

## Week 4. Static games with complete information III: Nash equilibria

### Exercise 1: Nash equilibrium vs dominance solvability

**Prove the following statements:**

- (i) If a pure strategy  $s_j^{(i)}$  is dominated by a pure strategy  $s_k^{(i)}$  and  $\sigma = (\sigma^{(1)}, \dots, \sigma^{(n)})$  is a Nash equilibrium, then  $\sigma_j^{(i)} = 0$ .
- (ii) If the game is dominance solvable such that the unique outcome of iterated elimination of dominated strategies is some pure strategy  $s = (s^{(1)}, \dots, s^{(n)})$ , then  $s$  is a Nash equilibrium.

[Suggestion: One could use contradiction to prove the above statements. For example, for (i) assume that there was a Nash equilibrium with  $\sigma_j^{(i)} > 0$ , and show that this would yield some contradiction.]

### Exercise 2: Best responses

Consider the stag hunt game:

		<u>player 2</u>	
		Stag	Hare
<u>player 1</u>	Stag	$(10, 10)$	$(0, 6)$
	Hare	$(6, 0)$	$(6, 6)$

Suppose player 1 uses the mixed strategy  $(x, 1 - x)$ , where  $x$  is player 1's probability to Stag. Similarly, player 2's strategy is  $(y, 1 - y)$ .

- (i) For given  $x, y$  compute the players' payoffs  $\pi^{(1)}(x, y), \pi^{(2)}(x, y)$  (see Remarks 2.6, 2.7).
- (ii) For a given  $y$  compute player 1's best response  $\text{BR}(y)$ . In particular, show that there is some  $y^*$  such that all  $x \in [0, 1]$  are a best response.
- (iii) Draw the two best response correspondences  $\text{BR}(x), \text{BR}(y)$  into a  $x - y$  plane. How often do they intersect? What does it mean if they intersect?

### Exercise 3: Finding games with a non-generic number of equilibria

Find an example of a symmetric 2 player game, with 2 actions per player, with:

- Exactly 2 Nash equilibria
- infinitely many Nash equilibria

[Note: These should include all equilibria, not just pure equilibria.]

### Exercise 4: Cournot Duopoly

The Cournot duopoly game is defined by:

- Players:  $N = \{\text{Firm 1, Firm 2}\}$
- Actions: Amount of good produced,  $x^{(i)} \in [0, \infty)$  for  $i \in \{1, 2\}$
- Payoffs:  $\pi^{(i)}(x^{(1)}, x^{(2)}) = [a - b(x^{(1)} + x^{(2)})]x^{(i)} - cx^{(i)}$

Show that there is a Nash equilibrium in pure strategies. For simplicity assume  $a = 10, b = 1, c = 1$ .

[Hint: For each  $x^{(i)}$  compute  $\text{BR}(x^{(-i)})$ . Then solve simultaneously:

$$\begin{aligned}x^{(1)} &= \text{BR}(x^{(2)}) \\x^{(2)} &= \text{BR}(x^{(1)})\end{aligned}$$

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### Exercise 5: Matching Pennies

Compute the Nash equilibria for the following two games, and interpret the result.

	Left	Right		Left	Right
Top	$(0.8, 0.4)$	$(0.4, 0.8)$	Top	$(3.2, 0.4)$	$(0.4, 0.8)$
Bottom	$(0.4, 0.8)$	$(0.8, 0.4)$	Bottom	$(0.4, 0.8)$	$(0.8, 0.4)$

### Bonus Exercise: Verifying NE in games with finitely many players & actions

Show that to verify whether a strategy profile  $\hat{\sigma} = (\hat{\sigma}^{(1)}, \dots, \hat{\sigma}^{(n)})$  is a Nash equilibrium, it is sufficient to check all deviations towards pure strategies.

**Specifically show that  $\hat{\sigma}$  is a Nash equilibrium if and only if for all players  $i$  the following two conditions hold:**

- (i) All actions that player  $i$  uses give the same payoff: if  $\sigma_j^{(i)} > 0$  and  $\sigma_k^{(i)} > 0$  then  $\pi^{(i)}(s_j^{(i)}, \hat{\sigma}^{(-i)}) = \pi^{(i)}(s_k^{(i)}, \hat{\sigma}^{(-i)})$ .
- (ii) Actions that are not played are not profitable: if  $\sigma_j^{(i)} = 0$  then  $\pi^{(i)}(s_j^{(i)}, \hat{\sigma}^{(-i)}) \leq \pi^{(i)}(\hat{\sigma}^{(i)}, \hat{\sigma}^{(-i)})$ .

[Hint: One way to prove the above is once again by contradiction.]