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# Problem 1

Prove that the sequence $\{\frac{n(-1)^n+2}{2n+4}\}$ is a bounded sequence

*Proof.* Lets start with the expression $|n(-1)^n+2|$ where $n$ is an arbitrary element of $\mathbb{N}$, then by the triangle inequality

$$|n(-n)^n+2| < |n(-1)^n| + |2|.$$

On the right hand side we see that absolute value of 2 is just 2, and $|n(-1)^n| = |n||(-1)^n|$, then we know that $|(-1)^n| = |-1|^n = 1^n = 1$. Thus the inequality evaluates to

$$|n(-n)^n+2| < |n \cdot 1| + 2.$$

Then $|n \cdot 1| = |n| = n$, so the inequality is equivalent to

$$|n(-n)^n+2| < n+2.$$

Then let us divide both sides of the inequality by $|2n+4|$, giving us equivalent to

$$\frac{|n(-n)^n+2|}{|2n+4|} < \frac{n+2}{|2n+4|}.$$

Then on the right hand side, the denominator can be evaluated by $|2n+4| = 2n+4 = 2(n+2)$, and on the left hand side we can combine the absolute value of the fraction to get

$$\left|\frac{n(-n)^n+2}{2n+4}\right| < \frac{n+2}{2(n+2)}.$$

Then on the right hand side we can divide the denominator and the numerator by $n+2$ to get

$$\left|\frac{n(-n)^n+2}{2n+4}\right| < \frac{1}{2}$$

Then since we know that $|x| < y \iff -y < x < y$, we can use this on our inequality above to get us,

$$-\frac{1}{2} < \frac{n(-n)^n+2}{2n+4} < \frac{1}{2} \qquad \forall n \in \mathbb{N}.$$

Thus, $\frac{1}{2}$ is a upper bound of sequence $\{\frac{n(-1)^n+2}{2n+4}\}$, and $-\frac{1}{2}$ is a lower bound of the sequence. Therefore, $\{\frac{n(-1)^n+2}{2n+4}\}$ is bounded since an upper and lower bound exist.

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# Problem 2

Prove that the sequence $\{n+(-1)^n(n-1)\}$ is unbounded (and thus divergent).

*Proof.* Suppose the sequence is bounded, then an upper bound exists so there exists a $u \in \mathbb{R}$ such that

$$x_n \leq u. \qquad \forall n \in \mathbb{N}$$

Let $n = 2k$ for arbitrary $k \in \mathbb{N}$. Then $x_{2k} = 2k + (-1)^{2k}(2k-1)$, then we know $2k$ is an even number by the definition of even, so $(-1)^{2k} = 1$, giving us

$$x_{2k} = 2k + (1)(2k-1) = 2k + 2k - 1 = 4k - 1$$

$$x_{2k} = 4k - 1$$

We can substitute $x_{2k}$ for $x_n$ into the inequality since $2k \in \mathbb{N}$ and an element of the sequence, giving us

$$x_{2k} \leq u$$

$$4k - 1 \leq u$$

Now, we can add 1 to both sides of the equality then divide by 4:

$$4k \leq u + 1$$

$$k \leq \frac{u+1}{4} \qquad \forall k \in \mathbb{N}$$

We know that $\frac{u+1}{4} \in \mathbb{R}$ since $\mathbb{R}$ is closed under addition and multiplication. Thus, there is a contradiction with the Archimedean Property and the inequality above that there exists a $\frac{u+1}{4} in \mathbb{R}$ greater than or equal to all elements of $\mathbb{N}$. Therefore, the sequence $\{n+(-1)^n(n-1)\}$ is unbounded and thus divergent. ∎

# Problem 3

Use the definition of limit to prove that

$$\lim_{x \to \infty} \frac{n}{n+3} = 1.$$

*Proof.* Let $n$ be an arbitrary element of $\mathbb{N}$, then $n > 0$. We also know that $3 > 0$, let us add $n$ to both sides of $3 > 0$ to get

$$n < n + 3$$

We can multiply both sides by 3 to get

$$3n < 3(n+3).$$

Then multiply both sides by $\frac{1}{n(n+3)}$ , $n(n+3) > 0$, to get

$$\frac{1}{n(n+3)} \cdot 3n < \frac{1}{n(n+3)} \cdot 3(n+3) \implies \frac{3}{n+3} < \frac{3}{n}.$$

$$\frac{3}{n+3} < \frac{3}{n} \qquad \text{Inequality (1)}$$

Then, let us use simplify the expression $\left|\frac{n}{n+3}-1\right|$ with arbitrary $n \in \mathbb{N}$. We see that one is equivalent to $\frac{n+3}{n+3}$, so

$$\left|\frac{n}{n+3}-1\right| = \left|\frac{n}{n+3}-\frac{n+3}{n+3}\right|$$

$$= \left|\frac{-3}{n+3}\right| \quad \text{by subtracting}$$

$$= \frac{3}{n+3} \quad \text{by definition of absolute value}$$

Thus we have shown that $\left|\frac{n}{n+3}-1\right| = \frac{3}{n+3}$. Now we can substitute in $\left|\frac{n}{n+3}-1\right|$ for $\frac{3}{n+3}$ on the left hand side of Inequality (1). Giving us

$$\left|\frac{n}{n+3}-1\right| < \frac{3}{n} \qquad \text{Inequality (2)}$$

Suppose an arbitrary $\varepsilon > 0 \in \mathbb{R}$, then $\frac{3}{\varepsilon} \in \mathbb{R}$ and $\frac{3}{\varepsilon} > 0$. Then by the Archimedean Property there exists a $K \in \mathbb{R}$ such that $K > \frac{3}{\varepsilon}$. When $n \geq K$, we have $n \geq K > \frac{3}{\epsilon} > 0$. Then by inversing both sides and flipping the direction of the inequality $n > \frac{3}{\epsilon}$ we get

$$\frac{1}{n} < \frac{1}{3/\epsilon} \qquad \forall n \geq K.$$

Then we can multiply both sides of the inequality by 3 to get

$$3(\frac{1}{n}) < 3(\frac{1}{3/\epsilon}) \implies \frac{3}{n} < \frac{1}{1/\epsilon}$$

Then on the right hand side, we can multiply by 1 in the form of $\frac{\epsilon}{\epsilon}$ to get

$$\frac{3}{n} < \frac{1}{1/\epsilon} \cdot \frac{\epsilon}{\epsilon} \implies \frac{3}{n} < \epsilon$$

Then we can combine Inequality (2) with $\frac{3}{n} < \epsilon$ to get

$$\left|\frac{n}{n+3}-1\right| < \epsilon \qquad \forall n \geq K$$

Therefore, by the definition of convergence and limit, the sequence $\{\frac{n}{n+3}\}$ converges to 1, and thus $\lim_{x\to\infty} \frac{n}{n+3} = 1$.

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## Problem 4

*Proof.* Use the definition of limit to prove that

$$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0.$$

Let $n$ be an arbitrary element of $\mathbb{N}$. Then $n > 0$ and $2 > 0$, thus $n^2 > 0$ and $2n^2 > 0$. So, let us now add $2n^2$ to both sides of $0 < 2$ to get

$$2n^2 < 2n^2 + 2$$

Then, let us distribute out a $n$ from the left hand side and a 2 from the right hand side to get

$$n \cdot 2n < 2(n^2 + 1)$$

Now, we can multiply both sides by $\frac{1}{n(n^2+1)}$, $n(n^2 + 1) > 0$, to get

$$\frac{1}{n(n^2+1)} \cdot n \cdot 2n < \frac{1}{n(n^2+1)} \cdot 2(n^2 + 1) \implies \frac{2n}{n^2+1} < \frac{2}{n}$$

$$\frac{2n}{n^2+1} < \frac{2}{n} \qquad \text{Inequality (a)}$$

Now, looking at the expression $\left|\frac{2n}{n^2+1} - 0\right|$, we can simplify it to

$$\left|\frac{2n}{n^2+1} - 0\right| = \left|\frac{2n}{n^2+1}\right|$$

Then by the definition of absolute value, since $\left|\frac{2n}{n^2+1}\right| > 0$, we get

$$\left|\frac{2n}{n^2+1}\right| = \frac{2n}{n^2+1}$$

Thus,

$$\left|\frac{2n}{n^2+1} - 0\right| = \frac{2n}{n^2+1}$$

So now we can substitute $\left|\frac{2n}{n^2+1} - 0\right|$ on the left hand side of Inequality (a) to get

$$\left|\frac{2n}{n^2+1} - 0\right| < \frac{2}{n} \qquad \text{Inequality (b)}$$

Now, suppose an arbitrary $\epsilon > 0$, then $2/\epsilon \in \mathbb{R}$ and $2/\epsilon > 0$. Then by the Archimedean Property, there exists $K \in \mathbb{N}$ such that $2/\epsilon < K$. When $n \geq K$ we get that $n \geq K > 2/\epsilon > 0$. Then we can invert both sides of the inequality $n > 2/\epsilon$ and swap the direction of the inequality to get

$$\frac{1}{n} < \frac{1}{2/\epsilon}$$

Multiplying both sides of 2 gets

$$\frac{2}{n} < \frac{1}{1/\epsilon}$$

Then by multiplying the right hand side by 1 in the form of $\frac{\epsilon}{\epsilon}$ to get

$$\frac{2}{n} < \frac{\epsilon}{\epsilon} \cdot \frac{1}{1/\epsilon} \implies \frac{2}{n} < \epsilon$$

Now, we can combine $\frac{2}{n} < \epsilon$ with Inequality (b) to get

$$\left|\frac{2n}{n^2+1} - 0\right| < \epsilon \qquad \forall n \geq K$$

Therefore by the definition of convergence and limit, the sequence $\{\frac{2n}{n^2+1}\}$ converges to 0, thus

$$\lim_{n\to\infty} \frac{2n}{n^2+1} = 0.$$

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# Problem 5

We proved in class that $2^n > n^2$ holds for $n \geq 5$. Use this inequality to prove that

$$\lim_{n\to\infty} \frac{n}{2^n} = 0.$$

*Proof.* Using $n^2 < 2^n$, we can divide both sides of the inequality by $n$, since $n \geq 5 > 0$, to get

$$\frac{n^2}{n} < \frac{2^n}{n} \implies n < \frac{2^n}{n}.$$

Then we can divide both sides of the inequality by $2^n$ to get

$$\frac{n}{2^n} < \frac{\frac{2^n}{n}}{2^n} \implies \frac{n}{2^n} < \frac{1}{n}.$$

Then we can see that

$$\left|\frac{n}{2^n} - 0\right| = \left|\frac{n}{2^n}\right| = \frac{n}{2^n}.$$

So now we can substitute the right hand side of $\frac{n}{2^n} < \frac{1}{n}$ with $|\frac{n}{2^n} - 0|$, giving us

$$\left|\frac{n}{2^n} - 0\right| < \frac{1}{n} \qquad \forall n \geq 5 \qquad \text{Inequality (i)}$$

Now, suppose arbitrary $\epsilon > 0$, $1/\epsilon \in \mathbb{R}$ and $1/\epsilon > 0$. Then by the Archimedean Property, there exists a $K_0 \in \mathbb{N}$ such that $1/\epsilon < K_0$. Then let $K = max\{K_0, 5\}$, then $K \geq K_0 > 1/\epsilon > 0$ and $K \geq 5$. Then when $n \geq K$ we have $n \geq K \geq K_0 > 1/\epsilon > 0$. Then by taking the inverse of both sides of the inequality $n > 1/\epsilon$ and switching the direction of the inequality we get

$$\frac{1}{n} < \frac{1}{1/\epsilon} \implies \frac{1}{n} < \epsilon.$$

Then by combining Inequality (i) with $\frac{1}{n} < \epsilon$ we get

$$\left|\frac{n}{2^n} - 0\right| < \epsilon \qquad \forall n \geq K \geq 5$$

Therefore by the definition of convergence and limit, the sequence $\{2^n > n^2\}$ converges to the limit 0, thus $\lim_{n\to\infty} \frac{n}{2^n} = 0$ for $n \geq 5$. ■

# Problem 6

Let $\{a_n\}$ be a sequence in $\mathbb{R}$ and $\alpha \in \mathbb{R}$. Suppose that $\{a_n\}$converges to $\alpha$. Prove that $\{|a_n|\}$ converges to $|\alpha|$.

*Proof.* Since $\{a_n\}$converges to $\alpha$, there exists a $K \in \mathbb{N}$ for arbitrary $\epsilon > 0$ such that

$$|a_n - \alpha| < \epsilon \quad \forall n \geq K \qquad \text{inequality (1)}$$

Now let us use a corollary of the triangle inequality on the expression $\big||a_n| - |\alpha|\big|$ to get

$$\big||a_n| - |\alpha|\big| \leq |a_n - \alpha| \qquad \forall n \in \mathbb{N}$$

When $n > K$, we can combine $\big||a_n| - |\alpha|\big| \leq |a_n - \alpha|$ and $|a_n - \alpha| < \epsilon$ to get

$$\big||a_n| - |\alpha|\big| < \epsilon \qquad \forall n \geq K$$

Thus by the definition of convergence, the sequence $\{|a_n|\}$ converges to $|\alpha|$. ■

# Problem 7

Let$t \in \mathbb{R}$. Prove that there exists a sequence $r_n$ such that $r_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} r_n = t$$

*Proof.* For an arbitrary $n \in \mathbb{N}$, since $n > 0$, $1/n > 0$ and $1/n \in \mathbb{R}$. Then $-1/n < 0$ and we can combine $-1/n < 0$ and $1/n > 0$ to get

$$-\frac{1}{n} < 0 < \frac{1}{n}.$$

$$-\frac{1}{n} < \frac{1}{n}$$

Then we can add $t$ to both sides of the inequality to get

$$t - \frac{1}{n} < t + \frac{1}{n} \qquad \forall n \in \mathbb{N}.$$

We know that $t - \frac{1}{n} \in \mathbb{R}$ and $t + \frac{1}{n} \in \mathbb{R}$ by the closure of the real numbers, thus by the density of the rationals we say that there exists a $r_n \in \mathbb{Q}$ such that

$$t - \frac{1}{n} < r_n < t + \frac{1}{n} \qquad \forall n \in \mathbb{N}$$

By subtracting $t$ from all sides we get

$$-\frac{1}{n} < r_n - t < \frac{1}{n}$$

Then by using "$|x| < y \iff -y < x < y$" with our inequality we get

$$|r_n - t| < \frac{1}{n}$$

Then for arbitrary $\epsilon > 0$, $1/\epsilon > 0$ and $1/\epsilon \in \mathbb{R}$. Thus by the Archimedean Property there exists a $K \in \mathbb{N}$ such that $K > 1/\epsilon$. Thus when $n \geq K$, $n \geq K > 1/\epsilon > 0$. Then we can take the inverse of both sides of $n > 1/\epsilon$ and switch the direction of the inequality to get

$$\frac{1}{n} < \frac{1}{1/\epsilon} \implies \frac{1}{n} < \epsilon$$

Then by combining $|r_n - t| < \frac{1}{n}$ and $\frac{1}{n} < \epsilon$ we get

$$|r_n - t| < \epsilon \qquad \forall n \geq K$$

Thus by the definition of convergence and limit, the sequence $\{r_n\}$ converges to $t$ and thus $\lim_{n\to\infty} r_n = t \quad \forall n \in \mathbb{N}$. ■