Linear Equations

A linear equation is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $x_1, x_2, \cdots, x_n$ are the variables (or unknowns) and $a_1, a_2, \cdots a_n$ are the coefficients, which are real numbers.

Example: $2x + 3y + 5z = -9$

Here x, y, z are variables and $2, 3, 5$ are coefficients.

System of linear equations:
It is a collection of one or more linear equations involving sem set of variables.

Example:

$$3x + 2y + z = 6$$
$$x - \frac{1}{2}y + \frac{2}{3}z = 7/6.$$
$$4x + 6y - 10z = 0$$

is a system of 3 equations in the 3 variables x, y, z. A solution to linear system is an assignment of the values to the variables such that all the equations are simeltaneously satisfied.

A solution to the system above is given by,

$$x = 1,\ y = 1,\ z = 1$$

General form of system of linear equations:

A general system of m linear equations with n unknowns can be written as,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Matrix Representation

A system of linear equations is equivalent to a matrix equation of the form

$$Ax = b$$

Here, A is an $m \times n$ matrix, x is a column vector with n entries and b is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}_{m \times n} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m \times 1}$$

The example mentioned can be expressed as,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1/2 & 2/3 \\ 4 & 6 & -10 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 7/6 \\ 0 \end{bmatrix}$$

Solution to a linear system

There are 3 possibilities for the solutions to a linear system of equations:

- The system has infinitely many solutions.
- The system has a single unique solution
- The system has no solution.

Consider the following example:

- Sheriff has ~~180~~ 180 km/h car
- Bank robber has 150 km/h car and 5 minute head start
- How long does it take to the sherrif to catch the robber?
- What distance will they have travelled at that point?
  (for simplicity lets ignore acceleration, traffic etc.)

→ 150 km/h = 2.5 km/min
180 km/h = 03 km/min

equation 1: $d = 2.5\,t$
equation 2: $d = 3\,(t-5)$

$\therefore \quad 2.5\,t = 3(t-5)$

$-0.5t = -15$

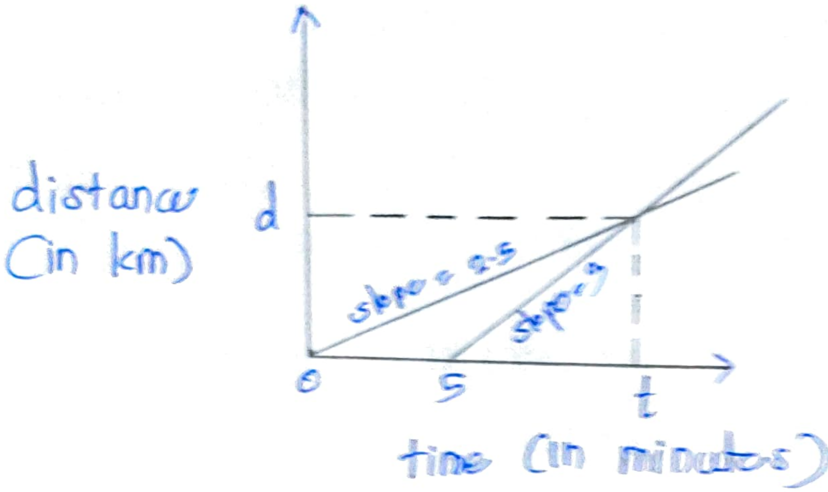
$t = -15/-0.5 = 30$ min

Put this value of t in equation d.

$d = 2.5\,t = 2.5 \times 30 = 75$ km

$d = 3\,(t-5) = 3(30-5) = 75$ km

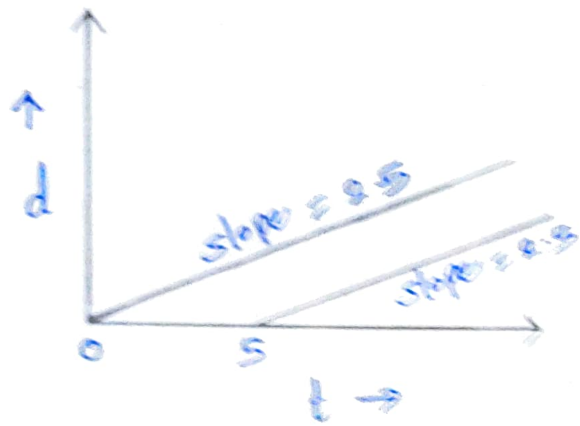
The above solution can be represented graphically as,



$d = 2.5t$

$d = 3(t-5)$

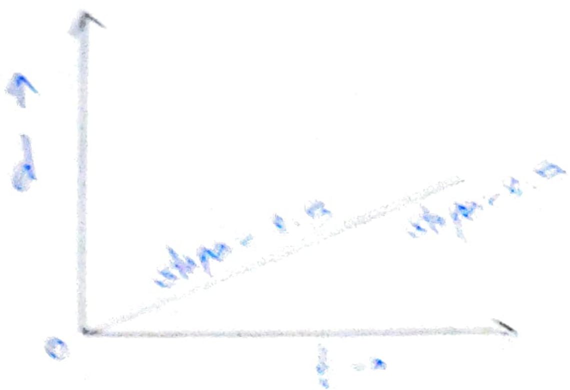
There will be a No-solution if sheriff's car is same speed as bank robber's.



$d = 2.5t$

$d = 2.5(t-5)$

There will be an infinite solutions if same speed and same starting point

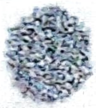


$d = 2.5t$

$d = 2.5t$

System of linear equations

Example 1:

Items	Buyer A	Buyer B	Buyer C
 Rice in Kg	8	12	3
 Dal in Kg	8	5	2
 Oil in Liter	4	7	5

Suppose A paid $Rs.1960$, B paid $Rs.2215$ and C paid $Rs.1135$. We want to find the price of each item using this data. Suppose price of Rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg., price of oil is $Rs.z$ per liter. Hence we have the following system of linear equations:

$$8x + 8y + 4z = 1960$$
$$12x + 5y + 7z = 2215$$
$$3x + 2y + 5z = 1135$$

After solving these equations, we get:

x=45 y= 125 z=150

Hence price of Rice is 45 Rs per kg, price of Dal is 125 Rs per kg and price of Oil is 150 Rs per liter.

Example of Infinitely many solutions:

Suppose A paid $Rs.215$, B paid $Rs.430$. We want to find the price of each items using this data. Suppose price of Rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg. Hence we have the following system of linear equations:

$$2x + y = 215$$
$$4x + 2y = 430$$

There are infinitely many x and y satisfying both the equations.

Example of a system of equations with no solutions:

Suppose A and B bought the same amount of items as in the previous example. But for some reason the seller gave a discount to B. Suppose A paid $Rs.215$ and B paid $Rs.400$. Now after returning home they decided to find out the price of each item by solving the linear system of equations as before. Suppose price of rice is $Rs.x$ per kg., price of dal is $Rs.y$ per kg. Hence we have the following system of linear equations:

$$2x + y = 215$$
$$4x + 2y = 400$$

Example 2:

Jill designs solar panels as a hobby.
On April 1st, Jill's "Mark I" design begins generating power: 1 kJ/day.
On May 1st, her "Mark II" design begins generating 4 kJ of power per day.

1. What day is it when Jill's Mark II design has generated as much total energy as the Mark I design?
2. How much total energy have both generated by that day?
3. What would the solutions to (1.) and (2). be if Mark II design generated 1kJ of power per day?

Solution:

Energy (Mark I) = $(T+30) * 1$

Energy (Mark II) = $T * 4$

$\therefore \quad (T+30) * 1 = T * 4$

$\therefore \quad T = 10$

1. After putting this value in mark I equation, we get, 40.
Hence after 40 days of 1st April, i.e. on 10th May, Jill's Mark II design will generate as much total energy as the Mark I design.

2. Each design will generate 40 kj energy. Hence total 80 kj energy will get generated by 10th May.

3. There will be no solution as,

$$T * 1 = (T * 30) * 1$$

$$\therefore \ T = 30T$$

(Not possible to get solution in above case.)

Contemporary applications:

- Solving for unknowns in ML algorithms including deep learning.
- Reduce dimensionality
- Ranking results (e.g. with eigenvector)
- Recommenders (e.g. singular value decomposition)
- Natural language processing
  - Topic modelling
  - Semantic analysis