

Table of Contents

1 Logarithms	3
1.1 Reciprical of the Logarithm	3
1.2 Divison of logs	3
1.2.1 General case	3
1.2.2 same Argument	3
2 Physics	4
2.1 Mix Temperature	4
3 Complex Numbers	5
3.1 Complex Number and their conjugate	5
3.1.1 Complex plus Conjugate: $z + z^*$	5
3.1.2 Complex minus Conjugate: $z - z^*$	5
3.1.3 Conjugate minus Complex: $z^* - z$	5
3.1.4 Complex times Conjugate: $z \cdot z^*$	6
3.1.5 Complex over Conjugate: $\frac{z}{z^*}$	6
3.1.6 Conjugate over Complex: $\frac{z^*}{z}$	6
4 Sums	7
4.1 Sum of the first n even numbers	7
4.2 Sum of the first n odd numbers	7
4.3 Sum of the first even numbers up to n	8
4.4 Sum of the first odd numbers up to n	8

5 Arctangent	9
5.1 $\arctan(x) + \arctan(1/x)$	9
5.1.1 over Triangle	9
5.1.2 over sin and cos	10
5.1.3 over derivative and limit	11
6 Signum Function	13
6.1 link to $ x $	13
6.2 Reciprocal of $\operatorname{sgn}(x)$	13
6.3 x and $\operatorname{sgn}(x)$	14
6.4 Looking at $\sqrt{\frac{a}{x^2}}$	14
6.5 Fun connection with $x \cdot \sqrt{\frac{a}{x^2}}$	14
7 Longer Derivations	15
7.1 Quadratic Formular	15
7.2 ln Triangle	15

1 Logarithms

1.1 Recipricol of the Logarithm

$$\frac{1}{\log_a(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} = \frac{\log_c(a)}{\log_c(b)} = \log_b(a)$$
$$\Rightarrow \boxed{\frac{1}{\log_a(b)} = \log_b(a)} \quad (1.1)$$

1.2 Divison of logs

1.2.1 General case

Using equation 1.1 we get:

$$\boxed{\frac{\log_a(b)}{\log_c(d)} = \log_a(b) \cdot \log_d(c)} \quad (1.2)$$

1.2.2 same Argument

Using Equation 1.2 and assuming that $b = d$, we get:

$$\frac{\log_a(b)}{\log_c(d)} = \frac{\log_a(b)}{\log_c(b)} = \log_a(b) \cdot \log_b(c) = \log_a(b^{\log_b(c)}) = \log_a(c)$$
$$\Rightarrow \boxed{\frac{\log_a(b)}{\log_c(b)} = \log_a(c)}$$

2 Physics

2.1 Mix Temperature

$$\vartheta_1 > \vartheta_2$$

$$\begin{aligned} c_1 \cdot m_1 \cdot (\vartheta_1 - \vartheta_m) &= c_2 \cdot m_2 \cdot (\vartheta_m - \vartheta_2) \\ c_1 \cdot m_1 \cdot \vartheta_1 - c_1 \cdot m_1 \cdot \vartheta_m &= c_2 \cdot m_2 \cdot \vartheta_m - c_2 \cdot m_2 \cdot \vartheta_2 \\ c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= c_1 \cdot m_1 \cdot \vartheta_m + c_2 \cdot m_2 \cdot \vartheta_m \\ c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 &= \vartheta_m \cdot (c_1 \cdot m_1 + c_2 \cdot m_2) \\ \Rightarrow \boxed{\vartheta_m = \frac{c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2}{c_1 \cdot m_1 + c_2 \cdot m_2}} \end{aligned}$$

3 Complex Numbers

3.1 Complex Number and their conjugate

$\forall z \in \mathbb{C}; \exists z^* \in \mathbb{C}$, such that:

$$\begin{aligned} z &= a + bi & a, b \in \mathbb{R} \\ z^* &= \text{conj}(z) = a - bi \end{aligned}$$

The real number a is called the *real part* of a complex number $z = a + bi$ $z \in \mathbb{C}$ $a, b \in \mathbb{R}$, while the real number b is called the *imaginary part* of that number. The real part of a complex number z is denoted with $\Re(z)$, the imaginary part with $\Im(z)$.

The magnitude (or absolute value) $|z|$ of a complex number $z \in \mathbb{C}$ is defined as:

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} = \sqrt{(\Re(z))^2 + (\Im(z))^2} \\ \implies |z|^2 &= a^2 + b^2 = (\Re(z))^2 + (\Im(z))^2 \end{aligned}$$

3.1.1 Complex plus Conjugate: $z + z^*$

$$\begin{aligned} z + z^* &= a + bi + a - bi = 2 \cdot a = 2 \cdot \Re(z) \\ \implies z + z^* &= 2 \cdot \Re(z) \end{aligned}$$

3.1.2 Complex minus Conjugate: $z - z^*$

$$\begin{aligned} z - z^* &= a + bi - a + bi = 2 \cdot bi = 2i \cdot \Im(z) \\ \implies z - z^* &= 2i \cdot \Im(z) \end{aligned} \tag{3.1}$$

3.1.3 Conjugate minus Complex: $z^* - z$

Using equation 3.1, we get:

$$\begin{aligned} z^* - z &= -(z - z^*) \\ \implies z^* - z &= -2i \cdot \Im(z) \end{aligned}$$

3.1.4 Complex times Conjugate: $z \cdot z^*$

$$\begin{aligned} z \cdot z^* &= (a + bi) \cdot (a - bi) = a^2 + a \cdot bi - a \cdot bi + b^2 = a^2 + b^2 \\ \implies &\boxed{z \cdot z^* = (\Re(z))^2 + (\Im(z))^2 = |z|^2} \end{aligned}$$

3.1.5 Complex over Conjugate: $\frac{z}{z^*}$

$$\begin{aligned} \frac{z}{z^*} &= \frac{a + bi}{a - bi} = \frac{a + bi}{a - bi} \cdot \frac{a + bi}{a + bi} = \frac{a^2 + 2i \cdot a \cdot b - b^2}{a^2 + b^2} \\ \implies &\boxed{\frac{z}{z^*} = \frac{(\Re(z))^2 - (\Im(z))^2 + 2i \cdot \Re(z) \cdot \Im(z)}{|z|^2}} \end{aligned}$$

3.1.6 Conjugate over Complex: $\frac{z^*}{z}$

$$\begin{aligned} \frac{z^*}{z} &= \frac{a - bi}{a + bi} = \frac{a - bi}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a^2 + b^2 - 2a \cdot bi}{a^2 + b^2} = 1 - \frac{2 \cdot a \cdot b}{a^2 + b^2} \cdot i \\ \implies &\boxed{\frac{z^*}{z} = 1 - i \frac{2 \cdot \Re(z) \cdot \Im(z)}{|z|^2}} \end{aligned}$$

4 Sums

4.1 Sum of the first n even numbers

Note: not my idea, but my way of showing this relation

Let m be the n th even number, then $m = 2n$, $m, n \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^n (2k) &= 2 \cdot \sum_{k=1}^n (k) = 2 \cdot \frac{n(n+1)}{2} = n^2 + n \\ \implies \boxed{\sum_{k=1}^n (2k) = n^2 + n} \end{aligned} \tag{4.1}$$

4.2 Sum of the first n odd numbers

Note: not my idea, but my way of showing this relation

Let m be the n th odd number, then $m = 2n - 1$, $m, n \in \mathbb{N}$

Using Equation 4.1

$$\begin{aligned} \sum_{k=1}^n (2k - 1) &= \sum_{k=1}^n (2k) - \sum_{k=1}^n (1) = n^2 + n - n \\ \implies \boxed{\sum_{k=1}^n (2k - 1) = n^2} \end{aligned} \tag{4.2}$$

4.3 Sum of the first even numbers up to n

If we let an even number $n \in \mathbb{N}$ have the form $n = 2k, k \in \mathbb{N}_0$ we need $2k = n$ for the upper bound of the sum, which implies $k = \frac{n}{2}$.

$$\begin{aligned} 2 + 4 + \dots + n &= \sum_{k=1}^{\frac{n}{2}} (2k) \\ \text{let } m &= \frac{n}{2} \\ \Rightarrow \sum_{k=1}^{\frac{n}{2}} (2k) &= \sum_{k=1}^m (2k) \end{aligned}$$

From equation 4.1 we get

$$\begin{aligned} \sum_{k=1}^m (2k) &= m^2 + m = \frac{n^2}{4} + \frac{n}{2} = \frac{n^2 + 2n}{4} \\ \Rightarrow \boxed{\sum_{k=1}^{\frac{n}{2}} (2k) = 2 + 4 + \dots + n = \frac{n(n+2)}{4}} \end{aligned}$$

4.4 Sum of the first odd numbers up to n

If we let an odd number $n \in \mathbb{N}$ have the form $n = 2k - 1, k \in \mathbb{N}_0$ we need $2k - 1 = n$ for the upper bound of the sum, which implies $k = \frac{(n+1)}{2}$.

$$\begin{aligned} 1 + 3 + \dots + n &= \sum_{k=1}^{\frac{(n+1)}{2}} (2k - 1) \\ \text{let } m &= \frac{n+1}{2}, m \in \mathbb{N} \\ \Rightarrow \sum_{k=1}^{\frac{(n+1)}{2}} (2k - 1) &= \sum_{k=1}^m (2k - 1) \end{aligned}$$

From equation 4.2 we get

$$\begin{aligned} \sum_{k=1}^m (2k - 1) &= m^2 = \left(\frac{n+1}{2}\right)^2 \\ \Rightarrow \boxed{\sum_{k=1}^{\frac{(n+1)}{2}} (2k - 1) = 1 + 3 + \dots + n = \left(\frac{n+1}{2}\right)^2} \end{aligned}$$

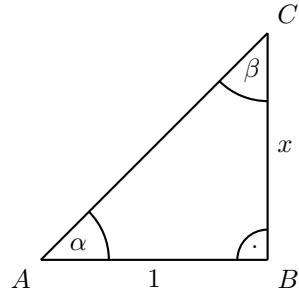
5 Arctangent

5.1 $\arctan(x) + \arctan(1/x)$

5.1.1 over Triangle

Note: Not my observation, nor my explanation, but I do love this

Consider the following right angled triangle $\triangle ABC$



All angles in a triangle sum up to $\pi \implies \alpha + \beta = \frac{\pi}{2}$

$$\begin{aligned}\tan(\alpha) &= x \Leftrightarrow \alpha = \arctan(x) \\ \tan(\beta) &= \frac{1}{x} \Leftrightarrow \beta = \arctan\left(\frac{1}{x}\right) \\ \alpha + \beta &= \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}\end{aligned}$$

From symmetry we know that $\arctan(-x) = -\arctan(x)$

$$\begin{aligned}\therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\ &= -\frac{\pi}{2}\end{aligned}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x) \cdot \frac{\pi}{2}}$$

5.1.2 over sin and cos

option 1:

Note: observation by D. J.

$$\boxed{x, \alpha, \beta > 0}$$

$$\alpha + \beta = \frac{\pi}{2} \implies \beta = \frac{\pi}{2} - \alpha$$

$$\text{let } \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = x$$

$$\tan(\beta) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)} = \frac{1}{x}$$

$$\implies \alpha = \arctan(x), \quad \beta = \frac{\pi}{2} - \alpha = \arctan\left(\frac{1}{x}\right)$$

$$\implies \arctan(x) + \arctan\left(\frac{1}{x}\right) = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$$

$$\tan(-\alpha) = -\tan(\alpha) = -x$$

$$\tan(-\beta) = -\tan(\beta) = -\frac{1}{x}$$

$$\implies \arctan(-x) = -\alpha, \quad \arctan\left(-\frac{1}{x}\right) = -\beta = -\frac{\pi}{2} + \alpha$$

$$\implies \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\alpha - \frac{\pi}{2} + \alpha = -\frac{\pi}{2}$$

$$\Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}}$$

option 2:

Note: observation by D. J.

$$\begin{aligned}
 \arctan(x) + \arctan\left(\frac{1}{x}\right) &= ? \\
 \tan(\alpha) &= \frac{\sin(\alpha)}{\cos(\alpha)} = x \\
 \tan(\beta) &= \frac{1}{x} = \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\sin(\beta)}{\cos(\beta)} \\
 \implies \beta &= \frac{\pi}{2} - \alpha \implies \alpha + \beta = \frac{\pi}{2} \\
 \implies \tan(\beta) &= \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan(\alpha)} = \frac{1}{x} \\
 \implies \arctan(x) + \arctan\left(\frac{1}{x}\right) &= \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}
 \end{aligned}$$

From symmetry we know that $\arctan(-x) = -\arctan(x)$

$$\begin{aligned}
 \therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\
 &= -\frac{\pi}{2} \\
 \implies \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right)} &= \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}
 \end{aligned}$$

5.1.3 over derivative and limit

$$\begin{aligned}
 f(x) &= \arctan(x) + \arctan\left(\frac{1}{x}\right) \quad D_f = \mathbb{R} \setminus \{0\} \\
 f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0
 \end{aligned}$$

If $f'(x) = 0$ then $f(x)$ has at least one constant value

Since 0 is excluded, we should check for values above, and below zero, because, this could be the only jump in the function, because it's continuous everywhere else (in \mathbb{R}^+ and in \mathbb{R}^-):

Case 1: $x > 0$

$$f(1) = \arctan(1) + \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Case 2: $x < 0$

$$f(-1) = \arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

$$\implies f(x) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}$$

6 Signum Function

The signum function yields the sign of a number:

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (6.1)$$

6.1 link to $|x|$

$$\begin{aligned} |x'|' &= (\sqrt{x^2})' = \frac{1}{2\sqrt{x^2}} \cdot (2x) = \frac{x}{\sqrt{x^2}} \\ &= \frac{x}{|x|} = \begin{cases} \frac{x}{x} & \text{if } x > 0 \\ \frac{x}{-x} & \text{if } x < 0 \end{cases} \\ &= \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \implies & \boxed{|x'|' = \operatorname{sgn}(x) = \frac{x}{|x|} \text{ if } x \neq 0} \end{aligned} \quad (6.2)$$

6.2 Reciprocal of $\operatorname{sgn}(x)$

From definition of the sgn function (see: 6.1) follows:

$$\begin{aligned} \frac{1}{\operatorname{sgn}(x)} &= \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \implies & \boxed{\frac{1}{\operatorname{sgn}(x)} = \operatorname{sgn}(x)} \end{aligned}$$

Using equation 6.2

$$\implies \boxed{\operatorname{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x} \text{ if } x \neq 0} \quad (6.3)$$

6.3 x and $\operatorname{sgn}(x)$

From equation 6.3 we get:

$$\boxed{\begin{aligned} |x| &= \operatorname{sgn}(x) \cdot x \\ x &= \operatorname{sgn}(x) \cdot |x| \end{aligned}}$$

when $x \neq 0$.

6.4 Looking at $\sqrt{\frac{a}{x^2}}$

$$\begin{aligned} \sqrt{\frac{a}{x^2}} &= \sqrt{\frac{1}{x^2}} \cdot \sqrt{a} = \frac{1}{\sqrt{x^2}} \sqrt{a} \\ \implies \boxed{\sqrt{\frac{a}{x^2}} &= \frac{1}{|x|} \sqrt{a}} \end{aligned} \tag{6.4}$$

6.5 Fun connection with $x \cdot \sqrt{\frac{a}{x^2}}$

From equation 6.4 we get:

$$\begin{aligned} x \cdot \sqrt{\frac{a}{x^2}} &= x \cdot \frac{1}{|x|} \cdot \sqrt{a} = \frac{x}{|x|} \cdot \sqrt{a} \\ \implies \boxed{x \cdot \sqrt{\frac{a}{x^2}} &= \operatorname{sgn}(x) \cdot \sqrt{a}} \end{aligned}$$

7 Longer Derivations

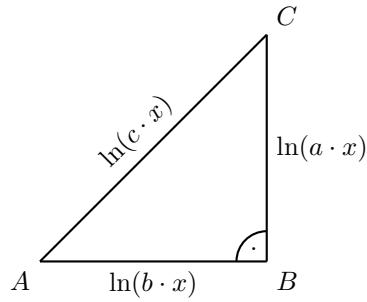
7.1 Quadratic Formular

$$\begin{aligned}
 ax^2 + bx + c &= a \left[x^2 + \frac{b}{a}x \right] + c = a \left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a} \right)^2 - \left(\frac{b}{2a} \right)^2 \right] + c \\
 &= a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a^2} \right] + c = a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c \\
 ax^2 + bx + c &= 0 \\
 a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2}{4a} + c &= 0 \\
 a \left(x + \frac{b}{2a} \right)^2 &= \frac{b^2}{4a} - c \\
 x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \\
 x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}} &= -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} \\
 \Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}
 \end{aligned}$$

7.2 ln Triangle

This section was inspired by the video of blackpenredpen about a similar triangle.

Consider the following triangle, with $a, b, c, x \in \mathbb{R}^+$:



We want to find x :

$$\begin{aligned}\ln(ax)^2 + \ln(bx)^2 &= \ln(cx)^2 \\ \ln(ax)^2 + \ln(bx)^2 - \ln(cx)^2 &= 0 \\ [\ln(a) + \ln(x)]^2 + [\ln(bx) + \ln(cx)] \cdot [\ln(bx) - \ln(cx)] &= 0 \\ \ln(a)^2 + 2\ln(a)\ln(x) + \ln(x)^2 + \ln(bc x^2) \ln\left(\frac{b}{c}\right) &= 0\end{aligned}$$

Let's focus on $\ln(bc x^2) \ln\left(\frac{b}{c}\right)$:

$$\ln(bc x^2) \ln\left(\frac{b}{c}\right) = [\ln(bc) + \ln(x^2)] \ln\left(\frac{b}{c}\right) = \ln(bc) \ln\left(\frac{b}{c}\right) + \ln \frac{b}{c} \ln(x^2)$$

Inserting back into the original equation:

$$\begin{aligned}\ln(a)^2 + 2\ln(a)\ln(x) + \ln(x)^2 + \ln(bc) \ln\left(\frac{b}{c}\right) + \ln\left(\frac{b}{c}\right) \ln(x^2) &= 0 \\ \ln(x)^2 + 2\ln(a)\ln(x) + 2\ln\left(\frac{b}{c}\right) \ln(x) + \ln(bc) \ln\left(\frac{b}{c}\right) + \ln(a)^2 &= 0 \\ \ln(x)^2 + 2\left[\ln(a) + \ln\left(\frac{b}{c}\right)\right] \ln(x) + \ln(bc) \ln\left(\frac{b}{c}\right) + \ln(a)^2 &= 0 \\ \ln(x)^2 + 2\ln\left(\frac{ab}{c}\right) \ln(x) + \ln(bc) \ln\left(\frac{b}{c}\right) + \ln(a)^2 &= 0 \\ \ln(x) = \frac{-2\ln\left(\frac{ab}{c}\right) \pm \sqrt{(2\ln\left(\frac{ab}{c}\right))^2 - 4(\ln(a)^2 + \ln(bc) \ln\left(\frac{b}{c}\right))}}{2} & \\ = \ln\left(\frac{c}{ab}\right) \pm \sqrt{\ln\left(\frac{ab}{c}\right)^2 - \left(\ln(a)^2 + \ln(bc) \ln\left(\frac{b}{c}\right)\right)} & \\ = \ln\left(\frac{c}{ab}\right) \pm \sqrt{\left(\ln\left(\frac{ab}{c}\right) - \ln(a)\right) \cdot \left(\ln\left(\frac{ab}{c}\right) + \ln(a)\right) - \ln(bc) \ln\left(\frac{b}{c}\right)} & \\ = \ln\left(\frac{c}{ab}\right) \pm \sqrt{\ln\left(\frac{b}{c}\right) \ln\left(\frac{a^2 b}{c}\right) - \ln(bc) \ln\left(\frac{b}{c}\right)} & \\ = \ln\left(\frac{c}{ab}\right) \pm \sqrt{\left(\ln\left(\frac{a^2 b}{c}\right) - \ln(bc)\right) \ln\left(\frac{b}{c}\right)} & \\ = \ln\left(\frac{c}{ab}\right) \pm \sqrt{\ln\left(\frac{a^2}{c^2}\right) \ln\left(\frac{b}{c}\right)} & \\ \implies \ln(x) = \ln\left(\frac{c}{ab}\right) \pm \sqrt{2\ln\left(\frac{a}{c}\right) \ln\left(\frac{b}{c}\right)} &\end{aligned}$$