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NEWTON'S WATER CLOCKS AND THE FLUID MECHANICS OF CLEPSYDRAE

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MOST biographies of Isaac Newton (1642–1727; F.R.S. 1672; P.R.S. 1703–1727) mention a water clock that he is reported to have constructed as a boy when living at Woolsthorpe Manor in Lincolnshire. The source of this and other anecdotes concerning Newton's boyhood (1) is William Stukeley, who made it his business to collect such memories upon retiring to live in nearby Grantham.

In a letter (2) dated 26 June 1727 (*i.e.* only some three months after Newton's death) he writes:

> Moreover Sir Isaac's water clock is much talked of. This he made out of a box he begged of Mr. Clark's (his landlord) wife's brother. As described to me, it resembled pretty much our common clocks and clock-cases, but less; for it was not above four feet in height, and of a proportionable breadth. There was a dial plate at top with figures of the hours. The index was turned by a piece of wood, which either fell or rose by water dropping. This stood in the room where he lay, and he took care every morning to supply it with its proper quantity of water; and the family upon occasion would go to see what was the hour by it. It was left in the house long after he went away to the University.

In a subsequent manuscript entitled *Memoirs of Sir Isaac Newton's Life* and dated 1752 (but not in fact published until 1936 (3)) we find the passage:

> He likewise made a good wooden clock, that went by weights, in the usual manner. This being a bare imitation, as before, was not sufficiently pleasing to him; but he made another clock which is much talked of still, on quite a new principle. It went by water dropping into a cistern, and was famous for its exactness. He constructed this out of a fir-deal box, which he beg'd of Mrs Clarks brother. It was in shape much like other clocks; a case about 4 foot high, a dyal plate, painted by himself with figures, and a wooden *index* that show'd the hour. This *index* was turn'd by a perpendicular piece of wood, rising as the water filled the cistern; I suppose either by rack-work, as they call it, or by a string winding its self round an *axis*.

> This clock always stood in his own garret where he lay, and he took care every morning to supply it with a proper quantity of water; and the family upon occasion went up thither, to be well informd upon the time of day; and it was left in the house long after he went to the University; destroyed probably when the house was pulled down and rebuilt.
>
> He lov'd to vary his operations, and to bring them by tryal to a greater simplicity. I was informed that he made another water-clock, which performed by dropping out of a cistern, the rod with the hours on it descending. I remember very well I have heard him speak of this himself, and at the Royal Society, particularly that time aforemention'd, when I was Dr Halley's deputy; on occasion of some paper read on water-clocks [4]. Sir Isaac spoke to it, and observ'd the exactness and usefulness of that kind of machine. He said 'the chief inconvenience attending it was this: the hole thro' which the water drops must necessarily be extremely small, therefore it was subject to be furr'd up by impuritys in the water. So hour-glasses made with sand will wear the hole thro' which it is transmitted bigger. These inconveniences in time spoil the use of both instruments'.

There appear then to have been *two* water clocks made at Woolsthorpe, and a vague association with a third is to be found in one of Newton's early notebooks in the Fitzwilliam Museum, Cambridge (5). An entry made before Whitsunday 1662 lists the 'sin' of 'Helping Pettit to make his water watch at 12 of the clock on Saturday night'.

Stukeley was, of course, wrong in supposing the principle of Newton's water clock to be a new invention: clepsydrae have been known since ancient times. In fact, it seems likely that even the design was not original, Andrade (6, 7, 8) having shown that it was probably based on a model described in *The Mysteryes of Nature and Art* by John Bate (9). This work was first published in London in 1634, but Andrade thinks it likely that Newton possessed (or at least had access to) the third edition of 1654, making him 12+ years old when he was reading and using it. He bases this proposition on the close concordance between certain recipes and rules of drawing and painting which appear in both Bate and another notebook compiled by the young Newton. Bate describes and illustrates a number of water clocks, including both 'rising rod' and 'float-and-dial' types (Figure 1). In the first a figure slowly rises as water flows into a reservoir; in the second a float descends as water drips from a nozzle in the base of a cistern, turning the single hour hand via a connecting cord and pulley. (Note the peculiar and unnecessary 'anticlockwise' dial!) It will be realized that in neither clock will a linear rate with equally spaced hour

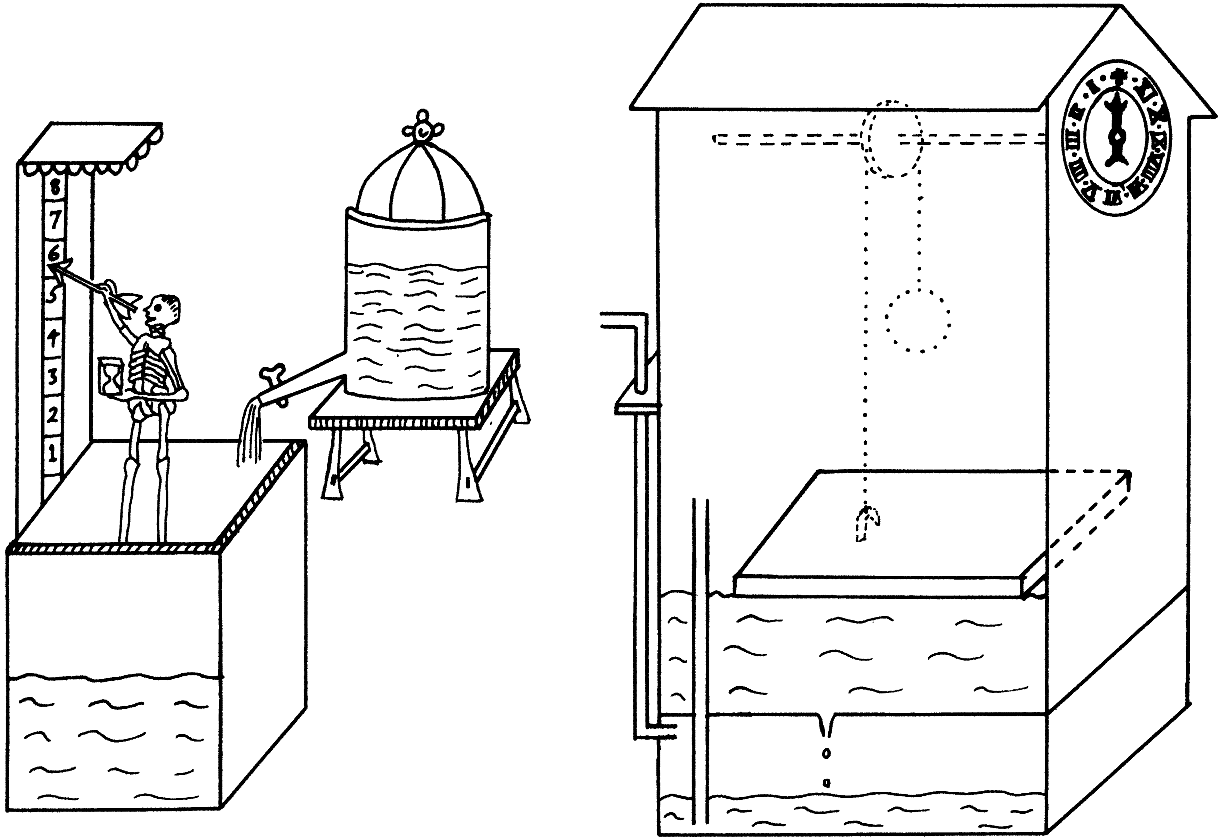

FIGURE I. Two water clocks traced (with minimal modifications for clarity) from crude woodcut illustrations in John Bate *'The Mysteryes of Nature and Art'* (London 1634). In the second clock water drips from a nozzle in the base of the upper reservoir, allowing the float to descend. It is intended that water should be returned to the upper reservoir by blowing with the mouth through the external tube.

divisions be achieved, due to the rate of flow decreasing as the head of water diminishes. Bate does not specifically state this fact, but tacitly recognizes it by directing that the clocks be initially calibrated by reference to an hour glass or mechanical clock.

It is not stated how Newton made the calibration, or determined the time to which to set his water clocks on refilling them each morning. However, it is known that he made a number of sundials in his boyhood, and use of one of these would undoubtedly have been the most accurate method available to him.

Ancient Clepsydrae

Water clocks, like sundials, have been known from remote antiquity, probably being invented in Mesopotamia (10). The oldest specimen to have survived sufficiently well to allow full reconstruction is a beautiful alabaster vessel excavated at Karnak in Upper Egypt, and dated to about 1400 B.C.

FIGURE 2. Plaster cast of the Karnak clepsydra of 1400 B.C. (Photograph courtesy of the Science Museum, London. Crown copyright reserved).

(11–13). It is illustrated in Figure 2, and is clearly a refined instrument indicative of many predecessors.

The Greeks of classical times knew the water clock as a *clepsydra*—literally 'water-thief', a term they originally applied to a device for retrieving samples of water, oil or wine from below the dusty and contaminated surface in a storage vessel (14, 15). The name is nowadays applied to water clocks of all periods and types (16–21).

Initially the clepsydra was used to measure intervals of time rather than the time of day, so being more allied to the modern stopwatch than the clock. In its simplest form it consisted of a vessel with a small hole at the bottom, which was filled with water to a fixed mark and then allowed to drain. If we neglect the effects of temperature and fine sediment a constant interval of time would be marked out. Clepsydrae of this form were employed in the lawcourts to curb long-winded orators (whose time would literally run-out!) and by doctors to quantitatively determine whether a patient had an abnormal pulse rate (20). Another variety was simply a perforated bowl which, placed upon water, was considered to take a constant time to fill-up and sink. This simple device remained in use even until quite recent times to delimit the period a man might take water from an irrigation canal (10). In these applications the *entire* period was always employed: there was no need to interpolate halves or quarters for example, so the non-linearity of the rate of descent would not be a problem.

However, there was always a need to tell time independently of the sundial, such as during cloudy weather or at night (10). It was natural that the clepsydra should be applied to this purpose too, having the significant advantage over calibrated lamps etc. that nothing was consumed!

The clepsydra as a clock: linear systems

It is a matter of common observation that water runs more slowly from a leaking container as the amount remaining decreases and the pressure or 'head'—the height of the free surface above the orifice—diminishes. Now a non-linear *clock* is a most unsatisfactory instrument, and so even in the Karnak vessel of 1400 B.C. we see that an attempt has been made to compensate the diminishing flow by decreasing the diameter, so producing a truncated straight-sided cone (Figure 2). The angle of the cone must have been determined by empirical experiment, for not until the seventeenth century was fluid mechanics sufficiently advanced to begin to allow mathematical derivation of a theoretical profile for a *linear outflow* clepsydra.

In the interim a number of alternative designs (Figure 3) were developed to circumvent the problem (20). Perhaps the most obvious—and certainly known by the time of the Egyptians—is the system shown at (a), where a constant-head reservoir provides a constant rate of flow. A cylinder (or indeed any receptacle of uniform cross-section) will therefore fill at a constant rate, giving a *linear inflow* clepsydra. If viewed from above, metal or pottery receivers could be graduated directly, but more convenient and accurate would be a remote indicator of some form operated by a float. Nowadays, the best-known reconstruction of a linear inflow clepsydra is probably that produced for the Tower of the Winds in Athens by Noble and Price (22).

The snag with a constant-head reservoir dependent on overflow is that it is wasteful of water: the proximity of a spring or aqueduct is almost a necessity. An alternative is a float-valve to regulate the supply in the manner of a modern ball-cock, and Ktesibios (3rd Century B.C.?) is credited with the invention of just such a device (20, 23, 24) (Figure 3, b). He also devised arrangements to cope with a complication that no longer troubles us; namely that the ancients took one daylight hour as one-twelfth of the period between sunrise and sunset, and similarly for the night, so the absolute length of their hours varied with the season (10, 20). Ktesibios' float valve appears to be the first genuine feed-back device in the history of mechanics (25).

The principle employed by Philon of Byzantium (23) for maintaining a constant level in an oil lamp found quite extensive application for that purpose (25), and is still to be seen as a drinking fountain for birds and small animals. Robert Hooke appears to have constructed a clepsydra on this principle (26).

What is now known as a Mariotte bottle (27) is another simple yet elegant scheme for achieving a constant rate of flow. It was in fact employed long before Mariotte described it, being the basis for an inflow clepsydra illustrated in the 13th Century *Libros del Saber* (28).

Particularly ingenious is the constant head achieved with a floating syphon (Figure 3, c). When operating in a vessel of uniform cross-section the entire assembly should descend at a constant rate, forming the basis of an indicator-rod or rotating-hand clepsydra. Such a device is illustrated by Hero of Alexandria (1st Century A.D.) (20, 23) but not explicitly as a clock; for this one has to await Isaac de Caus writing in 1644 (29).

A similar argument applies to a thin-walled sinking cup of uniform cross-section (Figure 3, d). Whilst it is filling, the difference between the levels of water inside and out must be sufficient to support the vessel through buoyancy. Thus a cylindrical tin can with a small hole in the bottom does (within experimental error) sink at a constant rate—provided it is intially loaded so that it floats vertically. A bowl does not suffer this flotation instability.

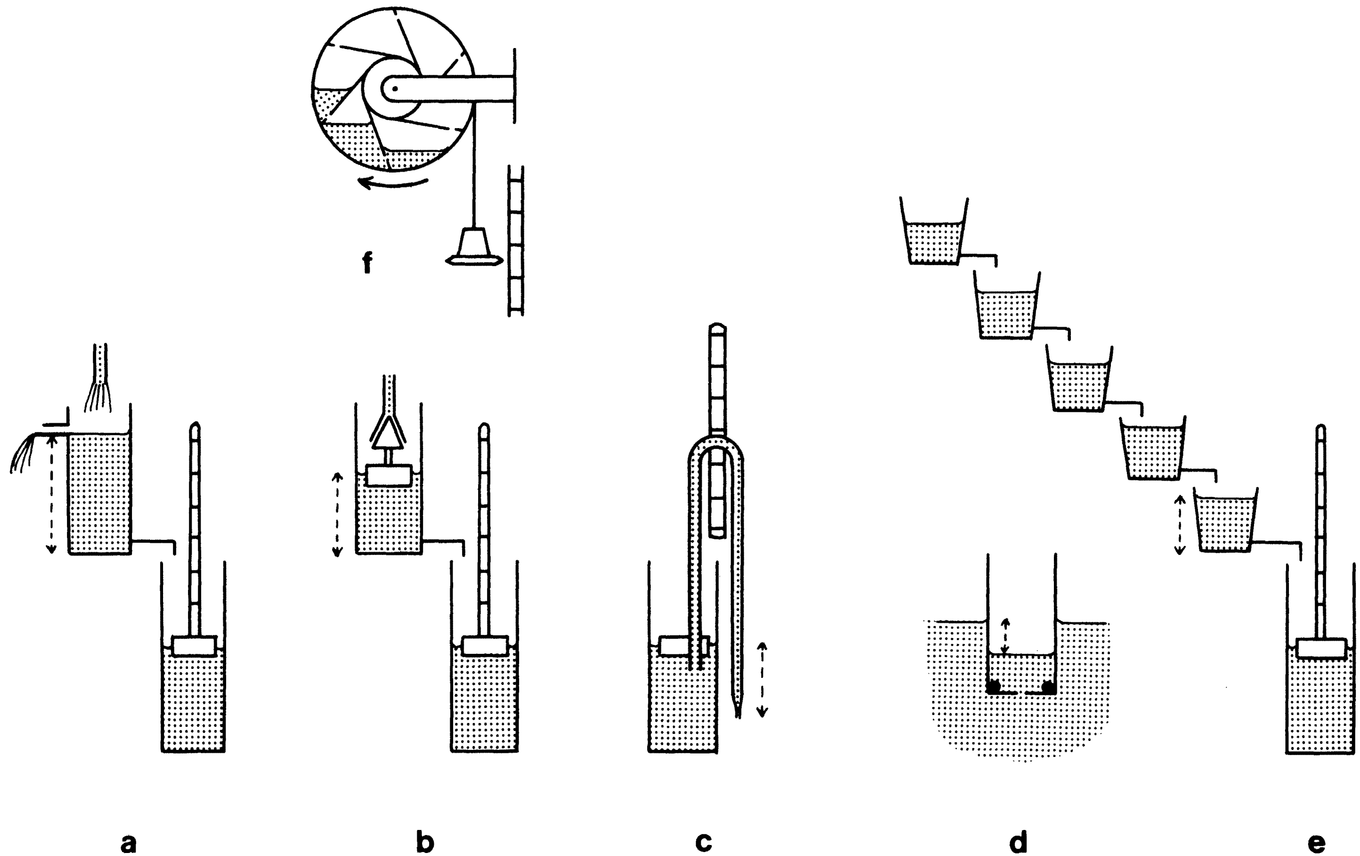

FIGURE 3. Various designs of linear clepsydrae employing a constant head, indicated by the dotted arrows.
(a) Overflow; (b) Float-valve; (c) Floating syphon; (d) Sinking perforated cylinder, weighted at the base; (e) Chinese 'staircase'; (f) Drum with perforated divisions.

The final form of linear water clock to appear in Europe is the drum clepsydra (Figure 3, f), first described in Spain in the 13th Century (28) but intermittently recurring in succeeding centuries in other parts of Europe as a 'new invention'. Bedini (30) thinks this design had an earlier Arab original, but otherwise Muslim science appears to have made no independent advances, taking-over the 'Karnak' form of conical outflow clepsydra (31) along with Greek constant-head inflow clepsydrae (25, 32, 33).

The Chinese (34) developed a technique for linearizing the clepsydra that never appeared in the western world. A series of bowls of equal size were arranged in staircase formation as shown in Figure 3, e. McNown (18) shows mathematically that a substantially constant head is maintained in the fifth bowl if the first is periodically refilled.

Consistent performance by clepsydrae required that the size of the orifice remained constant over a long period, for (as Newton pointed out) if it eroded the clock would run fast, but if it became partially blocked by corrosion products the clock would run slow. Vitruvius (24) reported that Ktesibios used:

> . . . an orifice in a piece of gold, or by perforating a gem, because these substances are not worn by the action of water, and do not collect dirt so as to get stopped up.

The Linear Outflow Clepsydra

Early Theories

At first sight the simplest clepsydra, the outflow form is in fact far from so because of the complex behaviour of moving fluids.

The Romans and their predecessors assumed that the rate of flow of water from an orifice was directly proportional to the head of water above it (35)—an error which ensnared even Leonardo da Vinci (36). Better understanding may be traced to Galileo's experiments on falling bodies, and his discovery (37) that their velocity varies directly as the *square root* of the distance fallen:

$$i.e. \qquad v \propto \sqrt[2]{h}$$

where v denotes the velocity after having fallen a distance h. Galileo writes that he employed a water clock for measuring the short intervals involved, collecting the water issuing as a thin jet from a large elevated vessel and weighing it to obtain the relative periods and ratios. The decrease of head during such brief periods would be negligible by comparison with other errors.

Galileo's work was extended to fluids by his students Evangelista Torricelli and Vincenzo Viviano. By 1644 the former was able to write (38):

> Liquids which issue with violence [from an opening in a vessel] have at the point of issue the same velocity which any heavy body would have, or any drop of the same liquid, if it were to fall from the upper surface of the liquid to the orifice from which it issues.

In symbolic form, we again have:

$$v \propto \sqrt[2]{h}$$
$$\text{or } v^2 = K.h$$

Nowadays this relation is usually written $v^2 = 2gh$ (where g is the acceleration due to gravity) and known as Torricelli's theorem, but the factor of proportionality $2g$ was not in fact defined until well after his time.

The stage was now set for calculation of the profile of a linear outflow clepsydra draining through an orifice, and this appears to have been first accomplished by Edme Mariotte some time before 1684. This is the date of his death: his book (27) was published posthumously in Paris in 1686, and was not translated into English until 1717. On page 185 of the latter, Desaguliers translates Mariotte as follows:

> It is proper enough in this Place to resolve a pretty curious Problem which Toricelly [sic] has not undertaken to resolve, tho' he propos'd it; this Problem is to find a Vessel of such a Figure that being pierc'd at the Bottom with a small Hole the Water should go out, its upper Surface descending from equal Heights in equal Times. . . .

In the absence of calculus Mariotte's solution is verbally tortuous, but his numerical example makes it clear that the desired profile is such that the square-squared of the radius is proportional to the height. In modern notation:

$$r^4 \propto h$$
$$\text{or, } r \propto h^{\frac{1}{4}}, \text{ a } \textit{quartic} \text{ curve.}$$

Unfortunately, Balmer (21) appears to have misunderstood the text, believing that Mariotte was recommending a *parabolic* profile with $r \propto h^{\frac{1}{2}}$ (Figure 4). He therefore incorrectly places the discovery of the quartic section much later in history (the 19th Century).

However, the parabola is indeed mentioned by a few authors as the theoretically correct shape, although no sound justification has been traced in the existing literature. The earliest of these appears to be Viviani, who about 1670 produced an extensive manuscript entitled *Treatise of Clepsydrae*, although it was never published (39). (Actually, the manuscript illustration shows a parabolic trough—not a paraboloid of revolution.) The parabolic profile has also been specified in some archaeological works (12).

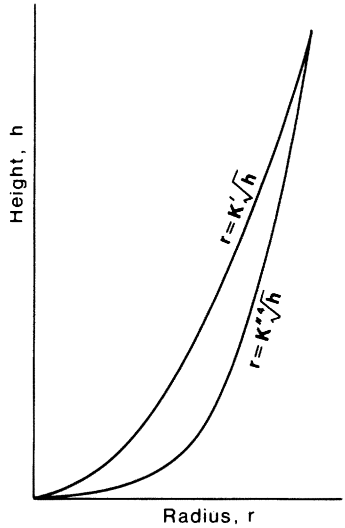

FIGURE 4. Graphs of the equations $r = K'h^{\frac{1}{2}}$, a parabola; and $r = K''h^{\frac{1}{4}}$, a quartic function.

THE QUARTIC PROFILE AND LATER THEORIES

Torricelli specifically states that his relationship applies to liquids *which issue with violence* from a vessel. It would seem that neglect of this condition, and a general lack of awareness of the influence of the design of outlet and regime of flow upon the rate of efflux, has caused much confusion and error.

Any textbook of physics or fluid mechanics (e.g. notes 37–38) will show that there are three possible forms of energy concerned with the flow of a fluid. These are referred to as pressure energy, potential energy and kinetic energy, and Bernoulli (44) proved (although not quite in these terms) that for an ideal liquid devoid of viscosity moving along a tube of flow their sum is always constant:

i.e. potential energy + pressure energy + kinetic energy = constant.

This is known as Bernoulli's equation, and may be written symbolically as:

$$hg + p/\rho + \frac{v^2}{2} = K$$

where the symbols have the meanings expressed in Figure 5, which represents a sharp-edged hole made in the side of a thin-walled tank:

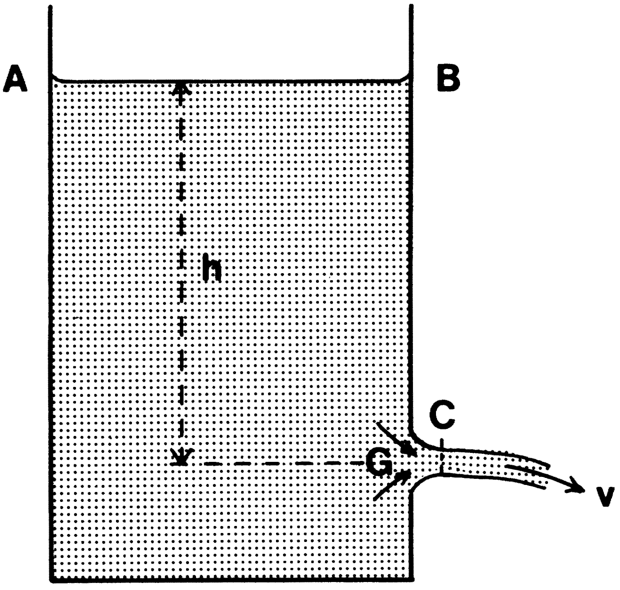

FIGURE 5. Liquid flowing from a sharp-edged orifice in a thin-walled tank. h represents the head, g the acceleration due to gravity, p the pressure, ρ the density and v the exit velocity (see text).

If the tank is sufficiently wide, the velocity at AB, the surface of the liquid, may be taken as zero. The pressure there is atmospheric, but as it is the same at the emerging jet the atmospheric pressure will not affect the flow. Consider a tube of flow which begins at AB and ends at G; at AB, $p = 0$ and $v = 0$, while at G, $p = 0$ and $h = 0$. Applying Bernoulli's equation to this tube of flow:

$$hg + 0 + 0 = 0 + 0 + v^2/2$$
$$\text{or, } v^2 = 2gh$$

which is Torricelli's theorem above. Now, however, the assumptions and approximations involved are more obvious:

a) The fluid is incompressible. This leads to no significant errors with water as the fluid.
b) The velocity of AB will not be zero in a working clepsydra.
c) The flow is steady and 'considerable'—a vague term, but not one associated with clepsydrae. This point will be taken up again later, and is related to (d) and (e).
d) The fluid is of zero viscosity. Since there is no such ideal, and some work must therefore be done to overcome viscous forces, the emergent velocity will be less in real situations.

e) The stream-lines in the plane of the aperture are not all horizontal, but converge on it as shown, the momentum of the emergent liquid causing a narrowing of the jet. The pressure at the outer surface of the jet is atmospheric, but is not uniform across it. In the plane of the aperture the pressure is below atmospheric, as the tubes of flow are narrowing and the velocity of the liquid is increasing. It is only at C, called the *vena contracta,* that the jet becomes uniform and the velocity becomes the same throughout. It is the velocity at the *vena contracta* that is given by Torricelli's equation.

It is found by experiment that, with water, the area of the jet from a circular orifice at the *vena contracta* is about 0.62 of the area of the orifice itself.

It will be appreciated why the apparently simple problem of water running-out of a leaking vessel is in fact so complex that a complete analytical solution has not yet been obtained. Newton himself—perhaps with memories of his boyhood water clocks—attempted the problem in the *Principia* (45) as Proposition 36, Problem 8. In the first edition he totally neglected the *vena contracta* and erroneously deduced that the efflux velocity was that due to only *half* the height of the water in the vessel. This mistake he afterwards corrected, but his treatment still remains open to very serious objections (46, 47).

However, let us temporarily forget these tedious assumptions and approximations, and go along with modern hydraulic and chemical engineers who generally find it quite satisfactory to follow Torricelli and Bernoulli in saying:

$$v \propto h^{\frac{1}{2}}$$

To calculate the profile of a linear outflow clepsydra in modern notation all that is necessary is to say:

Let the volume rate of outflow = Q.

Then $Q = Av$, where A is the area normal to the flow at any point in the vessel or jet, and v is the velocity of flow at that point.

As we also say $Q \propto \mathrm{d}V/\mathrm{d}t$, where V is the volume of water within the vessel, so:

$$\mathrm{d}V/\mathrm{d}t \propto v \propto h^{\frac{1}{2}}$$

Letting $\mathrm{d}V/\mathrm{d}t = \mathrm{d}V/\mathrm{d}h \, . \, \mathrm{d}h/\mathrm{d}t$

we have

$$h^{\frac{1}{2}} \propto \mathrm{d}V/\mathrm{d}h \,.\, \mathrm{d}h/\mathrm{d}t$$

For linear descent of the meniscus $\mathrm{d}h/\mathrm{d}t = \text{constant}$

$$\therefore\ h^{\frac{1}{2}} \propto \mathrm{d}V/\mathrm{d}h$$

For a body of revolution:

$$V = \int_{0}^{h} \pi r^2 \,\mathrm{d}h$$

$$\text{So } \mathrm{d}V/\mathrm{d}h = \pi r^2$$

$$\therefore\ h^{\frac{1}{2}} \propto \pi r^2$$

$$\text{or } h \propto r^4$$

This is equivalent to $r \propto h^{\frac{1}{4}}$, which is Mariotte's relation above. The expression has been found independently by many later workers (11, 18, 21, 48), but the assumptions implicit in the derivation are rarely noted. This profile is also closely approximated by the shape of the Karnak clepsydra (18), which appears to have emptied at a fairly rapid rate via a short metal tube (12, 49).

INFLUENCE OF DESIGN OF THE OUTLET

By neglecting viscosity in a practical situation we are really saying that, within desired limits, viscous forces are overwhelmed by inertial forces in the moving fluid. The quartic curve expresses this condition for unobstructed orifice flow of a jet of water into air.

Pipes, tubes, and other ducted flows

An obvious alternative to the simple hole in a thin metal diaphragm is a narrow pipe or tube. This form of outlet has therefore been employed in clepsydrae, but unfortunately without appreciating that ducted flow does not obey the same laws as flow from a sharp-edged orifice. Viscosity can no longer be neglected; indeed it is a vital factor in determining whether the flow in the tube will be turbulent or laminar. The end-member of the latter regime is that where viscous forces completely overwhelm inertial forces.

Nowadays, the ratio of inertial to viscous forces is assessed by the Reynolds number *Re* (refs. 40–43) defined as the ratio:

$$\frac{\text{Density} \times \text{Speed} \times \text{Size}}{\text{Viscosity}}$$

A small value means that viscous forces predominate; a large value that inertial forces are the most important.

The Reynolds number characterizing a given practical situation is normally found from the formula:

$$Re = \rho \bar{v} D/\eta$$

where D denotes the pipe diameter, $\bar{v}$ the average velocity of flow, ρ the density, and η the absolute viscosity, all in mutually compatible units. This expression reduces to:

$$Re = \bar{v} D/\upsilon$$

when υ represents the kinematic viscosity.

It has been found by experiment (50) that the character of the flow in a pipe may be predicted from the numerical value of the Reynolds number, the traditional boundaries being:

Re	*Type of flow*
> 4000	Turbulent
$4000 - 2000$	Transitional
< 2000	Laminar

Very slow laminar flow through a capillary

The situation where viscosity is overwhelmingly dominant is employed in a well-known method for measuring viscosity due to Poiseuille (40, 51), shown diagrammatically in Figure 6. Here the water is obliged to flow through a capillary so long by comparison with its bore and the available head that it issues as discrete drops from the far end. The virtual absence of kinetic energy on leaving the capillary is made apparent by positioning the tube horizontally and checking that drops fall vertically. Provided these conditions are observed, Poiseuille showed that the absolute viscosity η is given by the expression:

$$\eta = \pi hga^4/8lQ$$

where a is the radius of the capillary and Q is the volume flowing out per unit time. This is sometimes known as the Hagen-Poiseuille equation. Rewriting it as:

$$Q = \pi hga^4/8l\eta$$

we see that $Q \propto h$ if the other factors remain constant. *i.e.* the volume dripping out per unit time is directly proportional to the instantaneous head h at that

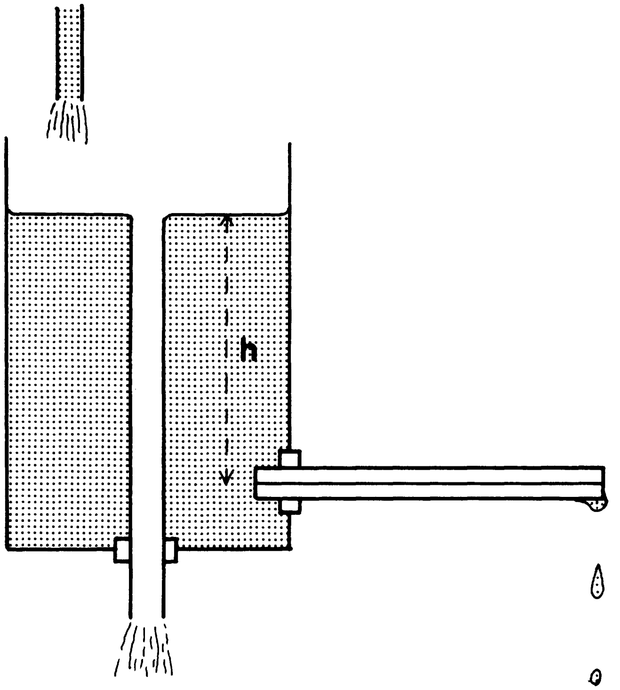

FIGURE 6. Poiseuille's apparatus for the measurement of viscosity. The constant head is h; the capillary tube is of length l and radius r.

time – *not* to $h^{\frac{1}{2}}$ as in Torricelli's equation for the opposite extreme of the flow regime. Putting $dV/dt \propto h$ in the above calculation of profile gives:

$$h \propto r^2$$

the equation of a *parabola.* Thus, the authors cited above could have arrived at this shape by simply assuming—like so many before them—that the rate of flow is proportional to the head. We now see that this is true only if the water leaves by very slow laminar flow in a long, narrow capillary.

Time, too, enters as a parameter in the sense that it takes a finite period to establish the velocity profiles associated with fully developed laminar or turbulent ducted flow. For a given velocity the exit tube must therefore exceed a certain minimum length—the pipe entrance length L_e—for the standard equations to apply. With laminar flow this is approximated by the expression (42, 52):

$$L_e \simeq 0.06\, Re\, D$$

In practice, it is usual to allow at least ten times this value to be sure of avoiding entrance effects—say 60 pipe diameters for very slow flow with $Re \simeq 100$, or 100 pipe diameters for somewhat faster flow. Turbulent momentum transfer causes the flow regime to become fully-developed in a shorter distance than laminar flow. A figure of 50 pipe diameters is commonly quoted (42), this distance being independent of the Reynolds number.

With this proviso of well-developed flow regimes well-removed from transitional regions, it has been found that the following relationships apply, where h represents the pressure or head:

	Rate of efflux
Laminar flow in a pipe:	$\propto h$ (if the liquid emerges with appreciable velocity a small quantity h' must be deducted from h (ref. 40))
Turbulent flow in a smooth pipe:	$\propto \sqrt[1.722]{h}$ (ref. 50—an expression not normally met nowadays, but apposite to the clepsydra)
Orifice flow:	$\propto \sqrt[2]{h}$

Yet another problem is that surface tension becomes significant with small orifices and tubes (42). This is a property notoriously liable to large changes following chance contamination with traces of surface-active materials.

To summarize, it appears that the theoretical shape of a linear outflow clepsydra has *two* possible boundary solutions according to the design of its outlet. The profile of a vessel from which water pours as a stream via a comparatively wide orifice in a thin diaphragm should follow the quartic curve $r \propto h^{\frac{1}{4}}$: the profile of a vessel from which water drips from a sufficiently long narrow tube should obey the parabolic relationship $r \propto h^{\frac{1}{2}}$. Between these boundary conditions there will be a grey area, with a turbulent regime in a tubular outlet apparently requiring a vessel with a profile given by $r \propto h^{\frac{1}{3.44}}$. The latter carries the additional complication of the distinct possibility of a transition from turbulent to laminar flow as the head diminishes and a critical Reynolds number of about 2000 is passed (42, 53). The rate of discharge should temporarily *increase* at this point.

No general analytical solution for the linear outflow clepsydra seems possible—it would be tantamount to solving the intractable Napier–Stokes equations (54).

The influence of temperature

Viscosity enters directly into Poiseuille's equation above, and any variation with temperature will be particularly apparent in a capillary-flow device intended as a clock, whether or not a constant head reservoir is employed. The Egyptians were probably aware of this (12), but only one inventor appears to have arranged for a specifically calibrated scale to be read in conjunction with a thermometer (55).

The variation of the viscosity of water with temperature (40, 56) is reproduced in Figure 7.

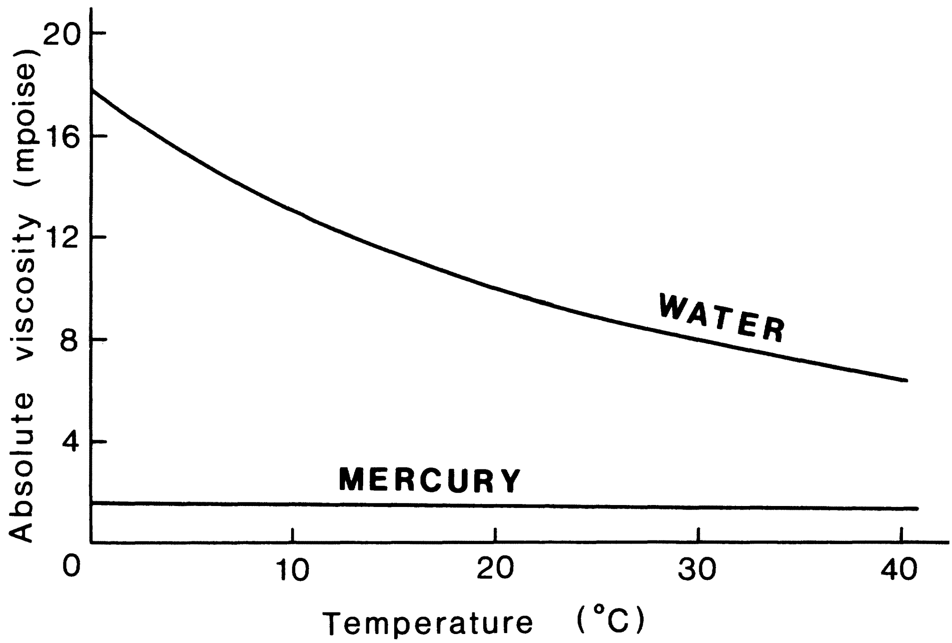

Figure 7. Variation with temperature of the viscosities of water and mercury.

It can be seen to be very great, averaging 2.5% per degree in the region of 20°C. It is hard to see how variations of at least a couple of degrees could be avoided in a clepsydra measuring hours, even if it were kept in a cellar. In a laminar flow device this would lead to changes of ±5% in time, equivalent to 3 minutes per hour, 36 minutes per 12 hours, or well over an hour per day. The expansion with increasing temperature of brass or bronze apertures, nozzles or tubes is quite negligible by comparison with this variation in viscosity.

A rapidly-emptying vessel should not be so affected by variations in temperature, it being established by Osborne Reynolds that when the flow is turbulent the pressure gradient required to produce a given velocity is independent of the viscosity but varies with the density of the fluid. For water around 20°C the variation of density with temperature averages only 0.02% per degree. Nevertheless, practical 'flowing' clepsydrae would be expected to embody sufficient viscous flow for temperature to exert a significant effect via viscosity as well as via the size of the orifice—but exactly how much could only be determined by experiment for each design (see below). Höppler (57) believed his tests with a vessel made to the shape of the Karnak clepsydra indicated that its 70° conical section closely compensated for an observed fall of temperature from 34°C at midday to 16°C at night in Upper Egypt. This seems rather *ad hoc*, and an error in mathematics renders his proposition doubly suspect.

Figure 7 also shows that the absolute viscosity of mercury is not only *less* than that of water, but changes much less with temperature, varying linearly over the range 10 to 30°C by only 0.32% per degree. The mercury drum clepsydra shown in the *Libros del Saber* (28) should therefore be a superior timekeeper.

It would also be expected that the sand clock should be much less sensitive to variations in temperature. This was confirmed by repeatedly timing the emptying period of an hermetically sealed all-glass egg-timer submerged in water at 0, 14, 30 and 100°C. Its mean period was 3 min 04 s, any variation with temperature being small enough to be obscured by a standard deviation of ± 6 s ($\pm 3\%$) characterizing every set of observations. Balmer (58) has pointed out other reasons why the sandglass first found extensive application at sea.

The replacement of the Clepsydra by the mechanical clock

In spite of Newton's advocacy, it is easy to see why the clepsydra was rapidly ousted when the mechanical clock appeared *ca* 1300 (59, 60). The medieval verge-and-foliot escapement kept time to within about $\frac{1}{4}$ hour per day—some 1%. Presumably this was the form of wooden clock made by the young Newton.

The isochronism of the pendulum was recognized by Galileo in 1582 and his son Vincenzio partially constructed a pendulum clock to his father's design in 1649 (61). Huygens made his first pendulum clock in 1656 (61, 62), and publication in the *Horologium* (62) and *Horologium Oscillatorium* (63) soon led to its widespread dissemination and a further order of magnitude improvement in the timekeeping ability of the mechanical clock.

Experimental tests

It was decided that the only way to delimit the 'grey areas' mentioned above was by experiment, using long narrow vessels to accentuate the movement of the water surface with time.

Apparatus

Translucent fibreglass was chosen as the constructional material; it being inert, readily moulded upon wooden formers, and allowing the water surface to be viewed directly by transmitted light.

To make a quartic vessel, the curve

$$r = 0.683\ h^{\frac{1}{4}}$$

where r is the radius in inches

and h the height in inches, with values from 1 to 38

was plotted on graph paper, mounted with rubber solution upon 0.08 inch thick aluminium sheet, and cut out. This template was then employed to control the profile of a hardwood pattern turned in a lathe. A corresponding paraboloidal former was made by employing the curve

$$r = 0.289\ h^{\frac{1}{2}}$$

Both patterns are illustrated in Figure 8.

The smooth, polyurethane-varnished patterns were treated with two coats of polyvinyl alcohol mould release agent, wax polished, and given an initial gel-coat of Uragel 259A thixotropic polyester resin. Fibreglass mat impregnated with Uralam 1346 polyester resin of matching refractive index was then applied to a thickness of 3/16 inch and, when set, detached to give translucent shells of the required internal profiles. These were trimmed off square top and bottom, and brass plates bearing a central hole epoxy-cemented to the lower ends. Brass extension pieces temporarily attached with sealing wax then enabled perforated diaphragms or capillary tubes to be held at points corresponding to the origins of the generating curves.

The 'sharp-edged apertures in a thin wall' were made by drilling 0.30, 0.50, 1.0, 2.0 and 3.0 mm diameter holes in 0.1 mm thick brass foil. Each was cemented to the corresponding extension piece when required with Faraday's wax. The 'capillaries' consisted of 100 mm lengths of glass precision tubing, with bores of 0.5, 1.0, 1.2, 2.0 and 3.0 mm. All were cleaned in chromic acid before use.

Tests were conducted in constant temperature rooms (± 1°C), using distilled water which had been allowed at least 24 h to achieve thermal equilibrium.

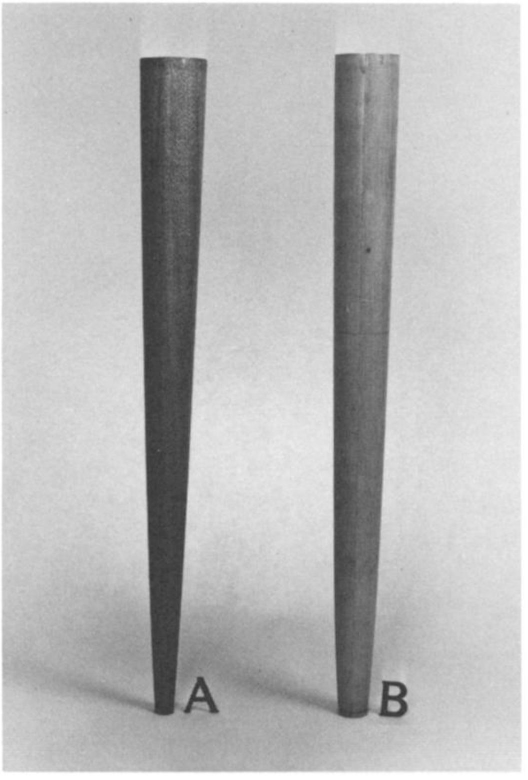

FIGURE 8. Patterns for experimental outflow clepsydrae. A: Parabolic profile; B: Quartic profile.

The temperature of a test was taken as that indicated by a thermometer placed in this water. A given vessel/aperture combination was supported vertically before a white fluorescent tube and, with the exit port temporarily sealed by adhesive tape, filled with water. After allowing a few minutes for bubbles to rise and the water to become still, the adhesive tape was carefully removed. A 'zero' mark was then made upon the fibreglass to indicate the position of the meniscus at the same time as a stopclock was started. Further marks were made at appropriate intervals until the vessel was empty. A metre scale was then clamped vertically alongside the vessel with its zero at the level of the outlet (*i.e.* coincident with the origin of the generating curve) and the height of each mark determined with the aid of a plastic set-square. Use of a fine-tipped fibre pen containing water soluble ink enabled the vessel to be readily cleaned before the next run.

Results

Figure 9 shows plots of the heights of meniscus versus time obtained for the quartic vessel discharging through various apertures, using water at 16°C. A 1 mm hole gave an excellent linear plot, but sizes larger than this tended to run-out some $3\frac{1}{2}$% too fast. A 0.5 mm aperture gave, of course, a considerably longer duration, and was linear above a 25 cm head. A 0.3 mm hole departed from linearity below 40 cm, where it ran too slow.

Figure 10 shows the corresponding plots for the paraboloidal vessel emptying through various tubes. Those of 3.0, 2.0 and 1.2 mm bore always ran as a continuous stream, much too fast to be linear. The 1.0 mm capillary showed interesting behaviour: it ran too fast from 93 to 25 cm, went through a

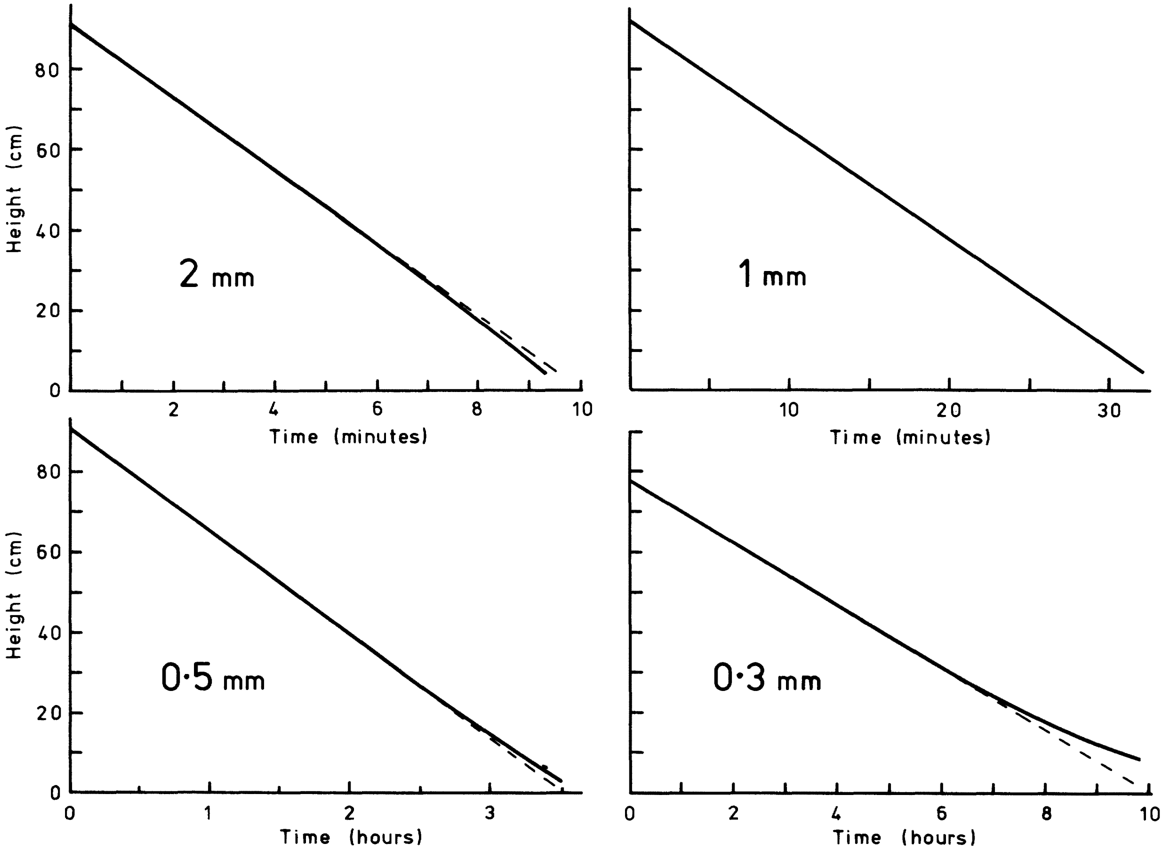

FIGURE 9. Rate of fall of water-level with time in a quartic vessel draining through holes of the indicated diameters.

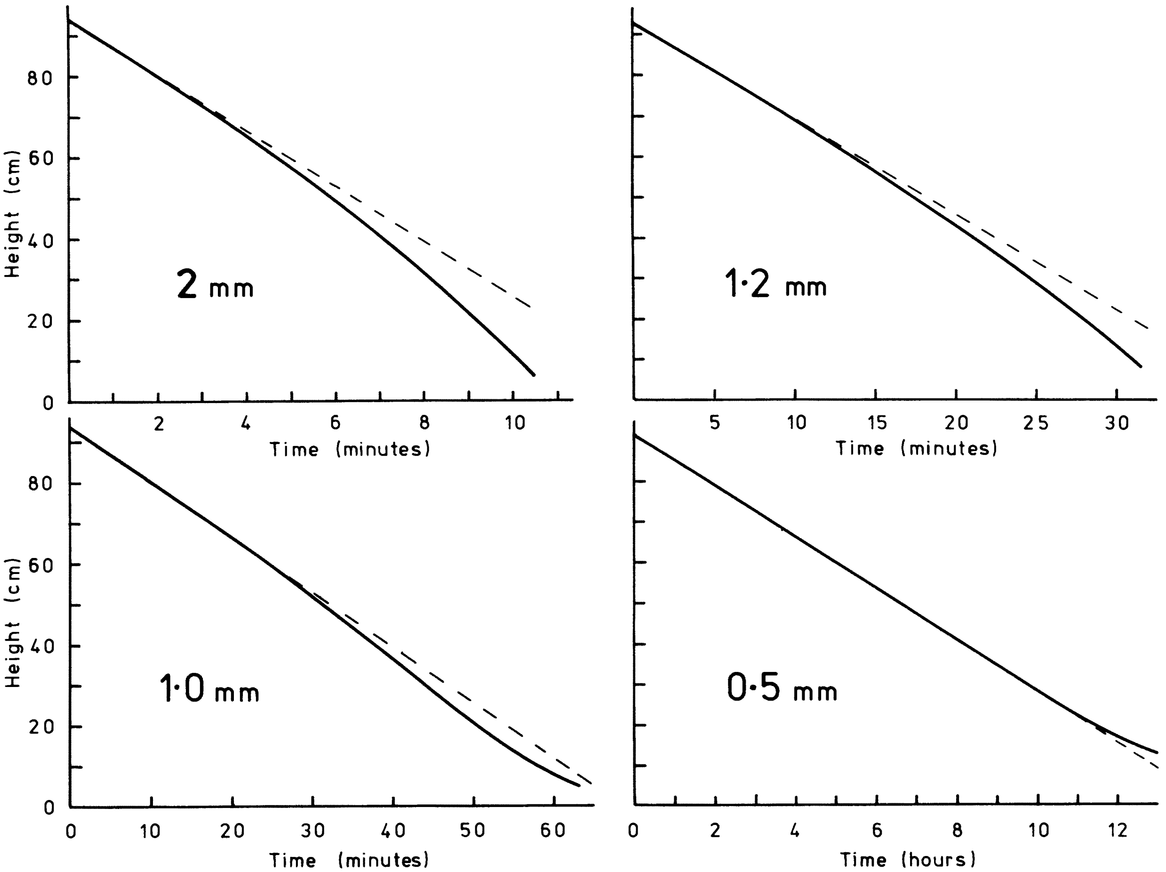

FIGURE 10. Rate of fall of water-level with time in a paraboloidal vessel draining through capillaries of the indicated internal diameters.

transitional region from the latter value (when discrete drops first appeared) to 20 cm, and then ran too slow for the remainder of the discharge. The tube of 0.5 mm internal diameter was sufficiently fine to ensure that the water issued as droplets even at maximum head, and the meniscus descended at a constant rate down to a 25 cm head, after which it began to run slow.

Fitting the 'inappropriate' outlets to the vessels produced the expected response: the quartic vessel always ran slow with capillary-type outlets (the deviation from linearity increasing as the diameter of the capillary diminished) while the paraboloidal vessel always ran fast with any of the orifice plates.

The effect of temperature was investigated by working in other constant temperature rooms, yielding data at 5, 16, 29 and 35°C. The paraboloidal vessel was fitted with the 0.5 mm capillary, and the rate of descent of the meniscus obtained from the linear portion of each graph. Plotting these values against temperature gave a mean variation of 2.1% per degree—not far from the 2.5% per degree derived above by considering only the variation of viscosity with temperature, and highly impractical for a timekeeping device. The quartic vessel fitted with the 0.5 mm aperture showed a variation of 0.3% per degree under similar circumstances—only 1/7 of that affecting nearly laminar flow through a capillary, and equivalent to ±4.3 minutes per day per degree change in temperature.

It is intended that further analysis of all the experimental results shall be submitted for publication elsewhere.

It is concluded that the most practical, accurate and reliable outflow clepsydrae would employ vessels of quartic profile, discharging through a hole between 0.5 and 1.0 mm diameter in a thin metal plate held at the origin of the curve. Linearity would be especially good if the final 25 cm of head did not form part of the calibrated section.

A water clock with a reservoir to the above design has been successfully constructed, its motion work and external appearance resembling Bate's illustration—just as Newton's clepsydra appears to have done.

Conclusions

According to Stukeley, Newton made two water clocks (clepsydrae) during his boyhood at Woolsthorpe Manor. These were most likely based on designs published by John Bate, and, as the head of water diminished with time, would have required calibration and given a scale or dial with unequal divisions.

Consideration of ancient clepsydrae shows that they were initially employed to delimit intervals of time, in the manner of a modern stopclock. However, there was a need for an alternative to the sundial for use during cloudy periods or at night, and the clepsydra was an obvious candidate for such an application. Now linearity is important in a clock, and the tapering conical shape of the Karnak vessel of 1400 B.C. proves the antiquity of attempts to linearize the outflow clepsydra by compensating for the decreasing rate of flow produced by a continually diminishing head. The problem was never satisfactorily solved, ancient peoples finding it preferable to employ cylindrical inflow clepsydrae filled by some form of constant-level device.

However, the question of the shape of the linear outflow clepsydra has continued to intrigue mathematicians for 3500 years. It is shown here that the longevity of this pursuit is due to failure to recognize and take account of the differing flow regimes associated with various types of orifice and rates of flow, *two* possible boundary solutions (and profiles) being end-members of a whole spectrum of shapes.

The simplest derivation ignores viscosity and other factors, considering only inertial forces by employing Torricelli's theorem (that the issuing velocity is proportional to the square root of the head) to arrive at a profile given by the quartic curve $r \propto h^{\frac{1}{4}}$. This formula was derived by Mariotte before 1684, and should be associated with discharge via a continuous stream issuing through a sharp-edged orifice in a thin metal diaphragm.

The opposite end-member is produced where viscous forces completely overwhelm inertial forces. In this situation the issuing velocity is directly proportional to the instantaneous head, leading to the parabolic profile $r \propto h^{\frac{1}{2}}$. This shape is associated with outflow through a capillary sufficiently long and fine to ensure laminar flow even at maximum head, with the water always issuing as discrete drops. Between these boundary conditions there is an ill-defined area where both regimes apply to varying extents.

The viscosity of water varies very greatly with temperature at ambient values, so it is expected that the latter class of laminar flow devices would be exceptionally temperature-sensitive. Turbulent flow designs should be much less susceptible to this effect.

Experimental tests with fibreglass vessels moulded to both quartic and parabolic profiles have confirmed all the above proposals. A quartic vessel discharging through a 0.5–1.0 mm hole in a thin metal foil proved to be the best design of linear outflow clepsydra.

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