GAME THEORY #8

Reminder

(*) Game theory is about strategic decision making

Elements: (-) Players
(-) Actions
(-) Information
(-) Order of play
(-) Payoffs

which action to pick

Strategies are rules that tell players what to do given the information they have

2 particular classes of games

(1) Static games with complete information

Only need to define players, actions, payoffs $\Gamma = (\mathcal{N}, \mathcal{A}, \pi)$

Solution concept: Nash equilibrium

(2) Dynamic games with complete information

Players make decisions at different stages

Order of moves becomes important

Solution concept: Subgame perfect equilibrium

§4 GAMES WITH INCOMPLETE INFORMATION

Remark 4.1 (Motivation)

So far we always assumed players have all relevant information (they know each others' payoffs, actions).

However, in many applications players lack crucial information

mainly

(1) If players have incomplete information about their co-players' payoffs

(-) Firm knows its own costs but does not precisely know competitor's cost

(-) Bargaining: How much is the item worth to the buyer? Seller does not know.

(2) Hiring processes: Applicant knows her true quality, the employer requires signals to find out the quality

§4.1 Static games with incomplete information (Static Bayesian games)

Example 4.2 (Volunteer's dilemma)

(1) Setup: Two players, each one needs to decide whether to cooperate by paying a cost $c^{(i)}$ for both players to get some benefit $b=1$.

	C	D
C	$1-c^{(1)}, 1-c^{(2)}$	$\boxed{1-c^{(1)}}, \boxed{1}$

$\rightarrow$ D $\boxed{1}, \boxed{1-c^{(2)}}$ $0, 0$

(2) Complete information : $c^{(1)}, c^{(2)}$ commonly known. $c^{(1)}, c^{(2)} < 1$

2 pure Nash equilibria (C,D), (D,C)

1 Mixed Nash equilibrium $\sigma^{(1)} = (x, 1-x)$

$\sigma^{(2)} = (y, 1-y)$

$$\pi^{(1)}(C, \sigma^{(2)}) = 1 - c^{(1)}$$

$$\parallel$$

$$\pi^{(1)}(D, \sigma^{(2)}) = 1 \cdot y + 0 \cdot (1-y)$$

$$y = 1 - c^{(1)}$$

$$x = 1 - c^{(2)}$$

$$\sigma^{(1)} = (1 - c^{(2)}, c^{(2)}) \qquad \sigma^{(2)} = (1 - c^{(1)}, c^{(1)})$$

2 weird things about the mixed equilibrium

(*) My play is independent of my cost

(*) Why randomize at all?

(3) With incomplete information

(*) Now suppose $c^{(1)}, c^{(2)}$ are random variables uniformly drawn from $[0,2]$ (independently)

Player know their own cost precisely.

Co-player's cost, only the distribution is known.

(*) Strategies: $s^{(i)}: [0,2] \to \{C, D\}$
$c^{(i)} \mapsto s^{(i)}(c^{(i)})$

(*) Ansatz: $\boxed{s^{(i)}(c^{(i)}) = \begin{cases} C & \text{if } c^{(i)} \leq \bar{c}^{(i)} \\ D & c^{(i)} > \bar{c}^{(i)} \end{cases}}$ $\quad \bar{c}^{(i)} \leq 1$

How should I choose $\bar{c}^{(1)}, \bar{c}^{(2)}$
(when is it better to cooperate?)

	C	D
C	$1-c^{(1)}, 1-c^{(1)}$	$1-c^{(1)}, 1$
D	$1, 1-c^{(2)}$	$0, 0$

$$\mathbb{E}_{c^{(2)}} \Pi^{(1)}(C, s^{(2)}, c_1, c_2) = 1 - c^{(1)}$$

$$\mathbb{E}_{c^{(2)}} \Pi^{(1)}(D, s^{(2)}, c_1, c_2) = 1 \cdot \mathbb{P}(\underbrace{\text{Player 2 cooperates}}_{\underbrace{c^{(2)} \leq \bar{c}^{(2)}}_{\bar{c}^{(2)}/2}}) + 0 \cdot \mathbb{P}(2 \text{ defects}$$

$\bar{c}^{(2)}$
0 — 2 c

$$= \bar{c}^{(2)}/2$$

Upper Bound:
$\bar{c}^{(2)}$
$1 - c^{(1)} \geq \bar{c}^{(2)}$

$c^{(1)} \leq 1 - \bar{c}$

$\begin{cases} \bar{c}^{(1)} = 1 - \bar{c} \\ \bar{c}^{(2)} = 1 - \bar{c} \end{cases}$

Cooperation is optimal whenever $1 - c^{(1)} \geq \frac{\bar{c}^{(2)}}{2}$

$$c^{(1)} \leq 1 - \frac{\bar{c}^{(2)}}{2} = \bar{c}^{(1)}$$

$\bar{c}^{(1)} = 1 - (1 - \bar{c}^{(1)}$

$\bar{c}^{(1)} = \bar{c}^{(1)}$

$\bar{c}^{(1)} = 1 - \bar{c}^{(2)}$

$$\bar{c}^{(1)} = 1 - \frac{\bar{c}^{(2)}}{2}$$

$$\bar{c}^{(2)} = 1 - \frac{\bar{c}^{(1)}}{2}$$

$$\bar{c}^{(1)} = 1 - \frac{1 - \bar{c}^{(1)}/2}{2} = \frac{2 - 1 + \bar{c}/2}{2}$$

$$\Leftrightarrow \quad 2\bar{c}^{(1)} = 1 + \bar{c}^{(1)}/2$$

$$\Leftrightarrow \quad \frac{3}{2}\bar{c}^{(1)} = 1$$

$$\Leftrightarrow \quad \bar{c}^{(1)} = 2/3 = \bar{c}^{(2)}$$

Interesting observation:

(*) From an outside perspective, it looks as if players use mixed strategies (they randomize between C and D); However, actually players use pure strategies here. It's the costs that are stochastic, not the strategies.

(*) Strategies are more intuitive
What I do depends on my cost.

Remark 4.3 (General setup of Static Bayesian Games)

(*) Players can be of different types $\theta^{(i)} \in \Theta^{(i)}$
[In the previous example $\theta^{(i)} = c^{(i)}$, $\Theta^{(i)} = [0,2]$]

(*) Player's strategy can be conditioned on her type $s^{(i)}: \Theta^{(i)} \to A^{(i)}$
$\sigma^{(i)}: \Theta^{(i)} \to \Sigma^{(i)}$

(*) Probability to observe a specific type profile $\theta = (\theta^{(1)}, \ldots, \theta^{(n)})$

is given by some distribution $F(\theta^{(1)}, \dots, \theta^{(n)})$

For most examples, we will assume types are drawn independently.

If types are correlated, by knowing my type I learn something about your type

Update probabilities $\mathbb{P}(\theta^{(-i)} / \theta^{(i)})$

↳ Exercise

Definition 4.4 (Bayesian Nash equilibria)

A strategy profile $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$ is a BNE if for each player i and for each type $\theta^{(i)}$:

$$\mathbb{E}_{\theta^{(-i)}} \pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)}, \theta) \geq \mathbb{E}_{\theta^{(-i)}} \pi^{(i)}(\sigma^{(i)}, \hat{\sigma}^{(-i)}, \theta) \quad \forall \sigma^{(i)}$$

Here, expectations need to be taken with respect to the posterior probabilities $P(\theta^{(-i)} / \theta^{(i)})$

Examples 4.5 (Auction theory)

(1) Setup: Suppose one item is sold to the highest bidder. (n players)

Each player's valuation $v^{(i)}$ of the item is uniformly & independently drawn from $[0,1]$ ↳ $\theta^{(i)}$

Each player determines a bid $b^{(i)} \in \mathbb{R}^+$ → $A^{(i)}$

Strategy is a function $s^{(i)}: [0,1] \to \mathbb{R}$
$v^{(i)} \mapsto b^{(i)}$

What is an equilibrium?

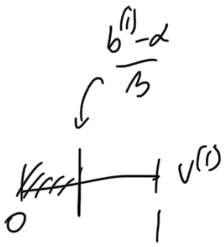
(2) "First-price sealed bid" auction : Highest bidder wins and pays her bid.

Payoffs $\quad \mathbb{E}\,\pi^{(i)} = (v^{(i)} - b^{(i)}) \cdot \mathbb{P}\left(b^{(i)} > \max_{j \neq i} b^{(j)}\right) \qquad (+0)$

Ansatz: (1) Let's assume strategies are symmetric

(2) Strategies are linear $b^{(i)} = \alpha + \beta v^{(i)}$

$$\mathbb{P}(b^{(i)} > b^{(j)}) = \mathbb{P}(b^{(i)} > \alpha + \beta v^{(j)})$$
$$= \mathbb{P}\left(v^{(j)} < \frac{b^{(i)} - \alpha}{\beta}\right)$$
$$= \frac{b^{(i)} - \alpha}{\beta}$$



$$\mathbb{P}\left(b^{(i)} > \max_{j \neq i} b^{(j)}\right) = \left(\frac{b^{(i)} - \alpha}{\beta}\right)^{n-1}$$

$$\mathbb{E}\,\pi^{(i)} = (v^{(i)} - b^{(i)}) \cdot \left(\frac{b^{(i)} - \alpha}{\beta}\right)^{n-1}$$

$$\frac{\partial\, \mathbb{E}\pi^{(i)}}{\partial b^{(i)}} = -\left(\frac{b^{(i)} - \alpha}{\beta}\right)^{n-1} + (v^{(i)} - b^{(i)}) \frac{n-1}{\beta} \left(\frac{b^{(i)} - \alpha}{\beta}\right)^{n-2}$$

$$\div \frac{(b^{(i)} - \alpha)^{n-2}}{\beta^{n-1}} \qquad -(b^{(i)} - \alpha) + (v^{(i)} - b^{(i)}) \cdot (n-1) = 0$$

$$-b^{(i)} + \alpha + (n-1)v^{(i)} - (n-1)b^{(i)} = 0$$

$$nb^{(i)} = (n-1)v^{(i)} + \alpha$$

$$b^{(i)} = \frac{n-1}{n}v^{(i)} + \frac{\alpha}{n} \; ?$$

$$b^{(i)} = \beta v^{(i)} + \alpha$$

$$b^{(i)} = \frac{n-1}{n}v^{(i)}$$

Bid is systematically <u>below</u> the valuation

(3) "second-price sealed bid" auction

Highest bid wins, but winner only has to pay second highest bid.

Seems counterintuitive from perspective of the seller

One major advantage:

<u>Claim</u>: Bidding $b^{(i)} = v^{(i)}$ is a weakly dominant strategy here.

"Vickrey truth serum" → Exercise

(4) Revenue equivalence theorem:

Both auction types give the same expected revenue to the seller.

↳ Exercise

(5) This theory is hugely important for the optimal design of auctions (e.g. auctions for electromagnetic spectra)

Nobel prizes: (*) Vickrey (1996)

(*) Wilson & Milgrom (2020)