PJ4: Car Tracking

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1 Problem 1: Bayesian network basics

First, we write the probability distribution in the bayes network as follows:

C_1	$P(C_1)$
0	0.5
1	0.5

C_t	C_{t+1}	$P(C_{t+1}\|C_t)$
0	0	1-ε
0	1	ε
1	0	ε
1	1	1-ε

C_t	D_t	$P(D_t\|C_t)$
0	0	1-η
0	1	η
1	0	η
1	1	1- η

1.1 a. Compute $P(C_2 = 1|D_2 = 0)$.

From above, we can calculate $P(C_2 = 1|D_2 = 0)$ as follows:

$$\begin{aligned}
P(C_2 = 1|D_2 = 0) &\propto P(C_2 = 1,\ D_2 = 0) \\
&= P(D_2 = 0|C_2 = 1)P(C_2 = 1) \\
&= P(D_2 = 0|C_2 = 1)\sum_{c_1} P(C_2 = 1|c_1)P(c_1) \\
&= \eta[0.5 \times \varepsilon + 0.5 \times (1-\varepsilon)] \\
&= 0.5\eta
\end{aligned}$$

$$\begin{aligned}
P(C_2 = 0|D_2 = 0) &\propto P(C_2 = 0,\ D_2 = 0) \\
&= P(D_2 = 0|C_2 = 0)P(C_2 = 0) \\
&= P(D_2 = 0|C_2 = 0)\sum_{c_1} P(C_2 = 0|c_1)P(c_1) \\
&= (1-\eta)[0.5 \times (1-\varepsilon) + 0.5 \times \varepsilon] \\
&= 0.5(1-\eta)
\end{aligned}$$

Then we normalize it and get:

$$P(C_2 = 1|D_2 = 0) = \frac{0.5\eta}{0.5\eta + 0.5(1-\eta)} = \eta$$

1.2 b. Compute $P(C_2 = 1|D_2 = 0, D_3 = 1)$.

From distribution tables, we can calculate $P(C_2 = 1|D_2 = 0, D_3 = 1)$ as follows:

$$
\begin{aligned}
P(C_2 = 1|D_2 = 0, D_3 = 1) &\propto P(C_2 = 1,\ D_2 = 0,\ D_3 = 1) \\
&= P(C_2 = 1) \cdot P(D_2 = 0|C_2 = 1) \cdot P(D_3 = 1|C_2 = 1, D_2 = 0) \\
&= 0.5\eta \cdot \sum_{c_3} P(c_3|C_2 = 1)P(D_3 = 1|c_3) \\
&= 0.5\eta \cdot [\varepsilon\eta + (1-\varepsilon)(1-\eta)]
\end{aligned}
$$

$$
\begin{aligned}
P(C_2 = 0|D_2 = 0, D_3 = 1) &\propto P(C_2 = 0,\ D_2 = 0,\ D_3 = 1) \\
&= P(C_2 = 0) \cdot P(D_2 = 0|C_2 = 0) \cdot P(D_3 = 1|C_2 = 0, D_2 = 0) \\
&= 0.5(1-\eta) \cdot \sum_{c_3} P(c_3|C_2 = 0)P(D_3 = 1|c_3) \\
&= 0.5(1-\eta) \cdot [\varepsilon(1-\eta) + (1-\varepsilon)\eta]
\end{aligned}
$$

Then we normalize it and get:

$$
\begin{aligned}
P(C_2 = 1|D_2 = 0, D_3 = 1) = &\ \frac{0.5\eta \cdot [\varepsilon\eta + (1-\varepsilon)(1-\eta)]}{0.5\eta \cdot [\varepsilon\eta + (1-\varepsilon)(1-\eta)] + 0.5(1-\eta) \cdot [\varepsilon(1-\eta) + (1-\varepsilon)\eta]} \\
= &\ \frac{\varepsilon\eta^2 + \eta(1-\varepsilon)(1-\eta)}{\varepsilon\eta^2 + 2\eta(1-\varepsilon)(1-\eta) + \varepsilon(1-\eta)^2}
\end{aligned}
$$

1.3 c. Suppose $\varepsilon = 0.1$ and $\eta = 0.2$.

1.3.1 (i) Compute and compare $P(C_2 = 1|D_2 = 0)$ and $P(C_2 = 1|D_2 = 0, D_3 = 1)$.

Substitute $\varepsilon = 0.1$ and $\eta = 0.2$ into the above result, we can calculate that:

$$
\begin{aligned}
P(C_2 = 1|D_2 = 0) &= \eta = 0.2 \\
P(C_2 = 1|D_2 = 0, D_3 = 1) &= \frac{\varepsilon\eta^2 + \eta(1-\varepsilon)(1-\eta)}{\varepsilon\eta^2 + 2\eta(1-\varepsilon)(1-\eta) + \varepsilon(1-\eta)^2} \approx 0.4157
\end{aligned}
$$

1.3.2 (ii) How did $D_3 = 1$ changes the result?

Because $\eta = 0.2$ is small, from the distribution table, we can know that it is more likely that $C_3 = 1$, where the sensor probability is $1 - \eta = 0.8$. And at the same time, because ε is also small, so $C_3 = 1$ implies that C_2 is more likely to be 1 to have a bigger transition probability $1 - \varepsilon$. So the observation $D_3 = 1$ implies it is more likely that $C_2 = 1$, which is the same as our results calculated above.

1.3.3 (iii) How did $D_3 = 1$ changes the result?

First, solve the following function:

$$\frac{\varepsilon\eta^2 + \eta(1-\varepsilon)(1-\eta)}{\varepsilon\eta^2 + 2\eta(1-\varepsilon)(1-\eta) + \varepsilon(1-\eta)^2} = \eta$$

Substitute $\eta = 0.2$ in, we can obtain $\varepsilon = 0.5$.

In this situation, we can't get more information from $D_3 = 1$, because although the sensor probability tells us C_3 is more likely to be 1, when it comes to the transition probability, either $C_2 = 1$ or $C_2 = 0$, have equal transition probability to arrive $C_3 = 1$.