

Topic 11

Heap, Set and Map

資料結構與程式設計
Data Structure and Programming

11/27/2019

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Linear Data Types

- ◆ In previous topic and Homework #5, we have learned linear data types like list and array
 - Tradeoffs between insert/delete/find operators
 - Memory overhead
 - ➔ Constant time for “push_back()” or “push_front()” operation
- ◆ The best way to use linear data types is ---
 - Data are recorded in a linear sequence (i.e. only push_back or push_front is needed)
 - Linearly traverse each element (i.e. for(...; li++))
 - No “find”, “insert any”, nor “delete any”
 - ➔ If needed, use “tree”?

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Consider the Scenario...

- ◆ Suppose we are assigning jobs sequentially to several machines ---
 - One job to one machine and we record the accumulated runtime for each machine.
 - Our machine selection criteria is to “even out” the runtime of the machines.
 - In other words, we would like to pick the machine with least accumulated runtime for the next job
- ➔ Do we need to sort ALL the elements?
- ➔ Need a priority queue

Priority Queue

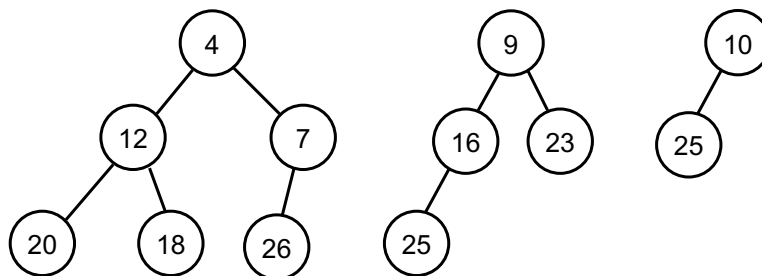
- ◆ An ADT that supports 2 operations
 - Insert
 - Delete min(or max)
- ◆ An element with arbitrary priority can be inserted to the queue
- ◆ At any time, it should take constant time to find the element with min(or max) priority and could efficiently remove it from the list
 - Need to figure out which is the one with next lowest(highest) priority efficiently

Using List or Array?

- ◆ Use linear ADT with an extra field to record the element with min(max) priority
 - Insert: $O(1)$
 - Delete min(max): $O(n)$
(why?)
- ◆ As we learn before, $O(n)$ is not good. We would prefer an ADT with $O(\log n)$ for both operations

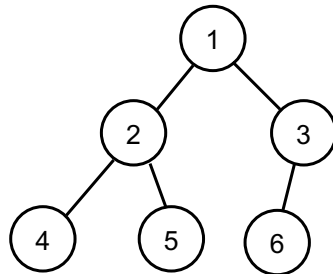
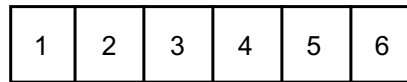
Min (Max) Heap

- ◆ A complete binary tree in which the key value in each node is no larger (smaller) than its children



Why complete binary tree?

Remember that we can use array to implement a complete binary tree...

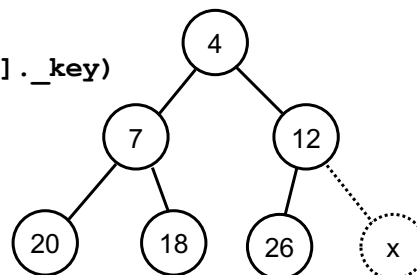


◆ Parent
= child / 2

◆ Child
= Parent * 2
or Parent * 2 + 1

MinHeap Insertion

```
// Let n be the index of the last element
void MinHeap::insert(const T& x)
{
    int t = ++n; // next to the last
    while (t > 1) {
        int p = t / 2;
        if (x._key >= _heap[p]._key)
            break;
        _heap[t] = _heap[p];
        t = p;
    }
    _heap[t] = x;
}
```

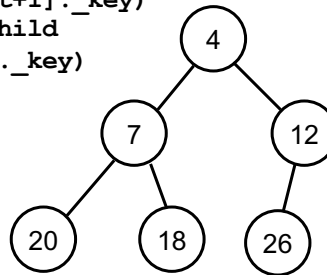


What's the time complexity?

Delete Min Element

```
T& MinHeap::deleteMin()
{
    T ret = _heap[1];
    int p = 1, t = 2 * p;
    while (t <= n) {
        if (t < n) // has right child
            if (_heap[t]._key > _heap[t+1]._key)
                ++t; // to the smaller child
        if (_heap[n]._key < _heap[t]._key)
            break;
        _heap[p] = _heap[t];
        p = t;
        t = 2 * p;
    }
    _heap[p] = _heap[n--];
    return ret;
}
```

- Remove root
- Recursively move the smaller child up
- Stop at the parent whose children are larger than the last node
- Replace the parent with the last node



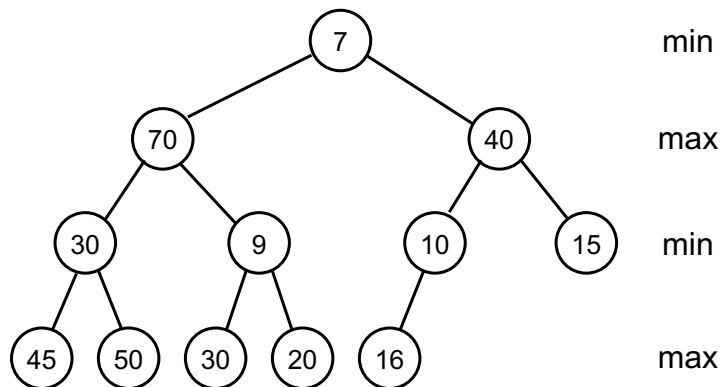
What's the time complexity?

Min(Max) Heap

- ◆ Simple implementation (just an array)
- ◆ Good insertion and deleteMin complexity
 - $O(\log n)$ vs. $O(n)$

What if you want to
delete min AND delete max?

Min-Max Heap



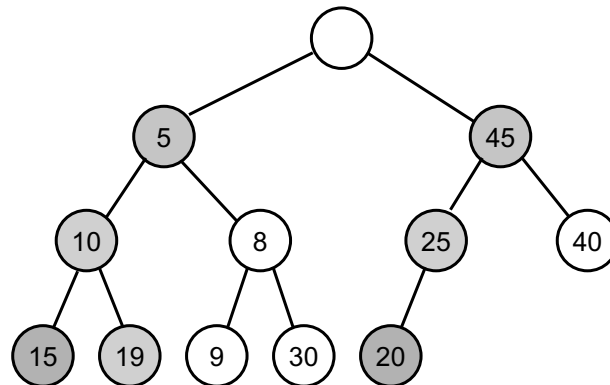
- Insert, delete min, delete max: all $O(\log n)$ (why?)

Deap

◆ Double-ended heap

1. The root contains no element
2. The left subtree is a min heap
3. The right subtree is a max heap
4. Let i be any node in the left subtree. Let j be the corresponding node in the right subtree. If such a j node does not exist, then let j be the corresponding parent of i .
→ The key in node i is less than or equal to that in j .

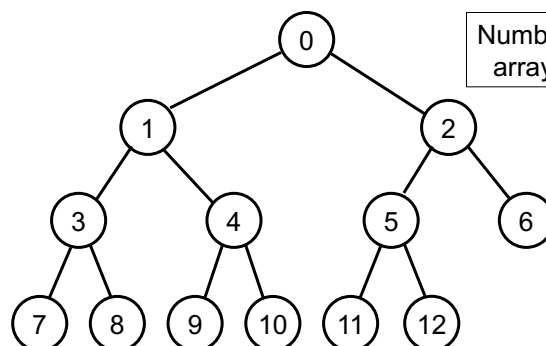
Deap Example



- Insert, delete min, delete max: all $O(\log n)$ (why?)
 - But faster than min-max heap by a constant factor
 - Algorithm is simpler

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Deap Implementation



Numbers are indices in array, not data value

- Given a node 'i', how to find the "corresponding parent" or "corresponding child"?
- When insertion or deletion, what should we do when the node value is greater/smaller than its corresponding parent/child?

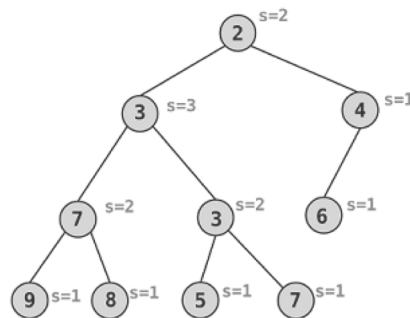
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Practice #1

- ◆ Write the pseudo codes for the “insert”, “delete min”, and “delete max” operations of the min-max heap.
- ◆ Write the pseudo codes for the “insert”, “delete min”, and “delete max” operations of the deap.

More Varieties of Heaps: Leftist Heap

- ◆ In contrast to a *binary heap*, a leftist heap attempts to be very unbalanced.
 - s-value(v):
the distance to the nearest leaf.
 - In addition to the heap property, the right child of each node has the lower s-value.
- ◆ Support
“combine(heap1, heap2)” in $O(\log n)$

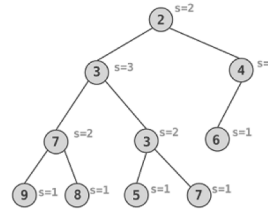


Leftist Heap: Huh?

- ◆ Remember: “combine(heap1, heap2)” in $O(\log n)$
 - Both “insert” and “deleteMin” operations can be realized by “combine”. (How?)

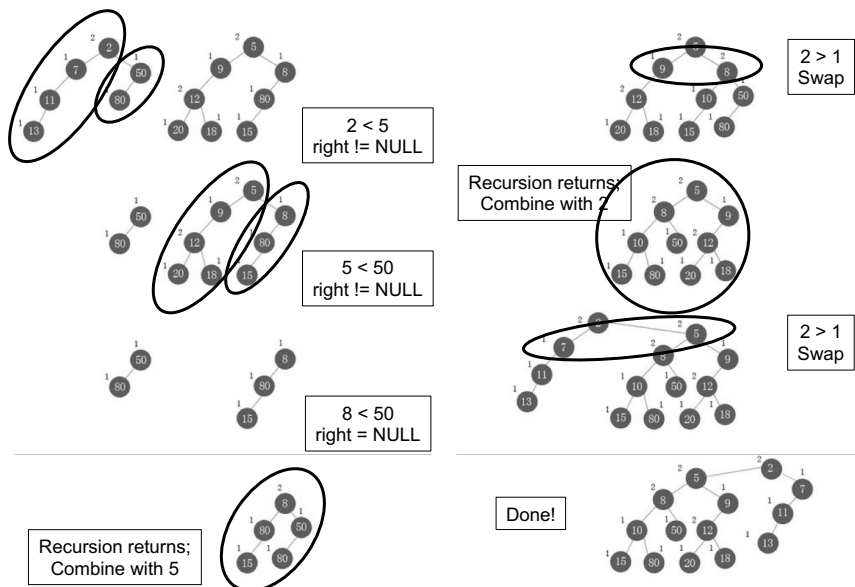
```

combine(h1, h2) {
  compare(min(h1), min(h2));
  // let min(hi) < min(hj)
  if (right(hi) == NULL)
    right(hi) = hj;
  else
    combine(right(hi), hj);
  // hj is now the combined heap
  if (s(right(hj)) > s(left(hj))
    swap(right(hj), left(hj));
}
    
```



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Leftist Heap: Combine [src] <http://blog.yam.com/rockmanray/article/44962825>



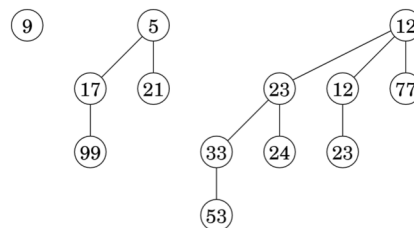
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Leftist Heap: Discussion

- ◆ How can “insert” and “delete” be implemented by “combine”?
- ◆ Can we implement “combine” efficiently in a min-heap?
- ◆ Can the leftist heap be implemented in an array, like a min-heap does?

More Varieties of Heaps: Binomial heap

- ◆ Binomial tree of order k
 - Binomial tree of order 0 is a single node
 - The root of a binomial tree of order k has k children, who are roots of binomial trees of order $k-1, k-2, \dots, 0$
 - Has exactly 2^k nodes; height = k
- ◆ Binomial heap
 - A collection of Binomial trees
 - Most operations have the complexity $O(\log n)$
 - But the amortized complexity is either $O(1)$ or $O(\log n)$



Binomial Heap: Properties

- ◆ Given a binomial heap with n nodes:
 - The node containing the min element is a root of $B_0, B_1, \dots, \text{or } B_k$.
 - It contains the binomial tree B_i iff $b_i = 1$, where $b_k \dots b_2 b_1 b_0$ is binary representation of n .
 - It has $\leq \lfloor \log_2 n \rfloor + 1$ binomial trees.
 - Its height $\leq \lfloor \log_2 n \rfloor$.

[src] <http://www.cs.princeton.edu/~wayne/kleinberg-tardos/pdf/BinomialHeaps.pdf>

Binomial Heap: Operations

- ◆ Similar to Leftist Heap, the operations of Binomial Heap can be realized by the “compose” (aka. “meld”) operation.
- ◆ Compose operation:
 - Binary addition
 - Given two binomial heaps

$$H_1 := \{ (B_3, B_2, B_1, B_0) = (1, 1, 0, 1) \}, \text{ and}$$

$$H_2 := \{ (B_4, B_3, B_2, B_1, B_0) = (1, 0, 1, 0, 1) \}.$$
 The composed binomial heap

$$H_m := \{ (B_5, B_4, B_3, B_2, B_1, B_0) = (1, 0, 0, 0, 1, 0) \}.$$

Binomial Heap: Compose Operation

- ◆ Atomic operation:
 - Given two binomial trees B_i, B_j , with the *same order* k , then $\text{compose}(B_i, B_j)$:
 1. Connect the roots r_i, r_j of B_i, B_j .
 2. Choose $\min(r_i, r_j)$ as the root of the composed tree
 3. The composed tree is of order $k+1$
 - ➔ What's the time complexity? $O(1)$
 - ➔ What if we have three binomial trees with the same order?
- ◆ The compose operation of two binomial heaps:
 1. Align the binomial trees of both heaps
 2. From the trees with the least order, perform tree composition
 3. Propagate to the next order of tree if necessary
- ◆ What's the time complexity? $O(\log n)$

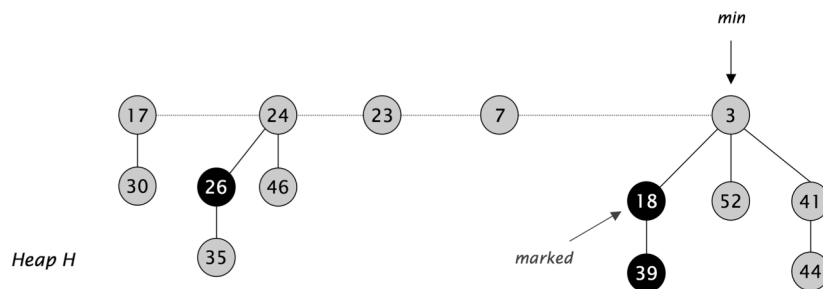
Binomial Heap: Other Operations

- ◆ FindMin
 - // remember: It has $\leq \lfloor \log_2 n \rfloor + 1$ binomial trees
 - $O(\log n)$
- ◆ DeleteMin
 - Note: after the "min" is removed, the corresponding binomial tree (of order k) is broken and becomes k binomial trees
 - It just becomes "compose" operations of some binomial trees // How many? Compose of 2 or 3?
 - $O(\log n)$
- ◆ DeleteNode(iterator pos)
 - $O(\log n)$
- ◆ Insert(x)
 - $O(\log n)$

More Varieties of Heaps: Fibonacci heap

◆ Fibonacci heap

- Especially useful when `deleteMin()` & `delete(n)` are rarely called → amortized $O(\log n)$
- All other operations are $O(1)$



Fibonacci Heap

◆ Basic idea

- Similar to binomial heaps, but less rigid structure
- Binomial heap: eagerly consolidate trees after each insert (maintain binomial structure)
- Fibonacci heap: lazily defer consolidation until next **delete-min**

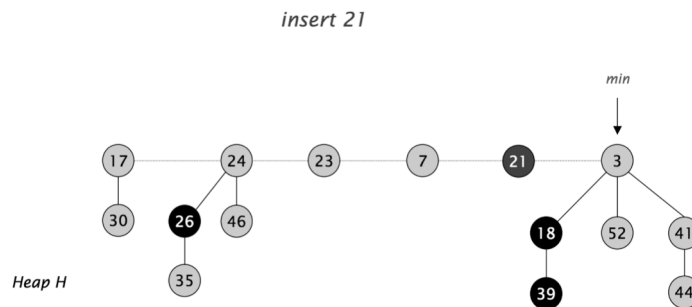
◆ Properties

- Set of heap-ordered trees.
- Maintain pointer to minimum element
- Set of marked nodes (if one of its children is removed)

(Ref) <https://www.cs.princeton.edu/~wayne/teaching/fibonacci-heap.pdf>

Fibonacci Heap: Insert Operation

- ◆ Create a new singleton tree.
- ◆ Add to root list; update min pointer (if necessary) → $O(1)$



(Ref) <https://www.cs.princeton.edu/~wayne/teaching/fibonacci-heap.pdf>

Fibonacci Heap: DeleteMin Operation

- ◆ Let H be a Fibonacci heap and x be a node
 - $\text{rank}(x)$: number of children of node x
 - $\text{rank}(H)$: max rank of any node in heap H
 - $\text{tree}(H)$: number of trees in heap H
- ◆ DeleteMin
 - Delete min; meld its children into root list; update min
 - Consolidate trees so that no two roots have same rank // What about the min pointer?
 - Time complexity: $O(\text{rank}(H)) + O(\text{trees}(H))$
 - Amortized cost: $O(\text{rank}(H))$

Heap Operations Supported in STL

- ◆ STL does not have a “heap” class
 - Instead, it support several operations that can operate on “array” like data structure
- ◆ Operations
 - `void make_heap(first, last[, comp]);`
 - `void push_heap(first, last[, comp]);`
 - `void pop_heap(first, last[, comp]);`
 - `void sort_heap(first, last[, comp]);`
 - `bool is_heap(first, last[, comp]);`
 - ➔ *first, last: RandomAccessIterator*
 - ➔ *comp: StrictWeakOrdering (optional)*

Heap Operations Supported in STL (Example)

```
{
    int size = 100;
    int arr[size];
    // write data to arr
    ...
    make_heap(arr, arr+size);
    int min = arr[0];
    pop_heap(arr, arr+(--size));
    arr[size++] = min*min;
    push_heap(arr, arr+size);
    sort_heap(arr, arr+size);
    // heap is now sorted
}
```

Summary: Heap Structures

◆ Pros:

1. Good complexity of “insert”, “delete min(max)”, ... operations
2. Simple data structure (low memory overhead)
3. Simpler algorithms (than BST)

◆ Con

1. Data are not sorted
→ Still have $O(n)$ for “find” operation

Review: Binary Search Trees

◆ Binary Search Trees (BSTs)

- Left subtree \leq this \leq right subtree
- Complexity depends on the height of the tree
- Worst case: can be degenerated as a tree with height $O(n)$

◆ Balanced BSTs

- The heights of left subtree and right subtree are somewhat balanced
 - Height $\sim O(\log n)$
- Examples: AVL, 2-3, 2-3-4, red-black, splay trees
- Algorithms for their operations are complicated

Sorted ADT in STL

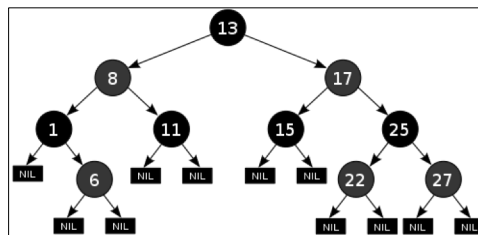
◆ Also classified as “Associative Containers”

1. set
2. multiset
3. map
4. multimap

➔ Implemented in “red black tree”

Red Black Tree (More to cover later)

- ◆ A node is either red or **black**. The root is **black**
- ◆ All leaves are black (i.e. All leaves are same color as the root.)
- ◆ Every red node must have two **black** child nodes.
- ◆ Every path from a given node to any of its descendant leaves contains the same number of **black** nodes.
- ◆ Memory efficient
- ◆ Although balancing is NOT perfect, $O(\log n)$ for insert, delete, and find



class set in STL

- ◆ To store elements in a set
 - e.g. { 2, 3, 5, 7, 9 }
- ◆ set<Key[, Compare, Alloc]>
 - class Key: element type
 - class Compare: how the elements are compared (optional; default = less<Key>)
 - class Alloc: used for internal memory management (optional; default = alloc)

Member Functions in class set

1. iterator begin() const;
iterator end() const;
2. pair<iterator, bool> insert(const value_type& x);
iterator insert(iterator pos, const value_type& x);
void insert(InputIterator, InputIterator);
3. void erase(iterator pos);
size_type erase(const key_type& k);
void erase(iterator first, iterator last);
4. iterator find(const key_type& k) const;
5. size_type count(const key_type& k) const;
6. iterator lower_bound(const key_type& k) const;
iterator upper_bound(const key_type& k) const;
pair<iterator, iterator> equal_range(const key_type& k) const;

Other Functions for class set

1. includes
 - Check if one set is included in another
2. set_union
3. set_intersection
4. set_difference
5. set_symmetric_difference
 - $(A - B) \cup (B - A)$

class multiset in STL

- ◆ Unlike “set”, where elements with same value are stored only once, in multiset, they can be stored repeatedly
 - e.g. { 2, 3, 5, 5, 6, 7, 7, 7 }
- ◆ multiset<Key[, Compare, Alloc]>
 - class Key: element type
 - class Compare: how the elements are compared (optional; default = less<Key>)
 - class Alloc: used for internal memory management (optional; default = alloc)

class map in STL

- ◆ In many applications, data are associated with keys (or id's)
 - For example, (id, student record)
 - e.g. { (Mary, 90), (John, 85), (Sam, 71) ... }
- ◆ class map<Key, Data[, Compare, Alloc]>
 - class Key: compared data type
 - class Data: value type
 - class Compare: how the elements are compared (optional; default = less<Key>)
 - class Alloc: used for internal memory management (optional; default = alloc)

Example of using class map (1)

```
map<string, unsigned> scoreMap;
scoreMap["Mary"] = 90;
scoreMap["John"] = 85;
scoreMap["Sam"] = 71;
unsigned maryScore = scoreMap["Mary"];
cout << "Mary's score = " << maryScore << endl;
map<string, unsigned>::iterator mi;
mi = scoreMap.find("John");
if (mi != scoreMap.end())
    cout << "John's score = " << (*mi).second << endl;
➔ How about "map<const char*, unsigned>"?
```

Comments about map::operator []

- ◆ Since operator[] might insert a new element into the map, it can't possibly be a const member function.
- ◆ Note that the definition of operator[] is somehow tricky: m[k] is equivalent to `((m.insert(value_type(k,data_type()))).first)).second`.
 - value_type = pair<Key, Data>
 - insert(value_type) returns a pair<map::iterator, bool>
- ◆ Strictly speaking, this member function is unnecessary: it exists only for convenience.

<http://www.sgi.com/tech/stl/Map.html>

Bad example of using class map

```
map<const char*, unsigned> mmm;  
map<const char*, unsigned>::iterator mi;  
char buf[1024];  
cin >> buf; mmm[buf] = 10;  
cin >> buf; mmm[buf] = 20;  
cin >> buf; unsigned s1 = mmm[buf];  
cout << buf << " = " << s1 << endl;  
cin >> buf; unsigned s2 = mmm[buf];  
cout << buf << " = " << s2 << endl;
```

Example of using class map (2)

```
string str;
for (int i = 0; i < 5; ++i) {
    cin >> str; mm.insert(pair<string, int>(str, i));
}
while (1) {
    cin >> str;
    map<string, int>::iterator mi = mm.find(str);
    if (mi == mm.end()) {
        cout << "Not found!!" << endl;
        break;
    }
    cout << (*mi).first << " = " << (*mi).second << endl;
}
```

Conclusion: Set and Map

- ◆ “set” and “map” are useful data structures when we need to perform efficient “insert”, “erase”, and “find” operations
 - Usually implemented by balanced binary search trees
 - Implementation efforts can be high
 - Using STL may be a good choice
- ◆ Remember, unbalanced BSTs may not be a bad choice
 - Most randomly inserted BSTs are somewhat balanced
- ◆ Remember, there’s no free lunch
 - Overhead in insert (vs. push_back)
 - If we don’t need to do “erase” or “find” during insertions... (what’s the alternative?)