SSY130 - Applied Signal Processing Hand in Problems

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1.1 a

In order to ensure a perfect reconstruction of the signal it is necessary that

$$\omega_{max} \leq \frac{\omega_s}{2}$$

If this criteria is not fulfilled there is a risk that aliasing might occur. In this case it is necessary that

$$\omega_s > 260\pi \times 10^3$$

1.2 b

The condition that has to be satisfied can be described with the inequality

$$20 \mid N_d(130\pi \times 10^3) \mid \leq X_d(0)$$

Where N_d is the DTFT of the filtered and sampled noise signal for $\mid \omega \mid < 130\pi \times 10^3$ and X_d the DTFT of the filtered and sampled desired signal at $\omega = 0$. Evaluating this inequality will lead to a condition that ω_s needs to satisfy. Notice that since

$$X_d(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X_f(\omega + \omega_s k)$$

then the right hand side of the inequality simply becomes $\frac{1}{\Delta t}$. Simplify:

$$\frac{20}{\Delta t} \frac{0.1}{\mid 1 + j\frac{130\pi \times 10^3 - \omega_s}{\omega_0} \mid} \leq \frac{1}{\Delta t}$$

$$\frac{2}{1 + (\frac{130\pi \times 10^3 - \omega_s}{150\pi \times 10^3})^2} \leq 1$$

$$\pm 1 \leq \frac{130\pi \times 10^3 - \omega_s}{150\pi \times 10^3}$$

$$\omega_s \geq (130 \pm 150)\pi \times 10^3$$

Since the sampling rate $\omega_s = -20\pi \times 10^3$ does not fulfill the previously stated requirement $\omega_s > 260\pi \times 10^3$ then the answer becomes $\omega_s > 280\pi \times 10^3$.

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Consider the signal

$$x_d(n) = sin(2\pi n \frac{f_0}{f_s})$$

This can be rewritten using Euler's formula as

$$x_d(n) = \frac{1}{2j}(e^{2\pi jn\frac{f_0}{f_s}} - e^{-2\pi jn\frac{f_0}{f_s}})$$

The DTFT of this equation is

$$x_d(n) \xrightarrow{DTFT} X_d(\omega) = \frac{1}{2j}(\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0))$$

where here the tilde over δ refers to the periodicity of the DTFT, more specifically that the function evaluates to 1 not only when 0 but also at $\omega_s k$, where $k = 0, \pm 1, \pm 2, \pm 3...$

Reconstruct the signal by using zero-order-hold method, which can be seen as continuous time filtering of a discrete time signal through the ZOH filter. In the Fourier domain we have

$$X(\omega) = H_{ZOH}(\omega)X_d(\omega)$$

where

$$H_{ZOH}(\omega) = \Delta t e^{-j\frac{\omega}{\omega_s}} \frac{sin(\pi\frac{\omega}{\omega_s})}{\pi\frac{\omega}{\omega_s}}$$

The function for evaluating the amplitude of the frequencies then becomes

$$X(\omega) = \frac{1}{2j}(\tilde{\delta}(\omega - \omega_0) - \tilde{\delta}(\omega + \omega_0))\Delta t e^{-j\pi\frac{\omega}{\omega_s}} \frac{sin(\pi\frac{\omega}{\omega_s})}{\pi\frac{\omega}{\omega_s}}$$

This can be used for finding the magnitude of the fundamental component and the three first harmonic components in the reconstructed signal.

$\Delta t^{-1} \mid X(\omega_s k) \mid$	k
0.0865	-3
0.1365	-2
0.3226	-1
0.8871	0
0.1868	1
0.1044	2
0.0724	3