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Problem 1

Let $a, b \in \mathbb{R}$ such that $b > a > 0$. Prove that

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = b.$$

Proof. Starting with the fact that $0 < a < b$, we can take the power of n on all sides to get

$$0^n < a^n < b^n \implies 0 < a^n < b^n.$$

Now we can add b^n to all sides to get

$$0 + b^n < a^n + b^n < b^n + b^n \implies b^n < a^n + b^n < 2b^n.$$

Now we can take the power of $1/n \in \mathbb{R}$ on all sides to get

$$(b^n)^{1/n} < (a^n + b^n)^{1/n} < (2b^n)^{1/n}$$

$$\implies b < (a^n + b^n)^{1/n} < (2)^{1/n}n. \qquad (1)$$

We know that $b \in R$ and the limit of a constant is said constant, so $\lim_{n\to\infty} b = b$. Then with $\lim_{n\to\infty}(2)^{1/n}n$, we can split the limit by multiplication with theorem 27 to get $\lim_{n\to\infty}(2)^{1/n} \cdot \lim_{n\to\infty} n$. Now, since $2 \in \mathbb{R}$ and $2 > 0$, by Theorem 30, $\lim_{n\to\infty}(2)^{1/n} = 1$ and the limit of the constant b is just b , so $\lim_{n\to\infty}(2)^{1/n} \cdot \lim_{n\to\infty} n = 1 \cdot b = b$. Now since we showed that $\lim_{n\to\infty} b = b$ and $\lim_{n\to\infty}(2)^{1/n}n = b$ we have

$$\lim_{n\to\infty} b = \lim_{n\to\infty} (2)^{1/n}n = b. \qquad (2)$$

Then by the squeeze theorem with (1) and (2),

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = b.$$

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Problem 2

Prove that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

You may use the inequality $n^{1/n} \leq 1 + 2\sqrt{\frac{1}{n}}$ for $n \in \mathbb{N}$ without proof.

Proof. For arbitrary $n \in \mathbb{N}$, $1 \leq n$ by definition of $\mathbb{N}$. We can take the power of $1/n$ on both sides to get

$$1^{1/n} \leq n^{1/n} \implies 1 \leq n^{1/n}.$$

Now we can combine $1 \leq n^{1/n}$ with $n^{1/n} \leq 1 + 2\sqrt{\frac{1}{n}}$ for $n \in \mathbb{N}$ to get

$$1 \leq n^{1/n} \leq 1 + 2\sqrt{\frac{1}{n}}. \qquad (1)$$

Now, we know that $\lim_{n\to\infty} 1 = 1$ since the limit of a constant is said constant. Then $\lim_{n\to\infty} 1 + 2\sqrt{\frac{1}{n}}$ can be split by addition with Theorem 27 to get

$$\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2\sqrt{\frac{1}{n}},$$

and then we can split the right limit by multiplication to get

$$\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2 \cdot \lim_{n\to\infty} \sqrt{\frac{1}{n}}.$$

Now, we can take the square root on the whole right limit by Theorem 27(iv) to get

$$\lim_{n\to\infty} 1 + \lim_{n\to\infty} 2 \cdot \sqrt{\lim_{n\to\infty} \frac{1}{n}}.$$

We know that $\lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} 2 = 2$, and the $\lim_{n\to\infty} 1/n = 0$, so our limit expression becomes

$$\begin{aligned} 1 + 2 \cdot \sqrt{0} &= 1 + 2 \cdot 0 \\ &= 1. \end{aligned}$$

Thus, we know that

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty} 1 + 2\sqrt{\frac{1}{n}} = 1. \tag{2}$$

Then by the Squeeze Theorem with (1) and (2),

$$\lim_{n\to\infty} n^{1/n} = 1.$$

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Problem 3

Prove that

$$\lim_{n\to\infty} \frac{n^4}{2^n} = 0.$$

Proof. Let us use the ratio test. Let $x_n = \frac{n^4}{2^n}$, then $x_{n+1} = \frac{(n+1)^4}{2^{n+1}}$, thus

$$\frac{x_{n+1}}{x_n} = \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}}.$$

Multiplying by $\frac{2^n}{2^n}$,

$$= \frac{\frac{(n+1)^4}{2^{n+1}}}{\frac{n^4}{2^n}} \cdot \frac{2^n}{2^n} = \frac{\frac{(n+1)^4}{2} \cdot \frac{2^n}{2^n}}{n^4 \cdot \frac{2^n}{2^n}} = \frac{\frac{(n+1)^4}{2}}{n^4} = \frac{(n+1)^4}{2n^4}$$

We can now move the 1/2 to the front and move the exponent to the whole term

$$\frac{(n+1)^4}{2n^4} = \frac{1}{2} \cdot \frac{(n+1)^4}{n^4} = \frac{1}{2} \cdot \left(\frac{(n+1)}{n}\right)^4.$$

Then we can divide n into $n+1$ so

$$\frac{1}{2} \cdot \left(\frac{(n+1)}{n}\right)^4 = \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right)^4$$

We can split the exponents into multiplication to get

$$\frac{1}{2} \cdot \left(1 + \frac{1}{n}\right)^4 = \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right).$$

Thus we have shown that

$$\frac{x_{n+1}}{x_n} = \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right). \tag{1}$$

Now, we know that the $\lim_{n\to\infty} 1/2 = 1/2$ since $1/2$ is a constant. Then with $\lim_{n\to\infty}(1+\frac{1}{n})$, we can split the limit by addition (Theorem 27) to get $\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n})$, we know that $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, so $\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1 + 0 = 1$.

Now, with $\lim_{n\to\infty}(\frac{1}{2} \cdot (1+\frac{1}{n}) \cdot (1+\frac{1}{n}) \cdot (1+\frac{1}{n}) \cdot (1+\frac{1}{n}))$, we can split this by multiplication to get the expression

$$\lim_{n\to\infty} \frac{1}{2} \cdot \lim_{n\to\infty} (1 + \frac{1}{n}) \cdot \lim_{n\to\infty} (1 + \frac{1}{n}) \cdot \lim_{n\to\infty} (1 + \frac{1}{n}) \cdot \lim_{n\to\infty} (1 + \frac{1}{n}).$$

Since we showed that $\lim_{n\to\infty} 1/2 = 1/2$ and $\lim_{n\to\infty}(1+\frac{1}{n}) = 1$, the expression above is equivalent to

$$\frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \frac{1}{2}.$$

Thus we have showed that

$$\lim_{n\to\infty} \left(\frac{1}{2} \cdot (1 + \frac{1}{n}) \cdot (1 + \frac{1}{n}) \cdot (1 + \frac{1}{n}) \cdot (1 + \frac{1}{n})\right) = \frac{1}{2}. \tag{2}$$

Now with $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}|$, we can use (1) as a substitution so that

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} \left|\frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right)\right|.$$

By Theorem 27 (V), we can take the absolve value outside the limit,

$$\lim_{n\to\infty} \left|\frac{1}{2} \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right) \cdot \left(1 + \frac{1}{n}\right)\right|$$

$$= \Big| \lim_{n\to\infty} \Big(\frac{1}{2} \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big)\Big)\Big|.$$

Then we can substitute into the expression above using (2) to get

$$\Big| \lim_{n\to\infty} \Big(\frac{1}{2} \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big) \cdot \Big(1+\frac{1}{n}\Big)\Big)\Big| = \Big|\frac{1}{2}\Big| = \frac{1}{2}$$

Thus,

$$\lim_{n\to\infty} \Big|\frac{x_{n+1}}{x_n}\Big| = \frac{1}{2}$$

Therefore, by the Ratio Test for Sequences, since $\lim_{n\to\infty} \Big|\frac{x_{n+1}}{x_n}\Big| = \frac{1}{2} < 1$,

$$\lim_{n\to\infty} \frac{n^4}{2^n} = 0.$$

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Problem 4

Prove that

$$\lim_{n\to\infty} \Big(\frac{2^{3n}-1}{3^{2n}+1}\Big) = 0.$$

Proof. Starting with $\frac{2^{3n}-1}{3^{2n}+1}$, we can take the exponent n out of the two terms to get

$$\frac{(2^3)^n - 1}{(3^2)^n + 1} = \frac{8^n - 1}{9^n + 1}$$

Then let us multiply the expression by $\frac{1/9^n}{1/9^n}$,

$$\frac{8^n-1}{9^n+1} \cdot \frac{1/9^n}{1/9^n} = \frac{(8^n-1)(1/9^n)}{(9^n+1)(1/9^n)} = \frac{\frac{8^n}{9^n} - \frac{1}{9^n}}{\frac{9^n}{9^n} + \frac{1}{9^n}} = \frac{(\frac{8}{9})^n - (\frac{1}{9})^n}{1 + (\frac{1}{9})^n}.$$

Thus we have shown that

$$\Big(\frac{2^{3n}-1}{3^{2n}+1}\Big) = \frac{(\frac{8}{9})^n - (\frac{1}{9})^n}{1 + (\frac{1}{9})^n}. \qquad (A)$$

We know that $9 > 8 > 0$, dividing all sides by 9 gives us $1 > 8/9 > 0$, thus $8/9 \in \mathbb{R}$ and $1 > |8/9|$ by definition of absolute value. Thus since $1 > |8/9|$, by Theorem 30,

$$\lim_{n\to\infty} (\frac{8}{9})^n = 0. \qquad (i)$$

We also know that $9 > 1 > 0$, then dividing all sides by 9 we get $1 > 1/9 > 0$, thus $1 > |1/9|$ and $1/9 \in \mathbb{R}$. Then since $1 > |1/9|$, by Theorem 30,

$$\lim_{n\to\infty} (\frac{1}{9})^n = 0. \qquad (ii)$$

Now, $\lim_{n\to\infty}\left((\frac{8}{9})^n - (\frac{1}{9})^n\right)$ can be spit by subtraction (Theorem 27) to get $\lim_{n\to\infty}(\frac{8}{9})^n - \lim_{n\to\infty}(\frac{1}{9})^n$. We can use (i) and (ii) to get $0-0=0$. Thus we have shown that

$$\lim_{n\to\infty}\left((\frac{8}{9})^n - (\frac{1}{9})^n\right) = 0. \qquad (B)$$

Now, $\lim_{n\to\infty}(1+(\frac{1}{9})^n)$ can be split by addition to get $\lim_{n\to\infty} 1 + \lim_{n\to\infty}(\frac{1}{9})^n$. We know that the limit of a constant is said constant and using (ii) we get $1+0=1$, thus we have shown that

$$\lim_{n\to\infty}\left(1+(\frac{1}{9})^n\right) = 1. \qquad (C)$$

Finally, with $\lim_{n\to\infty}\frac{(\frac{8}{9})^n-(\frac{1}{9})^n}{1+(\frac{1}{9})^n}$, we can split the limit by division (Theorem 27),

$$\frac{\lim_{n\to\infty}\left((\frac{8}{9})^n - (\frac{1}{9})^n\right)}{\lim_{n\to\infty}\left(1+(\frac{1}{9})^n\right)}.$$

Then by using (B) and (C) as substitutions we get

$$\frac{0}{1} = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{(\frac{8}{9})^n-(\frac{1}{9})^n}{1+(\frac{1}{9})^n} = 0.$$

Then by using (A) as a substitution we have

$$\lim_{n\to\infty}\left(\frac{2^{3n}-1}{3^{2n}+1}\right) = 0.$$

■

Problem 5

Define $\{x_n\}$ by $x_1 = 10$ and

$$x_{n+1} = \sqrt{6+x_n}$$

for $n \geq 1$. Prove that $\{x_n\}$ is convergent and find its limit.

Proof. Let $P(n)$ be "$3 \leq x_{n+1} \leq x_n$" $\forall n \in \mathbb{N}$
Base Case: When $n=1$. So $P(1)$ is $3 \leq x_{1+1} \leq x_1 \implies 3 \leq x_2 \leq x_1$. Using our recursive definition of x_n, we know $x_1 = 10$ and $x_2 = \sqrt{6+x_1} = \sqrt{6+10} = \sqrt{16} = 4$. So $2 \leq 4 \leq 10$ holds, then $P(1)$ holds, thus the base case holds.
Inductive Hypothesis: Suppose $P(k)$ holds for arbitrary $k \in \mathbb{N}$.
Inductive Step: Using our inductive step, we know that $P(k)$ holds, i.e. $3 \leq x_{k+1} \leq x_k$ holds for arbitrary $k \in \mathbb{N}$, we can add 6 to all sides of our inequality to get

$$9 \leq x_{k+1} + 6 \leq x_k + 6.$$

Then we can take the square root of all sides to get

$$3 \leq \sqrt{x_{k+1}+6} \leq \sqrt{x_k+6}.$$

Then using our inductive definition of $\{x_n\}$, $x_{k+2} = \sqrt{6+x_{k+1}}$ and x_n, $x_{k+1} = \sqrt{6+x_k}$, so using these as substitutions in our inequality we get

$$3 \leq x_{k+2} \leq x_{k+1}.$$

Thus we have shown that $P(k+1)$ holds for arbitrary $k \in \mathbb{N}$.
Therefore by the Principle of Mathematical Induction, $P(n)$ holds for all $n \in \mathbb{N}$.

Now, we know that "$3 \leq x_{n+1} \leq x_n$" holds $\forall n \in \mathbb{N}$. Then $x_{n+1} \leq x_n$, $\{x_n\}$ is a decreasing sequence and $3 \leq x_n$ $\forall n \in \mathbb{N}$, so 3 is a lower bound of $\{x_n\}$, thus by the Monotone Convergence Theorem, $\{x_n\}$ is convergent.
Therefore the limit of $\{x_n\}$ exists and let

$$x = \lim_{n\to\infty} x_n.$$

We know that $x_n \leq 3$ $\forall n \in \mathbb{N}$, so taking the $\lim_{n\to\infty}$ of both sides gives us

$$\lim_{n\to\infty} 3 \leq \lim_{n\to\infty} x_n \implies 3 \leq x.$$

Now let us take the limit of both sides of $x_{n+1} = \sqrt{6+x_n}$ to get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{6+x_n}.$$

Since we know that $\{x_{n+1}\}$ is a tail sequence and thus a subsequence of $\{x_n\}$, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$, we can use this as a substitution on the left side. Now we can also take out the square root from inside the limit (Theorem 27 VI) to get

$$x = \sqrt{\lim_{n\to\infty}(6+x_n)}.$$

We can now split the limit by addition (Theorem 27) to get

$$x = \sqrt{\lim_{n\to\infty} 6 + \lim_{n\to\infty} x_n}.$$

We know that the limit of a constant is said constant and $\lim_{n\to\infty} x_n = x$ so the equality becomes

$$x = \sqrt{6+x}.$$

We can now square both sides,

$$x^2 = \sqrt{6+x}^2 \implies x^2 = 6+x.$$

Moving subtracting the right hand side from both sides gives

$$x^2 - x - 6 = 0.$$

We know that $(x-3)(x+2) = x^2 - x - 6$ by distributing so using this as a substitution gives us

$$(x-3)(x+2) = 0.$$

So there are two cases where the equality holds when $x - 3 = 0$ and $x = 3$, or when $x + 2 = 0$ and $x = -2$, but since we showed that $x \geq 3$ the only possible value for x is 3. Thus we have showed that $\{x_n\}$ is convergent and its limit, x, is 3. ■

Problem 6

Define $\{a_n\}$ by $a_1 = 1$ and

$$a_{n+1} = \frac{6}{5 - a_n}$$

for $n \geq 1$. Prove that $\{a_n\}$ is convergent and find the value of $\lim_{n\to\infty} a_n$.

Proof. Let $P(n)$ be "$a_n \leq a_{n+1} \leq 2$" $\forall n \in \mathbb{N}$
Base Case: When $n = 1$. So $P(1)$ is $a_1 \leq a_2 \leq 2$. Using the recursive definition of $\{a_n\}$, we know $a_1 = 1$ and $a_2 = \frac{6}{5-a_1} = \frac{6}{4} = \frac{3}{2} = 1.5$, so substituting these into the inequality gets us $1 < 1.5 < 2$ which holds. Thus $P(1)$ holds and the base case holds.
Inductive Hypothesis: Suppose $P(k)$ holds for arbitrary $k \in \mathbb{N}$.
Inductive Step: Using our inductive step, we know that $a_k \leq a_{k+1} \leq 2$ holds for arbitrary $k \in \mathbb{N}$. We can then multiply all sides of the inequality by -1 and flip the sign to get

$$-a_k \geq -a_{k+1} \geq -2.$$

Then we can add 5 to all sides of the inequality to get

$$5 - a_k \geq 5 - a_{k+1} \geq 3.$$

Then we can take the multiplicative inverse of all sides and flip the sign to get

$$\frac{1}{5-a_k} \leq \frac{1}{5-a_{k+1}} \leq \frac{1}{3}.$$

We can now multiply all sides of the inequality by 6 to get

$$\frac{6}{5-a_k} \leq \frac{6}{5-a_{k+1}} \leq 2.$$

Now, we know from the recursive definition of $\{a_n\}$ that $a_{k+1} = \frac{6}{5-a_k}$ and $a_{k+2} = \frac{6}{5-a_{k+1}}$, and using these as substitutions into the inequality gives us

$$a_{k+1} \leq a_{k+2} \leq 2.$$

Therefore, we have showed that $P(k+1)$ holds and thus have finished our inductive step. By the Principle of Mathematical Induction, $P(n)$ hold for all $n \in \mathbb{N}$.
Since we now know that $a_n \leq a_{n+1} \leq 2 \; \forall n \in \mathbb{N}$, a_n is a increasing sequence since $a_n \leq a_{n+1}$, and a_n is bounded above by 2 since $a_n \leq 2$, thus by the Monotone Convergence Theorem, a_n is convergent. Therefore the limit of $\{a_n\}$ exists and let

$$a = \lim_{n\to\infty} a_n.$$

Then let us take the limit on both sides of $a_n < 2$ to give us

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} 2 \implies a \leq 2.$$

We since we know that $\{a_{n+1}\}$ is a tail sequence/a subsequence of $\{a_n\}$ and thus must converge to the same limit so $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$. We know that $\lim_{n\to\infty}(5-a_n)$ can be split by subtraction, $\lim_{n\to\infty} 5 - \lim_{n\to\infty} a_n = 5 - a > 0$.
Now let us take the limit of both sides of our recursive definition $a_{n+1} = \frac{6}{5-a_n}$ to get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{6}{5-a_n}.$$

We can then split the limit by division (Theorem 27),

$$\lim_{n\to\infty} a_{n+1} = \frac{\lim_{n\to\infty} 6}{\lim_{n\to\infty}(5-a_n)}$$

We showed that $\lim_{n\to\infty} a_{n+1} = a$, $\lim_{n\to\infty}(5-a_n) = 5-a$, and we know that $\lim_{n\to\infty} 6 = 6$, so using these as substitutions, our equality becomes

$$a = \frac{6}{5-a}.$$

Now, let us multiply both sides by $(5-a)$ then simplify to get

$$a \cdot (5-a) = \frac{6}{5-a} \cdot (5-a) \implies a \cdot (5-a) = 6.$$

Then we can distribute the left hand side to get

$$5a - a^2 = 6.$$

We then subtract 6 from both sides

$$-6 + 5a - a^2 = 0.$$

We multiply the both sides of the equality by -1

$$6 - 5a + a^2 = 0.$$

Now we know that $(a-6)(a+1) = 6 - 5a + a^2$ from distributing, and using this as a substitution we get

$$(a-6)(a+1) = 0.$$

Thus for the equality to hold, $a - 6 = 0$ and $a = 6$, or $a + 1 = 0$ and $a = -1$. We have shown that $a \leq 2$ so the the only valid value for a is -1. Therefore we have shown that $\{a_n\}$ is convergent and the value of $a = \lim_{n\to\infty} a_n = -1$. ■

Problem 7

Let $\{x_n\}$ be an arbitrary sequence. Define the sequence $\{y_n\}$ by

$$y_n = \frac{2x_n + (-1)^n}{|x_n| + 1}$$

for $n \in \mathbb{N}$. Prove that $\{y_n\}$ has a convergent subsequence.

Proof. Lets start with the expression $|2x_n + (11)^n|$ with arbitrary $n \in N$. By the triangle inequality we get

$$|2x_n + (-1)^n| \leq |2x_n| + |(-1)^n|.$$

So, no matter that n is, $|(-1)^n| = 1$, and we can split the absolute value of $|2x_n|$ by multiplication to get

$$|2x_n + (-1)^n| \leq |2||x_n| + 1.$$

First we know that $|2| = 2$.We also know that $0 < 1$ and we can add an "or equals" to get $0 \leq 1$. Since we know that $a \leq b$ and $c \leq d$ implies $a + c \leq b + d$, we can use this with $|2x_n + (-1)^n| \leq |2||x_n| + 1$. and $0 \leq 1$ to get

$$|2x_n + (-1)^n| + 0 \leq 2|x_n| + 1 + 1$$

$$\implies |2x_n + (-1)^n| \leq 2|x_n| + 2.$$

Then by the definition of absolute value $0 \leq |2x_n + (-1)^n|$ so

$$0 \leq |2x_n + (-1)^n| \leq 2|x_n| + 2$$

Now, let us divide both sides by $2|x_n| + 2$ to get

$$\frac{0}{2|x_n| + 2} \leq \frac{|2x_n + (-1)^n|}{2|x_n| + 2} \leq \frac{2|x_n| + 2}{2|x_n| + 2}$$

$$\implies 0 \leq \frac{|2x_n + (-1)^n|}{2|x_n| + 2} \leq 1.$$

We can distribute out a 2 on the denominator of the left hand side and multiply both sides by 2 to get

$$2 \cdot 0 \leq 2 \cdot \frac{|2x_n + (-1)^n|}{2(|x_n| + 1)} \leq 2 \cdot 1$$

$$\implies 0 \leq \frac{|2x_n + (-1)^n|}{|x_n| + 1} \leq 2.$$

Thus, by using the definition of $\{y_n\}$ as a substitution we get

$$0 \leq y_n \leq 2. \qquad \forall n \in \mathbb{N}$$

Therefore $\{y_n\}$ is bounded below by 0 and bounded above by 2, so $\{y_n\}$ is a bounded sequence, and by the Bolzano-Weierstrass Theorem there exists a convergent subsequence of $\{y_n\}$. ∎