Week 5. Sequential games with complete information I: Subgame perfection

Exercise 1: Sequential prisoner's dilemma with remorse

Revisit the prisoner's dilemma with remorse (Example 2.13 from the lecture). The payoff matrix is:

$$\begin{array}{cc} & \begin{array}{cc} \text{Silence} & \text{Confess} \end{array} \\ \begin{array}{c} \text{Silence} \\ \text{Confess} \end{array} & \begin{pmatrix} (3,3) & (0,0) \\ (4,0) & (1,1) \end{pmatrix} \end{array}$$

We already know that if both players move simultaneously, then the only Nash equilibrium is (Confess, Confess).

Now solve this game with backward induction, assuming that

1. the row player moves first,
2. the column player moves first,

and interpret the result.

Exercise 2: Ultimatum bargain in 2 rounds

There is a good that is worth 1€ to the buyer and 0€ to the seller. Sequence of moves:

1. Seller names a price $p \in \{0.01, 0.02, \ldots, 0.99\}$.
2. Buyer accepts or rejects the offer. If buyer accepts, the payoffs are p for the seller, $1 - p$ for the buyer and the game is over.
3. Otherwise, the buyer names a price $p \in \{0.01, 0.02, \ldots, 0.99\}$.
4. Seller accepts or rejects the offer. If the seller accepts, the payoffs are $p - \delta$ for the seller, and $1 - p - \delta$ for the buyer where $\delta = 0.045$ reflects the cost of having to go through a long negotiation. If the seller rejects, the payoff of both players is $-\delta$.

Solve by backward induction, and interpret the result.

Exercise 3: Stackelberg Duopoly

Similar to the Cournot duopoly game consider the following situation:

- Players: $N = \{\text{Firm } 1, \text{Firm } 2\}$
- Actions: Amount of good produced, $x^{(i)} \in [0, \infty)$ for $i \in \{1, 2\}$
- Payoffs: $\pi^{(i)}(x^{(1)}, x^{(2)}) = [a - b(x^{(1)} + x^{(2)})]x^{(i)} - cx^{(i)}$

However, now assume that Firm 1 decides first, and Firm 2 observes Firms 1's decision before choosing an action.

Solve the game by backward induction (for $a = 10, b = 1, c = 1$) and compare the result to the Nash equilibrium of the Cournot duopoly ($\hat{x}^{(1)} = \hat{x}^{(2)} = 3$).

[Hint: First you should find the best response of Firm 2. That is, for any given output $x^{(1)}$ of Firm 1, compute how Firm 2 would react if it wants to maximize its payoffs. Using this, compute how Firm 1 can maximize its own payoff when it takes into account that Firm 2 will react according to its best response.]

Bonus Exercise 1: Properties of backward induction

Prove that for any finite game with perfect information (i.e. in any game in which backward induction can be applied) the solution defined by backward induction is a Nash equilibrium.