Varieties of Minimal Rational Tangents on the Fano Varieties

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Preface

Chapter 1

Introduction to the Rational Curves

The main results here we follows the famous book [7].

1.1 Hilbert Schemes and Chow Schemes

1.1.1 Hilbert Schemes, a Basic Introduction

Definition 1.1.1. *Let X be an S-scheme, we define the* Hilbert functor $\mathscr{H}ilb_{X/S}$ *sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z.*

Fix a Polynomial P and a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{H}ilb^P_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z with Hilbert Polynomial P.

Theorem 1.1.2 (Grothendieck)**.** *Let S be a noetherian scheme, let $X \to S$ be a projective morphism, and $\mathscr{L}$ a relatively very ample line bundle on X. Then for any polynomial P, the Hilbert functor $\mathscr{H}ilb^P_{X/S}$ is representable by a projective S-scheme $\mathsf{Hilb}^P_{X/S}$. We also have $\mathsf{Hilb}_{X/S} = \coprod_P \mathsf{Hilb}^P_{X/S}$.*

Proof. Note that this notion of projectivity is much general than [5], but is the same when $S = \operatorname{Spec} k$. The proof is to embed it into Grassmannian. The original proof in [4] and we also refer [9], [7] and [3]. □

Remark 1.1.3. *In [2] we can remove the noetherian hypothesis, by instead assuming strong (quasi-)projectivity of $X \to S$. So also [1].*

Example 1.1.1. *Some examples and interesting results:*

(a) *We have* $\mathsf{Hilb}^1_{X/S} = X/S$.

(b) *Let* C *be a curve over a field* k*, then*

$$\mathsf{Hilb}^m_{C/k} \cong S^m C := \underbrace{C \times \cdots \times C}_{m} / \mathfrak{S}_m.$$

Hence if C *smooth, so is* $\mathsf{Hilb}^m_{C/k}$*. See also [3] Theorem 7.2.3(1) and Proposition 7.3.3.*

(c) *Let* S *be a smooth surface over a field* k*, then* $\mathsf{Hilb}^m_{S/k}$ *is also smooth of dimension* $2m$ *and hence* $\mathsf{Hilb}^m_{S/k} \to S^m X$ *(we will see this later for general settings) is a resolution of singularities. Note that* $S^m X$ *is smooth if and only if* X *is smooth and* $\dim X = 1$ *or* $m < 2$*. See [3] Theorem 7.2.3(2) and Theorem 7.3.4.*

(d) *Let* X *be a nonsingular variety. Then* $\mathsf{Hilb}^m_{X/k}$ *is nonsingular for* $m \leq 3$*. Moreover, for any nonsingular 3-fold the scheme* $\mathsf{Hilb}^4_{X/k}$ *is singular. See [3] Remark 7.2.5 and 7.2.6.*

(e) *Let* $\mathscr{E}$ *be a vector bundle of rank* $m+1$ *over* S *and let* $P_d(n) = \binom{m+n}{m} - \binom{m+n-d}{m}$*, then*

$$\mathsf{Hilb}^{P_d}_{\mathbb{P}(\mathscr{E})/S} \cong \mathbb{P}((\mathrm{Sym}^d \mathscr{E})^\vee).$$

(f) *Let* $Z \to S$*, we have* $\mathsf{Hilb}_{X \times_S Z/Z} \cong \mathsf{Hilb}_{X/S} \times_S Z$.

(g) Hartshorne's Connectedness Theorem: *for every connected noetherian scheme* S*,* $\mathsf{Hilb}^P_{\mathbb{P}^n_S/S}$ *is connected.*

(h) *Let* X *be a connected variety over* k*, then* $\mathsf{Hilb}^n_{X/k}$ *is connected for all* $n > 0$.

(i) Murphy's Law: *It has many singularities, that is, for every scheme* X *finite type over* $\mathbb{Z}$ *and point* $x \in X$*, there exists a point* $q \in \mathsf{Hilb}^P_{\mathbb{P}^n/k}$ *of some Hilbert scheme and an isomorphism*

$$\widehat{\mathscr{O}}_{X,p}[[x_1, ..., x_s]] \cong \widehat{\mathscr{O}}_{\mathsf{Hilb}^P_{\mathbb{P}^n/k}, q}[[y_1, ..., y_t]].$$

See [12]. In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ *for some* n*. It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.*

(j) *In [11] they gave a full classification of the situation where* $\mathsf{Hilb}^P_{\mathbb{P}^n/k}$ *smooth.*

Definition 1.1.4. *Let* $X/S, Y/S$ *are* S*-schemes, then we have a functor* $\mathscr{H}om_S(X, Y)$ *send* S*-scheme* T *into a set of* T*-morphisms* $X \times_S T \to Y \times_S T$.

For a subscheme $B \subset X$ *proper over* S *and* $g : B \to Y$*, we have a functor* $\mathscr{H}om_S(X, Y; g)$ *send* S*-scheme* T *into a set of* T*-morphisms* $X \times_S T \to Y \times_S T$ *such that* $f|_{B \times_S T} = g \times_S \mathrm{id}_T$.

Proposition 1.1.5. *If X/S and Y/S are both projective over S and X is flat over S, then $\mathscr{H}om_S(X,Y)$ represented by an open subscheme $\mathsf{Hom}_S(X,Y) \subset \mathsf{Hilb}_{X\times_S Y/S}$.*

Proof. Any $X \times_S T \to Y \times_S T$ correspond to its graph which is a closed immersion $\Gamma : X \times_S T \to X \times_S Y \times_S T$. As X is flat over S, then $X \times_S T$ is flat over T. Hence we get a morphism $\mathsf{Hom}_S(X,Y) \to \mathsf{Hilb}_{X\times_S Y/S}$. We omit the more details and refer Theorem I.1.10 in [7]. $\square$

Proposition 1.1.6. *If X/S and Y/S are both projective over S and X, B are both flat over S, then $\mathscr{H}om_S(X,Y;g)$ represented by a subscheme $\mathsf{Hom}_S(X,Y;g) \subset \mathrm{Hom}_S(X,Y)$.*

Proof. Consider the restriction map $R : \mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_S(B,Y)$, then $g : B \to Y$ gives a section $G : S \to \mathrm{Hom}_S(B,Y)$. Hence $\mathsf{Hom}_S(X,Y;g) := R^{-1}(G(S)) \subset \mathsf{Hom}_S(X,Y)$ represents $\mathscr{H}om_S(X,Y;g)$. $\square$

Now we state the deformation theory of Hilbert schemes. We only consider the simpler case that all schemes over a field k. For general case we refer Section 1.2 in [7].

Theorem 1.1.7. *Let Y be a projective scheme over a field k and $Z \subset Y$ is a subscheme. Then*

(a) We have

$$T_{[Z]}\mathsf{Hilb}_Y \cong \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

(b) The dimension of every irreducible components of Hilb_Y at $[Z]$ is at least

$$\dim \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) - \dim \mathrm{Ext}^1_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

Proof. See Theorem I.2.8 in [7]. For family case we refer Theorem I.2.15 in [7]. $\square$

Corollary 1.1.8. *Let X, Y are projective varieties over a field k with a morphism $f : X \to Y$. Let Y is smooth over k. Then*

(a) We have

$$T_{[f]}\mathsf{Hom}_k(X,Y) \cong \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

(b) The dimension of every irreducible components of $\mathsf{Hom}_k(X,Y)$ at $[f]$ is at least

$$\dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

Proof. Let $Z \subset X \times_k Y$ be the graph of f, we claim that $\mathscr{I}_Z/\mathscr{I}_Z^2 \cong f^*\Omega^1_Y$. Indeed we have an exact sequence $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega^1_{X\times_k Y}|_Z \to \Omega^1_Z \to 0$. This is split by $\mathscr{O}_Z \cong \mathscr{O}_X \xrightarrow{(\mathrm{id}_X,1)} \mathscr{O}_{X\times_k Y}$. Then we can show the claim. Hence the results follows from Theorem 1.1.7. The family version we refer Theorem I.2.17 in [7]. $\square$

1.1.2 Chow Schemes, a Basic Introduction

Here we only consider the schemes over a field k such that $\mathrm{char}(k) = 0$. The positive characteristic case is very complicated and we refer Section I.4 in [7].

Definition 1.1.9. *Let $g_i : U_i \to W$ be a proper morphism of schemes over W. Assume that W is reduced and U_i is irreducible. By generic flatness there is an open subset $W_i \subset g_i(U_i) \subset W$ such that g_i is flat of relative dimension d over W_i. Let $T = \mathrm{Spec}\,\Delta$ be the spectrum of a DVR Δ and $h : T \to W$ a morphism such that $h(T_g) \in W_i$ and $h(T_0) = w \in W$. Let $h^*U_i = U_i \times_h T$ and $\mathscr{J} \subset \mathscr{O}_{h^*U_i}$ the ideal of those sections whose support is contained in the special fiber of $h * U_i \to T$. Let $(U_i)'_T := \underline{\mathrm{Spec}}_T \mathscr{O}_{h^*U_i}/\mathscr{J}$ which is flat over T. Then we let $[Z_0]$ be the fundamental cycle of the central fiber of $(U_i)'_T \to T$, and define*

$$\lim_{h\to w}(U_i/U) := [Z_0] \in Z_d(g_i^{-1}(w) \times_{\kappa(w)} T_0)$$

which is called the cycle theoretic fiber of g_i at w along h.

Definition 1.1.10. *A well defined family of d-dimensional proper algebraic cycles over W is a pair $(g : U \to W)$ satisfying the following properties:*

(a) There is a reduced scheme $\mathrm{supp}\,U$ with irreducible components U_i such that $U = \sum_i m_i[U_i]$ is an algebraic cycle.

(b) W is a reduced scheme and $g : \mathrm{supp}\,U \to W$ is a proper morphism.

(c) Let $g_i := g|_{U_i}$. Then every g_i maps onto an irreducible component of W and every fiber of g_i is either empty or has dimension d. In particular there is a dense open subset $W_0 \subset W$ such that every g_i is flat over W_0.

(d) For every $w \in W$ there is a cycle $g^{[-1]}(w) \in Z_d(g^{-1}(w))$ such that for any $h : T \to W$ of spectrum of DVR such that $h(T_0) = w$ and $h(T_g) \in W_0$ we have

$$g^{[-1]}(w) =_{\mathrm{ess}} \sum_i m_i \lim_{h\to w}(U_i/W).$$

That is, both two cycles from a single cycle of $Z_d(g^{-1}(w))$.

Remark 1.1.11. *If W is normal, then (d) can be implied by (a)-(c). See Theorem I.3.17 in [7].*

Definition 1.1.12. *Let X be a scheme over S. A well defined family of proper algebraic cycles of X/S over W/S is a pair $(g : U/S \to W/S)$ satisfying the following properties:*

(a) $\mathrm{supp}\,U$ is a closed subscheme of $X \times_S W$ and g is the natural projection morphism.

(b) $(g : U \to W)$ *is a well defined family of* d*-dimensional proper algebraic cycles over* W *for some* d.

Proposition 1.1.13. *Assume that* $g : U \to W$ *is proper and flat of relative dimension* d *and* W *is reduced. Let* $\sum_i m_i[U_i]$ *be the fundamental cycle of* U*. Then* $g : [U] \to W$ *is a well defined family of algebraic cycles over* W.

Proof. See Lemma I.3.14 and Corollary I.3.15 in [7]. □

Definition 1.1.14 (Chow Schemes of Characteristic Zero)**.** *Let* X/S *and we define a functor* $\mathscr{C}how_{X/S}$ *sends* Z/S *to the set consists of well defined families of nonnegative proper algebraic cycles of* $X \times_S Z/Z$.

Let a relative ample line bundle $\mathscr{O}(1)$*, we can define* $\mathscr{C}how^{d,d'}_{X/S}$ *sends* Z/S *to the set consists of well defined families of nonnegative proper algebraic cycles of* $X \times_S Z/Z$ *which is of dimension* d *and degree* d'.

Theorem 1.1.15. *Let* X/S *be a scheme, projective over* S *and* $\mathscr{O}(1)$ *relatively ample. Then the functor* $\mathscr{C}how^{d,d'}_{X/S}$ *is representable by a semi-normal and projective* S*-scheme* $\mathsf{Chow}^{d,d'}_{X/S}$*. We also have* $\mathsf{Chow}_{X/S} = \coprod_{d,d'} \mathsf{Chow}^{d,d'}_{X/S}$.

Proof. Very complicated, we refer Theorem I.3.21 in [7]. □

Example 1.1.2. *Let* X *be a semi-normal variety, then* $\mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.

Proposition 1.1.16 (Hilbert-Chow)**.** *Let* X, Y *be* S*-schemes.*

(a) *We have a natural morphism* $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$*. This morphism can be factored by dimensions.*

(b) *If* X, Y *be projective* S*-schemes and* X/S *flat, then we have*

$$\mathsf{Hom}_S(X,Y)^{\mathrm{sn}} \to \mathsf{Chow}_{Y/S}.$$

Proof. For (a), consider $[\mathsf{Univ}^{\mathsf{Hilb}} \times_{\mathsf{Hilb}_{X/S}} \mathsf{Hilb}^{\mathrm{sn}}_{X/S}] \to \mathsf{Hilb}^{\mathrm{sn}}_{X/S}$, then by Proposition 1.1.13 this is a well defined family of algebraic cycles. This gives such morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$.

For (b), by (a) we have

$$\mathsf{Hom}_S(X,Y)^{\mathrm{sn}} \to \mathsf{Hilb}(X \times_S Y/S)^{\mathrm{sn}} \to \mathsf{Chow}_{X \times_S Y/S} \to \mathsf{Chow}_{Y/S}$$

and well done. □

Remark 1.1.17. *Let* X *be a semi-normal variety, hence we have* $(\mathsf{Hilb}^m_{X/k})^{\mathrm{sn}} \to \mathsf{Chow}^{0,m}_{X/k} \cong S^m X$.

1.1.3 Small Applications to Curves

For more applications we refer Section II.1 in [7]. Here we only need some easy case. We assume over a field k.

Theorem 1.1.18. *Let C be a proper curve and $f : C \to Y$ a morphism to a smooth variety Y of dimension n. Then*

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + n\chi(\mathscr{O}_C).$$

And equality holds if $H^1(C, f * T_Y) = 0$, in this case it is smooth at $[f]$.

Proof. By Corollary 1.1.8(b) we have

$$\begin{aligned}\dim_{[f]} \mathsf{Hom}(C, Y) &\geq \dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X)\\ &= h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y)\\ &= \deg f^*T_Y + n\chi(\mathscr{O}_C)\end{aligned}$$

by Riemann-Roch theorem. The final statement follows from Corollary 1.1.8(a). □

Proposition 1.1.19. *Assume that X/S is flat, B/S is flat and finite of degree m and Y/S is smooth of relative dimension n. Then* $\dim \mathsf{Hom}(X, Y; g) \geq \dim \mathsf{Hom}(X, Y) - kn$.

Proof. Let $p : B \to S$ be the projection. By Corollary 1.1.8 we find that $\mathsf{Hom}(B, Y)$ is smooth over S of relative dimension rank kn. Thus $g(S) \subset \mathrm{Hom}(B, Y)$ is locally defined by kn equations. Pulling back these equations by R we obtain local defining equations. □

Lemma 1.1.20. *Let $0 \in T$ be the spectrum of a local ring and let U/T be a flat and proper and V/T be a variety. Let $p : U \to V$ as a T-morphism. If $p_0 : U_0 \to V_0$ is a closed immersion (resp. an isomorphism), then so is p.*

Proof. See Lemma I.1.10.1 and Proposition I.7.4.1.2 in [7]. We omit this. □

Theorem 1.1.21. *Let C be a projective curve over k and Y a smooth variety over k. Let $B \subset C$ be a closed subscheme which is finite over k. Assume that C is smooth along B. Let $g : B \to Y$ be a morphism. Then*

(a) We have

$$T_{[f]}\mathsf{Hom}(C, Y; g) \cong H^0(C, f^*T_Y \otimes \mathscr{I}_B).$$

(b) The dimension of every irreducible component of $\mathsf{Hom}(C, Y; g)$ *at $[f]$ is at least*

$$h^0(C, f^*T_Y \otimes \mathscr{I}_B) - h^1(C, f^*T_Y \otimes \mathscr{I}_B).$$

Proof. The original proof we refer [8]. A simple case of family version we refer Theorem II.1.7 in [7]. Here we assume k is algebraically closed. Here $\mathscr{I}_B = \mathscr{O}_C(-s_1 - \ldots - s_m)$.

Let $X_0 := C \times_k Y$ and let $\gamma_0 : C \cong \Gamma_0 \subset X_0$ be the graph of f. Let $\pi_1 : X_1 := \mathrm{Bl}_{\{s_1\}} X_0 \to X_0$ and Γ_1 be the strict transform of Γ_0. Let $\gamma_1 : C \cong \Gamma_1 \subset X_1$ as C is smooth at s_1. Repeat the process and finally we get $\pi_m : X_m := \mathrm{Bl}_{\{s_m\}} X_{m-1} \to X_{m-1}$ and Γ_m be the strict transform of Γ_{m-1}. Let $\gamma_m : C \cong \Gamma_m \subset X_m$. Then we have $\gamma_0^*(\mathscr{I}_{\Gamma_0}/\mathscr{I}_{\Gamma_0}^2) \cong f^*\Omega_Y^1$ and $\gamma_{i+1}^*(\mathscr{I}_{\Gamma_{i+1}}/\mathscr{I}_{\Gamma_{i+1}}^2) \cong \gamma_i^*(\mathscr{I}_{\Gamma_i}/\mathscr{I}_{\Gamma_i}^2) \otimes \mathscr{O}_C(-s_{i+1})$. Hence we get $\gamma_m^*(\mathscr{I}_{\Gamma_m}/\mathscr{I}_{\Gamma_m}^2) \cong f^*\Omega_Y^1 \otimes \mathscr{I}_B$.

Now we claim that there is an open neighborhood $[\Gamma_m] \in U \subset \mathsf{Hilb}_{X_m}$ such that $\mathsf{Hom}(C, Y; g) \cong U$. Indeed, let $U \subset \mathsf{Hilb}_{X_m}$ be the open set parametrizing those 1-cycles D for which the projection $D \to C$ is an isomorphism. This is open by Lemma 1.1.20.

First, the universal family of U is contained in $\mathsf{Hom}(C, Y; g)(U)$. Conversely consider $[p_0 : C \times R \to Y \times R] \in \mathsf{Hom}(C, Y; g)(R)$. Let its graph is $G_0 \subset X_0 \times R$. As $\{s_1\} \times R \subset G_0$ and $G_0 \to R$ smooth along $\{s_1\} \times R$, we let $G_1 \subset X_1 \times R$ be the strict transform of G_0. Then $G_1 \cong G_0 \cong C \times R$. Repeat the process and finally we get $X_m \times R \supset C \times R \cong G_m \in \mathsf{Hilb}_{X_m}(R)$. Hence this give the isomorphism $\mathsf{Hom}(C, Y; g) \cong U$. Hence by Theorem 1.1.7 and we get the result. □

1.2 Families of Rational Curves

We may assume all schemes over a field k of characteristic zero locally of finite type. Note that there are also have the same results by some small modification in the case of positive characteristic, see Section II.2 in [7].

Proposition 1.2.1. *Let $f : X \to Y$ be a proper morphism of relative dimension one. Assume that if T is the spectrum of a DVR and $h : T \to Y$ a morphism, then every irreducible component of $T \times_Y X$ has dimension two (By Corollary I.3.16 in [7] this is always the case if f is a well defined family of proper algebraic 1-cycles). Then the subset*

$$\{y \in Y : f^{-1}(y) \text{ has geometrically rational components}\} \subset Y$$

is closed in Y.

Proof. See Proposition II.2.2 in [7]. □

Corollary 1.2.2. *Let $g : U \to V$ be a family of proper algebraic 1-cycles of X/S. Let $U' \subset U$ be the set of points $u \in U$ which are contained in a geometrically rational component of $g^{-1}(g(u))$. The image of the natural morphism $U' \to X$ is called the rational locus of g. It is denoted by $\mathsf{RatLocus}(g : U \to V)$.*

Now let $V \to S$ is proper, then $\mathsf{RatLocus}(g : U \to V)$ is proper over S.

Proof. WLOG we let V is irreducible. Let $U = \sum_i a_i U_i$, then we just need to consider every $g_i : U_i \to V$. Consider the generic fiber D_i of g_i which is a irreducible curve, then if D_i rational, then so is whole g_i by Proposition 1.2.1. Hence $\mathsf{RatLocus}(g_i : U_i \to V) = \mathrm{Im}(U_i \to X)$ is proper over S. If D_i is not rational, then there is an open subset $\emptyset \neq W \subset V$ such that the fibers of g_i over Ware irreducible and nonrational. Thus

$$\mathsf{RatLocus}(g_i : U_i \to V) = \mathsf{RatLocus}(g_i : g_i^{-1}(V\backslash W) \to V\backslash W).$$

Hence we can apply Noetherian induction. □

Definition 1.2.3. *Let* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *be a subscheme correspond to the morphisms* $\mathbb{P}^1 \to X$ *birational to its image. By Lemma 1.1.20 since* $\mathbb{P}^1 \to X$ *birational to its image if and only if it is a immersion at its generic point, then* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *is an open subscheme.*

Definition 1.2.4. *Let* X/S *be a scheme, projective over* S*.*

(a) Let $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} = \bigcup_i W_i$ *be the decomposition into irreducible subschemes of semi-normalization of* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$*. By Proposition 1.1.16 we have the Hilbert-Chow morphism* $\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} \to \mathsf{Chow}_{X/S}$*. Let* $V_i' = \overline{\mathrm{Im}(U_i \to \mathsf{Chow}_{X/S})}$*. By Proposition 1.2.1* V_i' *parametrizes 1-cycles with geometrically rational components, and the generic 1-cycle is irreducible. Let* $V_i \subset V_i'$ *be the open subscheme parametrizing irreducible 1-cycles.*

Let $\eta_i \in V_i$ *be the generic points correspond to curves* C_i*. By generic smoothness* C_i *is a smooth rational curve. Let* V_i^{n} *be the normalization of* V_i*. Then we define the family of rational curves on* X *is*

$$\mathsf{RatCurves}^{\mathrm{n}}(X/S) := \coprod_i V_i^{\mathrm{n}}.$$

with a normalization morphism $\mathsf{RatCurves}^{\mathrm{n}}(X/S) \to \mathsf{Chow}_{X/S}$*.*

If $\mathscr{L}$ *is ample on* X/S*, then we can define* $\mathsf{RatCurves}^{\mathrm{n}}(X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *where* $\mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *is quasi-projective over* S *for any* d*. We define its universal rational curve is*

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) := \left(\mathsf{RatCurves}^{\mathrm{n}}(X/S) \times_{\mathsf{Chow}_{X/S}} \mathsf{Univ}^{\mathsf{Chow}}_{X/S}\right)^{\mathrm{n}}$$

be the normalization.

(b) Fix a section $f : S \to X$*. Similar as (a) we can define* $\mathsf{RatCurves}^{\mathrm{n}}(f, X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(f, X/S)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(f, X/S)$*. This is called family of rational curves passing through* $\mathrm{Im}(f)$*.*

In particular if $S = \operatorname{Spec} k$ *where* k *is a field and* $f : (\operatorname{Spec} k) = x \in X$*, then we will use the notation* $\mathsf{RatCurves}^{\mathrm{n}}(x, X) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(x, X)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(x, X)$*.*

Theorem 1.2.5. *(a) Let $f : X \to Y$ be a proper and surjective morphism between irreducible and normal schemes. Assume that the dimension of every fiber is one (hence f is a well defined family of proper 1-cycles by Remark 1.1.11). Assume that for every $y \in Y$ the cycle theoretic fiber $f^{[-1]}(y)$ is an irreducible and reduced rational curve, then f is a $\mathbb{P}^1$-bundle.*

(b) In the case of the definition, the universal morphisms

$$\mathsf{Univ}^{\mathrm{rc}}(X/S) \to \mathsf{RatCurves}^{\mathrm{n}}(X/S) \text{ and } \mathsf{Univ}^{\mathrm{rc}}(x,X) \to \mathsf{RatCurves}^{\mathrm{n}}(x,X)$$

are $\mathbb{P}^1$-bundles.

Proof. (b) follows directly from (a), so we just need to prove (a).

One can show that f is smooth at the generic point of every fiber (see Theorem I.6.5 in [7]). For $y \in Y$ pick three different points $x_1, x_2, x_3 \in f^{-1}(y)$ such that f is smooth at x_i. Let $S_i \subset X$ be a Cartier divisor which intersects $f^{[-1]}(y)$ transversally at x_i (there may be other intersection points). Hence $S_i \to Y$ is étale at x_i. Let

$$Z = S_1 \times_Y S_2 \times_Y S_3, \quad z = (x_1, x_2, x_3) \in Z \text{ and } X_Z = X \times_Y Z.$$

So $Z \to Y$ is étale at z, thus X_Z is normal along $f_Z^{-1}(z)$ and f is smooth above y iff f_Z is smooth above z by some commutative algebra. Furthermore, f_Z has three sections $s_i : Z \to X_Z$ corresponding to the S_i. By shrinking Z we may assume that these sections are disjoint.

In $\mathbb{P}^1_Z \to Z$ we have three disjoint sections $p_i : Z \to \mathbb{P}^1_Z$ corresponding to $\{0, 1, \infty\}$. Our aim is the construct an isomorphism $q : \mathbb{P}^1_Z \cong X_Z$ such that $q \circ p_i = s_i$. Let $h : \mathbb{P}^1_Z \times_Z X_Z \to Z$ be the projection. In order to construct the graph of q let $\Gamma \subset \mathsf{Chow}_{\mathbb{P}^1_Z \times_Z X_Z/Z}$ be the closed subvariety parametrizing 1-cycles D with the following properties:

(1) $\deg \mathscr{O}_{\mathbb{P}^1}(1)|_D = 1$;

(2) $\deg \mathscr{O}(s_1(Z))|_D = 1$;

(3) $(p_i(h(D)), s_i(h(D))) \in D$ for $i = 1, 2, 3$.

Let $\mathsf{Univ}^\Gamma \to \Gamma$ be the universal family. We claim that the natural projections $\pi_1 : \mathsf{Univ}^\Gamma \to \mathbb{P}^1_Z$ and $\pi_2 : \mathsf{Univ}^\Gamma \to X_Z$ are isomorphisms.

For any $t \in Z$ consider $h^{-1}(t)$. By construction $(h^{-1}(t))_{\mathrm{red}} \cong \mathbb{P}^1_{\kappa(t)} \times C_t$ where C_t is an irreducible geometrically rational curve, smooth for general t. As D gives a 1-cycle on $(h^{-1}(t))_{\mathrm{red}}$ which has bidegree $(1, 1)$, thus D is either the graph of a birational morphism $q_t : \mathbb{P}^1_{\kappa(t)} \to C_t$ or the union of a vertical and of a horizontal section. In the latter case it can not contain all three points $(p_i(t), s_i(t))$. Hence D is the graph of the unique birational morphism q_t such that $q_t(p_i(t)) = s_i(t)$ for $i = 1, 2, 3$. Thus π_1, π_2 are both one-to-one. If C_t is smooth, then q_t is defined over $\kappa(t)$, thus π_1, π_2 are isomorphisms over the generic point of Z. Since X_Z and $\mathbb{P}^1_Z$ are normal, this implies that π_1, π_2 are isomorphisms. Well done. $\square$

Remark 1.2.6. *In positive characteristic, (a) is right if we assume generic-smoothness.*

Proposition 1.2.7. *Notation as above definitions, then*

(a) Let $m = \min\{d : \mathsf{RatCurves}^n_d(X/S) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(X/S)$ is proper over S for $k < 2m$.

(b) Let S be a field and let $m(x) = \min\{d : \mathsf{RatCurves}^n_d(x, X) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(x, X)$ is proper for $k < m + m(x)$.

Proof. (b) follows from the same proof of (a). For (a), as $\mathsf{Chow}^{1,k}_{X/S}$ is proper over S, we just need to show that $\bigcup_i V_i \subset \mathsf{Chow}^{1,k}_{X/S}$ is closed where $\mathsf{RatCurves}^n_k(X/S) = \bigcup_i V_i^{\mathrm{n}} \to \bigcup_i V_i$ is finite. Let $\sum_i a_i D_i \in \overline{\mathsf{RatCurves}^n_k(X/S)}$, then every D_i is rational by Proposition 1.2.1 and $\sum_i a_i \deg D_i = k < 2m$. By assumption $\deg D_i \geq m$, then $\sum_i a_i D_i$ is an irreducible and reduced rational curve. Hence $\mathsf{RatCurves}^n_k(X/S)$ closed. $\square$

Theorem 1.2.8. *Let $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}$ be the normalization of $\mathsf{Hom}_{\mathsf{bir}}$, then we have the following important results:*

(a) Let X/S projective scheme over S, then there is a natural commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{U} & \mathsf{Univ}^{\mathsf{rc}}(X/S) \\
\downarrow & & \downarrow \\
\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{u} & \mathsf{RatCurves}^{\mathrm{n}}(X/S)
\end{array}$$

where U and u are smooth of relative dimension 3 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles)

(b) Let X projective scheme over k with a k-point $x \in X(k)$, then there is a natural commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{U} & \mathsf{Univ}^{\mathsf{rc}}(x, X) \\
\downarrow & & \downarrow \\
\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{u} & \mathsf{RatCurves}^{\mathrm{n}}(x, X)
\end{array}$$

where U and u are smooth of relative dimension 2 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundles)

Proof. These are easy but boring since we consider the characteristic zero. See [7] Theorem II.2.15 and II.2.16. $\square$

Corollary 1.2.9. *Let X projective scheme over k with a k-point $x \in X(k)$, then*

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong H^0(\mathbb{P}^1, N_C), \quad T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(x, X) \cong H^0(\mathbb{P}^1, N_C \otimes \mathfrak{m}_x)$$

for general point $[C]$ where $f : \mathbb{P}^1 \to C \subset X$ is birational and $N_C = f^*T_X/T_{\mathbb{P}^1}$.

Proof. By Theorem 1.2.8, canonical morphism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) \to \mathsf{RatCurves}^{\mathrm{n}}(X/k)$ is a principal $\mathrm{Aut}(\mathbb{P}^1)$-bundle which is smooth. Hence we have

$$0 \to u^*\Omega^1_{\mathsf{RatCurves}^{\mathrm{n}}(X/k)} \to \Omega^1_{\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)} \to \Omega^1_u \to 0.$$

As $[C]$ general, we have $T_{[f]}\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k) = T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)$. Hence

$$T_{[C]}\mathsf{RatCurves}^{\mathrm{n}}(X/k) \cong T_{[f]}\mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1_k, X/k)/\mathrm{Aut}(\mathbb{P}^1) \cong H^0(\mathbb{P}^1, N_C)$$

by trivial reason. Similar for $\mathsf{RatCurves}^{\mathrm{n}}(x, X)$. □

1.3 Free and Minimal Rational Curves

We will assume all scheme over a algebraically closed field k of characteristic zero.

1.3.1 Free Rational Curves

Definition 1.3.1. *Let C be a proper curve, X a smooth variety and $f : C \to X$ a morphism. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. We call f is called* free over f *if f is nonconstant and $H^1(C, f^*T_X \otimes \mathscr{I}_B) = 0$ and $f^*T_X \otimes \mathscr{I}_B$ is generated by global sections. Therefore we can define $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ parameterizes the free rational curves.*

Proposition 1.3.2. *Being free is an open. Hence $\mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ is open.*

Proof. Trivial by definition. □

Theorem 1.3.3. *Let C be a proper curve and X a smooth variety. Let $B \subset C$ be a closed subscheme with ideal sheaf $\mathscr{I}_B$ and $g = f|_B$. Let $F : C \times \mathsf{Hom}(C, X; g) \to X$ be the universal morphism. Then $T_{\kappa(p,[f]), C\times \mathsf{Hom}(C,X;g)} = T_{\kappa(p),C} \oplus H^0(C, f^*T_X \otimes \mathscr{I}_B)$ if $p \notin B$ Consider the differential $df(s) : T_{\kappa(s),C} \to T_{\kappa(f(s)),X}$ and evaluation map*

$$\phi(p, f) : H^0(C, f^*T_X \otimes \mathscr{I}_B) \to f^*T_X \otimes \kappa(p),$$

then $dF(p, [f]) = df(p) + \phi(p, f)$. Furthermore If $\phi(p, f)$ is surjective, then F is smooth at $(p, [f])$. The converse also holds if $H^0(T_C \otimes \mathscr{I}_B) \to T_{\kappa(p),C}$ is surjective.

Proof. Trivial by definitions. □

Corollary 1.3.4. *If C is smooth and $f : C \to X$ is free over g, then $F : C \times \mathsf{Hom}(C, X; g) \to X$ is smooth along $(C \backslash B) \times [f]$. In particular $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth.*

Proposition 1.3.5. *Assume that $f : \mathbb{P}^1 \to X$, $g = f|_B$, $\mathrm{length} B \leq 2$ and write $f^* T_X \otimes \mathscr{I}_B = \sum_i \mathscr{O}(a_i)$. Then $\sharp\{i : a_i \geq 0\} = \mathrm{rank} dF(p, [f])$ for all $p \in \mathbb{P}^1 \backslash B$.*

In particular, if

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is smooth at $(p, [f])$ for some $p \in \mathbb{P}^1$, then f is free over g.

Proof. Note that $\mathrm{length} B \leq 2$ implies $H^0(T_{\mathbb{P}^1} \otimes \mathscr{I}_B) \to T_{\kappa(p), \mathbb{P}^1}$ is surjective for all $p \in \mathbb{P}^1 \backslash B$. Then these are trivial by arguments in Theorem 1.3.3. □

Theorem 1.3.6 (Kollár-Miyaoka-Mori, 1992)**.** *Let X be a smooth projective variety over k. Let $B \subset \mathbb{P}^1_k$ be a closed subscheme with $\mathrm{length} B \leq 2$ and $g : B \to X$. There are countably many subvarieties $V_i = V_i(B, g) \subset X$ such that if $f : \mathbb{P}^1 \to X$ is a nonconstant morphism such that $f|_B = g$ and $\mathrm{Im}(f) \nsubseteq \bigcup_i V_i$, then f is free over B.*

Proof. Let Z_i be the irreducible components of $\mathsf{Hom}(\mathbb{P}^1, X; g)$ with universal morphisms $F_i : \mathbb{P}^1 \times Z_i \to X$. Let $V_i = \overline{\mathrm{Im}(F_i)}$ if F_i is not dominant, and $V_i = X \backslash U_{F_i}$ if F_i is dominant, where $U_{F_i} \subset X$ is an open and dense subset such that $F_{i,\mathrm{red}} : \mathbb{P}^1 \times Z_{i,\mathrm{red}} \to X$ is smooth over U_{F_i} (this is where we use the char $= 0$ assumption). Then the result is trivial. □

Theorem 1.3.7. *Let X be a smooth proper variety over k, then the following statements equivalent.*

(1) X is uniruled.

(2) Generic rational curves of X are free.

(3) X has a free rational curve.

Proof. If X is uniruled then since the morphism

$$F_{\mathrm{red}} : \mathbb{P}^1 \times \mathsf{Hom}(\mathbb{P}^1, X; g)_{\mathrm{red}} \to X$$

is dominant, it is generic smooth. Hence by Proposition 1.3.5 the generic rational curves of X are free.

If the generic rational curves of X are free, then X has a free rational curve.

If X has a free rational curve, then the morphism $\mathbb{P}^1 \times \mathsf{Hom}^{\mathsf{free}}(\mathbb{P}^1, X) \to X$ is smooth by Corollary 1.3.4. Hence it has densed image. Hence X is uniruled. □

Remark 1.3.8. *More properties of uniruled varieties we refer Section IV.1 in [7].*

1.3.2 Minimal Rational Curves

Definition 1.3.9. *Let X be a smooth projective variety over k of dimension n.*

(a) A rational curve $f : \mathbb{P}^1 \to X$ is called standard *if*

$$f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(2) \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus p} \oplus \mathscr{O}_{\mathbb{P}^1}^{\oplus n-1-p}$$

where $p + 2 = -\deg f^*K_X$.

(b) Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a minimal free rational curve *if it is a free rational curve such that $-\deg f^*K_X$ is minimal.*

(c) Let X be a smooth Fano variety over k. A morphism $f : \mathbb{P}^1 \to X$ is called a minimal rational curve *if it is a deformation of the minimal free rational curves.*

Remark 1.3.10. *For any non-constant $f : \mathbb{P}^1 \to X$, it can be factored by $f : \mathbb{P}^1 \xrightarrow{g} \mathbb{P}^1 \xrightarrow{h} X$ where h is birational to its image, then it is a immersion at generic points. Hence $T_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{R}^1}(2) \subset h^*T_X$. Hence $\mathscr{O}_{\mathbb{P}^1}(2 \deg g) \subset f^*T_X$. So if we let $f^*T_X \cong \mathscr{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \mathscr{O}_{\mathbb{P}^1}(a_n)$ with $a_1 \geq \cdots \geq a_n$, then $a_1 \geq 2$.*

Proposition 1.3.11. *Let X be a smooth proper variety over k.*

(a) If X has a free rational curve, then generic free rational curves of X are standard.

(b) If X is Fano and $x \in X$ is a general point, for any irreducible component $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{p+2}(x, X)$ be of minimal degree $p + 2$. Then $\mathcal{K}_x$ is smooth variety of dimension p and the general points are minimal standard.

Proof. For (a), let that free rational curve is g, pick an irreducible component $V \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ containing $[g]$. Then by Theorem 1.3.7 V is dominated to X. Then by Theorem IV.2.4 and Corollary IV.2.9 in [7] there is a $W \subset \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X)$ such that dominated to X and general points in W is standard.

For (b), let $V \subset \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ be the irreducible component correspond to $\mathcal{K}_x$. Now since x is general, by Theorem 1.3.6 any members of V and hence $\mathcal{K}_x$ are free. Hence for any $[f] \in V$ we have $H^1(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = 0$. Then $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ is smooth at $[f]$ in this case. Hence by Theorem 1.1.21 V is also smooth at $[f]$ and of dimension $H^0(\mathbb{P}^1, f^*T_X \otimes \mathfrak{m}_0) = p + 2$. Hence by Theorem 1.2.8(b) the morephism $u : \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) \to \mathsf{RatCurves}^{\mathrm{n}}(x, X)$ is smooth and is an $\mathrm{Aut}(\mathbb{P}^1; 0)$-bundle, hence so is $V \to \mathcal{K}_x$. So $\mathcal{K}_x$ is smooth variety of dimension p. □

1.4 Bend and Break

Bend and Break is a classical method aiming to find the rational curves over the projective varieties which is first observed by S. Mori in [8]. Here we will give the main results proved in [7]. See also the first chapter in [6] for a brief introduction. Here we assume all schemes over a infinity field k.

1.4.1 Main Results of Bend and Break

Definition 1.4.1. *Let S be a proper surface and $B \subset S$ a proper curve. We say that B is* contractible in S *if there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional.*

Proposition 1.4.2 (Rigidity Lemma)**.** *Let $f : X \to Y$ be a proper morphism such that $f_*\mathscr{O}_X = \mathscr{O}_Y$. Let $g : X \to Z$ be a morphism. Assume that for some $y \in Y$ there is a factorization*

$$\begin{tikzcd} & & Z \\ X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(y) \arrow[l, hook'] \arrow[ur, "g|_{f^{-1}(y)}"] \arrow[d, "f_y"'] & \\ Y & \{y\} \arrow[l, hook'] \arrow[uur, "h_y"'] & \end{tikzcd}$$

Then there is an open neighborhood $y \in U \subset Y$ and a factorization

$$\begin{tikzcd} & & Z \\ X \arrow[urr, "g"] \arrow[d, "f"] & f^{-1}(U) \arrow[l, hook'] \arrow[ur, "g|_{f^{-1}(U)}"] \arrow[d, "f_U"'] & \\ Y & U \arrow[l, hook'] \arrow[uur, "h_U"'] & \end{tikzcd}$$

Proof. Let $\Gamma \subset Y \times Z$ be the image of (f, g). Then $p : \Gamma \to Y$ is proper and $p^{-1}(y) = (y, h_y(y))$ is finite over y. Thus there is an open neighborhood $y \in U \subset Y$ such that $p^{-1}(U) \to U$ is finite. Since

$$f_*\mathscr{O}_{f^{-1}(U)} \supset p_*\mathscr{O}_{p^{-1}(U)} \supset \mathscr{O}_U \supset f_*\mathscr{O}_{f^{-1}(U)}$$

which shows that $p^{-1}(U) \to U$ is an isomorphism. □

Corollary 1.4.3. *Let S be a proper surface and $B \subset S$ a contractible curve. Then $B \cdot B < 0$.*

In particular, let D be an irreducible and proper curve and C an arbitrary curve. Let $B_c = B \times \{c\} \subset B \times C$ where $c \in C$ is arbitrary. Then B_c is not contractible in $B \times C$.

Proof. Since $B \subset S$ a contractible, there is a surface S' and a dominant morphism $g : S \to S'$ such that $g(B)$ is zero dimensional. We prove this only for S smooth and S' projective. The general case works the same once the definition of intersection numbers is established in general.

Since S' projective, then we can find a finite morphism $f : S' \to \mathbb{P}^2$ since k is infinity. Let $\mathscr{O}(H) = f^*\mathscr{O}(1)$ which is ample and $H \cdot H > 0$ and $H \cdot B = 0$. By Hodge index theorem we have $B \cdot B < 0$.

For the final statement, note that $B_c \cdot B_c = 0$ hence B_c is not contractible. $\square$

Theorem 1.4.4 (Fundamental Bend and Break, Mori-Miyaoka 1979-1986)**.** *Let B be a smooth proper and irreducible curve over k and S an irreducible, proper and normal surface. Let $p : S \to B$ be a morphism. Assume that there is an open subset $B^0 \subset B$, a smooth projective curve C and an isomorphism*

$$f : [C \times B^0 \xrightarrow{\pi} B^0] \cong [p^{-1}(B^0) \xrightarrow{p} B^0].$$

We call a section $s : B \to S$ is called flat if $s(B^0) = \{c\} \times B^0$ under the above isomorphism.

(a) If there is a contractible flat section $s_1 : B \to S$, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve intersecting $s_1(B)$.

(b) If k algebraically closed, $g(C) = 0$ and there are two contmctible sections $s_1, s_2 : B \to S$, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ is either reducible or nonreduced.

(c) Let L be a nef $\mathbb{R}$-Cartier divisor on S. If there are $k \geq 1$ contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $L \cdot D \leq \frac{2}{k} L \cdot C$ where C be the general fiber of p.

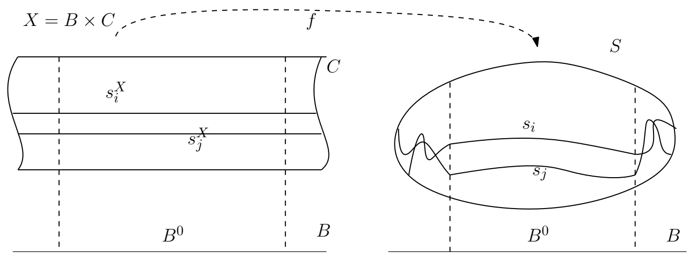
(d) Let L be a nef $\mathbb{R}$-Cartier divisor on S with $L^2 > 0$. If there are k contractible flat sections $s_i : B \to S$ such that $L \cdot s_i(B) = 0$ for every i, then for some $b \in B \backslash B^0$ the fiber $p^{-1}(b)$ contains a rational curve D intersecting a section $s_i(B)$ such that $0 < L \cdot D < \frac{2}{k} L \cdot C$ where C be the general fiber of p.

Proof. Let $X := C \times B$ and $\Gamma \subset X \times_B S$ be the closure of the graph of f. Consider projections p_X, p_S and every flat section s_i induces a flat section $s_i^X : B \to X$:

By Corollary 1.4.3 the rational map $f : X \dashrightarrow S$ is not defined some where along $s_i^X(B)$ if s_i contractible. Here we only prove (a) and (b). Actually (c) and (d) including the same idea with complicated computation and we refer Theorem II.5.4 in [7].

For (a), since $s_1 : B \to S$ is a contractible flat section, then $f : X \dashrightarrow S$ is not defined some where along $s_1^X(B)$. So we have a exceptional curve $D' \subset \Gamma$ of p_X. One can show that D' is rational, then take $D = p_S(D')$ and we get (a).

For (b), we assume that every fibres of p are integral, then $h^1(\mathscr{O}_{p^{-1}(b)}) = 1 - \chi(\mathscr{O}_{p^{-1}(b)})$ since k is algebraically closed. Then it is independent of $b \in B$ and every



fiber of p is isomorphic to $\mathbb{P}^1$. Since p has sections, then S is a minimal ruled surface over B. Now the matrix of intersection form of $s_1(B), s_2(B)$ and $C \times \{b\}$ is $\mathbf{M} = \begin{pmatrix} -a_1 & c & 1 \\ c & -a_2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ where $-a_i = s_i(B)^2 < 0$ by Corollary 1.4.3 and $c = s_1(B) \cdot s_2(B) \geq 0$. Hence $\det \mathbf{M} = 2c + a_1 a + 2 > 0$ which is impossible since $\dim N_1(S) = 2$ since $N_1(S)$ generated by $s_1(B)$ and $C \times \{b\}$. $\square$

Corollary 1.4.5. *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $p_1, ..., p_k \in C$ be k distinct points and $g : \{p_1, ..., p_k\} \to X$ a morphism. Assume that there is a smooth, irreducible, proper curve B, an open set $B^0 \subset B$ and a morphism*

$$[h^0 : C \times B^0 \to X \times B^0] \in \mathsf{Hom}(C, X; g)(B^0)$$

such that $h^0(C \times \{b\})$ and $p_X \circ h^0(\{c\} \times B^0)$ are one dimensional for some $b \in B^0$ and $c \in C$.

Then there is a unique normal compactification $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$. Let $p : S \to B$.

(a) If $k \geq 1$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D which passes through $g(p_1)$.

(b) If $C \cong \mathbb{P}^1$, $\dim \mathrm{Im}(p_X \circ h^0) = 2$ and $k \geq 2$, then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ is either reducible or nonreduced.

(c) Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 \leq L \cdot D \leq \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.

(d) Let L be a nef $\mathbb{R}$-Cartier divisor on X with $h^* L^2 > 0$ and $k \geq 1$. Then for some $b \in B \backslash B^0$ the 1-cycle $h_*(p^{-1}(b))$ contains a rational curve D such that $0 < L \cdot D < \frac{2}{k} L \cdot h_* C$ and $\{g(p_1), ..., g(p_k)\} \cap D \neq \emptyset$.

Proof. If $h^0(C \times \{b\})$ is a point for some $b \in B^0$, then by rigidity lemma $h^0(C \times \{b\})$ is a point for any $b \in B^0$, a contradiction. Thus h^0 is finite on every fiber of $C \times B^0 \to B^0$, hence the natural morphism h^0 is quasifinite. $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$.

If $\mathrm{Im}(p_X \circ h^0)$ is of dimension one, this is not hard to see. If $\mathrm{Im}(p_X \circ h^0)$ is of dimension two, then any p_i determines a contractible flat section of S given by $s_i : B^0 \to \{p_i\} \times B^0$. Then this follows from Theorem 1.4.4. $\square$

Theorem 1.4.6 (Bend and Break)**.** *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $f : C \to X$ be a nonconstant morphism.*

(a) If $\dim_{[f]} \mathsf{Hom}(C, X) \geq \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \mathrm{supp}(\sum_i a_i D_i)$ and

$$f_*[C] \sim_{\mathrm{alg}} (f_x)_*[C] + \sum_i a_i[D_i].$$

(b) If $g(C) = 0$ and $\dim_{[f]} \mathsf{Hom}(C, X) \geq 2\dim X + 2$ (holds if $-K_X \cdot C \geq n + 2$), then for every $x_1, x_2 \in f(C)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x_1, x_2 \in \mathrm{supp}(\sum_i a_i D_i)$ and

$$f_*[C] \sim_{\mathrm{alg}} \sum_i a_i[D_i].$$

(c) Let L be a nef $\mathbb{R}$-Cartier divisor on X and $k \geq 1$. If $\dim_{[f]} \mathsf{Hom}(C, X) \geq k \dim X + 1$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ ($a_1 > 0$) whose irreducible components are rational curves such that $x \in D_1$ and

$$f_*[C] \sim_{\mathrm{alg}} (f_x)_*[C] + \sum_i a_i[D_i], \quad L \cdot D_1 \leq \frac{2}{k} L \cdot f_* C.$$

Proof. $\square$

Theorem 1.4.7 (Smooth Bend and Break, Mori 1979-1982)**.** *Let X be a smooth projective variety.*

(a) Let $f : \mathbb{P}^1 \to X$ be a nonconstant morphism. Then for every $x \in f(\mathbb{P}^1)$ there is a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \mathrm{supp}(\sum_i a_i D_i)$ and

$$f_*[C] \sim_{\mathrm{alg}} \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

(b) *Let C be a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Assume that $\deg_C f^*(-K_X) > g(C)\dim X$, then for every $x \in f(C)$ there is a morphism $f_x : C \to X$ and a 1-cycle $\sum_i a_i D_i$ whose irreducible components are rational curves such that $x \in \operatorname{supp}(\sum_i a_i D_i)$ and $\deg_C f_x^*(-K_X) \leq g(C)\dim X$ and*

$$f_*[C] \sim_{\mathrm{alg}} (f_x)_*[C] + \sum_i a_i[D_i], \quad -K_X \cdot D_i \leq \dim X + 1.$$

Proof. □

1.4.2 Connection of Zero and Positive Characteristics

1.4.3 Applications of General Varieties and Fano Varieties

We assume that all varieties over an algebraically closed field k.

Theorem 1.4.8 (Kollár-Miyaoka-Mori, 1979-1982-1986-1991)**.** *Let X be a projective variety over k, let C a smooth, projective and irreducible curve, $f : C \to X$ a morphism and M any nef $\mathbb{R}$-divisor. Assume that X is smooth along $f(C)$ and $-K_X \cdot C > 0$.*

Then for every $x \in f(C)$ there is a rational curve $L_x \subset X$ containing x such that

$$M \cdot L_x \leq 2\dim X \frac{M \cdot C}{-K_X \cdot C}.$$

Proof. □

Theorem 1.4.9 (Smooth Case)**.** *Let X be a smooth projective variety, C a smooth, projective and irreducible curve and $f : C \to X$ a morphism. Let M be any nef $\mathbb{R}$-divisor. Assume that $-K_X \cdot C > 0$, then for any $x \in f(C)$ there is a rational curve $D_x \subset X$ containing x such that*

$$M \cdot D_x \leq 2\dim X \frac{M \cdot C}{-K_X \cdot C}, \quad -K_X \cdot D_x \leq \dim X + 1.$$

Proof. □

Corollary 1.4.10 (Fano Case)**.** *Let X be a smooth Fano variety, then for any x there is a rational curve $D_x \subset X$ containing x such that $-K_X \cdot D_x \leq \dim X + 1$. In particular any smooth Fano variety is uniruled.*

1.5 Application I: Basic Theory of Fano Manifolds

Some general theory of Fano varieties we refer [10]. Here we give some important basic theory of Fano manifolds. We consider any schemes over an algebraically closed field k.

1.5.1 Some General Properties

Theorem 1.5.1. *Let G be a reduced and connected linear algebraic group and X be a proper homogeneous space under the action of G. Pick $x \in X$ and stabilizer $G_x \subset G$. If G_x is reduced (always hold if $\mathrm{char} = 0$), then T_X is generated by global sections and $-K_X$ is very ample.*

Proof. □

Proposition 1.5.2. *Let X be a smooth Fano variety over an algebraically closed field k of characteristic zero.*

(a) *X is simply connected.*

(b) $\mathrm{Pic}(X)$ *is finite generated and torsion free.*

Proof. □

Theorem 1.5.3 (Cone Theorem)**.** *Let X be a smooth Fano variety over an algebraically closed field k. On X there are only finitely many families of rational curves C_μ such that $-K_X \cdot C_\mu \leq \dim X + 1$. Let $C_i : 1 \leq i \leq N$ be a set of representatives, then*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \sum_i \mathbb{R}^+[C_i].$$

Proof. □

Proposition 1.5.4. *Let $f : X \to Y$ be a smooth morphism between smooth projective varieties.*

(a) *If $\dim Y > 0$ then $-K_{X/Y}$ is not (absolutely) ample on X.*

(b) *If X is Fano, then Y is also Fano.*

Proof. □

1.5.2 Classifications Via Fano Index

1.6 Application II: Boundedness of Fano Manifolds

Theorem 1.6.1.

Theorem 1.6.2 (BAB Conjecture, Birkar 2021)**.**

Some Comments. □

1.7 Application III: Hartshorne's Conjecture

Hartshorne's Conjecture is first proved by S. Mori in his famous and important paper [8]. This paper is the beginning of the theory of VMRT.

Theorem 1.7.1 (Hartshorne's Conjecture, Mori 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k, if T_X is ample then $X \cong \mathbb{P}^n_k$.*

Proof. By Theorem 1.7.3 directly. □

This conjecture motivated by an important conjecture in complex geometry:

Theorem 1.7.2 (Frankel's Conjecture, Mori 1979 and Siu-Yau 1980)**.** *If X is a compact Kähler manifold of dimension n with everywhere positive holomorphic bisectional curvature, then $X \cong \mathbb{P}^n_{\mathbb{C}}$.*

Proof. By Kodaira embedding theorem to $-K_X$ we know that X is a projective manifold. Then by Theorem 1.7.1 we get the result. □

Our main result in this section is the following due to Mori which is much stronger than the Hartshorne's Conjecture as we mentioned above.

Theorem 1.7.3 (Mori, 1979)**.** *Consider n-dimensional smooth projective variety X over an algebraically closed field k. If*

(1) $-K_X$ is ample, that is, X is a Fano manifold;

(2) For any non-constant morphism $f : \mathbb{P}^1_k \to X$ the bundle f^*T_X is the sum of line bundles of positive degree.

Then $X \cong \mathbb{P}^n_k$.

Proof. We will use the following lemmas:

- **Lemma A.** For any $f : \mathbb{P}^1_k \to X$ such that bundle f^*T_X is the sum of line bundles of positive degree, we have $\deg f^*T_X \geq n+1$. If equality holds, then f is an closed embedding and is standard, that is, $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$.

  Proof of Lemma A. Let $f^*T_X \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$ where $a_1 \geq \cdots \geq a_n$. Then $a_i \geq 1$ and $a_1 \geq 2$ by Remark 1.3.10. Hence $\deg f^*T_X \geq n+1$. If equality holds, then the only possibility is $f^*T_X \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus n-1}$. To show f is an embedding, first we now that f is unramified by trivial reason. Others are also easy and we refer to Lemma V.3.7.3.2 in [7]. □

- **Lemma B.** In the case of Theorem, any rational curve can be deformed as a cycle to the sum of rational curves C such that $-K_X \cdot C = n+1$.

Proof of Lemma B. From bend and break directly. $\square$

Back to the theorem. We let $n \geq 2$. Pick $f : \mathbb{P}^1 \to X$ passing a general point $x \in X$ with $0 \mapsto x$ and with minimal degree $n+1$ by Lemma B. By Proposition 1.3.11 the components $V \subset \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) = \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x)$ containing $[f]$ is smooth of dimension $n+1$ and the correspond $\mathcal{K}_x \subset \mathsf{RatCurves}^{\mathrm{n}}_{n+1}(x, X)$ is also smooth of dimension $n-1$. Actually $\gamma : V \to \mathcal{K}_x$ is a principal $G := \mathrm{Aut}(\mathbb{P}^1; 0)$-bundle.

▶ **Step 1.** We claim that $\mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

Consider the tangent $\Phi : V \to \mathbb{V}(\Omega^1_{X,x})$ via $v \mapsto (dv)_0(\frac{d}{dt})$ for uniformizer $t \in \mathscr{O}_{\mathbb{P}^1,0}$ by Lemma A. First we claim that Φ is smooth. Easy to see that Φ is flat and we just need to show $\Phi^{-1}(\Phi(v))$ is smooth. Note that for any finite type k-scheme T and for any morphism $T \to V$ over k, it factors through $\Phi^{-1}(\Phi(v)) \to V$ if and only if the morphism $\mathbb{P}^1_T \to X_T$ coincides on $\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})$ with v_T. Hence

$$\Phi^{-1}(\Phi(v)) \cong V \cap \mathsf{Hom}_{\mathsf{bir}}(\mathbb{P}^1, X; v|_{\mathrm{Spec}(\mathscr{O}_{\mathbb{P}^1,0}/\mathfrak{m}^2_{\mathbb{P}^1,0})})$$

which is open and hence smooth with the same proof of Proposition 1.3.11.

Hence by Lemma A again we get a smooth morphism $\Phi : \mathcal{K}_x \to \mathbb{P}(\Omega^1_{X,x})$. Hence it is finite étale. Hence $\mathcal{K}_x \cong \mathbb{P}(\Omega^1_{X,x})$.

▶ **Step 2.** Let $F : V \times \mathbb{P}^1 \to \mathcal{K}_x \times X$ defined by $(v, x) \mapsto (\gamma(v), v(x))$, consider $Z := \underline{\mathrm{Spec}}_{\mathcal{K}_x \times X} F_* \mathscr{O}^G$ which is a geometrically quotient by G (can be checked along the principal bundle $V \to \mathcal{K}_x$). As $\psi : Z \to \mathcal{K}_x$ is a $\mathbb{P}^1$-bundle with a section $S \subset Z$ induced by $V \to V \times \mathbb{P}^1$ as $v \mapsto (v, 0)$, then $Z \cong \mathbb{P}(\psi_* \mathscr{O}_Z(S))$ is a projective bundle. Define a universal cycle map $\pi : Z \to X$ induced by G-invariant cycle morphism $V \times \mathbb{P}^1 \to X$. We claim that $\pi : Z \to X$ is étale on $Z \backslash S$ and $\pi(S) = x$.

Actually $\pi(S) = x$ is trivial, to show $\pi|_{Z \backslash S}$ is étale we just need to show $V \times \mathbb{P}^1 \to X$ is smooth. This follows from Corollary 1.3.4 and Theorem 1.3.6. Hence we get the claim.

▶ **Step 3.** Consider the Stein factorization we have $\pi : Z \xrightarrow{\phi} U \cong \underline{\mathrm{Spec}}_X \pi_* \mathscr{O}_Z \xrightarrow{\eta} X$. We claim that η is étale, $Z \backslash S \cong U \backslash \{r\}$ where $\phi(S) = r$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

In fact by Stein factorization η is étale outside a codimension ≥ 2 locus, by purity of branced locus we know that η is étale. Now $Z \backslash S \cong U \backslash \{r\}$ where $\phi(S) = r$ follows from Zariski main theorem. Finally we show that $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. Indeed, picka hyperplane $L \subset \mathcal{K}_x$ and a line $C \cong \mathbb{P}^1 \subset S$ such that $\psi(C) \nsubseteq L$. Let $D := \psi^{-1}(L)$, then $C \cdot D = 1$. As $r \in \phi(D)$, we have $\phi^{-1}\phi(D) = D + aS$ for some $a > 0$. So $C \cdot \phi^{-1}\phi(D) = \phi(D) \cdot D = 0$. Hence $C \cdot S = -1$ and $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$.

▶ **Step 4.** We claim that $U \cong \mathbb{P}^n$.

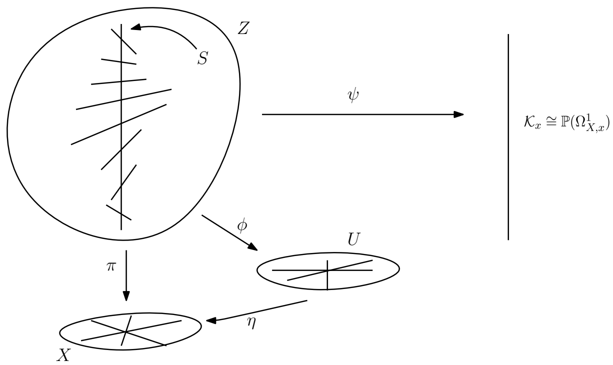
By Step 3 we have $\mathscr{O}_S(S) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(-1)$, hence

$$0 \to \mathscr{O}_Z \to \mathscr{O}_Z(S) \to \mathscr{O}_S(-1) \to 0$$

exact. Since $R^1\psi_*\mathscr{O}_Z = 0$, we get

$$0 \to \mathscr{O}_{\mathcal{K}_x} \to \psi_*\mathscr{O}_Z(S) \to \mathscr{O}_{\mathcal{K}_x}(-1) \to 0$$

exact. As $\mathrm{Ext}^1_{\mathbb{P}^{n-1}}(\mathscr{O}(-1), \mathscr{O}) = 0$, we get $\psi_*\mathscr{O}_Z(S) \cong \mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)$. Hence by Step 2 we have $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1))$.



Hence $Z \cong \mathbb{P}(\mathscr{O}_{\mathcal{K}_x} \oplus \mathscr{O}_{\mathcal{K}_x}(-1)) \cong \mathrm{Bl}_O\mathbb{P}^n$. We can have a contraction map $Z \to \mathrm{Bl}_O\mathbb{P}^n$ makes S to a point $O \in \mathbb{P}^n$ (in fact it is induced by $\psi^*\mathscr{O}(1) \otimes \mathscr{O}(S)$). Hence via $\mathbb{P}^n \leftarrow Z \to U$ we have a birational map $\mathbb{P}^n \dashrightarrow U$.

▶ **Step 5.** Finish the proof, that is, we have $X \cong \mathbb{P}^n$.

Since $\mathbb{P}^n$ is simply connected, $U \cong \mathbb{P}^n \to X$ is a Galois covering by Step 3 and 4. Thus $X \cong \mathbb{P}^n$ because any automorphism of $\mathbb{P}^n$ has a fixed point. □

Corollary 1.7.4 (Lazarsfeld, 1984)**.** *Let X be a smooth projective variety over an algebraically closed field k. Let there is a surjective separable morphism $p : \mathbb{P}^n_k \to X$, then $X \cong \mathbb{P}^n$.*

Proof. □

Chapter 2

Varieties of Minimal Rational Tangents

We will assume the base field is $\mathbb{C}$.

2.1 Basic Properties

2.2 Birationality of the Tangent Morphism

2.3 Examples of VMRT

2.4 Distributions and Its Properties

2.5 Cartan-Fubini Type Extension Theorem

Chapter 3

Some Basic Applications of VMRT

3.1 Stability of the Tangent Bundles

3.2 The Remmert-Van de Ven / Lazarsfeld Problem

3.3 Deformation Rigidity

3.4 Uniqueness of Contact Structures

Chapter 4

About Semiample Tangent Bundles

Chapter 5

Need to add

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