

# Note for the Virtual Fundamental Class

Xiaolong Liu

April 19, 2024

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Review of Basic Intersection Theory</b>	<b>2</b>
2.1	Basic Facts of Refined Gysin Pullback . . . . .	2
2.2	Localized Chern Class . . . . .	7
<b>3</b>	<b>A Brief of Cotangent Complexes</b>	<b>8</b>
<b>4</b>	<b>Foundations of Virtual Fundamental Class</b>	<b>9</b>
4.1	About Cones . . . . .	10
4.2	Cone Stack . . . . .	18
4.3	A Picard Stack of Special Type . . . . .	22
4.4	About Normal Cones . . . . .	27
4.5	Intrinsic Normal Cone . . . . .	29
4.6	Vistoli's Rational Equivalence . . . . .	31
4.7	Obstruction Theory and Virtual Class . . . . .	31
4.8	Examples . . . . .	31
<b>5</b>	<b>Atiyah-Bott Localization</b>	<b>31</b>
<b>6</b>	<b>Localization of Virtual Fundamental Class</b>	<b>32</b>
	<b>References</b>	<b>32</b>

## 1 Introduction

We will follow [BF97][AB84][GP99] and we will also use [Ric22].

We need [Har77][Ful98][EH16].

Here we will consider  $\mathbb{P}(-) = \mathbf{Proj} \operatorname{Sym}(-)^\vee$  for bundles and the vector bundle is both space and sheaf via  $\mathbf{Spec} \operatorname{Sym}(-)^\vee$ . For a cone  $C =$

$\mathrm{Spec}_X \mathcal{I}^*$ , we define  $\mathbb{P}(C) := \mathbf{Proj}_X \mathcal{I}^*$  and  $\mathbb{P}(C \oplus \mathcal{O}) := \mathbf{Proj}_X \mathcal{I}^*[z]$  which is the projective cone and projective completion, respectively. For more details we refer Appendix B.5 of [Ful98].

## 2 Review of Basic Intersection Theory

We will follow [Ful98]. We will omit the basic things such as Segre classes of bundles and cones, Chern classes of bundles and the technique of the deformation to the normal cone. We refer Chapter 1-5 in [Ful98]. We work over schemes of finite type over some field  $k$ .

### 2.1 Basic Facts of Refined Gysin Pullback

Here we will follow Chapter 6,8,9 of [Ful98]. We will state the results without the most of the proof.

**Definition 2.1** (Intersection Product). *Let  $i : X \hookrightarrow Y$  be a closed regular embedding of codimension  $d$  with normal bundle  $N_{X/Y}$ . Pick  $V$  be a scheme of pure dimension  $k$ . Consider the cartesian diagram*

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let  $\mathcal{I}$  be the ideal of  $i$  and  $\mathcal{J}$  be the ideal of  $j$ , then we have surjection

$$\bigoplus_n f^*(\mathcal{I}^n / \mathcal{I}^{n+1}) \rightarrow \bigoplus_n \mathcal{J}^n / \mathcal{J}^{n+1} \rightarrow 0$$

which induce embedding  $C_{W/V} \hookrightarrow g^*N_{X/Y}$ . Note that  $C_{W/V}$  is also a scheme of pure dimension  $k$  since  $\mathbb{P}(C_{W/V} \oplus \mathcal{O})$  is the exceptional divisor of  $\mathrm{Bl}_W(Y \times \mathbb{A}^1)$ . Let  $0 : W \rightarrow g^*N_{X/Y}$  be the zero-section of  $\pi : g^*N_{X/Y} \rightarrow W$ , then we define

$$X \cdot V := 0^*[C_{W/V}] := (\pi^*)^{-1}[C_{W/V}] \in \mathrm{CH}_{k-d}(W)$$

as the intersection class.

**Proposition 2.2.** *Consider the situation of Definition 2.1.*

- (a) *We have  $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$ .*
- (b) *Let  $\mathcal{Q}$  be the universal quotient bundle of  $q : \mathbb{P}(g^*N_{X/Y} \oplus \mathcal{O}) \rightarrow W$ , then*

$$X \cdot V = q_*(c_d(\mathcal{Q}) \cap [\mathbb{P}(C_{W/V} \oplus \mathcal{O})]).$$

(c) If  $j : W \hookrightarrow V$  is a regular embedding of codimension  $d'$ , then  $X \cdot V = c_{d-d'}(g^*N_{X/Y}/N_{W/V}) \cap [W]$ .

*Proof.* Easy, one omitted. See Proposition 6.1 and Example 6.1.7 in [Ful98].  $\square$

**Definition 2.3** (Refined Gysin Pullback). *Let  $i : X \hookrightarrow Y$  be a closed regular embedding of codimension  $d$  with normal bundle  $N_{X/Y}$ . Pick  $f : Y' \rightarrow Y$  be a morphism. Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xhookrightarrow{j} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Then we define  $i^! : \mathbf{Z}_k Y' \rightarrow \mathbf{CH}_{k-d} X'$  as  $\sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i$ . Now  $i^!$  can be decomposed as:

$$i^! : \mathbf{Z}_k Y' \xrightarrow{\sigma} \mathbf{Z}_k C_{X'/Y'} \rightarrow \mathbf{CH}_k(g^*N_{X/Y}) \xrightarrow{0^*} \mathbf{CH}_{k-d} X'$$

where  $\sigma : \mathbf{Z}_k Y' \rightarrow \mathbf{Z}_k C_{X'/Y'}$  given by  $[V] \mapsto [C_{V \cap X'/V}]$ . By the technique of deformation to the normal cone, this can be descend to the Chow-group level as  $\sigma : \mathbf{CH}_k Y' \rightarrow \mathbf{CH}_k C_{X'/Y'}$  (see Proposition 5.2 in [Ful98]) which is called the *specialization to the normal cone*. Hence this induce the refined Gysin pullback

$$i^! : \mathbf{CH}_k Y' \rightarrow \mathbf{CH}_{k-d} X', \quad \sum_i n_i [V_i] \mapsto \sum_i n_i X \cdot V_i.$$

**Proposition 2.4.** *Consider the situation of Definition 2.3. Consider*

$$\begin{array}{ccc} X'' & \xhookrightarrow{i''} & Y'' \\ q \downarrow & \lrcorner & p \downarrow \\ X' & \xhookrightarrow{i'} & Y' \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

- (a) If  $p$  proper and  $\alpha \in \mathbf{CH}_k(Y'')$ , then  $i^! p_*(\alpha) = q_* i^!(\alpha) \in \mathbf{CH}_{k-d}(X')$ .
- (b) If  $p$  is flat of relative dimension  $n$  and  $\alpha \in \mathbf{CH}_k(Y'')$ , then  $i^! p^*(\alpha) = q^* i^!(\alpha) \in \mathbf{CH}_{k+n-d}(X'')$ .
- (c) If  $i'$  is also a regular embedding of codimension  $d$  and  $\alpha \in \mathbf{CH}_k(Y'')$ , then  $i^! \alpha = (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X'')$ .

(d) If  $i'$  is a regular embedding of codimension  $d'$ , then for  $\alpha \in \mathbf{CH}_k(Y'')$  we have

$$i^!(\alpha) = c_{d-d'}(q^*(g^*N_{X/Y}/N_{X'/Y'})) \cap (i')^!(\alpha) \in \mathbf{CH}_{k-d}(X'').$$

We call  $g^*N_{X/Y}/N_{X'/Y'}$  the *excess normal bundle*.

(e) Let  $F$  be any vector bundle on  $Y'$ , then for  $\alpha \in \mathbf{CH}_k(Y'')$  we have

$$i^!(c_m(F) \cap \alpha) = c_m((i')^*F) \cap i^!(\alpha) \in \mathbf{CH}_{k-d-m}(X').$$

*Proof.* See Theorem 6.2, Theorem 6.3 and Proposition 6.3 in [Ful98].  $\square$

**Corollary 2.5.** Let  $i : X \hookrightarrow Y$  be a regular embedding of codimension  $d$ , then

$$i^*i_*(\alpha) = c_d(N_{X/Y}) \cap \alpha \in \mathbf{CH}_*(X).$$

*Proof.* By Proposition 2.4(d) directly.  $\square$

**Proposition 2.6.** The refined Gysin pullback have the following properties.

(a) Let  $i : X \hookrightarrow Y$  and  $j : S \hookrightarrow T$  are regular embeddings of codimension  $d, e$ , respectively. Consider cartesian:

$$\begin{array}{ccccc} X'' & \hookrightarrow & Y'' & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow j' & \lrcorner & \downarrow j \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{g} & T \\ \downarrow & \lrcorner & \downarrow f & & \\ X & \xrightarrow{i} & Y & & \end{array}$$

Then for any  $\alpha \in \mathbf{CH}_k(Y'')$ , we have

$$j^!i^!(\alpha) = i^!j^!(\alpha) \in \mathbf{CH}_{k-d-e}(X'').$$

(b) Let  $i : X \hookrightarrow Y$  and  $j : Y \hookrightarrow Z$  are regular embeddings of codimension  $d, e$ , respectively. Consider cartesian:

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{j'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{j} & Z \end{array}$$

Then  $ji$  is a regular embedding of codimension  $d + e$  and for all  $\alpha \in \mathbf{CH}_k(Z')$  we have

$$(ji)^!(\alpha) = i^!j^!(\alpha) \in \mathbf{CH}_{k-d-e}(X').$$

*Proof.* See Theorem 6.4 and Theorem 6.5 in [Ful98].  $\square$

**Proposition 2.7.** *Consider cartesians:*

$$\begin{array}{ccccc} X' & \xrightarrow{i'} & Y' & \xrightarrow{p'} & Z' \\ \downarrow h & \lrcorner & \downarrow g & \lrcorner & \downarrow f \\ X & \xrightarrow{i} & Y & \xrightarrow{p} & Z \end{array}$$

- (a) *If  $i$  is a regular embedding of codimension  $d$  and  $p$  and  $pi$  are flat of relative dimension  $n, n-d$ , respectively. Then  $i'$  is a regular embedding of codimension  $d$  and  $p', p'i'$  are flat, and for  $\alpha \in \text{CH}_k(Z')$  we have*

$$(p'i')^*(\alpha) = (i')^*((p')^*\alpha) = i^!((p')^*\alpha).$$

- (b) *If  $i$  is a regular embedding of codimension  $d$  and  $p$  is smooth of relative dimension  $n$ , and  $pi$  is a regular embedding of codimension  $d-n$ . Then for  $\alpha \in \text{CH}_k(Z')$  we have*

$$(pi)^!(\alpha) = i^!((p')^*\alpha).$$

*Proof.* See Proposition 6.5 in [Ful98].  $\square$

**Remark 2.8.** *Some remarks.*

- (a) *For local complete intersection morphism  $f : X \rightarrow Y$ , we can decompose it into  $f : X \xrightarrow{i} P \xrightarrow{p} Y$  where  $i$  is a closed regular embedding of constant codimension and  $p$  is smooth of constant relative dimension. Then we can define  $f^! := i^!(p')^*$ . See Section 6.6 in [Ful98] for more properties.*
- (b) *If  $Y$  is nonsingular of dimension  $n$ , then we can define the following intersection product: Let  $f : X \rightarrow Y$  and  $p : X' \rightarrow X$  and  $q : Y' \rightarrow Y$ . Let  $x \in \text{CH}_k(X')$  and  $y \in \text{CH}_l(Y')$ , consider the cartesian*

$$\begin{array}{ccc} X' \times_Y Y' & \longrightarrow & X' \times Y' \\ \downarrow & \lrcorner & \downarrow p \times q \\ X & \xrightarrow{\gamma_f} & X \times Y \end{array}$$

*and define  $x \cdot_f y := \gamma_f^!(x \times y) \in \text{CH}_{k+l-n}(X' \times_Y Y')$ .*

*So when  $x, y \in \text{CH}_*(Y)$ , then let  $X = Y$  and  $X' = |x|, Y' = |y|$ , then we get the new intersection product. Note that this is compactible as the definition before. See Chapter 8 in [Ful98] for more properties. In this case  $\text{CH}_*(Y)$  is a ring which is called Chow ring.*

Finally we will discuss something about equivalence and supportness.

**Definition 2.9.** Let  $i : X \hookrightarrow Y$  be a closed regular embedding of codimension  $d$  with normal bundle  $N_{X/Y}$ . Pick  $V$  be a scheme of pure dimension  $k$ . Consider the cartesian diagram

$$\begin{array}{ccc} W & \xhookrightarrow{j} & V \\ g \downarrow & \lrcorner & f \downarrow \\ X & \xhookrightarrow{i} & Y \end{array}$$

Let  $C_1, \dots, C_r$  be the irreducible components of  $C_{W/V}$ , then  $[C_{W/V}] = \sum_{i=1}^r m_i [C_i]$ . Let  $Z_i = \pi(C_i)$  where  $\pi : g^* N_{X/Y} \rightarrow W$  and we call them the **distinguished varieties** of the intersection of  $V$  by  $X$ . Let  $N_i := (g^* N_{X/Y})|_{Z_i}$  and let  $0_i : Z_i \rightarrow N_i$  be the zero-sections. Let  $\alpha_i := 0_i^*[C_i] \in \mathbf{CH}_{k-d}(Z_i)$  and hence we have  $X \cdot V = \sum_{i=1}^r m_i \alpha_i \in \mathbf{CH}_{k-d}(W)$ .

Pick any closed set  $S \subset W$ , we define

$$(X \cdot V)^S := \sum_{Z_i \subset S} m_i \alpha_i \in \mathbf{CH}_{k-d}(S)$$

as the part of  $X \cdot V$  supported on  $S$ .

**Definition 2.10.** Let  $X_i \hookrightarrow Y$  be closed regular embeddings of codimension  $d_i$ . Let  $V \subset Y$  be a  $k$ -dimensional subvariety. Consider

$$\begin{array}{ccc} \bigcap_i X_i \cap V & \hookrightarrow & V \\ \downarrow & \lrcorner & \downarrow \delta \\ X_1 \times \dots \times X_r & \hookrightarrow & Y \times \dots \times Y \end{array}$$

Then we can get  $X_1 \cdot \dots \cdot X_r \cdot V \in \mathbf{CH}_{\dim V - \sum_i d_i}(\bigcap_i X_i \cap V)$ .

Let  $Z$  be a connected component of  $\bigcap_i X_i \cap V$ , we will consider

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z \in \mathbf{CH}_{\dim V - \sum_i d_i}(Z)$$

as before.

**Proposition 2.11.** As in the previous situation, we have

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cap s(Z, V) \right\}_{\dim V - \sum_i d_i}.$$

If  $Z \hookrightarrow V$  is a regular embedding, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y}|_Z) \cdot c(N_{Z/V})^{-1} \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

If  $V, Z$  are both non-singular, then

$$(X_1 \cdot \dots \cdot X_r \cdot V)^Z = \left\{ \prod_{i=1}^r c(N_{X_i/Y|Z}) c(T_V|_Z)^{-1} c(T_Z) \cap [Z] \right\}_{\dim V - \sum_i d_i}.$$

*Proof.* See Proposition 9.1.1 in [Ful98].  $\square$

## 2.2 Localized Chern Class

Here we will follow Chapter 14.1 of [Ful98]. This is the most important part which is the local case of the virtual fundamental class.

**Definition 2.12.** Let  $E \rightarrow X$  be a vector bundle of rank  $e$  over a purely  $n$ -dimensional scheme  $X$ . Let  $s : X \rightarrow E$  be a section, consider the cartesian

$$\begin{array}{ccc} Z(s) & \longrightarrow & X \\ i \downarrow & \lrcorner & s \downarrow \\ X & \xrightarrow{0} & E \end{array}$$

with zero-section  $0 : X \rightarrow E$  which is a regular section by trivial reason. We define

$$c_{\text{loc}}(E, s) := 0^!([X]) = 0^*(C_{Z(s)/X}) \in \text{CH}_{n-e}(Z(s))$$

be the localized (top) Chern class of  $E$  with respect to  $s$ .

**Proposition 2.13.** Consider the situation of Definition 2.12.

- (a) We have  $i_*(c_{\text{loc}}(E, s)) = c_e(E) \cap [X]$ .
- (b) Each irreducible component of  $Z(s)$  has codimension at most  $e$  in  $X$ . If  $\text{codim}_{Z(s)} X = e$ , then  $c_{\text{loc}}(E, s)$  is a positive cycle whose support is  $Z(s)$ .
- (c) If  $s$  is a regular section, then  $c_{\text{loc}}(E, s) = [Z(s)]$ .
- (d) Let  $f : X' \rightarrow X$  be a morphism,  $s' = f^*s$  be a induced section of  $f^*E$ . Let  $g : Z(s') \rightarrow Z(s)$  be the induced morphism.
  - (d1) If  $f$  flat, then  $g^*c_{\text{loc}}(E, s) = c_{\text{loc}}(f^*E, s')$ .
  - (d2) If  $f$  is proper of varieties, then  $g_*c_{\text{loc}}(f^*E, s') = \deg(X'/X)c_{\text{loc}}(E, s)$ .

*Proof.* For (a), by Proposition 2.4(a) and Corollary 2.5, we have

$$i_*0^!([X]) = 0^*s_*[X] = s^*s_*[X] = c_e(E) \cap [X].$$

For (b),(c), these follows from the trivial arguments of intersection multiplicities, see Lemma 7.1 and Proposition 7.1 in [Ful98]. Finally (d) follows from the following cartesians

$$\begin{array}{ccc}
Z(s') & \longrightarrow & X' \\
\downarrow & \lrcorner & \downarrow s' \\
X' & \xrightarrow{0_{f^*E}} & f^*E \\
\downarrow & \lrcorner & \downarrow \\
X & \xrightarrow{0_E} & E
\end{array}$$

and Proposition 2.4. □

### 3 A Brief of Cotangent Complexes

Here we will give a quike introduction of cotangent complexes. We will consider Deligne-Mumford stacks locally of finite type over  $k$ . Morphisms are quasicompact and quasiseparated. We work over étale site.

**Theorem 3.1.** *For every morphism  $f : X \rightarrow Y$  of DM-stacks (resp. finite type morphism of noetherian DM-stacks), there exists a complex*

$$\mathbb{L}_{X/Y} : \cdots \rightarrow \mathbb{L}_{X/Y}^{-1} \rightarrow \mathbb{L}_{X/Y}^0 \rightarrow 0$$

*of flat  $\mathcal{O}_X$ -modules with quasi-coherent (resp., coherent) cohomology, whose image  $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$  (resp.  $\mathbf{D}_{\text{Coh}}^-(X_{\text{ét}})$ ) is also denoted by  $\mathbb{L}_{X/Y}$ . This is called the **cotangent complex** of  $f$ . It satisfies the following properties.*

- (a)  $H^0(X, \mathbb{L}_{X/Y}) = \Omega_{X/Y}^1$ .
- (b) *The morphism  $f$  is smooth if and only if  $f$  is locally of finite presentation and  $\mathbb{L}_{X/Y}$  is a perfect complex supported in degree 0. In this case, there is a quasi-isomorphism  $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}^1[0]$ .*
- (c) *If  $f$  factors as  $X \hookrightarrow Z$  defined by a sheaf of ideals  $\mathcal{I}$  and a smooth morphism  $Z \rightarrow Y$ , then*

$$\mathbb{L}_{X/Y} \cong [0 \rightarrow \mathcal{I}/\mathcal{I}^2 \rightarrow \Omega_{Z/Y}^1|_X \rightarrow 0]$$

*in  $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$  with  $\Omega_{X/Y}^1$  in degree 0. If in addition  $f$  is generically smooth, then  $\mathbb{L}_{X/Y} \cong \Omega_{X/Y}^1[0]$ . Moreover, if  $f$  is lci, then  $\mathbb{L}_{X/Y}$  is perfect of perfect amplitude contained in  $[-1, 0]$ .*



(d) If we have a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

then there is a morphism  $(\mathbf{L}g')^* \mathbb{L}_{X/Y} \rightarrow \mathbb{L}_{X'/Y'}$ . When  $f$  or  $g$  is flat, then it is a quasi-isomorphism.

(e) If  $X \xrightarrow{f} Y \rightarrow Z$  is a composition of morphisms of DM-stacks, then there is an exact triangle

$$\mathbf{L}f^* \mathbb{L}_{Y/Z} \rightarrow \mathbb{L}_{X/Z} \rightarrow \mathbb{L}_{X/Y} \rightarrow \mathbf{L}f^* \mathbb{L}_{Y/Z}[1]$$

in  $\mathbf{D}_{\text{Qcoh}}^-(X_{\text{ét}})$ . This induces a long exact sequence on cohomology

$$\cdots \rightarrow H^{-1}(\mathbb{L}_{X/Z}) \rightarrow H^{-1}(\mathbb{L}_{X/Y}) \rightarrow f^* \Omega_{Y/Z}^1 \rightarrow \Omega_{X/Z}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0.$$

*Proof.* In the level of ring maps  $A \rightarrow B$ , this constructed by standard simplicial free  $A$ -resolution  $B \rightarrow P(B)_*$  where  $P(B)_n = A[\cdots [A[B]] \cdots]$  as

$$\mathbb{L}_{B/A} := \Omega_{P(B)_*/A} \otimes_{P(B)_*} B.$$

See Tag 08UV Tag 0D0N Tag 0FK3 Tag 08QQ Tag 08T4. □

**Remark 3.2.** For the general algebraic stacks, any quasicompact and quasi-separated 1-morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  there exists a relative cotangent complex

$$\mathbb{L}_f \in \mathbf{D}_{\text{Coh}}^{\leq 1}(\mathcal{X}_{\text{lis-ét}})$$

over lisse-étale site of  $\mathcal{X}$ . Existence is good, but the fact that the cotangent complex trespasses to positive degree forces one to pay more attention when performing the cutoff. If the diagonal of  $f$  is unramified (as we consider now), then this problem goes away, in the sense that  $\mathbb{L}_f \in \mathbf{D}_{\text{Coh}}^{\leq 0}(\mathcal{X}_{\text{lis-ét}})$ . We refer section C.3 in [Ric22] for more comments about this and the generalization of the properties as above.

## 4 Foundations of Virtual Fundamental Class

We will follow [BF97]. Here an algebraic stack (or Artin stack) over a field  $k$  is assumed to be quasi-separated and locally of finite type over  $k$ .

## 4.1 About Cones

We will let  $X$  be a Deligne-Mumford stack now.

**Definition 4.1.** *Let  $X$  be a DM-stack.*

- (a) *We call an affine  $X$ -scheme  $C = \underline{\text{Spec}}_X \mathcal{S}$  is a **cone over  $X$**  if the quasi-coherent algebra  $\mathcal{S}$  is graded as  $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$  with  $\mathcal{S}^0 = \mathcal{O}_X$  and  $\mathcal{S}^1$  is coherent and  $\mathcal{S}$  is generated by  $\mathcal{S}^1$ .*
- (b) *A **morphism of cones over  $X$**  is an  $X$ -morphism induced by a graded morphism of graded sheaves of  $\mathcal{O}_X$ -algebras. A **closed subcone** is the image of a closed immersion of cones.*

**Remark 4.2.** (a) *The fiber product of cones over  $X$  is still a cone over  $X$ .*

- (b) *For every cone  $C \rightarrow X$ , it has a zero section  $0 : X \rightarrow C$  induced by  $\mathcal{S} \rightarrow \mathcal{S}^0$ .*
- (c) *For every cone  $C \rightarrow X$ , the grade induce a  $\mathbb{G}_m$ -action  $\mathbb{G}_m \times C = \underline{\text{Spec}}_X \mathcal{S}[t, t^{-1}] \rightarrow C$  induced by  $\mathcal{S} \rightarrow \mathcal{S}[t, t^{-1}]$  via  $s_0 + \dots s_d \mapsto \sum_i a_i t^i$  where  $s_i \in \mathcal{S}^i$ . Since no negative power of  $t$  occurs, we can in fact replace  $\mathbb{G}_m$  by  $\mathbb{A}^1$ . So we have the  $\mathbb{A}^1$ -action  $\gamma : \mathbb{A}^1 \times C \rightarrow C$  induced by  $\mathcal{S} \rightarrow \mathcal{S}[x]$  via  $\mathcal{S}^i \ni s \mapsto sx^i$ . Note that here  $\mathbb{A}^1$  is not a group scheme and the **action** here, as expected, to be the commutativity of the following diagrams:*

$$\begin{array}{ccc}
 C & \xrightarrow{(1, \text{id})/(0, \text{id})} & \mathbb{A}^1 \times C \\
 & \searrow \text{id}/0 & \downarrow \gamma \\
 & & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{A}^1 \times \mathbb{A}^1 \times C & \xrightarrow{\text{id} \times \gamma} & \mathbb{A}^1 \times \mathbb{A}^1 \times C \\
 m \times \text{id} \downarrow & & \downarrow \gamma \\
 \mathbb{A}^1 \times C & \xrightarrow{\gamma} & C
 \end{array}$$

where  $m(x, y) = xy$ .

- (d) *So a morphism of cones  $f : C \rightarrow D$  over  $X$  is just the  $\mathbb{A}^1$ -equivariant  $X$ -morphism respecting the zero section, that is, the following commutativity of the diagram:*

$$\begin{array}{ccccc}
 \mathbb{A}^1 \times C & \longrightarrow & C & \xleftarrow{0_C} & X \\
 \text{id} \times f \downarrow & & f \downarrow & \nearrow 0_D & \\
 \mathbb{A}^1 \times D & \longrightarrow & D & & 
 \end{array}$$

**Definition 4.3.** *Let  $\mathcal{F}$  be a coherent sheaf of  $X$ , then we can define  $C(\mathcal{F}) := \underline{\text{Spec}}_X \text{Sym}(\mathcal{F})$  which is a group scheme over  $X$  since it can be represented as  $C(\mathcal{F})(T) = \text{Hom}(\mathcal{F}_T, \mathcal{O}_T)$ . We call a cone of this form is an **abelian cone over  $X$** .*

- Remark 4.4.** (a) A fibered product of abelian cones is an abelian cone.  
(b) A vector bundle  $E = \underline{\mathrm{Spec}}_X \mathrm{Sym}(\mathcal{E}^\vee)$  is a special case.  
(c) Any cone  $C = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$  is canonically a closed subcone of an abelian cone  $A(C) = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{S}^1$ , called the **abelian hull** of  $C$ . The abelian hull is a vector bundle if and only if  $\mathcal{S}^1$  is locally free.  
(d) The **abelianization**  $C \mapsto A(C)$  is a functor has the forgetful functor as a right adjoint. So we have

$$\mathrm{Hom}_{\mathbf{AbCone}_X}(A(C), A) \cong \mathrm{Hom}_{\mathbf{Cone}_X}(C, A).$$

- (e) Let  $\mathbf{Alg}_X^o$  as the category of quasicoherent graded  $\mathcal{O}_X$ -algebras satisfying the condition in the definition of cones. So we have the following commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{Alg}_X^o & \xrightarrow{\underline{\mathrm{Spec}}_X} & \mathbf{Cone}_X^{\mathrm{op}} \\ \mathrm{Sym} \uparrow & & \uparrow \\ \mathbf{LocFree}_X & \xrightarrow{\underline{\mathrm{Spec}}_X \mathrm{Sym}(-)^\vee} & \mathbf{Vect}_X^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{Coh}_X & \xrightarrow{\underline{\mathrm{Spec}}_X \mathrm{Sym}} & \mathbf{AbCone}_X^{\mathrm{op}} \end{array}$$

**Example 4.5.** Two important examples. Let  $X \hookrightarrow Y$  be a closed immersion of ideal  $\mathcal{I}$ . Then  $C_{X/Y} := \underline{\mathrm{Spec}}_X \bigoplus_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$  is called the **normal cone** of  $X$  in  $Y$ . The associated abelian cone  $N_{X/Y} = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{I} / \mathcal{I}^2$  is called the **normal sheaf** of  $X$  in  $Y$ .

**Lemma 4.6.** About smoothness:

- (a) Let  $C = \underline{\mathrm{Spec}}_X \mathcal{S}$  be a cone over  $X$ . Then  $C_{X/C} \cong \mathcal{S}^1 \cong 0^* \Omega_{C/X}$ .  
(b) A cone  $C$  over  $X$  is a vector bundle if and only if it is smooth over  $X$ .  
(c) Let  $C \rightarrow D$  be a smooth morphism of cones of relative dimension  $n$  over  $X$ . Then the induced morphism  $A(C) \rightarrow A(D)$  is also smooth of relative dimension  $n$ .

*Proof.* For (a), note that  $C_{X/C} \cong \mathcal{S}^1$  is trivial by definition. Moreover,  $0 : X \rightarrow C$  is the zero section and we have  $0 \rightarrow C_{X/C} \rightarrow 0^* \Omega_{C/X} \rightarrow \Omega_{X/X} = 0$  exact (see Tag 0474). Well done.

For (b), let  $C = \underline{\mathrm{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$  and assume that  $C \rightarrow X$  has constant relative dimension  $r$ . Then  $\mathcal{S}^1 = 0^* \Omega_{C/X}$  is locally free of rank  $r$ . As  $C \hookrightarrow A(C)$  where  $A(C)$  is a vector bundle and  $\dim C = \dim A(C)$ , we know that  $C$  is a vector bundle.

For (c), apply the exact triangle of cotangent complex to  $X \rightarrow C \rightarrow D$  and (a), we have an exact sequence

$$0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow 0_C^* \Omega_{C/D} \rightarrow 0$$

where  $C = \operatorname{Spec}_X \mathcal{S}$  and  $D = \operatorname{Spec}_X \mathcal{T}$ . So locally we have  $A(C) = A(D) \times_X \operatorname{Spec}_X \operatorname{Sym}(0_C^* \Omega_{C/D})$ . Well done.  $\square$

**Definition 4.7.** A sequence of cone morphisms

$$0 \rightarrow E \xrightarrow{i} C \rightarrow D \rightarrow 0$$

is called **exact** if  $E$  is a vector bundle and locally over  $X$  there is a morphism of cones  $C \rightarrow E$  splitting  $i$  and inducing an isomorphism  $C \cong E \times_X D$ .

**Remark 4.8.** As  $E \rightarrow X$  is smooth and surjective by Lemma 4.6, if  $0 \rightarrow E \xrightarrow{i} C \rightarrow D \rightarrow 0$  then locally we have  $C \cong E \times_X D$  which force that  $C \rightarrow D$  is smooth and surjective! Similarly  $i : E \rightarrow C$  is a closed embedding.

**Lemma 4.9.** We have the following useful results.

- (a) Given a short exact sequence  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow 0$  of coherent sheaves on  $X$ , with  $\mathcal{E}$  locally free, then  $0 \rightarrow C(\mathcal{E}) \rightarrow C(\mathcal{F}) \rightarrow C(\mathcal{F}') \rightarrow 0$  is exact, and conversely is also true.
- (b) Let  $0 \rightarrow E \rightarrow F \xrightarrow{f} G \rightarrow 0$  be an exact sequence of abelian cones over  $X$  with  $E$  a vector bundle. Assume that  $D \subset G$  is a closed subcone, then the induced sequence  $0 \rightarrow E \rightarrow f^{-1}(D) =: C \rightarrow D \rightarrow 0$  is exact.
- (c) Let  $f : C \rightarrow D$  be a morphisms of cones over  $X$  which is smooth surjective, then the induced diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow & & \downarrow \\ A(C) & \xrightarrow{A(f)} & A(D) \end{array}$$

is cartesian. Moreover, we have  $D = [C/E]$  (see Lemma 4.12(a)) and  $A(D) = [A(C)/E]$ , where  $E := C \times_{D,0} X = A(C) \times_{A(D),0} X$ .

- (d) Let  $E$  be a vector bundle over  $X$  and then the sequence  $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$  is exact if and only if the abelian hulls  $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$  is exact and  $C \rightarrow D$  is smooth and surjective.

*Proof.* For (a), we refer Example 4.1.6 and Example 4.1.7 in [Ful98]. As exactness is local, we may assume  $\mathcal{E}$  is free. Then the first sequence is exact

if and only if  $\mathcal{F}' \oplus \mathcal{E} = \mathcal{F}$  if and only if the second sequence is exact as cones, since  $\text{Sym}(\mathcal{F}' \oplus \mathcal{E}) = \text{Sym}(\mathcal{F}') \otimes \text{Sym}(\mathcal{E}) = \text{Sym}(\mathcal{F})$ .

For (b), note that this can be checked locally, so we can let we can assume that  $\mathcal{F} = \mathcal{G} \oplus \mathcal{E}^\vee$  where  $E = \underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$  and  $F = \underline{\text{Spec}}_X \text{Sym} \mathcal{F}$  and  $G = \underline{\text{Spec}}_X \text{Sym} \mathcal{G}$ . Let  $D = \underline{\text{Spec}}_X \mathcal{T}$ , then we have surjection  $\text{Sym}(\mathcal{G}) \rightarrow \mathcal{T}$ . By definition, we have

$$\begin{aligned} C = F \times_G D &= \underline{\text{Spec}}_X (\text{Sym}(\mathcal{F}) \otimes_{\text{Sym}(\mathcal{G})} \mathcal{T}) \\ &= \underline{\text{Spec}}_X ((\text{Sym}(\mathcal{G}) \otimes \text{Sym} \mathcal{E}^\vee) \otimes_{\text{Sym}(\mathcal{G})} \mathcal{T}) \\ &= \underline{\text{Spec}}_X (\text{Sym} \mathcal{E}^\vee \otimes \mathcal{T}). \end{aligned}$$

This means locally  $C = E \oplus D$  and the splitting  $C \rightarrow E$  is induced by  $F \rightarrow E$ . Well done.

For (c), let  $E := C \times_{D,0} X$  and  $E' := A(C) \times_{A(D)} D$  with embedding  $E \hookrightarrow E'$ , then both of them are vector bundles by Lemma 4.6(b)(c) and hence  $E = E'$ . We have cartesians

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow \scriptstyle \ulcorner & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \begin{array}{ccc} E & \longrightarrow & X \\ \downarrow \scriptstyle \ulcorner & & \downarrow \\ A(C) & \longrightarrow & A(D) \end{array}$$

By the properties of commutative affine group schemes, we have  $A(D) = [A(C)/E]$ . But how about  $[C/E]$ ? Now we have

$$\begin{array}{ccccc} & & & & D \\ & & \nearrow & & \\ C & \xrightarrow{\quad} & [C/E] & \xrightarrow{\quad} & \\ \downarrow \scriptstyle \ulcorner & & \downarrow & \nearrow & \\ A(C) & \longrightarrow & A(D) & & \end{array}$$

Since  $C \rightarrow [C/E]$  and  $C \rightarrow D$  are both smooth and surjective, we know that  $[C/E] \rightarrow D$  is flat and surjective. But by closed embeddings  $[C/E] \rightarrow A(D)$  and  $D \rightarrow A(D)$ , we know that  $[C/E] \rightarrow D$  is also a closed embedding. Thus  $D = [C/E]$ , well done.

For (d), note that all the question is locally on  $X$ . First we assume  $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D \rightarrow 0$  is exact. Then by (a), to show that  $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$  is exact, we only need to show that  $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$  is exact where  $E = \underline{\text{Spec}}_X \text{Sym} \mathcal{E}^\vee$  and  $C = \underline{\text{Spec}}_X \mathcal{T}$  and  $D = \underline{\text{Spec}}_X \mathcal{T}$ . First since  $f$  is faithfully flat and quasi-compact, we know that  $\mathcal{T}^1 \rightarrow \mathcal{T}^1$  is injective. And since  $i$  is a closed embedding,  $\mathcal{T}^1 \rightarrow \mathcal{E}^\vee$  is surjective. Now

by local splitting, we know that locally we have  $\text{Sym}(E^\vee) \otimes \mathcal{T} = \mathcal{S}$ . In particular, we have  $\mathcal{T}^1 \oplus \mathcal{E}^\vee = \mathcal{S}^1$ . Thus the exactness of  $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$  is obtained. Conversely we assume that after taking abelian hull, the sequence is exact. Now the result follows from (a) and (c).  $\square$

**Proposition 4.10.** *Let  $C \rightarrow D$  be a smooth, surjective morphism of cones. If we let  $E = C \times_{D,0} X$ , then the sequence*

$$0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$$

*is exact. Conversely if  $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$  is exact, then  $E \cong C \times_{D,0} X$ .*

*Proof.* Let  $C = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{S}^i$  and  $D = \underline{\text{Spec}}_X \bigoplus_{i \geq 0} \mathcal{T}^i$ .

Let  $E = C \times_{D,0} X = \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$ , by Lemma 4.9(d) we just need to show that  $0 \rightarrow E \rightarrow A(C) \rightarrow A(D) \rightarrow 0$  is exact, that is,  $0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^1 \rightarrow \mathcal{E}^\vee \rightarrow 0$  is exact by Lemma 4.9(a). Note that  $\text{Sym } \mathcal{E}^\vee = \mathcal{S} \otimes_{\mathcal{T}} (\mathcal{T} / \mathcal{T}^{\geq 1})$  which force  $\mathcal{E}^\vee \cong \mathcal{S}^1 / \mathcal{T}^1$ . Well done.

Conversely, assume that the sequence  $0 \rightarrow E \rightarrow C \rightarrow D \rightarrow 0$  is exact and  $F = C \times_{D,0} X$ . Then by the universal property of fibre product, we get a morphism  $E \rightarrow F$ . From the construction, it is easy to see that  $\mathcal{F}^\vee \rightarrow \mathcal{E}^\vee$  is surjective. Since they are both bundles of the same rank over  $X$ , we know that  $E = F$ .  $\square$

**Definition 4.11.** (a) *If  $E$  is a vector bundle and  $f : E \rightarrow C(\mathcal{F})$  a morphism of abelian cones. Then there is an  $E$ -action as  $E \times_X C(\mathcal{F}) \rightarrow C(\mathcal{F})$  as  $(\nu, \gamma) \mapsto f\nu + \gamma$ .*

(b) *If  $E$  is a vector bundle and  $d : E \rightarrow C$  a morphism of cones, we say that  $C$  is an  $E$ -cone, if  $C$  is invariant under the action of  $E$  on  $A(C)$ .*

(c) *A morphism  $\phi$  from an  $E$ -cone  $C$  to an  $F$ -cone  $D$  is a commutative diagram of cones*

$$\begin{array}{ccc} E & \xrightarrow{d_E} & C \\ \downarrow \phi & & \downarrow \phi \\ F & \xrightarrow{d_F} & D \end{array}$$

(d) *If  $\phi : (E, d_E, C) \rightarrow (F, d_F, D)$  and  $\psi : (E, d_E, C) \rightarrow (F, d_F, D)$  are morphisms, we call them homotopic, if there exists a morphism of cones  $k : C \rightarrow F$ , such that  $kd_E = \psi - \phi = d_F k$ .*

**Lemma 4.12.** *Some useful lemmas:*

(a) *Let  $f : C \rightarrow D$  be a smooth surjective cone morphism with  $E = C \times_{D,0} X$ , then  $C$  is an  $E$ -cone.*

- (b) Let  $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D = [C/E] \rightarrow 0$  be a sequence of algebraic  $X$ -spaces with  $E$  a bundle,  $C$  is a  $E$ -cone,  $i$  a closed embedding and  $f : C \rightarrow D = [C/E]$  is the universal family. Then locally on  $X$ , there is a  $j : C \rightarrow E$  split  $i$  and induces an isomorphism  $(f, j) : C \rightarrow D \times_X E$ .
- (c) Let  $0 \rightarrow E \xrightarrow{i} C \xrightarrow{f} D \rightarrow 0$  be a sequence of algebraic  $X$ -spaces with sections and  $\mathbb{A}^1$ -actions such that  $E$  a bundle,  $C$  is a  $E$ -cone,  $i$  is a closed embedding and  $f$  is  $\mathbb{A}^1$ -equivariant. Then  $D$  is a cone with the sequence exact if and only if  $D \cong [C/E]$ .

*Proof.* For (a), this follows from directly check. We omit it.

For (b), since the question is local we can assume that  $E$  is a trivial bundle and  $X$  is a scheme. Let  $i' : E \rightarrow A(C)$  and  $C = \underline{\mathrm{Spec}}_X \mathcal{S}$  and  $E = \underline{\mathrm{Spec}}_X \mathrm{Sym} \mathcal{E}^\vee$ . Then the surjection  $\mathcal{S}^1 \rightarrow \mathcal{E}^\vee$  has a splitting  $\mathcal{E}^\vee \hookrightarrow \mathcal{S}^1$ , which gives  $j' : A(C) \rightarrow E$  such that  $j' \circ i' = \mathrm{id}_E$ . Then we just define  $j : C \rightarrow E$  as composition with  $C \rightarrow A(C)$  and  $j'$ . Hence  $j \circ i = \mathrm{id}_E$ .

Now since  $C \rightarrow D$  is also a principal  $E$ -bundle, and we have a  $E$ -equivariant  $D$ -morphism  $(f, j) : C \rightarrow D \oplus E$  from  $C$  to the trivial principal bundle. Since they are both  $E$ -principal bundle, we know that  $(f, j)$  is an isomorphism.

For (c), let  $D = [C/E]$ . We know that  $D \rightarrow X$  is affine since locally on  $X$  we have  $C \cong D \times_X E \rightarrow E$  is affine and (b) and faithfully flat descent. By construction we have  $E = C \times_{D,0} X$ , hence by Proposition 4.10 we just need to show  $D$  is a cone. Now as  $D \rightarrow X$  affine we have  $D = \underline{\mathrm{Spec}}_X \mathcal{T}$ . If  $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ , then  $\mathcal{T} \subset \mathcal{S}$  as  $C \rightarrow D$  is faithfully flat. Hence it has graded structure  $\mathcal{T} = \bigoplus_{i \geq 0} \mathcal{T} \cap \mathcal{S}^i$  as  $f$  is  $\mathbb{A}^1$ -equivariant. As it have zero section, we have  $\mathcal{T}^0 = \mathcal{O}_X$ . Finally we have  $\mathbb{A}^1$ -equivariant embedding  $D \hookrightarrow [A(C)/E]$  and  $[A(C)/E]$  is a cone by Lemma 4.9(c). Hence  $\mathcal{T}$  generated by the coherent sheaf  $\mathcal{T}^1$ .

Conversely, we assume  $D$  is a cone and that sequence is exact. Let  $D' = [C/E]$ . By the universal property of quotient, we have a natural map  $g : D' \rightarrow D$ . Since  $0 \rightarrow E \rightarrow C \rightarrow D' \rightarrow 0$  is also exact by the first case, by exactness we have locally  $C \cong E \times_X D \cong E \times_X D'$ . Note that these isomorphisms compatible with  $g : D' \rightarrow D$ , hence by faithfully flat descent we have  $g$  is an isomorphism.  $\square$

**Proposition 4.13.** *Let  $X$  be a DM-stack.*

- (a) Let  $E$  be a vector bundle. Consider the sequence of cone morphisms  $0 \rightarrow E \xrightarrow{i} C \xrightarrow{\phi} D \rightarrow 0$  with  $i$  a closed embedding. Then it is exact if

and only if  $C$  is a  $E$ -cone,  $\phi : C \rightarrow D$  is faithfully flat and the diagram

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ \downarrow p & \ulcorner & \downarrow \phi \\ C & \xrightarrow{\phi} & D \end{array}$$

is cartesian with projection  $p$  and action  $\sigma$ .

- (b) Let  $(C, 0, \gamma)$  and  $(D, 0, \gamma)$  be algebraic  $X$ -spaces with sections and  $\mathbb{A}^1$ -actions and let  $\phi : C \rightarrow D$  be an  $\mathbb{A}^1$ -equivariant  $X$ -morphism, which is smooth and surjective. Let  $E = C \times_{D,0} X$ . Assume that  $E$  is a vector bundle. Then  $C$  is an  $E$ -cone (resp. abelian cone, vector bundle) over  $X$  if and only if  $D$  is a cone (resp. abelian cone, vector bundle) over  $X$  and  $C$  is affine over  $X$ .

*Proof.* For (a), if it is exact, locally we have  $C \cong E \times_X D$ . So  $E$  act on  $C$  locally as  $E \times E \times_X D \rightarrow E \times_X D$  given by  $(f, (e, d)) \mapsto (i(f) + e, d)$ . So  $C$  is a  $E$ -cone. Now  $\phi : C \rightarrow D$  is trivially faithfully flat. The cartesian diagram follows from Lemma 4.12(c).

Conversely, since  $\phi$  is fppf, this diagram is also cocartesian by Proposition V.1.3.1 in [Li18] which force  $D = [C/E]$ . Hence the results follows from Lemma 4.12(c).

For (b), let  $C$  is an  $E$ -cone over  $X$ . Then we have  $g : [C/E] \rightarrow D$ . We claim that  $g$  is an isomorphism. Indeed, by the diagram in (a), we know that  $g$  induces an isomorphism  $g' : E \times_X C = C \times_{[C/E]} C \rightarrow C \times_D C$ . Note that we have a cartesian diagram:

$$\begin{array}{ccc} C \times_{[C/E]} C & \longrightarrow & C \times_D C \\ \downarrow & \ulcorner & \downarrow \\ [C/E] & \hookrightarrow & [C/E] \times_D [C/E] \end{array}$$

where  $C \times_D C \rightarrow [C/E] \times_D [C/E]$  is faithfully flat, hence  $[C/E] \hookrightarrow [C/E] \times_D [C/E]$  is an isomorphism. So  $g$  is a monomorphism. But since  $C \rightarrow [C/E]$  and  $C \rightarrow D$  are faithfully flat, hence epimorphism. Thus  $g$  is also an epimorphism, hence an isomorphism. Thus  $D \cong [C/E]$  and the result follows from Lemma 4.12(c).

Now assume that  $C = A(C)$  is an abelian cone, then taking hull to  $0 \rightarrow E \rightarrow C \rightarrow D = [C/E] \rightarrow 0$ . By Lemma 4.9(c)(d) we have  $A(D) = [A(C)/E] = [C/E] = D$ . Hence  $D$  is also an abelian cone.

Finally assume that  $C$  is a bundle. Then by the previous case we know that  $D$  is an abelian cone. The  $\mathcal{T}^1 = \ker(\mathcal{S}^1 \twoheadrightarrow \mathcal{E}^\vee)$  is clearly locally free since  $\mathcal{C}^1$  and  $\mathcal{E}$  are where  $C = \underline{\text{Spec}}_X \mathcal{S}$ ,  $D = \underline{\text{Spec}}_X \mathcal{T}$  and  $E = \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$ .



Conversely we let  $D$  is a cone and  $C$  is affine over  $X$ . Hence we have  $C = \underline{\text{Spec}}_X \mathcal{S}$  where  $\mathcal{S} = \bigoplus_{i \geq 0} \mathcal{S}^i$  and  $\mathcal{S}^1 = \mathcal{O}_X$ . By the same reason  $E$  is affine over  $X$ . Hence we have  $C = \underline{\text{Spec}}_X \mathcal{F}$  where  $\mathcal{F} = \bigoplus_{i \geq 0} \mathcal{F}^i$  and  $\mathcal{F}^1 = \mathcal{O}_X$ . If we let  $D = \underline{\text{Spec}}_X \mathcal{T}$ , then  $\mathcal{F} = \mathcal{S}/(\mathcal{T}^{\geq 1} \mathcal{S})$ .

Apply the exact triangle of cotangent complex to  $X \xrightarrow{0\zeta} C \rightarrow D$ , we have an exact sequence

$$0 \rightarrow \mathcal{T}^1 \rightarrow \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 = C_{X/C} \rightarrow \mathcal{E}^\vee := 0_C^* \Omega_{C/D} \rightarrow 0.$$

As  $\mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 = \mathcal{S}^1 \oplus \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2$ , we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{S}^1 & \longrightarrow & \mathcal{F}^1 \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{T}^1 & \longrightarrow & \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2 & \longrightarrow & \mathcal{E}^\vee \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2 & \xrightarrow{=} & \mathcal{E}^\vee/\mathcal{F} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

Locally on  $X$  we can assume that  $\mathcal{E}$  is free and  $\mathcal{T}^1 \oplus \mathcal{E}^\vee = \mathcal{S}^{\geq 1}/(\mathcal{S}^{\geq 1})^2$ . Then as  $\mathcal{F}^1 \subset \mathcal{E}^\vee$ , we know that  $\mathcal{F}^1$ . Since  $\mathcal{T}^1$  is also coherent, we know that so is  $\mathcal{S}^1$ . Finally we just need to show  $\mathcal{S}$  generated by  $\mathcal{S}^1$  as by Lemma 4.12(a) here  $C$  will be an  $E$ -cone.

Then locally on  $X$  we can choose generators of  $\mathcal{T}^1, \mathcal{F}^1, \mathcal{E}^\vee/\mathcal{F}^1 = \mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2$  such that gives a surjective  $\mathcal{O}_X$ -algebra morphism  $\phi : \mathcal{T} \oplus \text{Sym } \mathcal{E}^\vee \twoheadrightarrow \mathcal{S}$  which induce  $\mathcal{T} \oplus \text{Sym } \mathcal{F}^1 \rightarrow \mathcal{T} \oplus \text{Sym } \mathcal{E}^\vee \twoheadrightarrow \mathcal{S}$  is graded. Tensoring  $(-) \otimes_{\mathcal{T}} \mathcal{O}_X$  with  $\phi$  we get surjection  $\phi' : \text{Sym } \mathcal{E}^\vee \twoheadrightarrow \mathcal{F}$ . This induce the closed immersion  $E \hookrightarrow \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$ . Since they are both smooth of a same relative dimension over  $X$  and  $\underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$  is a vector bundle, hence  $E \cong \underline{\text{Spec}}_X \text{Sym } \mathcal{E}^\vee$  and  $\phi'$  is an isomorphism. Hence  $\mathcal{F} = \text{Sym}(\mathcal{F}^1)$  and  $\mathcal{F}^1$  is locally free. As  $\text{Sym}(\mathcal{F}^1) \subset \text{Sym } \mathcal{E}^\vee \xrightarrow{\phi'} \mathcal{F} = \text{Sym}(\mathcal{F}^1)$  is identity, this force  $\mathcal{E}^\vee = \mathcal{F}^1$ . As this can be check locally, we have  $\mathcal{E}^\vee = \mathcal{F}^1$  in whole  $X$ . By the diagram above, we have  $\mathcal{S}^{\geq 2}/(\mathcal{S}^{\geq 1})^2 = \mathcal{E}^\vee/\mathcal{F}^1 = 0$ . This means  $\mathcal{S}$  generated by  $\mathcal{S}^1$ . Well done.  $\square$

**Remark 4.14.** In the original paper [BF97] they claim (a) is enough for the surjectivity of  $f$ .

## 4.2 Cone Stack

Let  $X$  be a Deligne-Mumford stack.

**Definition 4.15.** Let  $\mathfrak{C}$  be an algebraic stack over  $X$ , together with a section  $0 : X \rightarrow \mathfrak{C}$ . An  $\mathbb{A}^1$ -action on  $(\mathfrak{C}, 0)$  is given by a morphism of  $X$ -stacks  $\gamma : \mathbb{A}^1 \times \mathfrak{C} \rightarrow \mathfrak{C}$  and three 2-isomorphisms  $\theta_1, \theta_0$  and  $\theta_\gamma$  between the 1-morphisms in the following diagrams.

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{(1, \text{id})/(0, \text{id})} & \mathbb{A}^1 \times \mathfrak{C} \\ & \searrow \text{id}/0 \quad \swarrow \gamma & \\ & \mathfrak{C} & \end{array}$$

$\text{id}/0 \quad \text{---} \theta_1/\theta_0 \quad \text{---} \gamma$

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\text{id} \times \gamma} & \mathbb{A}^1 \times \mathfrak{C} \\ \downarrow m \times \text{id} & \searrow \theta_\gamma & \downarrow \gamma \\ \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\gamma} & \mathfrak{C} \end{array}$$

The 2-isomorphisms  $\theta_1, \theta_0$  and  $\theta_\gamma$  are required to satisfy certain compatibilities.

**Definition 4.16.** Let  $(\mathfrak{C}, 0, \gamma)$  and  $(\mathfrak{D}, 0, \gamma)$  be  $X$ -stacks with sections and  $\mathbb{A}^1$ -actions. Then an  $\mathbb{A}^1$ -equivariant morphism  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a triple  $(\phi, \eta_0, \eta_\gamma)$ , where  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a morphism of algebraic  $X$ -stacks and  $\eta_0$  and  $\eta_\gamma$  are 2-isomorphisms between the morphisms in the following diagrams.

$$\begin{array}{ccc} X & \xrightarrow{0} & \mathfrak{C} \\ & \searrow \eta_0 & \downarrow \phi \\ & & \mathfrak{D} \end{array}$$

$0 \quad \text{---} \eta_\gamma \quad \text{---} \phi$

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathfrak{C} & \xrightarrow{\text{id} \times \phi} & \mathbb{A}^1 \times \mathfrak{D} \\ \downarrow \gamma & \searrow \eta_\gamma & \downarrow \gamma \\ \mathfrak{C} & \xrightarrow{\phi} & \mathfrak{D} \end{array}$$

Again, the 2-isomorphisms have to satisfy certain compatibilities.

**Definition 4.17.** Let  $(\phi, \eta_0, \eta_\gamma) : \mathfrak{C} \rightarrow \mathfrak{D}$  and  $(\psi, \eta'_0, \eta'_\gamma) : \mathfrak{C} \rightarrow \mathfrak{D}$  be two  $\mathbb{A}^1$ -equivariant morphisms. An  $\mathbb{A}^1$ -equivariant isomorphism  $\zeta : \phi \rightarrow \psi$  is a 2-isomorphism  $\zeta : \phi \rightarrow \psi$  such that the diagrams

$$\begin{array}{ccc} 0 & \xrightarrow{\eta_0} & \phi \circ 0 \\ & \searrow \eta'_0 & \downarrow \zeta \circ 0 \\ & & \psi \circ 0 \end{array} \quad \begin{array}{ccc} \phi \circ \gamma & \xrightarrow{\quad} & \gamma \circ (\text{id} \times \phi) \\ \downarrow \zeta \circ \gamma & & \downarrow \gamma \circ (\text{id} \times \zeta) \\ \psi \circ \gamma & \xrightarrow{\quad} & \gamma \circ (\text{id} \times \psi) \end{array}$$

commute.

**Example 4.18.** Let  $C$  be a  $E$ -cone, then consider the quotient stack  $[C/E]$ . We claim that  $[C/E]$  a zero section and an  $\mathbb{A}^1$ -action.

Indeed, the zero section  $0 : X \rightarrow [C/E]$  given by  $X \leftarrow E \rightarrow C$ . The  $\mathbb{A}^1$ -action of  $\alpha \in \mathbb{A}^1(T)$  on  $(P, f) \in [C/E](T)$  defined by  $(\alpha P, \alpha f)$  where  $\alpha P = P \times^{E, \alpha} E$  and  $\alpha f : P \times^{E, \alpha} E \rightarrow C$  given by  $[p, v] \mapsto \alpha f(p) + d(v)$  where  $d : E \rightarrow C$ .

Moreover, if  $\phi : (E, C) \rightarrow (F, D)$  is a morphism of vector bundle cones we get an induced  $\mathbb{A}^1$ -equivariant morphism  $\tilde{\phi} : [C/E] \rightarrow [D/F]$ .

**Lemma 4.19.** Some usrful results.

- (a) A homotopy  $k : \phi \rightarrow \psi$  of two morphisms of vector bundle cones  $\phi, \psi : (E, C) \rightarrow (F, D)$  gives rise to an  $\mathbb{A}^1$ -equivariant 2-isomorphism  $\tilde{k} : \tilde{\phi} \rightarrow \tilde{\psi}$  of  $\mathbb{A}^1$ -equivariant morphisms of stacks with  $\mathbb{A}^1$ -action.
- (b) Conversely, let two morphisms of vector bundle cones  $\phi, \psi : (E, C) \rightarrow (F, D)$  with an  $\mathbb{A}^1$ -equivariant 2-isomorphism  $\zeta : \tilde{\phi} \rightarrow \tilde{\psi}$  of  $\mathbb{A}^1$ -equivariant morphisms of stacks with  $\mathbb{A}^1$ -action. Then  $\zeta = k$  for unique homotopy  $k : \phi \rightarrow \psi$ .

*Proof.* For (a), samilar to Proposition 4.29. For (b) TBC...  $\square$

**Proposition 4.20.** Let  $C$  be an  $E$ -cone and  $D$  an  $F$ -cone and let  $\phi : (E, C) \rightarrow (F, D)$  be a morphism. If the diagram

$$\begin{array}{ccc} E & \longrightarrow & C \\ \downarrow & \ulcorner & \downarrow \phi \\ F & \xrightarrow{d} & D \end{array}$$

is cartesian and  $F \times_X C \rightarrow D$  by  $(\mu, \gamma) \mapsto d\mu + \phi(\gamma)$  is surjective, then  $[C/E] \rightarrow [D/F]$  is an isomorphism of algebraic  $X$ -stacks with  $\mathbb{A}^1$ -action.

*Proof.* For the same proof of Proposition 4.30.  $\square$

**Definition 4.21.** (a) We call an algebraic stack  $(\mathfrak{C}, 0, \gamma)$  over  $X$  with section and  $\mathbb{A}^1$ -action a **cone stack**, if, étale locally on  $X$ , there exists a cone  $C$  over  $X$  and an  $\mathbb{A}^1$ -equivariant morphism  $C \rightarrow \mathfrak{C}$  that is smooth and surjective and such that  $E = C \times_{\mathfrak{C}, 0} X$  is a vector bundle over  $X$ .

- (b) The morphism  $C \rightarrow \mathfrak{C}$  is called a **local presentation** of  $\mathfrak{C}$ . The section  $0 : X \rightarrow \mathfrak{C}$  is called the **vertex** of  $\mathfrak{C}$ .
- (c) Let  $\mathfrak{C}$  and  $\mathfrak{D}$  be cone stacks over  $X$ . A **morphism of cone stacks**  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is an  $\mathbb{A}^1$ -equivariant morphism of algebraic  $X$ -stacks. A 2-isomorphism of cone stacks is just an  $\mathbb{A}^1$ -equivariant 2-isomorphism.

- (d) A cone stack  $\mathfrak{C}$  over  $X$  is called **abelian cone stack** (resp. **vector bundle stack**), if, locally in  $X$ , one can find presentations  $C \rightarrow \mathfrak{C}$ , where  $C$  is an abelian cone (resp. vector bundle).

**Remark 4.22.** Some basic properties of cone stacks.

- (a) If  $C \rightarrow \mathfrak{C}$  is a global presentation with  $E = C \times_{\mathfrak{C},0} X$ , then  $C$  is an  $E$ -cone with  $\mathfrak{C} \cong [C/E]$  as stacks with  $\mathbb{A}^1$ -action. This follows from Proposition 4.10 and 4.13 and Lemma 4.12.
- (b) If  $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$  is a morphism of cone stacks, then, étale locally on  $X$ ,  $\phi$  is  $\mathbb{A}^1$ -equivariantly isomorphic to  $[C/E] \rightarrow [D/F]$ , where  $E \rightarrow F$  is a morphism of vector bundles over  $X$  and  $C \rightarrow D$  is a morphism from the  $E$ -cone  $C$  to the  $F$ -cone  $D$ .
- (c) A 2-isomorphism of cone stacks  $\zeta : \phi \rightarrow \psi$ , where  $\phi, \psi : \mathfrak{C} \rightarrow \mathfrak{D}$ , is étale locally over  $X$  given by a homotopy of morphisms of vector bundle cones. This follows from Lemma 4.19(b).
- (d) Let  $C \rightarrow \mathfrak{C}$  and  $D \rightarrow \mathfrak{D}$  be two local presentation of a cone stack  $\mathfrak{C}$  over  $X$ , then so is  $C \times_{\mathfrak{C}} D \rightarrow \mathfrak{C}$ .

Indeed, we only need to show that  $C \times_{\mathfrak{C}} D$  is a cone. Since  $C \rightarrow \mathfrak{C}$  and  $D \rightarrow \mathfrak{D}$  are affine, we know that  $C \times_{\mathfrak{C}} D \rightarrow D \rightarrow X$  is also affine. Then  $C \times_{\mathfrak{C}} D$  is a cone by Proposition 4.13(b) and the result follows.

- (e) Every fibered product of cone stacks is a cone stack.
- (f) If  $\mathfrak{C}$  is a representable cone stack over  $X$ , then it is a cone.

Indeed, locally on  $X$ ,  $\mathfrak{C} \rightarrow X$  is  $\mathbb{A}^1$ -isomorphic to a cone. In particular, as  $\mathfrak{C} \rightarrow X$  is representable, it is affine. Then we assume that  $C = \underline{\mathrm{Spec}}_X \mathcal{S}$ . Since there is a non-trivial  $\mathbb{A}^1$ -action on  $C$  and has a section, we know that  $\mathcal{S}$  is a graded algebra with  $\mathcal{S}^0 = \mathcal{O}_X$ . To show  $C$  is a cone, we only need to show that  $\mathcal{S}^1$  is coherent and  $\mathcal{S}$  is locally generated by  $\mathcal{S}^1$ . These are both local property, then they hold since locally  $\mathfrak{C} \rightarrow X$  is  $\mathbb{A}^1$ -isomorphic to a cone.

- (g) If  $\mathfrak{C}$  is abelian (a vector bundle stack), then for every local presentation  $C \rightarrow \mathfrak{C}$  the cone  $C$  will be abelian (a vector bundle).

**Example 4.23.** Note that all cones are cone stacks and all morphisms of cones are morphisms of cone stacks. For a vector bundle  $E$  on  $X$ , the classifying stack  $\mathbf{B}_X E$  is a cone stack. Every homomorphism of vector bundles  $\phi : E \rightarrow F$  gives rise to a morphism of cone stacks.

**Proposition 4.24.** Every cone stack is a closed subcone stack of an abelian cone stack. There exists a universal such abelian cone stack. It is called the **abelian hull**.

*Proof.* Just glue the stacks obtained from the abelian hulls of local presentations.  $\square$

**Definition 4.25.** (a) Let  $\mathfrak{E}$  be a vector bundle stack and  $\mathfrak{E} \rightarrow \mathfrak{C}$  a morphism of cone stacks. We say that  $\mathfrak{C}$  is an  $\mathfrak{E}$ -cone stack, if  $\mathfrak{E} \rightarrow \mathfrak{C}$  is locally isomorphic (as a morphism of cone stacks) to the morphism  $[E_1/E_0] \rightarrow [C/F]$  coming from a commutative diagram

$$\begin{array}{ccc} E_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & C \end{array}$$

where  $C$  is both  $E_1$ - and  $F$ -cone. The natural action  $\mathfrak{E} \times_X \mathfrak{C} \rightarrow \mathfrak{C}$  induced by  $E_1 \times C \rightarrow C$ .

(b) Let  $\mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D}$  be a sequence of morphisms of cone stacks where  $\mathfrak{C}$  is an  $\mathfrak{E}$ -cone stack. If

(b1)  $\mathfrak{C} \rightarrow \mathfrak{D}$  is a smooth epimorphism.

(b2) The diagram

$$\begin{array}{ccc} \mathfrak{E} \times_X \mathfrak{C} & \xrightarrow{\sigma} & \mathfrak{C} \\ p \downarrow & \ulcorner & \downarrow \\ \mathfrak{C} & \longrightarrow & \mathfrak{D} \end{array}$$

is cartesian where  $\sigma$  is action and  $p$  is projection.

Then we call  $0 \rightarrow \mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$  is a **short exact sequence of cone stacks**. As before, this is equivalent to  $\mathfrak{C}$  being locally isomorphic to  $\mathfrak{E} \times_X \mathfrak{D}$ .

**Proposition 4.26.** The sequence  $0 \rightarrow \mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D} \rightarrow 0$  of morphisms of cone stacks is exact if and only if locally in  $X$  there exist commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_0 & \longrightarrow & F & \longrightarrow & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & D \longrightarrow 0 \end{array}$$

where the top row is a short exact sequence of vector bundles and the bottom row is a short exact sequence of cones, such that  $\mathfrak{E} \rightarrow \mathfrak{C} \rightarrow \mathfrak{D}$  is isomorphic to  $[E_1/E_0] \rightarrow [C/F] \rightarrow [D/G]$ .

*Proof.* The statement is local on  $X$ . To prove the only if part we can assume  $\mathfrak{C} = \mathfrak{E} \times_X \mathfrak{D}$ , and then it is trivial. To prove the if part, note that both short exact sequences are locally split.  $\square$

### 4.3 A Picard Stack of Special Type

#### General Theory

First we will consider the case of complex of two terms.

**Definition 4.27.** *Let  $X$  be a topos.*

- (a) *Let  $d : E^0 \rightarrow E^1$  a homomorphism of abelian sheaves on  $X$ , which we shall consider as a complex of abelian sheaves on  $X$ . Via  $d$ , the abelian sheaf  $E^0$  acts on  $E^1$  and we may consider the quotient stack of this action, denoted*

$$\mathcal{H}^0/\mathcal{H}^1(E^\bullet) := [E^0/E^1]$$

*which is a Picard stack over  $X$ .*

- (b) *Now if  $d : F^0 \rightarrow F^1$  is another homomorphism of abelian sheaves on  $X$  and  $\phi : E^\bullet \rightarrow F^\bullet$  is a homomorphism of complexes, then we get an induced morphism of Picard stacks*

$$\mathcal{H}^0/\mathcal{H}^1(\phi) : \mathcal{H}^0/\mathcal{H}^1(E^\bullet) \rightarrow \mathcal{H}^0/\mathcal{H}^1(F^\bullet)$$

*given by  $(P, f) \mapsto (P \times^{E^0, \phi^0} F^0, \phi^1(f))$  where  $\phi^1(f)$  is the map*

$$\phi^1(f) : P \times^{E^0, \phi^0} F^0 \rightarrow F^1, \quad [p, \nu] \mapsto \phi^1(f(p) + d(\nu)).$$

- (c) *Now, if  $\psi : E^\bullet \rightarrow F^\bullet$  is another homomorphism of complexes, then the homotopy  $k : \phi \rightarrow \psi$  is a homomorphism of abelian sheaves  $k : E^1 \rightarrow F^0$ , such that  $kd = \psi^0 - \phi^0$  and  $dk = \psi^1 - \phi^1$ .*

**Remark 4.28.** *Note that roughly speaking, a Picard stack is a stack together with an ‘addition’ operation, that is both associative and commutative. For the precise definition of Picard stack see Sect. 1.4 of Exposé XVIII in [AGV73].*

*Here the quotient stack is similar as before: the groupoid  $\mathcal{H}^0/\mathcal{H}^1(E^\bullet)(U)$  is the category of pairs  $(P, f)$ , where  $P$  is an  $E^0$ -torsor over  $U$  and  $f : P \rightarrow E^1|_U$  is an  $E^0$ -equivariant morphism of sheaves on  $U$ .*

**Proposition 4.29.** *As in the condition of definition, if we have a homotopy  $k : \phi \rightarrow \psi$ , then this can induce isomorphism  $\theta : \mathcal{H}^0/\mathcal{H}^1(\phi) \rightarrow \mathcal{H}^0/\mathcal{H}^1(\psi)$  of morphisms of Picard stacks from  $\mathcal{H}^0/\mathcal{H}^1(E^\bullet)$  to  $\mathcal{H}^0/\mathcal{H}^1(F^\bullet)$ .*

*Proof.* Pick object  $U \in \text{ob}(X)$  and  $(P, f) \in \mathcal{H}^0/\mathcal{H}^1(E^\bullet)(U)$ , then  $\theta(U)(P, f) : \mathcal{H}^0/\mathcal{H}^1(\phi)(U)(P, f) \rightarrow \mathcal{H}^0/\mathcal{H}^1(\psi)(U)(P, f)$  in  $\mathcal{H}^0/\mathcal{H}^1(F^\bullet)(U)$  is the isomorphism of  $F^0|_U$ -torsors

$$\theta(U)(P, f) : P \times^{E^0, \phi^0} F^0 \rightarrow P \times^{E^0, \psi^0} F^0$$

given by  $[p, \nu] \mapsto [p, kf(p) + \nu]$  such that the diagram of  $F^0|_U$ -sheaves

$$\begin{array}{ccc} P \times^{E^0, \phi^0} F^0 & & \\ \theta(U)(P, f) \downarrow & \searrow \phi^1(f) & \\ P \times^{E^0, \psi^0} F^0 & \xrightarrow[\psi^1(f)]{} & F^1 \end{array}$$

commutes. □

**Proposition 4.30.** *Let  $\phi : E^\bullet \rightarrow F^\bullet$  is a homomorphism of complexes of abelian sheaves in the topos  $X$ . If  $\phi$  induces isomorphisms on kernels and cokernels (i.e. if  $\phi$  is a quasi-isomorphism), then*

$$\mathcal{H}^0/\mathcal{H}^1(\phi) : \mathcal{H}^0/\mathcal{H}^1(E^\bullet) \rightarrow \mathcal{H}^0/\mathcal{H}^1(F^\bullet)$$

*is an isomorphism of Picard stacks over  $X$ .*

*Proof.* First let us treat the case that  $\phi$  is a homotopy equivalence, that is, there is a homotopy inverse of  $\phi$  such that compositions are homotopic to  $\text{id}_{E^\bullet}$  and  $\text{id}_{F^\bullet}$ , respectively. By Proposition 4.29 well done.

Next we assume  $\phi$  is an epimorphism. In this case  $E^1 \rightarrow [F^1/F^0]$  is an epimorphism, so we just need to prove the diagram

$$\begin{array}{ccc} E^0 \times E^1 & \xrightarrow{d+\text{id}} & E^1 \\ \downarrow p & & \downarrow \\ E^1 & \longrightarrow & [F^1/F^0] \end{array}$$

is cartesian as in this case this will be a cocartesian diagram! This quickly reduces to proving that

$$\begin{array}{ccc} E^1 \times E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ E^1 \times F^0 & \longrightarrow & F^1 \end{array}$$

is cartesian, which, in turn, is equivalent to

$$\begin{array}{ccc} E^0 & \longrightarrow & E^1 \\ \downarrow & & \downarrow \\ F^0 & \longrightarrow & F^1 \end{array}$$

being cartesian, which is a consequence of the assumptions.

Finally in general case, let us note that a general  $\phi$  factors as a homotopy equivalence followed by an epimorphism, then well done. Indeed, consider  $E^\bullet \oplus F^0$ , which is homotopy equivalent to  $E^\bullet$ . Define a homomorphism  $\psi : E^\bullet \oplus F^0 \rightarrow F^\bullet$  by  $\psi^0(\nu, \mu) = \phi^0(\nu) + \mu$  and  $\psi^1(\xi, \mu) = \phi^1(\xi) + \mu$ . Then  $\psi$  is surjective and  $\phi = \psi \circ i$  where  $i : E^\bullet \hookrightarrow E^\bullet \oplus F^0$  is the canonical embedding.  $\square$

Now we consider the general case.

**Definition 4.31.** Let  $X$  be a topos and  $E^\bullet$  be a complex of abelian sheaves on  $X$ , then we define

$$\mathcal{H}^0/\mathcal{H}^1(E^\bullet) := \mathcal{H}^0/\mathcal{H}^1(\tau^{[0,1]}E^\bullet).$$

**Lemma 4.32.** Let  $X$  be a ringed topos with structure sheaf of rings  $\mathcal{O}_X$ .

- (a) We can define  $\mathcal{H}^0/\mathcal{H}^1(E^\bullet)$  and homomorphisms can be defined over  $\mathbf{D}(\mathcal{O}_X)$ .
- (b) Let  $\phi, \psi : E^\bullet \rightarrow F^\bullet$  be two morphisms in  $\mathbf{D}(\mathcal{O}_X)$ . Then, if for some choice of  $\mathcal{H}^0/\mathcal{H}^1(\phi)$  and  $\mathcal{H}^0/\mathcal{H}^1(\psi)$  we have  $\mathcal{H}^0/\mathcal{H}^1(\phi) \cong \mathcal{H}^0/\mathcal{H}^1(\psi)$  as morphisms of Picard stacks, then  $\phi = \psi$ .
- (c) Consider the zero morphism  $0(E, F) : \mathcal{H}^0/\mathcal{H}^1(E^\bullet) \rightarrow \mathcal{H}^0/\mathcal{H}^1(F^\bullet)$ . Then  $\text{Aut}(0(E, F)) = \text{Hom}_{\mathbf{D}(\mathcal{O}_X)}^{-1}(E^\bullet, F^\bullet)$ .

*Proof.* For (b)(c), see Sect. 1.4 of Exposé XVIII in [AGV73]. For (a), the quasi-isomorphism induce an isomorphism of Picard stacks, see Proposition 4.30.  $\square$

**Example 4.33.** Consider  $E^\bullet$  and we focus on  $d^0 : E^0 \rightarrow E^1$ .

- (1) If  $d^0$  is a monomorphism, then  $\mathcal{H}^0/\mathcal{H}^1(E^\bullet) = \text{coker}(d^0)$  is a sheaf.
- (2) If  $d^0$  is an epimorphism, then  $\mathcal{H}^0/\mathcal{H}^1(E^\bullet) = \mathbf{B}_X \ker(d^0)$  is a gerbe.

### Application

Come back to our case, let  $X$  be a DM-stack over a field  $k$ , then consider the big fppf topos  $X_{\text{fppf}}$  and small étale topos  $X_{\text{ét}}$ . Then we have the morphism of topoi

$$v : X_{\text{fppf}} \rightarrow X_{\text{ét}}.$$

- (a) Then we can get  $\mathbf{L}v^* : \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}}) \rightarrow \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}})$ . We may let  $M_{\text{fppf}}^\bullet := \mathbf{L}v^* M^\bullet$  for any  $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}})$ .
- (b) We also have  $\mathbf{R}\mathcal{H}om(-, \mathcal{O}_{X_{\text{fppf}}}) : \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}}) \rightarrow \mathbf{D}^+(\mathcal{O}_{X_{\text{fppf}}})$ . We may let  $M^{\bullet, \vee} := \mathbf{R}\mathcal{H}om(M^\bullet, \mathcal{O}_{X_{\text{fppf}}})$  for any  $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{fppf}}})$ .



**Remark 4.34.** We will consider the stack  $\mathcal{H}^0/\mathcal{H}^1(M_{\text{fppf}}^{\bullet,\vee})$  for  $M^\bullet \in \mathbf{D}^-(\mathcal{O}_{X_{\text{ét}}})$ . Note that in this case

$$\mathcal{H}^0/\mathcal{H}^1(M_{\text{fppf}}^{\bullet,\vee}) = \mathcal{H}^0/\mathcal{H}^1((\tau^{\geq -1} M_{\text{fppf}}^\bullet)^\vee).$$

**Remark 4.35.** For a complex  $E^\bullet$ , we define  $Z^i(E^\bullet) = \ker(E^i \rightarrow E^{i+1})$  and  $C^i(E^\bullet) = \text{coker}(E^{i-1} \rightarrow E^i)$ .

**Definition 4.36.** We call an object  $L^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$  satisfies Condition  $(*)$  if

- (1)  $H^i(L^\bullet) = 0$  for all  $i > 0$ .
- (2)  $H^i(L^\bullet)$  is coherent for all  $i = 0, -1$ .

Here are some fundamental results:

**Proposition 4.37.** Let  $X$  be a DM-stack.

- (a) Let  $L^\bullet \in \mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$  satisfy Condition  $(*)$ . Then the  $X$ -stack  $\mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet,\vee})$  is an abelian cone stack over  $X$ . Moreover, if  $L^\bullet$  is of perfect amplitude contained in  $[-1, 0]$ , then  $\mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet,\vee})$  is a vector bundle stack.
- (b) If  $\phi : E^\bullet \rightarrow L^\bullet$  is a homomorphism in  $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ , where  $E^\bullet$  and  $L^\bullet$  satisfy  $(*)$ , then we get an induced morphism of algebraic stacks

$$\phi^\vee : \mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet,\vee}) \rightarrow \mathcal{H}^0/\mathcal{H}^1(E_{\text{fppf}}^{\bullet,\vee}).$$

Then  $\phi^\vee$  is a morphism of abelian cone stacks. Moreover,  $H^0(\phi)$  is surjective if and only if  $\phi^\vee$  is representable.

- (c) The morphism  $\phi^\vee$  is a closed immersion if and only if  $H^0(\phi)$  is an isomorphism and  $H^{-1}(\phi)$  is surjective. Moreover,  $\phi^\vee$  is an isomorphism if and only if  $H^0(\phi)$  and  $H^{-1}(\phi)$  are isomorphisms.
- (d) Let  $E^\bullet \rightarrow F^\bullet \rightarrow G^\bullet \rightarrow E^\bullet[1]$  be a distinguished triangle in  $\mathbf{D}(\mathcal{O}_{X_{\text{ét}}})$ , where  $E^\bullet$  and  $F^\bullet$  satisfy  $(*)$  and  $G^\bullet$  is of perfect amplitude contained in  $[-1, 0]$ . Then the induced sequence

$$\mathcal{H}^0/\mathcal{H}^1(G_{\text{fppf}}^{\bullet,\vee}) \rightarrow \mathcal{H}^0/\mathcal{H}^1(F_{\text{fppf}}^{\bullet,\vee}) \rightarrow \mathcal{H}^0/\mathcal{H}^1(E_{\text{fppf}}^{\bullet,\vee})$$

is a short exact sequence of abelian cone stacks over  $X$ .

*Proof.* For (a), as the claim is étale local, we may assume  $L^\bullet$  consists of free  $\mathcal{O}_X$ -modules with  $L^i = 0$  for  $i > 0$  and  $L^0, L^{-1}$  have finite rank. Then  $L_{\text{fppf}}^\bullet = v^* L^\bullet$  and  $L_{\text{fppf}}^{\bullet,\vee}$  is taking dual of  $L_{\text{fppf}}^\bullet$  component-wise. Hence we have

$$\mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet,\vee}) = [Z^1(L^{\vee,\bullet})/L^{\vee,0}]$$

which is an abelian cone stack given by  $L^{\vee,0} \rightarrow Z^1(L^{\vee,\bullet}) = C(C^{-1}L^\bullet)$ .

When  $L^\bullet$  is of perfect amplitude contained in  $[-1, 0]$ , then  $\mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet, \vee})$  is a vector bundle stack since étale locally as above we have  $Z^1(L^\vee, \bullet) = L^{\vee, 1}$ .

For (b), the fact that  $\phi^\vee$  is a morphism of abelian cone stacks is immediate from the definition. The second question is étale local in  $X$ , so we may assume that  $E^\bullet$  and  $L^\bullet$  are complexes of free  $\mathcal{O}_X$ -modules and that  $E^i = L^i = 0$ , for  $i > 0$ , and that  $L^0, L^{-1}, E^0$  and  $E^{-1}$  are of finite rank. Consider the commutative diagram

$$\begin{array}{ccc} C^{-1}(E^\bullet) & & \\ & \searrow & \\ & F & \xrightarrow{\quad} E^0 \\ & \downarrow & \downarrow \\ & B^{-1}(L^\bullet) & \longrightarrow L^0 \end{array}$$

of coherent sheaves with fiber product  $F$ . This force  $0 \rightarrow F \rightarrow E^0 \oplus C^{-1}(L^\bullet) \rightarrow L^0$  exact. Then its easy to see that  $H^0(\phi)$  is surjective if and only if  $0 \rightarrow F \rightarrow E^0 \oplus C^{-1}(L^\bullet) \rightarrow L^0 \rightarrow 0$  exact. Hence taking duality we get  $0 \rightarrow L^{\vee, 0} \rightarrow E^{\vee, 0} \times_X Z^1(L^\vee, \bullet) \rightarrow C(F) \rightarrow 0$  exact. Then by Proposition 4.20 we get

$$[Z^1(L^\vee, \bullet)/L^{\vee, 0}] \cong [C(F)/E^{\vee, 0}].$$

This force the following cartesians

$$\begin{array}{ccc} C(F) & \xrightarrow{\quad} & Z^1(E^\vee, \bullet) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{H}^0/\mathcal{H}^1(L_{\text{fppf}}^{\bullet, \vee}) & \xrightarrow{\phi^\vee} & \mathcal{H}^0/\mathcal{H}^1(E_{\text{fppf}}^{\bullet, \vee}) \end{array}$$

hence  $\phi^\vee$  is representable.

For the converse, note that  $\phi^\vee : [Z^1(L^\vee, \bullet)/L^{\vee, 0}] \rightarrow [Z^1(E^\vee, \bullet)/E^{\vee, 0}]$  representable implies that  $[Z^1(L^\vee, \bullet)/L^{\vee, 0}] = [W/E^{\vee, 0}]$ . Then we have the commutative diagram:

$$\begin{array}{ccc} Z^1(L^\vee, \bullet) \times_X L^{\vee, 0} & \longrightarrow & Z^1(L^\vee, \bullet) \\ \downarrow & & \downarrow \\ Z^1(L^\vee, \bullet) \times_X E^{\vee, 0} & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z^1(L^\vee, \bullet) & \longrightarrow & [W/E^{\vee, 0}] \end{array}$$

such that the the whole diagram and the lower diagram are cartesian, then this force the upper square is cartesian. So we get cartesians

$$\begin{array}{ccccc}
L^{\vee,0} & \longrightarrow & Z^1(L^{\vee,\bullet}) \times_X L^{\vee,0} & \longrightarrow & Z^1(L^{\vee,\bullet}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
E^{\vee,0} & \longrightarrow & Z^1(L^{\vee,\bullet}) \times_X E^{\vee,0} & \longrightarrow & W
\end{array}$$

Hence  $L^{\vee,0} \cong E^{\vee,0} \times_W Z^1(L^{\vee,\bullet}) \rightarrow E^{\vee,0} \times_X Z^1(L^{\vee,\bullet})$  is a closed immersion. This implies that  $E^0 \oplus C^{-1}(L^\bullet) \rightarrow L^0$  is an epimorphism.

For (c), following the previous argument in (b),  $\phi^\vee$  is a closed immersion if and only if  $C(F) \rightarrow Z^1(E^{\vee,\bullet})$  is. This is equivalent to  $C^{-1}(E^\bullet) \rightarrow F$  being surjective. A simple diagram chase shows that this is equivalent to  $H^0(\phi)$  is an isomorphism and  $H^{-1}(\phi)$  is surjective. The ‘moreover’ follows similarly.

For (d), the question is étale local, so assume that  $E^i$  and  $F^i$  are 0 for  $i > 0$  and vector bundles for  $i = 0, -1$ , and that  $G^i = E^{i+1} \oplus F^i$ , that is,  $G^\bullet = \text{cone}(E^\bullet \rightarrow F^\bullet)$ . If we consider the small enough étale locally, we may let  $G^i = 0$  for  $i \leq -2$  as  $G^\bullet$  is of perfect amplitude contained in  $[-1, 0]$ . Now we have to prove that

$$0 \rightarrow [Z^1(G^{\vee,\bullet})/G^{\vee,0}] \rightarrow [Z^1(F^{\vee,\bullet})/F^{\vee,0}] \rightarrow [Z^1(E^{\vee,\bullet})/E^{\vee,0}] \rightarrow 0$$

is a short exact sequence of cone stacks. Now by directly check, we have the exact sequence of sheaves

$$0 \rightarrow C^{-1}(E^\bullet) \rightarrow C^{-1}(F^\bullet) \oplus E^0 \rightarrow C^{-1}(G^\bullet) \rightarrow 0.$$

Hence consider

$$\begin{array}{ccccccc}
0 & \longrightarrow & C^{-1}(E^\bullet) & \longrightarrow & C^{-1}(F^\bullet) \oplus E^0 & \longrightarrow & C^{-1}(G^\bullet) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E^0 & \longrightarrow & F^0 \oplus E^0 & \longrightarrow & G^0 = F^0 \longrightarrow 0
\end{array}$$

with exact rows. Finally by Proposition 4.26 we get the result.  $\square$

#### 4.4 About Normal Cones

Here we will consider some useful results about normal cones of DM-stacks.

Consider the commutative diagram of algebraic stacks

$$\begin{array}{ccc}
X' & \xrightarrow{j} & Y' \\
\downarrow u & & \downarrow v \\
X & \xrightarrow{i} & Y
\end{array}$$

with where  $i$  and  $j$  are local immersions. Then there is a natural morphism of cones over  $X'$

$$\alpha : C_{X'/Y'} \rightarrow C_{X/Y}.$$

If the diagram is cartesian, then  $\alpha$  is a closed immersion. If, moreover,  $v$  is flat, then  $\alpha$  is an isomorphism.

**Proposition 4.38.** *Consider a commutative diagram of DM-stacks*

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ & \searrow i & \downarrow f \\ & & Y \end{array}$$

where  $i$  and  $i'$  are local immersions and  $f$  is smooth. Then the sequence of morphisms of cones over  $X$

$$(i')^* T_{Y'/Y} \rightarrow C_{X/Y'} \rightarrow C_{X/Y}$$

is exact.

*Proof.* The question is local, so we can assume them are schemes and that  $i'$  and  $i$  are immersions. This is then Example 4.2.6 in [Ful98].  $\square$

**Lemma 4.39.** *Let  $f : U \rightarrow M$  be a local immersion of affine  $k$ -schemes of finite type, where  $M$  is smooth over  $k$ . Then the normal cone  $C_{U/M} \hookrightarrow N_{U/M}$  is invariant under the action of  $f^* T_M$  on  $N_{U/M}$ . In other words,  $C_{U/M}$  is an  $f^* T_M$ -cone.*

*Proof.* Consider projections  $p_i : M \times M \rightarrow M$ , we consider two diagrams:

$$\begin{array}{ccc} U & \xrightarrow{\Delta f} & M \times M \\ & \searrow f & \downarrow p_i \\ & & M \end{array} \quad \begin{array}{ccc} U & \xrightarrow{f} & M \\ & \searrow \Delta f & \downarrow \Delta \\ & & M \times M \end{array}$$

The first one give us exact sequence of abelian cones on  $U$ :

$$0 \rightarrow f^* T_M \xrightarrow{j_i} N_{U/M \times M} \xrightarrow{p_{i,*}} N_{U/M} \rightarrow 0$$

and the second one give us a homomorphism of abelian cones  $s : N_{U/M} \rightarrow N_{U/M \times M}$  which is a section of both  $p_{i,*}$ .

Using  $(j_1, p_{1,*})$  we make the identification  $N_{U/M \times M} = f^* T_M \times N_{U/M}$  and  $p_{2,*}$  is identified with the action of  $f^* T_M$  on  $N_{U/M}$ . Since the same functorialities of normal sheaves used so far are enjoyed by normal cones, we get that under the identification above the subcone  $C_{U/M \times M} \subset N_{U/M \times M}$  corresponds to  $f^* T_M \times C_{U/M}$  and the action  $p_{2,*} : f^* T_M \times N_{U/M} \rightarrow N_{U/M}$  restricts to  $p_{2,*} : f^* T_M \times C_{U/M} \rightarrow C_{U/M}$ .  $\square$

## 4.5 Intrinsic Normal Cone

Now let  $X$  be a Deligne-Mumford stack, locally of finite type over  $k$ . Now we will construct the intrinsic normal cone and intrinsic normal sheaf of  $X$  and their basic properties.

**Definition 4.40.** We denote the abelian cone stack

$$\mathfrak{N}_X := \mathcal{H}^1 / \mathcal{H}^0((\mathbb{L}_{X, \text{fppf}}^\bullet)^\vee)$$

and call it the *intrinsic normal sheaf* of  $X$  where  $\mathbb{L}_X^\bullet \in \mathbf{D}^{\leq 0}(\mathcal{O}_{X_{\text{ét}}})$  is the cotangent complex which satisfies the condition (\*).

**Definition 4.41.** (a) A *local embedding* of  $X$  is a pair  $(U, M)$  with a diagram  $U \xleftarrow{i} U \xrightarrow{f} M$  where

- (a1)  $U$  is an affine  $k$ -scheme of finite type;
- (a2)  $i : U \rightarrow X$  is an étale morphism;
- (a3)  $M$  is a smooth affine  $k$ -scheme of finite type;
- (a4)  $f : U \rightarrow M$  is a local immersion.

(b) A *morphism of local embeddings*  $\phi : (U', M') \rightarrow (U, M)$  is a pair of morphisms  $\phi_U : U' \rightarrow U$  and  $\phi_M : M' \rightarrow M$  such that

- (b1)  $\phi_U$  is an étale  $X$ -morphism;
- (b2)  $\phi_M$  is smooth morphism such that

$$\begin{array}{ccc} U' & \xrightarrow{f'} & M' \\ \downarrow \phi_U & & \downarrow \phi_M \\ U & \xrightarrow{f} & M \end{array}$$

commutes.

**Remark 4.42.** If  $(U', M')$  and  $(U, M)$  are local embeddings of  $X$ , then  $(U' \times_X U, M' \times M)$  is naturally a local embedding of  $X$  which we call the *product of local embeddings*, even though it may not be the direct product in the category of local embeddings of  $X$ .

Now we consider the local presentation of intrinsic normal sheaf  $\mathfrak{N}_X$ . Indeed, consider a local embedding  $U \xleftarrow{i} U \xrightarrow{f} M$  of  $X$ , then we have a natural homomorphism

$$\phi : \mathbb{L}_X^\bullet|_U \rightarrow [\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1]$$

in  $\mathbf{D}^{\leq 0}(\mathcal{O}_{U_{\text{ét}}})$  where  $\mathcal{I}$  be the ideal correspond to  $f$  and  $[\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1] \in \mathbf{D}^{[-1,0]}(\mathcal{O}_{U_{\text{ét}}})$ . Moreover, by Theorem 3.1(c) we know that  $\phi$  induces an isomorphism on  $H^{-1}$  and  $H^0$ . By Proposition 4.30 we get an induced isomorphism of cone stacks

$$\phi^\vee : [N_{U/M}/f^*T_M] \cong i^*\mathfrak{N}_X.$$

In other words,  $N_{U/M}$  is a local presentation of the abelian cone stack  $\mathfrak{N}_X$ .

**Theorem 4.43.** *There exists a unique closed subcone stack  $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$  such that for every local embedding  $(U, M)$  of  $X$  we have  $\mathfrak{C}_X|_U = [C_{U/M}/f^*T_M]$ , that is, the diagram*

$$\begin{array}{ccc} C_{U/M} & \hookrightarrow & N_{U/M} \\ \downarrow & \lrcorner & \downarrow \\ \mathfrak{C}_X|_U & \hookrightarrow & \mathfrak{N}_X|_U \end{array}$$

*Proof.* If  $\chi : (U', M') \rightarrow (U, M)$  is a morphism of local embeddings, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{L}_X^\bullet|_{U'} & & \\ \downarrow \phi|_{U'} & \searrow \phi' & \\ [\mathcal{I}/\mathcal{I}^2 \rightarrow f^*\Omega_M^1]|_{U'} & \xrightarrow{\tilde{\chi}} & [\mathcal{I}'/(\mathcal{I}')^2 \rightarrow (f')^*\Omega_{M'}^1] \end{array}$$

in  $\mathbf{D}^{\leq 0}(\mathcal{O}_{U'_{\text{ét}}})$  because of the naturality of  $\phi$  and thus induce the commutative diagram

$$\begin{array}{ccc} [N_{U'/M'}/(f')^*T_{M'}] & \xrightarrow{\tilde{\chi}^\vee} & [N_{U/M}/f^*T_M]|_{U'} \\ (\phi')^\vee, \cong \downarrow & \swarrow \phi^\vee|_{U'}, \cong & \\ \mathfrak{N}_X|_{U'} & & \end{array}$$

in  $\mathbf{D}^{\leq 0}(\mathcal{O}_{U'_{\text{ét}}})$ . In particular,  $\tilde{\chi}^\vee$  is an isomorphism of cone stacks over  $U'$ .

Now by Lemma 4.39  $\chi$  induce a morphism from the  $(f')^*T_{M'}$ -cone  $C_{U'/M'}$  to the  $f^*T_M|_{U'}$ -cone  $C_{U/M}|_{U'}$ . By Proposition 4.26 the pair  $(C_{U/M} \hookrightarrow N_{U/M})|_{U'}$  is the quotient of  $(C_{U'/M'} \hookrightarrow N_{U'/M'})$  by the action of  $(f')^*T_{M'/M}$  since the kernel of  $(f')^*T_{M'} \rightarrow f^*T_M|_{U'}$  is  $(f')^*T_{M'/M}$ . This implies that the isomorphism above

$$\tilde{\chi}^\vee : [N_{U'/M'}/(f')^*T_{M'}] \cong [N_{U/M}/f^*T_M]|_{U'}$$

identifies the closed subcone stack  $[C_{U'/M'}/(f')^*T_{M'}]$  with the closed subcone stack  $[C_{U/M}/f^*T_M]|_{U'}$ . This give us the unique closed subcone stack  $\mathfrak{C}_X \hookrightarrow \mathfrak{N}_X$  with the properties above.  $\square$

**Definition 4.44.** *This unique closed subcone stack  $\mathfrak{C}_X$  is called the intrinsic normal cone of  $X$ .*

**Theorem 4.45.** *The intrinsic normal cone  $\mathfrak{C}_X$  is of pure dimension zero which abelian hull is the intrinsic normal sheaf  $\mathfrak{N}_X$ .*

*Proof.* The second claim follows because the normal sheaf is the abelian hull of the normal cone, for any local embedding.

To prove the claim about the dimension of  $\mathfrak{C}_X$ , consider a local embedding  $(U, M)$  of  $X$ , giving rise to the local presentation  $C_{U/M}$  of  $\mathfrak{C}_X$ . Assume that  $M$  is of pure dimension. We then have a cartesian diagram of  $U$ -stacks

$$\begin{array}{ccc} C_{U/M} \times f^*T_M & \longrightarrow & C_{U/M} \\ \downarrow & \lrcorner & \downarrow \\ C_{U/M} & \longrightarrow & [C_{U/M}/f^*T_M] \end{array}$$

Thus  $C_{U/M} \rightarrow [C_{U/M}/f^*T_M]$  is a smooth epimorphism of relative dimension  $\dim M$ . So since  $C_{U/M}$  is of pure dimension  $\dim M$  (see the comments on the Definition 2.1), the stack  $[C_{U/M}/f^*T_M]$  has pure dimension  $\dim M - \dim M = 0$ . Well done.  $\square$

## 4.6 Vistoli's Rational Equivalence

Recall that if we consider the commutative diagram of algebraic stacks

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ \downarrow u & & \downarrow v \\ X & \xrightarrow{i} & Y \end{array}$$

with where  $i$  and  $j$  are local immersions. Then there is a natural morphism of cones over  $X'$

$$\alpha : C_{X'/Y'} \rightarrow C_{X/Y}.$$

If the diagram is cartesian, then  $\alpha$  is a closed immersion. If, moreover,  $v$  is flat, then  $\alpha$  is an isomorphism.

## 4.7 Obstruction Theory and Virtual Class

## 4.8 Examples

# 5 Atiyah-Bott Localization

We will follow [AB84].

## 6 Localization of Virtual Fundamental Class

We will follow [GP99].

### References

- [AB84] M. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23:1–28, 1984.
- [AGV73] M. Artin, A. Grothendieck, and J. Verdier. *Théorie des Topos et Cohomologie Etale des Schémas (SGA4)*. Springer, 1973.
- [BF97] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128:45–88, 1997.
- [EH16] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.
- [Ful98] William Fulton. *Intersection Theory*. Springer New York, 1998.
- [GP99] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135:487–518, 1999.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977.
- [Li18] Kezheng Li. *Group Schemes and Their Actions (in Chinese)*. Tsinghua University Press, 2018.
- [Ric22] Andrea Ricolfi. *An Invitation to Modern Enumerative Geometry*. Springer Cham, 2022.