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Problem 1

Let $A = \{a, b, c, d\}$ and $W = \{(a, b), (b, c), (c, d), (a, c), (a, d)\}$, where a,b,c,d are distinct. Define the relation $\prec$ on A by.

$$\forall x, y \in A, x \prec y \quad \text{if and only if } (x, y) \in W$$

Is the statement "$x \prec y$ and $y \prec z \implies x \prec z$" true for arbitrary $x, y, z \in A$.

Proof. Suppose the case when $x = b$, $y = c$ and $z = d$ where b and c are in A for the statement above, then $b \prec c$ and $c \prec d$ holds by the definition of the relation $\prec$ since both (b, c) and (c, d) are elements of W. But $b \prec d$ does not hold by the definition of $\prec$ since (b, d) is not an element of W.

Thus, the statement "$x \prec y$ and $y \prec z \implies x \prec z$" is not true for arbitrary $x, y, z \in A$ since there exists a $x, y, z \in A$, when $x = b$, $y = c$ and $z = d$, such that the left hand side of the inequality is true while the right hand side is false. Thus, making the implication false for this case and therefore the statement can not be said to be true for arbitrary $x, y, z \in A$. ■

Problem 2

Below is a proof of "If x $\in \mathbb{R}$ and $x < 0$, then $0 < -x$". Identify the item(s) from the list of axioms and theorems on Page 3 that are used in each step. Include only the ones that are actually needed.

Proof. $x < 0$
$\implies x + (-x) < 0 + (-x)$ (Axiom O2)
$\implies x + (-x) < -x$ (Axiom A4)
$\implies 0 < -x$ (Axiom A5)

■

Problem 3

Below is a proof of "$\forall a, b \in \mathbb{R}, (-a) + (-b) = -(a + b)$". Identify the item(s) from the list of axioms and theorems on Page 3 that are used in each step. Include only the ones that are actually needed.

Proof. $((-a) + (-b)) + (a + b) = ((-a) + (-b)) + (b + a)$ (Axiom A2)
$\implies ((-a) + (-b)) + (a + b) = (-a) + ((-b) + (b + a))$ (Axiom A3)
$\implies ((-a) + (-b)) + (a + b) = (-a) + ((-b) + b) + a)$ (Axiom A3)
$\implies ((-a) + (-b)) + (a + b) = (-a) + (0 + a)$ (Axiom A5)
$\implies ((-a) + (-b)) + (a + b) = (-a) + a$ (Axiom A4)
$\implies ((-a) + (-b)) + (a + b) = 0$ (Axiom A5)
$\implies (-a) + (-b) = -(a + b)$ (Theorem. 02) ■

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Problem 4

Below is a proof of "$\forall x \in \mathbb{R}, (-1)x = -x$". Identify the item(s) from the list of axioms and theorems on Page 3 that are used in each step. Include only the ones that are actually needed.

Proof. $(-1)x + x = (-1)x + 1x$ (Axiom M4)
$\implies (-1)x + x = ((-1) + 1)x$ (Axiom D)
$\implies (-1)x + x = 0x$ (Axiom A5)
$\implies (-1)x + x = 0$ (Theorem. 01)
$\implies (-1)x = -x$ (Theorem. 02)

■

Problem 5

Prove that, if $a, b \in \mathbb{R}$, then $ab = (-a)(-b)$. You may use the items from the list of axioms and theorems on Page 3 only. Identify the item(s) used in each step. Include only the ones that are actually needed.

Proof. Let us start this proof by using Theorem. 01 with a:

$$a0 = 0$$

Then we can substitute 0 on the left hand side using Theorem. A5 with b, giving us

$$a(b + (-b) = 0)$$

Now, using Axiom D we can distribute the left hand side, giving us

$$ab + a(-b) = 0$$

Here, we see this equation is in the form of Theorem. 02, and by using Theorem. 02 we get the equivalent equation of

$$ab = -(a(-b))$$

Finally, we can use Theorem. 03 on the right hand side to finish our proof

$$ab = (-a)(-b)$$

■

Problem 6

Prove that, if $a \in \mathbb{R}$ and $0 < a < 1$, then $aa < 1$. You may use the items from the list of axioms and theorems on Page 3 only. Identify the item(s) used in each step. Include only the ones that are actually needed.

Proof. Suppose $a > 0$ and $a < 1$
Now, we will use Theorem. 4 with $x = a$, $y = a$ and $z = 1$ to give us

$$aa < a1$$

Then by using Axiom M4 on the right hand side, we get

$$aa < a$$

Finally, by using Axiom O1(ii) with $aa < a$ and $a < 1$, we get

$$aa < 1$$

■

Problem 7

Let $E \subset R, \beta \in \mathbb{R}$ and $E + \beta = \{x + \beta : x \in E\}$. Suppose that E is bounded above. Prove that $E + \beta$ is also bounded above.

Proof. If we use the assumption that E is bounded above, then there exists a u in $\mathbb{R}$ such that $x \leq u$ for all x in E
Now, let $v = u + \beta$, and v must be in $\mathbb{R}$ since $\mathbb{R}$ is closed under addition and u and β are in $\mathbb{R}$ by Axiom A1
Then, let y be an arbitrary element of $E+\beta$, and by the definition of the set $E+\beta$, $y = x+\beta$ for some arbitrary $x \in E$
Since we know $x \leq u$ holds for all $x \in E$, we can use Theorem. 05 to add β to both sides of the inequality to get:

$$x + \beta \leq u + \beta$$

We can substitute the left hand side of the inequality with y by the definition of y and substitute the right hand side with v by the definition of v too get:

$$y \leq v$$

Thus, there is a v in $\mathbb{R}$ such that v is greater than or equal to all y in $E + \beta$.
Therefore, by the definition of upper bound, $E + \beta$ is bounded above.

■