Why would a pure mathmos care?

MATH50003
Numerical Analysis

II.3 Interval Arithmetic

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Part II
Representing Numbers

1. Reals via floating point
2. Floating point arithmetic and bounding errors
3. Interval arithmetic for rigorous computations

II.4 Interval Arithmetic

Use set operations with rounding to prove rigorous bounds

Here

$X = [a, b]$

$Y = [c, d]$

For sets $\mathrm{X}, \mathrm{Y} \subseteq \mathbb{R}$ consider the set operations

$$
\begin{aligned}
X + Y &:= \{x + y : x \in X, y \in Y\}, \\
XY &:= \{xy : x \in X, y \in Y\}, \\
X/Y &:= \{x/y : x \in X, y \in Y\}
\end{aligned}
$$

We will use floating point arithmetic to define operations so that

$$
\begin{aligned}
X + Y &\subseteq X \oplus Y, \\
XY &\subseteq X \otimes Y, \\
X/Y &\subseteq \underbrace{X \oslash Y}_{\text{definable on a computer}}
\end{aligned}
$$

Proposition 3 (interval bounds)**.** *For intervals $X = [a, b]$ and $Y = [c, d]$ satisfying $0 < a \le b$ and $0 < c \le d$, and $n > 0$, we have:*

$$\begin{aligned} X + Y &= [a + c, b + d] \\ X/n &= [a/n, b/n] \\ XY &= [ac, bd] \end{aligned}$$

Proof Just for $X + Y$.

Need to show

$$z \in X + Y \iff z \in [a+c, b+d]$$

$\Rightarrow$ $z = x + y \in X + Y$ satisfies

$x \in X$ sat. $a \le x \le b$

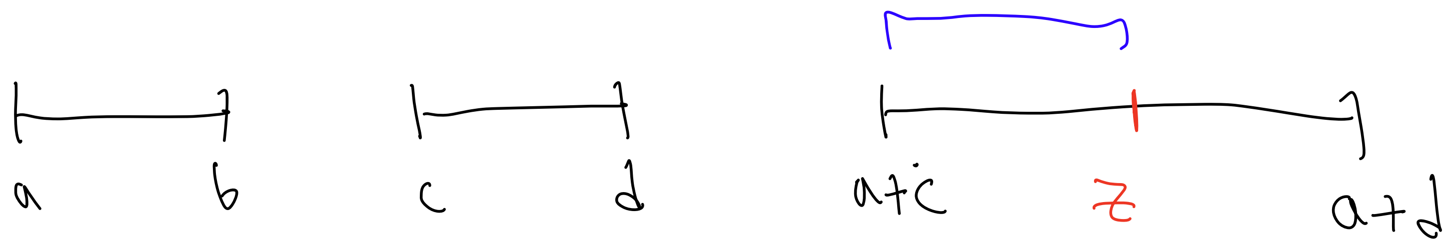
$c \le y \le d$

$$a + c \le x + y \le b + d \Rightarrow x + y \in [a+c, b+d]$$

$(\Longleftarrow)$ Use convexity:

$\exists\ 0 \le t \le 1$

$t[(b+d) - (a+c)]$



We have

$$z = a+c + t[(b+d) - (a+c)]$$

$$= \underbrace{(1-t)a + tb}_{\in X} + \underbrace{c(1-t) + td}_{\in Y}$$

$\Rightarrow\ z \in X + Y$

Definition 14 (floating point interval arithmetic)**.** For intervals $A = [a, b]$ and $B = [c, d]$ satisfying $0 < a \leq b$ and $0 < c \leq d$, and $n > 0$, define:

if $a, b, c, d \in \mathbb{F}$, computable on a computer.

$$[a, b] \oplus [c, d] := [\mathrm{fl}^{\mathrm{down}}(a + c), \mathrm{fl}^{\mathrm{up}}(b + d)]$$
$$[a, b] \ominus [c, d] := [\mathrm{fl}^{\mathrm{down}}(a - d), \mathrm{fl}^{\mathrm{up}}(b - c)]$$
$$[a, b] \oslash n := [\mathrm{fl}^{\mathrm{down}}(a/n), \mathrm{fl}^{\mathrm{up}}(b/n)]$$
$$[a, b] \otimes [c, d] := [\mathrm{fl}^{\mathrm{down}}(ac), \mathrm{fl}^{\mathrm{up}}(bd)]$$

Eg.

$$[a,b] + [c,d] = [\overbrace{a+c}^{\leq \mathrm{fl}^{\mathrm{down}}(a+c)}, \underbrace{b+d}_{\leq \mathrm{fl}^{\mathrm{up}}(b+d)}]$$

$$\subset [a,b] \oplus [c,d]$$

Example 11 (small sum). Can we compute digits of $e = 2.71\_\_$?

Consider

$$p(x) := 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = \sum_{k=0}^{3} \frac{x^k}{k!}$$

for $x = 1$, so that $p(1) \approx e$.

Let $X = [1, 1] = \{1\}$. We have

$$e \approx p(1) \in p(X) \subseteq ((1 \oplus X) \oplus ((X \,\textcircled{$\wedge$}\, 2) \oslash 2)) \oplus ((X \,\textcircled{$\wedge$}\, 3) \oslash 6)$$

Left-associative

Note

$$1 \oplus X = [\mathrm{fl}^{\mathrm{down}}(1+1), \mathrm{fl}^{\mathrm{up}}(1+1)]$$
$$= [2, 2]$$

Similarly:
$$X \,\textcircled{$\wedge$}\, 2 \oslash 2 = [1/2, 1/2]$$

and

$$(1 \oplus x) \oplus x \otimes 2 \oslash 2 =$$

$$[2, 2] \oplus [1/2, 1/2] = [5/2, 5/2]$$

But

$1/6 \in \quad \underbrace{\underset{[1,1]}{x} \otimes 3}_{[1,\, 1]} \oslash 6 = [fl^{down}(1/6), fl^{up}(1/6)]$

$$1/6 = \frac{1}{2} \times \frac{1}{3} = 2^{-3}(1.0101\ldots)_2$$

$$= [2^{-3} \times (1.0101010101)_2,$$
$$2^{-3} \times (1.0101010110)_2]$$

$$= [2 \times (0.0001010101\,0101)_2,$$

$$2 \times (0.0001010101\ 011)_2 \,\big]\,\Big) \quad ★$$

so that $\quad 5/2 = 2 + 2 \times \frac{1}{4}$

$$p(X) \subseteq \Big[\, \overbrace{2 \times (1.01)_2}, \; 2 \times (1.01)_2 \,\Big]$$

$$† \quad ★$$

$$= \Big[\, fl^{down}\big(2 \times (1.0101010101\ 0101)_2\big),$$

$$fl^{up}\big(2 \times (1.0101010101\ 011)_2\big) \Big]$$

$$= \Big[\, 2 \times (1.0101010101)_2,$$

$$2 \times (1.0101010110)_2 \Big] \approx [2.666, 2.668]$$

$\in p(1) = 2.6666\ldots$

Example 19 (exponential with intervals). $t \in [0,1]$

use this with interval arithmetic to bound

$$\exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\exp(t)\frac{x^{n+1}}{(n+1)!}}_{\delta_{x,n}}$$

$e := \exp(1)$

Here

$$e := \exp(1) \subseteq p(X) \oplus B$$

such that $\delta_{x,n} \in B$

Use $e \leq 3$ to show

$$|\delta_{x,n}| \leq \frac{e}{(n+1)!} \leq \frac{3}{(n+1)!} \Rightarrow$$

$$\delta_{x,n} \in B = \left[-3/(n+1)!\,,\; 3/(n+1)!\right]$$

$$\{e\} \subseteq \exp(X) \subseteq \overbrace{\left(\bigoplus_{k=0}^{n} X \mathbin{\bigtriangleup\!\!\!\!\bigcirc} k \oslash k!\right)}^{p(X)} \oplus \underbrace{\left[\mathrm{fl}^{\mathrm{down}}\left(-\frac{3}{(n+1)!}\right), \mathrm{fl}^{\mathrm{up}}\left(\frac{3}{(n+1)!}\right)\right]}_{\delta_{x,n} \in B \subseteq \tilde{B}}$$

$$\frac{3}{4!} = \frac{1}{2\times 4} = 2^{-3}$$

$$\overset{n=3}{=} \left[\simeq 2.666, \simeq \underline{2.668}\right] \oplus \left[-2^{-3}, \underbrace{2^{-3}}_{2\times(0.0001)_2}\right]$$

$$= \left[\cancel{\mathrm{fl}}^{\mathrm{down}}\left(2\times(1.0100010101)_2\right), \right.$$

$$\left. \cancel{\mathrm{fl}}^{\mathrm{up}}\left(2\times(1.0110010111)_2\right)\right]$$

exact float

$$\approx [2.541,\ 2.793] \ni e.$$

For $n > 3$: just use a computer.

Let’s implement Interval arithmetic in Lab 4.