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1 Logarithms

1.1 Recipricol of the Logarithm

$$\frac{1}{\log_a(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} = \frac{\log_c(a)}{\log_c(b)} = \log_b(a)$$

$$\implies \boxed{\frac{1}{\log_a(b)} = \log_b(a)} \tag{1.1}$$

1.2 Divison of logs

1.2.1 General case

Using equation 1.1 we get:

$$\boxed{\frac{\log_a(b)}{\log_c(d)} = \log_a(b) \cdot \log_d(c)} \tag{1.2}$$

1.2.2 same Argument

Using Equation 1.2 and assuming that $b = d$, we get:

$$\frac{\log_a(b)}{\log_c(d)} = \frac{\log_a(b)}{\log_c(b)} = \log_a(b) \cdot \log_b(c) = \log_a(b^{\log_b(c)}) = \log_a(c)$$

$$\implies \boxed{\frac{\log_a(b)}{\log_c(b)} = \log_a(c)}$$

2 Physics

2.1 Mix Temperature

$\vartheta_1 > \vartheta_2$

$$
\begin{aligned}
c_1 \cdot m_1 \cdot (\vartheta_1 - \vartheta_m) &= c_2 \cdot m_2 \cdot (\vartheta_m - \vartheta_2) \\
c_1 \cdot m_1 \cdot \vartheta_1 - c_1 \cdot m_1 \cdot \vartheta_m &= c_2 \cdot m_2 \cdot \vartheta_m - c_2 \cdot m_2 \cdot \vartheta_2 \\
c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= c_1 \cdot m_1 \cdot \vartheta_m + c_2 \cdot m_2 \cdot \vartheta_m \\
c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= \vartheta_m \cdot (c_1 \cdot m_1 + c_2 \cdot m_2) \\
\implies & \boxed{\vartheta_m = \frac{c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2}{c_1 \cdot m_1 + c_2 \cdot m_2}}
\end{aligned}
$$

3 Complex Numbers

3.1 Complex Number and their conjugate

$\forall z \in \mathbb{C};\ \exists z^* \in \mathbb{C}$, such that:

$$z = a + bi \qquad a, b \in \mathbb{R}$$
$$z^* = \mathrm{conj}(z) = a - bi$$

The real number a is called the *real part* of a complex number $z = a + bi$ $z \in \mathbb{C}$ $a, b \in \mathbb{R}$, while the real number b is called the *imaginary part* of that number. The real part of a complex number z is denoted with $\Re(z)$, the imaginary part with $\Im(z)$.

The magnitude (or absolut value) $|z|$ of a complex number $z \in \mathbb{C}$ is defined as:

$$|z| = \sqrt{a^2 + b^2} = \sqrt{(\Re(z))^2 + (\Im(z))^2}$$
$$\implies |z|^2 = a^2 + b^2 = (\Re(z))^2 + (\Im(z))^2$$

3.1.1 Complex plus Conjugate: $z + z^*$

$$z + z^* = a + bi + a - bi = 2 \cdot a = 2 \cdot \Re(z)$$
$$\implies \boxed{z + z^* = 2 \cdot \Re(z)}$$

3.1.2 Complex minus Conjugate: $z - z^*$

$$z - z^* = a + bi - a + bi = 2 \cdot bi = 2i \cdot \Im(z)$$
$$\implies \boxed{z - z^* = 2i \cdot \Im(z)} \tag{3.1}$$

3.1.3 Conjugate minus Complex: $z^* - z$

Using equation 3.1, we get:

$$z^* - z = -(z - z^*)$$
$$\implies \boxed{z^* - z = -2i \cdot \Im(z)}$$

3.1.4 Complex times Conjugate: $z \cdot z^*$

$$z \cdot z^* = (a+bi)\cdot(a-bi) = a^2 + a\cdot bi - a\cdot bi + b^2 = a^2 + b^2$$

$$\Longrightarrow \boxed{z \cdot z^* = (\Re(z))^2 + (\Im(z))^2 = |z|^2}$$

3.1.5 Complex over Conjugate: $\frac{z}{z^*}$

$$\frac{z}{z^*} = \frac{a+bi}{a-bi} = \frac{a+bi}{a-bi}\cdot\frac{a+bi}{a+bi} = \frac{a^2 + 2i\cdot a\cdot b - b^2}{a^2+b^2}$$

$$\Longrightarrow \boxed{\frac{z}{z^*} = \frac{(\Re(z))^2 - (\Im(z))^2 + 2i\cdot\Re(z)\cdot\Im(z)}{|z|^2}}$$

3.1.6 Conjugate over Complex: $\frac{z^*}{z}$

$$\frac{z^*}{z} = \frac{a-bi}{a+bi} = \frac{a-bi}{a+bi}\cdot\frac{a-bi}{a-bi} = \frac{a^2+b^2-2a\cdot bi}{a^2+b^2} = 1 - \frac{2\cdot a\cdot b}{a^2+b^2}\cdot i$$

$$\Longrightarrow \boxed{\frac{z^*}{z} = 1 - i\frac{2\cdot\Re(z)\cdot\Im(z)}{|z|^2}}$$

4 Sums

4.1 Sum of the first n even numbers

Note: not my idea, but my way of showing this relation
Let m be the nth even number, then $m = 2n, \quad m, n \in \mathbb{N}$

$$\sum_{k=1}^{n}(2k) = 2 \cdot \sum_{k=1}^{n}(k) = 2 \cdot \frac{n(n+1)}{2} = n^2 + n$$

$$\implies \boxed{\sum_{k=1}^{n}(2k) = n^2 + n} \tag{4.1}$$

4.2 Sum of the first n odd numbers

Note: not my idea, but my way of showing this relation
Let m be the nth odd number, then $m = 2n - 1, \quad m, n \in \mathbb{N}$
Using Equation 4.1

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n}(2k) - \sum_{k=1}^{n}(1) = n^2 + n - n$$

$$\implies \boxed{\sum_{k=1}^{n}(2k-1) = n^2} \tag{4.2}$$

4.3 Sum of the first even numbers up to n

If we let an even number $n \in \mathbb{N}$ have the form $n = 2k, k \in \mathbb{N}_0$ we need $2k = n$ for the upper bound of the sum, which implies $k = {}^{n}\!/_{2}$.

$$0 + 2 + 4 + \cdots + n = \sum_{k=1}^{n/2} (2k)$$
$$\text{let } m = \frac{n}{2}$$
$$\implies \sum_{k=1}^{n/2} (2k) = \sum_{k=1}^{m} (2k)$$
From equation 4.1 we get
$$\sum_{k=1}^{m} (2k) = m^2 + m = \frac{n^2}{4} + \frac{n}{2} = \frac{n^2 + 2n}{4}$$
$$\implies \boxed{\sum_{k=1}^{n/2} (2k) = 0 + 2 + 4 + \cdots + n = \frac{n(n+2)}{4}}$$

4.4 Sum of the first odd numbers up to n

If we let an odd number $n \in \mathbb{N}$ have the form $n = 2k - 1, k \in \mathbb{N}_0$ we need $2k - 1 = n$ for the upper bound of the sum, which implies $k = {}^{(n+1)}\!/_{2}$.

$$1 + 3 + \ldots + n = \sum_{k=1}^{(n+1)/2} (2k - 1)$$
$$\text{let } m = \frac{n+1}{2}, m \in \mathbb{N}$$
$$\implies \sum_{k=1}^{(n+1)/2} (2k - 1) = \sum_{k=1}^{m} (2k - 1)$$
From equation 4.2 we get
$$\sum_{k=1}^{m} (2k - 1) = m^2 = \left(\frac{n+1}{2}\right)^2$$
$$\implies \boxed{\sum_{k=1}^{(n+1)/2} (2k - 1) = 1 + 3 + \ldots + n = \left(\frac{n+1}{2}\right)^2}$$

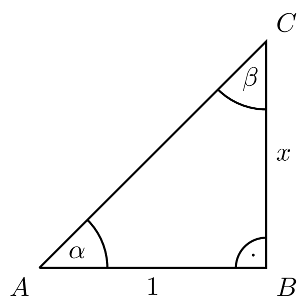
5 Arctangent

5.1 $\arctan(x) + \arctan(^1/_x)$

5.1.1 over Triangle

Note: Not my observation, nor my explanation, but I do love this

Consider the following right angled triangle $\triangle ABC$



All angles in a triangle sum up to $\pi \implies \alpha + \beta = \frac{\pi}{2}$

$$\tan(\alpha) = x \Leftrightarrow \alpha = \arctan(x)$$
$$\tan(\beta) = \frac{1}{x} \Leftrightarrow \beta = \arctan\left(\frac{1}{x}\right)$$
$$\alpha + \beta = \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

From symmetry we know that $\arctan(-x) = -\arctan(x)$

$$\begin{aligned} \therefore\ \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\ &= -\frac{\pi}{2} \end{aligned}$$

$$\Longrightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}}$$

$$\Longrightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x) \cdot \frac{\pi}{2}}$$

5.1.2 over sin and cos

option 1:

Note: observation by D. J.

$$\boxed{x, \alpha, \beta > 0}$$

$$\alpha + \beta = \frac{\pi}{2} \Longrightarrow \beta = \frac{\pi}{2} - \alpha$$

$$\text{let } \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = x$$

$$\tan(\beta) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)} = \frac{1}{x}$$

$$\Longrightarrow \alpha = \arctan(x), \quad \beta = \frac{\pi}{2} - \alpha = \arctan\left(\frac{1}{x}\right)$$

$$\Longrightarrow \arctan(x) + \arctan\left(\frac{1}{x}\right) = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$$

$$\tan(-\alpha) = -\tan(\alpha) = -x$$

$$\tan(-\beta) = -\tan(\beta) = -\frac{1}{x}$$

$$\Longrightarrow \arctan(-x) = -\alpha, \quad \arctan\left(-\frac{1}{x}\right) = -\beta = -\frac{\pi}{2} + \alpha$$

$$\Longrightarrow \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\alpha - \frac{\pi}{2} + \alpha = -\frac{\pi}{2}$$

$$\Longrightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}}$$

option 2:

Note: observation by D. J.

$$\arctan(x) + \arctan\left(\frac{1}{x}\right) = ?$$

$$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = x$$

$$\tan(\beta) = \frac{1}{x} = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\sin(\beta)}{\cos(\beta)}$$

$$\implies \beta = \frac{\pi}{2} - \alpha \implies \alpha + \beta = \frac{\pi}{2}$$

$$\implies \tan(\beta) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan(\alpha)} = \frac{1}{x}$$

$$\implies \arctan(x) + \arctan\left(\frac{1}{x}\right) = \cancel{\alpha} + \frac{\pi}{2} - \cancel{\alpha} = \frac{\pi}{2}$$

From symmetry we know that $\arctan(-x) = -\arctan(x)$

$$\therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$$

$$= -\frac{\pi}{2}$$

$$\implies \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}}$$

5.1.3 over derivative and limit

$$f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right) \quad D_f = \mathbb{R} \setminus \{0\}$$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$

If $f'(x) = 0$ then f(x) has at least one constant value

Since 0 is excluded, we should check for values above, and below zero, because, this could be the only jump in the function, because it's continuous everywhere else (in $\mathbb{R}^+$ and in $\mathbb{R}^-$):

Case 1: $x > 0$

$$f(1) = \arctan(1) + \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Case 2: $x < 0$

$$f(-1) = \arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

$$\Longrightarrow \boxed{f(x) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}}$$

6 Signum Function

The signum function yields the sign of a number:

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

6.1 link to $|x|$

$$\begin{aligned} |x|' &= \left(\sqrt{x^2}\right)' = \frac{1}{\cancel{2}\sqrt{x^2}} \cdot (\cancel{2}x) = \frac{x}{\sqrt{x^2}} \\ &= \frac{x}{|x|} = \begin{cases} \frac{x}{x} & \text{if } x > 0 \\ \frac{x}{-x} & \text{if } x < 0 \end{cases} \\ &= \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \Longrightarrow &\ \boxed{|x|' = \operatorname{sgn}(x) = \frac{x}{|x|} \text{ if } x \neq 0} \end{aligned} \tag{6.1}$$

6.2 Reciprocal of $\operatorname{sgn}(x)$

$$\begin{aligned} \operatorname{sgn}(x) &= \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \Longrightarrow &\ \boxed{\frac{1}{\operatorname{sgn}(x)} = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} = \operatorname{sgn}(x)} \end{aligned} \tag{6.2}$$

6.3 x and $\operatorname{sgn}(x)$

From equations 6.1 and 6.2 we get:

$$\boxed{x = \operatorname{sgn}(x) \cdot |x|}$$

$$\boxed{|x| = \operatorname{sgn}(x) \cdot x}$$

6.4 Fun connnection with $x \cdot \sqrt{\frac{a}{x^2}}$

$$x \cdot \sqrt{\frac{a}{x^2}} = x \cdot \sqrt{\frac{a}{x^2} \cdot \frac{x^2}{x^2}} = x \cdot \sqrt{a} \cdot \sqrt{\frac{1}{x^2}} = \frac{x}{\sqrt{x}} \sqrt{a} \implies \boxed{x \cdot \sqrt{\frac{a}{x^2}} = \operatorname{sgn}(x) \cdot \sqrt{a}}$$