Numerical Analysis MATH50003 (2024–25) Problem Sheet 1

Problem 1 Assuming f is differentiable on $[a, b]$ and its derivative is integrable, prove the left-point Rectangular rule error formula

$$\int_a^b f(x)\mathrm{d}x = h \sum_{j=0}^{n-1} f(x_j) + \delta$$

where $|\delta| \leq M(b-a)h$ for $M = \sup_{a \leq x \leq b} |f'(x)|$, $h = (b-a)/n$ and $x_j = a + jh$.

Problem 2(a) Assuming f is twice-differentiable on $[a, b]$ and its second derivative is integrable, prove a one-panel Trapezium rule error bound:

$$\int_a^b f(x)\mathrm{d}x = (b-a)\frac{f(a) + f(b)}{2} + \delta$$

where $|\delta| \leq M(b-a)^3$ for $M = \sup_{a \leq x \leq b} |f''(x)|$.

Hint: Recall from the notes

$$\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{b-a}\mathrm{d}x = (b-a)\frac{f(a) + f(b)}{2}$$

and you may need to use Taylor's theorem. Note that the bound is not sharp and so you may arrive at something sharper like $|\delta| \leq 3(b-a)^3 M/4$. The sharpest bound is $|\delta| \leq (b-a)^3 M/12$ but that would be a significantly harder challenge to show!

Problem 2(b) Assuming f is twice-differentiable on $[a, b]$ and its second derivative is integrable, prove a bound for the Trapezium rule error:

$$\int_a^b f(x)\mathrm{d}x = h\left[\frac{f(a)}{2} + \sum_{j=1}^{n-1} f(x_j) + \frac{f(b)}{2}\right] + \delta$$

where $|\delta| \leq M(b-a)h^2$ for $M = \sup_{a \leq x \leq b} |f''(x)|$.

Problem 3 Assuming f is twice-differentiable in $[x-h, x]$, for the left difference approximation

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \delta,$$

show that $|\delta| \leq Mh/2$ for $M = \sup_{x-h \leq t \leq x} |f''(t)|$.

Problem 4 Assuming f is thrice-differentiable in $[x-h, x+h]$, for the central differences approximation

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \delta,$$

show that $|\delta| \leq Mh^2/6$ for $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$.

Problem 5 Assuming f is thrice-differentiable in $[x-h, x+h]$, for the second-order derivative approximation

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + \delta$$

show that $|\delta| \leq Mh/3$ for $M = \sup_{x-h \leq t \leq x+h} |f'''(t)|$.