MATH50003 Numerical Analysis

III.2 LU and PLU factorisations

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Part III

Numerical Linear Algebra

1. Structured matrices such as banded
2. LU and PLU factorisations for solving linear systems
3. Cholesky factorisation for symmetric positive definite
4. Orthogonal matrices such as Householder reflections
5. QR factorisation for solving least squares

(Gauss Elimin)

LU factorisation:

Lower

$$A = LU$$

upper

So that

$$A^{-1}\vec{b} = (LU)^{-1}\vec{b} = \underbrace{U^{-1}\underbrace{(L^{-1}\vec{b})}_{\text{forward subs}}}_{\text{back sub.}}$$

PLU factorisation:

(Gauss Elim w/ row swaps)

$$A = P^{T}LU$$

Permutation matrix, note $P^T = P^{-1}$

$$\Rightarrow \quad A^{-1}\vec{b} = (P^T L U)^{-1}\vec{b} = U^{-1}L^{-1}P\vec{b}$$

swap rows

III.2.1 Outer products

Definition 15 (outer product)**.** Given $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ the *outer product* is:

$$\boldsymbol{x}\boldsymbol{y}^\top := [\boldsymbol{x}y_1|\cdots|\boldsymbol{x}y_n] = \begin{bmatrix} x_1y_1 & \cdots & x_1y_n \\ \vdots & \ddots & \vdots \\ x_my_1 & \cdots & x_my_n \end{bmatrix} \in \mathbb{F}^{m\times n}.$$

Equivalent to matrix mult viewing $\vec{x} \in \mathbb{F}^{m\times 1}$

$$\begin{bmatrix} x_1 \\ | \\ x_m \end{bmatrix} [y_1 \text{ — } y_n] = \begin{bmatrix} x_1y_1 & x_1y_2 & \text{—} & x_1y_n \\ | & \ddots & \ddots & | \\ x_my_1 & & \text{—} & x_my_n \end{bmatrix}$$

Proposition 4 (rank-1)**.** *A matrix $A \in \mathbb{F}^{m\times n}$ has rank 1 if and only if there exists $\boldsymbol{x} \in \mathbb{F}^m$ and $\boldsymbol{y} \in \mathbb{F}^n$ such that*

$$A = \boldsymbol{x}\boldsymbol{y}^\top.$$

since if $A = \vec{x}\,\vec{y}^T$ then $\operatorname{colspan}(A) = \operatorname{span}(\vec{x})$,

III.2.2 LU factorisation $\quad A = LU$

Gaussian elimination w/o pivoting computes an LU factorisation

Write

$\alpha \qquad \qquad \vec{w}^{\top}$

$$\vec{v} \quad A = \left[\begin{array}{c|cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right] = \left[\begin{array}{c|c} \alpha & \vec{w}^{\top} \\ \hline \vec{v} & K \end{array}\right]$$

K

Gauss Elimination:

$$\begin{bmatrix} 1 & & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & & \\ -\frac{a_{31}}{a_{11}} & & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ -\frac{a_{n1}}{a_{11}} & & & & \ddots \; 1 \end{bmatrix} A = \left[\begin{array}{c|c} 1 & \\ \hline -\vec{v}/\alpha & I \end{array}\right] \left[\begin{array}{c|c} \alpha & \vec{w}^T \\ \hline \vec{v} & K \end{array}\right]$$

$$= \left[\begin{array}{c|c} \alpha & \vec{w}^T \\ \hline & K - \dfrac{\vec{v}\,\vec{w}^T}{\alpha} \end{array}\right]$$

$$\Rightarrow \quad A = \left[\begin{array}{c|c} 1 & \\ \hline \vec{v}/\alpha & I \end{array}\right] \left[\begin{array}{c|c} \alpha & \vec{w}^T \\ \hline & A \end{array}\right] = \begin{bmatrix} 1 & \\ \vec{v}/\alpha & L \end{bmatrix} \begin{bmatrix} \alpha & \vec{w}^T \\ & U \end{bmatrix}$$

L_1

suppose $= \tilde{L}\tilde{U}$

L $\quad U$

Example 12 (LU by-hand).

α_1 $\quad \vec{w}_1^T$

$$A = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 2 & 4 & 8 \\ 1 & 4 & 9 \end{array}\right]$$

$\vec{v}_1$ $\quad K_1$

$$A = \left[\begin{array}{c|c} 1 & \\ \hline \vec{v}_1/\alpha_1 & I \end{array}\right]\left[\begin{array}{c|c} \alpha_1 & \vec{w}_1^T \\ \hline & A_2 \end{array}\right]$$

where

$$A_2 = K_1 - \frac{\vec{v}_1\vec{w}_1^T}{\alpha} = \begin{bmatrix} 4 & 8 \\ 4 & 9 \end{bmatrix} - \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{v}_1/\alpha_1}\underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\vec{w}_1^T}$$

α_2 $\quad w_2$

$$= \begin{bmatrix} 2 & 6 \\ \underset{v_2}{3} & \underset{k_2}{8} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \\ v_2/\alpha_2 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} \alpha_2 & w_2 \\ & A_3 \end{bmatrix}}_{U_2}$$

where

$$A_3 = k_2 - \frac{v_2 w_2}{\alpha_2} = 8 - \frac{3}{2} \times 6 = -1$$

so that

$$A = \begin{bmatrix} 1 & \\ \vec{v}_1/\alpha_1 & I \end{bmatrix} \begin{bmatrix} \alpha_1 & w_1^T \\ & \underbrace{A_2}_{L_2 U_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \\ \vec{v}_1/\alpha_1 & L_2 \end{bmatrix} \begin{bmatrix} \alpha_1 & w_1^T \\ & U_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}1 & & \\ 2 & 1 & \\ 1 & 3/2 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1 & 1 & 1\\ & 2 & 6\\ & & -1\end{bmatrix}}_{U}$$

III.2.3 PLU factorisation $\quad A = P^T LU$

Gaussian elimination w/ pivoting is a PLU factorisation

Permutation matrices:

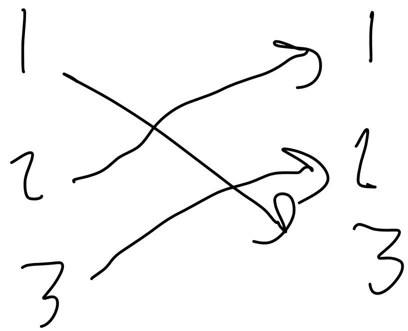
Permutations are $\overbrace{\text{bijections}}^{\text{1-to-1}}$ from $\{1, 2, \ldots, n\}$ to itself. A permutation σ can be written in Cauchy form

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix}$$

Eg

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

reprents



We can identify $\overset{\text{permutation}}{\sigma}$ w/ $\overset{\text{vector}}{\vec{\sigma}}$

$$\vec{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{bmatrix} \in \mathbb{Z}^n$$

this acts on vectors via

$$\underbrace{\vec{v}}_{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}[\vec{\sigma}] = \begin{bmatrix} v_{\sigma_1} \\ v_{\sigma_2} \\ \vdots \\ v_{\sigma_n} \end{bmatrix}$$

This is linear!

$$(\alpha \vec{v} + \beta \vec{w})[\vec{\sigma}] = \alpha \vec{v}[\vec{\sigma}] + \beta \vec{w}[\sigma]$$

$\Rightarrow \exists$ matrix P_σ s.t. $P_\sigma \vec{v} = \vec{v}[\vec{\sigma}]$

Eg, for $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ we have $P_\sigma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

$P_\sigma \vec{e}_1$

Theorem 5 (PLU). *A matrix $A \in \mathbb{C}^{n\times n}$ is invertible if and only if it has a PLU decomposition:*

$$A = P^\top LU$$

where the diagonal of L are all equal to 1 and the diagonal of U are all non-zero, and P is a permutation matrix.

Proof

Note $P^{-1} = P^T$ (see appendix),

PLU $\Rightarrow$ invertible

If $A = P^T L U$ then

$$A^{-1} = U^{-1} L^{-1} P.$$

Invert. $\Rightarrow$ PLU

$\exists\ k$ st $a_{k1} \neq 0$, eg

$k = \text{indmax}\ |a_{k1}|$.

Example 13 (PLU by-hand). Consider the matrix

a_{21} is largest

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

$P_1 = P_{(12)}$

α_1 $\qquad$ $\vec{w}_1^T$

$$\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} A = \left[\begin{array}{c|cc} 2 & 6 & 2 \\ \hline 0 & 2 & 1 \\ 1 & -1 & 5 \end{array}\right]$$

$\vec{v}_1$ $\qquad$ K_1

$$= \begin{bmatrix} 1 & \\ 0 & I \\ 1/2 & \end{bmatrix} \left[\begin{array}{c|cc} 2 & 6 & 2 \\ \hline & 2 & 1 \\ & -4 & 4 \end{array}\right]$$

A_2

Then

$$A_2 = \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix}$$

permute
largest entry

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P_2} A_2 = \begin{bmatrix} -4 & 4 \\ 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \\ -1/2 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} -4 & 4 \\ & 3 \end{bmatrix}}_{U_2}$$

$\Rightarrow$

$$\overbrace{\begin{bmatrix}1 & \\ & P_2\end{bmatrix} P_1}^{P} A = \begin{bmatrix}1 & \\ & P_2\end{bmatrix}\begin{bmatrix}1 & \\ \tilde{v}_1/\alpha_1 & I\end{bmatrix}\begin{bmatrix}\alpha_1 & \tilde{w}_1^T \\ & P_2^T L_2 U_2\end{bmatrix}$$

$$= \begin{bmatrix}1 & \\ P_2\tilde{v}_1/\alpha_1 & L_2\end{bmatrix}\begin{bmatrix}\alpha_1 & \tilde{w}_1^T \\ & U_2\end{bmatrix}$$

$$= \underbrace{\begin{bmatrix}1 & & \\ 1/2 & 1 & \\ 0 & -1/2 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}2 & 6 & 2 \\ & -4 & 4 \\ & & 3\end{bmatrix}}_{U}$$

where

$$P = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What's the fastest way to compute det?

$$\det A = \det(P^T L U) = \det P^T \underbrace{\det L}_{\text{prod of diag}} \underbrace{\det U}_{}$$

$O(n^3)$ operations.