Today:

- Solution via Green's function
- max-norm stability

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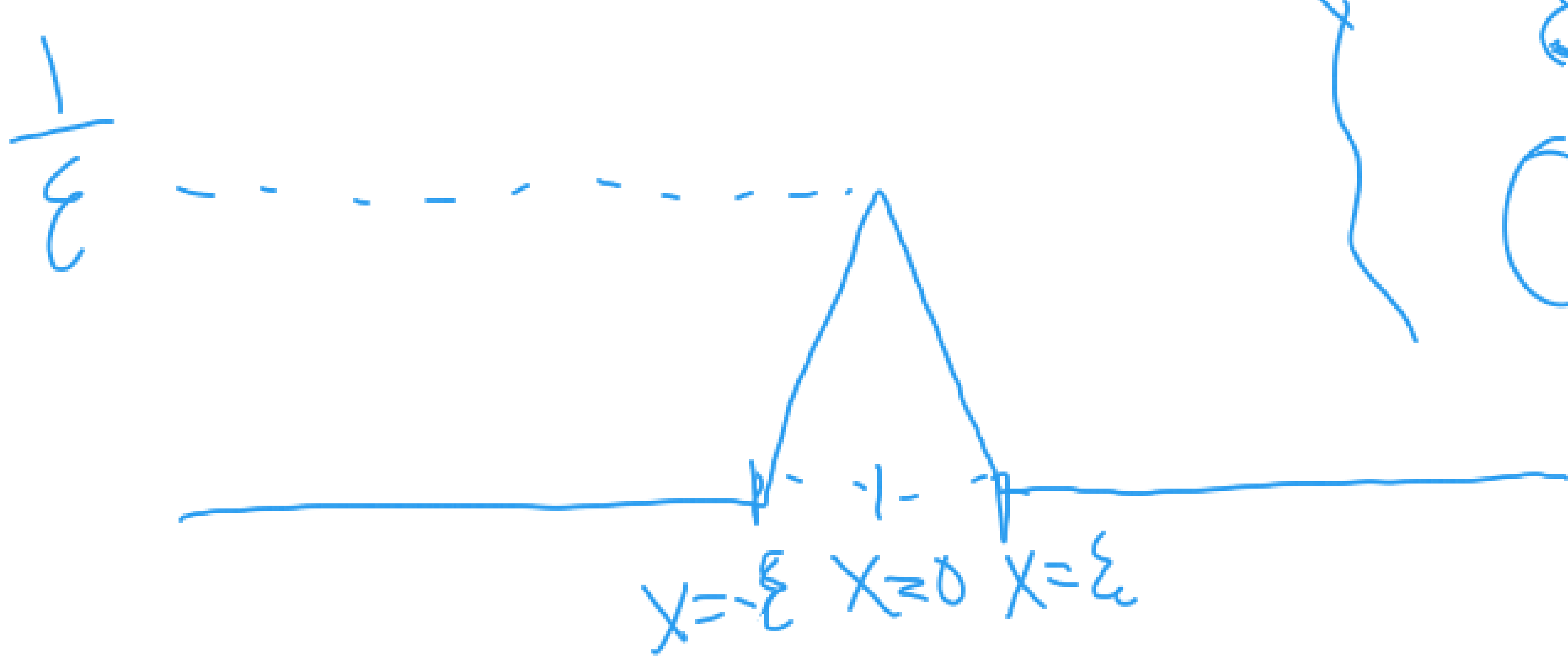
Dirac delta function

$$\phi_\varepsilon(x) = \begin{cases} \dfrac{\varepsilon + x}{\varepsilon^2} & -\varepsilon \le x \le 0 \\ \dfrac{\varepsilon - x}{\varepsilon^2} & 0 \le x \le \varepsilon \\ 0 & |x| > \varepsilon \end{cases}$$




$$\int_{-\infty}^{\infty} \phi_\varepsilon(x)\,dx = \int_{-\varepsilon}^{\varepsilon} \phi_\varepsilon(x)\,dx = 1$$

In the limit $\varepsilon \to 0$, this becomes $\delta(x)$ (Dirac $\delta$-functio

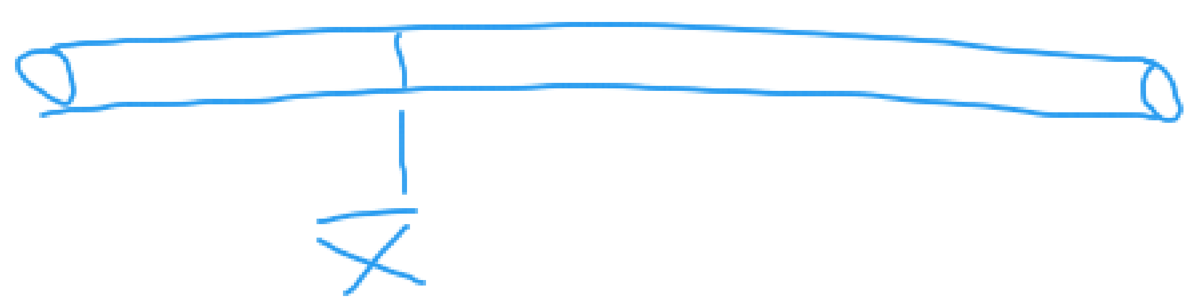
$\delta$ is an example of a "distribution"

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$u''(x) = f(x) \quad 0 \le x \le 1$$

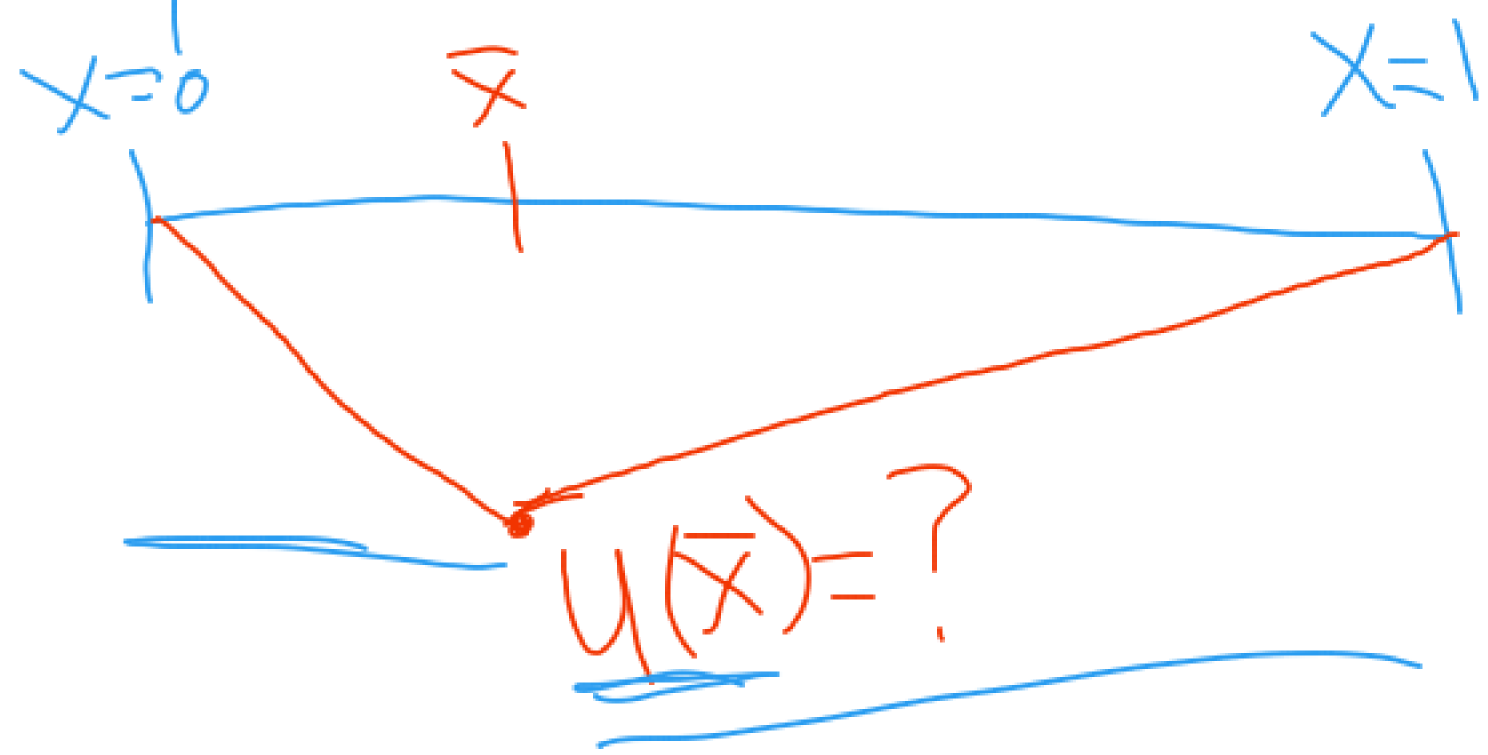
$$u(0) = \alpha \quad u(1) = \beta$$

Let's consider: $f(x) = \delta(x - \bar{x})$




For now, let $\alpha = \beta = 0$.

Away from $\bar{x}$, $u(x) = ax + b$.




Weak solution of the BVP

$$u(x) = \begin{cases} u_1(x) & 0 \le x \le \bar{x} \\ u_2(x) & \bar{x} \le x \le 1 \end{cases}$$

$$\underline{u'(\bar{x}+\varepsilon)} - \underline{u'(\bar{x}-\varepsilon)} = \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} u''(x)\,dx$$

$$= \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} f(x)\,dx$$

$$= \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \delta(\bar{x}-x)\,dx$$

$$= \underline{\underline{1}}$$

$u_1(0)=0$ $\qquad$ $u_2(1)=0$

So $u_1(x)=a_1x$ $\qquad$ So $u_2(x)=a_2(1-x)$

$$u_2'-u_1'=-a_2-a_1=1 \qquad a_1=-1-a_2$$

$$u_1(\bar{x})=u_2(\bar{x})=a_1\bar{x}=a_2(1-\bar{x})$$

$$-\bar{x}(1+a_2)=a_2(1-\bar{x})$$

$$-\bar{x}-\cancel{a_2\bar{x}}=a_2-\cancel{a_2\bar{x}} \Rightarrow a_2=-\bar{x}$$

$$a_1=\bar{x}-1$$

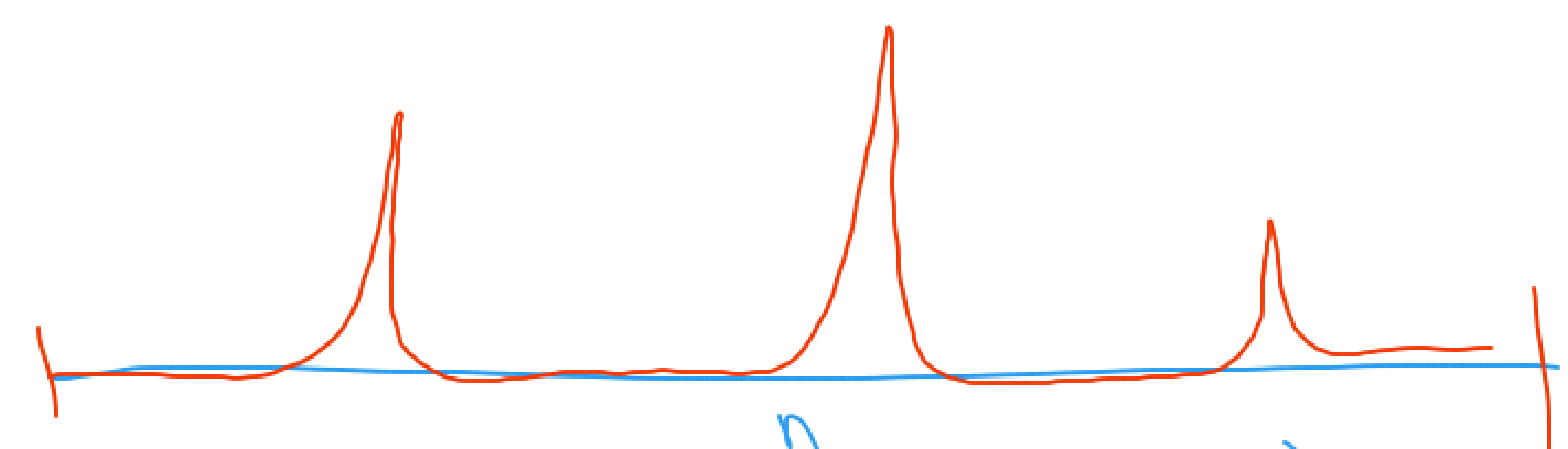
$$\boxed{\begin{aligned} u_2&=\bar{x}(x-1)\\ u_1&=x(\bar{x}-1)\end{aligned}}$$

$$\boxed{G(x;\bar{x})=\begin{cases} x(\bar{x}-1) & 0\le x\le \bar{x}\\ \bar{x}(x-1) & \bar{x}\le x\le 1\end{cases}}$$

This is the Green's function.

For $f(x)=\delta(x-\bar{x})$, the solution of the BVP is: $G(x;\bar{x})$.

What if $f(x)=\sum_{j=1}^{n}\alpha_j\delta(x-x_j)$?



Then $u(x)=\sum_{j=1}^{n}\alpha_j G(x;x_j)$

(Superposition)

In fact, any function $f(x)$ can be written:

$$f(x) = \int_0^1 \underline{f(\bar{x})}\,\underline{\delta(x-\bar{x})}\,\underline{d\bar{x}}$$

The general solution of $u''(x) = f(x)$ (still with $u(0) = u(1) = 0$) is

$$u(x) = \int_0^1 f(\bar{x})\, G(x;\bar{x})\, d\bar{x}$$

What about $u(0) = \alpha$, $u(1) = \beta$?

$u''(x) = 0 \quad u(0) = 1 \quad u(1) = 0$

Solution: $u(x) = 1 - x = G_0(x)$

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$u''(x) = 0 \quad u(0) = 0 \quad u(1) = 1$

Solution: $u(x) = x = G_1(x)$

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$u''(x) = f(x)$ $\left.\begin{array}{l} u(0) = \alpha \\ u(1) = \beta \end{array}\right\}$ Dirichlet BC

Solution:

$$\boxed{u(x) = \alpha G_0(x) + \beta G_1(x) + \int_0^1 f(\bar{x})\, G(x;\bar{x})\, d\bar{x}}$$

$$u''(x) = f(x)$$

$x_0 = 0$ $\qquad$ $x_{m+1} = 1$

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f(x_i) \qquad U_0 = \alpha \qquad U_{m+1} = \beta$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} h^2 & 0 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 0 & h^2 \end{bmatrix}}_{A^{(m+2)\times(m+2)}} \underbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_m \\ U_{m+1} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \alpha \\ f(x_1) \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}}_{F}$$

Let $B = A^{-1}$

$$AU = F \Rightarrow U = BF$$

$$B = \begin{bmatrix} B_0 \,\Big|\, B_1 \,\Big|\, \cdots \,\Big|\, B_m \,\Big|\, B_{m+1} \end{bmatrix}$$

Suppose $F = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← entry $k$


Then $U = B_k$.

$U = BF$ means

$$\boxed{U = \sum_j B_j f_j}$$

We may expect that $B_j$ approximates the Green's function $G(x_i; x_j)$

i.e.

$$\boxed{\begin{gathered} B_{ij} = h\,G(x_i; x_j) \\ 1 \le j \le m \\ 0 \le i \le m+1 \end{gathered}}$$

What about the BCs?

Let $f(x) = 0$, $\beta = 0$

$$F = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and } U = \alpha B_0$$

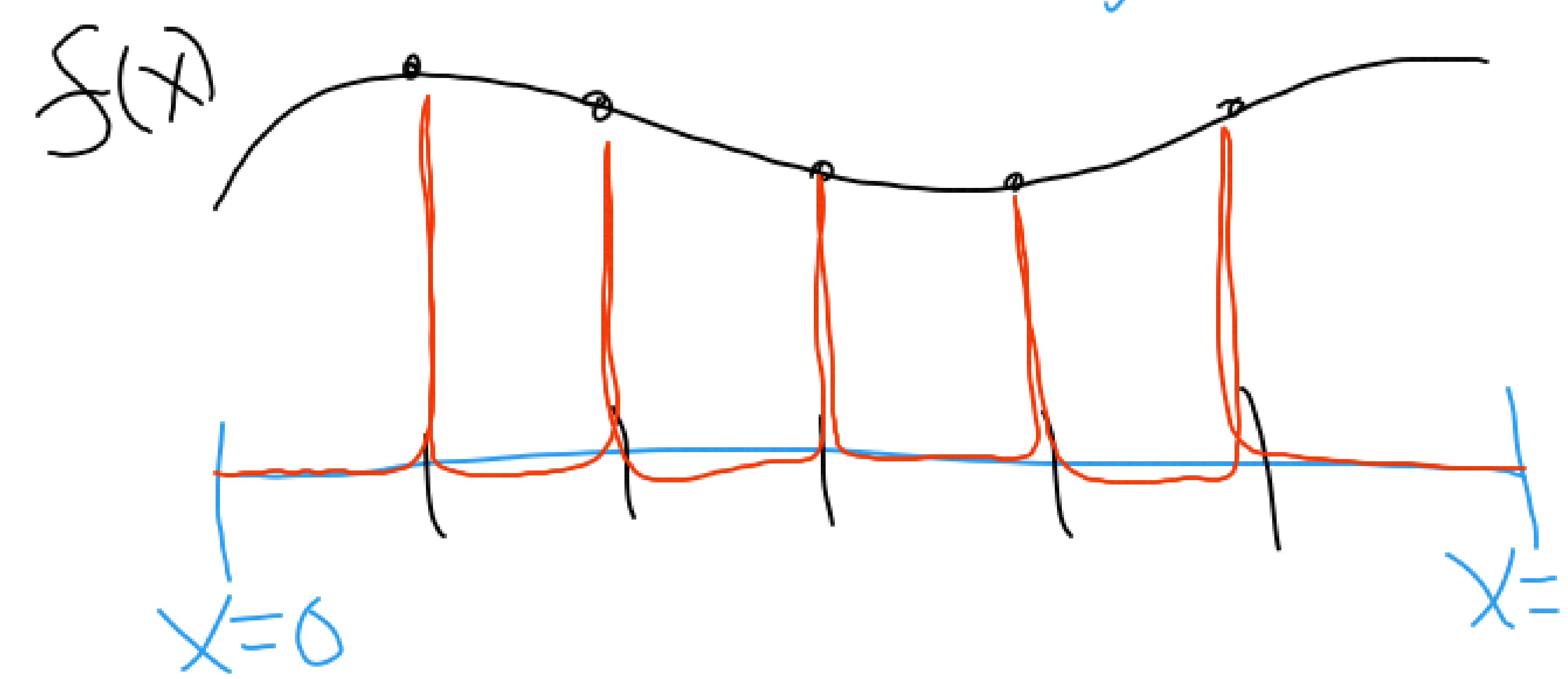
$$\boxed{B_{i0} = G_0(x_i)} = 1 - x$$

$$\boxed{B_{i,m+1} = G_1(x_i) = x}$$

$U = BF$

$$\Rightarrow U = \alpha B_0 + \beta B_{m+1} + \sum_{j=1}^{m} f(x_j) B_j$$

$U$ is the exact solution to the BVP with $f(x) = \sum_{j=1}^{m} f(x_j)\,\delta(x - x_j)$



$U = BF$

$E = B\tau$

$\|E\|_\infty \leq \|B\|_\infty \|\tau\|_\infty$

$\mathcal{O}(h^2)$

$\|B\|_\infty$ = maximum absolute row sum

$$\|B\|_\infty = \max_{0 \leq i \leq m+1} \sum_{j=0}^{m+1} |b_{ij}|$$

We want to show that $\|B\|_\infty < C \quad \forall h$.

$$\|B\|_\infty \leq \max_i |G_0(x_i)| + \max_i |G_1(x_i)| + \sum_{j=1}^{m} h$$

$$\leq 1 + 1 + \sum_{j=1}^{m} \frac{1}{m+1}$$

$$\leq 2 + \frac{m}{m+1} < 3$$