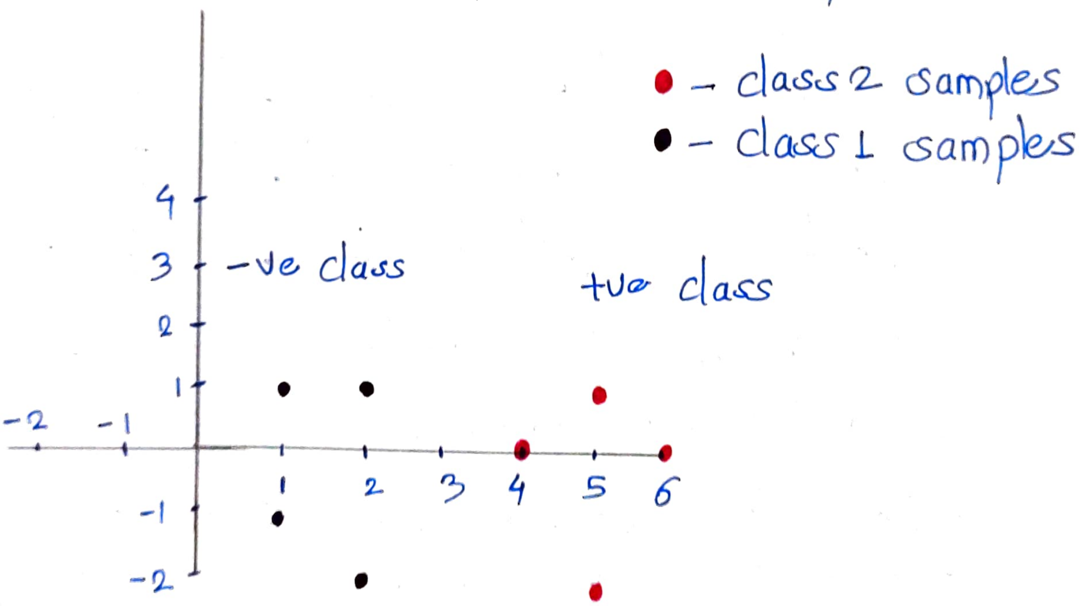
Support Vector Machine Numerical Example

Q.1 Find optimal hyperplane for the following data points.

class 1 : $\{(1,1), (2,1), (1,-1), (2,-1)\}$

class 2 : $\{(4,0), (5,1), (5,-1), (6,0)\}$

→ Plot all the data points in 2D space.



Select the support vectors s_1, s_2 and s_3

$s_1 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$ $\quad s_2 = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ $\quad s_3 = \begin{Bmatrix} 4 \\ 0 \end{Bmatrix}$

Augument above vectors with bias = 1

$$\tilde{S}_1 = \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} \qquad S_2 = \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} \qquad S_3 = \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix}$$

Now, we need to find 3 parameters $\alpha_1, \alpha_2, \alpha_3$ based on following 3 linear equations. These values will be used to find weight vector.

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_1 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_1 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_1 = -1 \text{ (-ve class)}$$

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_2 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_2 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_2 = -1 \text{ (-ve class)}$$

$$\alpha_1 \tilde{S}_1 + \tilde{S}_3 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_3 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_3 = +1 \text{ (+ve class)}$$

Now substitue the values of S_1, S_2 and S_3 in above equation.

$$\alpha_1 \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} = -1$$

$$\alpha_1 \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} = -1$$

$$\alpha_1 \begin{Bmatrix} 2 \\ 1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} 2 \\ -1 \\ 1 \end{Bmatrix} \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} \begin{Bmatrix} 4 \\ 0 \\ 1 \end{Bmatrix} = +1$$

$$\therefore \alpha_1 \{4+1+1\} + \alpha_2 \{4-1+1\} + \alpha_3 \{8+0+1\} = -1$$

$$\therefore \alpha_1 \{4-1+1\} + \alpha_2 \{4+1+1\} + \alpha_3 \{8+0+1\} = -1$$

$$\therefore \alpha_1 \{8+0+1\} + \alpha_2 \{8+0+1\} + \alpha_3 \{\cancel{8}^{16}+0+1\} = +1$$

$$\therefore \quad 6\alpha_1 + 4\alpha_2 + 9\alpha_3 = -1$$

$$4\alpha_1 + 6\alpha_2 + 9\alpha_3 = -1$$

$$9\alpha_1 + 9\alpha_2 + 17\alpha_3 = +1$$

Simplifying above 3 simettaneous equation, we get, $\alpha_1 = -3.25$, $\alpha_2 = -3.25$, $\alpha_3 = 3.5$

The hypeoplane that discriminate the positive class from the negative class is given by

$$\tilde{\omega} = \sum_i \alpha_i s_i$$

Here, $i = 3$ (as there are 3 support vectors)

$$\tilde{\omega} = \alpha_1 \tilde{s}_1 + \alpha_2 \tilde{s}_2 + \alpha_3 \tilde{s}_3$$

$$= -3.25 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + (-3.25) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 3.5 \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

$$\tilde{w} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

Above vector is augumented with bias.

$\therefore \quad w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and offset (bias) $= -3$

As, w is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, hyperplane line is parallel to y-axis with bias $= -3$.