Numerical Analysis MATH50003 (2023–24) Problem Sheet 10

Problem 1 What are the upper 3×3 sub-block of the multiplication matrix X / Jacobi matrix J for the monic and orthonormal polynomials with respect to the following weights on $[-1, 1]$:

$$1 - x, \sqrt{1 - x^2}, 1 - x^2$$

Problem 2 Compute the roots of the Legendre polynomial $P_3(x)$, orthogonal with respect to $w(x) = 1$ on $[-1, 1]$, by computing the eigenvalues of a 3×3 truncation of the Jacobi matrix.

Problem 3 Compute the 2-point interpolatory quadrature rule associated with roots of orthogonal polynomials for the weights $\sqrt{1 - x^2}$, 1, and $1 - x$ on $[-1, 1]$ by integrating the Lagrange bases.

Problem 4(a) For the matrix

$$J_n = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/2 \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots & 1/2 \\ & & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

use the relationship with the Jacobi matrix associated with $T_n(x)$ to prove that, for $x_j = \cos \theta_j$, and $\theta_j = (n - j + 1/2)\pi/n$,

$$J_n = Q_n \begin{bmatrix} x_1 \\ & \ddots \\ & & x_n \end{bmatrix} Q_n^\top$$

where

$$\boldsymbol{e}_1^\top Q_n \boldsymbol{e}_j = \frac{1}{\sqrt{n}}, \qquad \boldsymbol{e}_k^\top Q_n \boldsymbol{e}_j = \sqrt{\frac{2}{n}} \cos(k-1)\theta_j.$$

You may use without proof the sums-of-squares formula

$$1 + 2\sum_{k=1}^{n-1} \cos^2 k\theta_j = n.$$

Problem 4(b) Show for $w(x) = 1/\sqrt{1 - x^2}$ that the Gaussian quadrature rule is

$$Q_n^w[f] = \frac{\pi}{n} \sum_{j=1}^{n} f(x_j)$$

where $x_j = \cos \theta_j$ for $\theta_j = (j - 1/2)\pi/n$.

Problem 4(c) Give an explicit formula for the polynomial that interpolates $\exp x$ at the points $x_1, \ldots, x_n$ as defined above, in terms of Chebyshev polynomials with the coefficients defined in terms of a sum involving only exponentials, cosines and $\theta_j = (n - j + 1/2)\pi/n$.