Varieties of Minimal Rational Tangents on the Fano Varieties

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Preface

Chapter 1

Introduction to the Rational Curves

The main results here we follows the famous book [6].

1.1 Hilbert Schemes and Chow Schemes

1.1.1 Hilbert Schemes, a Basic Introduction

Definition 1.1.1. *Let X be an S-scheme, we define the* Hilbert functor $\mathscr{H}ilb_{X/S}$ *sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z.*

Fix a Polynomial P and a relative ample line bundle $\mathscr{O}(1)$, we can define $\mathscr{H}ilb^P_{X/S}$ sends an S-scheme Z to the set consists of subschemes $V \subset X \times_S Z$ which is proper and flat over Z with Hilbert Polynomial P.

Theorem 1.1.2 (Grothendieck). *Let S be a noetherian scheme, let $X \to S$ be a projective morphism, and $\mathscr{L}$ a relatively very ample line bundle on X. Then for any polynomial P, the Hilbert functor $\mathscr{H}ilb^P_{X/S}$ is representable by a projective S-scheme $\mathsf{Hilb}^P_{X/S}$. We also have $\mathsf{Hilb}_{X/S} = \coprod_P \mathsf{Hilb}^P_{X/S}$.*

Proof. Note that this notion of projectivity is much general than [5], but is the same when $S = \operatorname{Spec} k$. The proof is to embed it into Grassmannian. The original proof in [4] and we also refer [8], [6] and [3]. □

Remark 1.1.3. *In [2] we can remove the noetherian hypothesis, by instead assuming strong (quasi-)projectivity of $X \to S$. So also [1].*

Example 1.1.1. *Some examples and interesting results:*

(a) *We have* $\mathsf{Hilb}^1_{X/S} = X/S$.

(b) *Let* C *be a curve over a field* k*, then*

$$\mathsf{Hilb}^m_{C/k} \cong S^m C := \underbrace{C \times \cdots \times C}_{m} / \mathfrak{S}_m.$$

Hence if C *smooth, so is* $\mathsf{Hilb}^m_{C/k}$*. See also [3] Theorem 7.2.3(1) and Proposition 7.3.3.*

(c) *Let* S *be a smooth surface over a field* k*, then* $\mathsf{Hilb}^m_{S/k}$ *is also smooth of dimension* $2m$ *and hence* $\mathsf{Hilb}^m_{S/k} \to S^m X$ *(we will see this later for general settings) is a resolution of singularities. Note that* $S^m X$ *is smooth if and only if* X *is smooth and* $\dim X = 1$ *or* $m < 2$*. See [3] Theorem 7.2.3(2) and Theorem 7.3.4.*

(d) *Let* X *be a nonsingular variety. Then* $\mathsf{Hilb}^m_{X/k}$ *is nonsingular for* $m \leq 3$*. Moreover, for any nonsingular 3-fold the scheme* $\mathsf{Hilb}^4_{X/k}$ *is singular. See [3] Remark 7.2.5 and 7.2.6.*

(e) *Let* $\mathscr{E}$ *be a vector bundle of rank* $m+1$ *over* S *and let* $P_d(n) = \binom{m+n}{m} - \binom{m+n-d}{m}$*, then*

$$\mathsf{Hilb}^{P_d}_{\mathbb{P}(\mathscr{E})/S} \cong \mathbb{P}((\mathrm{Sym}^d \mathscr{E})^\vee).$$

(f) *Let* $Z \to S$*, we have* $\mathsf{Hilb}_{X \times_S Z/Z} \cong \mathsf{Hilb}_{X/S} \times_S Z$.

(g) Hartshorne's Connectedness Theorem: *for every connected noetherian scheme* S*,* $\mathsf{Hilb}^P_{\mathbb{P}^n_S/S}$ *is connected.*

(h) *Let* X *be a connected variety over* k*, then* $\mathsf{Hilb}^n_{X/k}$ *is connected for all* $n > 0$.

(i) Murphy's Law: *It has many singularities, that is, for every scheme* X *finite type over* $\mathbb{Z}$ *and point* $x \in X$*, there exists a point* $q \in \mathsf{Hilb}^P_{\mathbb{P}^n/k}$ *of some Hilbert scheme and an isomorphism*

$$\widehat{\mathscr{O}}_{X,p}[[x_1, ..., x_s]] \cong \widehat{\mathscr{O}}_{\mathsf{Hilb}^P_{\mathbb{P}^n/k}, q}[[y_1, ..., y_t]].$$

See [11]. In fact, it can be arranged that the Hilbert scheme parameterizes smooth curves in $\mathbb{P}^n$ *for some* n*. It turns out that various other moduli spaces also satisfy Murphy's Law: Kontsevich's moduli space of maps, moduli of canonically polarized smooth surfaces, moduli of curves with linear systems, and the moduli space of stable sheaves.*

(j) *In [10] they gave a full classification of the situation where* $\mathsf{Hilb}^P_{\mathbb{P}^n/k}$ *smooth.*

Definition 1.1.4. *Let* $X/S, Y/S$ *are* S*-schemes, then we have a functor* $\mathscr{H}om_S(X, Y)$ *send* S*-scheme* T *into a set of* T*-morphisms* $X \times_S T \to Y \times_S T$.

For a subscheme $B \subset X$ *proper over* S *and* $g : B \to Y$*, we have a functor* $\mathscr{H}om_S(X, Y; g)$ *send* S*-scheme* T *into a set of* T*-morphisms* $X \times_S T \to Y \times_S T$ *such that* $f|_{B \times_S T} = g \times_S \mathrm{id}_T$.

Proposition 1.1.5. *If X/S and Y/S are both projective over S and X is flat over S, then $\mathscr{H}om_S(X,Y)$ represented by an open subscheme $\mathsf{Hom}_S(X,Y) \subset \mathsf{Hilb}_{X\times_S Y/S}$.*

Proof. Any $X \times_S T \to Y \times_S T$ correspond to its graph which is a closed immersion $\Gamma : X \times_S T \to X \times_S Y \times_S T$. As X is flat over S, then $X \times_S T$ is flat over T. Hence we get a morphism $\mathsf{Hom}_S(X,Y) \to \mathsf{Hilb}_{X\times_S Y/S}$. We omit the more details and refer Theorem I.1.10 in [6]. $\square$

Proposition 1.1.6. *If X/S and Y/S are both projective over S and X, B are both flat over S, then $\mathscr{H}om_S(X,Y;g)$ represented by a subscheme $\mathsf{Hom}_S(X,Y;g) \subset \mathrm{Hom}_S(X,Y)$.*

Proof. Consider the restriction map $R : \mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_S(B,Y)$, then $g : B \to Y$ gives a section $G : S \to \mathrm{Hom}_S(B,Y)$. Hence $\mathsf{Hom}_S(X,Y;g) := R^{-1}(G(S)) \subset \mathsf{Hom}_S(X,Y)$ represents $\mathscr{H}om_S(X,Y;g)$. $\square$

Now we state the deformation theory of Hilbert schemes. We only consider the simpler case that all schemes over a field k. For general case we refer Section 1.2 in [6].

Theorem 1.1.7. *Let Y be a projective scheme over a field k and $Z \subset Y$ is a subscheme. Then*

(a) We have
$$T_{[Z]}\mathsf{Hilb}_Y \cong \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

(b) The dimension of every irreducible components of Hilb_Y at $[Z]$ is at least
$$\dim \mathrm{Hom}_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z) - \dim \mathrm{Ext}^1_Z(\mathscr{I}_Z/\mathscr{I}_Z^2, \mathscr{O}_Z).$$

Proof. See Theorem I.2.8 in [6]. For family case we refer Theorem I.2.15 in [6]. $\square$

Corollary 1.1.8. *Let X, Y are projective varieties over a field k with a morphism $f : X \to Y$. Let Y is smooth over k. Then*

(a) We have
$$T_{[f]}\mathsf{Hom}_k(X,Y) \cong \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

(b) The dimension of every irreducible components of $\mathsf{Hom}_k(X,Y)$ at $[f]$ is at least
$$\dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X).$$

Proof. Let $Z \subset X \times_k Y$ be the graph of f, we claim that $\mathscr{I}_Z/\mathscr{I}_Z^2 \cong f^*\Omega^1_Y$. Indeed we have an exact sequence $\mathscr{I}_Z/\mathscr{I}_Z^2 \to \Omega^1_{X\times_k Y}|_Z \to \Omega^1_Z \to 0$. This is split by $\mathscr{O}_Z \cong \mathscr{O}_X \overset{(\mathrm{id}_X,1)}{\to} \mathscr{O}_{X\times_k Y}$. Then we can show the claim. Hence the results follows from Theorem 1.1.7. The family version we refer Theorem I.2.17 in [6]. $\square$

1.1.2 Chow Schemes, a Basic Introduction

Here we only consider the schemes over a field k such that $\mathrm{char}(k) = 0$. The positive characteristic case is very complicated and we refer Section I.4 in [6].

Definition 1.1.9. *Let $g_i : U_i \to W$ be a proper morphism of schemes over W. Assume that W is reduced and U_i is irreducible. By generic flatness there is an open subset $W_i \subset g_i(U_i) \subset W$ such that g_i is flat of relative dimension d over W_i. Let $T = \operatorname{Spec}\Delta$ be the spectrum of a DVR Δ and $h : T \to W$ a morphism such that $h(T_g) \in W_i$ and $h(T_0) = w \in W$. Let $h^*U_i = U_i \times_h T$ and $\mathscr{J} \subset \mathscr{O}_{h^*U_i}$ the ideal of those sections whose support is contained in the special fiber of $h * U_i \to T$. Let $(U_i)'_T := \underline{\operatorname{Spec}}_T \mathscr{O}_{h^*U_i}/\mathscr{J}$ which is flat over T. Then we let $[Z_0]$ be the fundamental cycle of the central fiber of $(U_i)'_T \to T$, and define*

$$\lim_{h\to w}(U_i/U) := [Z_0] \in Z_d(g_i^{-1}(w) \times_{\kappa(w)} T_0)$$

which is called the cycle theoretic fiber of g_i at w along h.

Definition 1.1.10. *A well defined family of d-dimensional proper algebraic cycles over W is a pair $(g : U \to W)$ satisfying the following properties:*

(a) There is a reduced scheme $\operatorname{supp} U$ with irreducible components U_i such that $U = \sum_i m_i[U_i]$ is an algebraic cycle.

(b) W is a reduced scheme and $g : \operatorname{supp} U \to W$ is a proper morphism.

(c) Let $g_i := g|_{U_i}$. Then every g_i maps onto an irreducible component of W and every fiber of g_i is either empty or has dimension d. In particular there is a dense open subset $W_0 \subset W$ such that every g_i is flat over W_0.

(d) For every $w \in W$ there is a cycle $g^{[-1]}(w) \in Z_d(g^{-1}(w))$ such that for any $h : T \to W$ of spectrum of DVR such that $h(T_0) = w$ and $h(T_g) \in W_0$ we have

$$g^{[-1]}(w) =_{\mathrm{ess}} \sum_i m_i \lim_{h\to w}(U_i/W).$$

That is, both two cycles from a single cycle of $Z_d(g^{-1}(w))$.

Remark 1.1.11. *If W is normal, then (d) can be implied by (a)-(c). See Theorem I.3.17 in [6].*

Definition 1.1.12. *Let X be a scheme over S. A well defined family of proper algebraic cycles of X/S over W/S is a pair $(g : U/S \to W/S)$ satisfying the following properties:*

(a) $\operatorname{supp} U$ is a closed subscheme of $X \times_S W$ and g is the natural projection morphism.

(b) $(g : U \to W)$ *is a well defined family of* d*-dimensional proper algebraic cycles over* W *for some* d.

Proposition 1.1.13. *Assume that* $g : U \to W$ *is proper and flat of relative dimension* d *and* W *is reduced. Let* $\sum_i m_i[U_i]$ *be the fundamental cycle of* U*. Then* $g : [U] \to W$ *is a well defined family of algebraic cycles over* W*.*

Proof. See Lemma I.3.14 and Corollary I.3.15 in [6]. □

Definition 1.1.14 (Chow Schemes of Characteristic Zero)**.** *Let* X/S *and we define a functor* $\mathscr{C}how_{X/S}$ *sends* Z/S *to the set consists of well defined families of nonnegative proper algebraic cycles of* $X \times_S Z/Z$*.*

Let a relative ample line bundle $\mathscr{O}(1)$*, we can define* $\mathscr{C}how^{d,d'}_{X/S}$ *sends* Z/S *to the set consists of well defined families of nonnegative proper algebraic cycles of* $X \times_S Z/Z$ *which is of dimension* d *and degree* d'*.*

Theorem 1.1.15. *Let* X/S *be a scheme, projective over* S *and* $\mathscr{O}(1)$ *relatively ample. Then the functor* $\mathscr{C}how^{d,d'}_{X/S}$ *is representable by a semi-normal and projective* S*-scheme* $\mathsf{Chow}^{d,d'}_{X/S}$*. We also have* $\mathsf{Chow}_{X/S} = \coprod_{d,d'} \mathsf{Chow}^{d,d'}_{X/S}$*.*

Proof. Very complicated, we refer Theorem I.3.21 in [6]. □

Example 1.1.2. *Let* X *be a semi-normal variety, then* $\mathsf{Chow}^{0,m}_{X/k} \cong S^m X$*.*

Proposition 1.1.16 (Hilbert-Chow)**.** *Let* X, Y *be* S*-schemes.*

(a) *We have a natural morphism* $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$*. This morphism can be factored by dimensions.*

(b) *If* X, Y *be projective* S*-schemes and* X/S *flat, then we have*

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Chow}_{Y/S}.$$

Proof. For (a), consider $[\mathsf{Univ}^{\mathsf{Hilb}} \times_{\mathsf{Hilb}_{X/S}} \mathsf{Hilb}^{\mathrm{sn}}_{X/S}] \to \mathsf{Hilb}^{\mathrm{sn}}_{X/S}$, then by Proposition 1.1.13 this is a well defined family of algebraic cycles. This gives such morphism $\mathsf{Hilb}^{\mathrm{sn}}_{X/S} \to \mathsf{Chow}_{X/S}$.

For (b), by (a) we have

$$\mathsf{Hom}_S(X, Y)^{\mathrm{sn}} \to \mathsf{Hilb}(X \times_S Y/S)^{\mathrm{sn}} \to \mathsf{Chow}_{X \times_S Y/S} \to \mathsf{Chow}_{Y/S}$$

and well done. □

Remark 1.1.17. *Let* X *be a semi-normal variety, hence we have* $(\mathsf{Hilb}^m_{X/k})^{\mathrm{sn}} \to \mathsf{Chow}^{0,m}_{X/k} \cong S^m X$*.*

1.1.3 Small Applications to Curves

For more applications we refer Section II.1 in [6]. Here we only need some easy case. We assume over a field k.

Theorem 1.1.18. *Let C be a proper curve and $f : C \to Y$ a morphism to a smooth variety Y of dimension n. Then*

$$\dim_{[f]} \mathsf{Hom}(C, Y) \geq -C \cdot K_Y + n\chi(\mathscr{O}_C).$$

And equality holds if $H^1(C, f * T_Y) = 0$, in this case it is smooth at $[f]$.

Proof. By Corollary 1.1.8(b) we have

$$\begin{aligned}\dim_{[f]} \mathsf{Hom}(C, Y) &\geq \dim \mathrm{Hom}_X(f^*\Omega^1_Y, \mathscr{O}_X) - \dim \mathrm{Ext}^1_X(f^*\Omega^1_Y, \mathscr{O}_X)\\ &= h^0(C, f^*T_Y) - h^1(C, f^*T_Y) = \chi(C, f^*T_Y)\\ &= \deg f^*T_Y + n\chi(\mathscr{O}_C)\end{aligned}$$

by Riemann-Roch theorem. The final statement follows from Corollary 1.1.8(a). □

Proposition 1.1.19. *Assume that X/S is flat, B/S is flat and finite of degree m and Y/S is smooth of relative dimension n. Then $\dim \mathsf{Hom}(X, Y; g) \geq \dim \mathsf{Hom}(X, Y) - kn$.*

Proof. Let $p : B \to S$ be the projection. By Corollary 1.1.8 we find that $\mathsf{Hom}(B, Y)$ is smooth over S of relative dimension rank kn. Thus $g(S) \subset \mathrm{Hom}(B, Y)$ is locally defined by kn equations. Pulling back these equations by R we obtain local defining equations. □

Lemma 1.1.20. *Let $0 \in T$ be the spectrum of a local ring and let U/T be a flat and proper and V/T be a variety. Let $p : U \to V$ as a T-morphism. If $p_0 : U_0 \to V_0$ is a closed immersion (resp. an isomorphism), then so is p.*

Proof. See Lemma I.1.10.1 and Proposition I.7.4.1.2 in [6]. We omit this. □

Theorem 1.1.21. *Let C be a projective curve over k and Y a smooth variety over k. Let $B \subset C$ be a closed subscheme which is finite over k. Assume that C is smooth along B. Let $g : B \to Y$ be a morphism. Then*

(a) We have

$$T_{[f]}\mathsf{Hom}(C, Y; g) \cong H^0(C, f^*T_Y \otimes \mathscr{I}_B).$$

(b) The dimension of every irreducible component of $\mathsf{Hom}(C, Y; g)$ at $[f]$ is at least

$$h^0(C, f^*T_Y \otimes \mathscr{I}_B) - h^1(C, f^*T_Y \otimes \mathscr{I}_B).$$

Proof. The original proof we refer [7]. A simple case of family version we refer Theorem II.1.7 in [6]. Here we assume k is algebraically closed. Here $\mathscr{I}_B = \mathscr{O}_C(-s_1 - ... - s_m)$.

Let $X_0 := C \times_k Y$ and let $\gamma_0 : C \cong \Gamma_0 \subset X_0$ be the graph of f. Let $\pi_1 : X_1 := \mathrm{Bl}_{\{s_1\}} X_0 \to X_0$ and Γ_1 be the strict transform of Γ_0. Let $\gamma_1 : C \cong \Gamma_1 \subset X_1$ as C is smooth at s_1. Repeat the process and finally we get $\pi_m : X_m := \mathrm{Bl}_{\{s_m\}} X_{m-1} \to X_{m-1}$ and Γ_m be the strict transform of Γ_{m-1}. Let $\gamma_m : C \cong \Gamma_m \subset X_m$. Then we have $\gamma_0^*(\mathscr{I}_{\Gamma_0}/\mathscr{I}_{\Gamma_0}^2) \cong f^*\Omega_Y^1$ and $\gamma_{i+1}^*(\mathscr{I}_{\Gamma_{i+1}}/\mathscr{I}_{\Gamma_{i+1}}^2) \cong \gamma_i^*(\mathscr{I}_{\Gamma_i}/\mathscr{I}_{\Gamma_i}^2) \otimes \mathscr{O}_C(-s_{i+1})$. Hence we get $\gamma_m^*(\mathscr{I}_{\Gamma_m}/\mathscr{I}_{\Gamma_m}^2) \cong f^*\Omega_Y^1 \otimes \mathscr{I}_B$.

Now we claim that there is an open neighborhood $[\Gamma_m] \in U \subset \mathsf{Hilb}_{X_m}$ such that $\mathsf{Hom}(C, Y; g) \cong U$. Indeed, let $U \subset \mathsf{Hilb}_{X_m}$ be the open set parametrizing those 1-cycles D for which the projection $D \to C$ is an isomorphism. This is open by Lemma 1.1.20.

First, the universal family of U is contained in $\mathsf{Hom}(C, Y; g)(U)$. Conversely consider $[p_0 : C \times R \to Y \times R] \in \mathsf{Hom}(C, Y; g)(R)$. Let its graph is $G_0 \subset X_0 \times R$. As $\{s_1\} \times R \subset G_0$ and $G_0 \to R$ smooth along $\{s_1\} \times R$, we let $G_1 \subset X_1 \times R$ be the strict transform of G_0. Then $G_1 \cong G_0 \cong C \times R$. Repeat the process and finally we get $X_m \times R \supset C \times R \cong G_m \in \mathsf{Hilb}_{X_m}(R)$. Hence this give the isomorphism $\mathsf{Hom}(C, Y; g) \cong U$. Hence by Theorem 1.1.7 and we get the result. □

1.2 Families of Rational Curves

We may assume all schemes over a field k of characteristic zero locally of finite type. Note that there are also have the same results by some small modification in the case of positive characteristic, see Section II.2 in [6].

Proposition 1.2.1. *Let $f : X \to Y$ be a proper morphism of relative dimension one. Assume that if T is the spectrum of a DVR and $h : T \to Y$ a morphism, then every irreducible component of $T \times_Y X$ has dimension two (By Corollary I.3.16 in [6] this is always the case if f is a well defined family of proper algebraic 1-cycles). Then the subset*

$$\{y \in Y : f^{-1}(y) \text{ has geometrically rational components}\} \subset Y$$

is closed in Y.

Proof. See Proposition II.2.2 in [6]. □

Corollary 1.2.2. *Let $g : U \to V$ be a family of proper algebraic 1-cycles of X/S. Let $U' \subset U$ be the set of points $u \in U$ which are contained in a geometrically rational component of $g^{-1}(g(u))$. The image of the natural morphism $U' \to X$ is called the rational locus of g. It is denoted by $\mathsf{RatLocus}(g : U \to V)$.*

Now let $V \to S$ is proper, then $\mathsf{RatLocus}(g : U \to V)$ is proper over S.

Proof. WLOG we let V is irreducible. Let $U = \sum_i a_i U_i$, then we just need to consider every $g_i : U_i \to V$. Consider the generic fiber D_i of g_i which is a irreducible curve, then if D_i rational, then so is whole g_i by Proposition 1.2.1. Hence $\mathsf{RatLocus}(g_i : U_i \to V) = \mathrm{Im}(U_i \to X)$ is proper over S. If D_i is not rational, then there is an open subset $\emptyset \neq W \subset V$ such that the fibers of g_i over Ware irreducible and nonrational. Thus

$$\mathsf{RatLocus}(g_i : U_i \to V) = \mathsf{RatLocus}(g_i : g_i^{-1}(V \backslash W) \to V \backslash W).$$

Hence we can apply Noetherian induction. □

Definition 1.2.3. *Let* $\mathsf{Hom_{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *be a subscheme correspond to the morphisms* $\mathbb{P}^1 \to X$ *birational to its image. By Lemma 1.1.20 since* $\mathbb{P}^1 \to X$ *birational to its image if and only if it is a immersion at its generic point, then* $\mathsf{Hom_{bir}}(\mathbb{P}^1, X) \subset \mathsf{Hom}(\mathbb{P}^1, X)$ *is an* ***open subscheme****.*

Definition 1.2.4. *Let* X/S *be a scheme, projective over* S*.*

(a) Let $\mathsf{Hom_{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} = \bigcup_i W_i$ *be the decomposition into irreducible subschemes of semi-normalization of* $\mathsf{Hom_{bir}}(\mathbb{P}^1, X)$*. By Proposition 1.1.16 we have the Hilbert-Chow morphism* $\mathsf{Hom_{bir}}(\mathbb{P}^1, X)^{\mathrm{sn}} \to \mathsf{Chow}_{X/S}$*. Let* $V_i' = \overline{\mathrm{Im}(U_i \to \mathsf{Chow}_{X/S})}$*. By Proposition 1.2.1* V_i' *parametrizes* 1*-cycles with geometrically rational components, and the generic* 1*-cycle is irreducible. Let* $V_i \subset V_i'$ *be the open subscheme parametrizing irreducible* 1*-cycles.*

Let $\eta_i \in V_i$ *be the generic points correspond to curves* C_i*. By generic smoothness* C_i *is a smooth rational curve. Let* V_i^{n} *be the normalization of* V_i*. Then we define the* ***family of rational curves*** *on* X *is*

$$\mathsf{RatCurves}^{\mathrm{n}}(X/S) := \bigcup_i V_i^{\mathrm{n}}.$$

with a normalization morphism $\mathsf{RatCurves}^{\mathrm{n}}(X/S) \to \mathsf{Chow}_{X/S}$*.*

If $\mathscr{L}$ *is ample on* X/S*, then we can define* $\mathsf{RatCurves}^{\mathrm{n}}(X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *where* $\mathsf{RatCurves}^{\mathrm{n}}_d(X/S)$ *is quasi-projective over* S *for any* d*. We define its* ***universal rational curve*** *is*

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) := \left(\mathsf{RatCurves}^{\mathrm{n}}(X/S) \times_{\mathsf{Chow}_{X/S}} \mathsf{Univ}^{\mathsf{Chow}}_{X/S}\right)^{\mathrm{n}}$$

be the normalization.

(b) Fix a section $f : S \to X$*. Similar as (a) we can define* $\mathsf{RatCurves}^{\mathrm{n}}(f, X/S) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(f, X/S)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(f, X/S)$*. This is called* ***family of rational curves passing through*** $\mathrm{Im}(f)$*.*

In particular if $S = \mathrm{Spec}\, k$ *where* k *is a field and* $f : (\mathrm{Spec}\, k) = x \in X$*, then we will use the notation* $\mathsf{RatCurves}^{\mathrm{n}}(x, X) = \coprod_d \mathsf{RatCurves}^{\mathrm{n}}_d(x, X)$ *and* $\mathsf{Univ}^{\mathsf{rc}}(x, X)$*.*

Theorem 1.2.5. *(a) Let $f : X \to Y$ be a proper and surjective morphism between irreducible and normal schemes. Assume that the dimension of every fiber is one (hence f is a well defined family of proper 1-cycles by Remark 1.1.11). Assume that for every $y \in Y$ the cycle theoretic fiber $f^{[-1]}(y)$ is an irreducible and reduced rational curve, then f is a $\mathbb{P}^1$-bundle.*

(b) In the case of the definition, the universal morphisms

$$\mathsf{Univ}^{\mathsf{rc}}(X/S) \to \mathsf{RatCurves}^{\mathrm{n}}(X/S) \text{ and } \mathsf{Univ}^{\mathsf{rc}}(x,X) \to \mathsf{RatCurves}^{\mathrm{n}}(x,X)$$

are $\mathbb{P}^1$-bundles.

Proof. (b) follows directly from (a), so we just need to prove (a).

One can show that f is smooth at the generic point of every fiber (see Theorem I.6.5 in [6]). For $y \in Y$ pick three different points $x_1, x_2, x_3 \in f^{-1}(y)$ such that f is smooth at x_i. Let $S_i \subset X$ be a Cartier divisor which intersects $f^{[-1]}(y)$ transversally at x_i (there may be other intersection points). Hence $S_i \to Y$ is étale at x_i. Let

$$Z = S_1 \times_Y S_2 \times_Y S_3, \quad z = (x_1, x_2, x_3) \in Z \text{ and } X_Z = X \times_Y Z.$$

So $Z \to Y$ is étale at z, thus X_Z is normal along $f_Z^{-1}(z)$ and f is smooth above y iff f_Z is smooth above z by some commutative algebra. Furthermore, f_Z has three sections $s_i : Z \to X_Z$ corresponding to the S_i. By shrinking Z we may assume that these sections are disjoint.

In $\mathbb{P}^1_Z \to Z$ we have three disjoint sections $p_i : Z \to \mathbb{P}^1_Z$ corresponding to $\{0, 1, \infty\}$. Our aim is the construct an isomorphism $q : \mathbb{P}^1_Z \cong X_Z$ such that $q \circ p_i = s_i$. Let $h : \mathbb{P}^1_Z \times_Z X_Z \to Z$ be the projection. In order to construct the graph of q let $\Gamma \subset \mathsf{Chow}_{\mathbb{P}^1_Z \times_Z X_Z/Z}$ be the closed subvariety parametrizing 1-cycles D with the following properties:

(1) $\deg \mathscr{O}_{\mathbb{P}^1}(1)|_D = 1$;

(2) $\deg \mathscr{O}(s_1(Z))|_D = 1$;

(3) $(p_i(h(D)), s_i(h(D))) \in D$ for $i = 1, 2, 3$.

Let $\mathsf{Univ}^\Gamma \to \Gamma$ be the universal family. We claim that the natural projections $\pi_1 : \mathsf{Univ}^\Gamma \to \mathbb{P}^1_Z$ and $\pi_2 : \mathsf{Univ}^\Gamma \to X_Z$ are isomorphisms.

For any $t \in Z$ consider $h^{-1}(t)$. By construction $(h^{-1}(t))_{\mathrm{red}} \cong \mathbb{P}^1_{\kappa(t)} \times C_t$ where C_t is an irreducible geometrically rational curve, smooth for general t. As D gives a 1-cycle on $(h^{-1}(t))_{\mathrm{red}}$ which has bidegree $(1,1)$, thus D is either the graph of a birational morphism $q_t : \mathbb{P}^1_{\kappa(t)} \to C_t$ or the union of a vertical and of a horizontal section. In the latter case it can not contain all three points $(p_i(t), s_i(t))$. Hence D is the graph of the unique birational morphism q_t such that $q_t(p_i(t)) = s_i(t)$ for $i = 1, 2, 3$. Thus π_1, π_2 are both one-to-one. If C_t is smooth, then q_t is defined over $\kappa(t)$, thus π_1, π_2 are isomorphisms over the generic point of Z. Since X_Z and $\mathbb{P}^1_Z$ are normal, this implies that π_1, π_2 are isomorphisms. Well done. $\square$

Remark 1.2.6. *In positive characteristic, (a) is right if we assume generic-smoothness.*

Proposition 1.2.7. *Notation as above definitions, then*

(a) *Let $m = \min\{d : \mathsf{RatCurves}^n_d(X/S) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(X/S)$ is proper over S for $k < 2m$.*

(b) *Let S be a field and let $m(x) = \min\{d : \mathsf{RatCurves}^n_d(x, X) \neq \emptyset\}$. Then $\mathsf{RatCurves}^n_k(x, X)$ is proper for $k < m + m(x)$.*

Proof. (b) follows from the same proof of (a). For (a), as $\mathsf{Chow}^{1,k}_{X/S}$ is proper over S, we just need to show that $\bigcup_i V_i \subset \mathsf{Chow}^{1,k}_{X/S}$ is closed where $\mathsf{RatCurves}^n_k(X/S) = \bigcup_i V_i^{\mathrm{n}} \to \bigcup_i V_i$ is finite. Let $\sum_i a_i D_i \in \overline{\mathsf{RatCurves}^n_k(X/S)}$, then every D_i is rational by Proposition 1.2.1 and $\sum_i a_i \deg D_i = k < 2m$. By assumption $\deg D_i \geq m$, then $\sum_i a_i D_i$ is an irreducible and reduced rational curve. Hence $\mathsf{RatCurves}^n_k(X/S)$ closed. $\square$

Theorem 1.2.8. *Let $\mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}$ be the normalization of $\mathsf{Hom}_{\mathsf{bir}}$, then we have the following important results:*

(a) *Let X/S projective scheme over S, then there is a natural commutative diagram*

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{\quad U \quad} & \mathsf{Univ}^{\mathsf{rc}}(X/S) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1_S, X/S) & \xrightarrow{\quad u \quad} & \mathsf{RatCurves}^{\mathrm{n}}(X/S) \end{array}$$

where U and u are smooth of relative dimension 3 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(P^1)$-bundles)

(b) *Let X projective scheme over k with a k-point $x \in X(k)$, then there is a natural commutative diagram*

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{\quad U \quad} & \mathsf{Univ}^{\mathsf{rc}}(x, X) \\ \downarrow & & \downarrow \\ \mathsf{Hom}^{\mathrm{n}}_{\mathsf{bir}}(\mathbb{P}^1, X; 0 \mapsto x) & \xrightarrow{\quad u \quad} & \mathsf{RatCurves}^{\mathrm{n}}(x, X) \end{array}$$

where U and u are smooth of relative dimension 2 with connected fibers. (In fact both U and u are principal $\mathrm{Aut}(P^1; 0)$-bundles)

Proof. These are easy but boring since we consider the characteristic zero. See [6] Theorem II.2.15 and II.2.16. $\square$

1.3 Free and Minimal Rational Curves

1.4 Bend and Break

1.5 Application I: Basic Theory of Fano Manifolds

[9]

1.6 Application II: Boundedness of Fano Manifolds

1.7 Application III: Hartshorne's Conjecture

Chapter 2

Varieties of Minimal Rational Tangents

We will assume the base field is $\mathbb{C}$.

2.1 Basic Properties

2.2 Birationality of the Tangent Morphism

2.3 Examples of VMRT

2.4 Distributions and Its Properties

2.5 Cartan-Fubini Type Extension Theorem

Chapter 3

Some Basic Applications of VMRT

3.1 Stability of the Tangent Bundles

3.2 The Remmert-Van de Ven / Lazarsfeld Problem

3.3 Deformation Rigidity

3.4 Uniqueness of Contact Structures

Chapter 4

About Semiample Tangent Bundles

Chapter 5

Need to add

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