Algebraic Cycles and Hodge Theory

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July 26, 2023

Contents

1 Introduction

The reader of course need to be familiar with the book [3] including the basic theory and schemes, cohomology, curves and surfaces. We will also use the intersection theory frequently such as the main contents of [2] or [1] and the reader should familiar with these. Finally we will omit the most basic theory of complex Hodge theory, such as the first seven chapters in [5].

We will focus on the final part of the book [6]. There are three topics of Hodge theory in this book but we just discuss the final part of them.

2 Some Background of Mixed Hodge Theory

2.1 Basic Definition and Properties

Definition 2.1. *A rational (real) mixed Hodge structure of weight n is given by a $\mathbb{Q}$-vector space ($\mathbb{R}$-vector space) H equipped with an increasing filtration W_iH called the weight filtration , and a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$, called the Hodge filtration $F^kH_{\mathbb{C}}$. Such that the induced Hodge filtration on each $\mathrm{Gr}_i^W H$ make $\mathrm{Gr}_i^W H$ to be a Hodge structure of weight $n+i$.*

These filtrations are required to be bounded. Recall that a morphism $\alpha : (U, F) \to (V, G)$ is said to be strict if $\mathrm{Im}\alpha \cap G^pV = \alpha(F^pU)$. It's easy to show that the morphism of rational pure Hodge structures are strict for Hodge filtration (even in type (r, r), see [5] Lemma 7.23).

This is an analogue theory of Hodge decomposition of pure Hodge structures:

Lemma 2.2. *Let (H, W, F) be a mixed Hodge structure. Then there exists a decomposition*

$$H_{\mathbb{C}} = \bigoplus_{p,q} H^{p,q}$$

with $H^{p,q} \subset F^pH_{\mathbb{C}} \cap W_{p+q-n}H_{\mathbb{C}}$, such that via the projection $W_{p+q-n}H_{\mathbb{C}} \to \mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}$, the space $H^{p,q}$ can be identified with

$$H^{p,q}(\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}) := F^p\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}} \cap \overline{F^q\mathrm{Gr}^W_{p+q-n}H_{\mathbb{C}}}.$$

More generally, we have

$$W_iH_{\mathbb{C}} = \bigoplus_{p+q \leq n+i} H^{p,q}, F^iH_{\mathbb{C}} = \bigoplus_{p \geq i} H^{p,q}.$$

This decomposition is preserved by the morphisms of mixed Hodge structures.

Proof. This is pure linear algebra, we omit it and refer [6] Lemma 4.21. □

Remark 2.3. *Unlike the pure case, the decomposition above may satisfies* $H^{p,q} \neq \overline{H^{p,q}}$*, although this does become true after projection to* $\mathrm{Gr}^W_{p+q} H_{\mathbb{C}}$*.*

Theorem 2.4 (P. Deligne, 1971)**.** *The morphisms*

$$\alpha : (H, W, F) \to (H', W', F')$$

of (rational or real) mixed Hodge structures are strict for the filtrations W *and* F*.*

Proof. We will only show the statement for W since the statement for H is similar.

Pick $l' \in \alpha(H_{\mathbb{C}}) \cap W_i H'$ and we write $l' = \alpha(l)$ with $l = \sum_{p,q} l^{p,q}$ by Lemma 2.2. As $l' \in W_i' H_{\mathbb{C}}'$, then $\alpha(l^{p,q}) = 0$ for $p + q > n + i$ by Lemma 2.2 again. Hence $l' \in \alpha(W_i H_{\mathbb{C}})$ and well done. □

2.2 A Classical Example of Mixed Hodge Structure

We consider a smooth complex variety U with a compactification X such that $X \backslash U = D$, a effective normal crossing divisor.

Definition 2.5. *Define a subsheaf* $\Omega^k_X(\log D) \subset \Omega^k_X(*D)$ *such that* $\alpha \in \Gamma(V, \Omega^k_X(\log D))$ *if* α *is a meromorphic differential form on* V*, holomorphic on* $V \backslash D$ *and admits a pole of order at most 1 along (each component of)* D*, and the same holds for* $d\alpha$*. Hence* $d = \partial$ *in it and we call the complex* $(\Omega^*_X(\log D), \partial)$ *the* logarithmic de Rham complex *.*

Lemma 2.6. *Let* $z_1, ..., z_n$ *be local coordinates on an open set* $V \subset X$*, in which* $D \cap V$ *is defined by the equation* $z_1 \cdots z_r = 0$*. Then* $\Omega^k_X(\log D)|_V$ *is a sheaf of free* $\mathscr{O}|_U$*-modules with basis*

$$\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_m}$$

where $i_s \leq r$*,* $j_s > r$ *and* $l + m = k$*. In particular,* $\Omega^k_X(\log D)$ *is locally free.*

Proof. Almost trivial, see [5] Lemma 8.16. □

Proposition 2.7. *Let inclusion* $j : U \hookrightarrow X$*, then we have a canonical inclusion* $\Omega^k_X(\log D) \subset j_* \Omega^k_U \subset j_* \mathscr{A}^k_U$ *which give us a morphism of complex*

$$\Omega^*_X(\log D) \to j_* \mathscr{A}^*_U.$$

Then this is a quasi-isomorphism. In particular we have

$$H^k(U, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_X(\log D)).$$

Proof. This is not hard to see and we refer [5] Proposition 8.18. From this we have $\mathbb{H}^k(X, \Omega_X^*(\log D)) \cong \mathbb{H}^k(X, j_*\mathscr{A}_U^*)$. As $\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_U^\infty$-modules which is a resolution of $\mathbb{C}_U$, then $j_*\mathscr{A}_U^*$ is a sheaf of $\mathscr{C}_X^\infty$-modules, so it is acyclic and

$$\mathbb{H}^k(X, j_*\mathscr{A}_U^*) \cong H^k\Gamma(X, j_*\mathscr{A}_U^*) = H^k\Gamma(U, \mathscr{A}_U^*) = H^k(U, \mathbb{C}).$$

Hence we get the result. □

For now we will give $H^k(U, \mathbb{Q})$ (or $H^k(U, \mathbb{R})$) a mixed Hodge structure. First we will give two filtrations over $\Omega_X^*(\log D)$.

We define the Hodge filtration over $\Omega_X^*(\log D)$ to be

$$F^p\Omega_X^*(\log D) = \Omega_X^{\geq p}(\log D).$$

For weight filtration, we define $W_l\Omega_X^*(\log D)$ to be

$$W_l\Omega_X^*(\log D) = \begin{cases} \bigwedge^l \Omega_X^1(\log D) \wedge \Omega_X^{*-l}, & 0 \leq l \leq r, \\ 0, & l > r. \end{cases}$$

(We often let $W^k := W_{-k}$)

Now for simplicity, we let the divisor D is simply normal crossing with $D = \bigcup_i D_i$ where each $D_i \subset X$ is a smooth hypersurface, and the intersection of any l hypersurfaces $D_{i_1}, ..., D_{i_l}$ is transverse. We equip I with a total order. We let

$$D^{(k)} := \coprod_{K \subset I, |K|=k} D_K = \coprod_{K \subset I, |K|=k} \bigcap_{i \in K} D_i$$

with inclusions $j_k : D^{(k)} \to X$ and $j_M : D_M \to X$.

Proposition 2.8. *There exists a natural isomorphism*

$$W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

Proof. This morphism defined by Poincaré residue map . Give a local coordinates in $V \subset X$ we define $\mathrm{Res}^V : \Gamma(V, W_k\Omega_X^*(\log D)) \to \Gamma(V, j_{k,*}\Omega_{D^{(k)}}^{*-k})$ as

$$\begin{aligned} \alpha &= \sum_{K \subset \{1,\dots,r\} \subset I, |K| \leq k} \alpha_{K,L} dz_L \wedge \frac{dz_K}{z_K} \\ &\mapsto (\mathrm{Res}^V \alpha)_M = \left((2\pi\sqrt{-1})^k \sum_L \alpha_{M,L} dz_L|_{D_M \cap V} \right)_M . \end{aligned}$$

Note that this annihilates the sections of $W_{k-1}\Omega_X^*(\log D)$ and change coordinates only change the elements in $W_{k-1}\Omega_X^*(\log D)$, Hence we get a well-defined residue map:

$$\alpha : W_k\Omega_X^*(\log D)/W_{k-1}\Omega_X^*(\log D) \cong j_{k,*}\Omega_{D^{(k)}}^{*-k}.$$

This is an isomorphism is easy to see. We refer [5] Proposition 8.32. □

Now these two filtrations induce two filtrations over $R\Gamma(X, \Omega_X^*(\log D))$, and hence over $H^k(U, \mathbb{C})$ by Proposition 2.7. So the arguments in [5] is far from complete and we need some derived-version filtration of these, such as mixed Hodge complex. We omitted this and we refer section 3.3 in [4].

Theorem 2.9 (P. Deligne, 1971)**.** *The discussion above equip $H^k(U, \mathbb{C})$ a mixed Hodge structure which is independent with X, D.*

Proof. This follows from some analysis of the weight spectral sequence (induced by $W^* = W_{-*}$), here we give a sketch.

By the general theory of spectral sequence, we have

$$ {}_W E_1^{p,q} = \mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)). $$

By Proposition 2.8 we have $\mathrm{Gr}_W^p \Omega_X^*(\log D) \cong j_{-p,*}\Omega_{D^{(-p)}}^{*+p}$, hence

$$ \begin{aligned} \mathbb{H}^{p+q}(X, \mathrm{Gr}_W^p \Omega_X^*(\log D)) &= \mathbb{H}^{2p+q}(X, j_{-p,*}\Omega_{D^{(-p)}}^*) \\ &= \mathbb{H}^{2p+q}(D^{(-p)}, \Omega_{D^{(-p)}}^*) = H^{2p+q}(D^{(-p)}, \mathbb{C}). \end{aligned} $$

We can also get that the differential

$$ \begin{tikzcd} d_1 : & H^{2p+q}(D^{(-p)}, \mathbb{C}) \arrow[r] \arrow[d, "\cong"] & H^{2p+q+2}(D^{(-p-1)}, \mathbb{C}) \arrow[d, "\cong"] \\ & \bigoplus_{|K|=-p} H^{2p+q}(D_K, \mathbb{C}) \arrow[r] & \bigoplus_{|L|=-p-1} H^{2p+q+2}(D_L, \mathbb{C}) \end{tikzcd} $$

has component $d_{1,K}^L$ equal to zero for $L \not\subseteq K$, and equal to $(-1)^{q+s} j_{K,*}^L$ when $K = \{i_1 < \cdots < i_p\}$ and $L = K \backslash \{i_s\}$ where $j_K^L : D_K \to D_L$ (see Proposition 8.34 in [5]). Hence we can deduce any pages of weight spectral sequence! By some analysis we can get the result which omitted, we refer Theorem 3.4.7 and section 3.4.1.5 in [4]. $\square$

3 Cycle Classes and Abel–Jacobi Map

3.1 Cycle Classes and Hodge Classes

3.2 The Abel–Jacobi Map

4 Mumford's Theorem and its Generalizations

5 The Bloch Conjecture and its Generalizations

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References

[1] David Eisenbud and Joe Harris. *3264 and all that: A second course in algebraic geometry*. Cambridge University Press, 2016.

[2] William Fulton. *Intersection theory, 2nd*, volume 2. Springer Science & Business Media, 1998.

[3] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer, 1977.

[4] Lê Dung Tráng, Eduardo Cattani, Fouad El Zein, and Phillip A Griffiths. Hodge theory, 2014.

[5] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I*, volume 76. Cambridge University Press, 2002.

[6] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II*, volume 77. Cambridge University Press, 2003.