MATH50003
Numerical Analysis

II.3 Floating Point Arithmetic

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Part II

Representing Numbers

1. Reals via floating point
2. Floating point arithmetic and bounding errors
3. Interval arithmetic for rigorous computations

$$x = \pm 2^{q-\sigma}(1.b_1 b_2 \text{---} b_S b_{S+1} b_{S+2} \text{---})_2$$

Rounding

How does a computer round a real to a float?

Definition 7 (rounding). $\mathrm{fl}^{\mathrm{up}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$

$$\mathrm{fl}^{up}(x) := \min\{y \in F \text{ s.t. } y \geq x\}$$

Eg

$$\mathrm{fl}^{up}_{\underbrace{16}_{15,5,10}}(1/3) = \mathrm{fl}^{up}(2^{-2} \times (1.0101010101\,0101\text{---})_2)$$

$$= 2^{-2} \times (1.0101010110)_2$$

$\mathrm{fl}^{\mathrm{down}}_{\sigma,Q,S} : \mathbb{R} \to F_{\sigma,Q,S}$

$$\mathrm{fl}^{down}(x) := \max\{y \in F \text{ s.t. } y \leq x\}$$

Eg

$$\mathrm{fl}^{down}(1/3) = 2^{-2}(1.0101010101)_2$$

Note

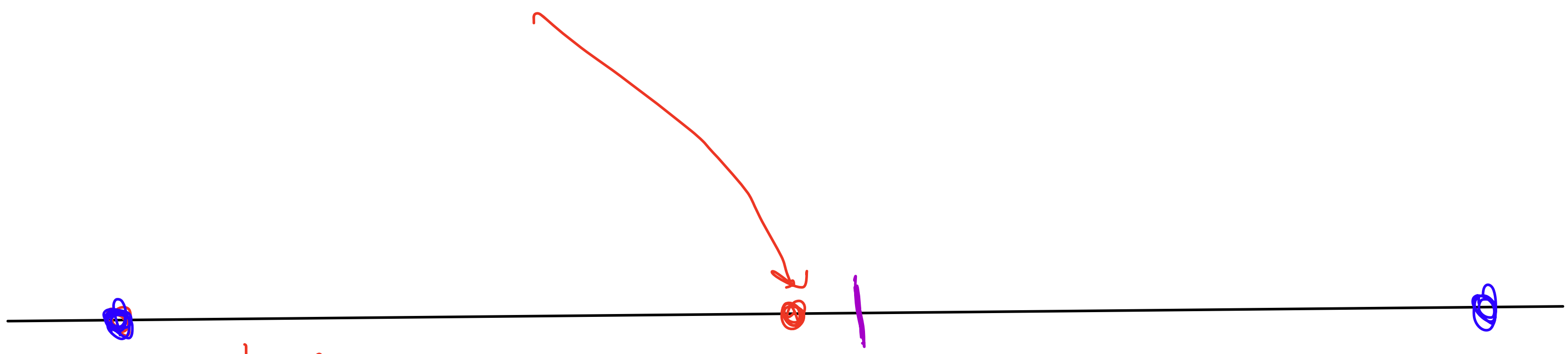
$$\mathrm{fl}^{up}(-x) = -\mathrm{fl}^{down}(x)$$

Default is round-to-nearest.

$\underbrace{\mathrm{fl}^{\text{nearest}}_{\sigma,Q,S}}_{=:\,\mathrm{fl}} : \mathbb{R} \to F_{\sigma,Q,S}$

Consider

$$x = 2^{q-\sigma}(1.b_1 b_2 \ldots b_S\, b_{S+1}\, b_{S+2} \ldots)_2 > 0$$



$$x_- = \mathrm{fl}^{\text{down}}(x) = 2^{q-\sigma}(1.b_1 \ldots b_S)_2$$

$$x_h = \frac{x_+ + x_-}{2} = x_- + 2^{q-\sigma-S-1} = 2^{q-\sigma}(1.b_1 \ldots b_S\, 1)_2$$

$$x_+ = \mathrm{fl}^{\text{up}}(x) = x_- + 2^{q-\sigma-S}$$

Different cases:

(down) $x_- \le x < x_h \Rightarrow \mathrm{fl}^{\text{near}}(x) = \mathrm{fl}^{\text{down}}(x) = x_-$

up $\quad x_b < x \le x_+ \Rightarrow fl^{near}(x) = fl^{up}(x) = x_+$

half $\quad x = x_b \Rightarrow fl^{near}(x) = \begin{cases} x_- & \text{if } b_s = 0 \\ x_+ & \text{otherwise} \end{cases}$

so that $fl^{near}(x)$ has last bit $\emptyset$.

$$\boxed{\begin{array}{l} \text{For } x < 0 \\ fl(x) = -fl(-x) \end{array}}$$

Arithmetic

Operations are exact up to rounding

$$x \oplus y := \mathrm{fl}(x + y)$$
$$x \ominus y := \mathrm{fl}(x - y)$$
$$x \otimes y := \mathrm{fl}(x * y)$$
$$x \oslash y := \mathrm{fl}(x / y)$$

"+" on a computer adds exactly, then rounds

Eg: in F_{16},

$$1 \oplus 2^{-11} = \mathrm{fl}(1 + 2^{-11})$$
$$= \mathrm{fl}((1.00000000001)_2)$$
$$= 1$$

$$x \,\hat{\circ}\, y := \mathrm{fl}(x^y)$$

Here, $x, y \in F$, not general reals!

Example 8 (decimal is not exact).

What's $1.1 + 0.1$ on a computer?

When we type "1.1" it makes $fl(1.1)$.

So this is

$$fl(1.1) \oplus fl(0.1)$$

We have

$b_{s+1} = 0 \Rightarrow$ round down

$$fl(1.1) = fl((1.0001100110\,01100\ldots)_2)$$

$$= (1.0001100110)_2$$

$b_{s+1} = 0 \Rightarrow$ round down

$$fl(0.1) = fl(2^{-4}(1.1001100110\,0110\ldots)_2)$$

$$= 2^{-4}(1.1001100110)_2$$

$\Rightarrow$

$$fl(1.1) \oplus fl(0.1) = fl(fl(1.1) + fl(0.1))$$

$$= fl((1.0011001100\,011)_2)$$

round down

$$= (1.0011001 1)_2$$

But

$$fl(1.2) = fl(1.0011001100\,11\,0\,0\,11\ldots)_2$$

$b_{s+1} = 1$

round up

$$= (1.0011001101)_2$$

II.2.1 Bounding errors
Analysis on rounding errors

Definition 8 (machine epsilon/smallest positive normal number/largest normal number)**.** *Machine epsilon* is denoted

$$\epsilon_{\mathrm{m},S} := 2^{-S}.$$

For F_{64}, $S = 52 \Rightarrow \epsilon_m = 2^{-52}$

$\approx 2.22 \times 10^{-16}$

Definition 9 (normalised range)**.** The *normalised range* $\mathcal{N}_{\sigma,Q,S} \subset \mathbb{R}$ is the subset of real numbers that lies between the smallest and largest normal floating-point number:

$$\mathcal{N}_{\sigma,Q,S} := \{x : \underbrace{\min |F^{\text{normal}}_{\sigma,Q,S}|}_{\text{smallest positive normal}} \leq |x| \leq \underbrace{\max F^{\text{normal}}_{\sigma,Q,S}}_{\text{largest normal}}\}$$

$$x \in N_{\sigma,Q,S} \Rightarrow fl^{\text{mode}}(x) \in F^{\text{normal}}$$

relative error

Proposition 2 (round bound). *If $x \in \mathcal{N}$ then*

$$\mathrm{fl}^{\mathrm{mode}}(x) = x(1 + \delta_x^{\mathrm{mode}}) = x + \underbrace{x\,\delta_x^{\mathrm{mode}}}$$

where the relative error *is bounded by:*

machine $\epsilon \approx 2.22 \times 10^{-16}$

$$|\delta_x^{\mathrm{nearest}}| \leq \frac{\epsilon_{\mathrm{m}}}{2}$$

$$|\delta_x^{\mathrm{up/down}}| < \epsilon_{\mathrm{m}}.$$

Proof Only show mode = near. Assume $x > 0$.

Suppose $fl(x) = fl^{\text{down}}(x)$, i.e., $x_- \le x \le x_h$.

(Round Down)

$x = 2^{q-\sigma}(1.b_1 b_2 \ldots b_S b_{S+1} \ldots)_2$

$2^{q-\sigma}$

$x_- := \mathrm{fl}^{\text{down}}(x)$
$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S)_2$

$x_h := \dfrac{x_+ + x_-}{2}$
$= x_- + 2^{q-\sigma-S-1}$
$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S 1)_2$

$x_+ := \mathrm{fl}^{\text{up}}(x)$
$= x_- + 2^{q-\sigma-S}$

We have

$$fl(x) = x_- = x\left(1 + \underbrace{\frac{x_- - x}{x}}_{\delta_x}\right)$$

Where

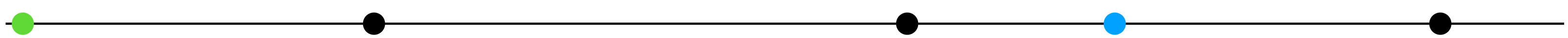
$$|\delta_x| \le \frac{x - x_-}{x} \underset{\text{since } x \le x_h}{\le} \frac{x_h - x_-}{x} \underset{x \ge x_-}{\le} \frac{x_h - x_-}{x_-}$$

$$\leq \frac{2^{q-\sigma-S-1}}{2^{q-\sigma}} = 2^{-S-1} = \frac{\epsilon_m}{2}$$

$\Uparrow$

(Round Up) $x_- \geq 2^{q-\sigma}$

$x = 2^{q-\sigma}(1.b_1 b_2 \ldots b_S b_{S+1} \ldots)_2$



$2^{q-\sigma}$

$x_- := \mathrm{fl}^{\mathrm{down}}(x)$

$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S)_2$

$x_{\mathrm{h}} := \dfrac{x_+ + x_-}{2}$

$= x_- + 2^{q-\sigma-S-1}$

$= 2^{q-\sigma}(1.b_1 b_2 \ldots b_S 1)_2$

$x_+ := \mathrm{fl}^{\mathrm{up}}(x)$

$= x_- + 2^{q-\sigma-S}$

Example 9 (bounding a simple computation).

$$(1.1 + 1.2) \times 1.3 = 2.99$$

What's a bound on error when done on computer?

$$\underbrace{[fl(1.1) \oplus fl(1.2)]}_{(*)} \otimes fl(1.3) = 2.99 + \underbrace{\delta}_{\text{absolute error}}$$

Show

$$|\delta| \le \underbrace{23\,\epsilon_m}_{\text{pecimistic bound}} \le 50 \times 10^{-16}$$

$$fl(\underbrace{1.1}_{\in N_{64,s}}) \overset{\text{round bound}}{=} 1.1\,(1 + \underbrace{\delta_1}_{|\delta_1| \le \frac{\epsilon_m}{2}})$$

$$fl(1.2) = 1.2(1+\delta_2)$$

$|\delta_2| \le \frac{\epsilon_m}{2}$

$$\Rightarrow$$

$$(*) = fl(\underbrace{1.1(1+\delta_1)}_{fl(1.1)} + 1.2(1+\delta_2))$$

from round bound

$$= (2.3 + 1.1\delta_1 + 1.2\delta_2)(1+\delta_3)$$

$|\delta_3| \le \frac{\epsilon_m}{2}$

$$= 2.3 + \underbrace{1.1\delta_1 + 1.2\delta_2 + 2.3\delta_3 + 1.1\delta_1\delta_3 + 1.2\delta_2\delta_3}_{=:\ \epsilon_1}$$

true

1.1 + 1.2

Where

$$|\epsilon_1| \le (\underbrace{1.1}_{\le 2} + \underbrace{1.2}_{\le 2} + \underbrace{2.3}_{\le 3})\frac{\epsilon_m}{2} + (\underbrace{1.1}_{\le 2} + \underbrace{1.2}_{\le 2})\underbrace{\frac{\epsilon_m^2}{4}}_{\le \frac{\epsilon_m}{4}2^{-5} \le \frac{\epsilon_m}{4}}$$

$$\le (2+2+3+1+1)\frac{\epsilon_m}{2} \le 5\epsilon_m.$$

$\Rightarrow$

error from $\otimes$

$$(*) \quad \otimes \quad \underbrace{fl(1.3)}_{1.3(1+\delta_4)} = (2.3+\epsilon_1)\,1.3(1+\delta_4)(1+\delta_5)$$

$$= 2.99 + \underbrace{1.3}_{\le 3/2}(\epsilon_1 + \underbrace{2.3}_{\le 3}\delta_4 + \underbrace{2.3}_{\le 3}\delta_5 + \underbrace{\epsilon_1\delta_4}_{\le \frac{5\epsilon_m^2}{2} \le \frac{5}{2}\epsilon_m} + \underbrace{\epsilon_1\delta_5}_{\le \frac{5\epsilon_m}{2}} + \underbrace{2.3}_{\le 3}\underbrace{\delta_4\delta_5}_{\le \frac{\epsilon_m}{4}} + \underbrace{\epsilon_1\delta_4\delta_5}_{\le \frac{5\epsilon_m}{4}})$$

δ

where $|\delta| \leq \cancel{2}\,3\,\epsilon_m$

II.2.2 Idealised floating point

A simplifed model for analysis

Definition 10 (idealised floating point)**.** An idealised mathematical model of floating point numbers for which the only subnormal number is zero can be defined as:

$$F_{\infty,S} := \{\pm 2^q \times (1.\underbrace{b_1 b_2 b_3 \ldots b_S}_{S \text{ digits}})_2 : q \in \mathbb{Z}\} \cup \{0\}$$

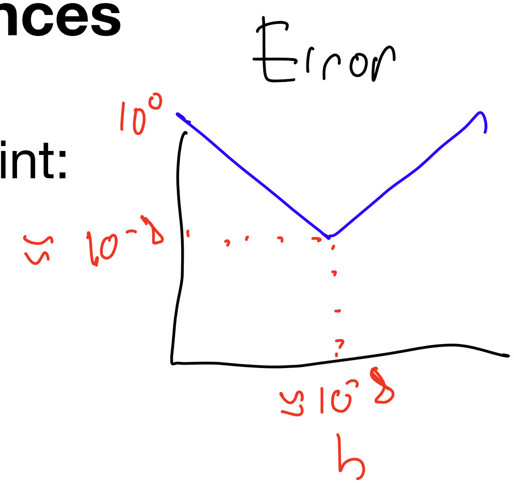
$Q=\infty$

anything

Round bound applies $\forall x \in F_{\infty,S}$

II.2.3 Divided differences floating point error bound

Explain the unexplained error in divided differences



General model of a function implemented in floating point:

$\forall x,$

$$f(x) = f^{\rm FP}(x) + \underbrace{\delta_x^f}_{\text{absolute error}}$$

such that

$$|\delta_x^f| \leq c\epsilon_{\rm m}$$

some reasonable c

Theorem 4 (divided difference error bound). For $x \in F_{\infty,S}$,

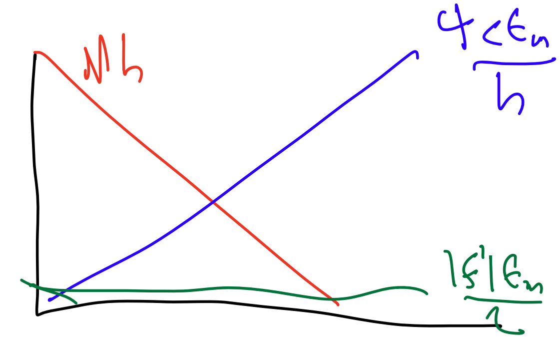
$$\frac{f^{\rm FP}(x+h) \ominus f^{\rm FP}(x)}{h} = f'(x) + \delta^{\rm FD}_{x,h}$$

abs error

where

$$|\delta^{\rm FD}_{x,h}| \leq \underbrace{\frac{|f'(x)|}{2}\epsilon_{\rm m}}_{\text{small } \approx 10^{-16}} + \underbrace{Mh}_{\to 0,\ h\to 0} + \underbrace{\frac{4c\epsilon_{\rm m}}{h}}_{\to \infty,\ h \to 0}$$

for $M = \sup_{x \leq t \leq x+h} |f''(t)|$.



Of course, bound is just upper bound so haven't shown it will blow up.

Proof

$$\frac{f^{FP}(x+h) \ominus f^{FP}(x)}{h} = \frac{\left(f(x+h) - \delta^f_{x+h} - f(x) + \delta^f_x\right)}{h}(1+\underbrace{\delta_1}_{|\delta_1| \leq \frac{\epsilon_m}{2}})$$

$$= \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{Prop 1}} (1+\delta_1) + \frac{\delta^f_x - \delta^f_{x+h}}{h} (1+\delta_1)$$

$$= f'(x) - \delta^{dd}$$

where

$$|\delta^{dd}| \leq \frac{Mh}{2}$$

$$= f'(x) + \underbrace{f'(x)\delta_1}_{|\cdot| \leq \frac{|f'(x)|}{2}\epsilon_m} - \underbrace{\delta^{dd}(1+\delta_1)}_{\leq |\delta^{dd}|(1+|\delta_1|) \leq Mh}$$

$$+ \underbrace{\frac{\delta^f_x - \delta^f_{x+h}}{h} (1+\delta_1)}$$

$$|\cdot| \leq \frac{|\delta_x^f| + |\delta_{x+h}^f|}{h}(1+|\delta_1|) \leq \frac{4 c \epsilon_m}{h}$$

Corollary 2 (divided differences in practice). *We have* (Non-examinable)

$$(f^{\mathrm{FP}}(x \oplus h) \ominus f^{\mathrm{FP}}(x)) \oslash h = \frac{f^{\mathrm{FP}}(x+h) \ominus f^{\mathrm{FP}}(x)}{h}$$

whenever $h = 2^{j-n}$ *for* $0 \leq n \leq S$ *and the last binary place of* $x \in F_{\infty,S}$ *is zero, that is* $x = \pm 2^j(1.b_1 \ldots b_{S-1}0)_2$.

No error in $\oslash$ since $\frac{\pm 2^q (1.b_1 \ldots b_S)_2}{2^{j-n}}$

$= \pm 2^{q+j-n}(1.b_1 \ldots b_S)_2 \in F_{\infty,S}$.

Also, since $b_S = 0$ we have

$x \oplus h = x + h$.

Heuristic (divided difference with floating-point step)

What's the best choice of h?

Balance

$$Mh \simeq \frac{4c\,\epsilon_m}{h} \Rightarrow$$

$$h^2 \simeq \frac{4c}{M}\epsilon_m$$

might not know

$$\Rightarrow \quad h = \text{const.}\ \sqrt{\epsilon_m}$$

$\hookrightarrow 1.49 \times 10^{-8}$

Now to Lab 3
To see rounding modes.