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Problem 1

Prove that

$$\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n}$$

is a convergent series and find its sum.

Proof. We can write $3^{3/2}$ as $(3^{1/2})^n$ so

$$\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n} = \sum_{n=1}^{\infty} \frac{(3^{1/2})^n}{2^n}.$$

We can take out an n from the exponent of the top and bottom of the fraction to get

$$\sum_{n=1}^{\infty} \frac{(3^{1/2})^n}{2^n} = \sum_{n=1}^{\infty} \Big(\frac{(3^{1/2})}{2}\Big)^n.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n} = \sum_{n=1}^{\infty} \Big(\frac{(3^{1/2})}{2}\Big)^n.$$

Now, we know that $4 > 3 > 0$ so $4^{1/2} > 3^{1/2} > 0^{1/2} \implies 2 > 3^{1/2} > 0$. Dividing both sides by two gets $\frac{2}{2} > \frac{3^{1/2}}{2} > \frac{0}{2} \implies 1 > \frac{3^{1/2}}{2} > 0$. Now since we showed that $\frac{3^{1/2}}{2} > 0$ and $1 > \frac{3^{1/2}}{2}$, then $1 > \left|\frac{3^{1/2}}{2}\right|$ by definition of absolute value. Then by the Geometric series test and since $1 > \left|\frac{3^{1/2}}{2}\right|$, $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n} = \sum_{n=1}^{\infty} (\frac{(3^{1/2})}{2})^n$ is a convergent series with the sum equal to $\dfrac{ar}{1-r}$ where $a = 1$ and $r = \frac{(3^{1/2})}{2}$. So,

$$\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n} = \frac{1 \cdot \frac{(3^{1/2})}{2}}{1 - \frac{(3^{1/2})}{2}}.$$

Multiplying the term by 2/2:

$$= \frac{(3^{1/2})}{2 - (3^{1/2})}.$$

We can multiply the fraction by $(2 + (3^{1/2}))/(2 + (3^{1/2}))$ to get

$$= \frac{(3^{1/2})(2 + (3^{1/2}))}{(2 - (3^{1/2}))(2 + (3^{1/2}))}.$$

Distributing the top and the bottom gets us

$$= \frac{2 \cdot 3^{1/2} + 3}{4 + 2 \cdot 3^{1/2} - 2 \cdot 3^{1/2} - 3}.$$

Simplifying the denominator gets

$$= \frac{2 \cdot 3^{1/2} + 3}{1} = 2 \cdot 3^{1/2} + 3 = 2\sqrt{3} + 3.$$

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n/2}}{2^n}$ is a convergent series and find its sum is $2\sqrt{3} + 3$. ■

Problem 2

Prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^3 - (n-1)^3}$$

is a divergent series.

Proof. Let $x_n = \dfrac{(-1)^n n^2}{(n+1)^3 - (n-1)^3}$ for arbitrary $n \in \mathbb{N}$. Then

$$x_{2k} = \frac{(-1)^{2k}(2k)^2}{(2k+1)^3 - (2k-1)^3}$$

for arbitrary $k \in \mathbb{N}$. Since $2k$ is always even, $(-1)^{2k} = 1$ for all k, then can factor out the the cubes in the denominator, and the the squared term in the numerator to get

$$\frac{1 \cdot 4k^2}{(8k^3 + 12k^2 + 6k + 1) - (8k^3 - 12k^2 + 6k - 1)}$$

$$= \frac{4k^2}{8k^3 + 12k^2 + 6k + 1 - 8k^3 + 12k^2 - 6k + 1}$$

$$= \frac{4k^2}{24k^2 + 2}.$$

Then we can multiply the term by $\frac{1/k^2}{1/k^2}$ to get

$$\frac{4k^2}{24k^2 + 2} \cdot \frac{1/k^2}{1/k^2}$$

$$= \frac{4 \cdot 1}{24 \cdot 1 + 2(1/k^2)} = \frac{4}{24 + 2(1/k^2)}.$$

Thus we have showed that

$$x_{2k} = \frac{4}{24 + 2(1/k^2)}.$$

Now lets take the limit of x_{2k}:

$$\lim_{k \to \infty} x_{2k} = \lim_{k \to \infty} \frac{4}{24 + 2(1/k^2)}.$$

Then can split $\lim_{k\to\infty} 24 + 2(1/k^2)$ by addition and then multiplication to get

$$\lim_{k\to\infty}(24 + 2(1/k^2)) = \lim_{k\to\infty} 24 + \lim_{k\to\infty} 2 \cdot \lim_{k\to\infty}(1/k^2). \qquad (1)$$

We know that $1/k^2 = (1/k)^2 = (1/k)(1/k)$, so $\lim_{k\to\infty} 1/k^2 = \lim_{k\to\infty}(1/k)(1/k)$ then splitting my multiplication gets $\lim_{k\to\infty}(1/k) \cdot \lim_{k\to\infty}(1/k)$, and since we know that $\lim_{k\to\infty} 1/k = 0$, we get $0 \cdot 0 = 0$. Thus we have shown that

$$\lim_{k\to\infty} 1/k^2 = 0. \qquad (2).$$

We know that the limit of a constant is said constant so $\lim_{k\to\infty} 4 = 4$ and $\lim_{k\to\infty} 24 = 24$, then using these and (2) as substitutions into (1) gets

$$24 + 2 \cdot 0 = 24.$$

Thus we showed that

$$\lim_{k\to\infty}(24 + 2(1/k^2)) = 24 \neq 0. \qquad (3)$$

Now, we can split the limit of x_{2k} by division to get

$$\frac{\lim_{k\to\infty} 4}{\lim_{k\to\infty}(24 + 2(1/k^2))}.$$

Now using the fact that $\lim_{k\to\infty} 4 = 4$ and (3) as substitutions to get

$$\frac{4}{24} = \frac{1}{6}.$$

Thus we have shown that $\lim_{k\to\infty} 2k = \frac{1}{6} \neq 0$ since x_{2k} is a subsequence of x_n and its limit not equal to zero, then $\lim_{n\to\infty} x_n \neq 0$ since all subsequences must converge to the same limit. Then by the n-th term test, because $\lim_{n\to\infty} x_n \neq 0$, $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^3 - (n-1)^3}$ is a divergent series. ∎

Problem 3

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{25n^2 + 5n - 6}$$

is a convergent series and find its sum.

Proof. To start, we can factor the denominator of $\frac{1}{25n^2 + 5n - 6}$ to get

$$\frac{1}{(5n-2)(5n+3)}.$$

Now, lets take the partial fraction decomposition of our expression so that

$$\frac{1}{(5n-2)(5n+3)} = \frac{A}{5n-2} + \frac{B}{5n+3}.$$

Lets multiply both sides of the equation by $(5n-2)(5n+3)$ to get

$$\frac{1}{(5n-2)(5n+3)} \cdot (5n-2)(5n+3) = \frac{A}{5n-2} \cdot (5n-2)(5n+3) + \frac{B}{5n+3} \cdot (5n-2)(5n+3)$$

$$\implies 1 = A(5n+3) + B(5n-2).$$

Now lets distribute in the A and B terms and reorder the terms to get

$$1 = 5A \cdot n + 3A + 5B \cdot n - 2B$$

$$\implies 1 = 5A \cdot n + 5B \cdot n + 3A - 2B.$$

Now, we know that $1 + 0 = 1$ and we can substitute 0 for $n \cdot 0$ so $1 + n \cdot 0 = 1$. We can also distribute out an n from the term on the right to get

$$n \cdot 0 + 1 = n(5A + 5B) + (3A - 2B).$$

We can match the terms on the left hand side and right had side so that $0 = (5A + 5B)$ and $1 = (3A - 2B)$. Starting with $0 = 5A + 5B$, we can divide both sides by 5 to get $0 = A + B$ and thus $A = -B$, then substituting this into $1 = (3A - 2B)$ we get $1 = -3B - 2B = -5B$, then dividing $1 = -5B$ by -5 gets us $-1/5 = B$. Now substituting $-1/5 = B$ into $A = -B$ we get $A = -(-1/5) = 1/5$.
Therefore we have found that $A = 1/5$ and $B = -1/5$. Thus we have

$$\frac{1}{(5n-2)(5n+3)} = \frac{1/5}{5n-2} + \frac{-1/5}{5n+3}$$

$$= \frac{1}{5(5n-2)} - \frac{1}{5(5n+3)}$$

$$= \frac{1}{25n-10} - \frac{1}{25n+15}.$$

Now we shown that

$$\sum_{n=1}^{\infty} \frac{1}{25n^2 + 5n - 6} = \sum_{n=1}^{\infty} \frac{1}{25n-10} - \frac{1}{25n+15}.$$

Now if $a_n = \frac{1}{25n-10} - \frac{1}{25n+15}$, let let the n-th partial sum be $s_n = a_1 + a_2 + ... + a_{n-1} + a_n$. Then we know that

$$s_n = \frac{1}{15} - \frac{1}{40} + \frac{1}{40} - \frac{1}{65} + \frac{1}{65} + \frac{1}{90} + ... + \frac{1}{25n-35} - \frac{1}{25-10} + \frac{1}{25n-10} - \frac{1}{25n+15}.$$

$$= \frac{1}{15} + (-\frac{1}{40} + \frac{1}{40}) + (-\frac{1}{65} + \frac{1}{65}) + (-\frac{1}{90} + \frac{1}{90}) +$$

$$\cdots$$

$$+\left(-\frac{1}{25n-35}+\frac{1}{25n-35}\right)+\left(-\frac{1}{25-10}+\frac{1}{25n-10}\right)-\frac{1}{25n+15}.$$

We see that this is a telescoping series, and all the terms expect the first and the last cancel in s_n to get

$$s_n = \frac{1}{15} - \frac{1}{25n+15}.$$

Multiplying the right term by $(1/n)/(1/n)$:

$$s_n = \frac{1}{15} - \frac{1/n}{25+15/n}.$$

We know by Theorem 27 that $\{s_n\}$ is a convergent sequence and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{15} - \frac{1/n}{25+15/n}\right)$$

which with Theorem 27 we can split by subtraction and then division then addition and take out a constant to get

$$= \lim_{n\to\infty}\frac{1}{15} - \frac{\lim_{n\to\infty}(1/n)}{\lim_{n\to\infty}25 + 15\lim_{n\to\infty}(1/n)}.$$

We know that $\lim\limits_{n\to\infty}(1/n) = 0$ and the limit of a constant is said constant so we have

$$= \frac{1}{15} - \frac{0}{25+15\cdot 0}$$

$$= \frac{1}{15} - 0$$

$$= \frac{1}{15}.$$

Therefore, by the definition of convergent series, since the partial sum sequence converges, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{25n^2+5n-6}$ converges and the sum is $\lim\limits_{n\to\infty} s_n = \dfrac{1}{15}$. ■

Problem 4

Suppose that $\displaystyle\sum_{n=1}^{\infty} x_n$ is a convergent series and $\displaystyle\sum_{n=1}^{\infty} y_n$ is a divergent series.

Prove that $\displaystyle\sum_{n=1}^{\infty}(x_n+y_n)$ is a divergent series.

Proof. Lets do a proof by contradiction. Suppose that $\sum_{n=1}^{\infty}(x_n + y_n)$ is convergent. Let $x_n + y_n = z_n$, so $\sum_{n=1}^{\infty} z_n$ is convergent. Now by the linearity of series, since $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} x_n$ are convergent, $\sum_{n=1}^{\infty}(z_n - x_n)$ is also convergent. Now with $z_n = x_n + y_n$, we can subtract x_n from both sides to get $z_n - x_n = y_n$, substituting this in to $\sum_{n=1}^{\infty}(z_n - x_n)$ gets us

$$\sum_{n=1}^{\infty}(z_n - x_n) = \sum_{n=1}^{\infty} y_n.$$

Thus, since $\sum_{n=1}^{\infty}(z_n - x_n)$ is convergent, $\sum_{n=1}^{\infty} y_n$ is convergent which is a contradiction with out our given statement that $\sum_{n=1}^{\infty} y_n$ is a divergent series.

Therefore $\sum_{n=1}^{\infty} y_n$ is a divergent series. ∎

Problem 5

Prove that

$$\sum_{n=1}^{\infty} \frac{n+2}{n^3 + 2n^2 + 5}$$

is a convergent series.

Proof. We know that $5 \geq 0$ then adding $n^3 + 2n^2$, for arbitrary $n \in \mathbb{N}$, to both sides gets $n^3 + 2n^2 + 5 \geq n^3 + 2n^2 + 0$ now taking the inverse of both sides and flipping the direction of the sign gets us

$$\frac{1}{n^3 + 2n^2 + 5} \leq \frac{1}{n^3 + 2n^2}.$$

Now we can multiply both sides by $n + 2$ to get

$$\frac{n+2}{n^3 + 2n^2 + 5} \leq \frac{n+2}{n^3 + 2n^2}.$$

Now we know that $0 \leq \frac{n+2}{n^3+2n^2+5}$ since $n > 0$ so we have

$$0 \leq \frac{n+2}{n^3 + 2n^2 + 5} \leq \frac{n+2}{n^3 + 2n^2}. \qquad \text{(I)}$$

Now, with the right hand side, $\frac{n+2}{n^3+2n^2}$, we can factor out n^2 form the denominator to get

$$\frac{n+2}{n^2(n+2)}.$$

Dividing out $(n+2)$ leaves us with

$$\frac{1}{n^2}.$$

So we have shown that

$$\frac{n+2}{n^3+2n^2} = \frac{1}{n^2} = \left(\frac{1}{n}\right)^2.$$

Now, by the p-series test, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2$ convergences since $2 > 1$, thus

$$\sum_{n=1}^{\infty}\frac{n+2}{n^3+2n^2} \text{ converges.} \qquad \text{(II)}$$

Then, by the comparison tests with (I) and (II),

$$\sum_{n=1}^{\infty}\frac{n+2}{n^3+2n^2+5}$$

is a convergent series.

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Problem 6

Is the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n^5+2n} - \sqrt{n^5+1}\right)$$

convergent or divergent? Justify your answer

Proof. We know that $2 > 1$, and with $n \geq 1$ we multiply both sides by 2 to get $2n \geq 2$ and combine this with $2 > 1$ gets us $2n > 1$. Then, adding n^5 to both sides gets us $n^5 + 2n > n^5 + 1$. Since $n > 0$ it must be that $n^5 + 2n > n^5 + 1 > 0$. Now we can use

$$0 \leq \sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

for $a > b > 0$ with our $a = n^5 + 2n$ and $b = n^5 + 1$ to get

$$0 \leq \sqrt{n^5+2n} - \sqrt{n^5+1} = \frac{n^5+2n-(n^5+1)}{\sqrt{n^5+2n}+\sqrt{n^5+1}} \qquad \text{(a)}$$

The right hand side simplifies to

$$\frac{2n-1}{\sqrt{n^5+2n}+\sqrt{n^5+1}}$$

Now we can make the numerator bigger and thus the faction bigger by taking out the -1 so,

$$\frac{2n-1}{\sqrt{n^5+2n}+\sqrt{n^5+1}} \leq \frac{2n}{\sqrt{n^5+2n}+\sqrt{n^5+1}}.$$

Now we can also take out the $2n$ and 1 under the square root in the denominator to make the fraction bigger to get

$$\frac{2n}{\sqrt{n^5+2n}+\sqrt{n^5+1}} \leq \frac{2n}{\sqrt{n^5}+\sqrt{n^5}}$$

Now, on the right hand side we can add to simplify the denominator to get

$$\frac{2n}{2\sqrt{n^5}}.$$

Dividing out the 2's:

$$= \frac{n}{\sqrt{n^5}}.$$

Rewrite the exponents:

$$= \frac{n^1}{n^{5/2}} = n^1 n^{-5/2}.$$

Combining the exponents :

$$= n^{-3/2} = \frac{1}{n^{3/2}}.$$

Thus we have,

$$\sqrt{n^5+2n} - \sqrt{n^5+1} = \frac{n^5+2n-(n^5+1)}{\sqrt{n^5+2n}+\sqrt{n^5+1}} = \frac{2n}{\sqrt{n^5+2n}+\sqrt{n^5+1}} \leq \frac{1}{n^{3/2}}$$

$$\implies \sqrt{n^5+2n} - \sqrt{n^5+1} = \frac{n^5+2n-(n^5+1)}{\sqrt{n^5+2n}+\sqrt{n^5+1}} \leq \frac{1}{n^{3/2}}$$

Combing the inequality above with (a) we get

$$0 \leq \sqrt{n^5+2n} - \sqrt{n^5+1} \leq \frac{1}{n^{3/2}}. \qquad \text{(b)}$$

Now, by the p-series test and since $3/2 > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} (\frac{1}{n})^{3/2}$ is a convergent series.

Therefore by (b) and the Comparison Test, $\sum_{n=1}^{\infty} \left(\sqrt{n^5+2n} - \sqrt{n^5+1}\right)$ is a convergent series.

■

Problem 7

Suppose that $a_n \geq 0$ for every $n \in \mathbb{N}$. Prove that, if $\sum_{n=1}^{\infty} a_n^2$ is a divergent series, then $\sum_{n=1}^{\infty} a_n$ is also a divergent series.

Proof. Lets do a proof by contrapositive, so let us assume that $\sum_{n=1}^{\infty} a_n$ is convergent and show that $\sum_{n=1}^{\infty} a_n^2$ is a convergent series. Since we know that $\sum_{n=1}^{\infty} a_n$ is convergent, then by the n-th term test, $\lim_{n\to\infty} a_n = 0$ and thus $\{a_n\}$ is a convergent sequence. Now, by Theorem 24, since $\{a_n\}$ is a convergent sequence, $\{a_n\}$ is bounded. Thus there exists a $u \in \mathbb{R}$ such that $a_n \leq u$ for all $n \in \mathbb{N}$, combing this with our given $a_n \geq 0$ for every $n \in \mathbb{N}$, we get

$$0 \leq a_n \leq u.$$

We multiply all sides by a_n to get

$$0 \leq a_n^2 \leq u \cdot a_n. \qquad (\heartsuit)$$

Now, we know from the linearity of series that since $\sum_{n=1}^{\infty} a_n$ is convergent, $\sum_{n=1}^{\infty} u \cdot a_n$ is also convergent. Now by the comparison test with ($\heartsuit$), we get that $\sum_{n=1}^{\infty} a_n^2$ is a convergent series. Therefore, by our proof by contrapositive, it holds that if $\sum_{n=1}^{\infty} a_n^2$ is a divergent series, then $\sum_{n=1}^{\infty} a_n$ is a divergent series. ■