PERVERSE SHEAVES

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Abstract. In this note we will introduce the basic theory of perverse sheaves, including constructible sheaves, perverse sheaves, nearby and vanishing cycles. Moreover we will also give a glimpse of $\mathscr{D}$-modules, the Riemann-Hilbert correspondence and mixed Hodge modules. Finally we will consider some applications of the theory, such as enumerative geometry and representation theory.

Contents

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1. Introduction

1.1. **Background/Motivation.** Perverse sheaves were discovered in the fall of 1980 by Beilinson-Bernstein- Deligne-Gabber in [BBDG18], sitting at the confluence of two major developments of the 1970s: the intersection homology theory of Goresky-MacPherson, and the Riemann-Hilbert correspondence, due to Kashiwara and Mebkhout.

We will first follows the book [Ach21] to learn the basic theory of perverse sheaves. We will focus the theory of algebraic varieties over $\mathbb{C}$ and using analytic topology. We will also give a quike discussion about pure-algebraic theory using étale topology and étale cohomology. See also original [BBDG18].

We will also discuss some applications of this theory. Such as representation theory and enumerative geometry, especially the relative DT conjecture.

The prerequisites are: familiarity with the language of derived and triangulated categories; familiarity with introductory algebraic topology and some topology of complex algebraic varieties; familiarity with basic algebraic geometry.

1.2. **Related works and some future direction.** Need to add.

Acknowledgments. Need to add.

2. Recollection of the basic theory of sheaves

2.1. **Six Functors.** Here we recollect some definitions of sheaves. Including six functors.

Definition 2.1. *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

- *Let $\mathscr{F} \in \mathsf{Sh}(Y, R)$, then the* pullback *$f^{-1}(\mathscr{F})$ of $\mathscr{F}$ is the sheafification of*
$$f^{-1}_{\mathrm{pre}}(\mathscr{F}) : U \mapsto \varinjlim_{V\subset Y \ open, V \supset f(U)} \mathscr{F}(V).$$
This is an exact functor.
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* pushforward *$f_*(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_*(\mathscr{F})(U) := \mathscr{F}(f^{-1}(U))$.*
- *Let $\mathscr{F} \in \mathsf{Sh}(X, R)$, then the* proper pushforward *$f_!(\mathscr{F})$ of $\mathscr{F}$ is defined by $f_!(\mathscr{F})(U) := \{s \in \mathscr{F}(f^{-1}(U)) : f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U$ is proper$\}$.*
- *We can define*
$$\mathbb{R}\mathscr{H}om(-,-) : \mathsf{D}^-(X, R)^{\mathrm{op}} \times \mathsf{D}^+(X, R) \to \mathsf{D}^+(X, R),$$
$$- \otimes^{\mathbf{L}} - : \mathsf{D}^{\pm}(X, R) \times \mathsf{D}^{\pm}(X, R) \to \mathsf{D}^{\pm}(X, R).$$

Here we recollect some useful and basic results about these functors.

Proposition 2.2 ([Ach21])**.** *Let $f : X \to Y$ be a continuous map between topological spaces and R be a commutative ring.*

(1) *f^{-1} is exact and $f_*, f_!$ are left exact functor. So we can define*
$$\mathbf{R}f_*, \mathbf{R}f_! : \mathsf{D}^+(X, R) \to \mathsf{D}^+(Y, R), \quad f^{-1} : \mathsf{D}(Y, R) \to \mathsf{D}(X, R).$$
Moreover, consider $f : X \to Y$ and $g : Y \to Z$, then we have $(g \circ f)^{-1} = f^{-1}g^{-1}$ and $(g \circ f)_* = g_* \circ f_*$. If X, Y, Z are Hausdorff and locally compact, then $(g \circ f)_! = g_! \circ f_!$.

(2) *If $h : Y \hookrightarrow X$ is a locally closed embedding, then for any $\mathscr{F} \in \mathsf{Sh}(Y, R)$ the sheaf $h_!(\mathscr{F})$ is the sheafification of $h_{!,\mathrm{pre}}\mathscr{F}$ which maps U to $\Gamma(U \cap Y, \mathscr{F})$ if $U \cap \overline{Y} \subset Y$ and 0 otherwise. Moreover in this case $h_!$ is exact. Note that $h_!(\mathscr{F})_x \cong \begin{cases} \mathscr{F}_x & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$*

(3) *We have*
$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om(f^{-1}\mathscr{F}, \mathscr{G}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}f_*\mathscr{G})$$
for any $\mathscr{F} \in \mathsf{D}^-(Y, R)$ and $\mathscr{G} \in \mathsf{D}^+(X, R)$.

(4) *We have*
$$\mathbf{R}\mathscr{H}om(\mathscr{F} \otimes^{\mathbf{L}} \mathscr{G}, \mathscr{H}) \cong \mathbf{R}\mathscr{H}om(\mathscr{F}, \mathbf{R}\mathscr{H}om(\mathscr{G}, \mathscr{H}))$$
for any $\mathscr{F}, \mathscr{G} \in \mathsf{D}^-(X, R)$ and $\mathscr{H} \in \mathsf{D}^+(X, R)$.

Theorem 2.3 (Proper base change, [Ach21] 1.2.13)**.** *Consider a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

(1) If all the spaces are Hausdorff and locally compact, then for any $\mathscr{F} \in \mathsf{D}^+(X, R)$ we have isomorphism

$$g^{-1} f_! \mathscr{F} \cong f'_! (g')^{-1} \mathscr{F}.$$

(2) If f is proper, then for any $\mathscr{F} \in \mathsf{D}^+(X, R)$ we have isomorphism

$$g^{-1} f_* \mathscr{F} \cong f'_* (g')^{-1} \mathscr{F}.$$

2.2. Local Systems.

3. Constructible sheaves

3.1. Preliminaries from algebraic geometry.

3.2. Stratifications and constructible sheaves.

3.3. Artin's vanishing theorem.

3.4. Constructibility theorem.

3.5. Verdier duality theorem.

3.6. More compatibilities.

3.7. Borel-Moore homology and fundamental classes.

4. Perverse sheaves

4.1. Perverse sheaves.

4.2. Intersection cohomology complexes.

4.3. Affine pushforward.

4.4. Smooth pullback and smooth descent.

4.5. Semismall maps.

4.6. The decomposition theorem and the hard Lefschetz theorem.

5. Nearby and vanishing cycles

5.1. Basic things.

5.2. Properties.

5.3. Beilinson's theorem.

6. A glimpse of the algebraic theory

7. About $\mathscr{D}$-modules and mixed Hodge modules

7.1. $\mathscr{D}$-modules and Riemann-Hilbert correspondence.

7.2. Mixed Hodge modules.

8. More Applications

8.1. Relative Donaldson-Thomas Theory for 4-folds.

8.2. For geometric representation theory.

References

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