MATH50003 Numerical Analysis

IV.2 Differential Equations via Finite Differences

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Part IV

Applications of Linear Algebra

1. Polynomial Interpolation and Regression for approximating data

2. Differential Equations via finite difference

IV.2.1 Indefinite Integration

Solve the simplest ODE replacing derivatives w/ divided differences

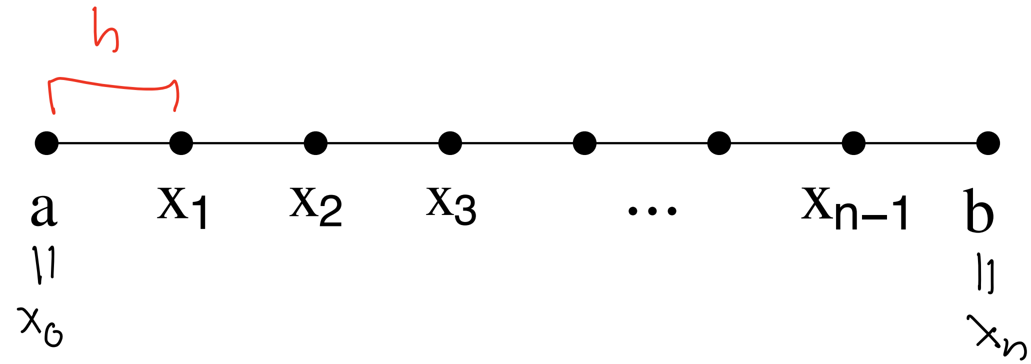
Indefinite integration can be thought of as an ODE:

$$
\begin{aligned}
u(a) &= c, \\
u'(x) &= f(x) \qquad \text{for} \qquad a \leq x \leq b.
\end{aligned}
$$

of course $u(x) = \int_a^x f(x)\,dx + c$, but let's ignore that since we want to generalise.

Idea: replace derivatives with divided differences.
Do so in 4 steps.

Step 1: ODE on interval → ODE on grid



$$\underbrace{\begin{aligned} u(a) &= c, \\ u'(x) &= f(x) \end{aligned}}_{\text{inf. dim problem}}$$

$$\underbrace{\begin{bmatrix} u(x_0) \\ u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}}_{\text{finite dim.}}$$

Stop at x_{n-1}.

Step 2: ODE on grid → Divided differences on grid

righted – sided

$$u'(x) \approx \frac{u(x+h) - u(x)}{h} \Rightarrow$$

$$u'(x_j) \approx \frac{u(x_j+h) - u(x_j)}{h} = \frac{u(x_{j+1}) - u(x_j)}{h}$$

$$\begin{bmatrix} u(x_0) \\ u'(x_0) \\ u'(x_1) \\ \vdots \\ u'(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}} \quad \Rightarrow \quad \begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h \\ (u(x_2) - u(x_1))/h \\ \vdots \\ (u(x_n) - u(x_{n-1})/h \end{bmatrix} \approx \boldsymbol{b}$$

Step 3: Divided differences on grid → Discrete system

New unknowns $\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}$ so that (hopefully) $u_i \approx u(x_i)$

$$\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h \\ (u(x_2) - u(x_1))/h \\ \vdots \\ (u(x_n) - u(x_{n-1})/h \end{bmatrix} \approx \boldsymbol{b} \quad \Longrightarrow \quad \begin{bmatrix} u_0 \\ (u_1 - u_0)/h \\ (u_2 - u_1)/h \\ \vdots \\ (u_n - u_{n-1})/h \end{bmatrix} = \boldsymbol{b}$$

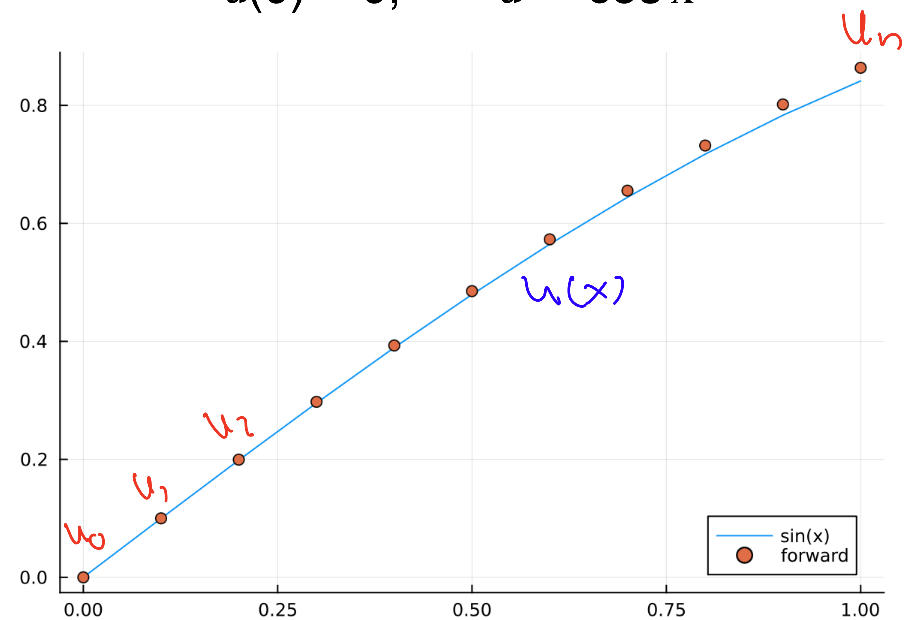
Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ (u_1 - u_0)/h \\ (u_2 - u_1)/h \\ \vdots \\ (u_n - u_{n-1})/h \end{bmatrix} = \boldsymbol{b}$$

$$\underbrace{\begin{bmatrix} 1 & & & \\ -1/h & 1/h & & \\ & \ddots & \ddots & \\ & & -1/h & 1/h \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$

$$\boldsymbol{b} = \begin{bmatrix} c \\ f_0 \\ \vdots \\ f_{n-1} \end{bmatrix}$$

u(0) = 0, u′ = cos x



Solve it using forward substition in $O(n)$ ops thereby find u_j

IV.2.1 Forward Euler

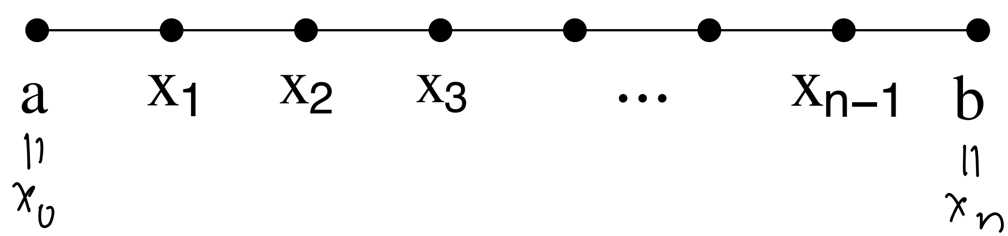
Generalise to first order linear ODEs

Consider general first order linear ODEs:

$$
\begin{aligned}
u(a) &= c \\
u'(x) - \omega(x)u(x) &= f(x)
\end{aligned}
$$

Repeat 4 steps as before.

Step 1: ODE on interval → ODE on grid



$$\underbrace{\begin{aligned} u(a) &= c \\ u'(x) - \omega(x)u(x) &= f(x) \end{aligned}}_{\infty\text{-dim}}$$

$$\underbrace{\begin{bmatrix} u(x_0) \\ u'(x_0) + \omega(x_0)u(x_0) \\ u'(x_1) + \omega(x_1)u(x_1) \\ \vdots \\ u'(x_{n-1}) + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}}}_{\text{finite dim}}$$

Step 2: ODE on grid → Divided differences on grid

$$u'(x_j) \approx \frac{u(x_{j+1}) - u(x_j)}{h}$$

$$\begin{bmatrix} u(x_0) \\ u'(x_0) + \omega(x_0)u(x_0) \\ u'(x_1) + \omega(x_1)u(x_1) \\ \vdots \\ u'(x_{n-1}) + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \end{bmatrix}}_{\boldsymbol{b}} \quad \Rightarrow \quad \begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h + \omega(x_0)u(x_0) \\ (u(x_2) - u(x_1))/h + \omega(x_1)u(x_1) \\ \vdots \\ \underbrace{(u(x_n) - u(x_{n-1}))/h}_{\text{right-sided divided diff}} + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} \approx \boldsymbol{b}$$

Step 3: Divided differences on grid → Discrete system

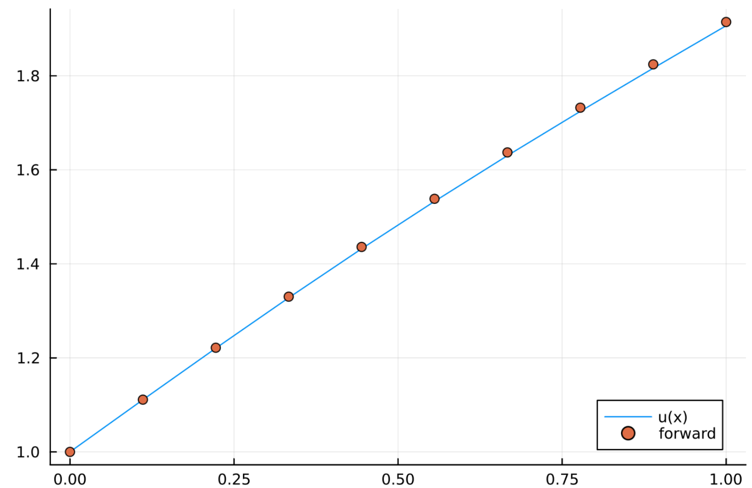
Now unknowns $u(x_j) \approx u_j$

$$\begin{bmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h + \omega(x_0)u(x_0) \\ (u(x_2) - u(x_1))/h + \omega(x_1)u(x_1) \\ \vdots \\ (u(x_n) - u(x_{n-1}))/h + \omega(x_{n-1})u(x_{n-1}) \end{bmatrix} \approx \boldsymbol{b} \quad \Rightarrow \quad \begin{bmatrix} u_0 \\ (u_1 - u_0)/h + \omega(x_0)u_0 \\ (u_2 - u_1)/h + \omega(x_1)u_1 \\ \vdots \\ (u_n - u_{n-1})/h + \omega(x_{n-1})u_{n-1} \end{bmatrix} = \boldsymbol{b}$$

Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ (u_1 - u_0)/h + \omega(x_0)u_0 \\ (u_2 - u_1)/h + \omega(x_1)u_1 \\ \vdots \\ (u_n - u_{n-1})/h + \omega(x_{n-1})u_{n-1} \end{bmatrix} = \boldsymbol{b}$$

$$u(0) = 1, u' + xu = \mathrm{e}^x$$



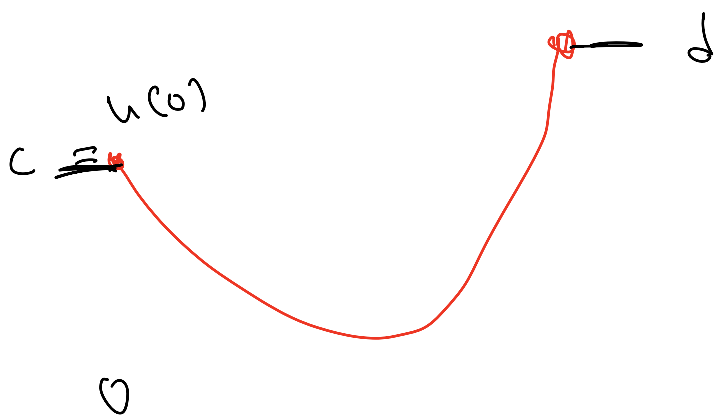
$$\underbrace{\begin{bmatrix} 1 & & & \\ \omega(x_0) - 1/h & 1/h & & \\ & \ddots & \ddots & \\ & & \omega(x_{n-1}) - 1/h & 1/h \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$

Then solve via forward subs.

IV.2.3 Poisson

Use second order divided differences

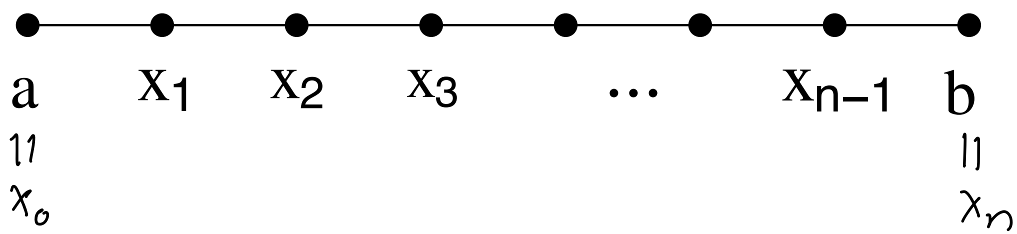
Consider the simplest second order ODE with boundary conditions:



$$
\begin{aligned}
u(0) &= c, \\
u''(x) &= f(x), \\
u(1) &= d
\end{aligned}
$$

Repeat 4 steps as before but with second-order divided differences and one more boundary condition.

Step 1: ODE on interval → ODE on grid



$$
\begin{aligned}
u(0) &= c, \\
u''(x) &= f(x), \\
u(1) &= d
\end{aligned}
$$

$$
\begin{bmatrix} u(x_0) \\ u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{n-1}) \\ u(x_n) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ d \end{bmatrix}}_{\boldsymbol{b}}
$$

just use interior x_i

Step 2: ODE on grid → Divided differences on grid

2nd order divided diff:

$$u''(x_j) \approx \frac{u(x_j-h)-2u(x_j)+u(x_j+h)}{h^2} = \frac{u(x_{j-1})-2u(x_j)+u(x_{j+1})}{h^2}$$

$$\begin{bmatrix} u(x_0) \\ u''(x_1) \\ u''(x_2) \\ \vdots \\ u''(x_{n-1}) \\ u(x_n) \end{bmatrix} = \underbrace{\begin{bmatrix} c \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \\ d \end{bmatrix}}_{\boldsymbol{b}} \quad \Longrightarrow \quad \begin{bmatrix} u(x_0) \\ \frac{u(x_0)-2u(x_1)+u(x_2)}{h^2} \\ \frac{u(x_1)-2u(x_2)+u(x_3)}{h^2} \\ \vdots \\ \frac{u(x_{n-2})-2u(x_{n-1})+u(x_n)}{h^2} \\ u(x_n) \end{bmatrix} \approx \boldsymbol{b}$$

Step 3: Divided differences on grid → Discrete system

Again $u_i \approx u(x_i)$

$$\begin{bmatrix} u(x_0) \\ \frac{u(x_0)-2u(x_1)+u(x_2)}{h^2} \\ \frac{u(x_1)-2u(x_2)+u(x_3)}{h^2} \\ \vdots \\ \frac{u(x_{n-2})-2u(x_{n-1})+u(x_n)}{h^2} \\ u(x_n) \end{bmatrix} \approx \boldsymbol{b} \quad \Longrightarrow \quad \begin{bmatrix} u_0 \\ \frac{u_0-2u_1+u_2}{h^2} \\ \frac{u_1-2u_2+u_3}{h^2} \\ \vdots \\ \frac{u_{n-2}-2u_{n-1}+u_n}{h^2} \\ u_n \end{bmatrix} = \boldsymbol{b}$$

Step 4: Discrete system → Linear system

$$\begin{bmatrix} u_0 \\ \frac{u_0-2u_1+u_2}{h^2} \\ \frac{u_1-2u_2+u_3}{h^2} \\ \vdots \\ \frac{u_{n-2}-2u_{n-1}+u_n}{h^2} \\ u_n \end{bmatrix} = \boldsymbol{b}$$

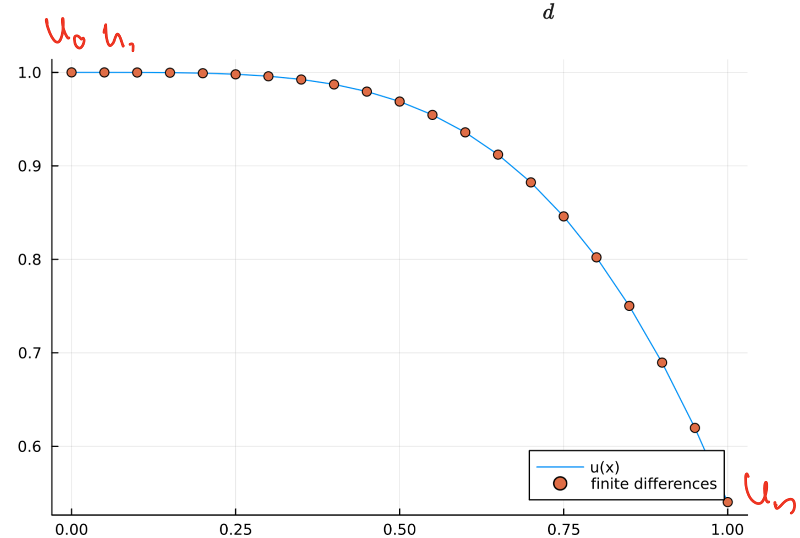
$$\Downarrow$$

$$\underbrace{\begin{bmatrix} 1 & & & & \\ 1/h^2 & -2/h^2 & 1/h & & \\ & \ddots & \ddots & \ddots & \\ & & 1/h^2 & -2/h^2 & 1/h \\ & & & & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}}_{\boldsymbol{u}} = \boldsymbol{b}$$

$$u(0) = \underbrace{1}_{c}$$

$$u''(x) = \underbrace{-4x^2\cos(x^2) - 2\sin(x^2)}_{f(x)}$$

$$u(1) = \underbrace{\cos 1}_{d}$$



A is tridiagonal, can compute LU, QR

and solve in $O(n)$ operations,

$$\overbrace{\hspace{10cm}}$$

Computer Examinable Material! Week 1 — today