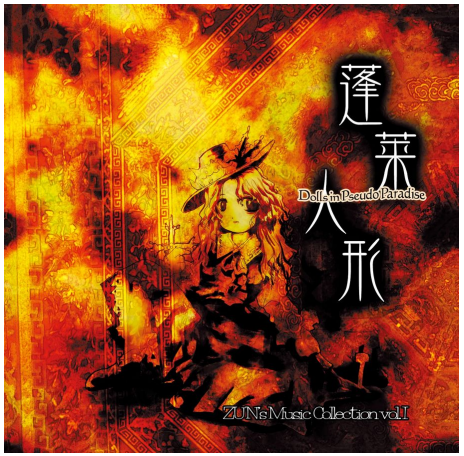
Our village of honest men originally consisted of only eight people.
We all picked up and moved to a mountain in the east. Two years of honest and boring daily life passed us by.
One day, one of us found a little hole by a peach tree.
Yes, after that we wandered into this paradise.
And right away, I quit being human.

— *Dolls in Pseudo Paradise*

蓬莱人形算法模板库

Reference Document for *Dolls in Pseudo Paradise*



2024-2025

Harbin Institute of Technology

目录

1 动态规划

1.1 斜率优化

1.1.1 形式

考虑一个经典的 dp 转移方程如下：

$$f_i = \max_{j<i}\{f(j) + w(j,i)\}$$

我们将式子拆成三个部分：只跟 i 有关或者与 i,j 均不相关的部分 $a(i)$，只跟 j 有关的部分 $b(j)$，跟 i,j 均有关的部分 $c(i,j)$：

$$f_i = a(i) + \max_{j<i}\{b(j) + c(i,j)\}$$

斜率优化可被用来解决这样一个情形：$c(i,j) = ic_j$。此时 $b(j) + c(i,j)$ 可视作关于 j 的一次函数。如果 c_j 随着 j 的增大而单调，那么可用单调栈维护；否则可以考虑 CDQ 分治或者在凸包上二分。在凸包上可以使用二分查询最高/最低点。

1.1.2 例题

玩具装箱。原始转移方程为：

$$f_i = \max_{j<i}\{f_j + (s_i - s_j - L')^2\}$$

其中 $s_i = i + \sum_{j\le i} c_i, L' = L + 1$。将其分类得到：

$$f_i = \max_{j<i}\{f_j + s_i^2 + s_j^2 + L'^2 - 2s_is_j + 2s_jL' - 2s_iL'\}$$
$$= (s_i^2 - 2s_iL' + L'^2) + \max_{j<i}\{(f_j + s_j^2 + 2s_jL') - 2s_is_j\}$$

在原始的玩具装箱中，s_j 单调增加，也就是斜率单调增加。因此可以直接使用单调栈维护凸包。同时 s_i 也单调增加，因此可以用指针维护。

```cpp
#include "../header.cpp"
int n, L, p, e, C[MAXN], Q[MAXN];
f80 S[MAXN], F[MAXN];
f80 gtx(int x){ return S[x]; }
f80 gty(int x){ return F[x] + S[x] * S[x]; }
f80 gtw(int x){ return -2.0 * (L - S[x]); }
f80 gtk(int x,int y){ return (gty(y) - gty(x)) / (gtx(y) - gtx(x)); }
int main(){
  cin >> n >> L;
  for(int i = 1;i <= n;++ i){
    cin >> C[i];
    S[i] = S[i - 1] + C[i];
  }
  for(int i = 1;i <= n;++ i){
    S[i] += i;
  }
  e = p = 1, L ++, Q[p] = 0;
  for(int i = 1;i <= n;++ i){
    while(e < p && gtk(Q[e], Q[e + 1]) < gtw(i))
      ++ e;
    int j = Q[e];
    F[i] = F[j] + pow(S[i] - S[j] - L, 2);
    while(1 < p && gtk(Q[p - 1], Q[p]) > gtk(Q[p], i))
      e -= (e == p), -- p;
    Q[++ p] = i;
  }
  printf("%.0Lf\n", F[n]);
  return 0;
}
```

2 数据结构

2.1 平衡树

2.1.1 无旋 Treap

```cpp
#include "../../header.cpp"
mt19937_64 MT(114514);
namespace Treap{
  const int SIZ = 1e6 + 1e5 + 3;
  int F[SIZ], C[SIZ], S[SIZ], W[SIZ], X[SIZ][2], sz;
  u64 H[SIZ];
  int  newnode(int w){
    W[++ sz] = w, C[sz] = S[sz] = 1; H[sz] = MT();
    return sz;
  }
  void pushup(int x){
    S[x] = C[x] + S[X[x][0]] + S[X[x][1]];
  }
  pair<int, int> split(int u, int x){
    if(u == 0)
      return make_pair(0, 0);
    if(W[u] > x){
      auto [a, b] = split(X[u][0], x);
      X[u][0] = b, pushup(u);
      return make_pair(a, u);
    } else {
      auto [a, b] = split(X[u][1], x);
      X[u][1] = a, pushup(u);
      return make_pair(u, b);
    }
  }
  int merge(int a, int b){
    if(a == 0 || b == 0)
      return a | b;
    if(H[a] < H[b]){
      X[a][1] = merge(X[a][1], b), pushup(a);
      return a;
    } else {
      X[b][0] = merge(a, X[b][0]), pushup(b);
      return b;
    }
  }
  void insert(int &root, int w){
    auto [p, q] = split(root, w  );
    auto [a, b] = split(   p, w - 1);
    if(b != 0){
      ++ S[b], ++ C[b];
    } else b = newnode(w);
    p  = merge(a, b);
    root = merge(p, q);
  }
  void erase(int &root, int w){
    auto [p, q] = split(root, w  );
    auto [a, b] = split(   p, w - 1);
    -- C[b], -- S[b];
    p  = C[b] == 0 ? a : merge(a, b);
    root = merge(p, q);
  }
  int  find_rank(int &root, int w){
    int x = root, o = x, a = 0;
    for(;x;){
      if(w <  W[x])
        o = x, x = X[x][0];
      else {
        a += S[X[x][0]];
        if(w == W[x]){
          o = x; break;
        }
        a += C[x];
        o = x, x = X[x][1];
      }
    }
    return a + 1;
  }
  int  find_kth(int &root, int w){
    int x = root, o = x, a = 0;
    for(;x;){
      if(w <= S[X[x][0]])
        o = x, x = X[x][0];
      else {
        w -= S[X[x][0]];
        if(w <= C[x]){
          o = x; break;
        }
        w -= C[x];
        o = x, x = X[x][1];
      }
    }
    return W[x];
  }
  int  find_pre(int &root, int w){
    return find_kth(root, find_rank(root, w) - 1);
  }
  int  find_suc(int &root, int w){
    return find_kth(root, find_rank(root, w + 1));
  }
}
```

2.2 珂朵莉树

```cpp
#include "../header.cpp"
namespace ODT {
  // <pos_type, value_type>
  map <int, long long> M;
  // 分裂为 [1, p) 和 [p, +inf)，返回后者迭代器
  auto split(int p) {
    auto it = prev(M.upper_bound(p));
    return M.insert(
      it,
      make_pair(p, it -> second)
    );
  }
  // 区间赋值
  void assign(int l, int r, int v) {
    auto it = split(l);
```

```
    split(r + 1);
    while (it -> first != r + 1) {
      it = M.erase(it);
    }
    M[l] = v;
  }
  // // 执行操作
  // void perform(int l, int r) {
  //   auto it = split(l);
  //   split(r + 1);
  //   while (it -> first != r + 1) {
  //     // Do something...
  //     it = next(it);
  //   }
  // }
};
```

2.3 可并堆

```
#include "../header.cpp"
namespace LeftHeap{
  const int SIZ = 1e5 + 3;
  int W[SIZ], D[SIZ], L[SIZ], R[SIZ], F[SIZ],
    s;
  bool E[SIZ];
  int merge(int u, int v){
    if(u == 0 || v == 0)
      return u | v;
    if(W[u] > W[v] || (W[u] == W[v] && u > v))
      swap(u, v);
    int &lc = L[u];
    int &rc = R[u];
    rc = merge(rc, v);
    if(D[lc] < D[rc])
      swap(lc, rc);
    D[u] = min(D[lc], D[rc]) + 1;
    if(lc != 0) F[lc] = u;
    if(rc != 0) F[rc] = u;
    return u;
  }
  void pop(int &root){
    int root0 = merge(L[root], R[root]);
    F[root0] = root0;
    F[root ] = root0;
    E[root ] = true;
    root = root0;
  }
  int top(int &root){
    return W[root];
  }
  int getfa(int u){
    return u == F[u] ? u : F[u] = getfa(F[u]);
  }
  int newnode(int w){
    ++ s;
    W[s] = w;
    F[s] = s;
    D[s] = 1;
    return s;
  }
}
```

2.4 Link Cut 树

```
#include "../header.cpp"
namespace LinkCutTree{
  const int SIZ = 1e5 + 3;
  int F[SIZ], C[SIZ], S[SIZ], W[SIZ], A[SIZ],
    X[SIZ][2], size;
  bool T[SIZ];
  bool is_root(int x){ return X[F[x]][0] != x
    && X[F[x]][1] != x;}
  bool is_rson(int x){ return X[F[x]][1] == x
    ;}
  int  new_node(int w){
    ++ size;
    W[size] = w, C[size] = S[size] = 1;
    A[size] = w, F[size] = 0;
    X[size][0] = X[size][1] = 0;
    return size;
  }
  void push_up(int x){
    S[x] = C[x] + S[X[x][0]] + S[X[x][1]];
    A[x] = W[x] ^ A[X[x][0]] ^ A[X[x][1]];
  }
  void push_down(int x){
    if(!T[x]) return;
    int lc = X[x][0], rc = X[x][1];
    if(lc) T[lc] ^= 1, swap(X[lc][0], X[lc
      ][1]);
    if(rc) T[rc] ^= 1, swap(X[rc][0], X[rc
      ][1]);
    T[x] = false;
  }
  void update(int x){
    if(!is_root(x)) update(F[x]); push_down(x)
      ;
  }
  void rotate(int x){
    int y = F[x], z = F[y];
    bool f = is_rson(x);
    bool g = is_rson(y);
    if(is_root(y)){
      F[x] = z, F[y] = x;
      X[z][ g] = x;
      X[y][ f] = X[x][!f], F[X[x][!f]] = y;
      X[x][!f] = y;
    } else {
      F[x] = z, F[y] = x;
      X[z][ g] = x;
      X[y][ f] = X[x][!f], F[X[x][!f]] = y;
      X[x][!f] = y;
    }
    push_up(y), push_up(x);
  }
  void splay(int x){
    update(x);
    for(int f = F[x];f = F[x], !is_root(x);
      rotate(x))
      if(!is_root(f)) rotate(is_rson(x) ==
        is_rson(f) ? f : x);
  }
  int  access(int x){
    int p;
    for(p = 0;x;p = x, x = F[x]){
      splay(x), X[x][1] = p, push_up(x);
    }
    return p;
  }
  void make_root(int x){
    x = access(x);
    T[x] ^= 1, swap(X[x][0], X[x][1]);
  }
  int find_root(int x){
    access(x), splay(x), push_down(x);
    while(X[x][0]) x = X[x][0], push_down(x);
    splay(x);
    return x;
  }
  void link(int x, int y){
    make_root(x), splay(x), F[x] = y;
  }
  void cut(int x, int p){
    make_root(x), access(p), splay(p), X[p][0]
      = F[x] = 0;
  }
  void modify(int x, int w){
    splay(x), W[x] = w, push_up(x);
  }
}
const int MAXN = 1e5 + 3;
map<pair<int, int>, bool> M;
int n, m;
int main(){
  cin >> n >> m;
  for(int i = 1;i <= n;++ i){
    int a; cin >> a;
```

```
    LinkCutTree :: new_node(a);
  }
  for(int i = 1;i ⩽ m;++ i){
    int o; cin >> o;
    if(o == 0){
      int u, v; cin >> u >> v;
      LinkCutTree :: make_root(u);
      int p = LinkCutTree :: access(v);
      printf("%d\n", LinkCutTree :: A[p]);
    } else if(o == 1){
      int u, v; cin >> u >> v;
      int a = LinkCutTree :: find_root(u);
      int b = LinkCutTree :: find_root(v);
      if(a ≠ b){
        LinkCutTree :: link(u, v);
        M[make_pair(min(u, v), max(u, v))] =
          true;
      }
    } else if(o == 2){
      int u, v; cin >> u >> v;
      if(M.count(make_pair(min(u, v), max(u, v
        )))){
        M.erase(make_pair(min(u, v), max(u, v)
          ));
        LinkCutTree :: cut(u, v);
      }
    } else {
      int u, w; cin >> u >> w;
      LinkCutTree :: modify(u, w);
    }
  }
  return 0;
}
```

2.5 线段树

2.5.1 李超树

```
#include "../../header.cpp"
struct Line{ int id; double k, b; Line() =
  default;};
namespace LCSeg{
  const int SIZ = 2e5 + 3;
  struct Line T[SIZ];
  #define lc(t) (t << 1)
  #define rc(t) (t << 1 | 1)
  bool cmp(int p, Line x, Line y){
    double w1 = x.k * p + x.b;
    double w2 = y.k * p + y.b;
    double d = w1 - w2;
    if(fabs(d) < 1e-8) return x.id > y.id;
    return d < 0;
  }
  void merge(int t, int a, int b, Line x, Line
     y){
    int c = a + b >> 1;
    if(cmp(c, x, y)) swap(x, y);
    if(cmp(a, y, x)){
      T[t] = x; if(a ≠ b) merge(rc(t), c + 1,
         b, T[rc(t)], y);
    } else {
      T[t] = x; if(a ≠ b) merge(lc(t), a,   c
        , T[lc(t)], y);
    }
  }
  // 插入线段 (l, f(l)) -- (r, f(r))
  void modify(int t, int a, int b, int l, int
    r, Line x){
    if(l ⩽ a && b ⩽ r) merge(t, a, b, T[t],
      x);
    else {
      int c = a + b >> 1;
      if(l ⩽ c) modify(lc(t), a,   c, l, r, x
        );
      if(r >  c) modify(rc(t), c + 1, b, l, r,
         x);
    }
  }
  // 查询 x = p 位置最高的线段（有多条取编号最
    小）
  void query(int t, int a, int b, int p, Line
    &x){
    if(cmp(p, x, T[t])) x = T[t];
    if(a ≠ b){
      int c = a + b >> 1;
      if(p ⩽ c) query(lc(t), a,   c, p, x);
      if(p >  c) query(rc(t), c + 1, b, p, x);
    }
  }
}
```

2.5.2 线段树 3

```
#include "../../header.cpp"
int A[MAXN];
struct Node{
  i64 sum; int len, max1, max2, max_cnt,
    his_mx;
  Node():
    sum(0), max1(-INF), max2(-INF), max_cnt(0)
      , his_mx(-INF), len(0) {}
  Node(int w):
    sum(w), max1(   w), max2(-INF), max_cnt(1)
      , his_mx(   w), len(1) {}
  bool update(int w1, int w2, int h1, int h2){
    his_mx = max({his_mx, max1 + h1});
    max1 += w1, max2 += w2;
    sum += 1ll * w1 * max_cnt + 1ll * w2 * (
      len - max_cnt);
    return max1 > max2;
  }
};
struct Tag{
  int max_add, max_his_add, umx_add,
    umx_his_add; bool have;
  void update(int w1, int w2, int h1, int h2){
    max_his_add = max(max_his_add, max_add +
      h1);
    umx_his_add = max(umx_his_add, umx_add +
      h2);
    max_add += w1, umx_add += w2, have = true;
  }
  void clear(){
    max_add = max_his_add = umx_add =
      umx_his_add = have = 0;
  }
};
struct Node operator +(Node a, Node b){
  Node t;
  t.max1 = max(a.max1, b.max1);
  if(t.max1 ≠ a.max1){
    if(a.max1 > t.max2) t.max2 = a.max1;
  } else{
    if(a.max2 > t.max2) t.max2 = a.max2;
    t.max_cnt += a.max_cnt;
  }
  if(t.max1 ≠ b.max1){
    if(b.max1 > t.max2) t.max2 = b.max1;
  } else{
    if(b.max2 > t.max2) t.max2 = b.max2;
    t.max_cnt += b.max_cnt;
  }
  t.sum = a.sum + b.sum, t.len = a.len + b.len
    ;
  t.his_mx = max(a.his_mx, b.his_mx);
  return t;
}
namespace Seg{
  const int SIZ = 2e6 + 3;
  struct Node W[SIZ]; struct Tag T[SIZ];
  #define lc(t) (t << 1)
  #define rc(t) (t << 1 | 1)
  void push_up(int t, int a, int b){
    W[t] = W[lc(t)] + W[rc(t)];
  }
  void push_down(int t, int a, int b){
```

```
    if(a == b) T[t].clear();
    if(T[t].have){
      int c = a + b >> 1, x = lc(t), y = rc(t)
        ;
      int w = max(W[x].max1, W[y].max1);
      int w1 = T[t].max_add, w2 = T[t].umx_add
        , w3 = T[t].max_his_add, w4 = T[t].
        umx_his_add;
      if(w == W[x].max1)
        W[x].update(w1, w2, w3, w4),
        T[x].update(w1, w2, w3, w4);
      else
        W[x].update(w2, w2, w4, w4),
        T[x].update(w2, w2, w4, w4);
      if(w == W[y].max1)
        W[y].update(w1, w2, w3, w4),
        T[y].update(w1, w2, w3, w4);
      else
        W[y].update(w2, w2, w4, w4),
        T[y].update(w2, w2, w4, w4);
      T[t].clear();
    }
  }
  void build(int t, int a, int b){
    if(a == b){W[t] = Node(A[a]), T[t].clear()
      ;} else {
      int c = a + b >> 1; T[t].clear();
      build(lc(t), a,   c);
      build(rc(t), c + 1, b);
      push_up(t, a, b);
    }
  }
  void modiadd(int t, int a, int b, int l, int
     r, int w){
    if(l <= a && b <= r){
      T[t].update(w, w, w, w);
      W[t].update(w, w, w, w);
    } else {
      int c = a + b >> 1; push_down(t, a, b);
      if(l <= c) modiadd(lc(t), a,   c, l, r,
        w);
      if(r >  c) modiadd(rc(t), c + 1, b, l, r
        , w);
      push_up(t, a, b);
    }
  }
  void modimin(int t, int a, int b, int l, int
     r, int w){
    if(l <= a && b <= r){
      if(w >= W[t].max1) return; else
      if(w >  W[t].max2){
        int k = w - W[t].max1;
        T[t].update(k, 0, k, 0);
        W[t].update(k, 0, k, 0);
      } else {
        int c = a + b >> 1;
        push_down(t, a, b);
        modimin(lc(t), a,   c, l, r, w);
        modimin(rc(t), c + 1, b, l, r, w);
        push_up(t, a, b);
      }
    } else {
      int c = a + b >> 1; push_down(t, a, b);
      if(l <= c) modimin(lc(t), a,   c, l, r,
        w);
      if(r >  c) modimin(rc(t), c + 1, b, l, r
        , w);
      push_up(t, a, b);
    }
  }
  Node query(int t, int a, int b, int l, int r
    ){
    if(l <= a && b <= r) return W[t];
    int c = a + b >> 1; Node ret; push_down(t,
      a, b);
    if(l <= c) ret = ret + query(lc(t), a,   c
      , l, r);
    if(r >  c) ret = ret + query(rc(t), c + 1,
      b, l, r);
    return ret;
  }
}
int qread();
int main(){
  int n = qread(), m = qread();
  for(int i = 1;i <= n;++ i)
    A[i] = qread();
  Seg :: build(1, 1, n);
  for(int i = 1;i <= m;++ i){
    int op = qread();
    if(op == 1){
      int l = qread(), r = qread(), w = qread
        ();
      Seg :: modiadd(1, 1, n, l, r, w);
    } else if(op == 2){
      int l = qread(), r = qread(), w = qread
        ();
      Seg :: modimin(1, 1, n, l, r, w);
    } else if(op == 3){
      int l = qread(), r = qread();
      auto p = Seg :: query(1, 1, n, l, r);
      printf("%lld\n", p.sum);
    } else if(op == 4){
      int l = qread(), r = qread();
      auto p = Seg :: query(1, 1, n, l, r);
      printf("%d\n", p.max1);
    } else if(op == 5){
      int l = qread(), r = qread();
      auto p = Seg :: query(1, 1, n, l, r);
      printf("%d\n", p.his_mx);
    }
  }
  return 0;
}
```

2.6 根号数据结构

3 树论

3.1 点分树

3.1.1 例题

给定 n 个点组成的树，点有点权 v_i。m 个操作，分为两种：

- `0 x k` 查询距离 x 不超过 k 的所有点的点权之和；
- `0 x y` 将点 x 的点权修改为 y。

```
#include "../header.cpp"
vector<int> E[MAXN];
namespace LCA{
  const int SIZ = 1e5 + 3;
  int D[SIZ], F[SIZ];
  int P[SIZ], Q[SIZ], o;
  void dfs(int u, int f){
    P[u] = ++ o;
    Q[o] = u;
    F[u] = f;
    D[u] = D[f] + 1;
    for(auto &v : E[u]) if(v != f){
      dfs(v, u);
    }
  }
  const int MAXH = 18 + 3;
  int h = 18;
  int ST[SIZ][MAXH];
  int cmp(int a, int b){
    return D[a] < D[b] ? a : b;
  }
  int T[SIZ], n;
  void init(int _n){
    n = _n;
    dfs(1, 0);
```

```
    for(int i = 1;i <= n;++ i)
      ST[i][0] = Q[i];
    for(int i = 2;i <= n;++ i)
      T[i] = T[i >> 1] + 1;
    for(int i = 1;i <= h;++ i){
      for(int j = 1;j <= n;++ j) if(j + (1 <<
        i - 1) <= n){
        ST[j][i] = cmp(ST[j][i - 1], ST[j + (1
          << i - 1)][i - 1]);
      }
    }
  }
  int lca(int a, int b){
    if(a == b)
      return a;
    int l = P[a];
    int r = P[b];
    if(l > r)
      swap(l, r);
    ++ l;
    int d = T[r - l + 1];
    return F[cmp(ST[l][d], ST[r - (1 << d) +
      1][d])];
  }
  int dis(int a, int b){
    return D[a] + D[b] - 2 * D[lca(a, b)];
  }
}
namespace BIT{
  void modify(int D[], int n, int p, int w){
    ++ p;
    while(p <= n)
      D[p] += w, p += p & -p;
  }
  int query(int D[], int n, int p){
    if(p < 0) return 0;
    p = min(n, p + 1);
    int r = 0;
    while(p >  0)
      r += D[p], p -= p & -p;
    return r;
  }
}
namespace PTree{
  const int SIZ = 1e5 + 3;
  bool V[SIZ];
  int  S[SIZ], L[SIZ];
  vector<int> EE[MAXN];
  int *D1[MAXN];
  int *D2[MAXN];
  void dfs1(int s, int &g, int u, int f){
    S[u] = 1;
    int maxsize = 0;
    for(auto &v : E[u]) if(v != f && !V[v]){
      dfs1(s, g, v, u);
      if(S[v] > maxsize)
        maxsize = S[v];
      S[u] += S[v];
    }
    maxsize = max(maxsize, s - S[u]);
    if(maxsize <= s / 2)
      g = u;
  }
  int n;
  void build(int s, int &g, int u, int f){
    dfs1(s, g, u, f);
    V[g] = true, L[g] = s;
    for(auto &u : E[g]) if(!V[u]){
      int h = 0;
      if(S[u] < S[g]) build(S[u], h, u, 0);
      else      build(s - S[g], h, u, 0);
      EE[g].push_back(h);
      EE[h].push_back(g);
    }
  }
  int F[SIZ];
  void dfs2(int u, int f){
    F[u] = f;
    for(auto &v : EE[u]) if(v != f){
      dfs2(v, u);
    }
  }
  void build(int _n){
    n = _n;
    int s = n, g = 0;
    dfs1(s, g, 1, 0);
    V[g] = true, L[g] = s;
    for(auto &u : E[g]){
      int h = 0;
      if(S[u] < S[g]) build(S[u], h, u, 0);
      else      build(s - S[g], h, u, 0);
      EE[g].push_back(h);
      EE[h].push_back(g);
    }
    dfs2(g, 0);
    for(int i = 1;i <= n;++ i){
      L[i] += 2;
      D1[i] = new int[L[i] + 3];
      D2[i] = new int[L[i] + 3];
      for(int j = 0;j < L[i] + 3;++ j)
        D1[i][j] = D2[i][j] = 0;
    }
  }
  void modify(int x, int w){
    int u = x;
    while(1){
      BIT :: modify(D1[x], L[x], LCA :: dis(u,
        x), w);
      int y = F[x];
      if(y != 0){
        int e = LCA :: dis(x, y);
        BIT :: modify(D2[x], L[x], LCA :: dis(
          u, y), w);
        x = y;
      } else break;
    }
  }
  int query(int x, int d){
    int ans = 0, u = x;
    while(1){
      ans += BIT :: query(D1[x], L[x], d - LCA
        :: dis(u, x));
      int y = F[x];
      if(y != 0){
        int e = LCA :: dis(x, y);
        ans -= BIT :: query(D2[x], L[x], d -
          LCA :: dis(u, y));
        x = y;
      } else break;
    }
    return ans;
  }
}
int W[MAXN];
int main(){
  ios :: sync_with_stdio(false);
  int n, m;
  cin >> n >> m;
  for(int i = 1;i <= n;++ i){
    cin >> W[i];
  }
  for(int i = 2;i <= n;++ i){
    int u, v;
    cin >> u >> v;
    E[u].push_back(v);
    E[v].push_back(u);
  }
  LCA :: init(n);
  PTree :: build(n);
  for(int i = 1;i <= n;++ i)
    PTree :: modify(i, W[i]);
  int lastans = 0;
  for(int i = 1;i <= m;++ i){
    int op; cin >> op;
    if(op == 0){
      int x, d;
```

```
      cin >> x >> d;
      x ^= lastans;
      d ^= lastans;
      cout << (lastans = PTree :: query(x, d))
         << endl;
    } else {
      int x, w;
      cin >> x >> w;
      x ^= lastans;
      w ^= lastans;
      PTree :: modify(x, -W[x]  );
      PTree :: modify(x,  W[x] = w);
    }
  }
  return 0;
}
```

3.2 树哈希

3.2.1 用法

给定大小为 n 的以 1 为根的树，计算 h_i 表示子树 i 的哈希值，计算有多少个本质不同的值。

```
#include "../header.cpp"
u64 xor_shift(u64 x);
u64 H[MAXN];
vector <int> E[MAXN];
void dfs(int u, int f){
  H[u] = 1;
  for(auto &v: E[u]) if(v != f){
    dfs(v, u);
    H[u] += H[v];
  }
  H[u] = xor_shift(H[u]); // !important
}
int main(){
  int n;
  cin >> n;
  for(int i = 2;i <= n;++ i){
    int u, v;
    cin >> u >> v;
    E[u].push_back(v);
    E[v].push_back(u);
  }
  dfs(1, 0);
  sort(H + 1, H + 1 + n);
  cout << (unique(H + 1, H + 1 + n) - H - 1)
    << endl;
  return 0;
}
```

3.3 Prufer 序列

```
#include "../header.cpp"
int D[MAXN], F[MAXN], P[MAXN];
vector<int> tree2prufer(int n){
  vector <int> P(n);
  for(int i = 1, j = 1;i <= n - 2;++ i, ++ j){
    while(D[j]) ++ j;
    P[i] = F[j];
    while(i <= n - 2 && !--D[P[i]] && P[i] < j
      )
      P[i + 1] = F[P[i]], i ++;
  }
  return P;
}
vector<int> prufer2tree(int n){
  vector <int> F(n);
  for(int i = 1, j = 1;i <= n - 1;++ i, ++ j){
    while(D[j]) ++ j;
    F[j] = P[i];
    while(i <= n - 1 && !--D[P[i]] && P[i] < j
      )
      F[P[i]] = P[i + 1], i ++;
  }
  return F;
}
```

3.4 虚树

```
#include "../header.cpp"
vector<pair<int, int> > E[MAXN];
namespace LCA{
  const int SIZ = 5e5 + 3;
  int D[SIZ], H[SIZ], F[SIZ], P[SIZ], Q[SIZ],
    o;
  void dfs(int u, int f){
    P[u] = ++ o, Q[o] = u, F[u] = f, D[u] = D[
      f] + 1;
    for(auto &[v, w] : E[u]) if(v != f){
      H[v] = H[u] + w, dfs(v, u);
    }
  }
  const int MAXH = 18 + 3;
  int h = 18;
  int ST[SIZ][MAXH];
  int cmp(int a, int b){
    return D[a] < D[b] ? a : b;
  }
  int T[SIZ], n;
  void init(int _n, int root);
  int lca(int a, int b);
  int dis(int a, int b);
}
using LCA :: init;
using LCA :: lca;
using LCA :: dis;
void LCA :: init(int _n, int root){
  n = _n;
  dfs(root, 0);
  for(int i = 1;i <= n;++ i) ST[i][0] = Q[i];
  for(int j = 1;j < h;++ j)
    for(int i = 1;i + (1 << j) - 1 <= n;++ i)
      ST[i][j] = cmp(ST[i][j - 1], ST[i + (1 <<
        j - 1)][j - 1]);
}
int LCA :: lca(int a, int b){
  if(a == b) return a;
  if((a = P[a]) == (b = P[b])) return a;
  if(a > b) swap(a, b);
  int l = __lg(b - a);
  return F[cmp(ST[a + 1][l], ST[b - (1 << l) +
    1][l])];
}
int LCA :: dis(int a, int b){
  return D[a] + D[b] - 2 * D[lca(a, b)];
}
```

```
}
bool cmp(int a, int b){
  return LCA :: P[a] < LCA :: P[b];
}
bool I[MAXN];
vector <int> E1[MAXN], V1;
void solve(vector <int> &V){
  using LCA :: lca; using LCA :: D;
  stack <int> S;
  sort(V.begin(), V.end(), cmp);
  S.push(1);
  int v, l;
  for(auto &u : V) I[u] = true;
  for(auto &u : V) if(u != 1){
    int f = lca(u, S.top());
    l = -1;
    while(D[v = S.top()] > D[f]){
      if(l != -1)
        E1[v].push_back(l);
      V1.push_back(l = v), S.pop();
    }
    if(l != -1)
      E1[f].push_back(l);
    if(f != S.top()) S.push(f);
    S.push(u);
  }
  l = -1;
  while(!S.empty()){
    v = S.top();
    if(l != -1) E1[v].push_back(l);
    V1.push_back(l = v), S.pop();
  }
  // dfs(1, 0); // SOLVE HERE !!!
  for(auto &u : V1)
    E1[u].clear(), I[u] = false;
  V1.clear();
}
```

4 图论

4.1 三元环计数

4.1.1 三元环计数

无向图：考虑将所有点按度数从小往大排序，然后将每条边定向，由排在前面的指向排在后面的，得到一个有向图。然后考虑枚举一个点，再枚举一个点，暴力数，具体见代码。结论是，这样定向后，每个点的出度是 $O(\sqrt{m})$ 的。复杂度 $O(m\sqrt{m})$。**有向图**：不难发现，上述方法枚举

了三个点，计算有向图三元环也就只需要处理下方向的事，这个由于算法够暴力，随便改改就能做了。

```cpp
// 无向图
ll n, m; cin >> n >> m;
vector<pair<ll, ll>> Edges(m);
vector<vector<ll>> G(n + 2);
vector<ll> deg(n + 2);
for (auto &[i, j] : Edges) cin >> i >> j, ++deg[i], ++deg[j];
for (auto [i, j] : Edges) {
    if (deg[i] > deg[j] || (deg[i] == deg[j] && i > j)) swap(i, j);
    G[i].emplace_back(j);
}
vector<ll> val(n + 2);
ll ans = 0;
for (ll i = 1; i <= n; ++i) {
    for (auto j : G[i]) ++val[j];
    for (auto j : G[i]) for (auto k : G[j]) ans += val[k];
    for (auto j : G[i]) val[j] = 0;
}
// 有向图
ll n, m; cin >> n >> m;
vector<pair<ll, ll>> Edges(m);
vector<vector<pll>> G(n + 2);
vector<ll> deg(n + 2);
for (auto &[i, j] : Edges) cin >> i >> j, ++deg[i], ++deg[j];
for (auto [i, j] : Edges) {
    ll flg = 0;
    if (deg[i] > deg[j] || (deg[i] == deg[j] && i > j)) swap(i, j), flg = 1;
    G[i].emplace_back(j, flg);
}
vector<ll> in(n + 2), out(n + 2);
ll ans = 0;
for (ll i = 1; i <= n; ++i) {
    for (auto [j, w] : G[i]) w ? (++in[j]) : (++out[j]);
    for (auto [j, w1] : G[i]) for (auto [k, w2] : G[j]) {
        if (w1 == w2) ans += w1 ? in[k] : out[k];
    }
    for (auto [j, w] : G[i]) in[j] = out[j] = 0;
}
cout << ans << '\n';
```

4.2 四元环计数

4.2.1 四元环计数

From zpk

- 无向图：类似，由于定向后出度结论过于强大，可以暴力。讨论了三种情况。
- 有向图：缺少题目，但应当类似三元环计数有向形式记录定向边和原边的正反关系。因为此法最强的结论是定向后出度 $O(\sqrt{m})$，实际上方法很暴力，应当不难数有向形式的。

```cpp
ll n, m; cin >> n >> m;
vector<pair<ll, ll>> Edges(m);
vector<vector<ll>> G(n + 2), iG(n + 2);
vector<ll> deg(n + 2);
for (auto &[i, j] : Edges) cin >> i >> j, ++deg[i], ++deg[j];
for (auto [i, j] : Edges) {
    if (deg[i] > deg[j] || (deg[i] == deg[j] && i > j)) swap(i, j);
    G[i].emplace_back(j), iG[j].emplace_back(i);
}
ll ans = 0;
vector<ll> v1(n + 2), v2(n + 2);
for (ll i = 1; i <= n; ++i) {
    for (auto j : G[i]) for (auto k : G[j]) ++v1[k];
    for (auto j : iG[i]) for (auto k : G[j]) ans += v1[k], ++v2[k];
    for (auto j : G[i]) for (auto k : G[j]) ans += v1[k] * (v1[k] - 1) / 2, v1[k] = 0;
    for (auto j : iG[i]) for (auto k : G[j]) {
        if (deg[k] > deg[i] || (deg[k] == deg[i] && k > i)) ans += v2[k] * (v2[k] - 1) / 2;
        v2[k] = 0;
    }
}
cout << ans << '\n';
```

4.3 2-SAT

4.3.1 例题

n 个变量 m 个条件，形如若 $x_i = a$ 则 $y_j = b$，找到任意一组可行解或者报告无解。

```cpp
#include "../header.cpp"
namespace SCC{
  const int MAXN= 2e6 + 3;
  vector <int> V[MAXN];
  stack  <int> S;
  int D[MAXN], L[MAXN], C[MAXN], o, s;
  bool F[MAXN], I[MAXN];
  void add(int u, int v){ V[u].push_back(v); }
  void dfs(int u){
    L[u] = D[u] = ++ o, S.push(u), I[u] = F[u] = true;
    for(auto &v : V[u]){
      if(F[v]){
        if(I[v]) L[u] = min(L[u], D[v]);
      } else {
        dfs(v),  L[u] = min(L[u], L[v]);
      }
    }
    if(L[u] == D[u]){
      int c = ++ s;
      while(S.top() != u){
        int v = S.top(); S.pop();
        I[v] = false;
        C[v] = c;
      }
      S.pop(), I[u] = false, C[u] = c;
    }
  }
}
const int MAXN = 1e6 + 3;
int X[MAXN][2], o;
int main(){
  ios :: sync_with_stdio(false);
  int n, m;
  cin >> n >> m;
  for(int i = 1;i <= n;++ i)
    X[i][0] = ++ o;
  for(int i = 1;i <= n;++ i)
    X[i][1] = ++ o;
  for(int i = 1;i <= m;++ i){
    int a, x, b, y;
    cin >> a >> x >> b >> y;
    SCC :: add(X[a][!x], X[b][y]);
    SCC :: add(X[b][!y], X[a][x]);
  }
  for(int i = 1;i <= o;++ i)
    if(!SCC :: F[i])
      SCC :: dfs(i);
  bool ok = true;
  for(int i = 1;i <= n;++ i){
    if(SCC :: C[X[i][0]] == SCC :: C[X[i][1]])
      ok = false;
```

```
  }
  if(ok){
    cout << "POSSIBLE" << endl;
    for(int i = 1;i ⩽ n;++ i){
      int a = SCC :: C[X[i][0]];
      int b = SCC :: C[X[i][1]];
      if(a < b)
        cout << 0 << " ";
      else
        cout << 1 << " ";
    }
    cout << endl;
  } else {
    cout << "IMPOSSIBLE" << endl;
  }
  return 0;
}
```

4.4 割点

```
#include "../header.cpp"
vector<int> V[MAXN];
int n, m, o, D[MAXN], L[MAXN];
bool F[MAXN], C[MAXN];
void dfs(int u, int g){
  L[u] = D[u] = ++ o, F[u] = true; int s = 0;
  for(auto &v : V[u]){
    if(!F[v]){
      dfs(v, g), ++ s;
      L[u] = min(L[u], L[v]);
      if(u ≠ g && L[v] ⩾ D[u]) C[u] = true;
    } else {
      L[u] = min(L[u], D[v]);
    }
  }
  if(u == g && s > 1) C[u] = true;
}
int main(){
  cin >> n >> m;
  for(int i = 1;i ⩽ m;++ i){
    int u, v;
    cin >> u >> v;
    V[u].push_back(v);
    V[v].push_back(u);
  }
  for(int i = 1;i ⩽ n;++ i)
    if(!F[i]) dfs(i, i);
  vector <int> ANS;
  for(int i = 1;i ⩽ n;++ i)
    if(C[i]) ANS.push_back(i);
  cout << ANS.size() << endl;
  for(auto &u : ANS)
    cout << u << " ";
  return 0;
}
```

4.5 边双连通分量

```
#include "../header.cpp"
vector <vector<int>> A;
vector <pair<int, int>> V[MAXN];
stack  <int> S;
int D[MAXN], L[MAXN], o;
bool I[MAXN];
void dfs(int u, int l){
  D[u] = L[u] = ++ o; I[u] = true, S.push(u);
    int s = 0;
  for(auto &p : V[u]) {
    int v = p.first, id = p.second;
    if(id ≠ l){
      if(D[v]){
        if(I[v])  L[u] = min(L[u], D[v]);
      } else {
        dfs(v, id), L[u] = min(L[u], L[v]), ++
           s;
      }
    }
  }
  if(D[u] == L[u]){
    vector <int> T;
    while(S.top() ≠ u){
      int v = S.top(); S.pop();
      T.push_back(v), I[v] = false;
    }
    T.push_back(u), S.pop(), I[u] = false;
    A.push_back(T);
  }
}
```

4.6 点双连通分量

```
#include "../header.cpp"
vector <vector<int>> A;
vector <int> V[MAXN];
stack  <int> S;
int D[MAXN], L[MAXN], o; bool I[MAXN];
void dfs(int u, int f){
  D[u] = L[u] = ++ o; I[u] = true, S.push(u);
    int s = 0;
  for(auto &v : V[u]) if(v ≠ f){
    if(D[v]){
      if(I[v])   L[u] = min(L[u], D[v]);
    } else {
      dfs(v, u), L[u] = min(L[u], L[v]), ++ s;
      if(L[v] ⩾ D[u]){
        vector <int> T;
        while(S.top() ≠ v){
          int t = S.top(); S.pop();
          T.push_back(t), I[t] = false;
        }
        T.push_back(v), S.pop(), I[v] = false;
        T.push_back(u);
        A.push_back(T);
      }
    }
  }
  if(f == 0 && s == 0){
    A.push_back({u});
  }
}
```

4.7 强连通分量

```
#include "../header.cpp"
vector <int> V[MAXN];
stack  <int> S;
int D[MAXN], L[MAXN], C[MAXN], o, s;
bool F[MAXN], I[MAXN];
void add(int u, int v){ V[u].push_back(v); }
void dfs(int u){
  L[u] = D[u] = ++ o, S.push(u), I[u] = F[u] =
     true;
  for(auto &v : V[u]){
    if(F[v]){
      if(I[v]) L[u] = min(L[u], D[v]);
    } else {
      dfs(v),  L[u] = min(L[u], L[v]);
    }
  }
  if(L[u] == D[u]){
    int c = ++ s;
    while(S.top() ≠ u){
      int v = S.top(); S.pop();
      I[v] = false;
      C[v] = c;
    }
    S.pop(), I[u] = false, C[u] = c;
  }
}
vector <int> ANS[MAXN];
int main(){
  int n, m;
  cin >> n >> m;
  for(int i = 1;i ⩽ m;++ i){
    int u, v;
```

```
    cin >> u >> v;
    V[u].push_back(v);
  }
  for(int i = 1;i <= n;++ i)
    if(!F[i])
      dfs(i);
  for(int i = 1;i <= n;++ i){
    ANS[C[i]].push_back(i);
  }
  cout << s << endl;
  for(int i = 1;i <= n;++ i) if(F[i]){
    int c = C[i];
    sort(ANS[c].begin(), ANS[c].end());
    for(auto &u : ANS[c])
      cout << u << " ", F[u] = false;
    cout << endl;
  }
  return 0;
}
```

5 网络流

5.1 费用流

```
#include "../header.cpp"
namespace MCMF{
  int H[MAXN], V[MAXM], N[MAXM], W[MAXM], F[
    MAXM], o = 1, n;
  void add(int u, int v, int f, int c){
    V[++ o] = v, N[o] = H[u], H[u] = o, F[o] =
       f, W[o] =  c;
    V[++ o] = u, N[o] = H[v], H[v] = o, F[o] =
       0, W[o] = -c;
    n = max(n, u);
    n = max(n, v);
  }
  void clear(){
    for(int i = 1;i <= n;++ i)
      H[i] = 0;
    n = 0, o = 1;
  }
  bool I[MAXN];
  i64 D[MAXN];
  bool spfa(int s, int t){
    queue <int> Q;
    Q.push(s), I[s] = true;
    for(int i = 1;i <= n;++ i)
      D[i] = INFL;
    D[s] = 0;
    while(!Q.empty()){
      int u = Q.front(); Q.pop(), I[u] = false
        ;
      for(int i = H[u];i;i = N[i]){
        const int &v = V[i];
        const int &f = F[i];
        const int &w = W[i];
        if(f && D[u] + w < D[v]){
          D[v] = D[u] + w;
          if(!I[v]) Q.push(v), I[v] = true;
        }
      }
    }
    return D[t] != INFL;
  }
  int C[MAXN]; bool T[MAXN];
  pair<i64, i64> dfs(int s, int t, int u, i64
    maxf){
    if(u == t)
      return make_pair(maxf, 0);
    i64 totf = 0;
    i64 totc = 0;
    T[u] = true;
    for(int &i = C[u];i;i = N[i]){
      const int &v = V[i];
      const int &f = F[i];
      const int &w = W[i];
      if(f && D[v] == D[u] + w && !T[v]){
        auto p = dfs(s, t, v, min(1ll * F[i],
          maxf));
        i64 f = p.first;
        i64 c = p.second;
        F[i  ] -= f;
        F[i ^ 1] += f;
        totf += f;
        totc += 1ll * f * W[i] + c;
        maxf -= f;
        if(maxf == 0){
          T[u] = false;
          return make_pair(totf, totc);
        }
      }
    }
    T[u] = false;
    return make_pair(totf, totc);
  }
  pair<i64, i64> mcmf(int s, int t){
    i64 ans1 = 0;
    i64 ans2 = 0;
    pair<i64, i64> r;
    while(spfa(s, t)){
      memcpy(C, H, sizeof(int) * (n + 3));
      r = dfs(s, t, s, INFL);
      ans1 += r.first;
      ans2 += r.second;
    }
    return make_pair(ans1, ans2);
  }
}
int qread();
int main(){
  int n = qread(), m = qread(), s = qread(), t
    = qread();
  for(int i = 1;i <= m;++ i){
    int u = qread(), v = qread(), f = qread(),
      c = qread();
    MCMF :: add(u, v, f, c);
  }
  pair<long long, long long> ans = MCMF ::
    mcmf(s, t);
  printf("%lld %lld\n", ans.first, ans.second)
    ;
  return 0;
}
```

5.2 最小割树

5.2.1 用法

给定无向图求出最小割树，点 u 和 v 作为起点终点的最小割为树上 u 到 v 路径上边权的最小值。

```
#include "../header.cpp"
namespace Dinic{
  const long long INF = 1e18;
  const int SIZ = 1e5 + 3;
  int n, m;
  int H[SIZ], V[SIZ], N[SIZ], F[SIZ], t = 1;
  int add(int u, int v, int f){
    V[++ t] = v, N[t] = H[u], F[t] = f, H[u] =
      t;
    V[++ t] = u, N[t] = H[v], F[t] = 0, H[v] =
      t;
    n = max(n, u);
    n = max(n, v);
    return t - 1;
  }
  void clear(){
    for(int i = 1;i <= n;++ i)
      H[i] = 0;
    n = m = 0, t = 1;
  }
  int D[SIZ];
  bool bfs(int s, int t){
    queue <int> Q;
    for(int i = 1;i <= n;++ i)
```

```
      D[i] = 0;
    Q.push(s), D[s] = 1;
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(int i = H[u];i;i = N[i]){
        const int &v = V[i];
        const int &f = F[i];
        if(f != 0 && !D[v]){
          D[v] = D[u] + 1;
          Q.push(v);
        }
      }
    }
    return D[t] != 0;
  }
  int C[SIZ];
  long long dfs(int s, int t, int u, long long maxf){
    if(u == t)
      return maxf;
    long long totf = 0;
    for(int &i = C[u];i;i = N[i]){
      const int &v = V[i];
      const int &f = F[i];
      if(D[v] == D[u] + 1){
        long long resf = dfs(s, t, v, min(maxf, 1ll * f));
        totf += resf;
        maxf -= resf;
        F[i  ] -= resf;
        F[i ^ 1] += resf;
        if(maxf == 0)
          return totf;
      }
    }
    return totf;
  }
  long long dinic(int s, int t){
    long long ans = 0;
    while(bfs(s, t)){
      memcpy(C, H, sizeof(int) * (n + 3));
      ans += dfs(s, t, s, INF);
    }
    return ans;
  }
}
namespace GHTree{
  const int MAXN =  500 + 5;
  const int MAXM = 1500 + 5;
  const int INF  = 1e9;
  int n, m, U[MAXM], V[MAXM], W[MAXM], A[MAXM], B[MAXM];
  void add(int u, int v, int w){
    ++ m;
    U[m] = u;
    V[m] = v;
    W[m] = w;
    A[m] = Dinic :: add(u, v, w);
    B[m] = Dinic :: add(v, u, w);
    n = max(n, u);
    n = max(n, v);
  }
  vector <pair<int, int> > E[MAXN];
  void build(vector <int> N){
    int s = N.front();
    int t = N.back();
    if(s == t) return;
    for(int i = 1;i <= m;++ i){
      int a = A[i]; Dinic :: F[a] = W[i], Dinic :: F[a ^ 1] = 0;
      int b = B[i]; Dinic :: F[b] = W[i], Dinic :: F[b ^ 1] = 0;
    }
    int w = Dinic :: dinic(s, t);
    E[s].push_back(make_pair(t, w));
    E[t].push_back(make_pair(s, w));
    vector <int> P;
    vector <int> Q;
    for(auto &u : N){
      if(Dinic :: D[u] != 0)
        P.push_back(u);
      else
        Q.push_back(u);
    }
    build(P), build(Q);
  }
  int D[MAXN];
  int cut(int s, int t){
    queue <int> Q; Q.push(s);
    for(int i = 1;i <= n;++ i)
      D[i] = -1;
    D[s] = INF;
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(auto &e : E[u]){
        int v = e.first;
        int w = e.second;
        if(D[v] == -1){
          D[v] = min(D[u], w);
          Q.push(v);
        }
      }
    }
    return D[t];
  }
}
```

5.3 最大流

```
#include "../header.cpp"
namespace Dinic{
  const i64 INF = 1e18;
  const int SIZ = 5e5 + 3;
  int n;
  int H[MAXN], V[MAXM], N[MAXM], F[MAXM], t = 1;
  void add(int u, int v, int f){
    V[++ t] = v, N[t] = H[u], F[t] = f, H[u] = t;
    V[++ t] = u, N[t] = H[v], F[t] = 0, H[v] = t;
    n = max(n, u);
    n = max(n, v);
  }
  void clear(){
    for(int i = 1;i <= n;++ i)
      H[i] = 0;
    n = 0, t = 1;
  }
  i64 D[MAXN];
  bool bfs(int s, int t){
    queue <int> Q;
    for(int i = 1;i <= n;++ i)
      D[i] = 0;
    Q.push(s), D[s] = 1;
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(int i = H[u];i;i = N[i]){
        const int &v = V[i];
        const int &f = F[i];
        if(f != 0 && !D[v]){
          D[v] = D[u] + 1;
          Q.push(v);
        }
      }
    }
    return D[t] != 0;
  }
  int C[MAXN];
  i64 dfs(int s, int t, int u, i64 maxf){
    if(u == t)
      return maxf;
    i64 totf = 0;
    for(int &i = C[u];i;i = N[i]){
      const int &v = V[i];
      const int &f = F[i];
```

```
      if(f && D[v] == D[u] + 1){
        i64 f = dfs(s, t, v, min(1ll * f, maxf
          ));
        F[i] -= f, F[i ^ 1] += f, totf += f,
          maxf -= f;
        if(maxf == 0)
          return totf;
      }
    }
    return totf;
  }
  i64 dinic(int s, int t){
    i64 ans = 0;
    while(bfs(s, t)){
      memcpy(C, H, sizeof(int) * (n + 3));
      ans += dfs(s, t, s, INFL);
    }
    return ans;
  }
}
```

5.4 上下界费用流

5.4.1 用法

- `add(u, v, l, r, c)`：连一条容量在 $[l,r]$ 的从 u 到 v 的费用为 c 的边；
- `solve()`：计算无源汇最小费用可行流；
- `solve(s, t)`：计算有源汇最小费用最大流。

```
#define add add0
#include "flow-cost.cpp"
#undef add
namespace MCMF{
  i64 cost0;
  int G[MAXN];
  void add(int u, int v, int l, int r, int c){
    G[v] += l;
    G[u] -= l;
    cost0 += 1ll * l * c;
    add0(u, v, r - l, c);
  }
  i64 solve(){
    int s = ++ n;
    int t = ++ n;
    i64 sum = 0;
    for(int i = 1;i <= n - 2;++ i){
      if(G[i] < 0)
        add0(i, t, -G[i], 0);
      else
        add0(s, i,  G[i], 0), sum += G[i];
    }
    auto res = mcmf(s, t);
    if(res.first != sum)
      return -1;
    return res.second + cost0;
  }
  i64 solve(int s0, int t0){
    add0(t0, s0, INF, 0);
    int s = ++ n;
    int t = ++ n;
    i64 sum = 0;
    for(int i = 1;i <= n - 2;++ i){
      if(G[i] < 0)
        add0(i, t, -G[i], 0);
      else
        add0(s, i,  G[i], 0), sum += G[i];
    }
    auto res = mcmf(s, t);
    if(res.first != sum)
      return -1;
    return res.second + cost0;
  }
}
```

5.5 上下界最大流

5.5.1 用法

- `add(u, v, l, r, c)`：连一条容量在 $[l,r]$ 的从 u 到 v 的边；
- `solve()`：检查是否存在无源汇可行流；
- `solve(s, t)`：计算有源汇最大流。

```
#define add add0
#include "flow-max.cpp"
#undef add
namespace Dinic{
  int G[MAXN];
  void add(int u, int v, int l, int r){
    G[v] += l;
    G[u] -= l;
    add0(u, v, r - l);
  }
  void clear(){
    for(int i = 1;i <= t;++ i){
      N[i] = F[i] = V[i] = 0;
    }
    for(int i = 1;i <= n;++ i){
      H[i] = G[i] = C[i] = 0;
    }
    t = 1, n = 0;
  }
  bool solve(){
    int s = ++ n;
    int t = ++ n;
    i64 sum = 0;
    for(int i = 1;i <= n - 2;++ i){
      if(G[i] < 0)
        add0(i, t, -G[i]);
      else
        add0(s, i,  G[i]), sum += G[i];
    }
    auto res = dinic(s, t);
    if(res != sum)
      return true;
    return false;
  }
  i64 solve(int s0, int t0){
    add0(t0, s0, INF);
    int s = ++ n;
    int t = ++ n;
    i64 sum = 0;
    for(int i = 1;i <= n - 2;++ i){
      if(G[i] < 0)
        add0(i, t, -G[i]);
      else
        add0(s, i,  G[i]), sum += G[i];
    }
    auto res = dinic(s, t);
    if(res != sum)
      return -1;
    return dinic(s0, t0);
  }
}
```

6 数学

6.1 线性代数

6.1.1 行列式

```
#include "../../header.cpp"
struct Mat{
  int n, m, W[MAXN][MAXN];
  Mat(int _n = 0, int _m = 0){
    n = _n, m = _m;
    for(int i = 1;i <= n;++ i)
      for(int j = 1;j <= m;++ j)
        W[i][j] = 0;
  }
};
int mat_det(Mat a){
```

```
  int ans = 1;
  const int &n = a.n;
  for(int i = 1;i ⩽ n;++ i){
    int f = -1;
    for(int j = i;j ⩽ n;++ j) if(a.W[j][i] ≠ 0){
      f = j; break;
    }
    if(f == -1) return 0;
    if(f ≠ i){
      for(int j = 1;j ⩽ n;++ j)
        swap(a.W[i][j], a.W[f][j]);
      ans = MOD - ans;
    }
    for(int j = i + 1;j ⩽ n;++ j) if(a.W[j][i]){
      while(a.W[j][i]){
        int u = a.W[i][i], v = a.W[j][i];
        if(u > v){
          for(int k = 1;k ⩽ n;++ k)
            swap(a.W[i][k], a.W[j][k]);
          ans = MOD - ans, swap(u, v);
        }
        int rate = v / u;
        for(int k = 1;k ⩽ n;++ k){
          a.W[j][k] = (a.W[j][k] - 1ll * rate
            * a.W[i][k] % MOD + MOD) % MOD;
        }
      }
    }
  }
  for(int i = 1;i ⩽ n;++ i)
    ans = 1ll * ans * a.W[i][i] % MOD;
  return ans;
}
int main(){
  int n; cin >> n;
  Mat A(n, n);
  for(int i = 1;i ⩽ n;++ i)
    for(int j = 1;j ⩽ n;++ j)
      cin >> A.W[i][j], A.W[i][j] %= MOD;
  cout << mat_det(A) << endl;
  return 0;
}
```

6.1.2 矩阵树

LGV 定理叙述 设 G 是一张有向无环图，边带权，每个点的度数有限。给定起点集合 $A = \{a_1, a_2, \cdots, a_n\}$，终点集合 $B = \{b_1, b_2, \cdots, b_n\}$。

- 一段路径 $p : v_0 \to^{w_1} v_1 \to^{w_2} v_2 \to \cdots \to^{w_k} v_k$ 的边权被定义为 $\omega(p) = \prod w_i$。
- 一对顶点 (a, b) 的权值定义为 $e(a, b) = \sum_{p:a\to b} \omega(p)$。

设矩阵 M 如下：

$$M = \begin{pmatrix} e(a_1,b_1) & e(a_1,b_2) & \cdots & e(a_1,b_n) \\ e(a_2,b_1) & e(a_2,b_2) & \cdots & e(a_2,b_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(a_n,b_1) & e(a_n,b_2) & \cdots & e(a_n,b_n) \end{pmatrix}$$

从 A 到 B 得到一个不交的路径组 $p = (p_1, p_2, \cdots, p_n)$，其中从 a_i 到达 b_{π_i}，π 是一个排列。定义 $\sigma(\pi)$ 是 π 逆序对的数量。

给出 LGV 的叙述如下：

$$\det(M) = \sum_{p:A\to B} (-1)^{\sigma(\pi)} \prod_{i=1}^{n} \omega(p_i)$$

可以将边权视作边的重数，那么 $e(a, b)$ 就可以视为从 a 到 b 的不同路径方案数。

矩阵树定理 对于无向图，

- 定义度数矩阵 $D_{i,j} = [i = j] \deg(i)$；
- 定义邻接矩阵 $E_{i,j} = E_{j,i}$ 是从 i 到 j 的边数个数；
- 定义拉普拉斯矩阵 $L = D - E$。

对于无向图的矩阵树定理叙述如下：

$$t(G) = \det(L_i) = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}$$

其中 L_i 是将 L 删去第 i 行和第 i 列得到的子式。

对于有向图，类似于无向图定义入度矩阵、出度矩阵、邻接矩阵 $D^{\text{in}}, D^{\text{out}}, E$，同时定义拉普拉斯矩阵 $L^{\text{in}} = D^{\text{in}} - E, L^{\text{out}} - E$。

$$t^{\text{leaf}}(G, k) = \det(L_k^{\text{in}})$$
$$t^{\text{root}}(G, k) = \det(L_k^{\text{out}})$$

其中 $t^{\text{leaf}}(G, k)$ 表示以 k 为根的叶向树，$t^{\text{root}}(G, k)$ 表示以 k 为根的根向树。

BEST 定理 对于一个有向欧拉图 G，记点 i 的出度为 out_i，同时 G 的根向生成树个数为 T。T 可以任意选取根。则 G 的本质不同的欧拉回路个数为：

$$T\prod_i(\text{out}_i - 1)!$$

```
#include "../../header.cpp"
struct Mat{
  int n, m;
  int W[MAXN][MAXN];
  Mat(int _n = 0, int _m = 0){
    n = _n;
    m = _m;
    for(int i = 1;i ⩽ n;++ i)
      for(int j = 1;j ⩽ m;++ j)
        W[i][j] = 0;
  }
};
int mat_det(Mat a){
  int ans = 1;
  const int &n = a.n;
  for(int i = 1;i ⩽ n;++ i){
    int f = -1;
    for(int j = i;j ⩽ n;++ j) if(a.W[j][i] ≠ 0){
      f = j;
      break;
    }
    if(f == -1){
      return 0;
    }
    if(f ≠ i){
      for(int j = 1;j ⩽ n;++ j)
        swap(a.W[i][j], a.W[f][j]);
      ans = MOD - ans;
    }
    for(int j = i + 1;j ⩽ n;++ j) if(a.W[j][i]){
      while(a.W[j][i]){
        int u = a.W[i][i];
        int v = a.W[j][i];
        if(u > v){
          for(int k = 1;k ⩽ n;++ k)
            swap(a.W[i][k], a.W[j][k]);
          ans = MOD - ans;
          swap(u, v);
        }
        int rate = v / u;
```

```
        for(int k = 1;k ⩽ n;++ k){
          a.W[j][k] = (a.W[j][k] - 1ll * rate
            * a.W[i][k] % MOD + MOD) % MOD;
        }
      }
    }
  }
  for(int i = 1;i ⩽ n;++ i)
    ans = 1ll * ans * a.W[i][i] % MOD;
  return ans;
}
int D[MAXN];
int W[MAXN][MAXN];
int main(){
  int n, m, t;
  cin >> n >> m >> t;
  for(int i = 1;i ⩽ m;++ i){
    int u, v, w;
    cin >> u >> v >> w;
    if(u ≠ v){
      if(t == 0){ // 无向图
        D[u] = (D[u] + w) % MOD;
        D[v] = (D[v] + w) % MOD;
        W[u][v] = (W[u][v] + w) % MOD;
        W[v][u] = (W[v][u] + w) % MOD;
      } else {  // 叶向树
        D[v] = (D[v] + w) % MOD;
        W[u][v] = (W[u][v] + w) % MOD;
      }
    }
  }
  Mat A(n - 1, n - 1);
  for(int i = 2;i ⩽ n;++ i)
    for(int j = 2;j ⩽ n;++ j)  // 以 1 为根的
      叶向树
      A.W[i - 1][j - 1] = MOD - W[i][j];
  for(int i = 2;i ⩽ n;++ i)
    A.W[i - 1][i - 1] = (D[i] + A.W[i - 1][i -
      1]) % MOD;
  cout << mat_det(A) << endl;
  return 0;
}
```

6.2 大步小步

6.2.1 用法

给定 a,p 求出 x 使得 $a^x = y \pmod p$，其中 p 为质数。

```
#include "../header.cpp"
namespace BSGS {
  unordered_map <int, int> M;
  int solve(int a, int y, int p){  // a ^ x =
    y (mod p)
    M.clear();
    int B = sqrt(p);
    int w1 = y, u1 = power(a, p - 2, p);
    int w2 = 1, u2 = power(a, B, p);
    for(int i = 0;i < B;++ i){
      M[w1] = i;
      w1 = 1ll * w1 * u1 % p;
    }
    for(int i = 0;i < p / B;++ i){
      if(M.count(w2)){
        return i * B + M[w2];
      }
      w2 = 1ll * w2 * u2 % p;
    }
    return -1;
  }
}
```

6.3 中国剩余定理

6.3.1 定理

对于线性方程：

$$\begin{cases} x \equiv a_1 \pmod{m_1} \\ x \equiv a_2 \pmod{m_2} \\ \cdots \\ x \equiv a_n \pmod{m_n} \end{cases}$$

如果 a_i 两两互质，可以得到 x 的解 $x \equiv L \pmod M$，其中 $M = \prod m_i$，而 L 由下式给出：

$$L = \left(\sum a_i m_i \times ((M/m_i)^{-1} \bmod m_i) \right) \bmod M$$

```
#include "../header.cpp"
i64 A[MAXN], B[MAXN], M = 1;
i64 exgcd(i64 a, i64 b, i64 &x, i64 &y);
int main(){
  int n; cin >> n;
  for(int i = 1;i ⩽ n;++ i){
    cin >> B[i] >> A[i];
    M = M * B[i];
  }
  i64 L = 0;
  for(int i = 1;i ⩽ n;++ i){
    i64 m = M / B[i], b, k;
    exgcd(m, B[i], b, k);
    L = (L + (__int128)A[i] * m * b) % M;
  }
  L = (L % M + M) % M;
  cout << L << endl;
  return 0;
}
```

6.4 狄利克雷前缀和

6.4.1 用法

计算：

$$s(i) = \sum_{d|i} f_d$$

```
#include "../header.cpp"
unsigned A[MAXN];
int p, P[MAXN]; bool V[MAXN];
void solve(int n){
  for(int i = 2;i ⩽ n;++ i){
    if(!V[i]){
      P[++ p] = i;
      for(int j = 1;j ⩽ n / i;++ j){ // 前缀
        和
        A[j * i] += A[j];
      }
    }
    for(int j = 1;j ⩽ p && P[j] ⩽ n / i;++ j
      ){
      V[i * P[j]] = true;
      if(i % P[j] == 0) break;
    }
  }
}
```

6.5 万能欧几里得

6.5.1 类欧几里得（万能欧几里得）

From zpk

一种神奇递归，对 $y = \left\lfloor \frac{Ax+B}{C} \right\rfloor$ 向右和向上走的每步进行压缩，做到 $O(\log V)$ 复杂度。其中 $A \geq C$ 就是直接压缩，向右之后必有至少 $\lfloor A/C \rfloor$ 步向上。$A < C$ 实际上切换 x,y 轴后，相当于压缩了一个上取整折线，而上取整下取整可以互化，便又可以递归。

代码中从 $(0,0)$ 走到 $(n,\lfloor(An+B)/C\rfloor)$，假设了 $A,B,C\geq 0,C\neq 0$（类欧基本都作此假设），U,R 矩阵是从右往左乘的，对列向量进行优化，和实际操作顺序恰好相反。快速幂的 log 据说可以被递归过程均摊掉，实际上并不会导致变成两个 log。

```
Matrix solve(ll n, ll A, ll B, ll C, Matrix R,
   Matrix U) {  // (0, 0) 走到 (n, (An+B)/C)
    if (A >= C) return solve(n, A % C, B, C, U
      .qpow(A / C) * R, U);
    ll l = B / C, r = (A * n + B) / C;
    if (l == r) return R.qpow(n) * U.qpow(l);
        // l = r -> l = r or A = 0 or n = 0.
    ll p = (C * r - B - 1) / A + 1;
    return R.qpow(n - p) * U * solve(r - l -
      1, C, C - B % C + A - 1, A, U, R) * U.
      qpow(l);
}
```

6.6 扩展欧几里得

6.6.1 内容

给定 a,b，求出 $ax+by=\gcd(a,b)$ 的一组 x,y。

```
int exgcd(int a, int b, int &x, int &y){
  if(a == 0){
    x = 0, y = 1; return b;
  } else {
    int x0 = 0, y0 = 0;
    int d = exgcd(b % a, a, x0, y0);
    x = y0 - (b / a) * x0;
    y = x0;
    return d;
  }
}
```

6.7 快速离散对数

6.7.1 用法

给定原根 g 以及模数 mod，T 次询问 x 的离散对数。

复杂度 $\mathcal{O}(\mathrm{mod}^{2/3}+T\log \mathrm{mod})$。

```
#include "../header.cpp"
namespace BSGS {
  unordered_map <int, int> M;
  int B, U, P, g;
  void init(int g, int P0, int B0);
  int solve(int y);
}
const int MAXN = 1e5 + 3;
int H[MAXN], P[MAXN], H0, p, h, g, mod;
bool V[MAXN];
int solve(int x){
  if(x <= h) return H[x];
  int v = mod / x, r = mod % x;
  if(r < x - r) return ((H0 + solve(r)) % (mod
     - 1) - H[v] + mod - 1) % (mod - 1);
  else          return (solve(x - r) - H[v +
    1] + mod - 1) % (mod - 1);
}
int main(){
  ios :: sync_with_stdio(false);
  cin.tie(nullptr);
  cin >> g >> mod;
  h = sqrt(mod) + 1;
  BSGS :: init(g, mod, sqrt(1ll * mod * sqrt(
    mod) / log10(mod)));
  H0 = BSGS :: solve(mod - 1);
  H[1] = 0;
  for(int i = 2;i <= h;++ i){
    if(!V[i]){
      P[++ p] = i;
      H[i] = BSGS :: solve(i);
    }
    for(int j = 1;j <= p && P[j] <= h / i;++ j
      ){
      int &p = P[j];
      H[i * p] = (H[i] + H[p]) % (mod - 1);
      V[i * p] = true;
      if(i % p == 0) break;
    }
  }
  int T; cin >> T;
  while(T --){
    int x; cin >> x;
    cout << solve(x) << "\n";
  }
  return 0;
}
```

6.8 原根

6.8.1 用法

计算 P 的最小原根。

原根表，其中 $P=r\times 2^k$，对应原根为 g。

Prime	g	Prime	g
104857601	3	7881299347898369	6
167772161	3	31525197391593473	3
469762049	3	180143985094819841	6
998244353	3	1945555039024054273	5
1004535809	3	4179340454199820289	3

```
#include "../header.cpp"
int getphi(int x){
  int t = x, r = x;
  for(int i = 2;i <= x / i;++ i){
    if(t % i == 0){
      r = r / i * (i - 1);
      while(t % i == 0)
        t /= i;
    }
  }
  if(t != 1){
    r = r / t * (t - 1);
  }
  return r;
}
vector <int> getprime(int x){
  vector <int> p;
  int t = x;
  for(int i = 2;i <= x / i;++ i){
    if(t % i == 0){
      p.push_back(i);
      while(t % i == 0)
        t /= i;
    }
  }
  if(t != 1)
    p.push_back(x);
  return p;
}
bool test(int g, int m, int mm, vector<int> &P
  ){
  for(auto &p: P){
    if(power(g, mm / p, m) == 1)
      return false;
  }
  return true;
}
int get_genshin(int m){
  int mm = getphi(m);
  vector <int> P = getprime(mm);
```

```
  for(int i = 1;;++ i){
    if(test(i, m, mm, P))
      return i;
  }
}
```

6.9 拉格朗日插值

6.9.1 定理

给定 n 个横坐标不同的点 (x_i, y_i)，可以唯一确定一个 $n-1$ 阶多项式如下：

$$f(x) = \sum_{i=1}^{n} \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)} \cdot y_i$$

6.10 min-max 容斥

6.10.1 定理

$$\max_{i\in S}\{x_i\} = \sum_{T\subseteq S}(-1)^{|T|-1}\min_{j\in T}\{x_j\}$$

$$\min_{i\in S}\{x_i\} = \sum_{T\subseteq S}(-1)^{|T|-1}\max_{j\in T}\{x_j\}$$

期望意义下上式依然成立。

另外设 $\max^k$ 表示第 k 大的元素，可以推广为如下式子：

$$\max_{i\in S}^{k}\{x_i\} = \sum_{T\subseteq S}(-1)^{|T|-k}\binom{|T|-1}{k-1}\min_{j\in T}\{x_j\}$$

此外在数论上可以得到：

$$\operatorname*{lcm}_{i\in S}\{x_i\} = \prod_{T\subseteq S}\left(\gcd_{j\in T}\{x_j\}\right)^{(-1)^{|T|-1}}$$

6.10.2 应用

对于计算 “n 个属性都出现的期望时间” 问题，设第 i 个属性第一次出现的时间是 t_i，所求即为 $\max(t_i)$，使用 min-max 容斥转为计算 $\min(t_i)$。

比如 n 个独立物品，每次抽中物品 i 的概率是 p_i，问期望抽多少次抽中所有物品。那么就可以计算 $\min_S$ 表示第一次抽中物品集合 S 内物品的时间，可以得到：

$$\max_{U} = \sum_{S\subseteq U}(-1)^{|S|-1}\min_{S} = \sum_{S\subseteq U}(-1)^{|S|-1}\cdot\frac{1}{\sum_{x\in S}p_x}$$

6.11 Barrett 取模

6.11.1 用法

调用 init 计算出 S 和 X，得到计算 $\lfloor x/P\rfloor = (x\times X)/2^{60+S}$。从而计算 $x \bmod P = x - P\times\lfloor x/P\rfloor$。

```
#include "../header.cpp"
i64 S = 0, X = 0;
void init(int MOD){
  while((1 << (S + 1)) < MOD) S ++;
  X = ((__int128)1 << 60 + S) / MOD + !!(((__
    int128)1 << 60 + S) % MOD);
  cerr << S << " " << X << endl;
}
int power(i64 x, int y, int MOD){
  i64 r = 1;
  while(y){
    if(y & 1){
      r = r * x;
      r = r - MOD * ((__int128)r * X >> 60 + S
        );
    }
    x = x * x;
    x = x - MOD * ((__int128)x * X >> 60 + S);
    y >>= 1;
  }
  return r;
}
```

6.12 Pollard's Rho

6.12.1 用法

- 调用 `test(n)` 判断 n 是否是质数；
- 调用 `rho(n)` 计算 n 分解质因数后的结果，不保证结果有序。

```
#include "../header.cpp"
i64 step(i64 a, i64 c, i64 m){
  return ((__int128)a * a + c) % m;
}
i64 multi(i64 a, i64 b, i64 m){
  return (__int128) a * b % m;
}
i64 power(i64 a, i64 b, i64 m){
  i64 r = 1;
  while(b){
    if(b & 1) r = multi(r, a, m);
    b >>= 1,  a = multi(a, a, m);
  }
  return r;
}
mt19937_64 MT;
bool test(i64 n){
  if(n < 3 || n % 2 == 0) return n == 2;
  i64 u = n - 1, t = 0;
  while(u % 2 == 0) u /= 2, t += 1;
  int test_time = 20;
  for(int i = 1; i <= test_time;++ i){
    i64 a = MT() % (n - 2) + 2;
    i64 v = power(a, u, n);
    if(v == 1) continue;
    int s;
    for(s = 0;s < t;++ s){
      if(v == n - 1) break;
      v = multi(v, v, n);
    }
    if(s == t) return false;
  }
  return true;
}
basic_string<i64> rho(i64 n){
  if(n == 1)  return { };
  if(test(n)) return {n};
  i64 a  = MT() % (n - 1) + 1;
  i64 x1 = MT() % (n - 1), x2 = x1;
  for(int i = 1;;i <<= 1){
    i64 tot = 1;
    for(int j = 1;j <= i;++ j){
      x2 = step(x2, a, n);
      tot = multi(tot, llabs(x1 - x2), n);
      if(j % 127 == 0){
        i64 d = __gcd(tot, n);
        if(d > 1)
          return rho(d) + rho(n / d);
      }
    }
    i64 d = __gcd(tot, n);
    if(d > 1)
      return rho(d) + rho(n / d);
    x1 = x2;
  }
}
```

6.13 polya 定理

6.13.1 Burnside 引理

记所有染色方案的集合为 X，其中单个染色方案为 x。一种**对称操作** $g\in X$ 作用于染色方案 $x\in X$ 上可以得到另外一种染色 x'。

将所有对称操作作为集合 G, 那么 $Gx = \{gx \mid g \in G\}$ 是与 x **本质相同的染色方案的集合**，形式化地称为 x 的轨道。统计本质不同染色方案数，就是**统计不同轨道个数**。

Burnside 引理说明如下：

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

其中 X^g 表示在 $g \in G$ 的作用下，**不动点**的集合。不动点被定义为 $x = gx$ 的 x。

6.13.2 Polya 定理

对于通常的染色问题, X 可以看作一个长度为 n 的序列，每个元素是 1 到 m 的整数。可以将 n 看作面数、m 看作颜色数。Polya 定理叙述如下：

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}\sum_{g\in G} m^{c(g)}$$

其中 $c(g)$ 表示对一个序列做轮换操作 g 可以**分解成多少个置换环**。

然而, 增加了限制 (比如要求某种颜色必须要多少个), 就无法直接应用 Polya 定理，需要利用 Burnside 引理进行具体问题具体分析。

6.13.3 应用

给定 n 个点 n 条边的环，现在有 n 种颜色，给每个顶点染色，询问有多少种本质不同的染色方案。

显然 X 是全体元素在 1 到 n 之间长度为 n 的序列, G 是所有可能的单次旋转方案，共有 n 种，第 i 种方案会把 1 置换到 i。于是：

$$\begin{aligned}\text{ans} &= \frac{1}{|G|}\sum_{i=1}^{n} m^{c(g_i)}\\ &= \frac{1}{n}\sum_{i=1}^{n} n^{\gcd(i,n)}\\ &= \frac{1}{n}\sum_{d|n}^{n} n^d \sum_{i=1}^{n}[\gcd(i,n)=d]\\ &= \frac{1}{n}\sum_{d|n}^{n} n^d \varphi(n/d)\end{aligned}$$

```cpp
#include "../header.cpp"
vector <tuple<int, int> > P;
void solve(int step, int n, int d, int f, int
  &ans){
  if(step == P.size()){
    ans = (ans + 1ll * power(n, n / d) * f) %
      MOD;
  } else {
    auto [w, c] = P[step];
    int dd = 1, ff = 1;
    for(int i = 0;i <= c;++ i){
      solve(step + 1, n, d * dd, f * ff, ans);
      ff = ff * (w - (i == 0));
      dd = dd * w;
    }
  }
}
int main(){
  int T; cin >> T;
  while(T --){
    int n, t;
    cin >> n;
    t = n;
    for(int i = 2;i * i <= n;++ i) if(n % i ==
       0){
      int w = i, c = 0;
      while(t % i == 0){
        t /= i, c ++;
      }
      P.push_back({ w, c });
    }
    if(t != 1){
      P.push_back({ t, 1 });
    }
    int ans = 0;
    solve(0, n, 1, 1, ans);
    ans = 1ll * ans * power(n, MOD - 2) % MOD;
    cout << ans << endl;
    P.clear();
  }
  return 0;
}
```

6.14 min25 筛

设有一个积性函数 $f(n)$，满足 $f(p^k)$ 可以快速求，考虑搞一个在质数位置和 $f(n)$ 相等的 $g(n)$，满足它有完全积性，并且单点和前缀和都可以快速求，然后通过第一部分筛出 g 在质数位置的前缀和，从而相当于得到 f 在质数位置的前缀和，然后利用它，做第二部分，求出 f 的前缀和。

第一部分：$G_k(n) = \sum_{i=1}^{n}[\text{mindiv}(i) > p_k \text{ or isprime}(i)]g(i)$ $(p_0 = 1)$，则有 $G_k(n) = G_{k-1}(n) - g(p_k)(G_{k-1}(n/p_k) - G_{k-1}(p_{k-1}))$，复杂度 $O(n^{3/4}/\log n)$。

第二部分：$F_k(n) = \sum_{i=1}^{n}[\text{mindiv}(i) \geq p_k]f(i)$，$F_k(n) = \sum_{\substack{h\geq k\\ p_h^2\leq n}}\sum_{\substack{c\geq 1\\ p_h^{c+1}\leq n}}(f(p_h^c)F_{h+1}(n/p_h^c) + f(p_h^{c+1})) + F_{\text{prime}}(n) - F_{\text{prime}}(p_{k-1})$，在 $n \leq 10^{13}$ 可以证明复杂度 $O(n^{3/4}/\log n)$。

常见细节问题：

- 由于 n 通常是 10^{10} 到 10^{11} 的数，导致 n 会爆 int，n^2 会爆 long long，而且往往会用自然数幂和，更容易爆，所以要小心。
- 记 $s = \lfloor\sqrt{n}\rfloor$，由于 F 递归时会去找 F_{h+1}，会访问到 s 以内最大的质数往后的一个质数，而已经证明对于所有 $n \in \mathbb{N}^+$，$[n+1, 2n]$ 中有至少一个质数，所以只需要筛到 $2s$ 即可。
- 注意补回 $f(1)$。

```cpp
// 预处理，$1$ 所在的块也算进去了
namespace init {
    ll init_n, sqrt_n;
    vector<ll> np, p, id1, id2, val;
    ll cnt;
    void main(ll n) {
        init_n = n, sqrt_n = sqrt(n);
        ll M = sqrt_n * 2; // 筛出一个 > floor
          (sqrt(n)) 的质数，避免后续讨论边界
        np.resize(M + 1), p.resize(M + 1);
        for (ll i = 2; i <= M; ++i) {
            if (!np[i]) p[++p[0]] = i;
            for (ll j = 1; j <= p[0]; ++j) {
                if (i * p[j] > M) break;
                np[i * p[j]] = 1;
                if (i % p[j] == 0) break;
            }
        }
        p[0] = 1;
        id1.resize(sqrt_n + 1), id2.resize(
          sqrt_n + 1);
        val.resize(1);
        for (ll l = 1, r, v; l <= n; l = r +
          1) {
```

```
            v = n / l, r = n / v;
            if (v ⩽ sqrt_n) id1[v] = ++cnt;
            else id2[init_n / v] = ++cnt;
            val.emplace_back(v);
        }
    }
    ll id(ll n) {
        if (n ⩽ sqrt_n) return id1[n];
        else return id2[init_n / n];
    }
}
using namespace init;
// 计算 $G_k$，两个参数分别是 $g$ 从 $2$ 开始的前缀和 $g$
auto calcG = [&] (auto&& sum, auto&& g) -> vector<ll> {
    vector<ll> G(cnt + 1);
    for (int i = 1; i ⩽ cnt; ++i) G[i] = sum(val[i]);
    ll pre = 0;
    for (int i = 1; p[i] * p[i] ⩽ n; ++i) {
        for (int j = 1; j ⩽ cnt; ++j) {
            if (p[i] * p[i] > val[j]) break;
            ll tmp = id(val[j] / p[i]);
            G[j] = (G[j] - g(p[i]) * (G[tmp] - pre)) % MD;
        }
        pre = (pre + g(p[i])) % MD;
    }
    for (int i = 1; i ⩽ cnt; ++i) G[i] = (G[i] % MD + MD) % MD;
    return G;
};
// 计算 $F_k$，直接搜，不用记忆化。`fp` 是 $F_{\text{prime}}$，`pc` 是 $p^c$，其中 `f(p[h]^c)` 要替换掉。
function<ll(ll, int)> calcF = [&] (ll m, int k) {
    if (p[k] > m) return 0;
    ll ans = (fp[id(m)] - fp[id(p[k - 1])]) % MD;
    for (int h = k; p[h] * p[h] ⩽ m; ++h) {
        ll pc = p[h], c = 1;
        while (pc * p[h] ⩽ m) {
            ans = (ans + calcF(m / pc, h + 1) * f(p[h] ^ c)) % MD;
            ++c, pc = pc * p[h], ans = (ans + f(p[h] ^ c)) % MD;
        }
    }
    return ans;
};
```

6.15 杜教筛

6.15.1 用法

对于积性函数 f，找到易求前缀和的积性函数 g,h 使得 $h=f*g$，根据递推式计算 $S(n)=\sum_{i=1}^{n}f(i)$：

$$S(n)=H(n)-\sum_{d=1}^{n}g(d)\times S(\lfloor\frac{n}{d}\rfloor)$$

6.15.2 例题

- 对于 $f=\varphi$，寻找 $g=1,h=\mathrm{id}$；
- 对于 $f=\mu$，寻找 $g=1,h=\varepsilon$。

```
#include "../header.cpp"
const int H = 1e7;
int P[MAXN], p; bool V[MAXN];
i64 ph[MAXN], sph[MAXN];
i64 mu[MAXN], smu[MAXN];
i64 tp[MAXN];
i64 solve_ph(i64 N){
  for(int d = N / H;d ⩾ 1;-- d){
    i64 n = N / d;
    i64 wh = 1ll * n * (n + 1) / 2;
    tp[d] = wh;
    for(i64 l = 2, r;l ⩽ n;l = r + 1){
      r = n / (n / l);
      i64 wg = r - l + 1;
      i64 ws = n / l ⩽ H ? sph[n / l] : tp[N / (n / l)];
      tp[d] -= wg * ws;
    }
  }
  return N ⩽ H ? sph[N] : tp[1];
}
i64 solve_mu(i64 N){
  for(int d = N / H;d ⩾ 1;-- d){
    i64 n = N / d;
    i64 wh = 1;
    tp[d] = wh;
    for(i64 l = 2, r;l ⩽ n;l = r + 1){
      r = n / (n / l);
      i64 wg = r - l + 1;
      i64 ws = n / l ⩽ H ? smu[n / l] : tp[N / (n / l)];
      tp[d] -= wg * ws;
    }
  }
  return N ⩽ H ? smu[N] : tp[1];
}
int main(){
  ios :: sync_with_stdio(false);
  cin.tie(nullptr);
  ph[1] = 1;
  mu[1] = 1;
  for(int i = 2;i ⩽ H;++ i){
    if(!V[i]){
      P[++ p] = i;
      ph[i] = i - 1;
      mu[i] = -1;
    }
    for(int j = 1;j ⩽ p && P[j] ⩽ H / i;++ j){
      int &p = P[j];
      V[i * p] = true;
      if(i % p == 0){
        ph[i * p] = ph[i] * p;
        mu[i * p] = 0;
        break;
      } else {
        ph[i * p] = ph[i] * (p - 1);
        mu[i * p] = -mu[i];
      }
    }
  }
  for(int i = 1;i ⩽ H;++ i){
    sph[i] = sph[i - 1] + ph[i];
    smu[i] = smu[i - 1] + mu[i];
  }
  int T; cin >> T;
  while(T --> 0){
    int n; cin >> n;
    cout << solve_ph(n) << " " << solve_mu(n) << "\n";
  }
  return 0;
}
```

6.16 PN 筛

6.16.1 用法

对于积性函数 $f(x)$，寻找积性函数 $g(x)$ 使得 $g(p)=f(p)$，且 g 易求前缀和 G。

令 $h=f*g^{-1}$，可以证明只有 PN 处 h 的函数值非 0，PN 指每个素因子幂次都不小于 2 的数。同时可以证明 n 以内的 PN 只有 $\mathcal{O}(\sqrt{n})$ 个，且可以暴力枚举质因子幂次得到所有 PN。

可利用下面公式计算 $h(p^c)$:

$$h(p^c) = f(p^c) - \sum_{i=1}^{c} g(p^i) \times h(p^{c-i})$$

6.16.2 例题

定义积性函数 $f(x)$ 满足 $f(p^k) = p^k(p^k - 1)$, 计算 $\sum f(i)$。

取 $g(p) = \mathrm{id}(p)\varphi(p) = f(p)$, 根据 $g * \mathrm{id} = \mathrm{id}_2$ 利用杜教筛求解。$h(p^c)$ 的值利用递推式进行计算。

```cpp
#include "../header.cpp"
const int H = 1e7;
const int MOD = 1e9 + 7;
const int DIV2 = 500000004;
const int DIV6 = 166666668;
int P[MAXN], p; bool V[MAXN];
int g[MAXN], le[MAXN], ge[MAXN];
int s1(i64 n){  // 1^1 + 2^1 + ... + n^1
  n %= MOD;
  return 1ll * n * (n + 1) % MOD * DIV2 % MOD;
}
int s2(i64 n){  // 1^2 + 2^2 + ... + n^2
  n %= MOD;
  return 1ll * n * (n + 1) % MOD * (2 * n + 1)
    % MOD * DIV6 % MOD;
}
int sg(i64 n, i64 N){
  return n <= H ? le[n] : ge[N / n];
}
int sieve_du(i64 N){
  for(int d = N / H;d >= 1;-- d){
    i64 n = N / d;
    int wh = s2(n);
    for(i64 l = 2, r;l <= n;l = r + 1){
      r = n / (n / l);
      int wg = (s1(r) - s1(l - 1) + MOD) % MOD
        ;
      int ws = sg(n / l, N);
      ge[d] = (ge[d] + 1ll * wg * ws) % MOD;
    }
    ge[d] = (wh - ge[d] + MOD) % MOD;
  }
  return N <= H ? le[N] : ge[1];
}
vector <int> hc[MAXM], gc[MAXM];
int ANS;
void sieve_pn(int last, i64 x, int h, i64 N){
  ANS = (ANS + 1ll * h * sg(N / x, N)) % MOD;
  for(i64 i = last + 1;x <= N / P[i] / P[i];++
    i){
    int c = 2;
    for(i64 t = x * P[i] * P[i];t <= N;t *= P[
      i], c ++){
      int hh = 1ll * h * hc[i][c] % MOD;
      sieve_pn(i, t, hh, N);
    }
  }
}
int main(){
  ios :: sync_with_stdio(false);
  cin.tie(nullptr);
  g[1] = 1;
  for(int i = 2;i <= H;++ i){
    if(!V[i]){
      P[++ p] = i, g[i] = 1ll * i * (i - 1) %
        MOD;
    }
    for(int j = 1;j <= p && P[j] <= H / i;++ j
      ){
      int &p = P[j];
      V[i * p] = true;
      if(i % p == 0){
        g[i * p] = 1ll * g[i] * p % MOD * p %
          MOD;
        break;
      } else {
        g[i * p] = 1ll * g[i] * p % MOD * (p -
          1) % MOD;
      }
    }
  }
  for(int i = 1;i <= H;++ i){
    le[i] = (le[i - 1] + g[i]) % MOD;
  }
  i64 N;
  cin >> N;
  for(int i = 1;i <= p && 1ll * P[i] * P[i] <=
    N;i ++){
    int &p = P[i];
    hc[i].push_back(1);
    gc[i].push_back(1);
    for(i64 c = 1, t = p;t <= N;t = t * p, ++
      c){
      if(c == 1){
        gc[i].push_back(1ll * p * (p - 1) %
          MOD);
      } else {
        gc[i].push_back(1ll * gc[i].back() * p
          % MOD * p % MOD);
      }
      int w = 1ll * (t % MOD) * ((t - 1) % MOD
        ) % MOD;
      int s = 0;
      for(int j = 1;j <= c;++ j){
        s = (s + 1ll * gc[i][j] * hc[i][c - j
          ]) % MOD;
      }
      hc[i].push_back((w - s + MOD) % MOD);
    }
  }
  sieve_du(N);
  sieve_pn(0, 1, 1, N);
  cout << ANS << "\n";
  return 0;
}
```

6.17 常用数表

6.17.1 大质数

10^{18} 级别:

- $P = 10^{18} + 3$, 好记。
- $P = 2924438830427668481$, 可以进行 NTT, $P = 174310137655 \times 2^24 + 1$, 原根为 3。

6.18 二次剩余

6.18.1 用法

多次询问, 每次询问给定奇素数 p 以及 y, 在 $\mathcal{O}(\log p)$ 复杂度计算 x 使得 $x^2 \equiv 0 \pmod p$ 或者无解。

```cpp
#include "../header.cpp"
bool check(int x, int p){
  return power(x, (p - 1) / 2, p) == 1;
}
struct Node {
  int real, imag;
};
Node mul(const Node a, const Node b, int p,
  int v){
  int nreal = (1ll * a.real * b.real + 1ll * a
    .imag * b.imag % p * v) % p;
  int nimag = (1ll * a.real * b.imag + 1ll * a
    .imag * b.real) % p;
  return { (nreal), nimag };
}
Node power(Node a, int b, int p, int v){
  Node r = { 1, 0 };
  while(b){
```

```
    if(b & 1) r = mul(r, a, p, v);
    b >>= 1,  a = mul(a, a, p, v);
  }
  return r;
}
mt19937 MT;
void solve(int n, int p, int &x1, int &x2){
  if(n == 0){
    x1 = x2 = 0;
    return;
  }
  if(!check(n, p)){
    x1 = x2 = -1;
    return;
  }
  int a, t;
  do {
    a = MT() % p;
  }while(check(t = (1ll * a * a - n + p) % p,
    p));
  Node u = { a, 1 };
  x1 = power(u, (p + 1) / 2, p, t).real;
  x2 = (p - x1) % p;
  if(x1 > x2) swap(x1, x2);
}
int main(){
  ios :: sync_with_stdio(false);
  cin.tie(nullptr);
  int T; cin >> T;
  while(T --){
    int n, p, x1, x2;
    cin >> n >> p;
    solve(n, p, x1, x2);
    if(x1 == -1){
      cout << "Hola!\n";
    } else {
      if(x1 == x2){
        cout << x1 << "\n";
      } else {
        cout << x1 << " " << x2 << "\n";
      }
    }
  }
  return 0;
}
```

6.19 单位根反演

6.19.1 定理

给出单位根反演如下:

$$[d \mid n] = \frac{1}{d}\sum_{i=0}^{d-1}\omega_d^{ni}$$

7 多项式

7.1 NTT 全家桶

7.1.1 用法

多项式全家桶。

- 包含基础多项式算法：快速傅里叶变换（FFT）及其逆变换（IFFT）、快速数论变换（NTT）及其逆变换（INTT）；
- 包含基于 NTT 的扩展多项式算法：多项式乘法（MUL）、多项式乘法逆元（INV）、多项式微分（DIF）、多项式积分（INT）、多项式对数（LN）、多项式指数（EXP）、多项式开根（SQT）、多项式平移（即计算 $G(x) = F(x + c)$，SHF）。

```
#include "../header.cpp"
int inv(int x);
const int MAX_ = (1 << 19) + 3;
using cplx = complex<double>;
const long double pi = acos(-1);
namespace Poly{
  void FFT(int n, cplx Z[]){
    static int W[MAX_];
    int l = 1; W[0] = 0;
    while (n >>= 1)
      up(0, l - 1, i)
        W[l++] = W[i] << 1 | 1, W[i] <<= 1;
    up(0, l - 1, i)
      if(W[i] > i) swap(Z[i], Z[W[i]]);
    for (n = l >> 1, l = 1;n;n >>= 1, l <<= 1)
      {
      cplx* S = Z, o(cos(pi / l), sin(pi / l))
        ;
      up(0, n - 1, i){
        cplx s(1, 0);
        up(0, l - 1, j){
          cplx x = S[j] + s * S[j + l];
          cplx y = S[j] - s * S[j + l];
          S[j] = x, S[j + l] = y, s = s * o;
        }
        S += l << 1;
      }
    }
  }
  void IFFT(int n, cplx Z[]){
    FFT(n, Z); reverse(Z + 1, Z + n);
    up(0, n - 1, i) Z[i] /= n;
  }
  void NTT(int n, int Z[]){
    static int W[MAX_];
    int g = 3, l = 1; W[0] = 0;
    while (n >>= 1)
      up(0, l - 1, i)
        W[l++] = W[i] << 1 | 1, W[i] <<= 1;
    up(0, l - 1, i)
      if (W[i] > i)swap(Z[i], Z[W[i]]);
    for (n = l >> 1, l = 1;n;n >>= 1, l <<= 1)
      {
      int* S = Z, o = power(g, (MOD - 1) / l /
        2);
      up(0, n - 1, i){
        int s = 1;
        up(0, l - 1, j){
          int x = (S[j] + 1ll * s * S[j + l] %
            MOD    ) % MOD;
          int y = (S[j] - 1ll * s * S[j + l] %
            MOD + MOD) % MOD;
          S[j] = x, S[j + l] = y;
          s = 1ll * s * o % MOD;
        }
        S += l << 1;
      }
    }
  }
  void INTT(int n, int Z[]){
    NTT(n, Z); reverse(Z + 1, Z + n);
    int o = inv(n);
    up(0, n - 1, i)
      Z[i] = 1ll * Z[i] * o % MOD;
  }
  void MUL(int n, int A[], int B[]){      //
    乘法
    NTT(n, A), NTT(n, B);
    up(0, n - 1, i)
      A[i] = 1ll * A[i] * B[i] % MOD;
    INTT(n, A);
  }
  void INV(int n, int Z[], int T[]){      //
    乘法逆
```

```
    static int A[MAX_];
    up(0, n - 1, i)
      T[i] = 0;
    T[0] = power(Z[0], MOD - 2);
    for (int l = 1;l < n;l <<= 1){
      up(  0, 2 * l - 1, i) A[i] = Z[i];
      up(2 * l, 4 * l - 1, i) A[i] = 0;
      NTT(4 * l, A), NTT(4 * l, T);
      up(0, 4 * l - 1, i)
        T[i] = (2ll * T[i] - 1ll * A[i] * T[i]
           % MOD * T[i] % MOD + MOD) % MOD;
      INTT(4 * l, T);
      up(2 * l, 4 * l - 1, i)
        T[i] = 0;
    }
  }
  void DIF(int n, int Z[], int T[]){      //
    微分
    up(0, n - 2, i)
      T[i] = 1ll * Z[i + 1] * (i + 1) % MOD;
    T[n - 1] = 0;
  }
  void INT(int n, int c, int Z[], int T[]){
    // 积分
    up(1, n - 1, i)
      T[i] = 1ll * Z[i - 1] * inv(i) % MOD;
    T[0] = c;
  }
  void LN(int n, int* Z, int* T){       // 求
    对数
    static int A[MAX_], B[MAX_];
    up(0, 2 * n - 1, i)
      A[i] = B[i] = 0;
    DIF(n, Z, A), INV(n, Z, B), MUL(2 * n, A,
      B), INT(n, 0, A, T);
  }
  void EXP(int n, int* Z, int* T){      // 求
    指数
    static int A[MAX_], B[MAX_];
    up(1, 2 * n - 1, i) T[i] = 0;
    T[0] = 1;
    for (int l = 1;l < n;l <<= 1){
      LN (2 * l, T, A);
      up(  0, 2 * l - 1, i)
        B[i] = (-A[i] + Z[i] + MOD) % MOD;
      B[0] = (B[0] + 1) % MOD;
      up(2 * l, 4 * l - 1, i)
        T[i] = B[i] = 0;
      MUL(4 * l, T, B);
    }
  }
  void SQT(int n, int* Z, int* T){      // 开
    根
    static int A[MAX_], B[MAX_];
    up(1, 2 * n - 1, i) T[i] = 0;
    T[0] = 1;
    int o = inv(2);
    for (int l = 1;l < n;l <<= 1){
      INV(2 * l, T, A);
      up(0, 2 * l - 1, i)
        B[i] = Z[i];
      up(2 * l, 4 * l - 1, i)
        A[i] = B[i] = 0;
      MUL(4 * l, A, B);
      up(0, 2 * l - 1, i)
        T[i] = 1ll * (T[i] + A[i]) * o % MOD;
    }
  }
  void SHF(int n, int c, int* Z, int* T){   //
    平移
    static int A[MAX_], B[MAX_], F[MAX_], G[
      MAX_];
    int o = 1;
    up(1, n - 1, i)
      F[i] = 1ll * F[i - 1] *   i  % MOD,
      G[i] = 1ll * G[i - 1] * inv(i) % MOD;
    up(0, n - 1, i)
      A[i] = 1ll * Z[n - 1 - i] * F[n - 1 - i]
        % MOD;
    up(0, n - 1, i){
      B[i] = 1ll * G[i] * o % MOD;
      o = 1ll * o * c % MOD;
    }
    int l = 1; while (l < 2 * n - 1) l <<= 1;
    up(n, l - 1, i)
      A[i] = B[i] = 0;
    MUL(l, A, B);
    up(0, n - 1, i)
      T[n - 1 - i] = 1ll * G[n - 1 - i] * A[i]
        % MOD;
  }
}
```

7.2 FWT 全家桶

7.2.1 用法

沃尔什全家桶。

包含与卷积、或卷积、异或卷积，定义分别为二进制与、或、异或带入下式：

$$b_k = \sum_{i \otimes j = k} a_i \times b_j$$

```
#include "../header.cpp"
namespace Solve1{   // or 卷积
  void FWT(int n, int *A){
    for(int l = 1 << n, u = 2, v = 1;u <= l;u
      <<= 1, v <<= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k)
          A[j + v + k] = (A[j + v + k] + A[j +
            k]) % MOD;
  }
  void IFWT(int n, int *A){
    for(int l = 1 << n, u = l, v = l / 2;u >
      1;u >>= 1, v >>= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k)
          A[j + v + k] = (A[j + v + k] - A[j +
            k] + MOD) % MOD;
  }
}
namespace Solve2{   // and 卷积
  void FWT(int n, int *A){
    for(int l = 1 << n, u = 2, v = 1;u <= l;u
      <<= 1, v <<= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k)
          A[j + k] = (A[j + k] + A[j + v + k])
            % MOD;
  }
  void IFWT(int n, int *A){
    for(int l = 1 << n, u = l, v = l / 2;u >
      1;u >>= 1, v >>= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k)
          A[j + k] = (A[j + k] - A[j + v + k]
            + MOD) % MOD;
  }
}
namespace Solve3{   // xor 卷积
  void FWT(int n, int *A){
    for(int l = 1 << n, u = 2, v = 1;u <= l;u
      <<= 1, v <<= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k){
          int a = A[j + k];
          int b = A[j + v + k];
          A[j + k    ] = (a + b + MOD) % MOD;
          A[j + v + k] = (a - b + MOD) % MOD;
        }
  }
  void IFWT(int n, int *A){
    int div2 = (MOD + 1) / 2;
```

```
    for(int l = 1 << n, u = l, v = l / 2;u >
      1;u >>= 1, v >>= 1)
      for(int j = 0;j < l;j += u)
        for(int k = 0;k < v;++ k){
          int a = A[j + k];
          int b = A[j + v + k];
          A[j + k    ] = 1ll * (a + b + MOD) *
            div2 % MOD;
          A[j + v + k] = 1ll * (a - b + MOD) *
            div2 % MOD;
        }
  }
}
```

7.3 任意模数 NTT

```
#include "poly-family.cpp"
const int BLOCK = 32768;
using cplx = complex<double>;
cplx A1[MAXN], A2[MAXN], B1[MAXN], B2[MAXN];
int n, m, L, mod;
cplx P[MAXN], Q[MAXN];
void FFTFFT(int L, cplx X[], cplx Y[]){
  for(int i = 0;i < L;++ i){
    P[i] = { X[i].real(), Y[i].imag() };
  }
  Poly :: FFT(L, P);
  for(int i = 0;i < L;++ i){
    Q[i] = (i == 0 ? P[0] : P[L - i]);
    Q[i].imag(-Q[i].imag());
  }
  for(int i = 0;i < L;++ i){
    X[i] = (P[i] + Q[i]);
    Y[i] = (Q[i] - P[i]) * cplx(0, 1);
    X[i] /= 2, Y[i] /= 2;
  }
}
int main(){
  ios :: sync_with_stdio(false);
  cin.tie(nullptr);
  cin >> n >> m >> mod;
  for(int i = 0;i <= n;++ i){
    int a; cin >> a; a %= mod;
    A1[i].real(a / BLOCK);
    A2[i].imag(a % BLOCK);
  }
  for(int i = 0;i <= m;++ i){
    int a; cin >> a; a %= mod;
    B1[i].real(a / BLOCK);
    B2[i].imag(a % BLOCK);
  }
  for(L = 1;L <= n + m;L <<= 1);
  FFTFFT(L, A1, A2), FFTFFT(L, B1, B2);
  for(int i = 0;i < L;++ i){
    P[i] = A1[i] * B1[i] + cplx(0, 1) * A2[i]
      * B1[i];
    Q[i] = A1[i] * B2[i] + cplx(0, 1) * A2[i]
      * B2[i];
  }
  Poly :: IFFT(L, P);
  Poly :: IFFT(L, Q);
  for(int i = 0;i < L;++ i){
    long long a1b1 = P[i].real() + 0.5;
    long long a2b1 = P[i].imag() + 0.5;
    long long a1b2 = Q[i].real() + 0.5;
    long long a2b2 = Q[i].imag() + 0.5;
    long long w = ((a1b1 % mod * (BLOCK *
      BLOCK % mod)) + ((a2b1 + a1b2) % mod) *
      BLOCK + a2b2) % mod;
    if(i <= n + m) cout << w << " ";
  }
  return 0;
}
```

8 字符串

8.1 AC 自动机

```
#include "../header.cpp"
namespace ACAM{
  int C[MAXN][MAXM], F[MAXN], o;
  void insert(char *S){
    int p = 0, len = 0;
    for(int i = 0;S[i];++ i){
      int e = S[i] - 'a';
      if(C[p][e]) p = C[p][e];
        else      p = C[p][e] = ++ o;
      ++ len;
    }
  }
  void build(){
    queue <int> Q; Q.push(0);
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(int i = 0;i < 26;++ i){
        int v = C[u][i];
        if(v == 0) continue;
        int p = F[u];
        while(!C[p][i] && p != 0) p = F[p];
        if(C[p][i] && C[p][i] != v)
          F[v] = C[p][i];
        Q.push(v);
      }
    }
  }
}
```

8.2 扩展 KMP

8.2.1 定义

$$z_i^{(1)} = |\mathrm{lcp}(b, \mathrm{suffix}(b, i))|$$
$$z_i^{(2)} = |\mathrm{lcp}(b, \mathrm{suffix}(a, i))|$$

```
#include "../header.cpp"
char A[MAXN], B[MAXN * 2];
int n, m, l, r, Z[MAXN * 2];
i64 ans1, ans2;
int main(){
  scanf("%s%s", A + 1, B + 1);
  n = strlen(A + 1);
  m = strlen(B + 1);
  l = 0, r = 0; Z[1] = 0, ans1 = m + 1;
  for(int i = 2;i <= m;++ i){
    if(i <= r) Z[i] = min(r - i + 1, Z[i - l +
      1]);
    else       Z[i] = 0;
    while(B[Z[i] + 1] == B[i + Z[i]])
      ++ Z[i];
    if(i + Z[i] - 1 > r)
      r = i + Z[i] - 1, l = i;
    ans1 ^= 1ll * i * (Z[i] + 1);
  }
  l = 0, r = 0;
  Z[1] = 0, B[m + 1] = '#', strcat(B + 1, A +
    1);
  for(int i = 2;i <= n + m + 1;++ i){
    if(i <= r) Z[i] = min(r - i + 1, Z[i - l +
      1]);
    else       Z[i] = 0;
    while(B[Z[i] + 1] == B[i + Z[i]])
      ++ Z[i];
    if(i + Z[i] - 1 > r)
      r = i + Z[i] - 1, l = i;
  }
  for(int i = m + 2;i <= n + m + 1;++ i){
    ans2 ^= 1ll * (i - m - 1) * (Z[i] + 1);
  }
  printf("%lld\n%lld\n", ans1, ans2);
  return 0;
}
```

8.3 回文自动机

```cpp
#include "../header.cpp"
namespace PAM{
  const int SIZ = 5e5 + 3;
  int n, s, F[SIZ], L[SIZ], D[SIZ];
  int M[SIZ][MAXM];
  char S[SIZ];
  void init(){
    S[0] = '$', n = 1;
    F[s = 0] = -1, L[0] = -1, D[0] = 0;
    F[s = 1] =  0, L[1] =  0, D[1] = 0;
  }
  void extend(int &last, char c){
    S[++ n] = c;
    int e = c - 'a', a = last;
    while(c != S[n - 1 - L[a]]) a = F[a];
    if(M[a][e]){
      last = M[a][e];
    } else {
      int cur = M[a][e] = ++ s;
      L[cur] = L[a] + 2;
      if(a == 0){
        F[cur] = 1;
      } else {
        int b = F[a];
        while(c != S[n - 1 - L[b]])
          b = F[b];
        F[cur] = M[b][e];
      }
      D[cur] = D[F[cur]] + 1;
      last = cur;
    }
  }
}
```

8.4 后缀数组（倍增）

```cpp
#include "../header.cpp"
int n, m, A[MAXN], B[MAXN];
int C[MAXN], R[MAXN], P[MAXN], Q[MAXN];
char S[MAXN];
int main(){
  scanf("%s", S), n = strlen(S), m = 256;
  for(int i = 0;i < n;++ i) R[i] = S[i];
  for (int k = 1;k <= n;k <<= 1){
    for(int i = 0;i < n;++ i){
      Q[i] = ((i + k > n - 1) ? 0 : R[i + k]);
      P[i] = R[i];
      m = max(m, R[i]);
    }
#define fun(a, b, c) \
    memset(C, 0, sizeof(int) * (m + 1)); \
    for(int i = 0;i <  n;++ i) C[a] +=    1; \
    for(int i = 1;i <= m;++ i) C[i] += C[i - 1];  \
    for(int i = n - 1;i >= 0;-- i) c[-- C[a]] = b;
    fun(Q[  i ],   i , B)
    fun(P[B[i]], B[i], A)
#undef fun
    int p = 1; R[A[0]] = 1;
    for(int i = 1;i <= n - 1;++ i){
      bool f1 = P[A[i]] == P[A[i - 1]];
      bool f2 = Q[A[i]] == Q[A[i - 1]];
      R[A[i]] = f1 && f2 ? R[A[i - 1]] : ++ p;
    }
    if (m == n) break;
  }
  for(int i = 0;i < n;++ i)
    printf("%u ", A[i] + 1);
  return 0;
}
```

8.5 广义后缀自动机（离线）

```cpp
#include "../header.cpp"
namespace SAM{
  const int SIZ = 2e6 + 3;
  int M[SIZ][MAXM];
  int L[SIZ], F[SIZ], S[SIZ];
  int s = 0, h = 25;
  void init(){
    F[0] = -1, s = 0;
  }
  void extend(int &last, char c){
    int e = c - 'a';
    int cur = ++ s;
    L[cur] = L[last] + 1;
    int p = last;
    while(p != -1 && !M[p][e])
      M[p][e] = cur, p = F[p];
    if(p == -1){
      F[cur] = 0;
    } else {
      int q = M[p][e];
      if(L[p] + 1 == L[q]){
        F[cur] = q;
      } else {
        int clone = ++ s;
        L[clone] = L[p] + 1;
        F[clone] = F[q];
        for(int i = 0;i <= h;++ i)
          M[clone][i] = M[q][i];
        while(p != -1 && M[p][e] == q)
          M[p][e] = clone, p = F[p];
        F[cur] = F[q] = clone;
      }
    }
    last = cur;
  }
  void solve(){
    i64 ans = 0;
    for(int i = 1;i <= s;++ i)
      ans += L[i] - L[F[i]];
    cout << ans << endl;
  }
}
namespace Trie{
  const int SIZ = 1e6 + 3;
  int M[SIZ][MAXM], s, h = 25;
  void insert(char *S){
    int p = 0;
    for(int i = 0;S[i];++ i){
      int e = S[i] - 'a';
      if(M[p][e]){
        p = M[p][e];
      } else
        p = M[p][e] = ++ s;
    }
  }
  int O[SIZ];
  void build_sam(){
    queue <int> Q;
    Q.push(0);
    while(!Q.empty()){
      int u = Q.front(); Q.pop();
      for(int i = 0;i <= h;++ i){
        char c = i + 'a';
        if(M[u][i]){
          int v = M[u][i];
          O[v] = O[u];
          SAM :: extend(O[v], c);
          Q.push(v);
        }
      }
    }
  }
}
```

8.6 广义后缀自动机（在线）

```cpp
#include "../header.cpp"
namespace SAM{
```

```
const int SIZ = 2e6 + 3;
int M[SIZ][MAXM];
int L[SIZ], F[SIZ], S[SIZ];
int s = 0, h = 25;
void init(){
  F[0] = -1, s = 0;
}
void extend(int &last, char c){
  int e = c - 'a';
  if(M[last][e]){
    int p = last;
    int q = M[last][e];
    if(L[q] == L[last] + 1){
      last = q;
    } else {
      int clone = ++ s;
      L[clone] = L[p] + 1;
      F[clone] = F[q];
      for(int i = 0;i <= h;++ i)
        M[clone][i] = M[q][i];
      while(p != -1 && M[p][e] == q)
        M[p][e] = clone, p = F[p];
      F[q] = clone;
      last = clone;
    }
  } else {
    int cur = ++ s;
    L[cur] = L[last] + 1;
    int p = last;
    while(p != -1 && !M[p][e])
      M[p][e] = cur, p = F[p];
    if(p == -1){
      F[cur] = 0;
    } else {
      int q = M[p][e];
      if(L[p] + 1 == L[q]){
        F[cur] = q;
      } else {
        int clone = ++ s;
        L[clone] = L[p] + 1;
        F[clone] = F[q];
        for(int i = 0;i <= h;++ i)
          M[clone][i] = M[q][i];
        while(p != -1 && M[p][e] == q)
          M[p][e] = clone, p = F[p];
        F[cur] = F[q] = clone;
      }
    }
    last = cur;
  }
}
void solve(){
  i64 ans = 0;
  for(int i = 1;i <= s;++ i)
    ans += L[i] - L[F[i]];
  cout << ans << endl;
 }
}
// 每次插入新字符串前将 last 清零
```

8.7 后缀自动机

```
#include "../header.cpp"
namespace SAM{
  const int SIZ = 2e6 + 3;
  int M[SIZ][MAXM];
  int L[SIZ], F[SIZ], S[SIZ];
  int last = 0, s = 0, h = 25;
  void init(){
    F[0] = -1, last = s = 0;
  }
  void extend(char c){
    int cur = ++ s, e = c - 'a';
    L[cur] = L[last] + 1;
    S[cur] = 1;
    int p = last;
    while(p != -1 && !M[p][e])
      M[p][e] = cur, p = F[p];
    if(p == -1){
      F[cur] = 0;
    } else {
      int q = M[p][e];
      if(L[p] + 1 == L[q]){
        F[cur] = q;
      } else {
        int clone = ++ s;
        L[clone] = L[p] + 1;
        F[clone] = F[q];
        S[clone] = 0;
        for(int i = 0;i <= h;++ i)
          M[clone][i] = M[q][i];
        while(p != -1 && M[p][e] == q)
          M[p][e] = clone, p = F[p];
        F[cur] = F[q] = clone;
      }
    }
    last = cur;
  }
  vector <int> E[SIZ];
  void build(){
    for(int i = 1;i <= s;++ i){
      E[F[i]].push_back(i);
    }
  }
  i64 ans = 0;
  void dfs(int u){
    for(auto &v : E[u]){
      dfs(v), S[u] += S[v];
    }
    if(S[u] > 1)
      ans = max(ans, 1ll * S[u] * L[u]);
  }
}
```

9 计算几何

10 其他

10.1 笛卡尔树

```
#include "../header.cpp"
// Li: 左儿子; Ri: 右儿子
int n, L[MAXN], R[MAXN], A[MAXN];
void build(){
  stack <int> S;
  A[n + 1] = -1e9;
  for(int i = 1;i <= n + 1;++ i){
    int v = 0;
    while(!S.empty() && A[S.top()] > A[i]){
      auto u = S.top();
      R[u] = v, v  = u, S.pop();
    }
    L[i] = v, S.push(i);
  }
}
```

10.2 CDQ 分治

10.2.1 例题

给定三元组序列 (a_i, b_i, c_i)，求解 $f(i) = \sum_j [a_j \le a_i \wedge b_j \le b_i \wedge c_j \le c_i]$。

```
#include "../header.cpp"
struct Node{
  int id, a, b, c;
}A[MAXN], B[MAXN];
bool cmp(Node a, Node b){
  if(a.a != b.a) return a.a < b.a;
  if(a.b != b.b) return a.b < b.b;
  if(a.c != b.c) return a.c < b.c;
  return a.id < b.id;
}
int K[MAXN], H[MAXN];
```

```
int qread();
int n, m, D[MAXM];
namespace BIT{
  void increase(int x, int w){
    while(x ⩽ m) D[x] += w, x += x & -x;
  }
  void decrease(int x, int w){
    while(x ⩽ m) D[x] -= w, x += x & -x;
  }
  void query(int x, int &r){
    while(x) r += D[x], x -= x & -x;
  }
}
void cdq(int l, int r){
  if(l ≠ r){
    int t = l + r >> 1; cdq(l, t), cdq(t + 1,
      r);
    int p = l, q = t + 1, u = l;
    while(p ⩽ t && q ⩽ r){
      if(A[p].b ⩽ A[q].b)
        BIT :: increase(A[p].c, 1), B[u ++] =
          A[p ++];
       else
        BIT :: query(A[q].c, K[A[q].id]), B[u
          ++] = A[q ++];
    }
    while(p ⩽ t) BIT :: increase(A[p].c, 1),
         B[u ++] = A[p ++];
    while(q ⩽ r) BIT :: query(A[q].c, K[A[q].
      id]), B[u ++] = A[q ++];
    up(l, t, i) BIT :: decrease(A[i].c, 1);
    up(l, r, i) A[i] = B[i];
  }
}
int main(){
  n = qread(), m = qread();
  up(1, n, i) A[i].id = i, A[i].a = qread(), A
    [i].b = qread(), A[i].c = qread();
  sort(A + 1, A + 1 + n, cmp), cdq(1, n);
  sort(A + 1, A + 1 + n, cmp);
  dn(n, 1, i){
    if(A[i].a == A[i + 1].a && A[i].b == A[i +
       1].b && A[i].c == A[i + 1].c)
      K[A[i].id] = K[A[i + 1].id];
    H[K[A[i].id]] ++;
  }
  up(0, n - 1, i) printf("%d\n", H[i]);
  return 0;
}
```

10.3 自适应辛普森

10.3.1 例题

计算

$$\int_0^{+\infty} x^{(a/x)-x}$$

```
#include "../header.cpp"
double simpson(double (*f)(double), double l,
  double r){
  double mid = (l + r) / 2;
  return (r - l) * (f(l) + 4 * f(mid) + f(r))
    / 6.0;
}
double adapt_simpson(double (*f)(double),
  double l, double r, double EPS, int step){
  double mid = (l + r) / 2;
  double w0 = simpson(f, l, r);
  double w1 = simpson(f, l, mid);
  double w2 = simpson(f, mid, r);
  if(fabs(w0 - w1 - w2) < EPS && step < 0)
    return w1 + w2;
  else
    return adapt_simpson(f, l, mid, EPS, step
      - 1) +
           adapt_simpson(f, mid, r, EPS, step
             - 1);
}
double a, l, r;
double fun(double x){
  return pow(x, a / x - x);
}
int main(){
  cin >> a;
  if(a < 0)
    cout << "orz" << endl;
  else {
    l = 1e-9, r = 150;
    cout << fixed << setprecision(5) <<
      adapt_simpson(fun, l, r, 1e-9, 15);
  }
}
```

10.4 模拟退火

10.4.1 例题

给定 n 个物品挂在洞下，第 i 个物品坐标 (x_i, y_i) 重量为 w_i。询问平衡点。

```
#include "../header.cpp"
const double T0 = 2e3, Tk = 1e-14, delta =
  0.993, R = 1e-3;
mt19937 MT(114514);
double distance(double x, double y, double a,
  double b){
  return sqrt(pow(a - x, 2) + pow(b - y, 2));
}
const int MAXN = 1e3 + 3;
double X[MAXN], Y[MAXN], W[MAXN]; int n;
double calculate(double x, double y){
  double gx, gy, a;
  for(int i = 0;i < n; ++i){
    a = atan2(y - Y[i], x - X[i]);
    gx += cos(a) * W[i];
    gy += sin(a) * W[i];
  }
  return pow(gx, 2) + pow(gy, 2);
}
double ex, ey, eans = 1e18;
void SA(){
  double T = T0, x = 0, y = 0, ans = calculate
    (x, y);
  double ansx, ansy;
  uniform_real_distribution<double> U;
  while(T > Tk){
    double nx, ny, nans;
    nx = x + 2 * (U(MT) - .5) * T;
    ny = y + 2 * (U(MT) - .5) * T;
    if((nans = calculate(nx, ny)) < ans){
      ans = nans;
      ansx = x = nx;
      ansy = y = ny;
    } else if(exp(-distance(nx, ny, x, y) / T
      / R) > U(MT)){
      x = nx, y = ny;
    }
    T *= delta;
  }
  if(ans < eans) eans = ans, ex = ansx, ey =
    ansy;
}
```

10.5 伪随机生成

```
#include "../header.cpp"
u32 xorshift32(u32 &x){
  x ^= x << 13, x ^= x >> 17, x ^= x << 5;
  return x;
}
u64 xorshift64(u64 &x){
  x ^= x << 13, x ^= x >> 7, x ^= x << 17;
  return x;
}
```