{X/Y \times M} \\ \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & X' & \xrightarrow{u} & X \\ \downarrow & & j \downarrow & & i \downarrow \\ N_{Y'/Y} \times M & \xrightarrow{\rho} & Y' \times M & \xrightarrow{v} & Y \times M \end{array}$$

Now we have Vistoli's rational equivalence $\beta(Y' \times M, X) \in W_*(N \times_X C_{X/Y \times M})$ such that

$$\partial\beta(Y' \times M, X) = [C_{u^*C_{X/Y \times M}/C_{X/Y \times M}}] - [N \times C_{X'/Y' \times M}].$$

By Proposition 4.60, $\beta(Y' \times M, X)$ is invariant under the action of $N \times u^*i^*T_{Y \times M}$ on $N \times_X C_{X/Y \times M}$. Hence in particular, $\beta(Y' \times M, X)$ is invariant under the subsheaf $N \times j^*T_{Y' \times M}$ and thus descends to $N \times [u^*C_{X/Y \times M}/j^*T_{Y' \times M}] = N \times \mathfrak{F} \times_{\mathfrak{C}} \mathfrak{C}_X$ which is a closed subcone stack of \mathfrak{F} . So pushing forward via this closed immersion, we get a rational equivalence on $N \times \mathfrak{F}$ which we denote it by $\beta(Y', X)$. Now we have

$$\partial\beta(Y', X) = [\phi^*C_{u^*\mathfrak{C}_X/\mathfrak{C}_X}] - [N \times \mathfrak{C}_{X'}]$$

as we need. By Proposition 4.59, we can glue them up and well done. \square

Theorem 4.76 (Pullback). *Let E^\bullet and F^\bullet be compatible perfect obstruction theories, as above. If E^\bullet and F^\bullet have global resolutions then*

$$v^![X]_{E^\bullet}^{\text{vir}} = [X']_{F^\bullet}^{\text{vir}}$$

holds in the following cases:

- (1) v is smooth.
- (2) Y' and Y are smooth.

Proof. First note that one may choose global resolutions $[E_0 \rightarrow E_1]$ of $E^{\bullet, \vee}$ and $[F_0 \rightarrow F_1]$ of $F^{\bullet, \vee}$ together with a pair of epimorphisms $\phi_0 : F_0 \rightarrow$

u^*E_0 and $\phi_1 : F_1 \rightarrow u^*E_1$ with kernels G_i such that the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G_0 & \longrightarrow & F_0 & \xrightarrow{\phi_0} & u^*E_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G_1 & \longrightarrow & F_1 & \xrightarrow{\phi_1} & u^*E_1 \longrightarrow 0 \end{array}$$

The induced short exact sequence of vector bundle stacks

$$[G_1/G_0] \rightarrow [F_1/F_0] \rightarrow [u^*E_1/u^*E_0]$$

is isomorphic to $g^*\mathfrak{N}_{Y'/Y} \rightarrow \mathfrak{F} \rightarrow u^*\mathfrak{E}$. Let $C_1 := \mathfrak{C}_X \times_{\mathfrak{E}} E_1$ and $D_1 := \mathfrak{C}_{X'} \times_{\mathfrak{F}} F_1$ and by definition we have $[X]_{E^\bullet}^{\text{vir}} = 0_{E_1}^! [C_1]$ and $[X']_{F^\bullet}^{\text{vir}} = 0_{F_1}^! [D_1]$.

For case (1), let v is smooth. Then by Proposition 4.46(c) we get the cartesian of the left following diagram:

$$\begin{array}{ccc} \mathfrak{C}_{X'} & \longrightarrow & u^*\mathfrak{C}_X \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{F} & \longrightarrow & u^*\mathfrak{E} \end{array} \quad \begin{array}{ccc} D_1 & \longrightarrow & u^*C_1 \\ \downarrow & \lrcorner & \downarrow \\ F_1 & \longrightarrow & u^*E_1 \end{array}$$

which imply the right one is also cartesian. This shows $0_{u^*E_1}^! [u^*C_1] = 0_{F_1}^! [D_1]$. Hence

$$v^! [X]_{E^\bullet}^{\text{vir}} = v^! 0_{E_1}^! [C_1] = 0_{u^*E_1}^! [u^*C_1] = 0_{F_1}^! [D_1] = [X']_{F^\bullet}^{\text{vir}}.$$

Well done.

Now we consider case (2) where Y' and Y are smooth. First we claim this is true for the case where v is a regular local immersion. Indeed, in this case we may choose F_1 as the fibered product

$$\begin{array}{ccc} F_1 & \longrightarrow & u^*E_1 \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{F} & \longrightarrow & u^*\mathfrak{E} \end{array}$$

Now lifting the rational equivalence in Lemma 4.75 to $N \times F_1$ we get $[N \times D_1] = \phi^* [C_{u^*C_1/C_1}]$. Hence we have

$$\begin{aligned} [X']_{F^\bullet}^{\text{vir}} &= 0_{F_1}^! [D_1] = 0_{N \times F_1}^! [N \times D_1] = 0_{N \times F_1}^! \phi^* [C_{u^*C_1/C_1}] \\ &= 0_{N \times u^*E_1}^! [C_{u^*C_1/C_1}] = 0_{u^*E_1}^! v^! [C_1] = v^! 0_{E_1}^! [C_1] = v^! [X]_{E^\bullet}^{\text{vir}}. \end{aligned}$$

Well done. Now we back to the general case.

In the general case factor v as $v : Y' \xrightarrow{\Gamma_Y} Y' \times Y \xrightarrow{p} Y$. Since Y' is smooth it has a canonical obstruction theory $\Omega_{Y'}$ and an obstruction theory on $Y' \times Y$ is $\Omega_{Y'} \boxplus E^\bullet$. Then combine the previous two cases and the fact that these obstruction theories are trivially compactible, well done. \square

4.9 Examples

Basic Case

Fibres of Morphism Between Smooth Stacks

Moduli Stacks of Projective Varieties

Space of Morphisms

5 Atiyah-Bott Localization

We will follow [AB84].

6 Localization of Virtual Fundamental Class

We will follow [GP99].

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