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For problems 1–4, determine whether the given series is absolutely convergent, conditionally convergent, or divergent. Justify your answers.

Problem 1

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$$

Proof. Let's test for divergence. So, let $x_n = \dfrac{(-1)^{n-1}n}{n+1}$ for arbitrary $n \in \mathbb{N}$, then we know that for arbitrary $k \in \mathbb{N}$,

$$x_{2k+1} = \frac{(-1)^{2k}(2k+1)}{2k+2}.$$

We know by the definition of even that $2k$ is even for all k and thus $(-1)^{2k} = 1$, so

$$x_{2k+1} = \frac{1 \cdot (2k+1)}{2k+2} = \frac{2k+1}{2k+2}.$$

We can multiply by $1 = (1/k)/(1/k)$ and distribute to get

$$x_{2k+1} = \frac{(2k+1)(1/k)}{(2k+2)(1/k)} = \frac{2+1/k}{2+2(1/k)}.$$

Now lets take the limit of x_{2k+1} to get

$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty} \frac{2+1/k}{2+2(1/k)}.$$

We can split the limit on the right hand side by division then addition then multiplication (Theorem 27) to get

$$\frac{\lim\limits_{k\to\infty} 2 + \lim\limits_{k\to\infty}(1/k)}{\lim\limits_{k\to\infty} 2 + \lim\limits_{n\to\infty} 2 \cdot \lim\limits_{n\to\infty}(1/k)}.$$

We know that the limit of a constant is said constant and that $\lim_{k\to\infty}(1/k) = 0$ so we have

$$\frac{2+0}{2+2\cdot 0} = \frac{2}{2} = 1.$$

Thus we have showed that $\lim\limits_{k\to\infty} x_{2k+1} = 1$. Now there are two cases, either the sequence $\{x_n\}$ is convergent or $\{x_n\}$ is divergent.

Case 1: $\{\mathbf{x_n}\}$ is convergent

If $\{x_n\}$ is convergent then all subsequences of $\{x_n\}$ and $\{x_n\}$ itself must converge to the same limit, and since we showed that $\lim\limits_{k\to\infty} x_{2k+1} = 1$, then $\lim\limits_{n\to\infty} x_n = 1$. Then by the n-th term test, since the limit of $\{x_n\}$ is non-zero, $\sum\limits_{n=1}^{\infty} x_n = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}n}{n+1}$ is a divergent series.

Case 2: $\{\mathbf{x_n}\}$ is divergent

If $\{x_n\}$ is divergent then the limit does not exist, and by the n-th term test $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ is a divergent series.

Therefore in all cases $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ is divergent, so $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$ ■

Problem 2

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{2^n}$$

Proof. Let $x_n = \frac{(-1)^{n-1}n}{2^n}$, then $x_{n+1} = \frac{(-1)^n(n+1)}{2^{n+1}}$, and now we know that

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{(-1)^n(n+1)}{2^{n+1}}}{\frac{(-1)^{n-1}n}{2^n}}\right| = \left|\frac{(-1)^n(n+1)}{2^{n+1}} \cdot \frac{2^n}{(-1)^{n-1}n}\right|$$

$$= \left|\frac{(-1)(n+1)}{2(n)}\right| = \left|-\frac{n+1}{2n}\right| = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}.$$

Thus we have shown that $\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}$. Now lets take the limit of $\left|\frac{x_{n+1}}{x_n}\right|$ to get

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}\right).$$

Then we can split the limit on the right hand side by addition then multiplication (Theorem 27) to get

$$\lim_{n\to\infty} \frac{1}{2} + \lim_{n\to\infty} \frac{1}{2} \cdot \lim_{n\to\infty} \frac{1}{n}.$$

We know that the limit of a constant is said constant and $\lim_{n\to\infty} \frac{1}{n} = 0$ so we get

$$\frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{2} + 0 = \frac{1}{2}.$$

Thus we have showed that $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{2}$. Then by the ratio test for series, since $\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{2} < 1$, $\sum_{n=1}^{\infty} x_n$ is an absolutely convergent series i.e. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{2^n}$ is an absolutely convergent series and thus convergent . ■

Problem 3

$$\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$$

Proof. Let $b_n = \sqrt{n+1} - \sqrt{n}$ for arbitrary $n \in \mathbb{N}$. Then let us use $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$ with b_n, where $a = n+1$ and $b = n$ to get

$$b_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}. \qquad (1)$$

Now,we know that $n > 0$ and $2 > 0$, adding n to both sides of $2 > 0$ gets us $n+2 > n > 0$, now taking the square root of all sides gets us $\sqrt{n+2} > \sqrt{n} > 0$. Then we can add $\sqrt{n+1}$, $n+1 > 0$, to the two left sides to get $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n} + \sqrt{n+1}$, we know that $\sqrt{n+2} + \sqrt{n+1} \neq 0$ and $\sqrt{n} + \sqrt{n+1} \neq 0$ so we can take the multiplicative inverse of both sides flip the direction of the inequality to get

$$\frac{1}{\sqrt{n+2}+\sqrt{n+1}} < \frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

Then, from our definition of b_n in (1) we have

$$b_{n+1} < b_n \qquad \forall n \in \mathbb{N}.$$

Thus, $\{b_n\}$ is a decreasing sequence by definition.
Now, with (1), lets multiply the right hand side by $1 = (1/\sqrt{n})/(1/\sqrt{n})$ to get

$$b_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{1/\sqrt{n}}{1/\sqrt{n}} = \frac{1/\sqrt{n}}{\frac{\sqrt{n+1}}{\sqrt{n}}+1} = \frac{\sqrt{1/n}}{\sqrt{1+1/n}+1}$$

$$b_n = \frac{\sqrt{1/n}}{\sqrt{1+1/n}+1}.$$

now lets take the limit of both sides to get

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{1/n}}{\sqrt{1+1/n}+1}.$$

Now let use split the limit on the right by division to get

$$\frac{\lim_{n\to\infty}\sqrt{1/n}}{\lim_{n\to\infty}\left(\sqrt{1+1/n}+1\right)}.$$

We can now take the square root outside the limit in the numerator (Theorem 27) and we know the $\lim_{n\to\infty}(1/n) = 0$ so we get

$$\frac{\sqrt{\lim\limits_{n\to\infty}(1/n)}}{\lim\limits_{n\to\infty}\left(\sqrt{1+1/n}+1\right)} = \frac{\sqrt{0}}{\lim\limits_{n\to\infty}\left(\sqrt{1+1/n}+1\right)} = \frac{0}{\lim\limits_{n\to\infty}\left(\sqrt{1+1/n}+1\right)} = 0.$$

We know that $\lim\limits_{n\to\infty}(\sqrt{1+1/n}+1) \neq 0$ since when we split the limit with theorem 27 we have $\sqrt{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty}(1/n)} + \lim\limits_{n\to\infty} 1 = 1 + 0 + 1 = 2.$
Thus, we have showed that

$$\lim_{n\to\infty} b_n = 0. \qquad (2)$$

Therefore by the Alternating Series Test, since $\{b_n\}$ is decreasing and $\lim\limits_{n\to\infty} b_n = 0$, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}b_n$ is a convergent series, i.e. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$ is a convergent series.
Now, for $|(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})|$, with (1) we know that

$$|(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})| = \left|(-1)^{n-1}\frac{1}{\sqrt{n+1}+\sqrt{n}}\right|.$$

then by the definition of absolute value we get

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Now, we see that we already have defined $b_n = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$, so

$$b_n = |(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})| \qquad (3)$$

and let $a_n = 1/n$, so

$$\begin{aligned}
\frac{a_n}{b_n} = \frac{1/n}{\dfrac{1}{\sqrt{n+1}+\sqrt{n}}} &= (1/n)(\sqrt{n+1}+\sqrt{n}) \\
&= \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n}}{n} \\
&= \frac{\sqrt{n+1}}{\sqrt{n^2}} + n^{-1/2} \\
&= \sqrt{\frac{n+1}{n^2}} + n^{-1/2} \\
&= \sqrt{\frac{1}{n} + \frac{1}{n^2}} + \frac{1}{\sqrt{n}}.
\end{aligned}$$

$$\frac{a_n}{b_n} = \sqrt{\frac{1}{n} + (\frac{1}{n})(\frac{1}{n})} + \sqrt{\frac{1}{n}}.$$

Then taking the limit of both sides gets us

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\sqrt{\frac{1}{n} + (\frac{1}{n})(\frac{1}{n})} + \sqrt{\frac{1}{n}}\right).$$

We can then split and modify the limits on the right hand side with Theorem 27 to get

$$\lim_{n\to\infty}\sqrt{\frac{1}{n} + (\frac{1}{n})(\frac{1}{n})} + \lim_{n\to\infty}\sqrt{\frac{1}{n}}$$

$$= \sqrt{\lim_{n\to\infty}\left(\frac{1}{n} + (\frac{1}{n})(\frac{1}{n})\right)} + \sqrt{\lim_{n\to\infty}\frac{1}{n}}$$

$$= \sqrt{\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}(\frac{1}{n})\cdot\lim_{n\to\infty}(\frac{1}{n})} + \sqrt{\lim_{n\to\infty}\frac{1}{n}}.$$

We know that $\lim_{n\to\infty}\frac{1}{n} = 0$, so

$$= \sqrt{\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}(\frac{1}{n})\cdot\lim_{n\to\infty}(\frac{1}{n})} + \sqrt{\lim_{n\to\infty}\frac{1}{n}} = \sqrt{0+0\cdot 0} + \sqrt{0} = 0 + 0 = 0.$$

Thus, we have shown that

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0.$$

Then, we also know that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n^1}$ is divergent by the p-series test since $1 \leq 1$.

Therefore, by the limit comparison test, since $\sum_{n=1}^{\infty} a_n$ is divergent and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$, $\sum_{n=1}^{\infty} b_n$ is also divergent, i.e. with (3) we have $\sum_{n=1}^{\infty}|(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})|$ is divergent.

Finally, since we showed that $\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$ is a convergent series and $\sum_{n=1}^{\infty}|(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})|$ is divergent, by the definition of conditionally convergent, $\sum_{n=1}^{\infty}(-1)^{n-1}(\sqrt{n+1}-\sqrt{n})$ is conditionally convergent. ■

Problem 4

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{1}+\sqrt{2}+...+\sqrt{n}}$$

Proof. Let $P(n)$ be "$\sqrt{1}+\sqrt{2}+...+\sqrt{n} \geq \frac{n\sqrt{n+1}}{2}$" for some $n \in \mathbb{N}$.
Base Case: When $n = 1$. On the left hand side of $P(1)$ we have $\sqrt{1} = 1$, and on the right hand side we have $\frac{1\sqrt{1+1}}{2} = \frac{\sqrt{2}}{2}$. So, we need to show that $1 \geq \frac{\sqrt{2}}{2}$ holds. We know that $4 \geq 2$, square rooting both sides gets us $2 \geq \sqrt{2}$ and dividing by 2 gets us $1 \geq \frac{\sqrt{2}}{2}$, thus $1 \geq \frac{\sqrt{2}}{2}$ holds, so $P(1)$ holds and the base case holds.
Inductive Hypothesis: Suppose $P(k)$ holds for arbitrary $k \in \mathbb{N}$.
Inductive Step: From our inductive hypothesis we know that $\sqrt{1}+\sqrt{2}+...+\sqrt{k} \geq \frac{k\sqrt{k+1}}{2}$ holds. Lets add $\sqrt{k+1}$ to both sides to get

$$\sqrt{1}+\sqrt{2}+...+\sqrt{k}+\sqrt{k+1} \geq \frac{k\sqrt{k+1}}{2}+\sqrt{k+1}.$$

The we can simplify the right hand side through the following steps:

$$\begin{aligned}\frac{k\sqrt{k+1}}{2}+\sqrt{k+1} &= \frac{k\sqrt{k+1}}{2}+\frac{2\sqrt{k+1}}{2}\\ &= \frac{k\sqrt{k+1}+2\sqrt{k+1}}{2}\\ &= \frac{\sqrt{k+1}\cdot(k+2)}{2}.\end{aligned}$$

Substituting our expression on the right hand side of the inequality gets us

$$\sqrt{1}+\sqrt{2}+...+\sqrt{k}+\sqrt{k+1} \geq \frac{\sqrt{k+1}\cdot(k+2)}{2}. \qquad \clubsuit$$

Now, we know that $2 \geq 1$, then adding k to both sides gets us $k+2 \geq k+1$. Dividing both sides by $k+1$ gets us

$$\frac{k+2}{k+1} \geq 1.$$

Now, taking the square root of both sides gets us

$$\sqrt{\frac{k+2}{k+1}} \geq 1.$$

Then we can split the left hand square root with the following steps

$$\frac{\sqrt{k+2}}{\sqrt{k+1}} \geq 1$$

$$\implies \frac{\sqrt{k+1}}{(k+1)^1}\cdot\frac{(k+2)^1}{\sqrt{k+2}} \geq 1.$$

Now, multiplying both sides by $\sqrt{k+2}$ and $k+1$ gets

$$\sqrt{k+1} \cdot (k+2) \geq (k+1)(\sqrt{k+2}).$$

Dividing both sides by 2:

$$\frac{\sqrt{k+1} \cdot (k+2)}{2} \geq \frac{(k+1)(\sqrt{k+2})}{2}. \qquad \spadesuit$$

Then by combining $\clubsuit$ and $\spadesuit$ we get

$$\sqrt{1} + \sqrt{2} + ... + \sqrt{k} + \sqrt{k+1} \geq \frac{(k+1)(\sqrt{k+2})}{2}.$$

Therefore, we have shown that $P(k+1)$ holds and thus our inductive step in complete. By the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Now, we can take the multiplicative inverse of both sides and flip the sign of $\sqrt{1} + \sqrt{2} + ... + \sqrt{n} \geq \dfrac{n\sqrt{n+1}}{2}$ to get

$$\frac{1}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}} \leq \frac{1}{\frac{n\sqrt{n+1}}{2}}$$

$$\implies \frac{1}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}} \leq \frac{2}{n\sqrt{n+1}}.$$

Now, we know $\left|\dfrac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}\right| = \dfrac{1}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}$ by the definition of absolute value since $|(-1)^{n-1}| = 1$ for all $n \in \mathbb{N}$, we also know from the definition of absolute value that $0 \leq \left|\dfrac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}\right|$, and combining these with the previous inequality gets us

$$0 \leq \left|\frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}\right| \leq \frac{2}{n\sqrt{n+1}}. \qquad \heartsuit$$

If we get rid of the 1 in the denominator of the right hand side, we make the expression bigger:

$$\frac{2}{n\sqrt{n+1}} \leq \frac{2}{n\sqrt{n}}$$

$$\implies \frac{2}{n\sqrt{n+1}} \leq \frac{2}{n^{3/2}}.$$

Combining the inequality above with $\heartsuit$ gives us

$$0 \leq \left|\frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}\right| \leq \frac{2}{n^{3/2}}. \qquad \diamondsuit$$

Now, we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by the p-series test since $3/2 > 1$. Then by the linearity of of series we know that $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ is convergent too. Now, by the comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ being convergent and $\diamondsuit$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{1}+\sqrt{2}+...+\sqrt{n}}$ is a convergent series. ■

Problem 5

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$ is a convergent series.

Proof. Let $x_n = \dfrac{3^n (n!)^2}{(2n)!}$ for arbitrary $n \in \mathbb{N}$, then

$$\begin{aligned}
\left|\frac{x_{n+1}}{x_n}\right| &= \left|\frac{\frac{3^{n+1}((n+1)!)^2}{(2n+2)!}}{\frac{3^n (n!)^2}{(2n)!}}\right| \\
&= \left|\frac{3^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n (n!)^2}\right| \\
&= \left|\frac{3}{(2n+2)(2n+1)} \cdot \frac{((n+1)!)^2}{(n!)^2}\right| \\
&= \left|\frac{3}{(2n+2)(2n+1)} \cdot \left(\frac{(n+1)!}{n!}\right)^2\right| \\
&= \left|\frac{3}{2(n+1)(2n+1)} \cdot (n+1)^2\right| \\
&= \left|\frac{3(n+1)}{2(2n+1)}\right| \\
&= \frac{3(n+1)}{2(2n+1)} \\
&= \frac{3}{2} \cdot \frac{n+1}{2n+1} \cdot \frac{1/n}{1/n} \\
&= \frac{3}{2} \cdot \frac{1+1/n}{2+1/n}
\end{aligned}$$

Thus, we have show that $\left|\dfrac{x_{n+1}}{x_n}\right| = \dfrac{3}{2} \cdot \dfrac{1+1/n}{2+1/n}$, now taking the limit of both sides gives us

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \left(\frac{3}{2} \cdot \frac{1+1/n}{2+1/n}\right)$$

We can split the limit on the right hand side by multiplication, then division and then addition (Theorem 27) to get

$$\lim_{n\to\infty} \frac{3}{2} \cdot \frac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} (1/n)}{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} (1/n)}.$$

We know that the limit of a constant is said constant and $\lim_{n\to\infty}(1/n) = 0$ so we have

$$\lim_{n\to\infty} \frac{3}{2} \cdot \frac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} (1/n)}{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} (1/n)} = \frac{3}{2} \cdot \frac{1+0}{2+0} = \frac{3}{4}.$$

Thus, we have shown that $\lim\limits_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \frac{3}{4}$. Therefore, by the ratio test for series, since $\frac{3}{4} < 1$, $\sum\limits_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$ is a convergent series. ■

Problem 6

Prove that $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is a divergent series.

Proof. We know that $0 < 1$, and with arability $n \in \mathbb{N}, n > 0$, so we can have $1/n > 0$. Then we can add $1/n$ to both sides of $0 < 1$ to get

$$1/n < 1 + 1/n.$$

Then since $n > 0$, if we take n to both sides of the inequality we can get

$$n^{1/n} < n^{1+1/n}.$$

And since $n > 0$, $n^{1/n} > 0$ so we have

$$0 < n^{1/n} < n^{1+1/n}.$$

$$\implies 0 \le n^{1/n} \le n^{1+1/n}. \qquad (I)$$

Now we know that $\lim\limits_{n\to\infty} n^{1/n} = 1$ from Problem 2 of HW 8, thus by the Test for Divergence/n-th Term Test, $\sum\limits_{n=1}^{\infty} n^{1/n}$ is a divergent series since $\lim\limits_{n\to\infty} n^{1/n} = 1 \neq 0$. Therefore, by the Comparison Test, since $\sum\limits_{n=1}^{\infty} n^{1/n}$ is a divergent series and (I), $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ is a divergent series. ■

Problem 7

Use the expression $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ to prove that $2\frac{43}{60} = 2\frac{430}{600} < e < 2\frac{431}{600}$.

Proof. We know $\frac{1}{n!} > 0$ $\forall n \in \mathbb{N}$ (even if $n = 0$), thus the infinite series would be bigger than a finite summation since the sum would only increase with more terms, so

$$\sum_{n=0}^{5} \frac{1}{n!} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies \frac{1}{1} + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies \frac{120}{120} + \frac{120}{120} + \frac{60}{120} + \frac{20}{120} + \frac{5}{120} + \frac{1}{120} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies \frac{326}{120} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies \frac{163}{60} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies 2\frac{43}{60} < \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$\implies 2\frac{43}{60} = 2\frac{430}{600} < \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Then we can substitute using series definition of e on the right hand side to get

$$2\frac{43}{60} = 2\frac{430}{600} < e. \qquad (A)$$

Now, we know that

$$\sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \ldots$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{6!}\left(1 + \frac{1}{7} + \frac{1}{8 \cdot 7} + \ldots\right)$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{720}\left(\left(1\right) + \left(\frac{1}{7}\right) + \left(\frac{1}{8}\cdot\frac{1}{7}\right) + ...\right)$$

We can make replace all the denominators in the fractions with 6 and would make the summation bigger since you are dividing by smaller numbers since all the all the denominators are greater than 6:

$$\sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{720}\left(\left(1\right) + \left(\frac{1}{7}\right) + \left(\frac{1}{8}\cdot\frac{1}{7}\right) + ...\right) < \frac{1}{720}\left(\left(1\right) + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\cdot\frac{1}{6}\right) + ...\right)$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720}\left(1 + \left(\frac{1}{6}\right)^1 + \left(\frac{1}{6}\right)^2 + ...\right).$$

The series, $1 + \left(\frac{1}{6}\right)^1 + \left(\frac{1}{6}\right)^2 + ...$, on the right is a geometric series with $a = 1$ and $r = 1/6$ and is thus convergent, the sum of the series is then $\frac{1}{1-\frac{1}{6}}$, so we have the inequality

$$\sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720}\left(\frac{1}{1-\frac{1}{6}}\right)$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720}\left(\frac{1}{\frac{5}{6}}\right)$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720}\left(\frac{6}{5}\right)$$

$$\implies \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{600}$$

Now, we know that $\sum_{n=0}^{5} \frac{1}{n!} + \sum_{n=6}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$ so let us add $\sum_{n=0}^{5} \frac{1}{n!}$ to both sides of the inequality to get

$$\sum_{n=0}^{5} \frac{1}{n!} + \sum_{n=6}^{\infty} \frac{1}{n!} < \sum_{n=0}^{5} \frac{1}{n!} + \frac{1}{600}.$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{n!} < \sum_{n=0}^{5} \frac{1}{n!} + \frac{1}{600}.$$

We know that from our calculation of (A) that $\sum_{n=0}^{5} \frac{1}{n!} = 2\frac{430}{600}$, so using this as a substitution on the right gives us

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 2\frac{430}{600} + \frac{1}{600}.$$

$$\implies \sum_{n=0}^{\infty} \frac{1}{n!} < 2\frac{431}{600}.$$

Now, we can substitute the left hand side with e by the series definition of e to get

$$e < 2\frac{431}{600}. \qquad (B)$$

Therefore, we can combine (A) and (B) to finally show that

$$2\frac{43}{60} = 2\frac{430}{600} < e < 2\frac{431}{600}.$$

■