

Week 6. Sequential games with complete information II: Subgame perfection and repeated games

Exercise 1: Battle of the sexes with an outside option

The battle of sexes is a game where two people would prefer to do something together, but each person likes a different activity best. The payoff matrix is,

$$\begin{array}{cc} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} (3, 1) & (0, 0) \\ (0, 0) & (1, 3) \end{pmatrix} \end{array}$$

Now suppose that before playing this game, player 1 can choose whether to play this game or to exit. If player 1 exits, both players obtain a payoff of 2.

Show that the battle of sexes with an outside option has two pure subgame perfect equilibria:

- (i) Player 1 plays, and both players choose activity a_1 .
- (ii) Player 1 exits, because if they were to play they would both choose activity a_2 .

Bonus question: Can you give a compelling argument why player 1 may be able to undermine the second equilibrium?

Exercise 2: Matching pennies with outside option

Consider the game rock, paper, scissors with the following matrix:

$$\begin{array}{ccc} & \begin{array}{ccc} \text{rock} & \text{paper} & \text{scissors} \end{array} \\ \begin{array}{c} \text{rock} \\ \text{paper} \\ \text{scissors} \end{array} & \begin{pmatrix} (0, 0) & (-1, 1) & (1, -1) \\ (1, -1) & (0, 0) & (-1, 1) \\ (-1, 1) & (1, -1) & (0, 0) \end{pmatrix} \end{array}$$

What is the subgame perfect equilibrium if the game is played twice?

Bonus 1: A finite repeated games

Consider the stag game with the payoff matrix,

$$\begin{array}{c|ccc} & C & D_1 & D_2 \\ \hline C & (3, 3) & (0, 4) & (-12, 0) \\ D_1 & (4, 0) & (1, 1) & (-10, 0) \\ D_2 & (0, -12) & (0, -10) & (-5, -5) \end{array}$$

Show that:

- (i) If the game is only played once, there is no Nash equilibrium in which C is played with a positive probability.
- (ii) If the game is played twice, there is a subgame perfect equilibrium in which C is played in the first round.

[Hint: Consider the strategy: Play C in the first round. If both players played C in the first round play D_1 in the second round, otherwise play D_2 .]

Bonus Exercise 2: Repeated prisoner's dilemma

Consider the infinitely repeated prisoner's dilemma with payoffs,

$$\begin{array}{c|cc} & C & D \\ \hline C & (3, 3) & (0, 4) \\ D & (4, 1) & (1, 1) \end{array}$$

Prove that there are sequences of actions such that players' average payoff,

$$\frac{1}{T+1} \sum_{t=0}^T u^{(i)}(a_t)$$

does not converge as $T \rightarrow \infty$.

[Hint: Consider the case that both players first play C for one round. Then they play D for 2 rounds. Then they play C for four rounds. Then they play D for 8 rounds.]