Problem 1

Let $x_n = (-1)^n\left(\frac{n+1}{n}\right)$ for $n \in \mathbb{N}$. Calculate $\limsup_{n\to\infty} x_n$.

Proof. Let $u_n = sup\{x_k : k \geq n\} = sup\{x_n, x_{n+1}, x_{n+2}, ...\}$. Let $y_n = \frac{n+1}{n} = 1 + \frac{1}{n}$ with arbitrary $n \in \mathbb{N}$.
We know that $1 > 0$, adding n to both sides gets $n + 1 > n$. Now, taking the multiplicative inverse of both sides and flipping the side gets us $\frac{1}{n+1} < \frac{1}{n}$. We can add one to both sides now to get $\frac{1}{n+1} + 1 < \frac{1}{n} + 1$, thus we have shown that $y_{n+1} < y_n$ $\forall n \in \mathbb{N}$ and therefore y_n is decreasing.
Now for u_n, we know $x_k = (-1)^k\left(\frac{k+1}{k}\right) = (-1)^k\left(1 + \frac{1}{k}\right) = (-1)^k \cdot y_k$ for $k \geq n$, so when k is odd $(-1)^k = -1$ then $x_k = -y_k$ and when k is odd we $(-1)^k = 1$ then $x_n = y_k$. So, when n is odd, the first term of x_k is x_k and since we know that y_n decreases then the first positive number, the first even k, in the sequence will the be supremum of u_n which would be x_{n+1}. Now, when n is even, that means x_n is already the largest positive number in the sequence being the first, so the supremum of $u_n = x_n$. Then if $u_n = sup\{x_k : k \geq n\}$ is x_{n+1} or x_n, $x_{n+1} \leq x_n$, and since we know that $x_{n+1} > 0$ and $x_n > 0$, then

$$1 \leq u_n \leq x_{n+1} \leq x_n$$

$$\implies 1 \leq u_n \leq x_n$$

$$\implies 1 \leq u_n \leq 1 + (1/n) \qquad \text{(i)}$$

We know that $\lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} 1/n = 0$, then $\lim_{n\to\infty}(1 + 1/n) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} 1/n = 1 + 0 = 1$ Thus,

$$\lim_{n\to\infty} 1 = \lim_{n\to\infty}(1 + 1/n) = 1 \qquad \text{(ii)}$$

Using (i) and (ii) with the squeeze theorem get us $\lim_{n\to\infty} u_n = 1$. Then by the definition of u_n, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} sup\{x_k : k \geq n\} = 1$ for some $n \in \mathbb{N}$. Then by the definition of *limsup*, $\limsup_{n\to\infty} x_n = 1$. ∎

Problem 2

Let $\{x_n\}$ be a sequence such that $\limsup_{n\to\infty}(|x_n|^{1/n}) < 1$. Prove that $\lim_{n\to\infty} x_n = 0$.

Proof. Let $L = \limsup_{n\to\infty}(|x_n|^{1/n}) = \lim_{\to\infty} sup\{|x_k|^{1/k} : k \geq n\}$, we can use this as a substitution in $\limsup_{n\to\infty}(|x_n|^{1/n}) < 1$ to get $L < 1$. Now, we can subtract both sides of the inequality by L to get $0 < 1 - L$ and then divide both sides by 2 to get $0 < \frac{1-L}{2}$.
Let $\epsilon_0 = \frac{1-L}{2}$m then $\epsilon_0 > 0$. Now, by the definition of limit with $\lim_{\to\infty} sup\{|x_k|^{1/k} : k \geq n\}$, $\exists K_0 \in \mathbb{N}$ such that

$$|sup\{|x_k|^{1/k} : k \geq n\} - L| < \varepsilon_0. \qquad n \geq K_0$$

$$\implies -\varepsilon_0 < sup\{|x_k|^{1/k} : k \geq n\} - L < \varepsilon_0.$$

$$\implies sup\{|x_k|^{1/k} : k \geq n\} - L < \varepsilon_0.$$

Adding L to both sides and substituting for ε_0:

$$\implies sup\{|x_k|^{1/k} : k \geq n\} < \frac{1-L}{2} + L.$$

$$\implies sup\{|x_k|^{1/k} : k \geq n\} < \frac{1-L}{2} + \frac{2L}{2} = \frac{2L-L+1}{2} = \frac{L+1}{2}.$$

So we have

$$sup\{|x_k|^{1/k} : k \geq n\} < \frac{L+1}{2}. \qquad \forall n \geq K_0 \qquad (1)$$

Now the by the definition of supremum and since $|x_n|^{1/n} \in \{|x_k|^{1/k} : k \geq n\}$, $\forall n \in \mathbb{N}$,

$$|x_n|^{1/n} \leq sup\{|x_k|^{1/k} : k \geq n\}. \qquad \forall n \in \mathbb{N} \qquad (2)$$

Then combining (1) and (2) we get

$$|x_n|^{1/n} \leq sup\{|x_k|^{1/k} : k \geq n\} < \frac{L+1}{2}. \qquad \forall n \geq K_0$$

Thus we have

$$|x_n|^{1/n} < \frac{L+1}{2}. \qquad \forall n \geq K_0$$

$$\implies |x_n - 0|^{1/n} < \frac{L+1}{2}.$$

Taking the power of n on both sides gets us

$$|x_n - 0| < \left(\frac{L+1}{2}\right)^n \qquad \forall n \geq K_0$$

We know that $|x_k| \geq$ the $|x_k|^{\frac{1}{k}} \geq 0$. Then know that $sup\{|x_k|^{1/k} : k \geq n\} \geq 0$ by definition of supremum and taking the limit of both sides gets

$$\lim_{n\to\infty} sup\{|x_k|^{1/k} : k \geq n\} \geq \lim_{n\to\infty} 0.$$

We know that $\lim_{n\to\infty} sup\{|x_k|^{1/k} : k \geq n\} = L$ and $\lim_{n\to\infty} 0 = 0$ so

$$L \geq 0.$$

Now, since we know that $0 \leq L < 1$, adding 1 to all sides gets $1 \leq L+1 < 2$. Dividing by 2 gets $1/2 \leq \frac{L+1}{2} < 1$. We know $0 < 1/2$ so $0 < \frac{L+1}{2} < 1$. Then by definition of absolute value $|\frac{L+1}{2}| < 1$. Thus by Theorem 30, $\lim_{n\to\infty}(\frac{L+1}{2})^n = 0$. Now by theorem 25 with $|x_n - 0| < 1 \cdot \left(\frac{L+1}{2}\right)^n$, $C_0 = 1$, and $\lim_{n\to\infty} \frac{L+1}{2} = 0$, $\lim_{n\to\infty} x_n = 0$. ∎

Problem 3

Let $\{a_n\}$ and $\{b_n\}$be sequences such that $\lim_{n\to\infty} b_n = 0$. Suppose that

$$|a_n - a_q| \leq b_q$$

holds whenever $n, q \in \mathbb{N}$ and $n \geq q$. Prove that $\{a_n\}$ is a Cauchy sequence.

Proof. Since we are given that $\lim_{n\to\infty} b_q = 0$, b_q is convergent and by the definition of limit, $\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that

$$|b_q - 0| < \varepsilon \qquad n \geq K.$$
$$\implies |b_q| < \varepsilon$$
$$\implies -\varepsilon < b_q < \varepsilon$$

Thus,

$$b_q < \varepsilon \qquad n \geq K \qquad (A.)$$

Case 1 (n $\geq$ m): Let $q = m$, using this with $|a_n - a_q| \leq b_q$ and (A.) gets us $|a_n - a_m| \leq b_m$ and $b_m < \varepsilon$, combing the two gets us

$$|a_n - a_m| \leq \varepsilon \qquad n, m \geq K.$$

Case 1 (n $\leq$ m): Let $q = n$, using this with $|a_n - a_q| \leq b_n$ gets us $|a_n - a_m| \leq b_n$, then by definition of of absolute value we have $|-(a_n - a_m)| = |-a_n + a_m| = |a_m - a_n|$. Now using $|a_m - a_n| \leq b_n$ with $b_n < \varepsilon$ gets us

$$|a_m - a_n| \leq \varepsilon \qquad n, m \geq K.$$

Thus for all n, m in $\mathbb{N}$, we have proved by the definition of a Cauchy sequence, $\{a_n\}$ is a Cauchy sequence. ■

Problem 4

Let $\{x_n\}$ be a sequence and $0 < \lambda < 1$. Suppose that

$$|x_n - x_{n-1}| \leq \lambda |x_{n-1} - x_{n-2}|$$

holds for all $n \geq 3$. Prove that $\{x_n\}$is a Cauchy sequence (and hence a convergent sequence). You may use the result of Extra Problem 1 below without proof, if needed.

Proof. Let $P(n)$ be "$|x_{n+1} - x_n| \leq \lambda^{n-1}|x_2 - x_1|$" for $n \in \mathbb{N}$.
Base Case: When $n = 1$. On the LHS of $P(1)$ we have $|x_2 - x_1|$ and on the RHS we have $\lambda^0|x_2 - x_1| = |x_2 - x_1|$ thus $|x_2 - x_1| \leq |x_2 - x_1|$ holds, $P(1)$ holds and the base case holds.
Inductive Hypothesis: Assume $P(k)$ holds for arbitrary $k \in \mathbb{N}$.
Inductive Step: We know that $|x_n - x_{n-1}| \leq \lambda|x_{n-1} - x_{n-2}|$ for $n \geq 3$, then we can have $n = k+2$, $k+2 \geq 3$, so that

$$|x_{k+2} - x_{k+1}| \leq \lambda |x_{k+1} - x_k|.$$

Dividing both sides by $|x_{k+1} - x_k|$ gives us

$$\frac{|x_{k+2} - x_{k+1}|}{|x_{k+1} - x_k|} \leq \lambda. \qquad (\clubsuit)$$

By our inductive hypothesis we know that

$$|x_{k+1} - x_k| \leq \lambda^{k-1}|x_2 - x_1|$$

holds, so multiplying the left hand side by the left hand side of $\clubsuit$ and the right and ride by the right hand of $\clubsuit$ gets us

$$\frac{|x_{k+2} - x_{k+1}|}{|x_{k+1} - x_k|} \cdot |x_{k+1} - x_k| \leq \lambda \cdot \lambda^{k-1}|x_2 - x_1|.$$

Simplifying both sides gets us

$$|x_{k+2} - x_{k+1}| \leq \lambda^k |x_2 - x_1|.$$

Thus we have shown that $P(k + 1)$ hold for all $k \in \mathbb{N}$. Therefore by the Principle of Mathematical induction, $P(n)$, i.e.

$$|x_{n+1} - x_n| \leq \lambda^{n-1}|x_2 - x_1| \qquad (\heartsuit)$$

hold for all $n \in \mathbb{N}$.
For $n, m \in \mathbb{N}$ and $n > m$

$$|x_n - x_m| = |(x_n - x_{n-1}) + (x_{n-1} - xn - 2) + ... + (x_{m+1} - x_m)|.$$

Using the triangle inequality gets us

$$|(x_n - x_{n-1}) + (x_{n-1} - xn - 2) + ... + (x_{m+1} - x_m)|$$

$$\leq |x_n - x_{n-1}| + |x_{n-1} - xn - 2| + ... + |x_{m+1} - x_m|.$$

Then by using $\heartsuit$ we get

$$|x_n - x_{n-1}| + |x_{n-1} - xn - 2| + ... + |x_{m+1} - x_m|$$

$$\leq \lambda^{n-2}|x_2 - x_1| + \lambda^{n-3}|x_2 - x_1| + ... + \lambda^{m-1}|x_2 - x_1|.$$

We can factor out a λ^{m-1} and a $|x_2 - x_1|$ to get

$$\lambda^{m-1}(1 + \lambda + ... + \lambda^{n-m-1})|x_2 - x_1|.$$

Thus

$$|x_n - x_m| \leq \lambda^{m-1}(1 + \lambda + ... + \lambda^{n-m-1})|x_2 - x_1|.$$

Let $s_n = 1 + \lambda + ... + \lambda^{n-m-1}$, then $\lambda s_n = \lambda + \lambda^2 + ... + \lambda^{n-m}$. Now, $s_n - \lambda s_n = 1 - \lambda^{n-m}$, factoring out a s_n on the left hand sides gets us $s_n(1-\lambda) = 1 - \lambda^{n-m}$, now dividing both sides by $(1-\lambda)$ gives us $s_n = \frac{1-\lambda^{n-m}}{(1-\lambda)}$. Substituting in for s_n in to the inequality gets us

$$|x_n - x_m| \leq \lambda^{m-1} \frac{1-\lambda^{n-m}}{(1-\lambda)} |x_2 - x_1|. \qquad \diamondsuit$$

Now, we know that $1 > \lambda \geq 0$, taking the power of $n - m > 0$ of both sides gets

$$\lambda^{n-m} \geq 0.$$

Multiplying by -1 gets us

$$-\lambda^{n-m} \leq 0,$$

adding one to both sides gets

$$1 - \lambda^{n-m} \leq 1.$$

Now we also know that $1 - \lambda > 0$, so $\frac{1}{1-\lambda} > 0$. So multiplying both sides of $1 - \lambda^{n-m} \leq 1$ by $\frac{1}{1-\lambda}$ gets us

$$\frac{1-\lambda^{n-m}}{1-\lambda} \leq \frac{1}{1-\lambda}.$$

Thus we can us the right hand side of the previous line as a substitution for $\frac{1-\lambda^{n-m}}{1-\lambda}$ in $\diamondsuit$ and the inequality still holds to get

$$|x_n - x_m| \leq \lambda^{m-1} \frac{1}{1-\lambda} |x_2 - x_1|.$$

Let $b_q = \lambda^{q-1} \frac{1}{1-\lambda} |x_2 - x_1|$ for arbitrary $q \in \mathbb{N}$, let $q = m$, and $n \geq q$, so we have

$$|x_n - x_q| \leq b_q.$$

Now lets take out a λ^{-1} from b_n and take the limit of b_n so that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \lambda^n \lambda^1 \frac{1}{1-\lambda} |x_2 - x_1|.$$

We can split the limit by multiplication:

$$\lim_{n\to\infty} \lambda^n \lim_{n\to\infty} \lambda^{-1} \lim_{n\to\infty} \frac{1}{1-\lambda} \lim_{n\to\infty} |x_2 - x_1|.$$

We know that $|\lambda| < 1$, so by Theorem 30 $\lim_{n\to\infty} \lambda^n = 0$. So

$$\lim_{n\to\infty} \lambda^n \lim_{n\to\infty} \lambda^{-1} \lim_{n\to\infty} \frac{1}{1-\lambda} \lim_{n\to\infty} |x_2 - x_1|$$

$$= 0 \cdot \lim_{n\to\infty} \lambda^{-1} \lim_{n\to\infty} \frac{1}{1-\lambda} \lim_{n\to\infty} |x_2 - x_1|$$
$$= 0.$$

Thus, $\lim_{n\to\infty} b_n = 0$. Then by Problem 3 of this homework $\{x_n\}$ is a Cauchy equence. $\blacksquare$

Problem 5

Let $\{x_n\}$ be defined by $x_1 = 2$, $x_2 = 7$ and

$$x_n = \frac{x_{n-1}}{3} + \frac{2x_{n-2}}{3} \; \forall \; n \geq 3$$

Prove that $\{x_n\}$is convergent.

Proof. Lets start with $|x_n - x_{n-1}|$, we can substitute x_n with the given definition of x_n to get

$$\Big| \Big(\frac{x_{n-1}}{3} + \frac{2x_{n-2}}{3} \Big) - x_{n-1} \Big|.$$

We can get a common denominator with the right term and add the two left terms to get

$$\Big| \frac{x_{n-1} + 2x_{n-2}}{3} - \frac{3x_{n-1}}{3} \Big|.$$

Adding the two resulting terms gets

$$\Big| \frac{-2x_{n-1} + 2x_{n-2}}{3} \Big|.$$

By the definition of absolute value we can have

$$\Big| - \Big(\frac{-2x_{n-1} + 2x_{n-2}}{3} \Big) \Big| = \Big| \frac{2x_{n-1} - 2x_{n-2}}{3} \Big|.$$

Distributing out a 2 and taking out the 1/3 get us

$$\big| (2/3)(2x_{n-1} + x_{n-2}) \big|.$$

Thus we showed that

$$|x_n - x_{n-1}| = \big| (2/3)(2x_{n-1} + x_{n-2}) \big|.$$

So then we can also say that

$$|x_n - x_{n-1}| \leq \big| (2/3)(2x_{n-1} + x_{n-2}) \big|.$$

Thus since $0 < \lambda = 2/3 < 1$, $\{x_n\}$ is contractive by the definition of contractive. Then by Theorem 35, contractive sequences are convergent thus $\{x_n\}$ is convergent. Alternatively, by problem 4, $\{x_n\}$ is a Cauchy Sequence ans thus convergent. ∎

Problem 6

Define the sequence x_n by $x_1 = 2$ and

$$x_{n+1} = \frac{3}{2 + x_n}$$

for $n \in \mathbb{N}$. Prove that $\{x_n\}$ is convergent and find its limit.

Proof. Let $P(n)$ be "$x_n > 0$" for all $n \in \mathbb{N}$.
Base Case: When $n = 1$. $x_1 = 1$ and $1 > 0$ thus base case holds.
Inductive Hypothesis: Suppose $P(k)$ holds for arbitrary $k \in \mathbb{N}$.
Inductive Step: From our inductive hypothesis we know that $x_k > 0$ holds. We know that $2 > 0$ and adding 2 to both sides of $x_k > 0$ gets us $2 + x_k > 2 > 0$. We know that $3 > 0$ so diving both sides by $2 + x_k$ gets us $\frac{3}{2+x_k} > 0$, then by your inductive definition we have $x_{k+1} > 0$. Thus $P(k+1)$ holds. Therefore $P(n)$ hold for all $n \in \mathbb{N}$ by the Principle of Mathematical induction. Now lets use $|x_n - x_{n-1}|$ to get

$$\Big|\frac{3}{2+x_{n-1}} - \frac{3}{2+x_{n-2}}\Big|$$

Multiply the terms to get a common denominator:

$$= \Big|\frac{3}{2+x_{n-1}} \cdot \frac{2+x_{n-2}}{2+x_{n-2}} - \frac{3}{2+x_{n-2}} \cdot \frac{2+x_{n-1}}{2+x_{n-1}}\Big| = \Big|\frac{3(2+x_{n-2})}{(2+x_{n-1})(2+x_{n-2})} - \frac{3(2+x_{n-1})}{(2+x_{n-1})(2+x_{n-2})}\Big|$$

Adding the terms together gets

$$= \Big|\frac{3(2+x_{n-2}) - 3(2+x_{n-1})}{(2+x_{n-1})(2+x_{n-2})}\Big|.$$

We can distribute out a 3 and subtract the numerator:

$$= |3|\Big|\frac{(2+x_{n-2}) - (2+x_{n-1})}{(2+x_{n-1})(2+x_{n-2})}\Big| = 3\Big|\frac{x_{n-2} - x_{n-1}}{(2+x_{n-1})(2+x_{n-2})}\Big|$$

Now lets also take out the denominator so that

$$= 3 \cdot \frac{1}{|(2+x_{n-1})(2+x_{n-2})|}|x_{n-2} - x_{n-1}| = \frac{3}{|2+x_{n-1}||2+x_{n-2}|}|x_{n-2} - x_{n-1}|$$

So we have got that

$$|x_n - x_{n-1}| = \frac{3}{|2+x_{n-1}||2+x_{n-2}|}|x_{n-2} - x_{n-1}|.$$

Now since we know that $x_n < 0$ for all $n \in \mathbb{N}$, lets add 2 to both sides to get $0 < x_n + 2 < 2$, taking the multiplicative inverse of both sides gets $\frac{1}{x_n+2} > \frac{1}{2}$. So we can say that $\frac{1}{x_n+2} \geq \frac{1}{2}$ and we can use this as substitutions for $|2 + x_{n-1}|$ and $|2 + x_{n-2}|$ to get the inequality

$$|x_n - x_{n-1}| \leq 3 \cdot \frac{1}{2} \cdot \frac{1}{2}|x_{n-2} - x_{n-1}|.$$

$$\implies |x_n - x_{n-1}| \leq \frac{3}{4}|x_{n-2} - x_{n-1}|.$$

Thus $\{x_n\}$ is contractive since $0 \leq \frac{3}{4} \leq 1$, then since contractive sequences converge, $\{x_n\}$ converges, so the limit of $\{x_n\}$ exists. Now let $L = \lim_{n\to\infty} x_n$, and since x_{n+1} is a subsequence of x_n, $\lim_{n\to\infty} x_{n+1} = L$. Now using the recursive definition, $x_{n+1} = \frac{3}{2+x_n}$, we can multiply both sides by $2 + x_n$ to get

$$x_{n+1}(2 + x_n) = 3.$$

Now taking the limit of both sides gets us

$$\lim_{n\to\infty} x_{n+1}(2+x_n) = \lim_{n\to\infty} 3.$$

Splitting by multiplication and them by addition (Theorem 27) get us

$$\lim_{n\to\infty} x_{n+1} \lim_{n\to\infty}(2+x_n) = \lim_{n\to\infty} 3.$$

$$\implies \lim_{n\to\infty} x_{n+1}(\lim_{n\to\infty} 2 + \lim_{n\to\infty} x_n) = \lim_{n\to\infty} 3.$$

Then using what we established before we substitute in our L's and the limit of a constant is said constant to get

$$L(2+L) = 3.$$

Distributing:

$$2L + L^2 = 3.$$

Subtracting 3 on both sides:

$$-3 + 2L + L^2 = 0.$$

Factoring:

$$(L+3)(L-1) = 0.$$

So either $L+3=0$ and $L=-3$, or $L-1=0$ and $L=1$. We know that $\{x_n\} > 0$ so taking the limit of both sides get us

$$\lim_{n\to\infty}\{x_n\} > \lim_{n\to\infty} 0.$$

$$\implies L > 0.$$

Thus the only possible solution is when $L=1$, therefore $L = \lim_{n\to\infty}\{x_n\} = 1$. ■

Problem 7

Find the limit of the sequence $\{x_n\}$ defined in Problem 5. Show details of your calculation/argument.

Proof. Let $P(n)$ be "$x_{n+1} = (-2/3)x_n + 25/3$" for all$n \in \mathbb{N}$
Base Case: When $n=1$, then on the LHS of $P(1)$ we have $x_2 = 7$ and on the LHS we have $(-2/3)x_1 + 25/3 = (-2/3)2 + 25/3 = -4/3 + 25/3 = 21/3 = 7$ thus the LHS and RHS equal so $P(1)$ holds and the base case holds.
Inductive Hypothesis: Assume for arbitrary $k \in \mathbb{N}$, $P(k)$ hold.
Inductive Step: By our inductive hypothesis we know that $x_{k+1} = (-2/3)x_k + 25/3$ holds i.e. $x_{k+1} = x_{k+1}$. We can multiply both sides by (-2/3) to get

$$(-2/3)x_{k+1} = (-2/3)x_{k+1}.$$

Then adding 25/3 on both sides gets

$$(-2/3)x_{k+1} + 25/3 = (-2/3)x_{k+1} + 25/3.$$

We know that $x_{k+2} = (-2/3)x_{k+1} + 25/3$, so using this to substitute on the right hand side to get $x_{k+2} = (-2/3)x_{k+1} + 25/3$. Thus $P(k+1)$ holds, and our inductive step is complete. Therefore by the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.
Now from problem 5, we know that the limit of $\{x_n\}$ exists, so let $x = \lim_{n\to\infty} x_n$. Now lets take the limit of both sides of $x_{n+1} = (-2/3)x_n + 25/3$ to get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}((-2/3)x_n + 25/3).$$

We can split the right hand side by addition and then the term by multiplication to get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(-2/3)x_n + \lim_{n\to\infty} 25/3$$

$$\implies \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(-2/3)\lim_{n\to\infty} x_n + \lim_{n\to\infty} 25/3.$$

We know that the limit of constants are said constant. And since x_{n+1} is a subsequence of x_n, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$, thus we get

$$x = (-2/3)x + 25/3.$$

Multiplying both sides by 3 gives us

$$3x = -2x + 25.$$

Adding $2x$ to both sides gets us

$$5x = 25.$$

Diving both sides gets us

$$x = 5.$$

Therefore $x = \lim_{n\to\infty} x_n = 5$. ■