Numerical Analysis MATH50003 (2024–25) Problem Sheet 7

Problem 1 Use Lagrange interpolation to interpolate the function $\cos x$ by a polynomial at the points $[0,2,3,4]$ and evaluate at $x=1$.

SOLUTION

- $$\ell_1(x) = \frac{(x-2)(x-3)(x-4)}{(0-2)(0-3)(0-4)} = -\frac{1}{24}(x-2)(x-3)(x-4)$$
- $$\ell_2(x) = \frac{(x-0)(x-3)(x-4)}{(2-0)(2-3)(2-4)} = \frac{1}{4}x(x-3)(x-4)$$
- $$\ell_3(x) = \frac{(x-0)(x-2)(x-4)}{(3-0)(3-2)(3-4)} = -\frac{1}{3}x(x-2)(x-4)$$
- $$\ell_4(x) = \frac{(x-0)(x-2)(x-3)}{(4-0)(4-2)(4-3)} = \frac{1}{8}x(x-2)(x-3)$$

So that $p(x) = \cos(0)\ell_1(x) + \cos(2)\ell_2(x) + \cos(3)\ell_3(x) + \cos(4)\ell_4(x)$. Note that $\ell_0(1) = 1/4$, $\ell_2(1) = 3/2$, $\ell_3(1) = -1$, $\ell_4(1) = 1/4$, so $p(1) = 1/4\cos(0) + 3/2\cos(2) - \cos(3) + 1/4\cos(4)$.

END

Problem 2 Compute the LU factorisation of the following transposed Vandermonde matrices:

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ x^3 & y^3 & z^3 & t^3 \end{bmatrix}$$

Can you spot a pattern? Test your conjecture with a 5×5 Vandermonde matrix.

SOLUTION (1)

$$\begin{bmatrix} 1 & 1 \\ x & y \end{bmatrix} = \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & y-x \end{bmatrix}$$

(2)

$$V := \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ x & 1 & \\ x^2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & y-x & z-x \\ & y^2-x^2 & z^2-x^2 \end{bmatrix}$$

We then have

$$\begin{bmatrix} y-x & z-x \\ y^2-x^2 & z^2-x^2 \end{bmatrix} = \begin{bmatrix} 1 & \\ y+x & 1 \end{bmatrix} \begin{bmatrix} y-x & z-x \\ & (z-y)(z-x) \end{bmatrix}$$

since $z^2 - x^2 - (z-x)(y+x) = z^2 + xy - zy = (z-y)(z-x)$. Thus we have

$$V = \begin{bmatrix} 1 & & \\ x & 1 & \\ x^2 & x+y & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & y-x & z-x \\ & & (z-y)(z-x) \end{bmatrix}$$

(3)

$$V := \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ x^3 & y^3 & z^3 & t^3 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ x^2 & & 1 & \\ x^3 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & y-x & z-x & t-x \\ & y^2-x^2 & z^2-x^2 & t^2-x^2 \\ & y^3-x^3 & z^3-x^3 & t^3-x^3 \end{bmatrix}$$

We then have

$$\begin{bmatrix} y-x & z-x & t-x \\ y^2-x^2 & z^2-x^2 & t^2-x^2 \\ y^3-x^3 & z^3-x^3 & t^3-x^3 \end{bmatrix} = \begin{bmatrix} 1 & & \\ y+x & 1 & \\ y^2+xy+x^2 & & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} y-x & z-x & t-x \\ & (z-y)(z-x) & (t-y)(t-x) \\ & (z-x)(z-y)(x+y+z) & (t-x)(t-y)(x+y+t) \end{bmatrix}$$

since

$$z^3 - x^3 - (z-x)(y^2+xy+x^2) = z^3 - zy^2 - xyz - zx^2 + xy^2 + x^2y = (x-z)(y-z)(x+y+z).$$

Finally we have

$$\begin{bmatrix} (z-y)(z-x) & (t-y)(t-x) \\ (z-x)(z-y)(x+y+z) & (t-x)(t-y)(x+y+t) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \\ x+y+z & 1 \end{bmatrix} \begin{bmatrix} (z-y)(z-x) & (t-y)(t-x) \\ & (t-x)(t-y)(t-z) \end{bmatrix}$$

since

$$(t-x)(t-y)(x+y+t) - (x+y+z)(t-y)(t-x) = (t-y)(t-x)(t-z).$$

Putting everything together we have

$$V = \begin{bmatrix} 1 & & & \\ x & 1 & & \\ x^2 & x+y & 1 & \\ x^3 & y^2+xy+x^2 & x+y+z & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ & y-x & z-x & t-x \\ & & (z-y)(z-x) & (t-y)(t-x) \\ & & & (t-y)(t-x)(t-z) \end{bmatrix}$$

We conjecture that $L[k,j]$ for $k > j$ contains a sum of all monomials of degree k of $x_1, \ldots, x_j$, and

$$U[k,j] = \prod_{s=1}^{k-1} (x_j - x_s)$$

for $1 < k \le j$. We can confirm that

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ x & y & z & t & s \\ x^2 & y^2 & z^2 & t^2 & s^2 \\ x^3 & y^3 & z^3 & t^3 & s^3 \\ x^4 & y^4 & z^4 & t^4 & s^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ x & 1 & & & \\ x^2 & x+y & 1 & & \\ x^3 & y^2+xy+x^2 & x+y+z & 1 & \\ x^4 & x^3+x^2y+xy^2+y^3 & x^2+y^2+z^2+xy+xz+yz & x+y+z+t & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & y-x & z-x & t-x & s-x \\ & & (z-y)(z-x) & (t-y)(t-x) & (s-x)(s-y) \\ & & & (t-y)(t-x)(t-z) & (s-y)(s-x)(s-z) \\ & & & & (s-y)(s-x)(s-z)(s-t) \end{bmatrix}$$

Multiplying it out we confirm that our conjecture is correct in this case.

END

Problem 3 Compute the interpolatory quadrature rule

$$\int_{-1}^{1} f(x)w(x)\mathrm{d}x \approx \sum_{j=1}^{n} w_j f(x_j)$$

for the points $[x_1, x_2, x_3] = [-1, 1/2, 1]$, for the weights $w(x) = 1$ and $w(x) = \sqrt{1-x^2}$.

SOLUTION

- $w(x) = 1$
- $w(x) = \sqrt{1-x^2}$

For the points $\boldsymbol{x} = \{-1, 1/2, 1\}$ we have the Lagrange polynomials:

$$\ell_1(x) = \left(\frac{x-1/2}{-1-1/2}\right) \cdot \left(\frac{x-1}{-1-1}\right) = \frac{1}{3}\left(x^2 - \frac{3}{2}x + \frac{1}{2}\right),$$

and

$$\ell_2(x) = -\frac{4}{3}x^2 + \frac{4}{3}, \ell_3(x) = x^2 + \frac{1}{2}x - \frac{1}{2},$$

similarly. We can then compute the weights,

$$w_j = \int_{-1}^{1} \ell_j(x)w(x)dx,$$

using,

$$\int_{-1}^{1} x^k \sqrt{1-x^2}dx = \begin{cases} \frac{\pi}{2} & k=0 \\ 0 & k=1 \\ \frac{\pi}{8} & k=2 \end{cases}$$

to find,

$$w_j = \begin{cases} \frac{\pi}{8} & j=1 \\ \frac{\pi}{2} & j=2 \\ -\frac{\pi}{8} & j=3, \end{cases}$$

so that the interpolatory quadrature rule is:

$$\Sigma_3^{w,\boldsymbol{x}}(f) = \frac{\pi}{2}\left(\frac{1}{4}f(-1) + f(1/2) - \frac{1}{4}f(1)\right)$$

END

Problem 4 Derive Backward Euler: use the left-sided divided difference approximation

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}$$

to reduce the first order ODE

$$u(a) = c, \qquad u'(x) + \omega(x)u(x) = f(x)$$

to a lower triangular system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$. Hint: only impose the ODE on the gridpoints $x_1, \ldots, x_n$ so that the divided difference does not depend on behaviour at x_{-1}.

SOLUTION

We go through all 4 steps (this is to help you understand what to do. In an exam I will still give full credit if you get the right result, even if you don't write down all 4 steps):

(Step 1) Since we need to avoid going off the left in step 2 we start the ODE discretisation at x_1:

$$\begin{pmatrix} u(x_0) \\ u'(x_1) + \omega(x_1)u(x_1) \\ \vdots \\ u'(x_n) + \omega(x_n)u(x_n) \end{pmatrix} = \underbrace{\begin{pmatrix} c \\ f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}}_{\boldsymbol{b}}$$

(Step 2) Replace with divided differences:

$$\begin{pmatrix} u(x_0) \\ (u(x_1) - u(x_0))/h + \omega(x_1)u(x_1) \\ \vdots \\ (u(x_n) - u(x_{n-1}))/h + \omega(x_n)u(x_n) \end{pmatrix} \approx \boldsymbol{b}$$

(Step 3) Replace with discrete system with equality:

$$\begin{pmatrix} u_0 \\ (u_1 - u_0)/h + \omega(x_1)u_1 \\ \vdots \\ (u_n - u_{n-1}))/h + \omega(x_n)u_n \end{pmatrix} = \boldsymbol{b}$$

(Step 4) Write as linear system:

$$\begin{bmatrix} 1 & & & \\ -1/h & 1/h + \omega(x_1) & & \\ & \ddots & \ddots & \\ & & -1/h & 1/h + \omega(x_n) \end{bmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix} = \boldsymbol{b}$$

END

Problem 5 Reduce a Schrödinger equation to a tridiagonal linear system by discretising on the grid $x_j = a + jh$ for $h = (b-a)/n$:

$$u(a) \ = c, \qquad u''(x) + V(x)u(x) = f(x), \qquad u(b) = d.$$

SOLUTION

(Step 1)

$$\begin{pmatrix} u(x_0) \\ u''(x_1) + V(x_1)u(x_1) \\ \vdots \\ u'(x_{n-1}) + V(x_{n-1})u(x_{n-1}) \\ u(x_n) \end{pmatrix} = \underbrace{\begin{pmatrix} c \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ d \end{pmatrix}}_{\boldsymbol{b}}$$

(Step 2) Replace with divided differences:

$$\begin{pmatrix} u(x_0) \\ (u(x_0) - 2u(x_1) + u(x_2))/h^2 + V(x_1)u(x_1) \\ \vdots \\ (u(x_{n-2} - 2u(x_{n-1}) + u(x_n))/h^2 + V(x_{n-1})u(x_{n-1}) \\ u(x_n) \end{pmatrix} \approx \boldsymbol{b}$$

(Step 3) Replace with discrete system with equality:

$$\begin{pmatrix} u_0 \\ (u_0 - 2u_1 + u_2)/h^2 + V(x_1)u_1 \\ \vdots \\ (u_{n-2} - 2u_{n-1} + u_n))/h^2 + V(x_{n-1})u_{n-1} \\ u_n \end{pmatrix} = \boldsymbol{b}$$

(Step 4) Write as a tridiagonal linear system:

$$\begin{bmatrix} 1 & & & & \\ 1/h^2 & V(x_1) - 2/h^2 & 1/h^2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/h^2 & V(x_{n-1}) - 2/h^2 & 1/h^2 \\ & & & & 1 \end{bmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix} = \boldsymbol{b}$$

END