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For problems 1–4, determine whether the given series is absolutely convergent, conditionally convergent, or divergent. Justify your answers.

## Problem 1

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$$

*Proof.* Let's test for divergence. So, let  $x_n = \frac{(-1)^{n-1}n}{n+1}$  for arbitrary  $n \in \mathbb{N}$ , then we know that for arbitrary  $k \in \mathbb{N}$ ,

$$x_{2k+1} = \frac{(-1)^{2k}(2k+1)}{2k+2}.$$

We know by the definition of even that  $2k$  is even for all  $k$  and thus  $(-1)^{2k} = 1$ , so

$$x_{2k+1} = \frac{1 \cdot (2k+1)}{2k+2} = \frac{2k+1}{2k+2}.$$

We can multiply by  $1 = (1/k)/(1/k)$  and distribute to get

$$x_{2k+1} = \frac{(2k+1)(1/k)}{(2k+2)(1/k)} = \frac{2 + 1/k}{2 + 2(1/k)}.$$

Now let's take the limit of  $x_{2k+1}$  to get

$$\lim_{k \rightarrow \infty} x_{2k+1} = \lim_{k \rightarrow \infty} \frac{2 + 1/k}{2 + 2(1/k)}.$$

We can split the limit on the right hand side by division then addition then multiplication (Theorem 27) to get

$$\frac{\lim_{k \rightarrow \infty} 2 + \lim_{k \rightarrow \infty} (1/k)}{\lim_{k \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} 2 \cdot \lim_{n \rightarrow \infty} (1/k)}.$$

We know that the limit of a constant is said constant and that  $\lim_{k \rightarrow \infty} (1/k) = 0$  so we have

$$\frac{2 + 0}{2 + 2 \cdot 0} = \frac{2}{2} = 1.$$

Thus we have showed that  $\lim_{k \rightarrow \infty} x_{2k+1} = 1$ . Now there are two cases, either the sequence  $\{x_n\}$  is convergent or  $\{x_n\}$  is divergent.

### Case 1: $\{x_n\}$ is convergent

If  $\{x_n\}$  is convergent then all subsequences of  $\{x_n\}$  and  $\{x_n\}$  itself must converge to the same limit, and since we showed that  $\lim_{k \rightarrow \infty} x_{2k+1} = 1$ , then  $\lim_{n \rightarrow \infty} x_n = 1$ . Then by the n-th

term test, since the limit of  $\{x_n\}$  is non-zero,  $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$  is a divergent series.

### Case 2: $\{x_n\}$ is divergent

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If  $\{x_n\}$  is divergent then the limit does not exist, and by the n-th term test  $\sum_{n=1}^{\infty} x_n =$

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$  is a divergent series.

Therefore in all cases  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$  is divergent, so  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n+1}$  ■

## Problem 2

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{2^n}$$

*Proof.* Let  $x_n = \frac{(-1)^{n-1}n}{2^n}$ , then  $x_{n+1} = \frac{(-1)^n(n+1)}{2^{n+1}}$ , and now we know that

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} \right| &= \left| \frac{\frac{(-1)^n(n+1)}{2^{n+1}}}{\frac{(-1)^{n-1}n}{2^n}} \right| = \left| \frac{(-1)^n(n+1)}{2^{n+1}} \cdot \frac{2^n}{(-1)^{n-1}n} \right| \\ &= \left| \frac{(-1)(n+1)}{2(n)} \right| = \left| -\frac{n+1}{2n} \right| = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}. \end{aligned}$$

Thus we have shown that  $\left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n}$ . Now lets take the limit of  $\left| \frac{x_{n+1}}{x_n} \right|$  to get

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \left( \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{n} \right).$$

Then we can split the limit on the right hand side by addition then multiplication (Theorem 27) to get

$$\lim_{n \rightarrow \infty} \frac{1}{2} + \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \lim_{n \rightarrow \infty} \frac{1}{n}.$$

We know that the limit of a constant is said constant and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  so we get

$$\frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{2} + 0 = \frac{1}{2}.$$

Thus we have showed that  $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{2}$ . Then by the ratio test for series, since  $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| =$

$\frac{1}{2} < 1$ ,  $\sum_{n=1}^{\infty} x_n$  is an absolutely convergent series i.e.  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{2^n}$  is an absolutely convergent series and thus convergent . ■

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### Problem 3

$$\sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$$

*Proof.* Let  $b_n = \sqrt{n+1} - \sqrt{n}$  for arbitrary  $n \in \mathbb{N}$ . Then let us use  $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$  with  $b_n$ , where  $a = n+1$  and  $b = n$  to get

$$b_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$b_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}. \quad (1)$$

Now, we know that  $n > 0$  and  $2 > 0$ , adding  $n$  to both sides of  $2 > 0$  gets us  $n+2 > n > 0$ , now taking the square root of all sides gets us  $\sqrt{n+2} > \sqrt{n} > 0$ . Then we can add  $\sqrt{n+1}$ ,  $n+1 > 0$ , to the two left sides to get  $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n} + \sqrt{n+1}$ , we know that  $\sqrt{n+2} + \sqrt{n+1} \neq 0$  and  $\sqrt{n} + \sqrt{n+1} \neq 0$  so we can take the multiplicative inverse of both sides flip the direction of the inequality to get

$$\frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n} + \sqrt{n+1}}.$$

Then, from our definition of  $b_n$  in (1) we have

$$b_{n+1} < b_n \quad \forall n \in \mathbb{N}.$$

Thus,  $\{b_n\}$  is a decreasing sequence by definition.

Now, with (1), let's multiply the right hand side by  $1 = (1/\sqrt{n})/(1/\sqrt{n})$  to get

$$b_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{1/\sqrt{n}}{1/\sqrt{n}} = \frac{1/\sqrt{n}}{\frac{\sqrt{n+1}}{\sqrt{n}} + 1} = \frac{\sqrt{1/n}}{\sqrt{1 + 1/n} + 1}$$

$$b_n = \frac{\sqrt{1/n}}{\sqrt{1 + 1/n} + 1}.$$

now let's take the limit of both sides to get

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{\sqrt{1/n}}{\sqrt{1 + 1/n} + 1}.$$

Now let us split the limit on the right by division to get

$$\frac{\lim_{n \rightarrow \infty} \sqrt{1/n}}{\lim_{n \rightarrow \infty} (\sqrt{1 + 1/n} + 1)}.$$

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We can now take the square root outside the limit in the numerator (Theorem 27) and we know the  $\lim_{n \rightarrow \infty} (1/n) = 0$  so we get

$$\frac{\sqrt{\lim_{n \rightarrow \infty} (1/n)}}{\lim_{n \rightarrow \infty} (\sqrt{1 + 1/n} + 1)} = \frac{\sqrt{0}}{\lim_{n \rightarrow \infty} (\sqrt{1 + 1/n} + 1)} = \frac{0}{\lim_{n \rightarrow \infty} (\sqrt{1 + 1/n} + 1)} = 0.$$

We know that  $\lim_{n \rightarrow \infty} (\sqrt{1 + 1/n} + 1) \neq 0$  since when we split the limit with theorem 27 we have  $\sqrt{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} (1/n)} + \lim_{n \rightarrow \infty} 1 = 1 + 0 + 1 = 2$ .

Thus, we have showed that

$$\lim_{n \rightarrow \infty} b_n = 0. \quad (2)$$

Therefore by the Alternating Series Test, since  $\{b_n\}$  is decreasing and  $\lim_{n \rightarrow \infty} b_n = 0$ ,  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

is a convergent series, i.e.  $\sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{n+1} - \sqrt{n})$  is a convergent series.

Now, for  $|(-1)^{n-1} (\sqrt{n+1} - \sqrt{n})|$ , with (1) we know that

$$|(-1)^{n-1} (\sqrt{n+1} - \sqrt{n})| = \left| (-1)^{n-1} \frac{1}{\sqrt{n+1} + \sqrt{n}} \right|.$$

then by the definition of absolute value we get

$$\frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Now, we see that we already have defined  $b_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$ , so

$$b_n = |(-1)^{n-1} (\sqrt{n+1} - \sqrt{n})| \quad (3)$$

and let  $a_n = 1/n$ , so

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{1/n}{\frac{1}{\sqrt{n+1} + \sqrt{n}}} = (1/n)(\sqrt{n+1} + \sqrt{n}) \\ &= \frac{\sqrt{n+1}}{n} + \frac{\sqrt{n}}{n} \\ &= \frac{\sqrt{n+1}}{\sqrt{n^2}} + n^{-1/2} \\ &= \sqrt{\frac{n+1}{n^2}} + n^{-1/2} \\ &= \sqrt{\frac{1}{n} + \frac{1}{n^2}} + \frac{1}{\sqrt{n}}. \end{aligned}$$

$$\frac{a_n}{b_n} = \sqrt{\frac{1}{n} + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)} + \sqrt{\frac{1}{n}}.$$

Then taking the limit of both sides gets us

$$\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \lim_{n \rightarrow \infty} \left( \sqrt{\frac{1}{n} + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)} + \sqrt{\frac{1}{n}} \right).$$

We can then split and modify the limits on the right hand side with Theorem 27 to get

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sqrt{\frac{1}{n} + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)} + \lim_{n \rightarrow \infty} \sqrt{\frac{1}{n}} \\ &= \sqrt{\lim_{n \rightarrow \infty} \left( \frac{1}{n} + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) \right)} + \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n}} \\ &= \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)} + \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n}}. \end{aligned}$$

We know that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , so

$$= \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \cdot \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)} + \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n}} = \sqrt{0 + 0 \cdot 0} + \sqrt{0} = 0 + 0 = 0.$$

Thus, we have shown that

$$\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = 0.$$

Then, we also know that  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^1}$  is divergent by the p-series test since  $1 \leq 1$ .

Therefore, by the limit comparison test, since  $\sum_{n=1}^{\infty} a_n$  is divergent and  $\lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = 0$ ,  $\sum_{n=1}^{\infty} b_n$

is also divergent, i.e. with (3) we have  $\sum_{n=1}^{\infty} |(-1)^{n-1}(\sqrt{n+1} - \sqrt{n})|$  is divergent.

Finally, since we showed that  $\sum_{n=1}^{\infty} (-1)^{n-1}(\sqrt{n+1} - \sqrt{n})$  is a convergent series and  $\sum_{n=1}^{\infty} |(-1)^{n-1}(\sqrt{n+1} - \sqrt{n})|$

is divergent, by the definition of conditionally convergent,  $\sum_{n=1}^{\infty} (-1)^{n-1}(\sqrt{n+1} - \sqrt{n})$  is conditionally convergent. ■

## Problem 4

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}$$

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*Proof.* Let  $P(n)$  be “ $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \frac{n\sqrt{n+1}}{2}$ ” for some  $n \in \mathbb{N}$ .

**Base Case:** When  $n = 1$ . On the left hand side of  $P(1)$  we have  $\sqrt{1} = 1$ , and on the right hand side we have  $\frac{1\sqrt{1+1}}{2} = \frac{\sqrt{2}}{2}$ . So, we need to show that  $1 \geq \frac{\sqrt{2}}{2}$  holds. We know that  $4 \geq 2$ , square rooting both sides gets us  $2 \geq \sqrt{2}$  and dividing by 2 gets us  $1 \geq \frac{\sqrt{2}}{2}$ , thus  $1 \geq \frac{\sqrt{2}}{2}$  holds, so  $P(1)$  holds and the base case holds.

**Inductive Hypothesis:** Suppose  $P(k)$  holds for arbitrary  $k \in \mathbb{N}$ .

**Inductive Step:** From our inductive hypothesis we know that  $\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} \geq \frac{k\sqrt{k+1}}{2}$  holds. Lets add  $\sqrt{k+1}$  to both sides to get

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \geq \frac{k\sqrt{k+1}}{2} + \sqrt{k+1}.$$

The we can simplify the right hand side through the following steps:

$$\begin{aligned} \frac{k\sqrt{k+1}}{2} + \sqrt{k+1} &= \frac{k\sqrt{k+1}}{2} + \frac{2\sqrt{k+1}}{2} \\ &= \frac{k\sqrt{k+1} + 2\sqrt{k+1}}{2} \\ &= \frac{\sqrt{k+1} \cdot (k+2)}{2}. \end{aligned}$$

Substituting our expression on the right hand side of the inequality gets us

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \geq \frac{\sqrt{k+1} \cdot (k+2)}{2}. \quad \clubsuit$$

Now, we know that  $2 \geq 1$ , then adding  $k$  to both sides gets us  $k+2 \geq k+1$ . Dividing both sides by  $k+1$  gets us

$$\frac{k+2}{k+1} \geq 1.$$

Now, taking the square root of both sides gets us

$$\sqrt{\frac{k+2}{k+1}} \geq 1.$$

Then we can split the left hand square root with the following steps

$$\begin{aligned} \frac{\sqrt{k+2}}{\sqrt{k+1}} &\geq 1 \\ \implies \frac{\sqrt{k+1}}{(k+1)^1} \cdot \frac{(k+2)^1}{\sqrt{k+2}} &\geq 1. \end{aligned}$$

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Now, multiplying both sides by  $\sqrt{k+2}$  and  $k+1$  gets

$$\sqrt{k+1} \cdot (k+2) \geq (k+1)(\sqrt{k+2}).$$

Dividing both sides by 2:

$$\frac{\sqrt{k+1} \cdot (k+2)}{2} \geq \frac{(k+1)(\sqrt{k+2})}{2}. \quad \spadesuit$$

Then by combining  $\clubsuit$  and  $\spadesuit$  we get

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{k} + \sqrt{k+1} \geq \frac{(k+1)(\sqrt{k+2})}{2}.$$

Therefore, we have shown that  $P(k+1)$  holds and thus our inductive step is complete. By the Principle of Mathematical Induction,  $P(n)$  holds for all  $n \in \mathbb{N}$ .

Now, we can take the multiplicative inverse of both sides and flip the sign of  $\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \frac{n\sqrt{n+1}}{2}$  to get

$$\begin{aligned} \frac{1}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} &\leq \frac{1}{\frac{n\sqrt{n+1}}{2}} \\ \implies \frac{1}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} &\leq \frac{2}{n\sqrt{n+1}}. \end{aligned}$$

Now, we know  $\left| \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} \right| = \frac{1}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}$  by the definition of absolute value since  $|(-1)^{n-1}| = 1$  for all  $n \in \mathbb{N}$ , we also know from the definition of absolute value that  $0 \leq \left| \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} \right|$ , and combining these with the previous inequality gets us

$$0 \leq \left| \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} \right| \leq \frac{2}{n\sqrt{n+1}}. \quad \heartsuit$$

If we get rid of the 1 in the denominator of the right hand side, we make the expression bigger:

$$\begin{aligned} \frac{2}{n\sqrt{n+1}} &\leq \frac{2}{n\sqrt{n}} \\ \implies \frac{2}{n\sqrt{n+1}} &\leq \frac{2}{n^{3/2}}. \end{aligned}$$

Combining the inequality above with  $\heartsuit$  gives us

$$0 \leq \left| \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}} \right| \leq \frac{2}{n^{3/2}}. \quad \diamond$$

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Now, we know that  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  is convergent by the p-series test since  $3/2 > 1$ . Then by the linearity of series we know that  $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$  is convergent too. Now, by the comparison test with  $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$  being convergent and  $\diamond$ ,  $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}$  is a convergent series. ■

## Problem 5

Prove that  $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$  is a convergent series.

*Proof.* Let  $x_n = \frac{3^n (n!)^2}{(2n)!}$  for arbitrary  $n \in \mathbb{N}$ , then

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} \right| &= \left| \frac{\frac{3^{n+1}((n+1)!)^2}{(2n+2)!}}{\frac{3^n (n!)^2}{(2n)!}} \right| \\ &= \left| \frac{3^{n+1}((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{3^n (n!)^2} \right| \\ &= \left| \frac{3}{(2n+2)(2n+1)} \cdot \frac{((n+1)!)^2}{(n!)^2} \right| \\ &= \left| \frac{3}{(2n+2)(2n+1)} \cdot \left( \frac{(n+1)!}{n!} \right)^2 \right| \\ &= \left| \frac{3}{2(n+1)(2n+1)} \cdot (n+1)^2 \right| \\ &= \left| \frac{3(n+1)}{2(2n+1)} \right| \\ &= \frac{3(n+1)}{2(2n+1)} \\ &= \frac{3}{2} \cdot \frac{n+1}{2n+1} \cdot \frac{1/n}{1/n} \\ &= \frac{3}{2} \cdot \frac{1+1/n}{2+1/n} \end{aligned}$$

Thus, we have show that  $\left| \frac{x_{n+1}}{x_n} \right| = \frac{3}{2} \cdot \frac{1+1/n}{2+1/n}$ , now taking the limit of both sides gives us

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \left( \frac{3}{2} \cdot \frac{1+1/n}{2+1/n} \right)$$



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We can split the limit on the right hand side by multiplication, then division and then addition (Theorem 27) to get

$$\lim_{n \rightarrow \infty} \frac{3}{2} \cdot \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} (1/n)}{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} (1/n)}.$$

We know that the limit of a constant is said constant and  $\lim_{n \rightarrow \infty} (1/n) = 0$  so we have

$$\lim_{n \rightarrow \infty} \frac{3}{2} \cdot \frac{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} (1/n)}{\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} (1/n)} = \frac{3}{2} \cdot \frac{1 + 0}{2 + 0} = \frac{3}{4}.$$

Thus, we have shown that  $\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{3}{4}$ . Therefore, by the ratio test for series, since  $\frac{3}{4} < 1$ ,  $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$  is a convergent series. ■

## Problem 6

Prove that  $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$  is a divergent series.

*Proof.* We know that  $0 < 1$ , and with arbitrariness  $n \in \mathbb{N}, n > 0$ , so we can have  $1/n > 0$ . Then we can add  $1/n$  to both sides of  $0 < 1$  to get

$$1/n < 1 + 1/n.$$

Then since  $n > 0$ , if we take  $n$  to both sides of the inequality we can get

$$n^{1/n} < n^{1+1/n}.$$

And since  $n > 0$ ,  $n^{1/n} > 0$  so we have

$$\begin{aligned} 0 < n^{1/n} < n^{1+1/n}. \\ \implies 0 \leq n^{1/n} \leq n^{1+1/n}. \end{aligned} \quad (I)$$

Now we know that  $\lim_{n \rightarrow \infty} n^{1/n} = 1$  from Problem 2 of HW 8, thus by the Test for Divergence/ $n$ -th Term Test,  $\sum_{n=1}^{\infty} n^{1/n}$  is a divergent series since  $\lim_{n \rightarrow \infty} n^{1/n} = 1 \neq 0$ . Therefore, by the Comparison Test, since  $\sum_{n=1}^{\infty} n^{1/n}$  is a divergent series and (I),  $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$  is a divergent series. ■

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## Problem 7

Use the expression  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$  to prove that  $2^{\frac{43}{60}} = 2^{\frac{430}{600}} < e < 2^{\frac{431}{600}}$ .

*Proof.* We know  $\frac{1}{n!} > 0 \forall n \in \mathbb{N}$  (even if  $n = 0$ ), thus the infinite series would be bigger than a finite summation since the sum would only increase with more terms, so

$$\begin{aligned}
 & \sum_{n=0}^5 \frac{1}{n!} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & \frac{1}{1} + \frac{1}{1} + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & \frac{120}{120} + \frac{120}{120} + \frac{60}{120} + \frac{20}{120} + \frac{5}{120} + \frac{1}{120} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & \frac{326}{120} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & \frac{163}{60} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & 2^{\frac{43}{60}} < \sum_{n=0}^{\infty} \frac{1}{n!} \\
 \implies & 2^{\frac{43}{60}} = 2^{\frac{430}{600}} < \sum_{n=0}^{\infty} \frac{1}{n!}.
 \end{aligned}$$

Then we can substitute using series definition of  $e$  on the right hand side to get

$$2^{\frac{43}{60}} = 2^{\frac{430}{600}} < e. \quad (A)$$

Now, we know that

$$\begin{aligned}
 & \sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \dots \\
 \implies & \sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{6!} \left( 1 + \frac{1}{7} + \frac{1}{8 \cdot 7} + \dots \right)
 \end{aligned}$$

$$\Rightarrow \sum_{n=6}^{\infty} \frac{1}{n!} = \frac{1}{720} \left( (1) + \left(\frac{1}{7}\right) + \left(\frac{1}{8} \cdot \frac{1}{7}\right) + \dots \right)$$

We can make replace all the denominators in the fractions with 6 and would make the summation bigger since you are dividing by smaller numbers since all the all the denominators are greater than 6:

$$\begin{aligned} \sum_{n=6}^{\infty} \frac{1}{n!} &= \frac{1}{720} \left( (1) + \left(\frac{1}{7}\right) + \left(\frac{1}{8} \cdot \frac{1}{7}\right) + \dots \right) < \frac{1}{720} \left( (1) + \left(\frac{1}{6}\right) + \left(\frac{1}{6} \cdot \frac{1}{6}\right) + \dots \right) \\ &\Rightarrow \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720} \left( 1 + \left(\frac{1}{6}\right)^1 + \left(\frac{1}{6}\right)^2 + \dots \right). \end{aligned}$$

The series,  $1 + \left(\frac{1}{6}\right)^1 + \left(\frac{1}{6}\right)^2 + \dots$ , on the right is a geometric series with  $a = 1$  and  $r = 1/6$  and is thus convergent, the sum of the series is then  $\frac{1}{1 - \frac{1}{6}}$ , so we have the inequality

$$\begin{aligned} \sum_{n=6}^{\infty} \frac{1}{n!} &< \frac{1}{720} \left( \frac{1}{1 - \frac{1}{6}} \right) \\ &\Rightarrow \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720} \left( \frac{1}{\frac{5}{6}} \right) \\ &\Rightarrow \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{720} \left( \frac{6}{5} \right) \\ &\Rightarrow \sum_{n=6}^{\infty} \frac{1}{n!} < \frac{1}{600} \end{aligned}$$

Now, we know that  $\sum_{n=0}^5 \frac{1}{n!} + \sum_{n=6}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}$  so let us add  $\sum_{n=0}^5 \frac{1}{n!}$  to both sides of the inequality to get

$$\begin{aligned} \sum_{n=0}^5 \frac{1}{n!} + \sum_{n=6}^{\infty} \frac{1}{n!} &< \sum_{n=0}^5 \frac{1}{n!} + \frac{1}{600}. \\ &\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} < \sum_{n=0}^5 \frac{1}{n!} + \frac{1}{600}. \end{aligned}$$

We know that from our calculation of (A) that  $\sum_{n=0}^5 \frac{1}{n!} = 2\frac{430}{600}$ , so using this as a substitution on the right gives us

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 2\frac{430}{600} + \frac{1}{600}.$$

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$$\implies \sum_{n=0}^{\infty} \frac{1}{n!} < 2\frac{431}{600}.$$

Now, we can substitute the left hand side with  $e$  by the series definition of  $e$  to get

$$e < 2\frac{431}{600}. \quad (B)$$

Therefore, we can combine (A) and (B) to finally show that

$$2\frac{43}{60} = 2\frac{430}{600} < e < 2\frac{431}{600}.$$

■