Hermitian of Matrix:

The hermitian of matrix A is denoted by A^{θ} (or A^{*}) and is defined as conjugate transpose of A

i.e. $A^{\theta} = \overline{(A')} = (\bar{A})'$

Hermitian Matrix:

A square matrix A is said to be Hermitian Matrix if

$$A = A^{\theta}$$

Skew-Hermitian Matrix:

The square matrix A is said to be skew-Hermitian Matrix if

$$A = -A^{\theta}$$

"Every square matrix can be uniquely expressed as a sum of a Hermitian Matrix and a Skew-Hermitian Matrix"

{ Hint: ① Replace ($'$) by (θ)
② $(A^\theta)^\theta = A$ }

Unitary Matrix:

A square matrix A is said to be unitary matrix if $AA^\theta = A^\theta A = I$

Remark: If A is unitary matrix then $A^{-1} = A^\theta$

Ex. Prove that $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is unitary and hence find A^{-1}

Sol^n: Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$A' = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$

$A^\theta = \overline{(A')} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

we have

$$AA^{\theta} = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$\left\{ \begin{array}{l} (x+iy)(x-iy) = x^2+y^2 \\ i^2 = -1 \end{array} \right\}$$

$$= \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1+i)(1-i)+(-1+i)(-1-i) & (1+i)(1-i)+(-1+i)(1+i) \\ (1+i)(1-i)+(1-i)(-1-i) & (1+i)(1-i)+(1-i)(1+i) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1+1+1+1 & 1+i-i-i^2+i+i^2-1-i \\ 1+1-1-i+i+i^2 & 1+1+1+1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore AA^{\theta} = I \text{ ——— } (1)$$

Now consider

$$A^{\theta}A = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\therefore A^{\theta}A = I \text{ ——— } (2)$$

From eqn (1) and (2), we have

$$AA^{\theta} = A^{\theta}A = I$$

$\therefore$ A is unitary matrix.

we have

$$A^{-1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$