MATH50003
Numerical Analysis

III.2 Cholesky factorisation

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Part III

Numerical Linear Algebra

1. Structured matrices such as banded
2. LU and PLU factorisations for solving linear systems
3. Cholesky factorisation for symmetric positive definite
4. Orthogonal matrices such as Householder reflections
5. QR factorisation for solving least squares

LU factorisation:

$$A = LU$$

PLU factorisation:

$$A = P^T LU$$

Cholesky factorisation: for sym. pos. def!

$$A = L\underbrace{L^T}_{U}$$

but diag of L is not all 1.

III.2.4 Cholesky factorisations $A = LL^\top$

Symmetric positive definite matrices have Cholesky factorisations

SPD

Definition 16 (positive definite)**.** A square matrix $A \in \mathbb{R}^{n\times n}$ is *positive definite* if for all $\boldsymbol{x} \in \mathbb{R}^n, x \neq 0$ we have

$$\boldsymbol{x}^\top A\boldsymbol{x} > 0$$

Motivation: How to prove A is SPD?

Proposition 5 (conjugating positive definite)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite and $V \in \mathbb{R}^{n\times n}$ is non-singular then*

$$V^\top A V$$

is positive definite.

Proof

since V is nonsingular

For $\vec{x} \neq 0$, $\vec{w} := V\vec{x} \neq 0$

we have

$$\vec{x}\; V^\top A (V\vec{x}) = \vec{w}^\top A \vec{w} > 0$$

Proposition 6 (diag positivity)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite then its diagonal entries are positive: $a_{kk} > 0$.*

since

$$a_{kk} = e_k^T A e_k > 0$$

Lemma 4 (subslice positive definite)**.** *If $A \in \mathbb{R}^{n\times n}$ is positive definite then $\underbrace{A[2:n, 2:n]}_{K} \in \mathbb{R}^{(n-1)\times(n-1)}$ is also positive definite.*

Proof write $A = \begin{bmatrix} \alpha & \vec{w}^T \\ \vec{v} & K \end{bmatrix}$ then for $\vec{x} \neq 0$

$$\vec{x}^T K \vec{x} = \begin{bmatrix} 0 & \vec{x}^T \end{bmatrix} \begin{bmatrix} \alpha & \vec{w}^T \\ \vec{v} & K \end{bmatrix} \begin{bmatrix} 0 \\ \vec{x} \end{bmatrix} > 0$$

Theorem 6 (Cholesky and SPD). *A matrix A is symmetric positive definite if and only if it has a Cholesky factorisation*

$$A = LL^\top$$

where L is lower triangular with positive diagonal entries.

<u>Proof</u> Cholesky $\Rightarrow$ SPD If $A = LL^T$ then

$A^T = A$ Sym

and for $\vec{x} \neq 0$

$$\vec{x}^T A \vec{x} = \vec{x}^T L^T (L\vec{x}) = \| L\vec{x} \|^2 > 0 .$$

SPD $\Rightarrow$ Cholesky Induction. If $A \in \mathbb{R}^{1\times 1}$ SPD then

$$A = [\underset{\substack{\uparrow \\ a_{11} > 1}}{\alpha}] = \underbrace{[\sqrt{\alpha}]}\ \underbrace{[\sqrt{\alpha}]}$$

$L \qquad L^T$

For $n > 1$, write

$$A = \begin{bmatrix} \alpha & \vec{v}^T \\ \vec{v} & K \end{bmatrix} = \underbrace{\begin{bmatrix} \sqrt{\alpha} & \\ \vec{v}/\sqrt{\alpha} & I \end{bmatrix}}_{L_1} \begin{bmatrix} \sqrt{\alpha} & \vec{v}^T/\sqrt{\alpha} \\ & \underbrace{A_2}_{K - \frac{\vec{v}\vec{v}^T}{\alpha}} \end{bmatrix}$$

$$= L_1 \begin{bmatrix} 1 & \\ & \underbrace{A_2}_{= \tilde{L}\tilde{L}^T} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{\alpha} & \vec{v}^T/\sqrt{\alpha} \\ & I \end{bmatrix}}_{L_1^T}$$

by induction

$$= \underbrace{\begin{bmatrix} \sqrt{\alpha} & \\ \vec{v}/\sqrt{\alpha} & \tilde{L} \end{bmatrix}} \underbrace{\begin{bmatrix} \sqrt{\alpha} & \vec{v}^T/\sqrt{\alpha} \\ & \tilde{L}^T \end{bmatrix}}$$

Example 14 (Cholesky by hand)**.**

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$