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Problem 1

Prove that

$$\lim_{x\to\infty} \frac{2n+(-1)^n}{3n+1} = \frac{2}{3}$$

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Proof. Let's begin with an arbitrary $n \in N$ in the expression

$$\left|\frac{2n+(-1)^n}{3n+1} - \frac{2}{3}\right|.$$

Then we will multiply the left term by $\frac{3}{3}$ and the right term by $\frac{3n+1}{3n+1}$ to get a common denominator,

$$\left|\left(\frac{3}{3}\right) \cdot \frac{2n+(-1)^n}{3n+1} - \left(\frac{3n+1}{3n+1}\right) \cdot \frac{2}{3}\right|$$

Then multiplying the fractions we get

$$\left|\frac{6n+3(-1)^n}{9n+3} - \frac{6n+2}{9n+3}\right|.$$

Now, let us add the two terms together and simplify the numerator to get

$$\left|\frac{6n+3(-1)^n-6n-2}{9n+3}\right| = \left|\frac{3(-1)^n-2}{9n+3}\right|$$

Then let use split the fraction into two terms to get

$$\left|\frac{3(-1)^n}{9n+3} + \frac{-2}{9n+3}\right|$$

Then by the triangle inequality,

$$\left|\frac{3(-1)^n}{9n+3} + \frac{-2}{9n+3}\right| \leq \left|\frac{3(-1)^n}{9n+3}\right| + \left|\frac{-2}{9n+3}\right|$$

We can then take the absolute values of each individual component of the fraction so that

$$\left|\frac{3(-1)^n}{9n+3}\right| + \left|\frac{-2}{9n+3}\right| = \frac{|3(-1)^n|}{|9n+3|} + \frac{|-2|}{|9n+3|} = \frac{|3||(-1)^n|}{|9n+3|} + \frac{|-2|}{|9n+3|}$$

We can now by the definition of absolute value we can simplify the expression where $9n+3 > 0$, $3 > 0$, and $2 > 0$, it also must be the case that $|(-1)^n| = 1$, since when n is odd $|(-1)^n| = |-1| = 1$, when n is even $|(-1)^n| = |1| = 1$, thus

$$\frac{|3||(-1)^n|}{|9n+3|} + \frac{|-2|}{|9n+3|} = \frac{3\cdot 1}{9n+3} + \frac{2}{9n+3}$$

Now adding the two fractions together we get,

$$\frac{3}{9n+3} + \frac{2}{9n+3} = \frac{5}{9n+3}$$

Thus we have showed that

$$(1) \qquad \left|\frac{2n+(-1)^n}{3n+1} - \frac{2}{3}\right| \leq \frac{5}{9n+3} \qquad \forall n \in \mathbb{N}.$$

Now, we know for arbitrary $n \in \mathbb{N}$, $n > 0, 9 > 0$ so $9n > 0$. We also know that $3 > 0$, and by adding $9n$ to both sides of $3 > 0$ we get:

$$9n + 3 > 9n$$

Now, we also know $9n + 3 > 0$ so we can take the inverse and swap the direction of the inequality to get

$$\frac{1}{9n+3} < \frac{1}{9n}.$$

Then multiplying both sides by 5 gives us

$$(2) \qquad \frac{5}{9n+3} < \frac{5}{9n}.$$

By combining (1) and (2) we get the inequality we get

$$(3) \qquad \left|\frac{2n+(-1)^n}{3n+1} - \frac{2}{3}\right| < \frac{5}{9n} \qquad \forall n \in \mathbb{N}$$

Now, for arbitrary $\varepsilon > 0$, we know that $\frac{1}{\varepsilon} > 0 \in \mathbb{R}$ and thus $\frac{5}{9\varepsilon} > 0 \in \mathbb{R}$. Then by the Archimedean Property, there exists a $K \in \mathbb{N}$ such that $K > \frac{5}{9\varepsilon}$. So when $n \geq K$, we have $n \geq K > \frac{5}{9\varepsilon} > 0$. Then by using $n > \frac{5}{9\varepsilon}$, can multiply both sides by ε and $\frac{1}{n}$ to get

$$\varepsilon \cdot n \cdot \frac{1}{n} > \varepsilon \cdot \frac{5}{9\varepsilon} \cdot \frac{1}{n} \implies \varepsilon \cdot 1 > \frac{5}{9} \cdot \frac{1}{n} \implies \varepsilon > \frac{5}{9n}$$

Now, we can combine $\varepsilon > \frac{5}{9n}$ and (3) to get

$$\left|\frac{2n+(-1)^n}{3n+1} - \frac{2}{3}\right| < \varepsilon \qquad \forall n \geq K$$

Therefore by the definition of convergence and limit, $\frac{2n+(-1)^n}{3n+1}$ converges to $\frac{2}{3}$, thus $\lim_{x\to\infty} \frac{2n+(-1)^n}{3n+1} = \frac{2}{3}$. ■

Problem 2

Prove that $\left\{\frac{2+(-1)^n}{4+(-1)^{n+1}}\right\}$ is divergent.

Proof. Let $x_n = \left\{\frac{2+(-1)^n}{4+(-1)^{n+1}}\right\}$.
Now lets look at the subsequence $\{x_{2k}\}$, $k \in \mathbb{N}$:

$$x_{2k} = \frac{2+(-1)^{2k}}{4+(-1)^{2k+1}}$$

By the definition of even $2k$ is an even number, thus $(-1)^{2k} = 1$, and by the definition of odd $2k+1$ is an odd number, thus $(-1)^{2k+1} = -1$, therefore

$$x_{2k} = \frac{2+1}{4-1} = \frac{3}{3} = 1$$

Then, by Theorem 23 since $1 \in \mathbb{R}$, the limit of a sequence of a constant is said constant, so $\{x_{2k}\}$ converges to 1.
Now lets look at the subsequence $\{x_{2k+1}\}$, $k \in \mathbb{N}$:

$$x_{2k+1} = \frac{2+(-1)^{2k+1}}{4+(-1)^{2k+1+1}} = \frac{2+(-1)^{2k+1}}{4+(-1)^{2k+2}} = \frac{2+(-1)^{2k+1}}{4+(-1)^{2(k+1)}}$$

By the definition of odd $2k+1$ is an odd number, thus $(-1)^{2k+1} = -1$, and by the definition of even $2(k+1)$ is an even number, therefore

$$x_{2k+1} = \frac{2-1}{4+1} = \frac{1}{5}$$

Then, by Theorem 23 since $\frac{1}{5} \in \mathbb{R}$, the limit of a sequence of a constant is said constant, so $\{x_{2k+1}\}$ converges to $\frac{1}{5}$.
Therefore by Theorem 26 since two subsequences of x_n do not converge to the same limit, x_n does not converge and is thus diverges. ∎

Problem 3

Prove that

$$\lim_{n\to\infty} \frac{n^2+2n}{4n^2-\pi} = \frac{1}{4}.$$

Proof. Lets start with $\frac{n^2+2n}{4n^2-\pi}$, we can multiply this expression by $\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$ to get

$$\frac{(n^2+2n)\cdot\frac{1}{n^2}}{(4n^2-\pi)\cdot\frac{1}{n^2}} = \frac{n^2/n^2+2n/n^2}{4n^2/n^2-\pi/n^2} = \frac{1+\frac{2}{n}}{4-\frac{\pi}{n^2}} = \frac{1+2\cdot\frac{1}{n}}{4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}}.$$

Now let us find $\lim_{n\to\infty} 4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}$, we can split the limit by addition with Theorem 27 to give us

$$\lim_{n\to\infty} 4 + \lim_{n\to\infty} -\pi\cdot\frac{1}{n}\cdot\frac{1}{n}.$$

Then we can split the limit of the right term by multiplication to give us

$$\lim_{n\to\infty} 4 + \lim_{n\to\infty} -\pi \cdot \lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{1}{n}.$$

Now, the limit of a constant is said constant and we have established in class, notes and recitation that the $\lim_{n\to\infty}\frac{1}{n} = 0$, so we have

$$4-\pi\cdot 0\cdot 0 = 4.$$

Thus we have shown that $\lim_{n\to\infty} 4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}$ is non zero and equals 4.
Now let us find $\lim_{n\to\infty} 1+2\cdot\frac{1}{n}$, we can split this limit by addition with Theorem 27 to get

$$\lim_{n\to\infty} 1+\lim_{n\to\infty} 2\cdot\frac{1}{n}.$$

Then we can split the right limit by multiplication with Theorem 27 to get

$$\lim_{n\to\infty} 1+\lim_{n\to\infty} 2\cdot\lim_{n\to\infty}\frac{1}{n}.$$

We know that the limit of a constant is said constant and $\lim_{n\to\infty}\frac{1}{n}=0$,so we get

$$1+2\cdot 0=1$$

Thus we have shown that $\lim_{n\to\infty} 1+2\cdot\frac{1}{n}=1$
Now since we found the limits of the numerator and non zero denominator of $\frac{1+2\cdot\frac{1}{n}}{4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}}$ we can say that

$$\lim_{n\to\infty}\frac{1+2\cdot\frac{1}{n}}{4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}}=\frac{\lim_{n\to\infty} 1+2\cdot\frac{1}{n}}{\lim_{n\to\infty} 4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}}.$$

And since we showed that $\lim_{n\to\infty} 4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}=4$ and $\lim_{n\to\infty} 1+2\cdot\frac{1}{n}=1$:

$$\lim_{n\to\infty}\frac{1+2\cdot\frac{1}{n}}{4-\pi\cdot\frac{1}{n}\cdot\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2+2n}{4n^2-\pi}=\frac{1}{4}.$$

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Problem 4

Use

$$\sqrt{n^2+n}-n=\frac{1}{2}-\frac{n}{2(\sqrt{n^2+n}+n)^2} \qquad (*)$$

to show that $\lim_{n\to\infty}(\sqrt{n^2+n}-n)=\frac{1}{2}$.

Proof. Let us start with arbitrary $n\in\mathbb{N}$ with the expression

$$\left|\sqrt{n^2+n}-n-\frac{1}{2}\right|.$$

We see that $(*)$ can also be written as

$$\sqrt{n^2+n}-n-\frac{1}{2}=-\frac{n}{2(\sqrt{n^2+n}+n)^2}.$$

So lets substitute into our expression using the rearranged $(*)$ to get

$$\left|-\frac{n}{2(\sqrt{n^2+n}+n)^2}\right|=\frac{n}{2(\sqrt{n^2+n}+n)^2}.$$

Thus we have shown that

$$\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| = \frac{n}{2(\sqrt{n^2+n}+n)^2}. \qquad (1)$$

Then, we know that $n > 0$ so $\sqrt{2} \cdot \sqrt{n^2+n} > 0$, and since $\sqrt{2} > 1$ then $\sqrt{2} \cdot n > n$.
Now, if $a > b$ and $c > d$ we add c to both sides of $a > b$ to get $a+c > b+c$, then we add b to both sides of $c > d$ to get $b+c > b+d$, thus by combining $a+c > b+c$ and $b+c > b+d$ we get $a+c > b+c > b+d$, so $a+c > b+d$ lets call this Theorem T. Now by using Theorem T with $\sqrt{2} \cdot \sqrt{n^2+n} > 0$ and $\sqrt{2} \cdot n > n$ we get

$$\sqrt{2} \cdot \sqrt{n^2+n} + \sqrt{2} \cdot n > 0 + n \implies \sqrt{2} \cdot \sqrt{n^2+n} + \sqrt{2} \cdot n > n.$$

Then we can distribute $\sqrt{2}$ from the left hand side,

$$\sqrt{2}(\sqrt{n^2+n}+n) > n.$$

Now we square both sides of the inequality,

$$(\sqrt{2}(\sqrt{n^2+n}+n))^2 > n^2 \implies \sqrt{2}^2(\sqrt{n^2+n}+n)^2 > n^2 \implies 2(\sqrt{n^2+n}+n)^2 > n^2.$$

Now we take the multiplicative inverse of both sides and flip the sign to get

$$\frac{1}{2(\sqrt{n^2+n}+n)^2} < \frac{1}{n^2}.$$

Now multiplying both sides by n gives us

$$n \cdot \frac{1}{2(\sqrt{n^2+n}+n)^2} < n \cdot \frac{1}{n^2} \implies \frac{n}{2(\sqrt{n^2+n}+n)^2} < \frac{1}{n}.$$

Now let use substitute the left hand side of the inequality with (1) to get us

$$\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| < \frac{1}{n} \qquad (2).$$

Now for arbitrary $\varepsilon > 0$, $1/\varepsilon > 0$ and $1/\varepsilon \in \mathbb{R}$, thus by the Archimedean Property there exists a $K \in \mathbb{N}$ such so that $K > 1/\varepsilon$. Then when $n \geq K$, we have $n \geq K > 1/\varepsilon > 0$. Now by using $n > 1/\varepsilon$ we can take the multiplicative inverse of both sides and flip the sign to get

$$\frac{1}{n} < \frac{1}{1/\varepsilon} \implies \frac{1}{n} < \varepsilon \qquad \forall n \geq K.$$

Then by combining (2) and $\frac{1}{n} < \varepsilon$, we get

$$\left|\sqrt{n^2+n}-n-\frac{1}{2}\right| < \varepsilon \qquad \forall n \geq K$$

Therefore by the definition of convergence and limit, $\sqrt{n^2+n}-n$ converges to $\frac{1}{2}$ and thus $\lim_{n\to\infty}(\sqrt{n^2+n}-n) = \frac{1}{2}$. ∎

Problem 5

Prove the following statement:
If $\{a_n\}$ is a convergent sequence and $\lim_{n\to\infty} a_n > 0$, then there exists a $K \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq K$.

Proof. Let L be represent the limit $\{a_n\}$, thus $L = \lim_{n\to\infty} a_n$ and $L > 0$. By the definition of limit we can fix ε to be L since $L > 0$, by doing so and by the definition of a limit there exists a $K \in \mathbb{N}$ such that

$$\Big|a_n - L\Big| < L \qquad \forall n \geq K.$$

Then by using "$|x| < y \iff -y < x < y$" with our inequality we get

$$-L < a_n - L < L.$$

Then we can add L to all sides of the inequality to get

$$-L + L < a_n - L + L < L + L \implies 0 < a_n < 2L.$$

Therefore, there exists a $K \in \mathbb{N}$ so that

$$0 < a_n \qquad \forall n \geq K.$$

∎

Problem 6

Suppose that $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty} y_n = 0$. Prove that $\{x_n y_n\}$ is convergent and

$$\lim_{n\to\infty} (x_n y_n) = 0.$$

Proof. Because $\{x_n\}$ is bounded, due to the definition of a bounded sequence, $\exists C \in \mathbb{R}$ such that

$$|x_n| \leq C \qquad \forall n \in \mathbb{N}.$$

Then we can multiply both sides by $|y_n|$ to get

$$|y_n||x_n| \leq |y_n|C \implies |x_n||y_n| \leq C|y_n| \implies |x_n y_n| \leq C|y_n| \qquad \forall n \in \mathbb{N}.$$

Then by Theorem 27 (V), we know that $\lim_{n\to\infty} |y_n| = |\lim_{n\to\infty} y_n|$, and since we are given that $\lim_{n\to\infty} y_n = 0$:

$$\lim_{n\to\infty} |y_n| = |\lim_{n\to\infty} y_n| = |0| = 0.$$

Now, since we know that $|x_n y_n| \leq C|y_n| \quad \forall n \geq n_0$ (where n_0 can be fixed to be 1) and $\lim_{n\to\infty} |y_n| = 0$, by Theorem 25, $\{|x_n y_n|\}$ is convergent, converging to 0, and thus $\lim_{n\to\infty}(x_n y_n) = 0$. ∎

Problem 7

The **Fibonacci sequence** $\{f_n\}_{n=1}^{\infty}$ is given by the inductive definition

$$f_1 = 1,\ f_2 = 1,\ f_{n+2} = f_{n+1} + f_n \text{ for } n \geq 1$$

It is known that $f_n \geq 1$ for all $n \in N$ and $\lim_{n\to\infty}(f_n + 1/f_n)$ exists as a positive real number (you may use this without proof). Calculate the value of $\lim_{n\to\infty}(f_n + 1/f_n)$. Each step of your argument should be justified.

Proof. Let $a_n = f_{n+1}/f_n$. We know that

$$a_{n+1} = f_{n+1+1}/f_{n+1} = f_{n+2}/f_{n+1}.$$

Then by recursive definition of the Fibonacci sequence $f_{n+2} = f_{n+1} + f_n$, we can substitute this to get

$$a_{n+1} = (f_{n+1} + f_n)/f_{n+1} = f_{n+1}/f_{n+1} + f_n/f_{n+1} = 1 + f_n/f_{n+1}.$$

Since $a_n = f_{n+1}/f_n$, we know then $\frac{1}{a_n} = 1/(f_{n+1}/f_n) = f_n/f_{n+1}$, and using $\frac{1}{a_n}$ as a substitution gets us

$$a_{n+1} = 1 + \frac{1}{a_n} \qquad (\star).$$

Now, let $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} f_{n+1}/f_n$. Then we know that the $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$ since $\{a_{n+1}\}$ is a subsequence of $\{a_n\}$ and thus must converge to the same limit. We know that $L = \lim_{n\to\infty} a_n$ and the limit of a constant is said constant so, $\lim_{n\to\infty} 1 = 1$ and we know that $\lim_{n\to\infty} a_n = L$. Now we can take the limits of both sides of $(\star)$ to get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(1 + \frac{1}{a_n}).$$

We can split the limit oh the right hand side by addition to get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{a_n}.$$

Now, we showed that $\lim_{n\to\infty} a_{n+1} = L$, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{L}$ and the $\lim_{n\to\infty} 1 = 1$, so substituting these in gives us

$$L = 1 + \frac{1}{L}.$$

Subtracting 1 and $\frac{1}{L}$ from both sides gives us

$$L - 1 - \frac{1}{L} = 0.$$

Then multiplying both sides by L gives us

$$L^2 - L - 1 = 0.$$

Now by using the quadratic formula we get

$$L = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus, $L = \frac{1+\sqrt{5}}{2}$ or $L = \frac{1-\sqrt{5}}{2}$. Since $1 < \sqrt{5}$, $1 - \sqrt{5} < 0$ and thus $\frac{1-\sqrt{5}}{2} < 0$, also know that $\frac{1+\sqrt{5}}{2} > 0$. Therefore, since we are given that $\lim_{n\to\infty}(f_n + 1/f_n)$,i.e. $\lim_{n\to\infty} L$, exists as a positive real number, the $L \neq \frac{1-\sqrt{5}}{2}$, so $L = \frac{1+\sqrt{5}}{2}$. Thus $\lim_{n\to\infty}(f_n + 1/f_n) = \frac{1+\sqrt{5}}{2}$.

$\blacksquare$