Numerical Analysis MATH50003 (2024–25) Problem Sheet 7

Problem 1 Use Lagrange interpolation to interpolate the function $\cos x$ by a polynomial at the points $[0, 2, 3, 4]$ and evaluate at $x = 1$.

Problem 2 Compute the LU factorisation of the following transposed Vandermonde matrices:

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ x^3 & y^3 & z^3 & t^3 \end{bmatrix}$$

Can you spot a pattern? Test your conjecture with a 5×5 Vandermonde matrix.

Problem 3 Compute the interpolatory quadrature rule

$$\int_{-1}^1 f(x)w(x)\mathrm{d}x \approx \sum_{j=1}^n w_j f(x_j)$$

for the points $[x_1, x_2, x_3] = [-1, 1/2, 1]$, for the weights $w(x) = 1$ and $w(x) = \sqrt{1 - x^2}$.

Problem 4 Derive Backward Euler: use the left-sided divided difference approximation

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}$$

to reduce the first order ODE

$$u(a) = c, \qquad u'(x) + \omega(x)u(x) = f(x)$$

to a lower triangular system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$. Hint: only impose the ODE on the gridpoints $x_1, \ldots, x_n$ so that the divided difference does not depend on behaviour at x_{-1}.

Problem 5 Reduce a Schrödinger equation to a tridiagonal linear system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$:

$$u(a) = c, \qquad u''(x) + V(x)u(x) = f(x), \qquad u(b) = d.$$