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(ARTIFICIAL INTELLIGENCE & MACHINE LEARNING)**

Singular Value Decomposition

step 1: $U = A A^T$

step 2: Calculate eigenvalue and eigenvector of U .

step 3: Normalize and get U .

step 4: $V = A^T A$

step 5: Calculate eigenvalue and eigenvector of V .

step 6: Normalize and get V and V^T .

step 7: Get Σ by square root of eigenvalues of U .

Example: $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$$\begin{aligned} \rightarrow U &= A \cdot A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \end{aligned}$$

Calculate eigenvalue of above matrix.

$$\therefore |A A^T - \lambda I| = 0$$



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$$\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = 0$$

$$\therefore (17-\lambda)^2 - 64 = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$\therefore (\lambda - 25)(\lambda - 9) = 0$$

$$\therefore \lambda_1 = 25 \quad \lambda_2 = 9$$

Get the eigenvectors of above eigenvalues.
 $\lambda = 25$

$$\begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore 8x_1 - 8x_2 = 0$$

$$\therefore x_1 = x_2$$

$$\therefore \text{vector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



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$$\lambda = 9$$

$$\begin{bmatrix} 17-9 & 8 \\ 8 & 17-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$8x_1 + 8x_2 = 0$$

$$\therefore x_1 = -x_2$$

$$\therefore \text{Vector} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \text{eigenvector} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Normalize this vector by taking square root of every column.

$$\therefore \text{Normalized vector} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\because \sqrt{1^2 + 1^2} = \sqrt{2} \quad \&$$

$$\sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\therefore U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$



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Now calculate $A^T A$

$$\therefore A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$\therefore \lambda^3 - s_1 \lambda^2 + s_2 \lambda - |A| = 0$$

Here, $s_1 = \text{Trace of matrix}$

$$= 13 + 13 + 8$$
$$= 34$$

$s_2 = \text{addition of minor elements}$

$$= [(13 \times 8) - (4)] + [(13 \times 8) - 4] + [(13 \times 13) - (12 \times 12)]$$
$$= (104 - 4) + (104 - 4) + (169 - 144)$$
$$= 225$$

$$|A| = 0$$

$$\therefore \lambda^3 - 34\lambda^2 + 225\lambda = 0$$



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From above equation, calculate eigenvalues.

$$\lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0$$

Calculate eigenvectors for each eigenvalue.

\therefore For $\lambda = 25$

$$\begin{bmatrix} 13-25 & 12 & 2 \\ 12 & 13-25 & -2 \\ 2 & -2 & 8-25 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-12x + 12y + 2z = 0$$

$$12x - 12y - 2z = 0$$

$$2x - 2y - 17z = 0$$

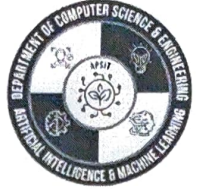
Using Cramer's Rule,

$$\frac{x}{\begin{vmatrix} -12 & -2 \\ -2 & -17 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & -2 \\ 2 & -17 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & -12 \\ 2 & -2 \end{vmatrix}}$$

$$\therefore \frac{x}{200} = \frac{-y}{-200} = \frac{z}{0}$$

$$\therefore x = 1, y = 1, z = 0$$

$$\therefore \text{vector is } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



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For $\lambda = 9$, calculate eigenvector.

$$\begin{bmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore 4x + 12y + 2z = 0$$

$$12x + 4y - 2z = 0$$

$$2x - 2y - z = 0$$

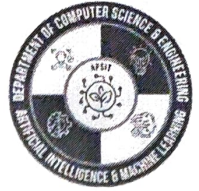
Using Cramer's rule,

$$\frac{x}{\begin{vmatrix} 4 & -2 \\ -2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & -2 \\ 2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & 4 \\ 2 & -2 \end{vmatrix}}$$

$$-\frac{x}{8} = \frac{y}{8} = -\frac{z}{32}$$

$$\therefore x = 1, y = -1, z = 1$$

$$\therefore \text{vector} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$



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For $\lambda = 0$, calculate eigenvalue.

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$13x + 12y + 2z = 0$$

$$12x + 13y - 2z = 0$$

$$2x - 2y + 8z = 0$$

Using, Cramer's Rule,

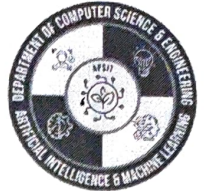
$$\frac{x}{\begin{vmatrix} 13 & -2 \\ -2 & 8 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 12 & -2 \\ 2 & 8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & 13 \\ 2 & -2 \end{vmatrix}}$$

$$\frac{x}{100} = \frac{-y}{100} = \frac{-z}{50}$$

$$x = 2, y = -2, z = -1$$

$$\therefore \text{vector} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$\therefore \text{Eigenvectors are} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$



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After Normalization, above vectors
will be

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}$$

Transpose above vector like below

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & -2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$

Next calculate Σ .

$$\therefore \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \underline{SVD = U \Sigma V^T}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & -2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$