

# Table of Contents

<b>1 Logarithms</b>	<b>3</b>
1.1 Reciprical of the Logarithm . . . . .	3
1.2 Divison of logs . . . . .	3
1.2.1 General case . . . . .	3
1.2.2 same Argument . . . . .	3
<b>2 Physics</b>	<b>4</b>
2.1 Mix Temperature . . . . .	4
<b>3 Complex Numbers</b>	<b>5</b>
3.1 Complex Number and their conjugate . . . . .	5
3.1.1 Complex plus Conjugate: $z + z^*$ . . . . .	5
3.1.2 Complex minus Conjugate: $z - z^*$ . . . . .	5
3.1.3 Conjugate minus Complex: $z^* - z$ . . . . .	5
3.1.4 Complex times Conjugate: $z \cdot z^*$ . . . . .	6
3.1.5 Complex over Conjugate: $\frac{z}{z^*}$ . . . . .	6
3.1.6 Conjugate over Complex: $\frac{z^*}{z}$ . . . . .	6
<b>4 Sums</b>	<b>7</b>
4.1 Sum of the first $n$ even numbers . . . . .	7
4.2 Sum of the first $n$ odd numbers . . . . .	7
4.3 Sum of the first even numbers up to $n$ . . . . .	8
4.4 Sum of the first odd numbers up to $n$ . . . . .	8

<b>5 Arctangent</b>	<b>9</b>
5.1 $\arctan(x) + \arctan(1/x)$ . . . . .	9
5.1.1 over Triangle . . . . .	9
5.1.2 over sin and cos . . . . .	10
5.1.3 over derivative and limit . . . . .	11

# 1 Logarithms

## 1.1 Recipricol of the Logarithm

$$\frac{1}{\log_a(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} = \frac{\log_c(a)}{\log_c(b)} = \log_b(a)$$
$$\Rightarrow \boxed{\frac{1}{\log_a(b)} = \log_b(a)} \quad (1.1)$$

## 1.2 Divison of logs

### 1.2.1 General case

Using equation 1.1 we get:

$$\boxed{\frac{\log_a(b)}{\log_c(d)} = \log_a(b) \cdot \log_d(c)} \quad (1.2)$$

### 1.2.2 same Argument

Using Equation 1.2 and assuming that  $b = d$ , we get:

$$\frac{\log_a(b)}{\log_c(d)} = \frac{\log_a(b)}{\log_c(b)} = \log_a(b) \cdot \log_b(c) = \log_a(b^{\log_b(c)}) = \log_a(c)$$
$$\Rightarrow \boxed{\frac{\log_a(b)}{\log_c(b)} = \log_a(c)}$$

## 2 Physics

### 2.1 Mix Temperature

$$\vartheta_1 > \vartheta_2$$

$$\begin{aligned} c_1 \cdot m_1 \cdot (\vartheta_1 - \vartheta_m) &= c_2 \cdot m_2 \cdot (\vartheta_m - \vartheta_2) \\ c_1 \cdot m_1 \cdot \vartheta_1 - c_1 \cdot m_1 \cdot \vartheta_m &= c_2 \cdot m_2 \cdot \vartheta_m - c_2 \cdot m_2 \cdot \vartheta_2 \\ c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= c_1 \cdot m_1 \cdot \vartheta_m + c_2 \cdot m_2 \cdot \vartheta_m \\ c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= \vartheta_m \cdot (c_1 \cdot m_1 + c_2 \cdot m_2) \\ \Rightarrow \boxed{\vartheta_m = \frac{c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2}{c_1 \cdot m_1 + c_2 \cdot m_2}} \end{aligned}$$

### 3 Complex Numbers

#### 3.1 Complex Number and their conjugate

$\forall z \in \mathbb{C}; \exists z^* \in \mathbb{C}$ , such that:

$$\begin{aligned} z &= a + bi & a, b \in \mathbb{R} \\ z^* &= \text{conj}(z) = a - bi \end{aligned}$$

The real number  $a$  is called the *real part* of a complex number  $z = a + bi$   $z \in \mathbb{C}$   $a, b \in \mathbb{R}$ , while the real number  $b$  is called the *imaginary part* of that number. The real part of a complex number  $z$  is denoted with  $\Re(z)$ , the imaginary part with  $\Im(z)$ .

The magnitude (or absolute value)  $|z|$  of a complex number  $z \in \mathbb{C}$  is defined as:

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} = \sqrt{(\Re(z))^2 + (\Im(z))^2} \\ \implies |z|^2 &= a^2 + b^2 = (\Re(z))^2 + (\Im(z))^2 \end{aligned}$$

##### 3.1.1 Complex plus Conjugate: $z + z^*$

$$\begin{aligned} z + z^* &= a + bi + a - bi = 2 \cdot a = 2 \cdot \Re(z) \\ \implies z + z^* &= 2 \cdot \Re(z) \end{aligned}$$

##### 3.1.2 Complex minus Conjugate: $z - z^*$

$$\begin{aligned} z - z^* &= a + bi - a + bi = 2 \cdot bi = 2i \cdot \Im(z) \\ \implies z - z^* &= 2i \cdot \Im(z) \end{aligned} \tag{3.1}$$

##### 3.1.3 Conjugate minus Complex: $z^* - z$

Using equation 3.1, we get:

$$\begin{aligned} z^* - z &= -(z - z^*) \\ \implies z^* - z &= -2i \cdot \Im(z) \end{aligned}$$

### 3.1.4 Complex times Conjugate: $z \cdot z^*$

$$\begin{aligned} z \cdot z^* &= (a + bi) \cdot (a - bi) = a^2 + a \cdot bi - a \cdot bi + b^2 = a^2 + b^2 \\ \implies &\boxed{z \cdot z^* = (\Re(z))^2 + (\Im(z))^2 = |z|^2} \end{aligned}$$

### 3.1.5 Complex over Conjugate: $\frac{z}{z^*}$

$$\begin{aligned} \frac{z}{z^*} &= \frac{a + bi}{a - bi} = \frac{a + bi}{a - bi} \cdot \frac{a + bi}{a + bi} = \frac{a^2 + 2i \cdot a \cdot b - b^2}{a^2 + b^2} \\ \implies &\boxed{\frac{z}{z^*} = \frac{(\Re(z))^2 - (\Im(z))^2 + 2i \cdot \Re(z) \cdot \Im(z)}{|z|^2}} \end{aligned}$$

### 3.1.6 Conjugate over Complex: $\frac{z^*}{z}$

$$\begin{aligned} \frac{z^*}{z} &= \frac{a - bi}{a + bi} = \frac{a - bi}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a^2 + b^2 - 2a \cdot bi}{a^2 + b^2} = 1 - \frac{2 \cdot a \cdot b}{a^2 + b^2} \cdot i \\ \implies &\boxed{\frac{z^*}{z} = 1 - i \frac{2 \cdot \Re(z) \cdot \Im(z)}{|z|^2}} \end{aligned}$$

## 4 Sums

### 4.1 Sum of the first $n$ even numbers

Note: not my idea, but my way of showing this relation

Let  $m$  be the  $n$ th even number, then  $m = 2n$ ,  $m, n \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^n (2k) &= 2 \cdot \sum_{k=1}^n (k) = 2 \cdot \frac{n(n+1)}{2} = n^2 + n \\ \implies \boxed{\sum_{k=1}^n (2k) = n^2 + n} \end{aligned} \tag{4.1}$$

### 4.2 Sum of the first $n$ odd numbers

Note: not my idea, but my way of showing this relation

Let  $m$  be the  $n$ th odd number, then  $m = 2n - 1$ ,  $m, n \in \mathbb{N}$

Using Equation 4.1

$$\begin{aligned} \sum_{k=1}^n (2k - 1) &= \sum_{k=1}^n (2k) - \sum_{k=1}^n (1) = n^2 + n - n \\ \implies \boxed{\sum_{k=1}^n (2k - 1) = n^2} \end{aligned} \tag{4.2}$$

### 4.3 Sum of the first even numbers up to $n$

If we let an even number  $n \in \mathbb{N}$  have the form  $n = 2k, k \in \mathbb{N}_0$  we need  $2k = n$  for the upper bound of the sum, which implies  $k = n/2$ .

$$\begin{aligned} 0 + 2 + 4 + \cdots + n &= \sum_{k=1}^{n/2} (2k) \\ \text{let } m &= \frac{n}{2} \\ \Rightarrow \sum_{k=1}^{n/2} (2k) &= \sum_{k=1}^m (2k) \end{aligned}$$

From equation 4.1 we get

$$\begin{aligned} \sum_{k=1}^m (2k) &= m^2 + m = \frac{n^2}{4} + \frac{n}{2} = \frac{n^2 + 2n}{4} \\ \Rightarrow \boxed{\sum_{k=1}^{n/2} (2k)} &= 0 + 2 + 4 + \cdots + n = \frac{n(n+2)}{4} \end{aligned}$$

### 4.4 Sum of the first odd numbers up to $n$

If we let an odd number  $n \in \mathbb{N}$  have the form  $n = 2k - 1, k \in \mathbb{N}_0$  we need  $2k - 1 = n$  for the upper bound of the sum, which implies  $k = (n+1)/2$ .

$$\begin{aligned} 1 + 3 + \dots + n &= \sum_{k=1}^{(n+1)/2} (2k - 1) \\ \text{let } m &= \frac{n+1}{2}, m \in \mathbb{N} \\ \Rightarrow \sum_{k=1}^{(n+1)/2} (2k - 1) &= \sum_{k=1}^m (2k - 1) \end{aligned}$$

From equation 4.2 we get

$$\begin{aligned} \sum_{k=1}^m (2k - 1) &= m^2 = \left(\frac{n+1}{2}\right)^2 \\ \Rightarrow \boxed{\sum_{k=1}^{(n+1)/2} (2k - 1)} &= 1 + 3 + \dots + n = \left(\frac{n+1}{2}\right)^2 \end{aligned}$$

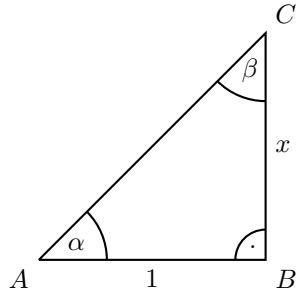
## 5 Arctangent

### 5.1 $\arctan(x) + \arctan(1/x)$

#### 5.1.1 over Triangle

Note: Not my observation, nor my explanation, but I do love this

Consider the following right angled triangle  $\triangle ABC$



All angles in a triangle sum up to  $\pi \implies \alpha + \beta = \frac{\pi}{2}$

$$\begin{aligned}\tan(\alpha) &= x \Leftrightarrow \alpha = \arctan(x) \\ \tan(\beta) &= \frac{1}{x} \Leftrightarrow \beta = \arctan\left(\frac{1}{x}\right) \\ \alpha + \beta &= \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}\end{aligned}$$

From symmetry we know that  $\arctan(-x) = -\arctan(x)$

$$\begin{aligned}\therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\ &= -\frac{\pi}{2}\end{aligned}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x) \cdot \frac{\pi}{2}}$$

### 5.1.2 over sin and cos

**option 1:**

Note: observation by D. J.

$$\boxed{x, \alpha, \beta > 0}$$

$$\alpha + \beta = \frac{\pi}{2} \implies \beta = \frac{\pi}{2} - \alpha$$

$$\text{let } \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = x$$

$$\tan(\beta) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)} = \frac{1}{x}$$

$$\implies \alpha = \arctan(x), \quad \beta = \frac{\pi}{2} - \alpha = \arctan\left(\frac{1}{x}\right)$$

$$\implies \arctan(x) + \arctan\left(\frac{1}{x}\right) = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$$

$$\tan(-\alpha) = -\tan(\alpha) = -x$$

$$\tan(-\beta) = -\tan(\beta) = -\frac{1}{x}$$

$$\implies \arctan(-x) = -\alpha, \quad \arctan\left(-\frac{1}{x}\right) = -\beta = -\frac{\pi}{2} + \alpha$$

$$\implies \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\alpha - \frac{\pi}{2} + \alpha = -\frac{\pi}{2}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}}$$

**option 2:**

Note: observation by D. J.

$$\begin{aligned}
 \arctan(x) + \arctan\left(\frac{1}{x}\right) &= ? \\
 \tan(\alpha) &= \frac{\sin(\alpha)}{\cos(\alpha)} = x \\
 \tan(\beta) &= \frac{1}{x} = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\sin(\beta)}{\cos(\beta)} \\
 \implies \beta &= \frac{\pi}{2} - \alpha \implies \alpha + \beta = \frac{\pi}{2} \\
 \implies \tan(\beta) &= \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan(\alpha)} = \frac{1}{x} \\
 \implies \arctan(x) + \arctan\left(\frac{1}{x}\right) &= \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}
 \end{aligned}$$

From symmetry we know that  $\arctan(-x) = -\arctan(x)$

$$\begin{aligned}
 \therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\
 &= -\frac{\pi}{2} \\
 \implies \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right)} &= \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}
 \end{aligned}$$

### 5.1.3 over derivative and limit

$$\begin{aligned}
 f(x) &= \arctan(x) + \arctan\left(\frac{1}{x}\right) \quad D_f = \mathbb{R} \setminus \{0\} \\
 f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0
 \end{aligned}$$

If  $f'(x) = 0$  then  $f(x)$  has at least one constant value

Since 0 is excluded, we should check for values above, and below zero, because, this could be the only jump in the function, because it's continuous everywhere else (in  $\mathbb{R}^+$  and in  $\mathbb{R}^-$ ):

Case 1:  $x > 0$

$$f(1) = \arctan(1) + \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Case 2:  $x < 0$

$$f(-1) = \arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

$$\implies f(x) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}$$