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Problem 1

Let $\alpha, \beta \in \mathbb{R}$. Prove that

$$\alpha\beta \leq \frac{\alpha^2 + \beta^2}{2}$$

Proof. We know that $(\alpha - \beta)^2 \geq 0$ holds since $(\alpha - \beta)^2$ can be written as $|(\alpha - \beta|^2|$ by theorem 19(iv), and thus by the definition of absolute value, $(\alpha - \beta)^2 \geq 0$, we know $\alpha - \beta$ is in $\mathbb{R}$ since the real numbers are closed under addition.
Now we can use $(\alpha - \beta)^2 \geq 0$ and simplify the left hand side of this expression with the binomial theorem to get:

$$\alpha^2 - 2\alpha\beta + \beta^2 \geq 0$$

Then by the commutativity of addition (Axiom A2) we can change the order of the left hand side to get:

$$-2\alpha\beta + \alpha^2 + \beta^2 \geq 0$$

Then we can group $\alpha^2 + \beta^2$ together by the associativity of addition (Theorem A3) to get:

$$-2\alpha\beta + (\alpha^2 + \beta^2) \geq 0$$

Now we can use Theorem O2 for the inequality, we can add the term $-(\alpha^2 + \beta^2)$ to both sides:

$$-2\alpha\beta + (\alpha^2 + \beta^2) + (-(\alpha^2 + \beta^2)) \geq 0 - (\alpha^2 + \beta^2)$$

We can simplify by canceling $(\alpha^2 + \beta^2)$ with $-(\alpha^2 + \beta^2)$ on the left hand side and use the identity property of addition on the right hand side:

$$-2\alpha\beta \geq -(\alpha^2 + \beta^2)$$

Then divide by -2 which cancels the negative on the right hand side and swaps the direction of our inequality:

$$\alpha\beta \leq \frac{\alpha^2 + \beta^2}{2}$$

Therefore by we have proved that $\alpha\beta \leq \frac{\alpha^2+\beta^2}{2}$ holds true. ∎

Problem 2

(i) Let $a, b \in \mathbb{R}$ such that $0 < a < b$. Prove that

$$a^2 < b^2$$

Proof. From our assumption that $0 < a < b$, we know that $a > 0$ and $a < b$, and by using $a > 0$, $a < b$ and Theorem 8 (ii) we get:

$$a^2 < ab \qquad \text{(inequality 1)}$$

Also from our assumption that $0 < a < b$ we can get that $b > 0$. So now by using Theorem 8(ii) again but with $b > 0$ and $a < b$, we get:

$$ba < b^2$$

Then we can swap a and b on the left hand side since multiplication with real numbers is commutative, giving us:

$$ab < b^2 \qquad \text{(inequality 2)}$$

Now we can combine inequalities (1) and (2) with Axiom O2 to get

$$a^2 < b^2$$

and we have therefore completed our proof, showing that $a^2 < b^2$ holds. ∎

(ii) Let $a, b \in \mathbb{R}$ such that $a < b < 0$. Prove that

$$b^2 < a^2$$

Proof. From our assumption that $a < b < 0$. We get $0 > b$ and $b > a$, we can now use $0 > b$ with Theorem 8(i) to get a $0 < -b$, and by multiplying the inequality $b > a$ by -1 we get $-b < -a$.
Then by Axiom O2 we can combine $0 < -b$ and $-b < -a$ to get $0 < -b < -a$.
Now we can get $-a > 0$ from $0 < -b < -a$, then by using $-a > 0$ and $-b < -a$ with Theorem 8 (ii), we get:

$$(-a)(-b) < (-a)(-a)$$

And by simplifying the negative signs and combings the right hand side into an exponent we get:

$$ab < a^2 \qquad \text{(inequality A)}$$

Now, we know $-b > 0$ from $0 < -b < -a$, and by using Theorem 8 (ii) again with $-b > 0$ and $-b < -a$ we get

$$(-b)(-b) < (-b)(-a)$$

Then by simplifying all the negatives and combining the left hand side into an exponent gets us $b^2 < ba$, and through Axiom M2, the commutativity of multiplication, we can swap the order of a and b on the right hand side:

$$b^2 < ab \qquad \text{(inequality B)}$$

Then we finish our proof by using inequality A and inequality B with Axiom O2, combining them into:

$$b^2 < a^2$$

Therefore, we have showed that $b^2 < a^2$ holds and completed our proof. ∎

Problem 3

Let $\delta \in \mathbb{R}$ and $\delta > 0$. Prove that there exists an $n \in \mathbb{N}$ such that

$$\frac{2}{n+2} < \delta$$

Proof. Since we know that $\delta > 0$, this means that $\delta \neq 0$ and thus by Axiom M5 (the existence of a multiplicative inverses), $\frac{1}{\delta} \in \mathbb{R}$ and since $\delta > 0$ then $\frac{1}{\delta} > 0$.
Then we know that $2(\frac{1}{\delta} - 1) \in \mathbb{R}$ since the real numbers are closed under addition and multiplication. So adding a real number by -1 and multiplying it by 2 would still be a real number.
Now using the Archimedean Property, we can say that there exists a $n \in \mathbb{N}$ such that $nx > y$, $xy \in \mathbb{R}$ where our x would be 1 and our y would be $2(\frac{1}{\delta} - 1)$ which gives us:

$$n > 2(\frac{1}{\delta} - 1)$$

Then through Axiom D, we can distribute the right hand side of the inequality to $2\frac{1}{\delta} - 2$, and then by adding 2 to both sides of the inequality, the -2 on the right hand side cancels out and the inequality becomes:

$$n + 2 > 2\frac{1}{\delta}$$

Now we know that $2\frac{1}{\delta} > 0$ since the product of two positive numbers remains positive. Then combining $n + 2 > 2\frac{1}{\delta}$ and $2\frac{1}{\delta} > 0$ we can get the inequalities:

$$n + 2 > 2\frac{1}{\delta} > 0$$

Thus using the fact that "$x > y > 0$ implies $1/x < 1/y$" with our inequality $n + 2 > 2\frac{1}{\delta} > 0$, we get the inequality:

$$\frac{1}{n+2} < \frac{1}{2(1/\delta)}$$

Then multiplying both sides of the inequality by 2 we get

$$\frac{2}{n+2} < \frac{1}{(1/\delta)}$$

Then we finish our proof after we multiply the right hand side by $1 = \frac{\delta}{\delta}$ which simplifies to:

$$\frac{2}{n+2} < \delta$$

Thus, we have finished our proof by showing that $\frac{2}{n+2} < \delta$ holds true. ■

Problem 4

Let $S = \{\frac{n}{n+2} : n \in \mathbb{N}\}$. Prove that $\sup(S) = 1$.

Proof. Let us first prove that 1 is a upper bound of S. We know that S is not empty since $\frac{1}{1+2} = \frac{1}{3}$ is an element of S when $n \in \mathbb{N} = 1$. We know that $0 < 2$, and also $n > 0$ since $n \in \mathbb{N}$.
Thus let us add n to both sides of $0 < 2$ to get $n < n + 2$ where n is an arbitrary element of $\mathbb{N}$, and by combining $n > 0$ and $n < n + 2$ we get the inequalities:

$$0 < n < n + 2$$

This inequality is equivalent to $n + 2 > n > 0$, then by using "$x > y > 0$ implies $1/x < 1/y$" with $n + 2 > n > 0$, we can get the inequality of

$$\frac{1}{n+2} < \frac{1}{n}$$

Then we can multiply both sides of the inequality by n, giving us:

$$\frac{n}{n+2} < 1$$

Thus, for all n in $\mathbb{N}$, 1 is greater than $\frac{n}{n+2}$ for an for all elements of $\mathbb{N}$. Now let u be an arbitrary element of set S, then $u = \frac{n}{n+2}$ for an arbitrary $n \in \mathbb{N}$ by the definition of the set S . We substitute u into the inequality $\frac{n}{n+2} < 1$ so that $u < 1$, thus for all u in S, less u is less than 1. Thus by the definition of a upper bound, 1 is an upper bound of S.

Now, let us show that 1 is the least upper bound. Let v be in $\mathbb{R}$ such that $v < 1$. We can now subtract v from both sides of the inequality to get $0 < 1 - v = 1 - v > 0$, thus we know that $1 - v \neq 0$. Since real numbers are closed under addition, then $1 - v$ is a real number. Thus since $1 - v \neq 0$, by the existence of a multiplicative inverse, $\frac{1}{1-v}$ is in $\mathbb{R}$. Then we know that $2(\frac{1}{1-v} - 1)$ is in $\mathbb{R}$ since the real numbers are closed under multiplication and addition. Now we can use the Archimedean Property to say that there exists some n in $\mathbb{N}$ such that

$$n(1) > 2(\frac{1}{1-v} - 1)$$

We can use the identity property on the left and distribute the 2 on the right hand side to get

$$n > \frac{2}{1-v} - 2$$

Then we can add 2 to both sides to get

$$n + 2 > \frac{2}{1-v}$$

Now we can divide both sides by (n+2) and multiply both sides by (1-v) to get

$$1 - v > \frac{2}{n+2}$$

Then we can subtract 1 to both sides in the form of $\frac{n+2}{n+2}$ to get

$$-v > \frac{2}{n+2} - \frac{n+2}{n+2}$$

Subtracting the factions together on the right hand side gives us:

$$-v > \frac{2-n-2}{n+2}$$

Then simplifying the numerator of the right hand side and taking out a -1 gives us:

$$-v > -\frac{n}{n+2}$$

Then we can mulifpy both sides by -1 which flips the inequality giving us:

$$v < \frac{n}{n+2}$$

Thus there is an element of S in the form of $\frac{n}{n+2}$ (the definition of an element in S) for some n in $\mathbb{N}$ such that v is less than it. Therefore by the definition of upper bound, v is not an upper bound for S. Then there is no $v < 1$ such that v is an upper bound of S. Thus by the definition of supremum, since 1 is an upper bound and there is no $v < 1$ that is an upper bound, 1 is the supremum of set S. ■

Problem 5

Let $v \in \mathbb{R}$ and S be a nonempty subset of $\mathbb{R}$. Suppose that $v \in S$ and v is an upper bound of S . Prove that $\sup(S) = v$.

Proof. We can assume that the supremum of S exists because S is non empty and S is bounded above.
Then, since v is in S and $\sup(S)$ is an upper bound, by the definition of upper bound

$$v \leq sup(S)$$

Now we also know that v is an upper bound and $\sup(S)$ is the supremum and thus the least upper bound, then by the definition of supremum

$$v \geq sup(S)$$

Therefore, since $v \leq sup(S)$ and $v \geq sup(S)$ then v must be equal to $sup(S)$ ■

Problem 6

Let $\emptyset \neq A \subset R$ and $B = \{2x + x^2 : x \in A\}$. Suppose that A is bounded below. Prove that B is also bounded below and

$$\inf(B) \geq 2inf(A)$$

Proof. Let arbitrary y be an arbitrary element in B, and by the definition of set B, $y = 2x+x^2$ for some x in A.
Continuing, we can say that for all $x \in A$,

$$x \geq inf(A)$$

by the definition of infimum since A is not empty and bounded below, the infimum of A exists.
Now since we know that $2 > 0$, we can multiply both sides of the inequality by two to get

$$2x \geq 2inf(A)$$

Now, $x^2 \geq 0$ holds since x^2 can be written as $|x^2|$ by theorem 19(iv), then by the definition of absolute value, $x^2 \geq 0$. x^2 is also in $\mathbb{R}$ since $\mathbb{R}$ is closed under multiplication.
Then we add x^2 to both sides of $2x \geq 2inf(A)$ to get:

$$2x + x^2 \geq 2inf(A) + x^2$$

Then let us add $2inf(A)$ to both sides of $x^2 \geq 0$ to get:

$$2inf(A) + x^2 \geq 2inf(A)$$

Then we can combine "$2x + x^2 \geq 2inf(A) + x^2$" and "$2inf(A) + x^2 \geq 2inf(A)$" to get the inequality:

$$2x + x^2 \geq 2inf(A) + x^2 \geq 2inf(A)$$

And thus it must hold that

$$2x + x^2 \geq 2inf(A)$$

Then by substituting y into the left hand side of the equation we get

$$y \geq 2inf(A)$$

And thus there is a $2inf(A)$ in $\mathbb{R}$ greater than all y in B, then by the definition of a lower bound, $2inf(A)$ is a lower bound of B. We know that $2inf(A)$ is a real number since $inf(A)$ is a real number by the definition of infimum of A since $A \subset \mathbb{R}$ and the real numbers are closed under multiplication.
We also know that B is non-empty, since A is non-empty, and B is bounded thus $inf(B)$ exists. Then by the definition of infimum, the infimum is the greatest lower bound thus

$$inf(B) \geq 2inf(A)$$

Therefore we have proved that B is bounded and that the $inf(B)$ is greater than or equal to the lower bound of $2inf(A)$.

■

Problem 7

Let $\emptyset \neq A \subset \mathbb{R}$ such that A is bounded above. Prove that, for every $n \in \mathbb{N}$, there exists a $u_n \in A$ such that

$$u_n \leq sup(A) < u_n + 1/n$$

Proof. We know that A is not empty and bounded above, thus the $sup(A)$ exists. Let n be an arbitrary element of the natural numbers, thus $n > 0$. Then the multiplicative inverse of n is also greater than zero since $n > 0$, giving us $\frac{1}{n} > 0$. Then we can add $sup(A)$ to both sides of $\frac{1}{n} > 0$ to get:

$$\frac{1}{n} + sup(A) > sup(A)$$

Then we can subtract $\frac{1}{n}$ from both sides to get $sup(A) > sup(A) - \frac{1}{n}$, this inequality is also equivalent to $sup(A) - \frac{1}{n} < sup(A)$. Now we know that $sup(A) - \frac{1}{n}$ is not an upper bound since $sup(A)$ is the least upper bound by the definition of supremum, so there is no upper bound less than it. Then because $sup(A) - \frac{1}{n}$ is not an upper bound there exists a $u_n \in A$ such that

$$u_n > sup(A) - \frac{1}{n}$$

Then adding $\frac{1}{n}$ to both sides gives us $u_n + \frac{1}{n} > sup(A)$ which is equivalent to

$$sup(A) < u_n + \frac{1}{n}$$

Now, since u_n is in A and $sup(A)$ is an upper bound of A, by the definition of upper bound

$$u_n \leq sup(A)$$

Finally, we can combine $u_n \leq sup(A)$ with $sup(A) < u_n + \frac{1}{n}$ to get

$$u_n \leq sup(A) < u_n + \frac{1}{n}$$

Therefore we have finished our proof, showing that $u_n \leq sup(A) < u_n + \frac{1}{n}$ holds. ∎