Problem 1

Let A be a bounded nonempty subset of $\mathbb{R}$, $\beta \in \mathbb{R}$ and $\beta < 0$. Let $\beta A = \{\beta a : a \in A\}$. Prove that

$$sup(\beta A) = \beta inf(A).$$

Do not use Proposition 1.2.6.

Proof. The $inf(A)$ exists since A is bounded and non-empty, which also means the set βA is non-empty. Now, for all $y \in \beta A$, $\exists x \in A$, such that $y = \beta x$. Then since $inf(A)$ is a lower bound of A and x is in A, we get

$$x \geq inf(A).$$

Since we are given that $\beta < 0$, when we multiply by β to the inequality, the sign flips sides, giving us

$$\beta x \leq \beta inf(A).$$

Then we can replace the right hand side with $y = \beta x$ which gives us

$$y \leq \beta inf(A).$$

Thus we can say that for all $y \in \beta A$, $y \leq \beta inf(A)$ and thus by the definition of upper bound, $\beta inf(A)$ is an upper bound of βA. Then, since $sup(\beta A)$ is the least upper bound of βA, and $\beta inf(A)$ is an upper bound of βA we can say that

$$sup(\beta A) \leq \beta inf(A). \qquad \text{Inequality 1}$$

Now for all $x \in A$, $\beta x \in \beta A$ by the definition of set βA. Since $\beta x \in \beta A$ and $sup(\beta A)$ is an upper bound of βA, we can say that

$$\beta x \leq sup(\beta A).$$

We also know that $\frac{1}{\beta} < 0$ since $\beta < 0$, and thus when we multiply the inequality by $\frac{1}{\beta}$ we flip the direction of the inequality sign, giving us

$$\frac{1}{\beta}\beta x \geq \frac{1}{\beta} sup(\beta A).$$

Now by the existence of a multiplicative inverse and then by identity property of multiplication we can simplify the $\frac{1}{\beta}\beta$ on the left hand side to get

$$x \geq \frac{1}{\beta} sup(\beta A).$$

Thus, for all $x \in A$, $x \geq \frac{1}{\beta} sup(\beta A)$ and by the definition of lower bound, $\frac{1}{\beta} sup(\beta A)$ is a lower bound of A. Now since $inf(A)$ is the greatest lower bound of A and $\frac{1}{\beta} sup(\beta A)$ is a lower bound of A, it means that

$$\frac{1}{\beta} sup(\beta A) \leq inf(A)$$

Then since $\beta < 0$, when we multiply the inequality by β we flip the direction of the inequity to get

$$\beta \frac{1}{\beta} sup(\beta A) \geq \beta inf(A)$$

Now by the existence of a multiplicative and then by identity property of multiplication we can simplify the left hand side $\beta \frac{1}{\beta} sup(\beta A)$ to get

$$sup(\beta A) \geq \beta inf(A) \qquad \text{Inequality 2}$$

Then when we combine Inequality 1 with Inequality 2 it must hold that $sup(\beta A) = \beta inf(A)$. ■

Problem 2

Let $x, y, t \in \mathbb{R}$ such that $x < y$ and $t > 0$. Prove that there exists a $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$x < \frac{\lambda}{t} < y.$$

Proof. First we can can take the inequality $x < y$ and multiply both sides by t since we know that $t > 0$ and $t \in \mathbb{R}$ and thus the direction the inequality will hold:

$$tx < ty$$

We know that tx and ty are in $\mathbb{R}$ since the real numbers are closed under multiplication. Then we we can say by the density of the irrationals (Theorem 18) that if tx and ty are in $\mathbb{R}$ and $tx < ty$ then there exists a λ in $\mathbb{R} \setminus \mathbb{Q}$ such that

$$tx < \lambda < ty$$

We also know that $\frac{1}{t} > 0$ holds since $t > 0$. Thus we can multiply all sides of the inequality by $\frac{1}{t}$ and the sign holds, giving us

$$\frac{1}{t} tx < \frac{1}{t} \lambda < \frac{1}{t} ty$$

Now by we can simplify the $\frac{1}{t} t$ with the existence of multiplicative inverse, making it 1, and then multiplying one into the right and left hand sides by the identity property of multiplication. The middle terms can just combined into a single fraction. Doing these operations gives us

$$x < \frac{\lambda}{t} < y$$

Therefore we showed that there exists a $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \frac{\lambda}{t} < y$ for $x, y, t \in \mathbb{R}$. ■

Problem 3

Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Prove that there exists an $r \in Q$ such that $r > 0$ and $a < \sqrt{r} < b$.

Proof. To start we know that a^2 and b^b are in $\mathbb{R}$ and are greater than 0 since the square of any real number is greater than 0 and real numbers are closed under multiplication. We also know that since $0 < a < b$ we can square both sides of $a < b$ and the direction of the inequality is maintained as proved in problem 2 in homework 3, thus

$$a^2 < b^2.$$

Now by the density of rationals (Theorem 16), since a^2 and b^b are in $\mathbb{R}$ and $a^2 < b^2$, there exits a r in $\mathbb{Q}$ such that

$$a^2 < r < b^2.$$

Then $r > 0$, and since $a^2 > 0$ and $r > a^2$, we know $a^2 < r$ and $r < b^2$. Now by citing the first additional problem at the end of the page, since $0 < a^2 < r < b^2$ we can square root both sides of $a^2 < r$ and $r < b^2$ to get

$$a < \sqrt{r} \qquad \text{and} \qquad \sqrt{r} < b$$

Thus, by combining the two inequalities we get

$$a < \sqrt{r} < b$$

Therefore we have showed that there exists a $r \in Q$ such that $r > 0$ and $a < \sqrt{r} < b$. ∎

Problem 4

Find all the real numbers x which satisfy $x^2 < 3|x| + 10$. Justify your answer.

Proof. We know that x^2 is equivalent to $|x|^2$, and thus by substituting the left hand side of $x^2 < 3|x| + 10$ we get

$$|x|^2 < 3|x| + 10.$$

Now by subtracting $3|x| + 10$ to both sides of the inequality we get

$$|x|^2 - (3|x| + 10) < 0.$$

Then we can distribute the -1 into $(3|x| + 10)$ to give us

$$|x|^2 - 3|x| - 10 < 0.$$

Now we can we can make $-3|x|$ into $2|x| - 5|x|$:

$$|x|^2 + 2|x| - 5|x| - 10 < 0.$$

Then we can now use Axiom D, the distributive law, to distribute a $|x|$ out of $|x|^2 + 2|x|$ and a -5 out of $-5|x| - 10$, giving us the inequality

$$|x|(|x| + 2) - 5(|x| + 2) < 0$$

Now using the distributive law to take out a $(|x|+2)$ from the whole left side of the inequality to give us

$$(|x| + 2)(|x| - 5) < 0$$

We know that $|x| > 0$ by definition of absolute value and $2 > 0$, thus $|x| + 2 > 0$ since the sum of two positive numbers is also positive. Thus, we know that $\frac{1}{|x|+2} > 0$ since $|x| + 2 > 0$, then its multiplicative inverse is also greater than zero. So now we can multiply both sides of $(|x| + 2)(|x| - 5) < 0$ by $\frac{1}{|x|+2}$ and the direction of the sign is maintained, giving us

$$\frac{1}{|x| + 2}(|x| + 2)(|x| - 5) < \frac{1}{|x| + 2}0$$

Then the $\frac{1}{|x|+2}(|x|+2)$ on the left hand side reduced to one by the existence of a multiplicative inverse and multiplies into $(|x| - 5)$ by the identity property of multiplication, and on the right hand side the multiplication of any real number gives us 0, thus

$$|x| - 5 < 0$$

Now we can add 5 to both sides which cancels the -5 on the left hand side:

$$|x| < 5$$

Thus, by Theorem 19(V), since $|x| < 5$ then

$$-5 < x < 5$$

Therefore, $x \in \mathbb{R}$ satisfies $x^2 < 3|x| + 10$ when $-5 < x < 5$. ∎

Problem 5

Let $\alpha, \beta \in \mathbb{R}$ and let $min\{\alpha, \beta\}$ denote the smaller of the two. Prove that

$$min\{\alpha, \beta\} = (\alpha + \beta - |\alpha - \beta|)/2.$$

Proof. We can prove this using two cases for $\alpha, \beta \in \mathbb{R}$:

$$\underline{Case\ 1:}\ \alpha \geq \beta \qquad\qquad \underline{Case\ 2:}\ \alpha \leq \beta$$

Case 1: We can subtract β to both sides of $\alpha \geq \beta$ to get $\alpha - \beta \geq 0$. Then $|\alpha - \beta| = \alpha - \beta$ by the definition of absolute value since $\alpha - \beta$ is greater than zero. Now, we will use the expression $(\alpha + \beta - |\alpha - \beta|)/2$, if we substitute $\alpha - \beta$ for $|\alpha - \beta|$ we get

$$(\alpha + \beta - (\alpha - \beta))/2.$$

Simplifying by using the distributive law on -1 with $-(\alpha-\beta)$ gives us

$$(\alpha+\beta-\alpha+\beta))/2.$$

Then the α and $-\alpha$ cancel out and the two $\beta's$ can be added together into 2β, giving us

$$(2\beta)/2.$$

Then we can divide out the 2 from the 2β to give

$$\beta.$$

Thus, since $\alpha \geq \beta$, the min between α and β is β, and when we simplify $(\alpha+\beta-\alpha+\beta))/2$ for this case we have shown that this expression equals β, thus we have shown that for Case 1 $min\{\alpha,\beta\} = (\alpha+\beta-|\alpha-\beta|)/2$.

Case 2: We can subtract β from both sides of $\alpha < \beta$ to get get $\alpha-\beta < 0$. Then $|\alpha-\beta|$ would equal to $-(\alpha-\beta)$ by the definition of absolute value since $\alpha-\beta<0$. Then by simplifying $-(\alpha-\beta)$ we get $-\alpha+\beta$, thus $|\alpha-\beta|$ is show to equal $-\alpha+\beta$ and we can substitute $-\alpha+\beta$ for $|\alpha-\beta|$ into $(\alpha+\beta-|\alpha-\beta|)/2$ to get

$$(\alpha+\beta-(-\alpha+\beta))/2.$$

Then we can distribute the negative one into $(-\alpha+\beta)$ to get

$$(\alpha+\beta+\alpha-\beta)/2.$$

The β's cancel out and we can add the α's to get

$$2\alpha/2.$$

Then dividing the 2 out from 2α gives us

$$\alpha.$$

Thus, since $\alpha < \beta$, the min between α and β is α, and we showed that in this case that $(\alpha+\beta-\alpha+\beta))/2$ simplifies into α, thus for Case 2 we have shown that the $min\{\alpha,\beta\} = (\alpha+\beta-|\alpha-\beta|)/2$.
Therefore, we have finished our proof since we have shown for both cases, which together covers all real numbers, that $min\{\alpha,\beta\} = (\alpha+\beta-|\alpha-\beta|)/2$. ■

Problem 6

Prove that, for every $x \in \mathbb{R}$,

$$2|x-1|+|2x+1| \geq 3$$

Proof. Let $a = -2(x-1)$ and $b = 2x+1$. We know that a and b are real numbers since x is a real number and the real numbers are closed under addition and multiplication. Now using the Triangle Inequality with a and b we get

$$|-2(x-1)+2x+1| \leq |-2(x-1)| + |2x+1|.$$

Then, on the left hand side of the inequality you can distribute -2 into $x-1$, and on the right hand side we can use Theorem 19, $|xy| = |x||y|$, on $|-2(x-1)|$, giving us

$$|-2x+2+2x+1| \leq |-2||(x-1)| + |2x+1|.$$

Then, on the left hand side the $-2x$ and $2x$ add to 0 and the 2 and 1 add together to become 3, and on the right hand side the $|-2| = --2$, by the definition of absolute value, and simplifies to 2, giving us

$$|3| \leq 2|(x-1)| + |2x+1|.$$

Then the $|3| = 3$ by the definition of absolute value giving us $3 \leq 2|(x-1)| + |2x+1|$, and by reversing the terms and signs we get the equivalent inequality of

$$2|(x-1)| + |2x+1| \geq 3.$$

Therefore, we have shown that for every $x \in \mathbb{R}$, $2|x-1| + |2x+1| \geq 3$.

■

Problem 7

For each $n \in \mathbb{N}$, let $I_n = (-\infty, -n]$. Show that $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Proof. We will prove this through contradiction.
Suppose that that $\bigcap_{n=1}^{\infty} I_n$ is not empty. Thus by the definition of $\bigcap_{n=1}^{\infty} I_n$ there exists a x in I_n for all n in $\mathbb{N}$, and by the definition of $I_n = (-\infty, -n]$, there exists an x in $\mathbb{R}$ such that

$$x \leq -n \qquad \text{for all n in } \mathbb{N}$$

Then we multiply both sides by -1 which flips the sign of the inequality and gives us

$$-x \geq n \qquad \text{for all n in } \mathbb{N}$$

We know that $-x$ is in $\mathbb{R}$ since x is in $\mathbb{R}$. So there exists a $-x$ in $\mathbb{R}$ for all $n \in N$ such that $-x \geq n$, this is a clear contradiction to the Archimedean Property which states that for all t in $\mathbb{R}$ there exists an n_t in $\mathbb{N}$ such that $n_t > t$. Thus since $-x$ is in $\mathbb{R}$, there is a contraction with $-x \geq n$ for all n in $\mathbb{N}$, and there exists an n_t in $\mathbb{N}$ such that $n_t > -x$. Therefore it must hold that $\bigcap_{n=1}^{\infty} I_n = \emptyset$. ■