Inner Product

Dot Product:

Dot product of 2 vectors $x = \langle 2, -3 \rangle$ & $y = \langle 5, 1 \rangle$ is,

$$x \cdot y = 2 \cdot 5 + (-3 \cdot 1) = 7$$

Dot product can also be expressed as,

$$x^T y = [2, -3] \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 7$$

Length:

The length of vector x is denoted as $\|x\|$.

Example: $x = [3, 4]^T$ then,

$$\|x\| = \sqrt{3^2 + 4^2}$$
$$= \sqrt{25}$$
$$= 5$$

Note that this is same as

$$x^T x = [3 \; 4] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \sqrt{3^2 + 4^2} = 5$$

$$\boxed{\therefore \|x\| = \sqrt{x^T x}}$$

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General Inner Products

Inner Product:

Let V be a vector space and $\Omega : V \times V \to R$ be a bilinear mapping that takes 2 vectors and maps them onto a real number. Then,

- A positive definite, symmetric bilinear mapping $\Omega : V \times V \to R$ is called an inner product on V.
  We typically write $\langle x, y \rangle$ instead of $\Omega \langle x, y \rangle$.
- The pair $(V, \langle \cdot, \cdot \rangle)$ is called inner product space or vector space of inner product.
- Ω is called symmetric if

  $$\Omega(x, y) = \Omega(y, x) \text{ for all } x, y \in V$$

  order of the arguments doesn't matter.
- Ω is called positive definite if

  $$\forall x \in V \setminus \{0\} : \Omega(x, x) > 0, \quad \Omega(0, 0) = 0$$
- The positive definiteness of the inner product implies that,

  $$\forall x \in V \setminus \{0\} : x^T A x > 0$$

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(ARTIFICIAL INTELLIGENCE & MACHINE LEARNING)

Example:

$$A_1 = \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 9 & 6 \\ 6 & 3 \end{bmatrix}$$

A_1 is symmetric as, $A_1 = A_1^T$

Now check for the positive definiteness.

$$\therefore x^T A_1 x = [x_1 \ x_2] \begin{bmatrix} 9 & 6 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [9x_1 + 6x_2 \quad 6x_1 + 5x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 9x_1^2 + 6x_1x_2 + 6x_1x_2 + 5x_2^2$$

$$= 9x_1^2 + 12x_1x_2 + 5x_2^2$$

$$= (3x_1 + 2x_2)^2 + x_2^2 > 0$$

for all $x \in V \setminus \{0\}$. $\therefore$ A_1 is positive definite.

A_2 is symmetric.

$$x^T A_2 x = [x_1 \ x_2] \begin{bmatrix} 9 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [9x_1 + 6x_2 \quad 6x_1 + 3x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 9x_1^2 + 6x_1x_2 + 6x_1x_2 + 3x_2^2$$

$$= 9x_1^2 + 12x_1x_2 + 3x_2^2$$

If we put $x = [2 \ -3]$ then

$$= 36 - 72 + 9$$

$$= -27$$

$x^T A_2 x < 0$

$\therefore$ A_1 is symmetric and positive definite.
Also A_2 is symmetric but not positive definite.

If $A \in R$ is symmetric, positive definite then

~~$\langle x, y \rangle = x^T A y$~~

$$\langle x, y \rangle = x^T A y$$

defines inner product where x and y are the coordinate representations of $x, y \in V$.

Note: Not every norm is ~~id~~ induced by an inner product. (l_1 norm)

Inner product induces a norm

$$\|x\| = \sqrt{x^T x} = \sqrt{\langle x, x \rangle} \quad (l_2 \text{ norm})$$