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# 1 Logarithms

## 1.1 Recipricol of the Logarithm

$$\frac{1}{\log_a(b)} = \frac{1}{\frac{\log_c(b)}{\log_c(a)}} = \frac{\log_c(a)}{\log_c(b)} = \log_b(a)$$
$$\Rightarrow \boxed{\frac{1}{\log_a(b)} = \log_b(a)} \quad (1.1)$$

## 1.2 Divison of logs

### 1.2.1 General case

Using equation 1.1 we get:

$$\boxed{\frac{\log_a(b)}{\log_c(d)} = \log_a(b) \cdot \log_d(c)} \quad (1.2)$$

### 1.2.2 same Argument

Using Equation 1.2 and assuming that  $b = d$ , we get:

$$\frac{\log_a(b)}{\log_c(d)} = \frac{\log_a(b)}{\log_c(b)} = \log_a(b) \cdot \log_b(c) = \log_a(b^{\log_b(c)}) = \log_a(c)$$
$$\Rightarrow \boxed{\frac{\log_a(b)}{\log_c(b)} = \log_a(c)}$$

## 2 Physics

### 2.1 Mix Temperature

$$\vartheta_1 > \vartheta_2$$

$$c_1 \cdot m_1 \cdot (\vartheta_1 - \vartheta_m) = c_2 \cdot m_2 \cdot (\vartheta_m - \vartheta_2)$$

$$c_1 \cdot m_1 \cdot \vartheta_1 - c_1 \cdot m_1 \cdot \vartheta_m = c_2 \cdot m_2 \cdot \vartheta_m - c_2 \cdot m_2 \cdot \vartheta_2$$

$$c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 = c_1 \cdot m_1 \cdot \vartheta_m + c_2 \cdot m_2 \cdot \vartheta_m$$

$$c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2 = \vartheta_m \cdot (c_1 \cdot m_1 + c_2 \cdot m_2)$$

$$\Rightarrow \boxed{\vartheta_m = \frac{c_1 \cdot m_1 \cdot \vartheta_1 + c_2 \cdot m_2 \cdot \vartheta_2}{c_1 \cdot m_1 + c_2 \cdot m_2}}$$

## 3 Complex Numbers

### 3.1 Complex Number and their conjugate

$\forall z \in \mathbb{C}; \exists z^* \in \mathbb{C}$ , such that:

$$\begin{aligned} z &= a + bi & a, b \in \mathbb{R} \\ z^* &= \text{conj}(z) = a - bi \end{aligned}$$

The real number  $a$  is called the *real part* of a complex number  $z = a + bi$   $z \in \mathbb{C}$   $a, b \in \mathbb{R}$ , while the real number  $b$  is called the *imaginary part* of that number. The real part of a complex number  $z$  is denoted with  $\Re(z)$ , the imaginary part with  $\Im(z)$ .

The magnitude (or absolut value)  $|z|$  of a complex number  $z \in \mathbb{C}$  is defined as:

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} = \sqrt{(\Re(z))^2 + (\Im(z))^2} \\ \implies |z|^2 &= a^2 + b^2 = (\Re(z))^2 + (\Im(z))^2 \end{aligned}$$

#### 3.1.1 Complex plus Conjugate: $z + z^*$

$$\begin{aligned} z + z^* &= a + bi + a - bi = 2 \cdot a = 2 \cdot \Re(z) \\ \implies &\boxed{z + z^* = 2 \cdot \Re(z)} \end{aligned}$$

#### 3.1.2 Complex minus Conjugate: $z - z^*$

$$\begin{aligned} z - z^* &= a + bi - a + bi = 2 \cdot bi = 2i \cdot \Im(z) \\ \implies &\boxed{z - z^* = 2i \cdot \Im(z)} \end{aligned} \tag{3.1}$$

#### 3.1.3 Conjugate minus Complex: $z^* - z$

Using equation 3.1, we get:

$$\begin{aligned} z^* - z &= -(z - z^*) \\ \implies &\boxed{z^* - z = -2i \cdot \Im(z)} \end{aligned}$$

### 3.1.4 Complex times Conjugate: $z \cdot z^*$

$$z \cdot z^* = (a + bi) \cdot (a - bi) = a^2 + a \cdot bi - a \cdot bi + b^2 = a^2 + b^2$$

$$\implies \boxed{z \cdot z^* = (\Re(z))^2 + (\Im(z))^2 = |z|^2}$$

### 3.1.5 Complex over Conjugate: $\frac{z}{z^*}$

$$\frac{z}{z^*} = \frac{a + bi}{a - bi} = \frac{a + bi}{a - bi} \cdot \frac{a + bi}{a + bi} = \frac{a^2 + 2i \cdot a \cdot b - b^2}{a^2 + b^2}$$

$$\implies \boxed{\frac{z}{z^*} = \frac{(\Re(z))^2 - (\Im(z))^2 + 2i \cdot \Re(z) \cdot \Im(z)}{|z|^2}}$$

### 3.1.6 Conjugate over Complex: $\frac{z^*}{z}$

$$\frac{z^*}{z} = \frac{a - bi}{a + bi} = \frac{a - bi}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a^2 + b^2 - 2a \cdot bi}{a^2 + b^2} = 1 - \frac{2 \cdot a \cdot b}{a^2 + b^2} \cdot i$$

$$\implies \boxed{\frac{z^*}{z} = 1 - i \frac{2 \cdot \Re(z) \cdot \Im(z)}{|z|^2}}$$

## 4 Sums

### 4.1 Sum of the first $n$ even numbers

Note: not my idea, but my way of showing this relation

Let  $m$  be the  $n$ th even number, then  $m = 2n$ ,  $m, n \in \mathbb{N}$

$$\begin{aligned}\sum_{k=1}^n (2k) &= 2 \cdot \sum_{k=1}^n (k) = 2 \cdot \frac{n(n+1)}{2} = n^2 + n \\ \implies \boxed{\sum_{k=1}^n (2k) &= n^2 + n}\end{aligned}\tag{4.1}$$

### 4.2 Sum of the first $n$ odd numbers

Note: not my idea, but my way of showing this relation

Let  $m$  be the  $n$ th odd number, then  $m = 2n - 1$ ,  $m, n \in \mathbb{N}$

Using Equation 4.1

$$\begin{aligned}\sum_{k=1}^n (2k - 1) &= \sum_{k=1}^n (2k) - \sum_{k=1}^n (1) = n^2 + n - n \\ \implies \boxed{\sum_{k=1}^n (2k - 1) &= n^2}\end{aligned}\tag{4.2}$$

### 4.3 Sum of the first even numbers up to $n$

If we let an even number  $n \in \mathbb{N}$  have the form  $n = 2k, k \in \mathbb{N}_0$  we need  $2k = n$  for the upper bound of the sum, which implies  $k = n/2$ .

$$0 + 2 + 4 + \dots + n = \sum_{k=1}^{n/2} (2k)$$

$$\text{let } m = \frac{n}{2}$$

$$\implies \sum_{k=1}^{n/2} (2k) = \sum_{k=1}^m (2k)$$

From equation 4.1 we get

$$\sum_{k=1}^m (2k) = m^2 + m = \frac{n^2}{4} + \frac{n}{2} = \frac{n^2 + 2n}{4}$$

$$\implies \boxed{\sum_{k=1}^{n/2} (2k) = 0 + 2 + 4 + \dots + n = \frac{n(n+2)}{4}}$$

### 4.4 Sum of the first odd numbers up to $n$

If we let an odd number  $n \in \mathbb{N}$  have the form  $n = 2k - 1, k \in \mathbb{N}_0$  we need  $2k - 1 = n$  for the upper bound of the sum, which implies  $k = (n+1)/2$ .

$$1 + 3 + \dots + n = \sum_{k=1}^{(n+1)/2} (2k - 1)$$

$$\text{let } m = \frac{n+1}{2}, m \in \mathbb{N}$$

$$\implies \sum_{k=1}^{(n+1)/2} (2k - 1) = \sum_{k=1}^m (2k - 1)$$

From equation 4.2 we get

$$\sum_{k=1}^m (2k - 1) = m^2 = \left(\frac{n+1}{2}\right)^2$$

$$\implies \boxed{\sum_{k=1}^{(n+1)/2} (2k - 1) = 1 + 3 + \dots + n = \left(\frac{n+1}{2}\right)^2}$$



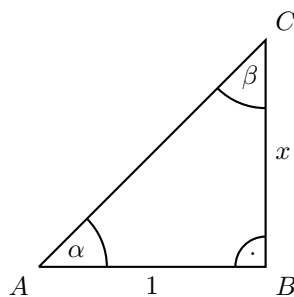
## 5 Arctangent

### 5.1 $\arctan(x) + \arctan(1/x)$

#### 5.1.1 over Triangle

Note: Not my observation, nor my explanation, but I do love this

Consider the following right angled triangle  $\triangle ABC$



All angles in a triangle sum up to  $\pi \implies \alpha + \beta = \frac{\pi}{2}$

$$\tan(\alpha) = x \Leftrightarrow \alpha = \arctan(x)$$

$$\tan(\beta) = \frac{1}{x} \Leftrightarrow \beta = \arctan\left(\frac{1}{x}\right)$$

$$\alpha + \beta = \arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

From symmetry we know that  $\arctan(-x) = -\arctan(x)$

$$\begin{aligned} \therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\ &= -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow & \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}} \\ \Rightarrow & \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \operatorname{sgn}(x) \cdot \frac{\pi}{2}} \end{aligned}$$

### 5.1.2 over sin and cos

#### option 1:

Note: observation by D. J.

$$\begin{aligned} & \boxed{x, \alpha, \beta > 0} \\ & \alpha + \beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \alpha \\ & \text{let } \tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)} = x \\ & \tan(\beta) = \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{1}{\tan(\alpha)} = \frac{1}{x} \\ & \Rightarrow \alpha = \arctan(x), \quad \beta = \frac{\pi}{2} - \alpha = \arctan\left(\frac{1}{x}\right) \\ & \Rightarrow \arctan(x) + \arctan\left(\frac{1}{x}\right) = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2} \\ & \tan(-\alpha) = -\tan(\alpha) = -x \\ & \tan(-\beta) = -\tan(\beta) = -\frac{1}{x} \\ & \Rightarrow \arctan(-x) = -\alpha, \quad \arctan\left(-\frac{1}{x}\right) = -\beta = -\frac{\pi}{2} + \alpha \\ & \Rightarrow \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\alpha - \frac{\pi}{2} + \alpha = -\frac{\pi}{2} \\ & \Rightarrow \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}} \end{aligned}$$

**option 2:**

Note: observation by D. J.

$$\begin{aligned}
 \arctan(x) + \arctan\left(\frac{1}{x}\right) &= ? \\
 \tan(\alpha) &= \frac{\sin(\alpha)}{\cos(\alpha)} = x \\
 \tan(\beta) = \frac{1}{x} &= \frac{1}{\tan(\alpha)} = \frac{\cos(\alpha)}{\sin(\alpha)} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\sin(\beta)}{\cos(\beta)} \\
 \implies \beta &= \frac{\pi}{2} - \alpha \implies \alpha + \beta = \frac{\pi}{2} \\
 \implies \tan(\beta) &= \tan\left(\frac{\pi}{2} - \alpha\right) = \frac{1}{\tan(\alpha)} = \frac{1}{x} \\
 \implies \arctan(x) + \arctan\left(\frac{1}{x}\right) &= \cancel{\alpha} + \frac{\pi}{2} - \cancel{\alpha} = \frac{\pi}{2}
 \end{aligned}$$

From symmetry we know that  $\arctan(-x) = -\arctan(x)$

$$\begin{aligned}
 \therefore \arctan(-x) + \arctan\left(-\frac{1}{x}\right) &= -\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right) \\
 &= -\frac{\pi}{2}
 \end{aligned}$$

$$\implies \boxed{\arctan(x) + \arctan\left(\frac{1}{x}\right) = \begin{cases} \frac{\pi}{2} & \text{if } x > 0 \\ -\frac{\pi}{2} & \text{if } x < 0 \end{cases}}$$

**5.1.3 over derivative and limit**

$$\begin{aligned}
 f(x) &= \arctan(x) + \arctan\left(\frac{1}{x}\right) \quad D_f = \mathbb{R} \setminus \{0\} \\
 f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0
 \end{aligned}$$

If  $f'(x) = 0$  then  $f(x)$  has at least one constant value

Since 0 is excluded, we should check for values above, and below zero, because, this could be the only jump in the function, because it's continuous everywhere else (in  $\mathbb{R}^+$  and in  $\mathbb{R}^-$ ):

Case 1:  $x > 0$

$$f(1) = \arctan(1) + \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

Case 2:  $x < 0$

$$f(-1) = \arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$$

$$\Rightarrow f(x) = \begin{cases} \frac{\pi}{2} & \text{für } x > 0 \\ -\frac{\pi}{2} & \text{für } x < 0 \end{cases}$$

## 6 Signum Function

The signum function yields the sign of a number:

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \quad (6.1)$$

### 6.1 link to $|x|$

$$\begin{aligned} |x|' &= \left(\sqrt{x^2}\right)' = \frac{1}{2\sqrt{x^2}} \cdot (2x) = \frac{x}{\sqrt{x^2}} \\ &= \frac{x}{|x|} = \begin{cases} \frac{x}{x} & \text{if } x > 0 \\ \frac{x}{-x} & \text{if } x < 0 \end{cases} \\ &= \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \Rightarrow &\boxed{|x|' = \operatorname{sgn}(x) = \frac{x}{|x|} \text{ if } x \neq 0} \end{aligned} \quad (6.2)$$

### 6.2 Reciprocal of $\operatorname{sgn}(x)$

From definition of the  $\operatorname{sgn}$  function (see: 6.1) follows:

$$\begin{aligned} \frac{1}{\operatorname{sgn}(x)} &= \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \\ \Rightarrow &\boxed{\frac{1}{\operatorname{sgn}(x)} = \operatorname{sgn}(x)} \end{aligned}$$

Using equation 6.2

$$\Rightarrow \boxed{\operatorname{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}} \quad (6.3)$$

### 6.3 $x$ and $\text{sgn}(x)$

From equation 6.3 we get:

$$\boxed{|x| = \text{sgn}(x) \cdot x}$$

$$\boxed{x = \text{sgn}(x) \cdot |x|}$$

### 6.4 Fun connection with $x \cdot \sqrt{\frac{a}{x^2}}$

$$\begin{aligned} x \cdot \sqrt{\frac{a}{x^2}} &= x \cdot \sqrt{\frac{a}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{x^2}} = x \cdot \sqrt{a} \cdot \sqrt{\frac{1}{x^2}} = \frac{x}{\sqrt{x^2}} \sqrt{a} = \frac{x}{|x|} \sqrt{a} \\ &\Rightarrow \boxed{x \cdot \sqrt{\frac{a}{x^2}} = \text{sgn}(x) \cdot \sqrt{a}} \end{aligned}$$

### 6.5 Similarly $\sqrt{\frac{a}{x^2}}$

$$\begin{aligned} \sqrt{\frac{a}{x^2}} &= \sqrt{\frac{1}{x^2}} \cdot \sqrt{a} = \frac{1}{\sqrt{x^2}} \sqrt{a} \\ &\Rightarrow \boxed{\sqrt{\frac{a}{x^2}} = \frac{1}{|x|} \sqrt{a}} \end{aligned}$$