第二章 递归状态估计

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1.

这道题目与2.4.2 的例题很相似，练习离散贝叶斯滤波的使用。根据题意可知：

$$
\begin{aligned}
bel(X_0 = broken) &= 0.01 \quad (1)\\
bel(X_0 = not\_broken) &= 0.99 \quad (2)\\
p(Z_t < 1|X_t = broken) &= 1 \quad (3)\\
p(Z_t < 1|X_t = not\_broken) &= 1/3 \quad (4)
\end{aligned}
$$

其中 Z_t 表示 t 时刻距离传感器的测量值，在 0 ~ $3m$ 范围内连续取值；X_t 表示距离传感器的状态，取值为离散的 $broken$，not_broken。

下面根据贝叶斯滤波算法进行计算，由于这道题中没有对距离传感器进行任何控制，因此贝叶斯滤波的第一步（预测）并不改变系统状态的置信度：

$$
\bar{bel}(X_t) = bel(X_{t-1}) \quad (5)
$$

贝叶斯滤波的第二步（观测），题干中介绍到$Z_t < 1$，于是有以下递推式

$$
\begin{aligned}
bel(X_t = broken) &= \eta p(Z_t < 1|X_t = broken)\bar{bel}(X_t = broken) \quad (6)\\
bel(X_t = not\_broken) &= \eta p(Z_t < 1|X_t = not\_broken)\bar{bel}(X_t = not\_broken) \quad (7)
\end{aligned}
$$

其中 η 为归一化因子

$$
\begin{aligned}
&\eta = [p(Z_t < 1|X_t = broken)\bar{bel}(X_t = broken) + \\
&p(Z_t < 1|X_t = not\_broken)\bar{bel}(X_t = not\_broken)]^{-1} \quad (8)
\end{aligned}
$$

对以上5、6、7式代入先验初值1、2进行迭代，即可得到各个时刻距离传感器故障的概率如下

时刻	$bel(X_t = broken \vert Z_t < 1)$	$bel(X_t = not\_broken \vert Z_t < 1)$	η
0	0.01	0.99	
1	0.029411764705882353	0.9705882352941176	2.941176470588235
2	0.08333333333333333	0.9166666666666666	2.833333333333333
3	0.2142857142857143	0.7857142857142857	2.5714285714285716
4	0.45000000000000007	0.55	2.1
5	0.7105263157894737	0.2894736842105263	1.5789473684210524
6	0.8804347826086958	0.11956521739130438	1.2391304347826089
7	0.9566929133858267	0.043307086614173235	1.0866141732283463
8	0.9851351351351351	0.014864864864864866	1.0297297297297296
9	0.9949954504094631	0.0050045495905368526	1.0100090991810737
10	0.9983262325015215	0.0016737674984783934	1.0033475349969567

最后推导公式:

$$\begin{aligned} bel(X_t = broken) &= \eta_t \times 1 \times bel(X_{t-1} = broken) \\ &= \eta_t \eta_{t-1} \times 1 \times bel(X_{t-2} = broken) \\ &= \textstyle\prod_{i=0}^{n} \eta_i bel(X_0 = broken) \end{aligned} \tag{9}$$

$$\begin{aligned} bel(X_t = not\_broken) &= \eta_t \times \tfrac{1}{3} \times bel(X_{t-1} = not\_broken) \\ &= \eta_t \eta_{t-1} \times \tfrac{1}{3} \times bel(X_{t-2} = broken) \\ &= \textstyle\prod_{i=0}^{n} \eta_i \times \tfrac{1}{3^n} \times bel(X_0 = not\_broken) \end{aligned} \tag{10}$$

其中 $\prod_{i=0}^{n} \eta_i$ 满足归一化要求，即

$$\prod_{i=0}^{n} \eta_i = [bel(X_0 = broken) + \frac{1}{3^n} \times bel(X_0 = not\_broken)]^{-1} \tag{11}$$

因此，传感器失效的概率为

$$bel(X_t = broken) = \frac{bel(X_0 = broken)}{bel(X_0 = broken) + \frac{1}{3^n} \times bel(X_0 = not\_broken)} \tag{12}$$

2.

3.

4.

5.

6.