Week 6. Sequential games with complete information II: Subgame perfection and repeated games

Exercise 1: Battle of the sexes with an outside option

The battle of sexes is a game where two people would prefer to do something together, but each person likes a different activity best. The payoff matrix is,

$$\begin{array}{cc} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_1 \end{array} & \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \end{array}$$

Now suppose that before playing this game, player 1 can choose whether to play this game or to exit. If player 1 exits, both players obtain a payoff of 2.

Show that the battle of sexes with an outside option has two pure subgame perfect equilibria:

(i) Player 1 plays, and both players choose activity a_1.

(ii) Player 1 exits, because if they were to play they would both choose activity a_2.

Bonus question: Can you give a compelling argument why player 1 may be able to undermine the second equilibrium?

Exercise 2: Matching pennies with outside option

Consider the matching pennies games with the following matrix (Example 3.12),

$$\begin{array}{cc} & \begin{array}{cc} \text{Left} & \text{Right} \end{array} \\ \begin{array}{c} \text{Up} \\ \text{Down} \end{array} & \begin{pmatrix} (0.8,0.4) & (0.4,0.8) \\ (0.4,0.8) & (0.8,0.4) \end{pmatrix} \end{array}$$

and assume that before playing this game player 1 can choose whether to play this game or to exit.

- What are the players' possible pure strategies?
- Write the games as a payoff matrix. Is there a pure Nash equilibrium? If yes why is the mixed equilibrium derived in Example 3.12 more compelling?

Exercise 3: A finite repeated games

Consider the stag game with the payoff matrix,

$$\begin{array}{c} \\ C \\ D_1 \\ D_2 \end{array} \begin{array}{c} \begin{array}{ccc} C & D_1 & D_2 \end{array} \\ \begin{pmatrix} (3,3) & (0,4) & (-12,0) \\ (4,0) & (1,1) & (-10,0) \\ (0,-12) & (0,-10) & (-5,-5) \end{pmatrix} \end{array}$$

Show that:

(i) If the game is only player once, there is <u>no</u> Nash equilibrium in which C is player with a positive probability.

(ii) If the game is played twice, there is a subgame perfect equilibrium in which C is player in the first round.

[Hint: Consider the strategy: Play C in the first round. If both players played C in the first round play D_1 in the second round, otherwise play D_2.]

Bonus Exercise 1

Consider the infinitely repeated prisoner's dilemma with payoffs,

$$\begin{array}{c} \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cc} C & D \end{array} \\ \begin{pmatrix} (3,3) & (0,4) \\ (4,1) & (1,1) \end{pmatrix} \end{array}$$

Prove that there are sequences of actions such that players' average payoff,

$$\frac{1}{T+1}\sum_{t=0}^{T} u^{(i)}(a_t)$$

does not converge as $T \to \infty$.

[Hint: Consider the case that both players first play C for one round. Then they play D for 2 rounds. Then they play C for four rounds. Then they play D for 8 rounds.]