Relations in Discrete Mathematics

FFHS - Fernfachhochschule Schweiz
BsC in Cyber Security

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Relation

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1 Cartesian product

The Cartesian product of two sets A and B is the set of all ordered pairs (a, b) where a is an element of A and b is an element of B.
$A * B = \{(a, b) \mid a \in A \land b \in B\}$

The Cartesian product of the sets $A = \{1, 2\}$ and $B = \{3, 4\}$ is:
$A * B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.

2 Relation

A relation R from a set A to a set B is a subset of the Cartesian product $A * B$.
$R \subseteq A * B$

Let $A = \{1, 2\}$ and $B = \{3, 4\}$. The relation $R = \{(1, 3), (2, 4)\}$ is a relation from A to B.

For $(a, b) \in R$, we write aRb, and say that a is in relation R to b.

3 Inverse relation

The inverse relation $R^{\{-1\}}$ of a relation R is the relation that contains the ordered pairs of R in reverse order.
$R^{\{-1\}} = \{(b, a) \mid (a, b) \in R\}$

Let $R = \{(1, 3), (2, 4)\}$. The inverse relation $R^{\{-1\}}$ is:
$R^{\{-1\}} = \{(3, 1), (4, 2)\}$.

4 Composition of relations

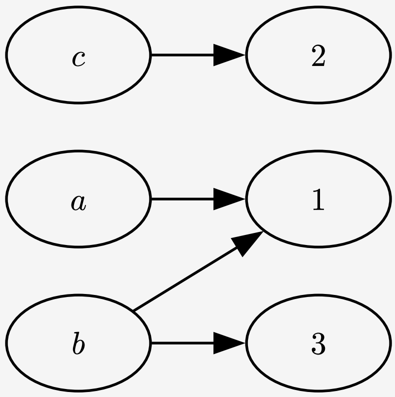
Given the relation $R \subseteq A * B$ and $S \subseteq B * C$, the composition of $R \circ S$ is the relation from A to C defined by:

$R \circ S = \{(a, c) \mid \exists b \in B, (a, b) \in R \land (b, c) \in S\}$

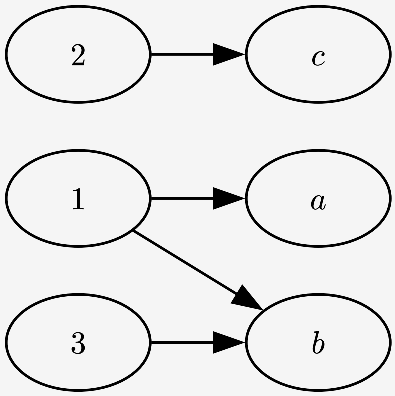
5 Representation of relations

Relations can be represented in different ways, one way is by using a directed graph.

$R = \{(a, 1), (b, 1), (b, 3), (c, 2)\} \subseteq A * B$ when $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.



or $R^{\{-1\}} = \{(1, a), (1, b), (3, b), (2, c)\}$.

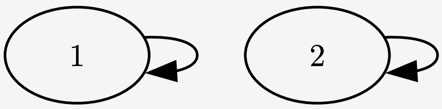


6 Relations $R \subseteq A * A$

Relations that are subsets of the Cartesian product of a set with itself are called relations on the set. They can have the following properties:

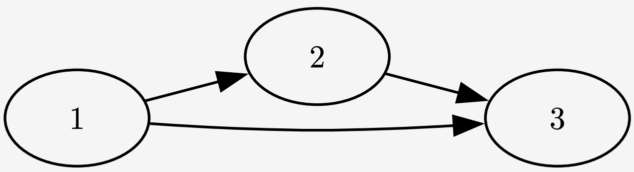
Reflexive: $(a, a) \in R \forall a \in A$.

The relation $R = \{(1, 1), (2, 2)\} \subseteq A * A$ is reflexive.



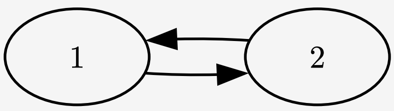
Transitive: $\forall a, b, c \in A, (a, b) \in R \land (b, c) \in R \Rightarrow (a, c) \in R$.

The relation $R = \{(1, 2), (2, 3), (1, 3)\} \subseteq A * A$ is transitive.



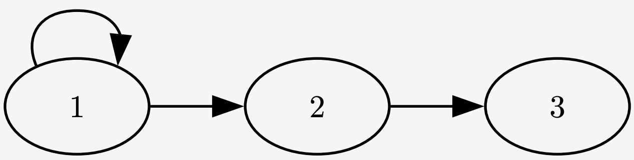
Symmetric: $\forall a, b \in A, (a, b) \in R \Rightarrow (b, a) \in R$.

The relation $R = \{(1, 2), (2, 1)\} \subseteq A * A$ is symmetric.



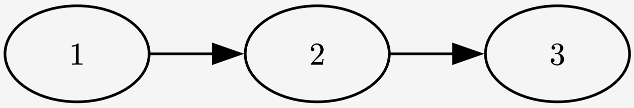
Antisymmetric: $\forall a, b \in A, (a, b) \in R \land (b, a) \in R \Rightarrow a = b$.
or equivalently: $\forall a, b \in A, (a, b) \in R \land a \neq b \Rightarrow (b, a) \notin R$.

The relation $R = \{(1, 2), (2, 3), (1, 1)\} \subseteq A * A$ is antisymmetric.



Asymmetric: $\forall a, b \in A, (a, b) \in R \Rightarrow (b, a) \notin R$.

The relation $R = \{(1,2),(2,3)\} \subseteq A * A$ is asymmetric.



A relation R on a set A is called an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.
For $(a,b) \in R$, we say that a is **equivalent** to b and write $a \equiv b$.

7 Equivalence classes

Given an equivalence relation R on a set A, the equivalence class of an element $a \in A$ is the set of all elements in A that are equivalent to a.
$[a]_R = \{b \in A \mid a \equiv b\}$

Given the relation R is an equivalence relation on the set A then the following properties hold:

1. The equivalence classes of R form a partition of A.

2. A partition of a set A is a collection of nonempty, mutually disjoint subsets of A whose union is A.

8 Order Relations

A relation R on a set A is called a order(relation) if it is **reflexive**, **antisymmetric** and **transitive**.
Often denoted by $a \leq b$.

For each order there also exists a strict order. A strict order is the result of removing the reflexive property from the order relation.

8.1 Strict order

A relation R on a set A is called a strict order if it is **antisymmetric** and **transitive** and **not reflexive**.

From each order relation R there exists a strict order relation S such that $aRb \Longleftrightarrow aSb \wedge a \neq b$. From each strict order a order relation can be derived by adding the reflexive property.

$A \leq B$ is a order relation on the set A.
$A < B$ is a strict order relation on the set A.

9 Comparability

Two elements a and b in a set A are said to be comparable with respect to a relation R if either aRb or bRa.

9.1 Total order

A relation R on a set A is called a total order if it is a order relation and for all $a, b \in A$ either aRb or bRa.

Total means that for any elements a and b in A, they are always related (they can always be compared) with respect to $R \iff aRb \vee bRa$.

9.2 Partial order

A relation R on a set A is called a partial order if it is a order relation and for all $a, b \in A$ if aRb then bRa.

10 closures

Closure of a relation R is the smallest relation that contains R and has a certain property.

10.1 Reflexive closure

The reflexive closure of a relation R on a set A is the smallest relation that contains R and is reflexive.
A relation R is reflexive if for all $a \in A$, $(a, a) \in R$.

The reflexive closure of a relation R is $R \cup \{(a, a) \mid a \in A\}$.
Often denoted by $[R]^{\mathrm{refl}}$.

Let $R = \{(1, 2), (2, 3)\} \subseteq A * A$.
The reflexive closure of R is $R \cup \{(1, 1), (2, 2), (3, 3)\}$.

10.2 Transitive closure

The transitive closure of a relation R on a set A is the smallest relation that contains R and is transitive.

A relation R is transitive if for all $a, b, c \in A$, $(a, b) \in R \land (b, c) \in R \Rightarrow (a, c) \in R$.

The transitive closure of a relation R is the intersection of all transitive relations that contain R.
Often denoted by $[R]^{\text{trans}}$.
$[R]^{\text{trans}} = R \cup \{(a, c) \mid \exists b \in A, (a, b) \in R \land (b, c) \in R\}$.

Let $R = \{(1, 2), (2, 3)\} \subseteq A * A$.
The transitive closure of R is $R \cup \{(1, 3)\}$.

10.3 Symmetric closure

The symmetric closure of a relation R on a set A is the smallest relation that contains R and is symmetric.
A relation R is symmetric if for all $a, b \in A$, $(a, b) \in R \Rightarrow (b, a) \in R$.

The symmetric closure of a relation R is $R \cup \{(b, a) \mid (a, b) \in R\}$.
Often denoted by $[R]^{\text{sym}}$.

Let $R = \{(1, 2), (2, 3)\} \subseteq A * A$.
The symmetric closure of R is $R \cup \{(2, 1), (3, 2)\}$.