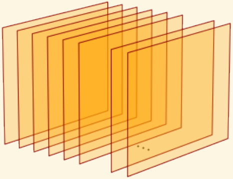
Reactive-n strategies of direct reciprocity

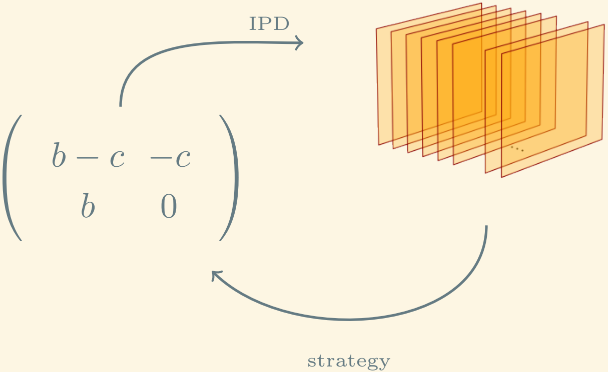
Theory Seminar

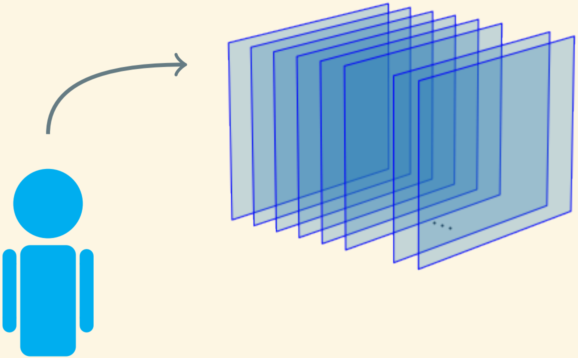
Nikoleta Glynatsi, Christian Hilbe

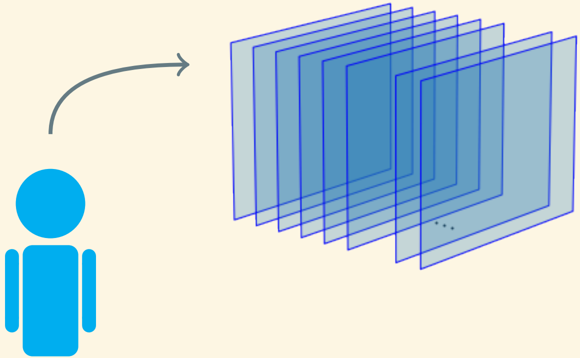
$$\begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}$$

$$\begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix} \xrightarrow{\text{IPD}}$$




IPD
$b-c$ $-c$
b 0
strategy





memory-n strategies

- Press, W.H. and Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent. Proceedings of the National Academy of Sciences

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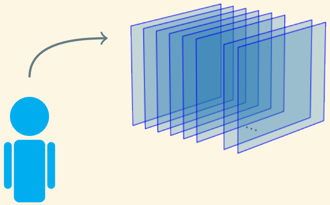
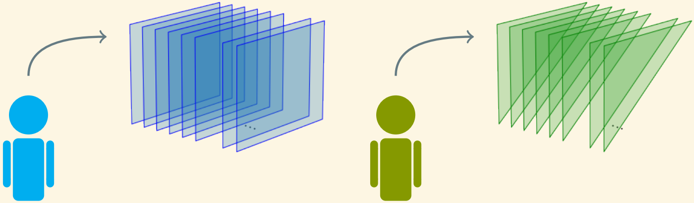
- Hilbe, C., Martinez-Vaquero, L.A., Chatterjee, K. and Nowak, M.A., 2017. Memory-n strategies of direct reciprocity. Proceedings of the National Academy of Sciences
- Stewart, A.J. and Plotkin, J.B., 2016. Small groups and long memories promote cooperation. Scientific reports





Akin, E., 2016. The iterated prisoner's dilemma: good strategies and their dynamics. Ergodic Theory, Advances in Dynamical Systems

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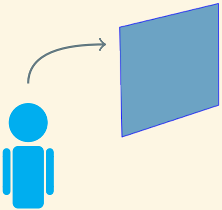
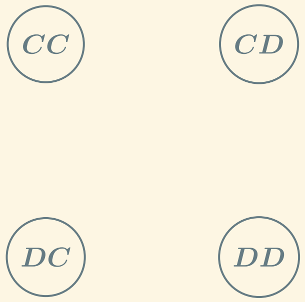
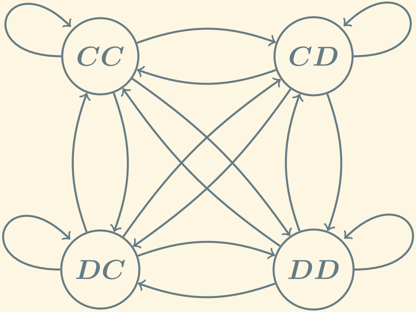
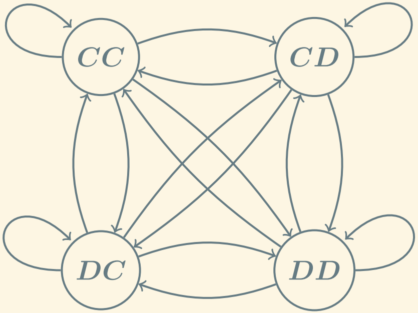
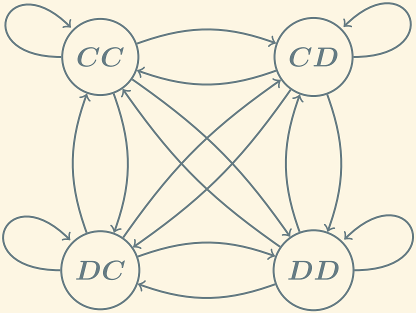


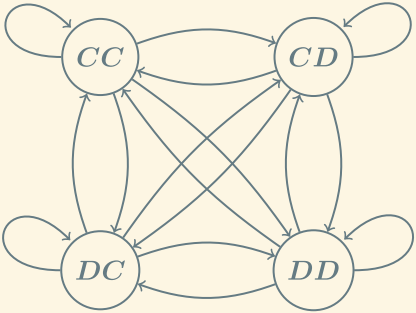

CC
CD
DC
DD


CC
CD
DC
DD


CC
CD
DC
DD

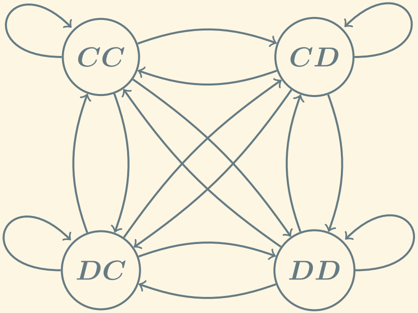


$$\mathbf{p} = (p_1, p_2, p_3, p_4) \ \& \ \mathbf{q} = (q_1, q_2, q_3, q_4)$$



$$\mathbf{p} = (p_1, p_2, p_3, p_4) \text{ \& } \mathbf{q} = (q_1, q_2, q_3, q_4)$$

$$\mathbf{v}M = \mathbf{v}$$



$$\mathbf{p} = (p_1, p_2, p_3, p_4) \ \& \ \mathbf{q} = (q_1, q_2, q_3, q_4)$$

$$\mathbf{v}M = \mathbf{v}$$

$$\mathbf{S}_p = (b - c, -c, b, 0) \ \& \ \pi(\mathbf{p}, \mathbf{q}) = \mathbf{v} \cdot \mathbf{S}_p$$

Akin, E., 2016. The iterated prisoner's dilemma: good strategies and their dynamics. Ergodic Theory, Advances in Dynamical Systems

Definition

A memory-one strategy is **agreeable** if it always cooperates following a mutual cooperation, thus $p_1 = 1$.

Definition

A strategy is called **good** if (i) it is agreeable, and (ii) if for any general strategy chosen by the co-player against it the expected payoffs satisfy:

$$s_{\mathbf{q}} \geq (b - c) \Rightarrow s_{\mathbf{q}} = s_{\mathbf{p}} = (b - c). \tag{1}$$

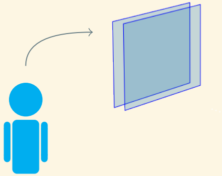
The strategy is of **Nash type** if (i) it is agreeable and (ii) if the expected payoffs against any general strategy satisfy:

$$s_{\mathbf{q}} \geq R \Rightarrow s_{\mathbf{q}} = (b - c). \tag{2}$$

Theorem

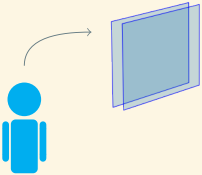
Akin's Theorem.

$$(1 - p_1) \cdot v_1 + (1 - p_2) \cdot v_2 = p_3 \cdot v_3 + p_4 \cdot v_4$$



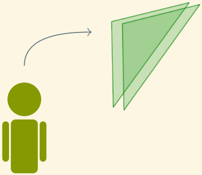
$p = (p_1, p_2, p_3, \ldots, p_{15}, p_{16})$

memory-two



$p = (p_1, p_2, p_3, \ldots, p_{15}, p_{16})$

memory-two



$p = (p_1, p_2, p_1, \ldots, p_3, p_4)$

$\tilde{p} = (p_1, p_2, p_3, p_4)$

two-bit reactive

Definition

A $n-$bit reactive strategy is agreeable if it cooperates with a probability one given that the co-player has consecutively cooperated in that last n rounds.

Lemma

Extension of Akin's Theorem

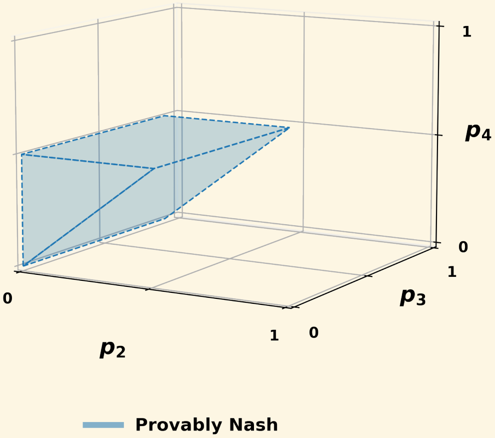
$$\begin{aligned}(v_1 + v_9)(1 - p_1) + (v_2 + v_{10})(1 - p_2) + (v_5 + v_{13})(1 - p_3) + (v_6 + v_{14})(1 - p_4)\\ = (v_3 + v_{11})p_1 + (v_4 + v_{12})p_2 + (v_7 + v_{15})p_3 + (v_8 + v_{16})p_4.\end{aligned}$$

Theorem

Let the two-bit reactive strategy $\hat{\mathbf{p}} = (p_1, p_2, p_3, p_4)$ be an ***agreeable strategy****; that is $p_1 = 1$. Strategy $\hat{\mathbf{p}}$ is* ***Nash*** *if the following inequalities hold:*

$$p_4 \leq 1 - \frac{c}{b} \qquad p_2 \leq p_4 \qquad p_3 \leq 1 \qquad (1 + p_2) \leq \frac{b}{c} - \frac{p_4(b-c)}{c}$$

The agreeable strategy $\hat{\mathbf{p}}$ is good if and only if all inequalities above are strict.



0
1
p_2
0
1
p_3
0
1
p_4
Provably Nash

Theorem

Let the memory-two strategy $\mathbf{p} = (p_1, p_2, \ldots, p_{16})$ *be an* ***agreeable strategy****; that is* $p_1 = 1$*. Strategy* $\mathbf{p}$ *is* ***Nash*** *if the following inequalities hold:*

$$p_2, p_{10}, p_{14} \leq p_6 \qquad p_5, p_9, p_{13} \leq 1 \qquad \frac{c}{b} p_{11} \leq 1 - p_6 \qquad \frac{c}{b} p_{15} \leq 1 - p_6$$

$$\frac{c}{b} p_3 \leq 1 - p_6 \qquad p_6 \leq 1 + \frac{c}{b-c} p_{12} \qquad p_6 \leq 1 + \frac{c}{b-c} p_{16}$$

$$\frac{c}{b} p_7 \leq 1 - p_6 \qquad \frac{c}{b-c} p_8 < 1 - p_6$$

The agreeable strategy $\mathbf{p}$ *is good if and only if all inequalities above are strict.*

Theorem

Let the memory-two strategy $\mathbf{p} = (p_1, p_2, \ldots, p_{16})$ *be an* ***agreeable strategy****; that is* $p_1 = 1$*. Strategy* $\mathbf{p}$ *is* ***Nash*** *if the following inequalities hold:*

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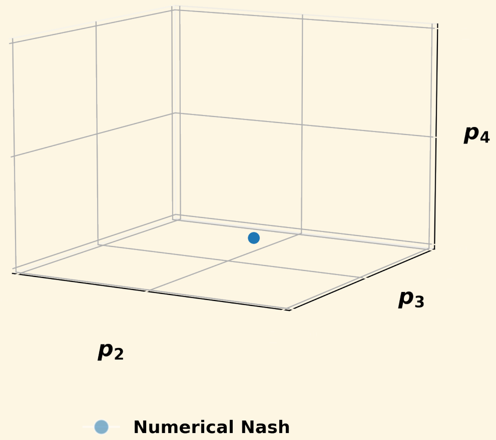
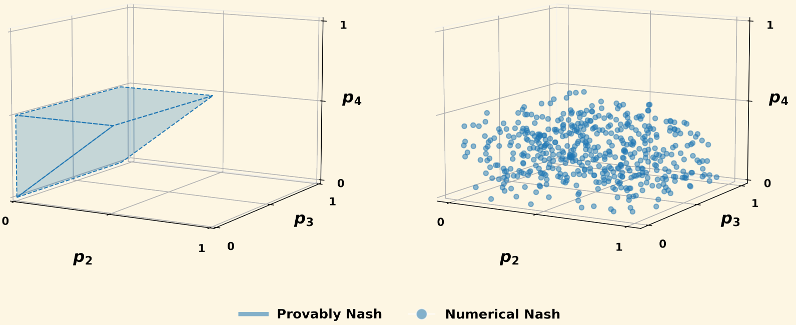
$$\frac{c}{b} p_3 \leq 1 - p_6 \qquad p_6 \leq 1 + \frac{c}{b-c} p_{12} \qquad p_6 \leq 1 + \frac{c}{b-c} p_{16}$$

$$\frac{c}{b} p_7 \leq 1 - p_6 \qquad \frac{c}{b-c} p_8 < 1 - p_6$$

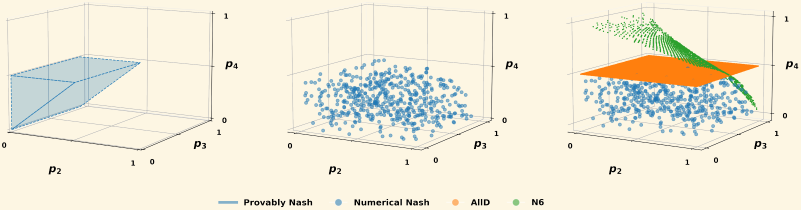
The agreeable strategy $\mathbf{p}$ *is good if and only if all inequalities above are strict.*

Numerical Evaluation of Nash

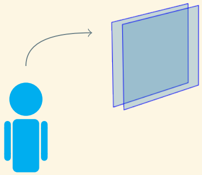

p_2
p_3
p_4
Numerical Nash



0
1
p_2
0
1
p_3
0
1
p_4
0
1
p_2
0
1
p_3
0
1
p_4
Provably Nash
Numerical Nash



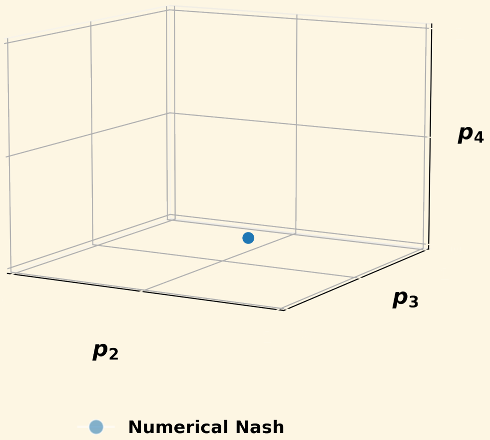
0
1
0
1
0
1
p_2
p_3
p_4
Provably Nash
Numerical Nash
AllD
N6

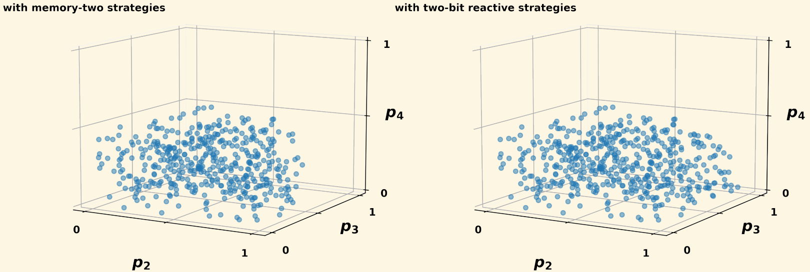


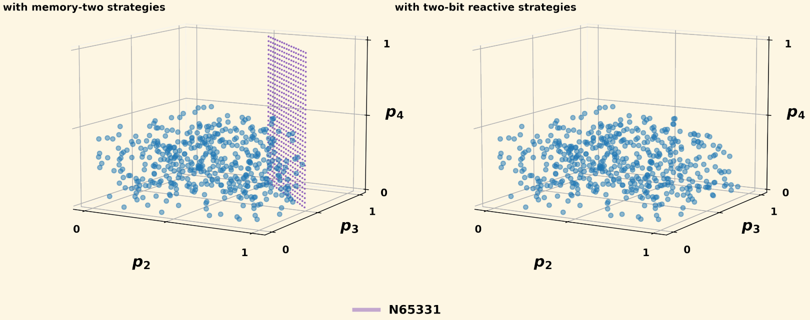
$$p = (p_1, p_2, p_3, \ldots, p_{15}, p_{16})$$

memory-two

1. Anti Press and Dyson


p_2
p_3
p_4
Numerical Nash


with memory-two strategies
with two-bit reactive strategies
p_4
p_3
p_2
0
1


with memory-two strategies
with two-bit reactive strategies
p_2
p_3
p_4
0
1
N65331

Let X play a short memory strategy, and Y play with longer memory of the past outcomes. In the perspective of the forgetful strategy X, X's score is exactly the same as if Y had played a certain shorter-memory strategy.

Let X play a memory-one strategy, and Y play a longer memory-n strategy. In the perspective of X, there is a memory-one representation of Y's strategy such that X's score is exactly the same.

Let X play a one-bit reactive strategy, and Y play a longer memory-n strategy. In the perspective of X, there is a memory-one representation of Y's strategy such that X's score is exactly the same.

2. Pure Nash

Theorem

Let $\mathbf{p} = (p_1, p_2, p_3, p_4)$ be an agreeable memory-one strategy. That is, $p_1 = 1$. The strategy $\mathbf{p}$ is of Nash type iff the following inequalities hold:

$$\frac{cp_3}{b} \leq (1 - p_2) \qquad \frac{cp_4}{(b - c)} \leq (1 - p_2)$$

The strategy $\mathbf{p}$ is good iff, in addition, both inequalities are strict.

Table 1. Stable and cooperative memory-2 strategies for the prisoner's dilemma

Strategy description	Previous two rounds: player 1, player 2																Cooperation rate against itself	Minimum b/c ratio
	CC, CC	CC, CD	CD, CC	CD, CD	CC, DC	CC, DD	CD, DC	CD, DD	DC, CC	DC, CD	DD, CC	DD, CD	DC, DC	DC, DD	DD, DC	DD, DD		
Strategies	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0.952	1.526
equivalent to	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0.951	1.526
AON_2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	0.951	1.526
	1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	1	0.952	1.526
Strategies	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0.971	2.041
equivalent to	1	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	0.971	2.041
AON_1	1	0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	0.970	2.041
	1	0	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0.971	2.041
Delayed	1	0	0	1	0	0	0	1	0	0	0	1	1	0	0	1	0.952	2.083
versions of	1	0	0	1	1	0	0	0	1	0	0	0	1	0	0	1	0.970	2.021
AON_1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0.971	2.041

All strategies prescribe to cooperate if (*i*) both players cooperated in both rounds, (*ii*) both players defected in both rounds, and (*iii*) both players cooperated in the last round, but defected in the second-to-last round. Moreover, all strategies defect if the actions of the two players were different from each other in both previous rounds. The last column gives the threshold that the b/c ratio needs to exceed for the respective strategy to be a strict Nash equilibrium among the finite set of pure memory-2 strategies. The table suggests the existence of a tradeoff: Strategies equivalent to AON_2 require a lower b/c ratio to be stable, but they also have a slightly lower cooperation rate against themselves.

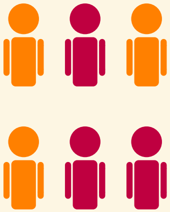
Hilbe, C., Martinez-Vaquero, L.A., Chatterjee, K. and Nowak, M.A., 2017. Memory-n strategies of direct reciprocity. Proceedings of the National Academy of Sciences

	Strategy	ρ (self coop. rate)	Min. $\frac{b}{c}$ ratio	Max. $\frac{b}{c}$ ratio
One-bit reactive	$p_1 = 0, p_2 = 0$	0	0	0
Two-bit reactive	$p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0$	0.0	None	None
	$p_1 = 0, p_2 = 1, p_3 = 0, p_4 = 0$	0.255	1.04	None
	$p_1 = 0, p_2 = 0, p_3 = 1, p_4 = 0$	0.255	1.04	None
Three-bit reactive	$p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 0, p_7 = 0, p_8 = 0$	0.0	None	None
	$p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, p_5 = 0, p_6 = 1, p_7 = 0, p_8 = 0$	0.182	1.0590	1.0592
	$p_1 = 0, p_2 = 0, p_3 = 1, p_4 = 0, p_5 = 0, p_6 = 0, p_7 = 1, p_8 = 0$	0.255	1.041	1.042

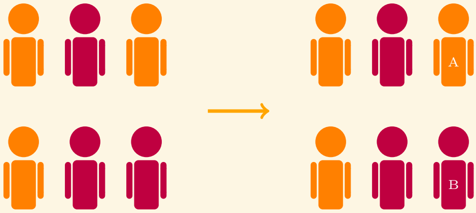
Pure one, two and three bit(s) reactive strategies ($\epsilon = 0.01$).

@NikoletaGlyn
@chilbe3

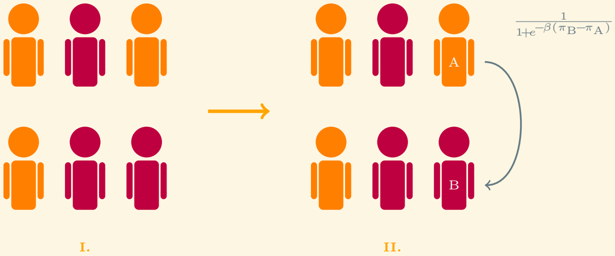
Nikoleta - v3
http://web.evolbio.mpg.de/social-behaviour/

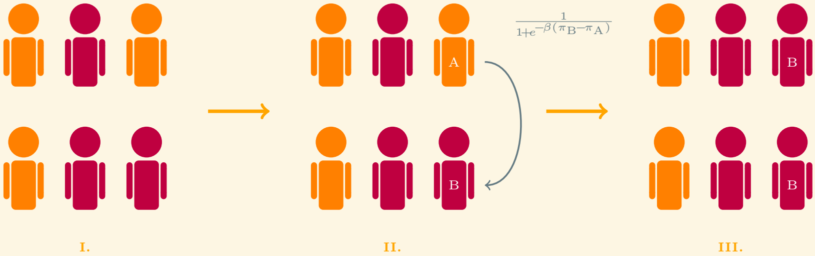


I.

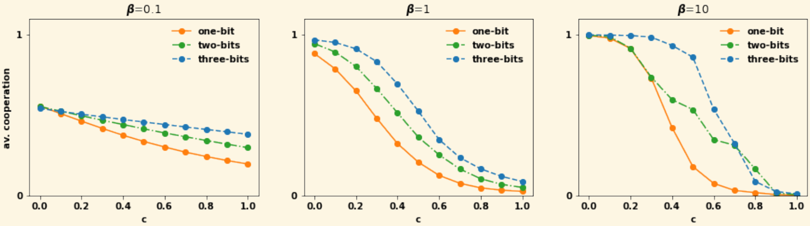


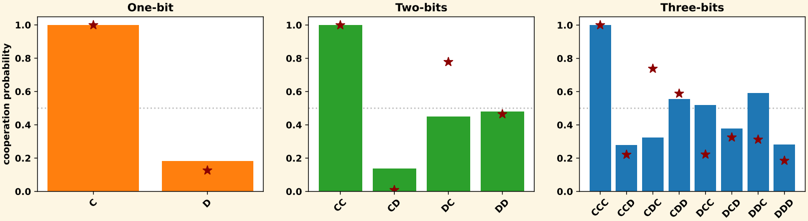
I. II.


$\frac{1}{1+e^{-\beta(\pi_B-\pi_A)}}$
A
B
I.
II.



$\frac{1}{1+e^{-\beta(\pi_B-\pi_A)}}$
A
B
B
B
B
I.
II.
III.


β=0.1
β=1
β=10
one-bit
two-bits
three-bits
av. cooperation
c
0
1
0.0
0.2
0.4
0.6
0.8
1.0



One-bit
Two-bits
Three-bits
cooperation probability
1.0
0.8
0.6
0.4
0.2
0.0
C
D
CC
CD
DC
DD
CCC
CCD
CDC
CDD
DCC
DCD
DDC
DDD