

## Numerical Analysis MATH50003 (2024–25) Problem Sheet 2

**Problem 1** Using dual number arithmetic, compute the following polynomials evaluated at the dual number  $2 + \epsilon$  and use this to deduce their derivative at 2:

$$2x^2 + 3x + 4, (x + 1)(x + 2)(x + 3), (2x + 1)x^3.$$

**Problem 2** What should the following functions applied to dual numbers return for  $x = a + b\epsilon$ :

$$f(x) = x^{100} + 1, g(x) = 1/x, h(x) = \tan x.$$

**Problem 3(a)** What is the correct definition of division on dual numbers, i.e., for what choice of  $s$  and  $t$  does the following hold:

$$(a + b\epsilon)/(c + d\epsilon) = s + t\epsilon.$$

**Problem 3(b)** A *field* is a commutative ring such that  $0 \neq 1$  and all nonzero elements have a multiplicative inverse, i.e., there exists  $a^{-1}$  such that  $aa^{-1} = 1$ . Can we use the previous part to define  $a^{-1} := 1/a$  to make  $\mathbb{D}$  a field? Why or why not?

**Problem 4** Use dual numbers to compute the derivative of the following functions at  $x = 0.1$ :

$$\exp(\exp x \cos x + \sin x), \prod_{k=1}^3 \left( \frac{x}{k} - 1 \right), \text{ and } f_2^s(x) = 1 + \frac{x-1}{2 + \frac{x-1}{2}}$$

Consider a 2D analogue of dual numbers  $a + b\epsilon_x + c\epsilon_y$  defined by the relationship  $\epsilon_x\epsilon_y = \epsilon_x^2 = \epsilon_y^2 = 0$ .

**Problem 5(a)** Derive the formula for writing the product of two 2D dual numbers  $(a + a_x\epsilon_x + a_y\epsilon_y)(b + b_x\epsilon_x + b_y\epsilon_y)$  where  $a, a_x, a_y, b, b_x, b_y \in \mathbb{R}$  as a 2D dual number.

**Problem 5(b)** Show for all 2D polynomials

$$p(x, y) = \sum_{k=0}^n \sum_{j=0}^m c_{kj} x^k y^j$$

that

$$p(x + a\epsilon_x, y + b\epsilon_y) = p(x, y) + a \frac{\partial p}{\partial x} \epsilon_x + b \frac{\partial p}{\partial y} \epsilon_y.$$

**Problem 5(c)** Use 2D dual numbers to compute the gradient of  $p(x, y) = (1 + x + 3xy)(1 + y)$  at  $x = 1$  and  $y = 2$ .

**Problem 6** Suppose  $f$  is twice-differentiable in a neighbourhood of  $B$  of  $r$  such that  $f(r) = f'(r) = 0$ , where  $f''$  does in  $B$ . Show that the error of the  $k$ -th Newton iteration  $\epsilon_k := x - x_k$  satisfies

$$|\epsilon_{k+1}| \leq M |\epsilon_k|$$

where

$$M \doteq \frac{1}{2} \sup_{y \in B} |f''(y)| \sup_{y \in B} \frac{1}{|f''(y)|}.$$