# A guide to the numerical zoo

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David Ketcheson

# The purpose of this lecture

- Introduce some general classifications of numerical methods

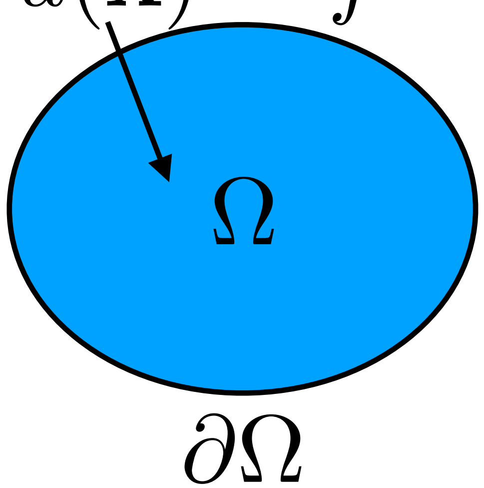
- Give you a basic idea of how each kind works and what their advantages are

- We won’t go into details

# Three types of problems

- Steady state
  - boundary -value problem
  - Solution doesn't change in time

$$\nabla^2 u(\mathbf{x}) = f$$




$$u(\mathbf{x}) = g(\mathbf{x}) \quad (\mathbf{x} \in \partial\Omega)$$

- Time-dependent
  - Initial-value problem
  - Solution changes in time

$$u'(t) = f(t, u)$$
$$u(0) = u_0$$

- Initial boundary value problem

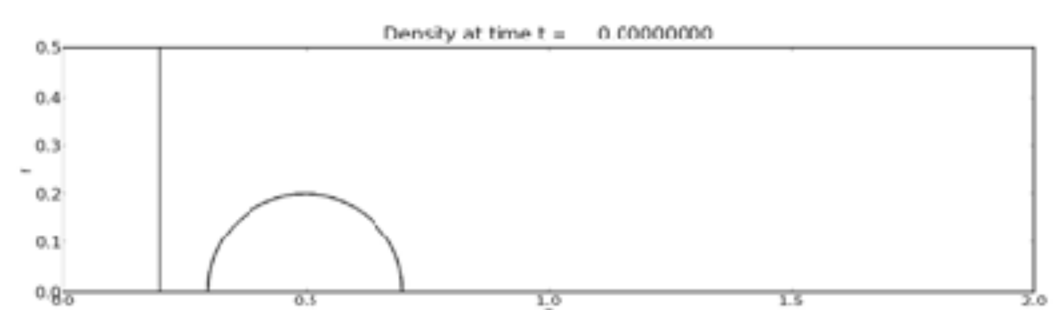
$$u_t = \nabla^2 u(\mathbf{x}) - f$$
$$u(t, \mathbf{x} \in \partial\Omega) = g(\mathbf{x})$$
$$u(t = 0, \mathbf{x}) = u_0(\mathbf{x})$$

# Linear vs. Nonlinear problems

- Linear problems
  - May be exactly solvable
  - Can use techniques like superposition
  - Discretizations lead to linear algebra
  - Examples:
    - diffusion of heat
    - electromagnetic waves
    - acoustic waves
    - gravitational potential

# Linear vs. Nonlinear problems

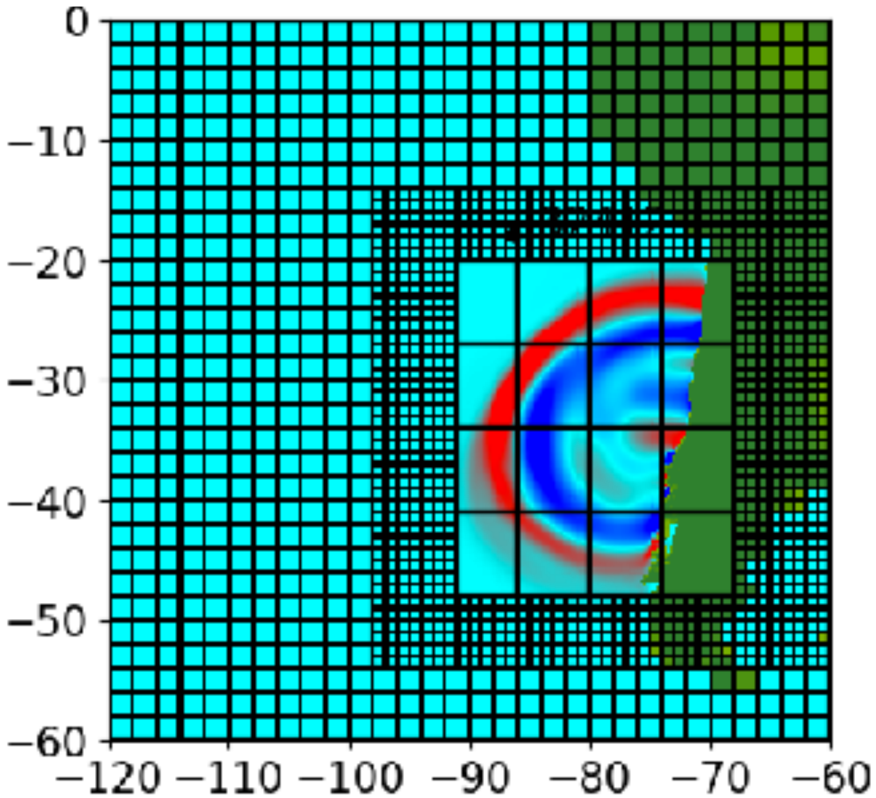
- Nonlinear problems
  - Rarely have exact solutions
  - No superposition
  - Discretizations lead to nonlinear algebra
  - Solution may not be unique
  - Examples:
    - Pendulum
    - Spread of disease
    - Water waves
    - Fluid dynamics



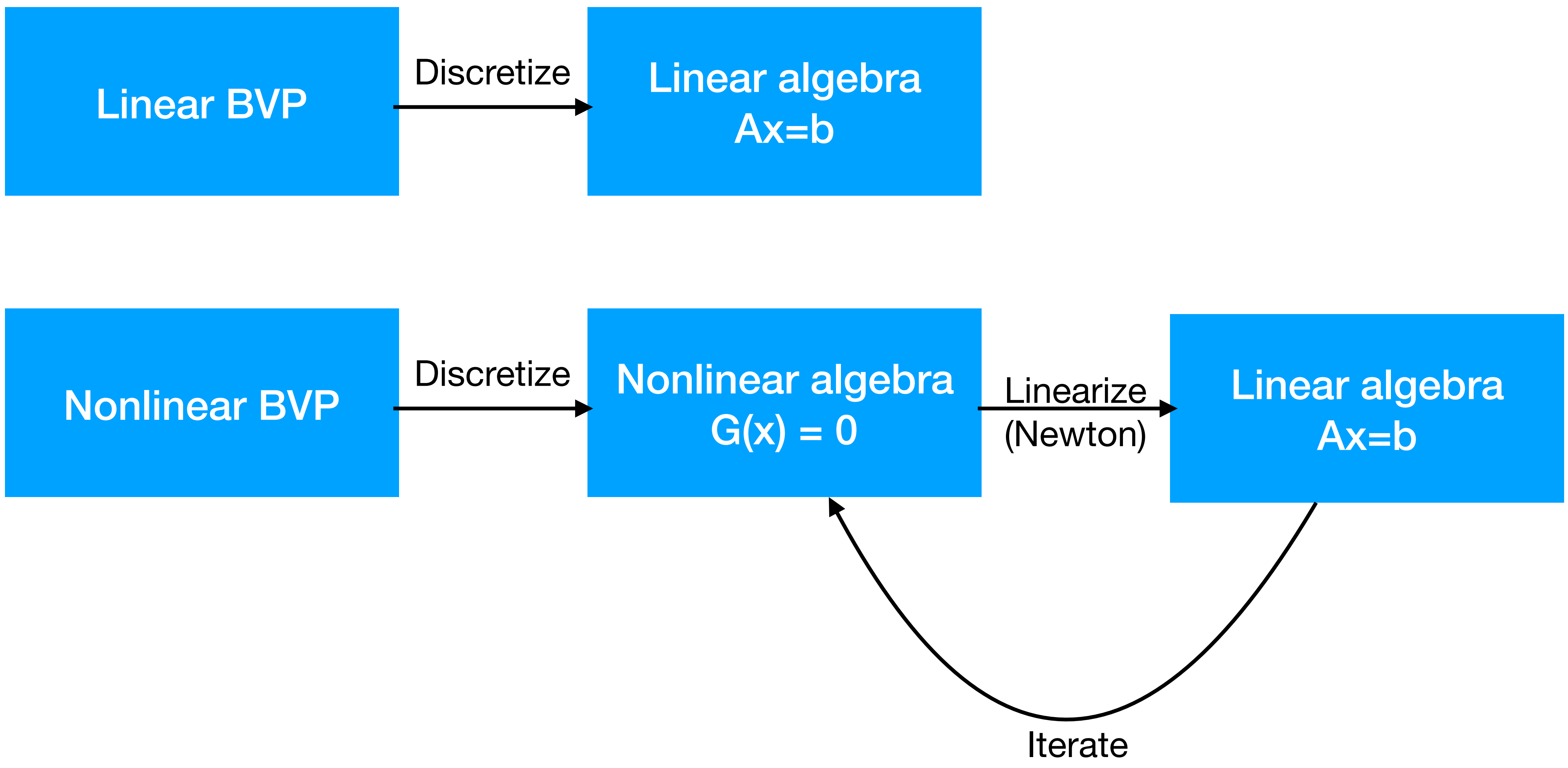
# Discretization

- Solutions of ODEs and PDEs live in continuous, infinite-dimensional spaces!

- To compute with them, we must replace those with discrete spaces that are finite-dimensional



# Discretization of boundary value problems



# 3 “finite”s

- Finite differences

- Finite volumes

- Finite elements

# 3 “finite”s

### FDM

- Easier to code
- Lower computational cost
- Works best with simple geometries
- Simplest to understand

### FVM

Falls somewhere in-between

### FEM

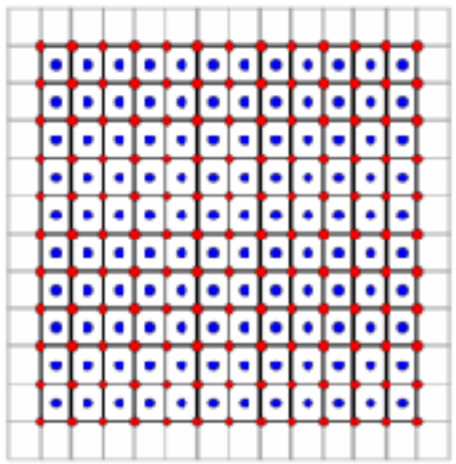
- Harder to code
- Higher computational cost
- Works easily with complex geometries
- More sophisticated mathematical foundation

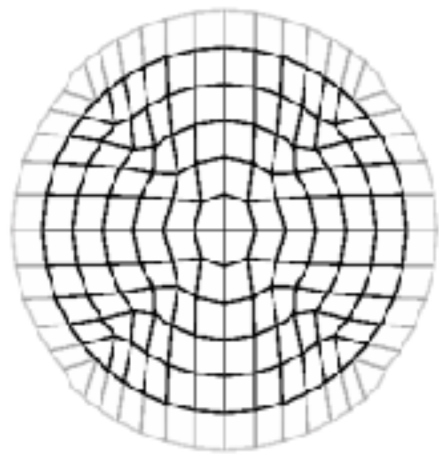
# Basis functions

- Spectral methods
  - Basis functions have global support
  - More accurate for smooth solutions
  - Work best in simple geometries
  - Dense linear algebra

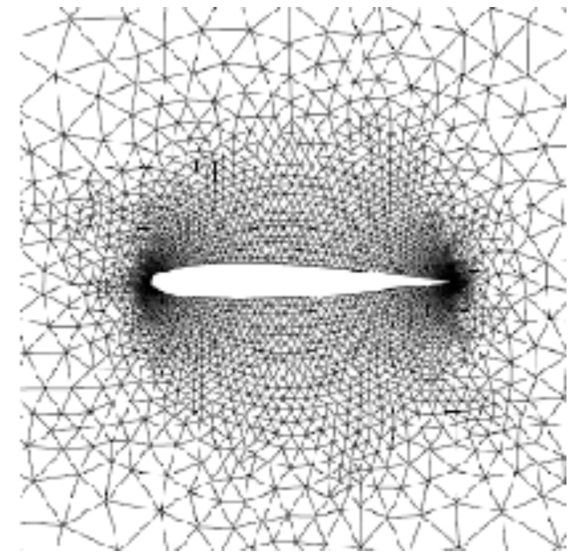
- “Local” methods
  - Basis functions have compact support
  - More suited to non-smooth solutions
  - More suited to other geometries
  - Sparse linear algebra

# Grids





- Structured grid
  - Natural ordering
  - Neighbors are obvious
  - Computationally efficient



- Unstructured grids
  - Need lookup table for neighbors
  - Arbitrary geometries
  - Local refinement

# Explicit vs. Implicit

$$u'(t) = f(u)$$

$$u_{n+1} = u_n + \Delta t f(u_n)$$

$$u_{n+1} = u_n + \Delta t f(u_{n+1})$$

- Explicit
  - Easier to program
  - Low cost per step
  - Efficient for wave equations (hyperbolic PDEs)
- Implicit
  - Harder to program
  - Higher cost per step
  - Efficient for stiff probems (elliptic, parabolic PDEs)