Introduction to Statistical Physics for Evolutionary Game Theorists

Agenda

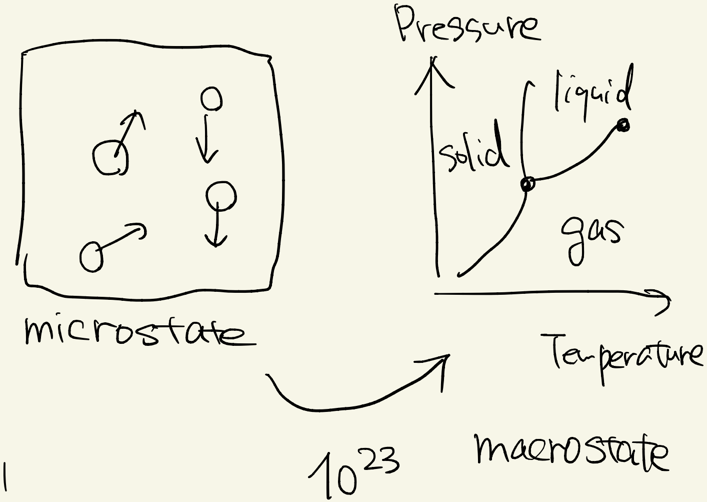
1. Intro to Stat. Phys.

2. Markov Chain Monte Carlo

3. Analogy between Physics and Evolutionary Game Theory

What is Statistical Physics

- a subfield of Physics
- NOT "statistics"
- explain "macroscopic phenomena"
  based on "microscopic interactions"



What can we explain by Stat. Phys.?

- property of material
  - solid (ice) $\overset{0°C}{\Longleftrightarrow}$ liquid (water) $\overset{100°C}{\Longleftrightarrow}$ gas (vapor)
  - magnets

    https://www.youtube.com/watch?v=haVX24hOwQI

    https://www.complexity-explorables.org/explorables/i-sing-well-tempered/
  - elasticity (metal v.s. rubber)
  - electronics (semiconductor, superconductivity)
  - protein folding (drug design)

not limited to "physical" system

- social systems
  - traffic
  - pandemic
  - social networks
  - economy

Goal : Calculate macrostate at temperature T

Workflow

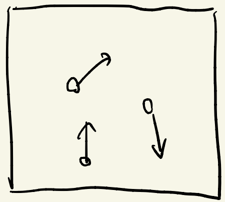
given $\mathcal{H}$ and T

$\Downarrow$

probability distribution of microstates

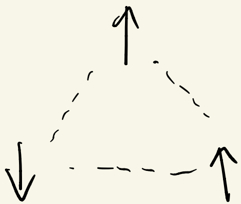
$\Downarrow$

macroscopic property at T



$\mathcal{H}$: microstate $\to$ energy

Ising model



$$H = -J \sum \sigma_i \sigma_j \qquad \sigma_i \in \{-1, 1\}$$

2^3 configurations

$$\begin{array}{cc} \uparrow\ \uparrow\ \uparrow & -3J \end{array}$$

$$\left.\begin{array}{cc} \uparrow\ \uparrow\ \downarrow & J \\ \uparrow\ \downarrow\ \uparrow & J \\ \vdots & \vdots \end{array}\right\} \times 6$$

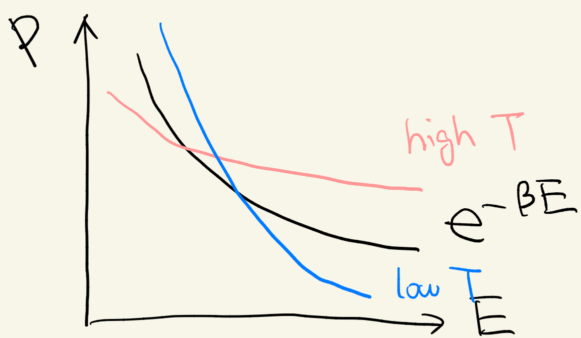
$$\begin{array}{cc} \downarrow\ \downarrow\ \downarrow & -3J \end{array}$$

at temperature T

each configuration realizes with

$$P_i \propto e^{-\beta E_i}$$

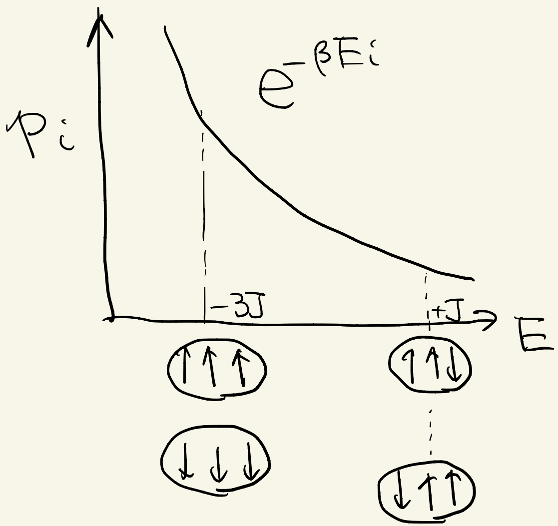
$$\beta \equiv \frac{1}{k_B T}$$ inverse temperature



$T \to 0$: Only lowest energy states realize

$T \to \infty$: Any configuration occur randomly.

for the Ising model

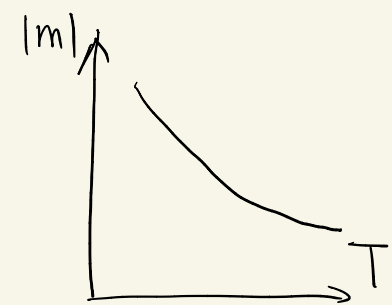


$$P(\uparrow\uparrow\uparrow) = \frac{e^{3\beta J}}{\underbrace{2e^{3\beta J} + 6e^{-\beta J}}_{= Z}}$$

$$= \frac{e^{3\beta J}}{Z}$$

$$P(\uparrow\uparrow\downarrow) = \frac{e^{-\beta J}}{Z}$$

$$\langle |m| \rangle = \langle |\#(\uparrow) - \#(\downarrow)| \rangle$$
$$= 3 \times \frac{e^{3\beta J}}{Z} + 3 \times \frac{e^{3\beta J}}{Z} + \frac{e^{-\beta J}}{Z} + \cdots + \frac{e^{-\beta J}}{Z}$$



Summary so far

- microstates realizes with $e^{-\beta E}$

low T : strongly biased towards low energy states

high T : any states are equally likely

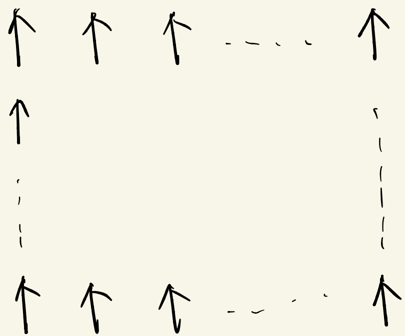
the number of microstates matters

- calculate the expectation of macroscopic variables

(e.g. magnetization, total energy, ...)

2. Markov Chain Monte Carlo

Principle is easy but its calculation is not



Ising model on 10×10 square lattice

# of possible microstates $= 2^{100}$

$$P(\sigma) = \frac{1}{Z} e^{-\beta E(\sigma)}$$

$$\langle |m| \rangle = \sum |m(\sigma)| P(\sigma)$$

impossible!

sum is taken all over possible configurations

Monte Calro sampling

draw configuration i with probability $\frac{e^{-\beta E_i}}{Z}$

$i \in \{1, 2, \cdots, 2^{100}\}$

$$\overline{m} = (m_{15} + m_{37} + \cdots + m_{256})/\,n$$

$$\to \langle m \rangle \qquad \text{as } n \to \infty$$

(static) Monte Carlo

But! This is also hard.

$$P_i = \frac{e^{-\beta E_i}}{Z}$$

this is not easy to calculate

We can draw samples even without Z

$\Rightarrow$ **Markov Chain Monte Carlo**

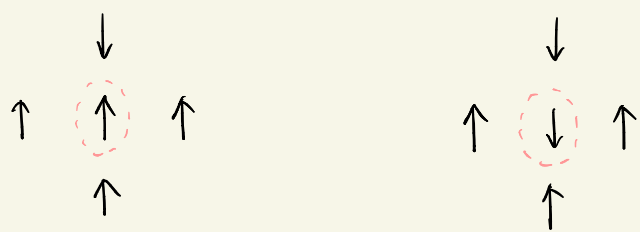
Point! We don't know P_i.

But, we know the relative frequency P_i / P_j

MCMC algorithm

1. Start from an arbitrary configuration

2. Choose a spin randomly. Make a "trial" state.

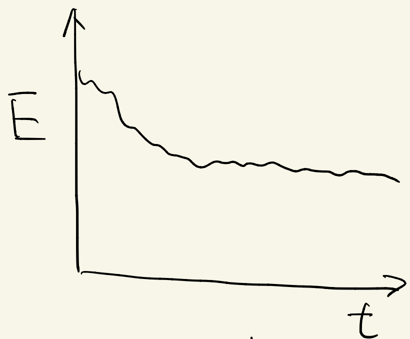


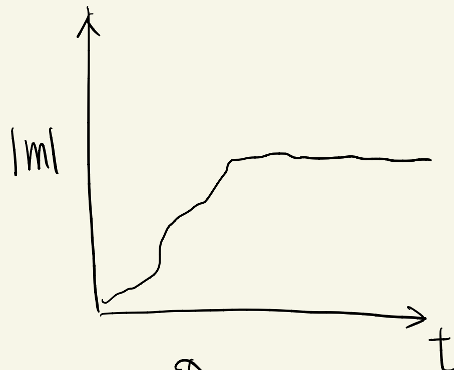
trial

3. Accept "trial" with $\dfrac{e^{-\beta E_{trial}}}{e^{-\beta E_{org}} + e^{-\beta E_{trial}}} = \dfrac{1}{1 + e^{\beta \Delta E}}$

Otherwise, keep the old state.

4. Repeat 2 & 3 sufficiently long.





long-time average converges to $\langle E \rangle$ and $\langle m \rangle$

Working principle of MCMC

Design Markov Process satisfying

1. irreducible
2. aperiodic
3. detailed-balance condition

$\Rightarrow$ Markov process converges the expected prob. dist.

$$P(A)\,W(A \to B) = P(B)\,W(B \to A)$$

$$A \rightleftarrows B$$

several choices for transition prob.

$$P(A)\, W(A \to B) = P(B)\, W(B \to A)$$

$e^{-\beta E_A}$ $\qquad$ $e^{-\beta E_B}$

- heat-bath method $\left(\dfrac{e^{-\beta E_B}}{e^{-\beta E_A} + e^{-\beta E_B}},\ \dfrac{e^{-\beta E_A}}{e^{-\beta E_A} + e^{-\beta E_B}} \right)$

- Metropolis method $\left(1,\ e^{-\beta (E_A - E_B)} \right)$ (if $E_A > E_B$)

As long as the detailed balance condition is satisfied, the long-term consequences are same.

① calculate all possible candidates, and choose one

$$\frac{e^{-\beta E_i}}{e^{-\beta E_1} + e^{-\beta E_2} + \cdots + e^{-\beta E_n}}$$

only for discrete systems

(heat-bath method, logit model)

② randomly choose "trial". Accept with $p = \frac{1}{1+e^{\beta \Delta E}}$

(introspection)

③ randomly choose "trial". Accept with $p = \min\{1, e^{-\beta \Delta E}\}$

(Metropolis)

Another application of MCMC : Bayes inference

$$p(\theta \mid x) = \frac{p(x \mid \theta)\, p(\theta)}{p(x)}$$

$$p(x) = \int p(x \mid \theta)\, p(\theta_1)\, d\theta_1\, p(\theta_2)\, d\theta_2 \cdots$$

not easy to calculate

$$\frac{p(\theta_1 \mid x)}{p(\theta_2 \mid x)} = \frac{p(x \mid \theta_1)\, p(\theta_1)}{p(x \mid \theta_2)\, p(\theta_2)}$$

easy to calculate

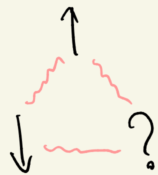
We update θ by MCMC

1. θ_{trial}
2. $W(\theta \to \theta_{trial})$
3. if random $< W$, $\theta \leftarrow \theta_{trial}$

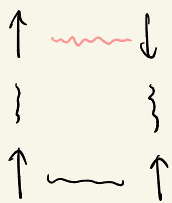
In the long run, distribution of θ converges to the posterior distribution.

3. Analogy between Physics and Evolutionary Game Theory

Physical system also has "dilemma"



anti-ferromagnetic interaction on triangular lattice



ferro & anti-ferro interactions are mixed (spin glass)

"frustration"

Analogy

phys	game
energy	$-$payoff
local minimum	Nash eq.
frustration	dilemma
inverse temperature	selection strength

Difference

phys	game
the whole system has a single E	each player has its own payoff function
⇓	⇓
the global minimum is a local minimum	Maximum of payoff sum is not always Nash eq.

(exception "potential game")

Summary

physics	game
◦ sample microstates with $e^{-\beta E}$ using MCMC	◦ define players' strategy updating dynamics
$\Downarrow$	$\Downarrow$
stationary distribution (macroscopic property)	stationary distribution (fraction of cooperators)
◦ single objective function, E	◦ multiple objective function