Chapter 8

8-1) **(a)** **State variables:** $x_1 = y, \quad x_2 = \dfrac{dy}{dt}$

State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1$$

(b) **State variables:** $x_1 = y, \quad x_2 = \dfrac{dy}{dt}, \quad x_3 = \dfrac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$$

(c) **State variables:** $x_1 = \int_0^t y(\tau)d\tau, \quad x_2 = \dfrac{dx_1}{dt}, \quad x_3 = \dfrac{dy}{dt}, \quad x_4 = \dfrac{d^2y}{dt^2}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

(d) **State variables:** $x_1 = y, \quad x_2 = \dfrac{dy}{dt}, \quad x_3 = \dfrac{d^2y}{dt^2}, \quad x_4 = \dfrac{d^3y}{dt^3}$

State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1$$

8-2) a) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$

⇨ $(s^2 + 3s + 2)Y(s) = (s + 3)U(s)$

⇨ $sY(s) + 3Y(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s) + V(s)$

Let $X(s) = -\frac{2}{s}Y(s) + \frac{3}{5}U(s)$

Then $\begin{cases} sY(s) = X(s) + U(s) + 3Y(s) \\ sX(s) = -2Y(s) + 3U(s) \end{cases}$ ⇨ $\begin{cases} \dot{y} = -3y + x + u \\ \dot{x} = -2y + 3u \end{cases}$

If $y = x_1$ and $x = x_2$, then

$$\begin{cases} \dot{x}_1 = -3x_1 + x_2 + u \\ \dot{x}_2 = -2x_1 + 3u \end{cases}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & +1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $G(s) = \frac{Y(s)}{U(s)} = \frac{6}{s^3+6s^2+11s+6}$

⇨ $Y(s)(s^3 + 6s^2 + 11s + 6) = 6U(s)$

⇨$sY(s) + 6Y(s) = -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s^2}U(s)$

Let X(s) $= -\frac{6}{s^2}Y(s) - \frac{11}{s}Y(s) + \frac{6}{s}U(s)$, therefore $sX(s) = -\frac{6}{s}Y(s) - 11Y(s) + \frac{6}{s}U(s)$ and Let $Z(s) = -\frac{6}{s}Y(s) + \frac{6}{s}U(s)$, then $sZ(s) = -6Y(s) + 6U(s)$. As a result:

$$\begin{cases} sY(s) = -6Y(s) + X(s) \\ sX(s) = -11Y(s) + Z(s) \\ sZ(s) = -6Y(s) + 6U(s) \end{cases}$$

or

$$\begin{cases} \dot{y} = -6y + x \\ \dot{x} = -11y + z \\ \dot{z} = -6y + 6u \end{cases}$$

If $y = x_1,\ x = x_2\ and\ z = x_3$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} \text{u}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) $G(s) = \frac{Y(s)}{U(s)} = \frac{s+2}{s^2+7s+12}$

⇨$Y(s)(s^2 + 7s + 12) = (s + 2)U(s)$

⇨$sY(s) = -7Y(s) - \frac{12}{s}Y(s) + U(s) + \frac{2}{s}U(s)$

Let $sX(s) = -\frac{12}{s}Y(s) + \frac{2}{s}U(s)$, then $sX(s) = -12Y(s) + 2U(s)$. As a result:

$$\begin{cases} \dot{y} = -7y + x + u \\ \dot{x} = -12y + 2u \end{cases}$$

Let $y = x_1$ and $x = x_2$, then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

d) $G(s) = \frac{Y(s)}{U(s)} = \frac{s^3+11s^2+35s+250}{s^2(s^3+4s^2+39s+108)}$

⇨ $(s^3 + 4s + 39s + 108)Y(s) = \left[s + 11 + \frac{35}{s} + \frac{250}{s^2}\right]u(s)$

⇨ $sY(s) = -4Y(s) + \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$

Let $X_2(s) = \frac{39}{s}Y(s) + \frac{108}{s^2}Y(s) + \left[\frac{1}{s} + \frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$, then

$sX_2(s) = 39Y(s) + \frac{108}{s}Y(s) + U(s) + \left[\frac{11}{s^2} + \frac{35}{s^3} + \frac{250}{s^4}\right]U(s)$

Now, let $X_3(s) = \frac{108}{s}Y(s) + \frac{11}{s^2}U(s) + \frac{35}{s^3}U(s) + \frac{250}{s^4}U(s)$, therefore

$$\begin{cases} sX_2(s) = 39Y(s) + X_3(s) + U(s) \\ sX_3(s) = 108Y(s) + \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s) \end{cases}$$

Let $X_4(s) = \frac{11}{s}U(s) + \frac{35}{s^2}U(s) + \frac{250}{s^3}U(s)$, then $sX_4(s) = 11U(s) + \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$

Let $X_5(s) = \frac{35}{s}U(s) + \frac{250}{s^2}U(s)$, or $sX_5(s) = 35U(s) + \frac{250}{s}U(s)$

Let $X_6(s) = \frac{250}{s}U(s)$, then $sX_6(s) = 250U(s)$. If $Y(s) = X_1(s)$, then:

$$\begin{cases} \dot{x}_1 = -4x_1 + x_2 \\ \dot{x}_2 = 39x_1 + x_2 + u \\ \dot{x}_3 = 108x_1 + x_4 \\ \dot{x}_4 = 11u + x_5 \\ \dot{x}_5 = 35u + 36x_6 \\ \dot{x}_6 = 250u \end{cases}$$

or

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\\ \dot{x}_5\\ \dot{x}_6\end{bmatrix} = \begin{bmatrix}-4 & 1 & 0 & 0 & 0 & 0\\ 39 & 0 & 1 & 0 & 0 & 0\\ 108 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 0\\ 11\\ 35\\ 250\end{bmatrix}u$$

8-3) (a) Alternatively use equations (8-225), (8-232) and (8-233)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

The state variables are defined as

$$x_1(t) = y(t)$$
$$x_2(t) = \frac{dy(t)}{dt}$$

Then the state equations are represented by the vector-matrix equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where **x**(*t*) is the 2 × 1 state vector, *u*(*t*) the scalar input, and

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix}3\\ 1\end{bmatrix}$$ (Also see section 2-3-3 or 8-6)

$$\mathbf{C} = \begin{bmatrix}1 & 0\end{bmatrix} \qquad \mathbf{D} = 0$$

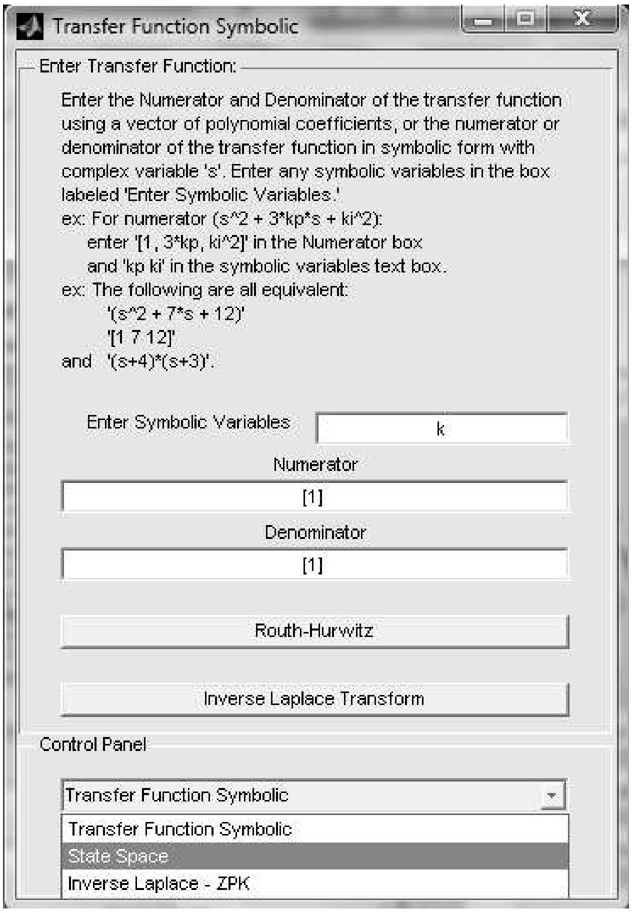
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

MATLAB

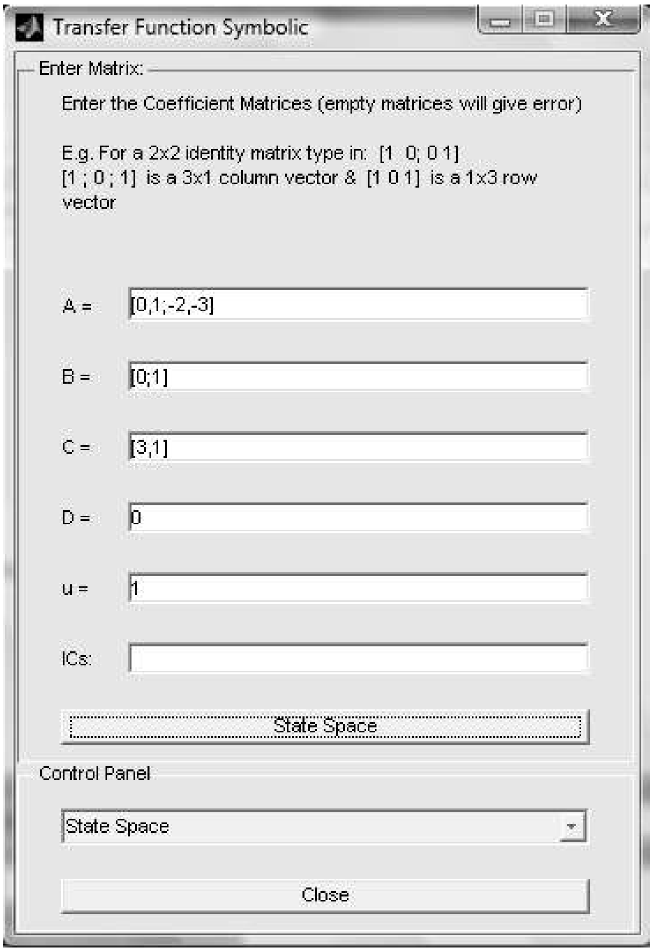
```
>> clear all
>> syms s
>> A=[0,1;-2,-3]
A =
     0     1
    -2    -3
>> B=[0;1]
B =
     0
     1
>> C=[3,1]
C =
     3     1
>> s*eye(2)-A
 ans =
 [   s,  -1]
[   2, s+3]
  >> inv(ans)
 ans =
 [ (s+3)/(s^2+3*s+2),     1/(s^2+3*s+2)]
[    -2/(s^2+3*s+2),     s/(s^2+3*s+2)]
  >> C*ans*B
 ans =
 3/(s^2+3*s+2)+s/(s^2+3*s+2)
```

Use ACSYS as demonstrate in section 8-19-2

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "Transfer Function Symbolic" pushbutton
5) Enter the transfer function
6) Use the "State Space" option as shown below:



You get the next window. Enter the A,B,C, and D values.



State Space Analysis

Inputs:

A=| 0 1| B=|0|

 |-2 -3| |1|

C=|3 1| D=|0|

State Space Representation:

Dx = | 0 1|x + |0|u

  |-2 -3| |1|

y = |3 1|x + |0|u

Determinant of (s*I-A):

$$s^2 + 3s + 2$$

Characteristic Equation of the Transfer Function:

$$s^2 + 3s + 2$$

The eigen values of A and poles of the Transfer Function are:

-1

-2

Inverse of (s*I-A) is:

$$\begin{bmatrix} \dfrac{s+3}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ & \\ -\dfrac{2}{s^2+3s+2} & \dfrac{s}{s^2+3s+2} \end{bmatrix}$$

State transition matrix (phi) of A:

$$\begin{bmatrix} 2\exp(-t) - \exp(-2t) & \exp(-t) - \exp(-2t) \\ & \\ -2\exp(-t) + 2\exp(-2t) & -\exp(-t) + 2\exp(-2t) \end{bmatrix}$$

Transfer function between u(t)and y(t) is:

```
     s + 3
   ------------
   2
   s  + 3 s + 2
```

No Initial Conditions Specified

States (X) in Laplace Domain:

```
[       1       ]
[---------------]
[(s + 2) (s + 1)]
[               ]
[       s       ]
[---------------]
[(s + 2) (s + 1)]
```

Inverse Laplace x(t):

```
[ exp(-t) - exp(-2 t)  ]
[                      ]
[-exp(-t) + 2 exp(-2 t)]
```

Output Y(s):

```
      s + 3
  ---------------
  (s + 2) (s + 1)
```

Inverse Laplace y(t):

```
2 exp(-t) - exp(-2 t)
```

Use the same procedure for parts b, c and d.

8-4) a) $x_1 = \frac{-x_4+u}{s+2}$ ➔ $(s+2)X_1 = -X_4 + U,$ ➔ $\dot{x}_1 = -x_4 - 2x_1 + u$

$x_4 = \frac{x_3+x_1}{s}$ ➔ $sX_4 = X_3 + X_1$ ➔ $\dot{x}_4 = x_1 + x_3$

$x_2 = \frac{0.5}{s}x_1$ ➔ $sX_2 = 0.5X_1$ ➔ $\dot{x}_2 = 0.5x_1$

$x_3 = \frac{x_2}{s}$ ➔ $sX_3 = X_2$ ➔ $\dot{x}_3 = x_2$

$y = x_1 + x_2 + x_3$

As a result:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0 & -1\\ 0.5 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 1 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

b) $X_1(s) = \frac{1}{s+2}U(s)$ ➔ $sX_1(s) = -2X_1(s) + U(s)$➔ $\dot{x}_1 = -2x_1 + u$

$X_2 = \frac{s+4}{s+3}X_1$➔ $sX_2(s) = sX_1 + 4X_1 - 3X_2$➔$\dot{x}_2 = \dot{x}_1 + 4x_1 - 3x_2 = 2x_1 - 3x_2 + u$

$X_3 = \frac{X_2+X_1-6X_3}{s}$ ➔ $sX_3(s) = X_2 + X_1 - 6X_3$➔$\dot{x}_3 = x_2 + x_1 - x_3$

$y = x_3 + x_1$

As a result:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\ 2 & -3 & 0\\ 1 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

c) $X_1(s) = \frac{1}{s+5}U(s)$ ➔ $sX_1 = -5X_1 + U$➔ $\dot{x}_1 = -5x_1 + u$

$X_2 = \frac{X_1+U-X_3}{s+2}$ ➔ $sX_2 = X_1 - 2X_2 - X_3 + U$➔$\dot{x}_2 = x_1 - 2x_2 - x_3 + u$

$X_3 = \frac{X_2}{s+4}$ ➔$sX_3 = X_2 - 4X_3$ ➔ $\dot{x}_3 = x_2 - 4x_3$

$X_4 = \frac{2X_3}{s}$ ➔ $sX_4 = 2X_3$ ➔ $\dot{x}_4 = 2x_3$

$y = x_2 + x_4$

As a result:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{u}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

8-5) We shall first show that

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{1}{2!}\frac{\mathbf{A}^2}{s^2} + \cdots$$

We multiply both sides of the equation by $\left(s\mathbf{I} - \mathbf{A}\right)$, and we get **I** = **I**. Taking the inverse Laplace transform on both sides of the equation gives the desired relationship for $\phi(t)$.

8-6)

(a) USE MATLAB

```
Amat=[0 1;-2 -1]
[mA,nA]=size(Amat);
rankA=rank(Amat);
disp(' Characteristic Polynomial: ')
chareq=poly(Amat);
[mchareq,nchareq]=size(chareq);
syms s;
poly2sym(chareq,s)
[evecss,eigss]=eig(Amat);
disp(' Eigenvalues of A = Diagonal Canonical Form of A is:');
Abar=eigss,
disp('Eigen Vectors are  ')
T=evecss
% state transition matrix
ilaplace(inv([s 0;0 s]-Amat))
```

```
Results in MATLAB COMMAND LINE
```

Amat =

    0   1

-2 -1

Characteristic Polynomial:

ans =

s^2+s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
  -0.5000 + 1.3229i        0
       0           -0.5000 - 1.3229i
```

Eigen Vectors are

T =

```
  -0.2041 - 0.5401i  -0.2041 + 0.5401i
   0.8165             0.8165
```

phi=ilaplace(inv([s 0;0 s]-Amat))

phi =

[1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)+7^(1/2)*sin(1/2*7^(1/2)*t)), 2/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t)]

[-4/7*7^(1/2)*exp(-1/2*t)*sin(1/2*7^(1/2)*t), 1/7*exp(-1/2*t)*(7*cos(1/2*7^(1/2)*t)-7^(1/2)*sin(1/2*7^(1/2)*t))]

% use vpa to convert to digital format. Use digit(#) to adjust level of precision if necessary.

vpa(phi)

ans =

[.1428571*exp(.5000000*t)*(7.*cos(1.322876*t)+2.645751*sin(1.322876*t)),

.7559289*exp(-.5000000*t)*sin(1.322876*t)]

[-1.511858*exp(-.5000000*t)*sin(1.322876*t),

.1428571*exp(-.5000000*t)*(7.*cos(1.322876*t)-2.645751*sin(1.322876*t))]

ANALYTICAL SOLUTION:

Characteristic equation: $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + s + 2 = 0$

Eigenvalues: $s = -0.5 + j1.323, \quad -0.5 - j1.323$

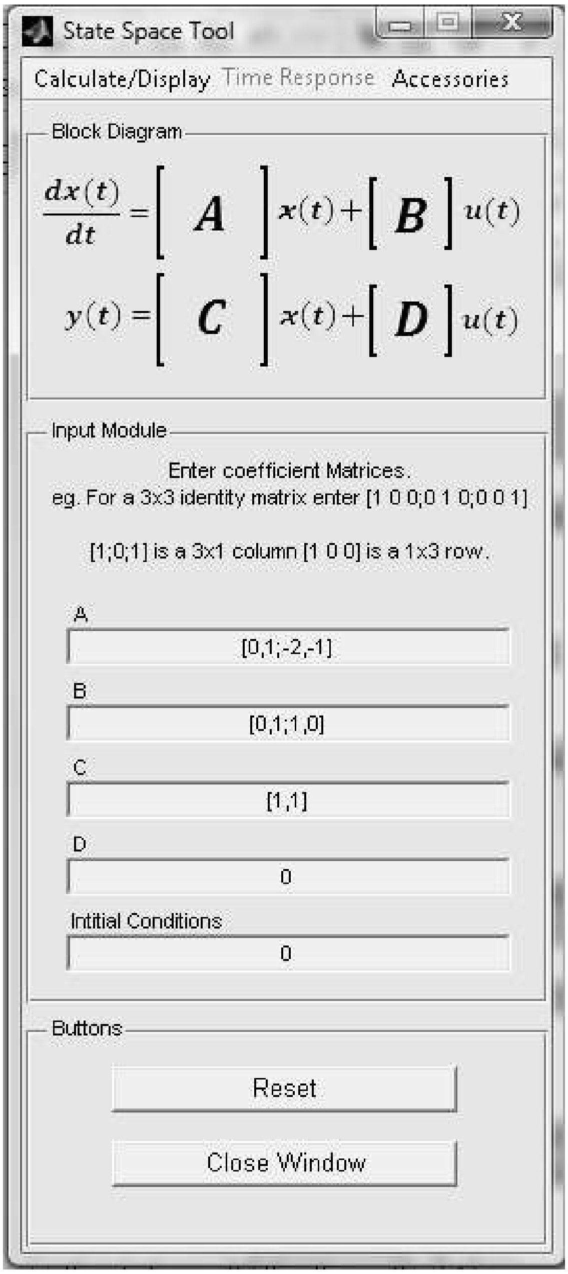
State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 1.323t + 0.378 \sin 1.323t & 0.756 \sin 1.323t \\ -1.512 \sin 1.323t & -1.069 \sin\left(1.323t - 69.3^\circ\right) \end{bmatrix} e^{-0.5t}$$

Alternatively

USE ACSYS as illustrated in section 8-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "State Space" pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the "Calculate/Display" menu and find the eigenvalues.



From MATLAB Command Window:

The A matrix is:

Amat =

0 1

-2 -1

Characteristic Polynomial:

ans =

s^2+s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-0.5000 + 1.3229i 0

0 -0.5000 - 1.3229i

Eigen Vectors are

T =

-0.2041 - 0.5401i -0.2041 + 0.5401i

0.8165 0.8165

THE REST ARE SAME AS PART A.

(b) **Characteristic equation:** $\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 4 = 0$ **Eigenvalues:** $s = -4, \quad -1$

State transition matrix:

$$\phi(t) = \begin{bmatrix} 1.333e^{-t} - 0.333e^{-4t} & 0.333e^{-t} - 0.333e^{-4t} \\ -1.333e^{-t} - 1.333e^{-4t} & -0.333e^{-t} + 1.333e^{-4t} \end{bmatrix}$$

(c) **Characteristic equation:** $\Delta(s) = (s+3)^2 = 0$ **Eigenvalues:** $s = -3, \quad -3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(d) **Characteristic equation:** $\Delta(s) = s^2 - 9 = 0$ **Eigenvalues:** $s = -3, \quad 3$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

(e) Characteristic equation: $\Delta(s) = s^2 + 4 = 0$ **Eigenvalues:** $s = j2, \ -j2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

(f) Characteristic equation: $\Delta(s) = s^3 + 5s^2 + 8s + 4 = 0$ **Eigenvalues:** $s = -1, \ -2, \ -2$

State transition matrix:

$$\phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

(g) Characteristic equation: $\Delta(s) = s^3 + 15s^2 + 75s + 125 = 0$ **Eigenvalues:** $s = -5, \ -5, \ -5$

$$\phi(t) = \begin{bmatrix} e^{-5t} & te^{-5t} & 0 \\ 0 & e^{-5t} & te^{-5t} \\ 0 & 0 & e^{-5t} \end{bmatrix}$$

State transition equation: $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{Br}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

8-7)In MATLAB USE ilaplace to find $\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right]$ see previous problem for codes.

(a)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{s\left(s^2+s+2\right)} \\ \dfrac{s-2}{s\left(s^2+s+2\right)} \end{bmatrix} = \begin{bmatrix} 1 + 0.378\sin 1.323t - \cos 1.323t \\ -1 + 1.134\sin 1.323t + \cos 1.323t \end{bmatrix} \qquad t \geq 0$$

(b)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+6}{s(s+1)(s+2)} \\ \dfrac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{s} - \dfrac{1.67}{s+1} + \dfrac{0.167}{s+4} \\ \dfrac{-1}{s} + \dfrac{1.67}{s+1} - \dfrac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5 - 1.67e^{-t} + 0.167e^{-4t} \\ -1 + 1.67e^{-t} - 0.667e - 4t \end{bmatrix} \qquad t \geq 0$$

(c)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(d)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s-3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(e)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s^2+4} & 2 \\ \dfrac{-2}{s^2+4} & \dfrac{s}{s^2+4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{2}{s} \\ \dfrac{1}{\left(s^2+4\right)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5\sin 2t \end{bmatrix} \qquad t \geq 0$$

(f)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\left(1-e^{-2t}\right) \\ 0 \end{bmatrix} \qquad t \geq 0$$

(g)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+5} & \frac{1}{(s+5)^2} & 0 \\ 0 & \frac{1}{s+5} & \frac{1}{(s+5)^2} \\ 0 & 0 & \frac{1}{s+5} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+5)^2} \\ \frac{1}{s(s+5)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{0.04}{s} - \frac{0.04}{s+5} - \frac{0.2}{(s+5)^2} \\ \frac{0.2}{s} - \frac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

8-8) **State transition equation:** $\mathbf{x}(t) = \phi(t)\mathbf{x}(t) + \int_0^t \phi(t-\tau)\mathbf{Br}(\tau)d\tau$ $\phi(t)$ for each part is given in Problem 5-3.

(a)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{s\left(s^2+s+2\right)} \\ \dfrac{s-2}{s\left(s^2+s+2\right)} \end{bmatrix} = \begin{bmatrix} 1+0.378\sin 1.323t - \cos 1.323t \\ -1+1.134\sin 1.323t + \cos 1.323t \end{bmatrix} \qquad t \geq 0$$

(b)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\frac{1}{\Delta(s)}\begin{bmatrix} s+5 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+6}{s(s+1)(s+2)} \\ \dfrac{s-4}{s(s+1)(s+4)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1.5}{s}-\dfrac{1.67}{s+1}+\dfrac{0.167}{s+4} \\ \dfrac{-1}{s}+\dfrac{1.67}{s+1}-\dfrac{0.667}{s+4} \end{bmatrix} = \begin{bmatrix} 1.5-1.67e^{-t}+0.167e^{-4t} \\ -1+1.67e^{-t}-0.667e-4t \end{bmatrix} \qquad t \geq 0$$

(c)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s+3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(d)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{1}{s-3} & 0 \\ 0 & \dfrac{1}{s+3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \dfrac{1}{s\left(s+3\right)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.333\left(1-e^{-3t}\right) \end{bmatrix} \qquad t \geq 0$$

(e)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s^2+4} & 2 \\ \frac{-2}{s^2+4} & \frac{s}{s^2+4} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{2}{s} \\ \frac{1}{(s^2+4)} \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5\sin 2t \end{bmatrix} \qquad t \geq 0$$

(f)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\left(1-e^{-2t}\right) \\ 0 \end{bmatrix} \qquad t \geq 0$$

(g)

$$\int_0^t \phi(t-\tau)\mathbf{B}r(\tau)d\tau = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}R(s)\right] = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+5} & \frac{1}{(s+5)^2} & 0 \\ 0 & \frac{1}{s+5} & \frac{1}{(s+5)^2} \\ 0 & 0 & \frac{1}{s+5} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{1}{s(s+5)^2} \\ \frac{1}{s(s+5)} \end{bmatrix} = \mathcal{L}^{-1}\begin{bmatrix} 0 \\ \frac{0.04}{s} - \frac{0.04}{s+5} - \frac{0.2}{(s+5)^2} \\ \frac{0.2}{s} - \frac{0.2}{s+5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.04 - 0.04e^{-5t} - 0.2te^{-5t} \\ 0.2 - 0.2e^{-5t} \end{bmatrix} u_s(t)$$

8-9) (a) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(b) Not a state transition matrix, since $\phi(0) \neq \mathbf{I}$ (identity matrix).

(c) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$ and

$$[\phi(t)]^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{-t} & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1-e^{t} & e^{t} \end{bmatrix} = \phi(-t)$$

(d) $\phi(t)$ is a state transition matrix, since $\phi(0) = \mathbf{I}$, and

$$[\phi(t)]^{-1} = \begin{bmatrix} e^{2t} & -te^{2t} & t^2e^{2t}/2 \\ 0 & e^{2t} & -te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} = \phi(-t)$$

8-10) a) $\dot{x} = Ax + Bu$ ➔$sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$ and $(sI-A)^{-1} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

Therefore:

$$\Phi(t) = L^{-1}\{(sI-A)^{-1}\} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

If $x(0) = 0$, then $x(t) = \int_0^t \Phi(t-\tau)Bu(t)d\,\tau = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$

b) $\Phi(t) = L^{-1}\{(sI-A)^{-1}\}$

$$= L^{-1}\left\{\frac{1}{s^2+s+0.5}\begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 5t + \sin 0.5t) \end{bmatrix}$$

If $\mathrm{x}(0) = 0$, then

$$x(t) = A^{-1}(e^{At} - I)B = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t} - \sin 0.5t \end{bmatrix}$$
$$= \begin{bmatrix} e^{-0.5t}\sin 5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}$$

and

$$\mathrm{y(t)} = [1 \quad 0]\begin{bmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{bmatrix} = \mathrm{x}_1 = \mathrm{e}^{-0.5\mathrm{t}}\sin 0.5\mathrm{t}$$

8-11) (a) (1) Eigenvalues of A: $2.325, \quad -0.3376 + j0.5623, \quad -0.3376 - j0.5623$

(2) Transfer function relation:

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2+3s+2 & s+3 & 1 \\ -1 & s\left(s+3\right) & s \\ -s & -2s-1 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s)$$

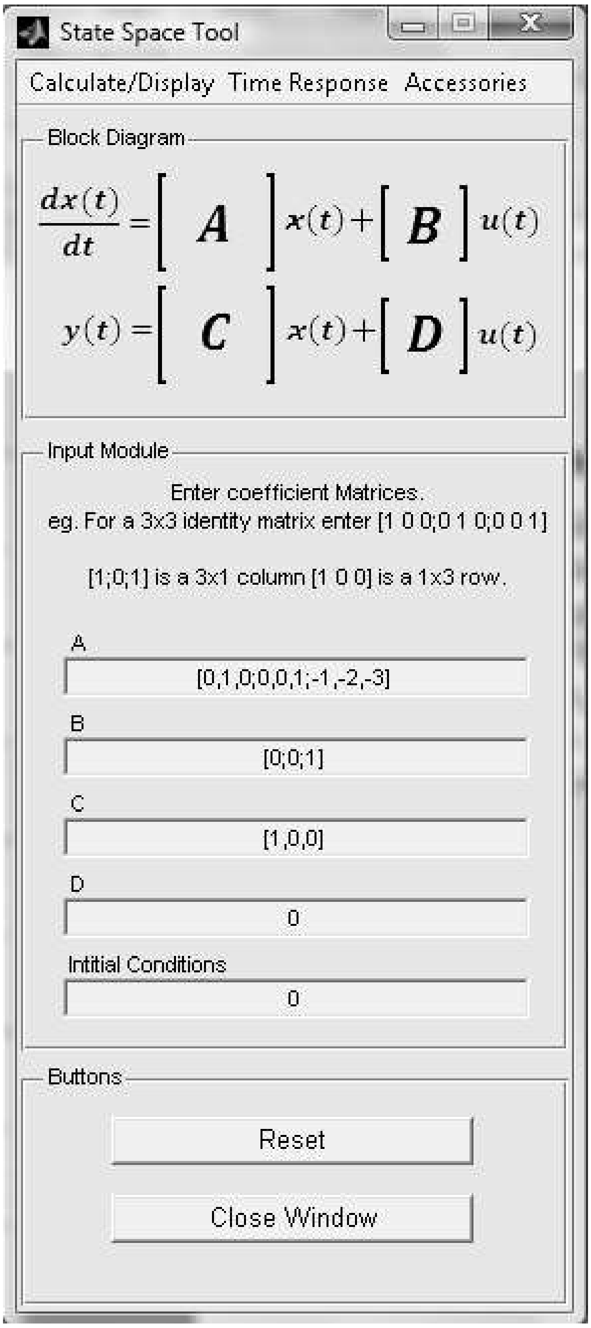
$$\Delta(s) = s^3 + 3s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{1}{s^3 + 3s^2 + 2s + 1}$$

USE ACSYS as illustrated in section 8-19-1

7) Activate MATLAB
8) Go to the folder containing ACSYS
9) Type in Acsys
10) Click the "State Space" pushbutton
11) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
12) Use the "Calculate/Display" menu and find the eigenvalues and other State space calculations.



The A matrix is:

Amat =

0	**1**	**0**
0	**0**	**1**
-1	**-2**	**-3**

Characteristic Polynomial:

ans =

s^3+3*s^2+2*s+1

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-2.3247	0	0
0	-0.3376 + 0.5623i	0
0	0	-0.3376 - 0.5623i

Eigen Vectors are

T =

0.1676	0.7868	0.7868
-0.3896	-0.2657 + 0.4424i	-0.2657 - 0.4424i
0.9056	-0.1591 - 0.2988i	-0.1591 + 0.2988i

State-Space Model is:

a =

	x1	x2	x3
x1	0	1	0
x2	0	0	1
x3	-1	-2	-3

b =

	u1
x1	0
x2	0
x3	1

```
c =

      x1  x2  x3

  y1   1   0   0


d =

      u1

  y1   0


Continuous-time model.

 Characteristic Polynomial:

 ans =

 s^3+3*s^2+2*s+1


  Equivalent Transfer Function Model is:

 Transfer function:

1.776e-015 s^2 + 6.661e-016 s + 1

---------------------------------

     s^3 + 3 s^2 + 2 s + 1


 Pole, Zero Form:


Zero/pole/gain:

1.7764e-015 (s^2  + 0.375s + 5.629e014)

---------------------------------------

  (s+2.325) (s^2  + 0.6753s + 0.4302)
```

The numerator is basically equal to 1

Use the same procedure for other parts.

(b) (1) Eigenvalues of A: $-1, \quad -1.$

(2) Transfer function relation:

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}U(s) = \begin{bmatrix} \dfrac{1}{\left(s+1\right)^2} \\ \dfrac{1}{\left(s+1\right)} \end{bmatrix}U(s) \qquad \Delta(s) = s^2 + 2s + 1$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{\left(s+1\right)^2} \\ \dfrac{1}{s+1} \end{bmatrix} = \frac{1}{\left(s+1\right)^2} + \frac{1}{s+1} = \frac{s+2}{\left(s+1\right)^2}$$

(c) (1) Eigenvalues of A: $0, \quad -1, \quad -1.$

(2) Transfer function relation:

$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} s^2 + 2s = 1 & s+2 & 1 \\ 0 & s\left(s+2\right)\big) & s \\ 0 & -s & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}U(s) = \frac{1}{\Delta(s)}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}U(s) \qquad \Delta(s) = s\left(s^2 + 2s + 1\right)$$

(3) Output transfer function:

$$\frac{Y(s)}{U(s)} = \mathbf{C}(s)\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = \frac{s+1}{s\left(s+1\right)^2} = \frac{1}{s\left(s+1\right)}$$

8-12) We write $\dfrac{dy}{dt} = \dfrac{dx_1}{dt} + \dfrac{dx_2}{dt} = x_2 + x_3 \qquad \dfrac{d^2y}{dt} = \dfrac{dx_2}{dt} + \dfrac{dx_3}{dt} = -x_1 - 2x_2 - 2x_3 + u$

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (1)$$

$$\overline{\mathbf{x}} = \begin{bmatrix} x_1 \\ y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} \qquad \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \overline{\mathbf{x}} \qquad (2)$$

Substitute Eq. (2) into Eq. (1), we have

$$\frac{d\overline{\mathbf{x}}}{dt} = \mathbf{A}_1 \overline{\mathbf{x}} + \mathbf{B}_1 u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -2 \end{bmatrix} \overline{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

8-13) For MATLAB Codes see 8-15

(a)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-2 & 0 \\ 1 & 0 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 2 & 6 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ -4 & -2 & 1 \end{bmatrix}$$

(b)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s & -2 & 0 \\ -1 & s-1 & 0 \\ 1 & -1 & s-1 \end{vmatrix} = s^3 - 3s^2 + 2 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 2, \quad a_1 = 0, \quad a_2 = -3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 3 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+2 & -1 & 0 \\ 0 & s+2 & 0 \\ 1 & 2 & s+3 \end{vmatrix} = s^3 + 7s^2 + 16s + 12 = s^3 + a_2 s^2 + a_1 s + a_0 \quad a_0 = 12, \quad a_1 = 16, \quad a_2 = 7$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 4 \\ 1 & -6 & 23 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix}$$

(d)

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s+1 & -1 & 0 \\ 0 & s=1 & -1 \\ 0 & 0 & s+1 \end{vmatrix} = s^3 + 3s^2 + 3s + 1 = s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_0 = 1, \quad a_1 = 3, \quad a_2 = 3$$

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & 1 \\ a_2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

(e)

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-1 & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 2s - 1 = s^2 + a_1 s + a_0 \qquad a_0 = -1, \quad a_1 = 2$$

$$\mathbf{M} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\mathbf{P} = \mathbf{SM} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

8-14) For MATLAB codes see 8-15

(a) From Problem 8-13(a),

$$\mathbf{M} = \begin{bmatrix} 0 & -3 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.5 & 1 & 3 \\ 0.5 & 1.5 & 4 \\ -0.5 & -1 & -2 \end{bmatrix}$$

(b) From Problem 8-13(b),

$$\mathbf{M} = \begin{bmatrix} 16 & 7 & 1 \\ 7 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 1 \\ 2 & 5 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0.2308 & 0.3077 & 1.0769 \\ 0.1538 & 0.5385 & 1.3846 \\ -0.2308 & -0.3077 & -0.0769 \end{bmatrix}$$

(c) From Problem 8-13(c),

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 0 \end{bmatrix}$$

Since **V** is singular, the OCF transformation cannot be conducted.

(d) From Problem 8-13(d),

$$\mathbf{M} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(e) From Problem 8-13(e),

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$ Then, $$\mathbf{V} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{Q} = (\mathbf{MV})^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

8-15) (a) Eigenvalues of **A**: 1, 2.7321, −0.7321

$$\mathbf{T} = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3] = \begin{bmatrix} 0 & 0.5591 & 0.8255 \\ 0 & 0.7637 & -0.3022 \\ 1 & -0.3228 & 0.4766 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvectors.

(b) Eigenvalues of **A**: 1, 2.7321, –0.7321

$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0.5861 & 0.7546 \\ 0 & 0.8007 & -0.2762 \\ 1 & 0.1239 & 0.5952 \end{bmatrix}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are the eigenvalues.

(c) Eigenvalues of **A**: -3, -2, -2. A nonsingular DF transformation matrix **T** cannot be found.

(d) Eigenvalues of **A**: –1, –1, –1

The matrix **A** is already in Jordan canonical form. Thus, the DF transformation matrix **T** is the identity matrix **I**.

(e) Eigenvalues of **A**: 0.4142, –2.4142

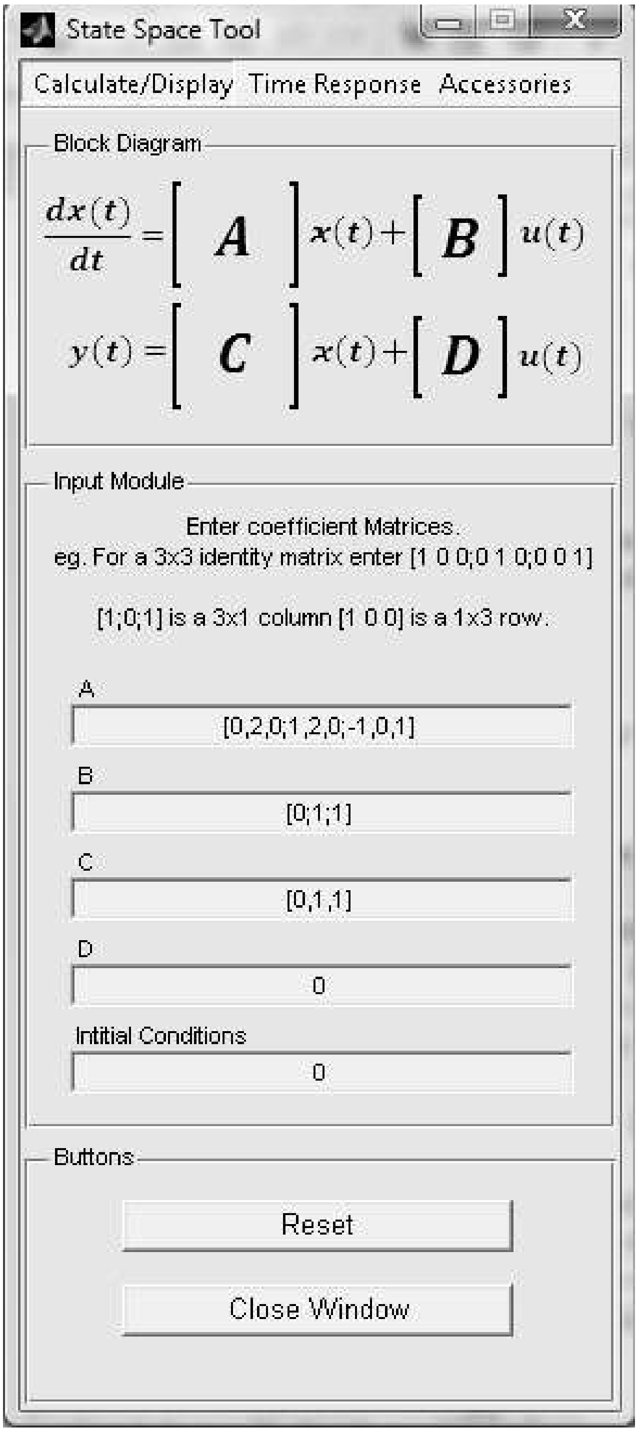
$$\mathbf{T} = \begin{bmatrix} \mathbf{p}_2 & \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} 0.8629 & -0.2811 \\ -0.5054 & 0.9597 \end{bmatrix}$$

USE ACSYS as illustrated in section 8-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the “State Space” pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the “Calculate/Display” menu and find the eigenvalues.
7) Next use the “Calculate/Display” menu and conduct State space calculations.
8) Next use the “Calculate/Display” menu and conduct Controlability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.

SOLVE PART (a)



The A matrix is:

Amat =

0	**2**	**0**
1	**2**	**0**
-1	**0**	**1**

Characteristic Polynomial:

ans =

s^3-3*s^2+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

1.0000	**0**	**0**
0	**2.7321**	**0**
0	**0**	**-0.7321**

Eigen Vectors are

T =

0	**0.5591**	**0.8255**
0	**0.7637**	**-0.3022**
1.0000	**-0.3228**	**0.4766**

State-Space Model is:

a =

	x1	x2	x3
x1	0	2	0
x2	1	2	0
x3	-1	0	1

b =

	u1
x1	0
x2	1
x3	1

c =

	x1	x2	x3
y1	0	1	1

```
d =

       u1

   y1   0
```

Continuous-time model.

Characteristic Polynomial:

ans =

s^3-3*s^2+2

Equivalent Transfer Function Model is:

Transfer function:

```
       2 s^2 - 3 s - 4
------------------------------
s^3 - 3 s^2 + 8.882e-016 s + 2
```

Pole, Zero Form:

Zero/pole/gain:

```
  2 (s-2.351) (s+0.8508)
--------------------------
```

(s-2.732) (s-1) (s+0.7321)

The Controllability Matrix [B AB A^2B ...] is =

Smat =

0	**2**	**4**
1	**2**	**6**
1	**1**	**-1**

The system is therefore Controllable, rank of S Matrix is =

rankS =

3

Mmat =

0	**-3**	**1**
-3	**1**	**0**
1	**0**	**0**

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

-2	**2**	**0**
0	**-1**	**1**
-4	**-2**	**1**

The transformed matrices using CCF are:

Abar =

0	**1.0000**	**0.0000**
0	**-0.0000**	**1.0000**
-2.0000	**0.0000**	**3.0000**

Bbar =

0
0
1

Cbar =

-4	**-3**	**2**

Dbar =

0

8-16) a) $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$

Consider:

$$Y(s) = (s^{-2} - s^{-4})X(s)$$
$$X(s) = U(s) — 2s^{-2}X(s) = U(s) + 2s^{-2}X(s)$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

As $\frac{Y(s)}{U(s)} = \frac{s^2-1}{s^2(s^2-2)}$, therefore $\mathrm{sY(s)} = \frac{2}{\mathrm{s}}\mathrm{Y(s)} + \frac{\mathrm{U(s)}}{\mathrm{s}} - \frac{\mathrm{U(s)}}{\mathrm{s}^3}$

Let $\mathrm{X_2(s)} = \frac{2}{\mathrm{s}}\mathrm{Y_2(s)} + \frac{\mathrm{U(s)}}{\mathrm{s}} - \frac{\mathrm{U(s)}}{\mathrm{s}^3}$. If $\mathrm{y = x_1}$, then $\mathrm{sY(s) = sX_1(s) = X_2}$, or $\dot{\mathrm{x}}_1 = \mathrm{x}_2$. As a result:

$$sX_2 = 2X_1 + U(s) - \frac{U(s)}{s^2}$$

Now consider $\mathrm{X_3} = -\frac{U(s)}{s^2}$, and $sX_3 = \frac{U}{s} = X_4$, then

$$\dot{x}_2 = 2x_1 - x_3 + u$$

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = u$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \mathrm{u}$$

b) $\frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$

Consider:

$$Y(s) = (2s^{-1} + s^{-2})X(s)$$

$$X(s) = U(s) - (4s^{-1} + 4s^{-2})X(s)$$

Therefore:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$As\ \frac{Y(s)}{U(s)} = \frac{2s+1}{s^2+4s+4}$, therefore $sY(s) = -4Y(s) + \frac{4}{s}Y(s) + 2U(s) + \frac{U(s)}{s}$. As a result:

$$\begin{cases} y = x_1 \quad \rightarrow \quad \dot{x}_1 = -4x_1 + 2u + x_2 \\ X_2 = \dfrac{21}{s}Y(s) + \dfrac{U(s)}{s} \dashrightarrow sX_2 = 4Y(s) + U(s) \rightarrow \dot{x}_2 = 4x_1 + u \end{cases}$$

8-17) For MATLAB codes see 8-15

(a)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$$

S is singular.

(b)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 3 & -3 & 3\end{bmatrix}$$

S is singular.

(c)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}2 & 2+2\sqrt{2}\\ \sqrt{2} & 2+\sqrt{2}\end{bmatrix}$$

S is singular.

(d)

$$\mathbf{S} = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix} = \begin{bmatrix}1 & -2 & 4\\ 0 & 0 & 0\\ 1 & -4 & 14\end{bmatrix}$$

S is singular.

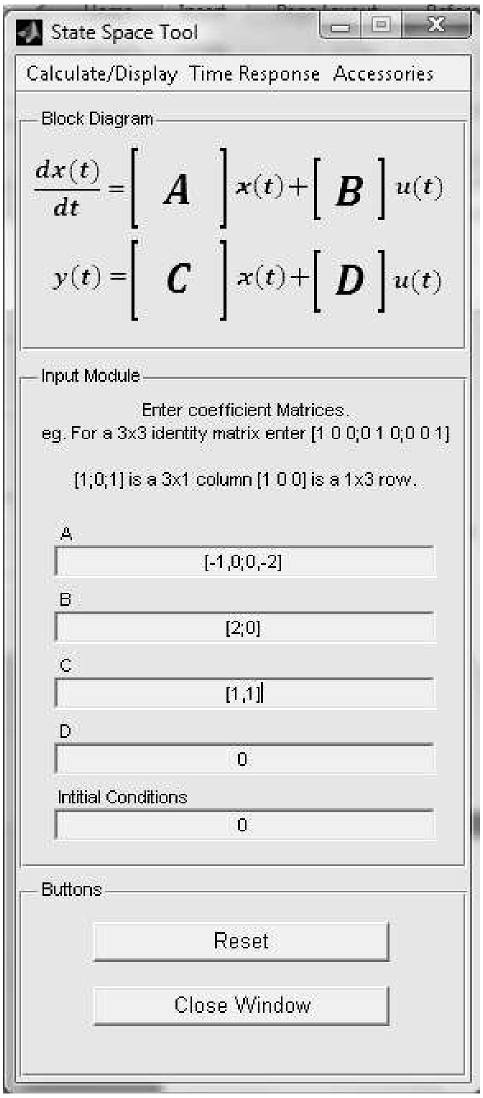
8-18) a, d and e are controllable

b, c, and f are not controllable

USE ACSYS as illustrated in section 8-19-1

9) Activate MATLAB
10) Go to the folder containing ACSYS
11) Type in Acsys
12) Click the "State Space" pushbutton
13) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
14) Use the "Calculate/Display" menu and find the eigenvalues.
15) Next use the "Calculate/Display" menu and conduct State space calculations.
16) Next use the "Calculate/Display" menu and conduct Controlability calculations.

NOTE: the above order of calculations MUST BE followed in the order stated, otherwise you will get an error.



For part b, the system is not Controllable because [B AB] is singular (rank is less than 2):

The A matrix is:

Amat =

```
   -1     0
    0    -2
```

Characteristic Polynomial:

ans =

s^2+3*s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
   -2     0
    0    -1
```

Eigen Vectors are

T =

```
     0     1
     1     0
```

Characteristic Polynomial:

ans =

s^2+3*s+2

Equivalent Transfer Function Model is:

Transfer function:

```
  2
-----
s + 1
```

Pole, Zero Form:

Zero/pole/gain:

2

(s+1)

The Controllability Matrix [B AB A^2B ...] is =

Smat =

2 -2

0 0 **←Rank is 1, and this is a singular matrix**

The system is therefore Not Controllable, rank of S Matrix is =

rankS =

1

Mmat =

3 1

1 0

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

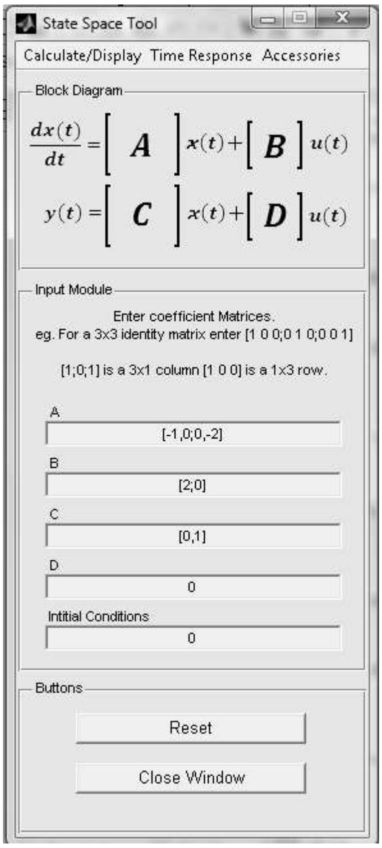
4 2

0 0

8-19) a, d, and e are observable

b, c, and f are not observable

Using ACSYS (also see the previous problem for more details):



For part b, the system is not observable. Note: you must choose a B matrix arbitrarily.

The A matrix is:

Amat =

-1 0

0 -2

Characteristic Polynomial:

ans =

s^2+3*s+2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-2 0

0 -1

Eigen Vectors are

T =

0 1

1 0

Characteristic Polynomial:

ans =

s^2+3*s+2

Equivalent Transfer Function Model is:

Transfer function:

0

Pole, Zero Form:

Zero/pole/gain:

0

The Observability Matrix (transpose:[C CA CA^2 ...]) is =

Vmat =

0 1

0 -2

The System is therefore Not Observable, rank of V Matrix is =

rankV =

1

Mmat =

3 1

1 0

8-20) (a) Rewrite the differential equations as:

$$\frac{d^2\theta_m}{dt^2} = -\frac{B}{J}\frac{d^2\theta_m}{dt^2} - \frac{K}{J}\theta_m + \frac{K_i}{J}i_a \qquad \frac{di_a}{dt} = -\frac{K_b}{L_a}\frac{d\theta_m}{dt} - \frac{R_a}{L_a}i_a + \frac{K_a K_s}{L_a}\left(\theta_r - \theta_m\right)$$

State variables: $x_1 = \theta_m, \quad x_2 = \dfrac{d\theta_m}{dt}, \quad x_3 = i_a$

State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \\ \dfrac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{K}{J} & -\dfrac{B}{J} & \dfrac{K_i}{J} \\ -\dfrac{K_a K_s}{L_a} & -\dfrac{K_b}{L_a} & -\dfrac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_a K_s}{L_a} \end{bmatrix} \theta_r$$

Output equation:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x} = x_1$$

(b) Forward-path transfer function:

$$G(s) = \frac{\Theta_m(s)}{E(s)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ \dfrac{K}{J} & s + \dfrac{B}{J} & -\dfrac{K_i}{J} \\ 0 & \dfrac{K_b}{L_a} & s + \dfrac{R_a}{L_a} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \dfrac{K_a}{L_a} \end{bmatrix} = \frac{K_i K_a}{\Delta_o(s)}$$

$$\Delta_o(s) = JL_a s^3 + \left(BL_a + R_a J\right)s^2 + \left(KL_a + K_i K_b + R_a B\right)s + KR_a = 0$$

Closed-loop transfer function:

$$M(s)=\frac{\Theta_m(s)}{\Theta_r(s)}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix} s & -1 & 0 \\ \frac{K}{J} & s+\frac{B}{J} & -\frac{K_i}{J} \\ \frac{K_aK_s}{L_a} & \frac{K_b}{L_a} & s+\frac{R_a}{L_a}\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\ \frac{K_aK_s}{L_a}\end{bmatrix}=\frac{K_sG(s)}{1+K_s(s)}$$

$$=\frac{K_iK_aK_s}{JL_as^3+\left(BL_a+R_aJ\right)s^2+\left(KL_a+K_iK_b+R_aB\right)s+K_iK_aK_s+KR_a}$$

8-21) (a)

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\qquad \mathbf{A}^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\qquad \mathbf{A}^3=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\qquad \mathbf{A}^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-\frac{t^2}{2!}+\frac{t^4}{4!}-\cdots & t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots \\ -t+\frac{t^3}{3!}-\frac{t^5}{5!}+\cdots & 1-\frac{t^2}{2!}+\frac{t^4}{4!}\cdots\end{bmatrix}=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s)=\left(s\mathbf{I}-\mathbf{A}\right)^{-1}=\begin{bmatrix}s & -1\\ 1 & s\end{bmatrix}^{-1}=\frac{1}{s^2+1}\begin{bmatrix}s & 1\\ -1 & s\end{bmatrix}\qquad \phi(t)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$$

(b)

$$\mathbf{A}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\qquad \mathbf{A}^2=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\qquad \mathbf{A}^3=\begin{bmatrix}-1 & 0\\ 0 & -8\end{bmatrix}\qquad \mathbf{A}^4=\begin{bmatrix}-1 & 0\\ 0 & 16\end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t)=\mathbf{I}+\mathbf{A}t+\frac{1}{2!}\mathbf{A}^2t^2+\cdots=\begin{bmatrix}1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots & 0\\ 0 & 1-2t+\frac{4t^2}{2!}-\frac{8t^3}{3!}+\cdots\end{bmatrix}=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ 0 & \dfrac{1}{s+2} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(1) Infinite series expansion:

$$\phi(t) = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots = \begin{bmatrix} 1 + \dfrac{t^2}{2!} + \dfrac{t^4}{4!} + \cdots & t + \dfrac{t^3}{3!} + \dfrac{t^5}{5!}\cdots \\ t + \dfrac{t^3}{3!} + \dfrac{t^5}{5!}\cdots & 1 + \dfrac{t^2}{2!} + \dfrac{t^4}{4!} + \cdots \end{bmatrix} = \begin{bmatrix} e^{-t} + e^{t} & -e^{-t} + e^{t} \\ -e^{-t} + e^{t} & e^{-t} + e^{t} \end{bmatrix}$$

(2) Inverse Laplace transform:

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - 1}\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \dfrac{0.5}{s+1} - \dfrac{0.5}{s-1} & \dfrac{-0.5}{s+1} + \dfrac{0.5}{s-1} \\ \dfrac{-0.5}{s+1} + \dfrac{0.5}{s-1} & \dfrac{0.5}{s+1} + \dfrac{0.5}{s-1} \end{bmatrix}$$

$$\phi(t) = 0.5\begin{bmatrix} e^{-t} + e^{t} & -e^{-t} + e^{t} \\ -e^{-t} + e^{t} & e^{-t} + e^{t} \end{bmatrix}$$

8-22) (a) $e = K_s\left(\theta_r - \theta_y\right) \qquad e_a = e - e_s \qquad e_s = R_s i_a \qquad e_u = Ke_a$

$$i_a = \frac{e_u - e_b}{R_a + R_s} \qquad e_b = K_b\frac{d\theta_y}{dt} \qquad T_m = K_i i_a = \left(J_m + J_L\right)\frac{d^2\theta_y}{dt^2}$$

Solve for i_a in terms of θ_y and $\dfrac{\mathrm{d}\theta_y}{dt}$, we have

$$i_a = \frac{KK_s\left(\theta_r - \theta_y\right) - K_b\dfrac{d\theta_y}{dt}}{R_s + R_s + KR_s}$$

Differential equation:

$$\frac{d^2\theta_y}{dt^2} = \frac{K_i i_a}{J_m + J_L} = \frac{K_i}{(J_m + J_L)(R_a + R_s + KR_s)}\left(-K_b \frac{d\theta_y}{dt} - KK_s\theta_y + KK_s\theta_y\right)$$

State variables: $x_1 = \theta_y, \quad x_2 = \dfrac{d\theta_y}{dt}$

State equations:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-KK_sK_i}{(J_m + J_L)(R_a + R_s + KR_s)} & \dfrac{-K_bK_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{-KK_sK_i}{(J_m + J_L)(R_a + R_s + KR_s)} \end{bmatrix}\theta_r$$

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix}\theta_r$$

We can let $v(t) = 322.58\theta_r$, then the state equations are in the form of CCF.

(b)

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 322.58 & s + 80.65 \end{bmatrix}^{-1} = \frac{1}{s^2 + 80.65s + 322.58}\begin{bmatrix} s + 80.65 & 1 \\ -322.58 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-0.06}{s + 76.42} - \dfrac{1.059}{s + 4.22} & \dfrac{-0.014}{s + 76.42} + \dfrac{0.014}{s + 4.22} \\ \dfrac{4.468}{s + 76.42} - \dfrac{4.468}{s + 4.22} & \dfrac{1.0622}{s + 76.42} - \dfrac{0.0587}{s + 4.22} \end{bmatrix}$$

For a unit-step function input, $u_s(t) = 1/s$.

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\frac{1}{s} = \begin{bmatrix} \dfrac{322.2}{s(s + 76.42)(s + 4.22)} \\ \dfrac{322.2}{s(s + 76.42)(s + 4.22)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s} + \dfrac{0.0584}{s + 76.42} - \dfrac{1.058}{s + 4.22} \\ \dfrac{-4.479}{s + 76.42} + \dfrac{4.479}{s + 4.22} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} -0.06e^{-76.42t} - 1.059e^{-4.22t} & -0.014e^{-76.42t} + 0.01e^{-4.22t} \\ 4.468e^{-76.42t} - 4.468e^{-4.22t} & 1.0622e^{-76.42t} - 0.0587e^{-4.22t} \end{bmatrix}\mathbf{x}(0)$$

$$= \begin{bmatrix} 1 + 0.0584e^{-76.42t} - 1.058e^{-4.22t} \\ -4.479e^{-76.42t} + 4.479e^{-4.22t} \end{bmatrix} \qquad t \geq 0$$

(c) Characteristic equation: $\Delta(s) = s^2 + 80.65s + 322.58 = 0$

(d) From the state equations we see that whenever there is R_a there is $(1+K)R_s$. Thus, the purpose of R_s is to increase the effective value of R_a by $(1+K)R_s$. This improves the time constant of the system.

8-23) (a) State equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-KK_sK_i}{J(R+R_s+KR_s)} & \frac{-K_bK_i}{J(R+R_s+KR_s)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{KK_sK_i}{J(R+R_s+KR_s)} \end{bmatrix} \theta_r$$

$$= \begin{bmatrix} 0 & 1 \\ -818.18 & -90.91 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 818.18 \end{bmatrix} \theta_r$$

Let $v = 818.18\theta_r$. The equations are in the form of CCF with *v* as the input.

(b) $$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 818.18 & s+90.91 \end{bmatrix}^{-1} = \frac{1}{(s+10.128)(s+80.782)} \begin{bmatrix} s+90.91 & 1 \\ -818.18 & s \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 1.143e^{-10.128t} - 0.142e^{-80.78t} & 0.01415e^{-10.128t} - 0.0141e^{-80.78t} \\ -11.58e^{-10.128t} + 0.1433e^{-80.78t} & -0.1433e^{-10.128t} + 1.143e^{-80.78t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+ \begin{bmatrix} 11.58e^{-10.128t} - 11.58e^{-80.78t} \\ 1 - 1.1434e^{-10.128t} + 0.1433e^{-80.78t} \end{bmatrix} \qquad t \geq 0$$

(c) Characteristic equation: $\Delta(s) = s^2 + 90.91s + 818.18 = 0$

Eigenvalues: –10.128, –80.782

(d) Same remark as in part (d) of Problem 5-14.

8-24) If $\dot{x} = Ax$ and P diagonalizing A, let consider $x = P\hat{x}$, therefore $\dot{x} = P\dot{\hat{x}}$ or $\dot{\hat{x}} = P^{-1}AP\hat{x} = D\hat{x}$

The solution for $\hat{x}$ is $\hat{x} = e^{Dt}\hat{x}(0)$, therefore

$$x(t) = P\hat{x}(t) = Pe^{Dt}P^{-1}x(0) \qquad (1)$$

on the other hand

$$x(t) = e^{At}x(0) \qquad (2)$$

From equation (1) and (2):

$$e^{At} = Pe^{Dt}P^{-1}$$

8-25) Consider $\dot{x} = Ax$ and $s^{-1}As = J$. If $x = S\hat{x}$, then $\dot{x} = S\dot{\hat{x}}$ or $\dot{\hat{x}} = s^{-1}As\hat{x} = J\hat{x}$

The solution for x̂ is $\hat{x}(t) = e^{Jt}\hat{x}(0)$, therefore:

$$x(t) = s\hat{x}(t) = se^{Jt}s^{-1}x(0) \qquad (1)$$

On the other hand:

$$x(t) = e^{At}x(0) \qquad (2)$$

From equation (1) and (2):

$$e^{At} = se^{Jt}s^{-1}$$

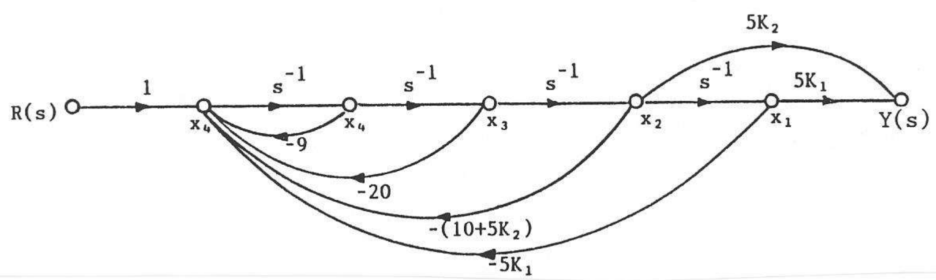
8-26 (a) Forward-path transfer function:

$$G(s) = \frac{Y(s)}{E(s)} = \frac{5(K_1 + K_2 s)}{s[s(s+4)(s+5)+10]}$$

Closed-loop transfer function:

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{5(K_1 + K_2 s)}{s^4 + 9s^3 + 20s^2 + (10 + 5K_2)s + 5K_1}$$

(b) State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5K & -(10+5K_2) & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r$$

Output equation:

$$y = \begin{bmatrix} 5K_1 & 5K_2 & 0 \end{bmatrix} \mathbf{x}$$

(c) Final value: $r(t) = u_s(t), \quad R(s) = \frac{1}{s}.$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{5\left(K_1 + K_2 s\right)}{s^4 + 9s^3 + 20s^2 + \left(10 + 5K_2\right)s + 5K_1} = 1$$

8-27 In CCF form,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 \\ 0 & s & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & s + a_n \end{bmatrix}$$

$$\left|s\mathbf{I} - \mathbf{A}\right| = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$$

Since **B** has only one nonzero element which is in the last row, only the last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)$ is going to contribute to $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)B$. The last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)$ is obtained from the cofactors of the last row of $\left(s\mathbf{I} - \mathbf{A}\right)$. Thus, the last column of $\mathrm{adj}\left(s\mathbf{I} - \mathbf{A}\right)B$ is $\begin{bmatrix} 1 & s & s^2 & \cdots & s^{n-1} \end{bmatrix}'$.

8-28 (a) State variables: $x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad x_3 = \frac{d^2y}{dt^2}$

State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) State transition matrix:

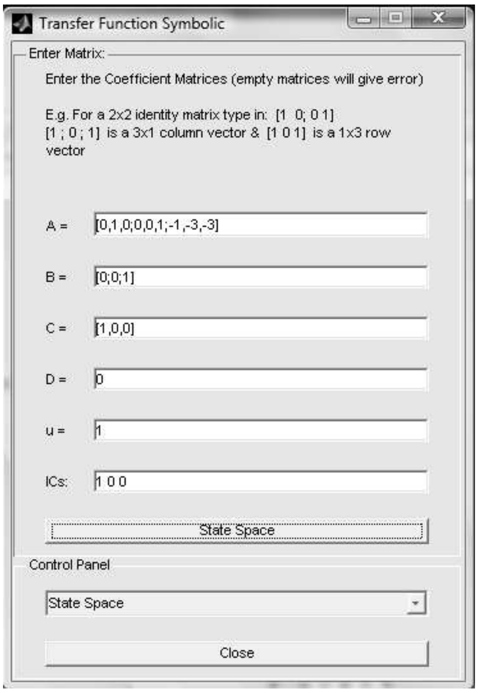
$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 3 & s+3 \end{bmatrix}^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s^2+3s+3 & s+3 & 1 \\ -1 & s^2+3s & s \\ -s & -3s-1 & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{1}{(s+1)^3} & \dfrac{1}{(s+1)^2} + \dfrac{2}{(s+1)^3} & \dfrac{1}{(s+1)^3} \\ \dfrac{-1}{(s+1)^3} & \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} - \dfrac{2}{(s+1)^3} & \dfrac{s}{(s+1)^3} \\ \dfrac{-s}{(s+1)^3} & \dfrac{-3}{(s+1)^2} + \dfrac{2}{(s+1)^3} & \dfrac{s^2}{(s+1)^3} \end{bmatrix}$$

$$\Delta(s) = s^3 + 3s^2 + 3s + 1 = (s+1)^3$$

$$\phi(t) = \begin{bmatrix} \left(1+t+t^2/2\right)e^{-t} & \left(t+t^2\right)e^{-t} & t^2e^{-t}/2 \\ -t^2e^{-t}/2 & \left(1+t-t^2\right)e^{-t} & \left(t-t^2/2\right)e^{-t} \\ \left(-t+t^2/2\right)e^{-t} & t^2e^{-t} & \left(1-2t+t^2/2\right)e^{-t} \end{bmatrix}$$

(c) Use ACSYS or MATLAB and follow the procedure shown in solution to 8-3.




```
--------------------------------------------------------------
      State Space Analysis
--------------------------------------------------------------
Inputs:
A=| 0  1  0|     B=|0|
  | 0  0  1|       |0|
  |-1 -3 -3|       |1|

C=|1  0  0|     D=|0|

State Space Representation:
     | 0  1  0|    |0|
Dx = | 0  0  1|x + |0|u
     |-1 -3 -3|    |1|

 y = |1  0  0|x + |0|u

Determinant of (s*I-A):

                    3     2
                   s  + 3 s  + 3 s + 1
Characteristic Equation of the Transfer Function:

                    3     2
```

```
              s  + 3 s  + 3 s + 1
```

The eigen values of A and poles of the Transfer Function are:

```
                   -1
                   -1
                   -1
```

Inverse of (s*I-A) is:

```
        [ 2                        ]
        [s  + 3 s + 3     s + 3      1  ]
        [------------     -----      ----]
        [    %1           %1        %1 ]
        [                          ]
        [     1        s (s + 3)     s  ]
        [   - ----      ---------    ----]
        [     %1          %1        %1 ]
        [                          ]
        [                         2 ]
        [     s         3 s + 1     s  ]
        [   - ----      - -------    ----]
        [     %1          %1        %1 ]

                  3     2
              %1 := s  + 3 s  + 3 s + 1
```

State transition matrix (phi) of A:

```
   [                    2         2               2       ]
   [1/2 exp(-t) (2 + 2 t + t ) , (t + t ) exp(-t) , 1/2 t  exp(-t)]

   [      2                  2                           2
   [- 1/2 t  exp(-t) , -(-t - 1 + t ) exp(-t) , - 1/2 exp(-t) (-2 t + t )

   ]
   ]

   [                 2                  2
   [1/2 exp(-t) (-2 t + t ) , exp(-t) (-3 t + t ) ,

                    2 ]
   1/2 exp(-t) (2 - 4 t + t )]
```

Transfer function between u(t)and y(t) is:

```
                   1
              -------------------
               3     2
              s  + 3 s  + 3 s + 1
```

No Initial Conditions Specified

States (X) in Laplace Domain:

```
[   1    ]
[--------]
[       3]
[(s + 1) ]
[        ]
[   s    ]
[--------]
[       3]
[(s + 1) ]
[        ]
[    2   ]
[   s    ]
[--------]
[       3]
[(s + 1) ]
```

Inverse Laplace x(t):

```
[           2              ]
[      1/2 t  exp(-t)      ]
[                          ]
[                        2 ]
[- 1/2 exp(-t) (-2 t + t ) ]
[                          ]
[                        2 ]
[1/2 exp(-t) (2 - 4 t + t )]
```

Output Y(s):

```
    1
 --------
       3
 (s + 1)
```

Inverse Laplace y(t):

```
     2
1/2 t  exp(-t)
```

```
-------------------------------------------------------------
      State Space Analysis
-------------------------------------------------------------
```

Inputs:

```
A=| 0  1  0|     B=|0|
  | 0  0  1|       |0|
  |-1 -3 -3|       |1|

C=|1  0  0|     D=|0|
```

State Space Representation:

$$Dx = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{vmatrix} x + \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} u$$

$$y = |1 \;\; 0 \;\; 0| x + |0| u$$

Determinant of (s*I-A):

$$s^3 + 3 s^2 + 3 s + 1$$

Characteristic Equation of the Transfer Function:

$$s^3 + 3 s^2 + 3 s + 1$$

The eigen values of A and poles of the Transfer Function are:

$$-1$$
$$-1$$
$$-1$$

Inverse of (s*I-A) is:

$$\begin{bmatrix} \dfrac{s^2 + 3 s + 3}{\%1} & \dfrac{s + 3}{\%1} & \dfrac{1}{\%1} \\ -\dfrac{1}{\%1} & \dfrac{s (s + 3)}{\%1} & \dfrac{s}{\%1} \\ -\dfrac{s}{\%1} & -\dfrac{3 s + 1}{\%1} & \dfrac{s^2}{\%1} \end{bmatrix}$$

$$\%1 := s^3 + 3 s^2 + 3 s + 1$$

State transition matrix (phi) of A:

$$[1/2 \exp(-t)\, (2 + 2 t + t^2),\ (t + t^2) \exp(-t),\ 1/2\, t^2 \exp(-t)]$$

$$[- 1/2\, t^2 \exp(-t),\ -(t - 1 + t^2) \exp(-t),\ - 1/2 \exp(-t)\, (-2 t + t^2)$$

]
]

```
[                  2                  2
[1/2 exp(-t) (-2 t + t ) , exp(-t) (-3 t + t ) ,

                  2 ]
1/2 exp(-t) (2 - 4 t + t )]
```

Transfer function between u(t)and y(t) is:

```
              1
     -------------------
      3      2
     s  + 3 s  + 3 s + 1
```

Initial Conditions:

```
     x(0)= 1
           0
           0
```

States (X) in Laplace Domain:

```
     [ 2          ]
     [s  + 3 s + 4]
     [------------]
     [        3   ]
     [  (s + 1)   ]
     [            ]
     [   s - 1    ]
     [  --------  ]
     [        3   ]
     [  (s + 1)   ]
     [            ]
     [ s (s - 1)  ]
     [ ---------  ]
     [       3    ]
     [ (s + 1)    ]
```

Inverse Laplace x(t):

```
     [          2          ]
     [ (t + 1 + t ) exp(-t)  ]
     [                     ]
     [         2           ]
     [  -(-t + t ) exp(-t)   ]
     [                     ]
     [            2        ]
     [(-3 t + 1 + t ) exp(-t)]
```

Output Y(s) (with initial conditions):

```
      2
     s  + 3 s + 4
     ------------
```

```
           3
     (s + 1)
Inverse Laplace y(t):

          2
    (t + 1 + t ) exp(-t)
```

(d) Characteristic equation: $\Delta(s) = s^3 + 3s^2 + 3s + 1 = 0$

Eigenvalues: $-1,\ -1,\ -1$

8-29 (a) State variables: $x_1 = y, \quad x_2 = \dfrac{dy}{dt}$

State equations:

$$\begin{bmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t)$$

State transition matrix:

$$\Phi(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{1}{(s+1)^2} \\ \dfrac{-1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Characteristic equation: $\Delta(s) = (s+1)^2 = 0$

(b) State variables: $x_1 = y, \quad x_2 = y + \dfrac{dy}{dt}$

State equations:

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2 - y = x_2 - x_1 \qquad \frac{dx_2}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt} = -y - \frac{dy}{dt} + r = -x_2 + r$$

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

State transition matrix:

$$\Phi(s) = \begin{bmatrix} s+1 & -2 \\ 0 & s+1 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{-2}{(s+1)^2} \\ 0 & \dfrac{1}{s+1} \end{bmatrix} \qquad \phi(t) = \begin{bmatrix} e^{-t} & -te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

(c) Characteristic equation: $\Delta(s) = (s+1)^2 = 0$ which is the same as in part (a).

8-30 (a) State transition matrix:

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix} \qquad (s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s-\sigma & -\omega \\ \omega & s-\sigma \end{bmatrix} \qquad \Delta(s) = s^2 - 2\sigma + (\sigma^2 + \omega^2)$$

$$\phi(t) = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right] = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} e^{\sigma t}$$

(b) Eigenvalues of A: $\sigma + j\omega, \quad \sigma - j\omega$

8-31 (a)

$$\frac{Y_1(s)}{U_1(s)} = \frac{s^{-3}}{1+s^{-1}+2s^{-2}+3s^{-3}} = \frac{1}{s^3+s^2+2s+3}$$

$$\frac{Y_2(s)}{U_2(s)} = \frac{s^{-3}}{1+s^{-1}+2s^{-2}+3s^{-3}} = \frac{1}{s^3+s^2+2s+3} = \frac{Y_1(s)}{U_1(s)}$$

(b) State equations [Fig. 5-21(a)]: $\dot{\mathbf{x}} = \mathbf{A}_1\mathbf{x} + \mathbf{B}_1 u_1$ **Output equation:** $\mathbf{y}_1 = \mathbf{C}_1\mathbf{x}$

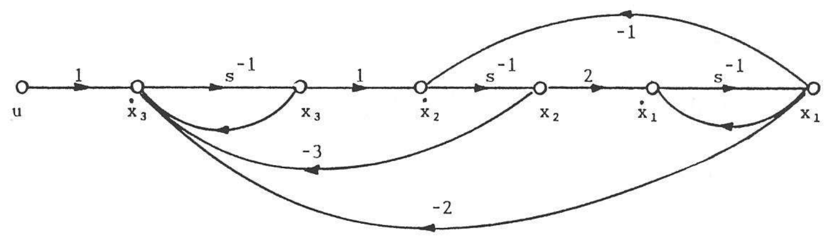
$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -1 \end{bmatrix} \qquad \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

State equations [Fig. 5-21(b)]: $\dot{\mathbf{x}} = \mathbf{A}_2\mathbf{x} + \mathbf{B}_2 u_2$ **Output equation:** $y_2 = \mathbf{C}_2\mathbf{x}$

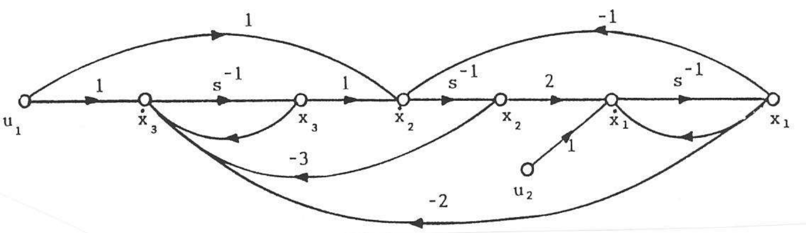
$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathbf{A}_2 = \mathbf{A}_1'$

8-32 (a) State diagram:



(b) State diagram:

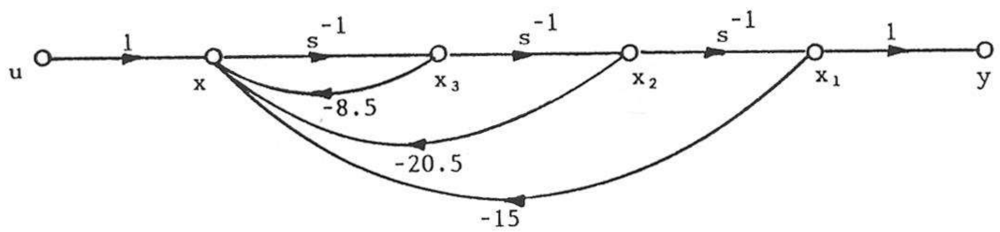


8-33 (a)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{10s^{-3}}{1+8.5s^{-1}+20.5s^{-2}+15s^{-3}}\frac{X(s)}{X(s)} \qquad Y(s)=10X(s)$$

$$X(s)=U(s)-8.5s^{-1}X(s)-20.5s^{-2}X(s)-15s^{-3}X(s)$$

State diagram:



State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -15 & -20.5 & -8.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

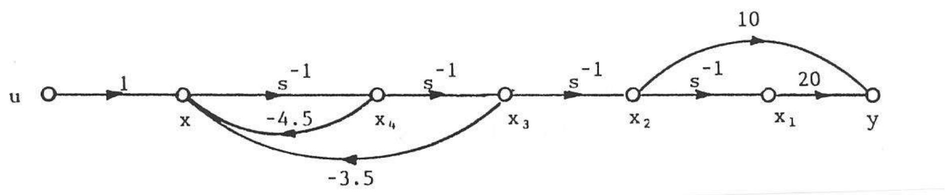
A and **B** are in CCF

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-3} + 20s^{-4}}{1 + 4.5s^{-1} + 3.5s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 10s^{-3}X(s) + 20s^{-4}X(s) \qquad X(s) = -4.5s^{-1}X(s) - 3.5s^{-2}X(s) + U(s)$$

State diagram:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

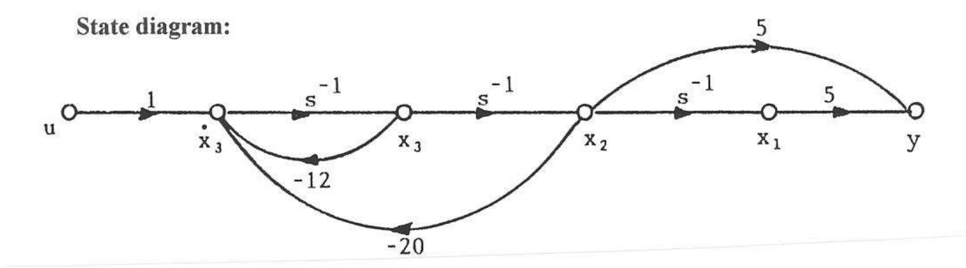
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3.5 & -4.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A and **B** are in CCF

(c)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{5s^{-2} + 5s^{-3}}{1 + 12s^{-1} + 20s^{-2}} \frac{X(s)}{X(s)}$$

$$Y(s) = 5s^{-2}X(s) + 5s^{-3}X(s) \qquad X(s) = U(s) - 12s^{-1}X(s) - 20s^{-2}X(s)$$



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

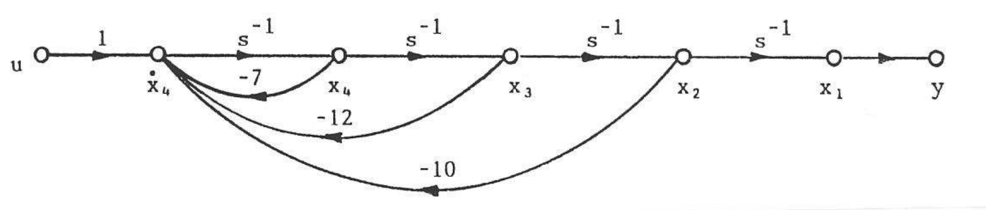
A and **B** are in CCF

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)} = \frac{s^{-4}}{1+7s^{-1}+12s^{-2}+10s^{-3}} \frac{X(s)}{X(s)}$$

$$Y(s) = s^{-4}X(s) \qquad X(s) = U(s) - 7s^{-1}X(s) - 12s^{-2}X(s) - 10s^{-3}$$

State diagram:



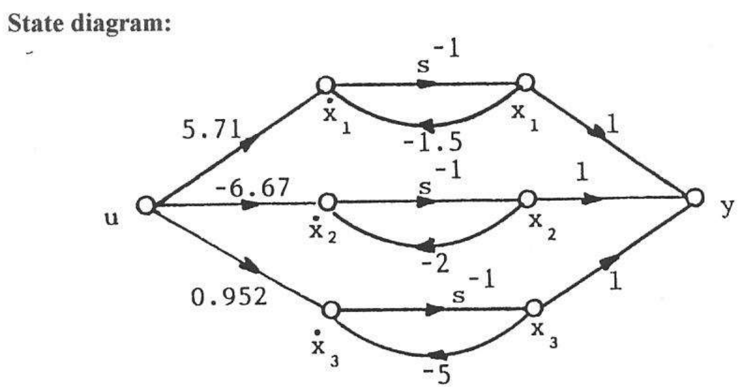
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -10 & -12 & -7 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A and **B** are in CCF

8-34 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^3 + 8.5s^2 + 20.5s + 15} = \frac{5.71}{s+15} - \frac{6.67}{s+2} + \frac{0.952}{s+5}$$



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

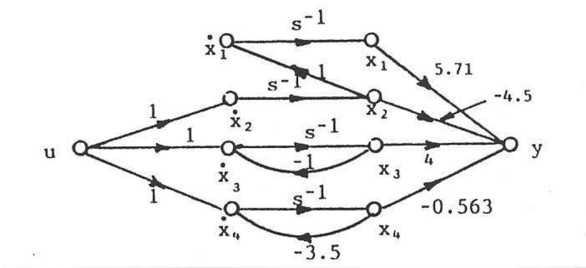
$$\mathbf{A} = \begin{bmatrix} -1.5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5.71 \\ -6.67 \\ 0.952 \end{bmatrix}$$

The matrix **B** is not unique. It depends on how the input and the output branches are allocated.

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10(s+2)}{s^2(s=1)(s+3.5)} = \frac{-4.5}{s} + \frac{0.49}{s+3.5} + \frac{4}{s+1} + \frac{5.71}{s^2}$$

State diagram:

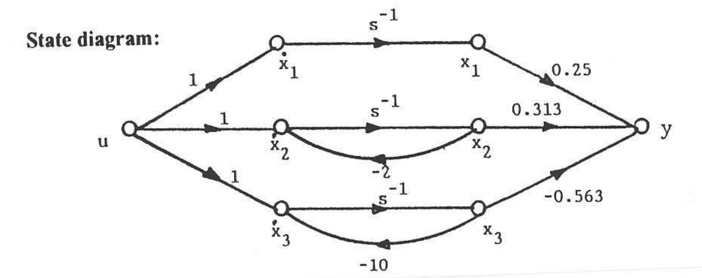


State equation: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

(b)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+1)}{s(s+2)(s+10)} = \frac{2.5}{s} + \frac{0.313}{s+2} - \frac{0.563}{s+10}$$

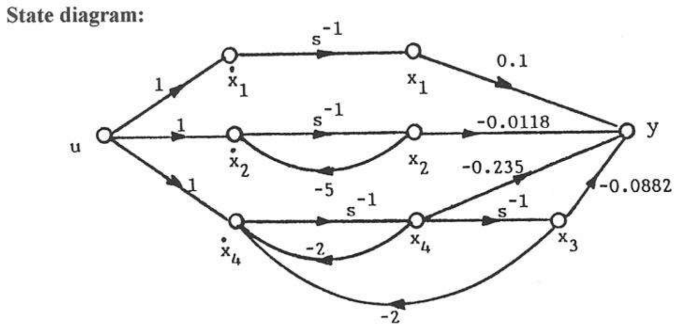


State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)\left(s^2+2s+2\right)} = \frac{0.1}{s} - \frac{0.0118}{s+5} - \frac{0.0882s + 0.235}{s^2+2s+2}$$



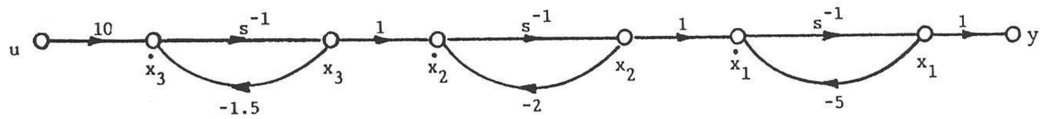
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

8-35 (a)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1.5)(s+2)(s+5)}$$

State diagram:



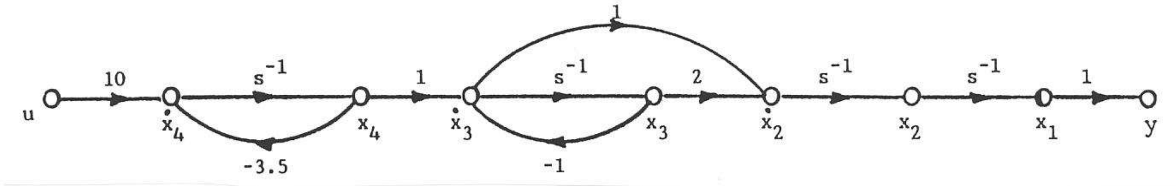
State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1.5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

(b)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{10(s+2)}{s^2(s+1)(s+3.5)}=\left(\frac{10}{s^2}\right)\left(\frac{s+2}{s+1}\right)\left(\frac{1}{s+3.5}\right)$$

State diagram:



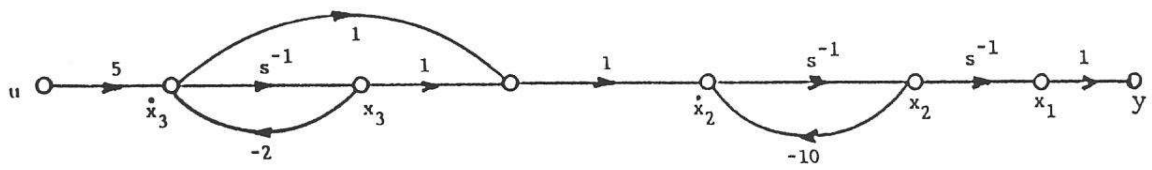
State equations: $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)$

$$\mathbf{A}=\begin{bmatrix}0&1&0&0\\0&0&1&1\\0&0&-1&1\\0&0&0&-3.5\end{bmatrix}\qquad \mathbf{B}=\begin{bmatrix}0\\0\\0\\10\end{bmatrix}$$

(c)

$$G(s)=\frac{Y(s)}{U(s)}=\frac{59s+1)}{s(s+2)(s+10)}=\left(\frac{5}{s}\right)\left(\frac{s+1}{s+2}\right)\left(\frac{1}{s+10}\right)$$

State diagram:



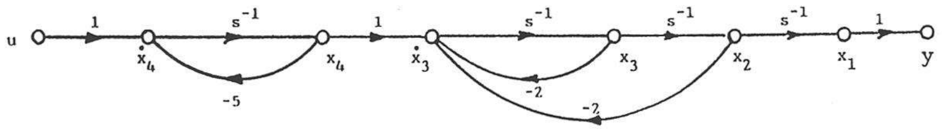
State equations: $\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}u(t)$

$$\mathbf{A}=\begin{bmatrix}0&1&0\\0&-10&-1\\0&0&-2\end{bmatrix}\qquad \mathbf{B}=\begin{bmatrix}0\\0\\5\end{bmatrix}$$

(d)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+5)(s^2+2s+2)}$$

State diagram:

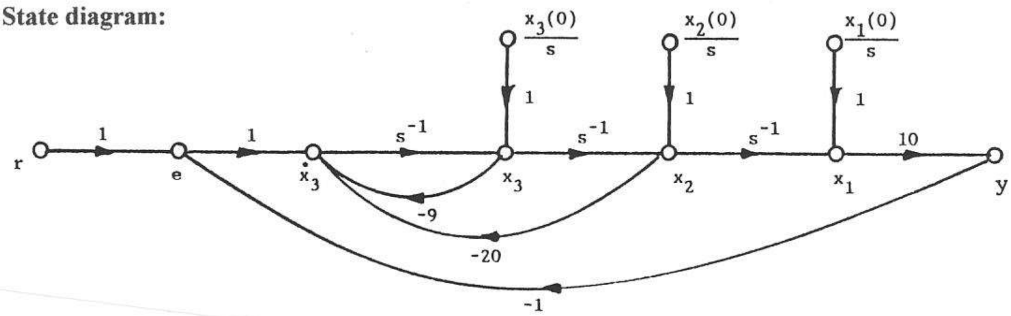


State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

8-36 (a)

$$G(s) = \frac{Y(s)}{E(s)} = \frac{10}{s(s+4)(s+5)} = \frac{10s^{-3}}{1+9s^{-1}+20s^{-2}} \frac{X(s)}{X(s)}$$



(b) Dynamic equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r \qquad y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \mathbf{x}$$

(c) State transition equation:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix} = \frac{1}{\Delta(s)} \begin{bmatrix} s^{-1}\left(1+9s^{-1}+20s^{-2}\right) & s^{-2}\left(1+9s^{-1}\right) & s^{-3} \\ -10s^{-3} & s^{-1}\left(1+9s^{-1}\right) & s^{-2} \\ -10s^{-2} & -20s^{-2} & s^{-1} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta(s)} \begin{bmatrix} s^{-3} \\ s^{-2} \\ s^{-1} \end{bmatrix} \frac{1}{s}$$

$$= \frac{1}{\Delta_c(s)} \begin{bmatrix} s^2+9s+20 & s+9 & 1 \\ -10 & s(s+9) & s \\ -10s & -10(2s+1) & s^2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \frac{1}{\Delta_c(s)} \begin{bmatrix} \frac{1}{s} \\ 1 \\ s \end{bmatrix}$$

$$\Delta(s) = 1+9s^{-1}+20s^{-2}+10s^{-3} \qquad \Delta_c(s) = s^3+9s^2+20s+10$$

$$\mathbf{x}(t) = \left\{ \begin{bmatrix} 1.612 & 0.946 & 0.114 \\ -1.14 & -0.669 & -0.081 \\ 0.807 & 0.474 & 0.057 \end{bmatrix} e^{-0.708t} + \begin{bmatrix} -0.706 & -1.117 & -0.169 \\ 1.692 & 2.678 & 4.056 \\ -4.056 & -6.420 & -0.972 \end{bmatrix} e^{-2.397t} + \begin{bmatrix} 0.0935 & 0.171 & 0.055 \\ -0.551 & -1.009 & -0.325 \\ 3.249 & 5.947 & 1.915 \end{bmatrix} e^{-5.895t} \right\} \mathbf{x}(0)$$

$$+ \begin{bmatrix} 0.1-0.161e^{-0.708t}+0.0706e^{-2.397t}-0.00935e^{-5.895t} \\ 0.114e^{-0.708t}-0.169e^{-2.397t}+0.055e^{-5.895t} \\ -0.087e^{-0.708t}+0.406e^{-2.397t}-0.325e^{-5.895t} \end{bmatrix} \qquad t \ge 0$$

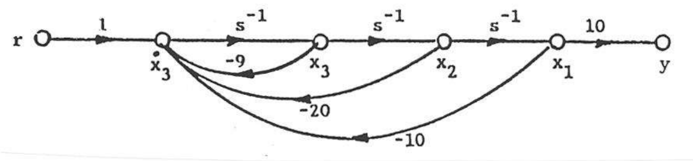
(d) Output:

$$y(t) = 10x_1(t) = 10\left(1.612e^{-0.708t} - 0.706e^{-2.397t} + 0.0935e^{-5.897t}\right)x_1(0) + 10\left(0.946e^{-0.708t} - 1.117e^{-2.397t} + 0.1711e^{-5.895t}\right)x_2(0)$$
$$+10\left(1.141e^{-0.708t} - 0.169e^{-2.397t} + 0.0550e^{-5.895t}\right)x_3(0) + 1 - 1.61e^{-0.708t} + 0.706e^{-2.397t} - 0.0935e^{-5.895t} \qquad t \ge 0$$

8-37(a) Closed-loop transfer function:

$$\frac{Y(s)}{R(s)} = \frac{10}{s^3+9s^2+20s+10}$$

(b) State diagram:



(c) **State equations:**

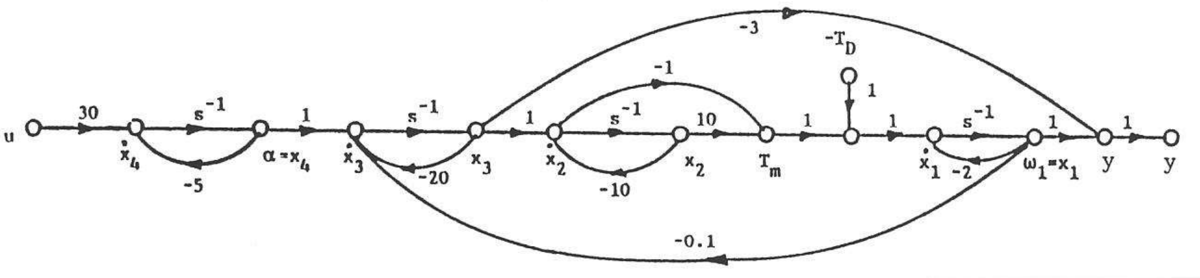
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -20 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

(d) **State transition equations:**

[Same answers as Problem 5-26(d)]

(e) **Output:** [Same answer as Problem 5-26(e)]

8-38 (a) **State diagram:**



(b) **State equations:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 20 & -1 & 0 \\ 0 & -10 & 1 & 0 \\ -0.1 & 0 & -20 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \end{bmatrix} \begin{bmatrix} u \\ T_D \end{bmatrix}$$

(c) **Transfer function relations:**

From the system block diagram,

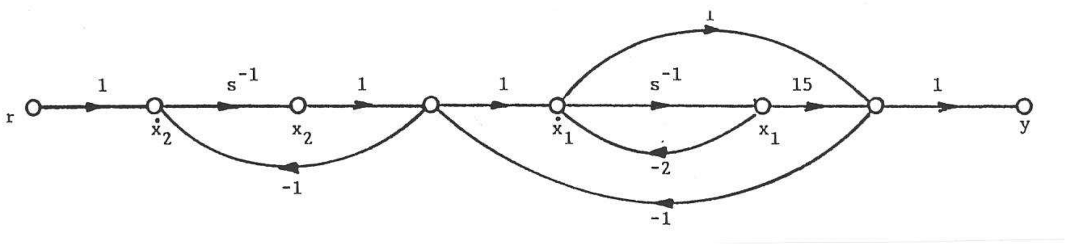
$$Y(s) = \frac{1}{\Delta(s)} \left(\frac{-1}{s+2} T_D(s) + \frac{0.3}{(s+2)(s+20)} T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+2)(s+5)(s+20)} + \frac{90U(s)}{(s+5)(s+20)} \right)$$

$$\Delta(s) = 1 + \frac{0.1e^{-0.2s}}{(s+2)(s+20)} = \frac{(s+2)(s+20) + 0.1e^{-0.2s}}{(s+2)(s+20)}$$

$$Y(s) = \frac{-(s+19.7)}{(s+2)(s+20)+0.1e^{-0.2s}}T_D(s) + \frac{30e^{-0.2s}+90(s+2)U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

$$\Omega(s) = \frac{-(s+20)}{(s+2)(s+20)+0.1e^{-0.2s}}T_D(s) + \frac{30e^{-0.2s}U(s)}{(s+5)\left[(s+2)(s+20)+0.1e^{-0.2s}\right]}$$

8-39 (a) There should not be any incoming branches to a state variable node other than the s^{-1} branch. Thus, we should create a new node as shown in the following state diagram.



(b) State equations: Notice that there is a loop with gain –1 after all the s^{-1} branches are deleted, so Δ = 2.

$$\frac{dx_1}{dt} = \frac{17}{2}x_1 + \frac{1}{2}x_2 \qquad \frac{dx_2}{dt} = \frac{15}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}r$$

Output equation: $y = 6.5x_1 + 0.5x_2$

8-40 (a) Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{Ks^2+5s+1}{(s+1)(s^2+11s+2)}$$

(b) Characteristic equation:

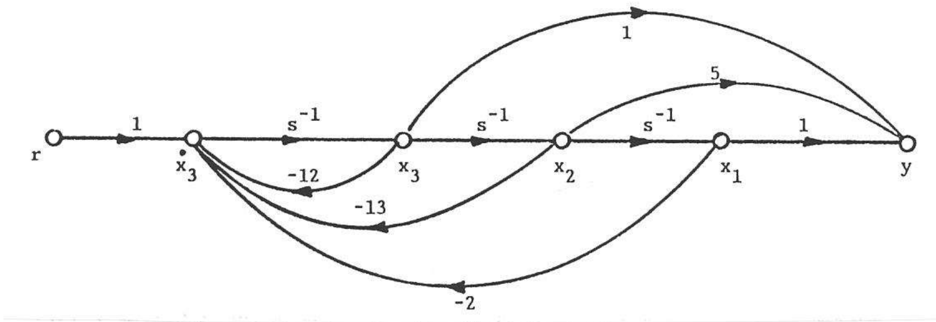
$$(s+1)(s^2+11s+2) = 0$$

Roots of characteristic equation: –1, –0.185, –-10.82. These are not functions of *K*.

(c) When *K* = 1:

$$\frac{Y(s)}{R(s)} = \frac{s^2+5s+1}{s^3+12s^2+13s+2}$$

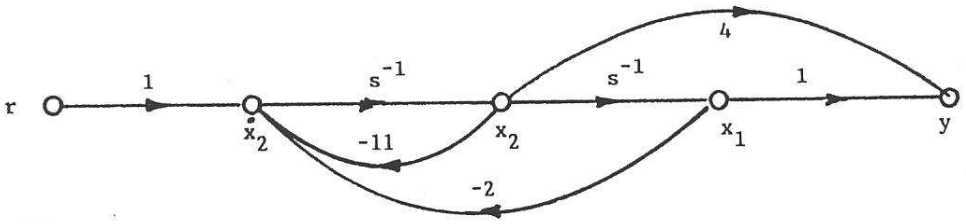
State diagram:



(d) When *K* = 4:

$$\frac{Y(s)}{R(s)}=\frac{4s^2+5s+1}{\left(s+1\right)\left(s^2+11s+2\right)}=\frac{(s+1)(4s+1)}{\left(s+1\right)\left(s^2+11s+2\right)}=\frac{4s+1}{s^2+11s+2}$$

State diagram:



(e)

$$\frac{Y(s)}{R(s)}=\frac{Ks^2+5s+1}{\left(s+1\right)\left(s^2+11s+2\right)} \qquad \left(s+1\right)\left(s^2+11s+2\right)=0$$

MATLAB

```
solve(s^2+11*s+2)
 ans =  -11/2+1/2*113^(1/2)
 -11/2-1/2*113^(1/2)
 >> vpa(ans)
 ans =
   -.20
 -10.8
```

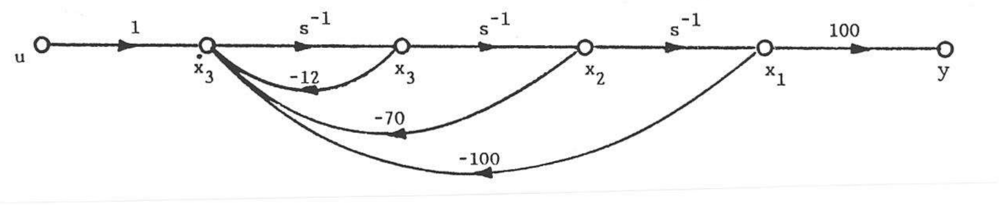
$$\frac{Y(s)}{R(s)}=\frac{Ks^2+5s+1}{(s+1)(s+0.2)(s+10.82)}$$
$$K=4,2.1914,0.4536$$

Pole zero cancelation occurs for the given values of K.

8-41 (a)

$$G_p(s)=\frac{Y(s)}{U(s)}=\frac{1}{(1+0.5s)\left(1+0.2s+0.02s^2\right)}=\frac{100}{s^3+12s^2+70s+100}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-100&-70&-12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

(b) **Characteristic equation of closed-loop system:**

$$s^3+12s^2+70s+200=0$$

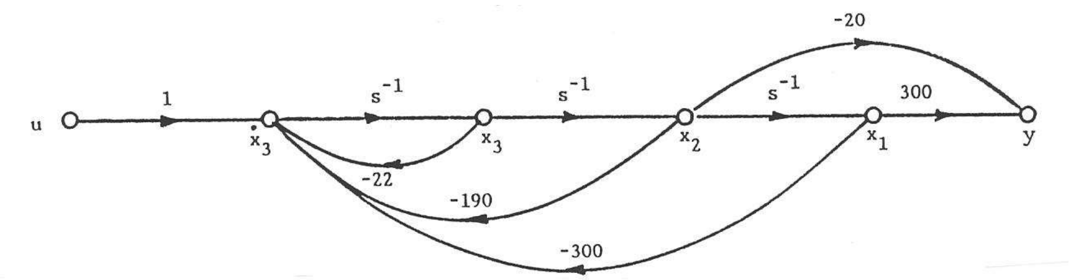
Roots of characteristic equation:

$$-5.88,\quad -3.06+j4.965,\quad -3.06-j4.965$$

8-42 (a)

$$G_p(s)=\frac{Y(s)}{U(s)}\cong\frac{1-0.066s}{(1+0.5s)\left(1+0.133s+0.0067s^2\right)}=\frac{-20(s-15)}{s^3+22s^2+190s+300}$$

State diagram by direct decomposition:



State equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -300 & -190 & -22 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Characteristic equation of closed-loop system:

$$s^3 + 22s^2 + 170s + 600 = 0$$

Roots of characteristic equation:

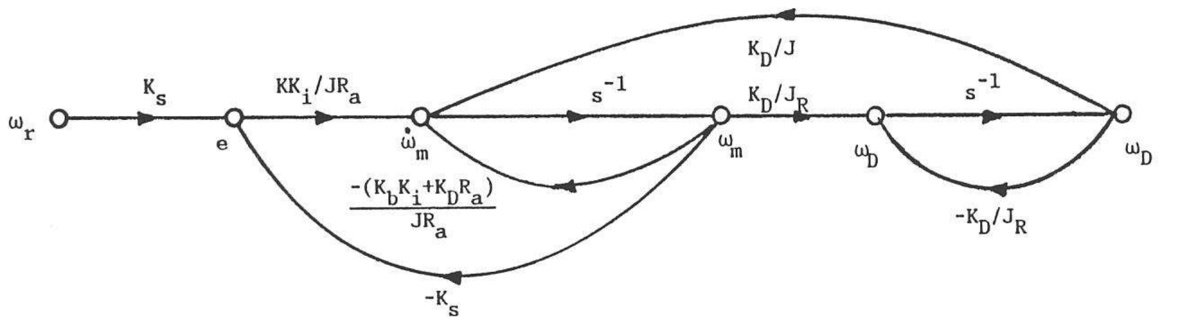
$-12,\ -5 + j5,\ -5 - j5$

8-43 (a) State variables: $x_1 = \omega_m$ and $x_2 = \omega_D$

State equations:

$$\frac{d\omega_m}{dt} = -\frac{K_b K_i + K_b R_a}{JR_a} + \frac{K_D}{J}\omega_D + \frac{KK_i}{JR_a}e \qquad \frac{d\omega_D}{dt} = \frac{K_D}{J_R}\omega_m - \frac{K_D}{J_R}\omega_D$$

(b) State diagram:



(c) Open-loop transfer function:

$$\frac{\Omega_m(s)}{E(s)} = \frac{KK_i\left(J_R s + K_D\right)}{JJ_R R_a s^2 + \left(K_b J_R K_i + K_D R_a J_R + K_D J R_a\right)s + K_D K_b K_i}$$

Closed-loop transfer function:

$$\frac{\Omega_m(s)}{\Omega_r(s)} = \frac{K_s K K_i \left(J_R s + K_D\right)}{J J_R R_a s^2 + \left(K_b J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R\right)s + K_D K_b K_i + K_s K K_i K_D}$$

(d) **Characteristic equation of closed-loop system:**

$$\Delta(s) = J J_R R_a s^2 + \left(K_D J_R K_i + K_D R_a J_R + K_D J R_a + K_s K K_i J_R\right)s + K_D K_b K_i + K_s K K_i K_D = 0$$

$$\Delta(s) = s^2 + 1037s + 20131.2 = 0$$

Characteristic equation roots: -19.8, -1017.2

8-44 (a) **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} -b & d \\ c & -a \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Since **S** is nonsingular, the system is controllable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & d \\ 1 & -a \end{bmatrix}$$

The system is controllable for $d \neq 0$.

8-45 (a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

S is singular. The system is uncontrollable.

(b)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix}$$

S is nonsingular. The system is controllable.

8-46 (a) **State equations:** $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

S is singular. The system is uncontrollable.

Output equation: $y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} = \mathbf{Cx}$ $\qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

V is nonsingular. The system is observable.

(b) **Transfer function:**

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+2s-3} = \frac{1}{s-1}$$

Since there is pole-zero cancellation in the input-output transfer function, the system is either uncontrollable or unobservable or both. In this case, the state variables are already defined, and the system is uncontrollable as found out in part (a).

8-47 (a) $\alpha = 1, 2, \text{or } 4$. These values of α will cause pole-zero cancellation in the transfer function.

(b) The transfer function is expanded by partial fraction expansion,

$$\frac{Y(s)}{R(s)} = \frac{\alpha-1}{3(s+1)} - \frac{\alpha-2}{2(s+2)} + \frac{\alpha-4}{6(s+4)}$$

By parallel decomposition, the state equations are: $\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{B}r(t)$, output equation: $y(t) = \mathbf{Cx}(t)$.

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \alpha-1 \\ \alpha-2 \\ \alpha-4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

The system is uncontrollable for α = 1, or α = 2, or α = 4.

(c) Define the state variables so that

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{2} \\ \frac{-1}{6} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \alpha-1 & \alpha-2 & \alpha-4 \end{bmatrix}$$

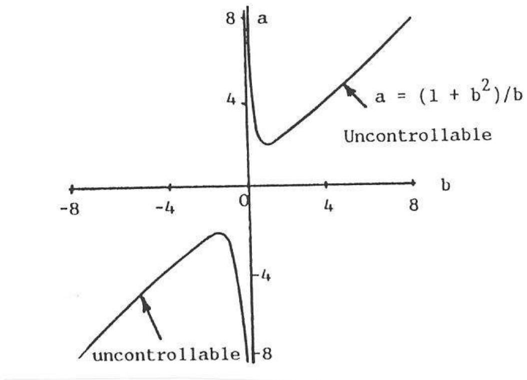
The system is unobservable for α = 1, or α = 2, or α = 4.

8-48

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & b \\ b & ab-1 \end{bmatrix} \qquad |\mathbf{S}| = ab - 1 - b^2 \neq 0$$

The boundary of the region of controllability is described by $ab-1-b^2 = 0.$

Regions of controllability:



8-49

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} b_1 & b_1 + b_2 \\ b_2 & b_2 \end{bmatrix} \qquad |\mathbf{S}| = 0 \text{ when } b_1 b_2 - b_1 b_2 - b_2^2 = 0, \text{ or } b_2 = 0$$

The system is completely controllable when $b_2 \neq 0.$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} d_1 & d_2 \\ d_2 & d_1 + d_2 \end{bmatrix} \qquad |\mathbf{V}| = 0 \text{ when } d_1 \neq 0.$$

The system is completely observable when $d_2 \neq 0.$

8-50 (a) State equations:

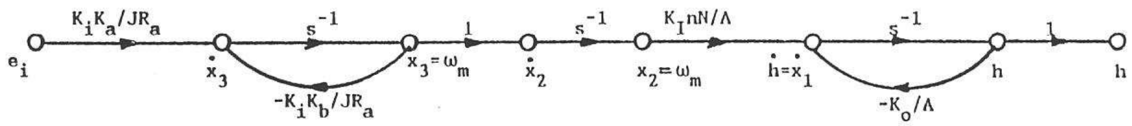
$$\frac{dh}{dt} = \frac{1}{A}\left(q_i - q_o\right) = \frac{K_I nN}{A}\theta_m - \frac{K_o}{A}h \qquad \frac{d\theta_m}{dt} = \omega_m \qquad \frac{d\omega_m}{dt} = -\frac{K_i K_b}{JR_a}\omega_m + \frac{K_i K_a}{JR_a}e_i \qquad J = J_m + n^2 J_L$$

State variable: $x_1 = h, \quad x_2 = \theta_m, \quad x_3 = \frac{d\theta_m}{dt} = \omega_m$

State equations: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}e_i$

$$\mathbf{A} = \begin{bmatrix} \frac{-K_o}{A} & \frac{K_I nN}{A} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{K_i K_b}{JR_a} \end{bmatrix} = \begin{bmatrix} -1 & 0.016 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -11.767 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{K_i K_a}{JR_a} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8333.33 \end{bmatrix}$$

State diagram:



(b) Characteristic equation of A:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s + \frac{K_o}{A} & \frac{-K_I nN}{A} & 0 \\ 0 & s & -1 \\ 0 & 0 & s + \frac{K_i K_b}{JR_a} \end{vmatrix} = s\left(s + \frac{K_o}{A}\right)\left(s + \frac{K_i K_b}{JR_a}\right) = s(s+1)(s+11.767)$$

Eigenvalues of A: 0, −1, −11.767.

(c) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 133.33 \\ 0 & 8333.33 & -98058 \\ 8333.33 & -98058 & 1153848 \end{bmatrix}$$ $|\mathbf{S}| \neq 0.$ The system is controllable.

(d) Observability:

(1) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0.016 & -0.016 \\ 0 & 0 & 0.016 \end{bmatrix}$$ **V** is nonsingular. The system is observable.

(2) $\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -11.767 \end{bmatrix}$$ **V** is singular. The system is unobservable.

(3) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$**:**

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' & (\mathbf{A}')^2\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -11.767 & 138.46 \end{bmatrix}$$ **V** is singular. The system is unobservable.

8-51 (a) Characteristic equation: $\Delta(s) = \left| s\mathbf{I} - \mathbf{A}^* \right| = s^4 - 25.92s^2 = 0$

Roots of characteristic equation: −5.0912, 5.0912, 0, 0

(b) Controllability:

$$\mathbf{S} = \begin{bmatrix} \mathbf{B}^* & \mathbf{A}^*\mathbf{B}^* & \mathbf{A}^{*2}\mathbf{B}^* & \mathbf{A}^{*3}\mathbf{B}^* \end{bmatrix} = \begin{bmatrix} 0 & -0.0732 & 0 & -1.8973 \\ -0.0732 & 0 & -1.8973 & 0 \\ 0 & 0.0976 & 0 & 0.1728 \\ 0.0976 & 0 & 0.1728 & 0 \end{bmatrix}$$

S is nonsingular. Thus, $\left[\mathbf{A}^*, \quad \mathbf{B}^* \right]$ is controllable.

(c) Observability:

(1) $\mathbf{C}^* = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 25.92 & 0 \\ 0 & 1 & 0 & 25.92 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

(2) $\mathbf{C}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & 25.92 & 0 & 671.85 \\ 1 & 0 & 25.92 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

(3) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & -2.36 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

S is nonsingular. The system is observable.

(4) $\mathbf{C}^* = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}^{*'} & \mathbf{A}^{*'}\mathbf{C}^{*'} & (\mathbf{A}^{*'})^2\mathbf{C}^{*'} & (\mathbf{A}^{*'})^3\mathbf{C}^{*'} \end{bmatrix} = \begin{bmatrix} 0 & -2.36 & 0 & -61.17 \\ 0 & 0 & -2.36 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

S is singular. The system is unobservable.

8-52 The controllability matrix is

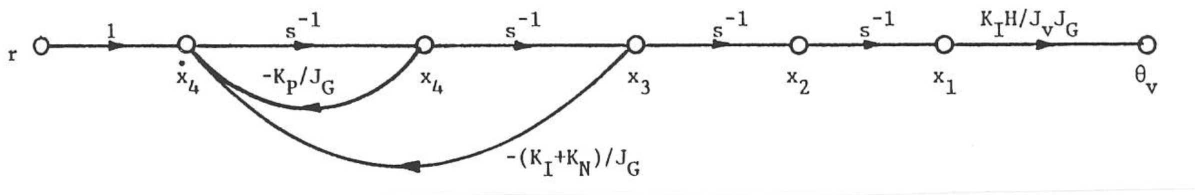
$$\mathbf{S} = \begin{bmatrix} 0 & -1 & 0 & -16 & 0 & -384 \\ -1 & 0 & -16 & 0 & -384 & 0 \\ 0 & 0 & 0 & 16 & 0 & 512 \\ 0 & 0 & 16 & 0 & 512 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of **S** is 6. The system is controllable.

8-53 (a) Transfer function:

$$\frac{\Theta_v(s)}{R(s)} = \frac{K_I H}{J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right)}$$

State diagram by direct decomposition:



State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}r(t)$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{-(K_I + K_N)}{J_G} & \dfrac{-K_P}{J_G} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Characteristic equation: $J_v s^2 \left(J_G s^2 + K_P s + K_I + K_N \right) = 0$

8-54 (a) State equations: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

S is nonsingular. $\begin{bmatrix} \mathbf{A}, & \mathbf{B} \end{bmatrix}$ is controllable.

Output equation: $y_2 = \mathbf{C}\mathbf{x}$ $\qquad \mathbf{C} = \begin{bmatrix} -1 & 1 \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -3 \end{bmatrix}$$

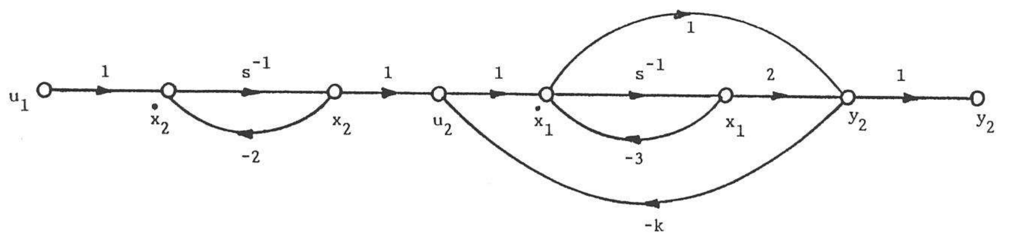
V is singular. The system is unobservable.

(b) With feedback, $u_2 = -kc_2$, the state equation is: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u_1(t)$.

$$\mathbf{A} = \begin{bmatrix} \dfrac{-3-2k}{1+g} & \dfrac{1}{1+k} \\ 0 & -2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{S} = \begin{bmatrix} 0 & \dfrac{1}{1+k} \\ 1 & -2 \end{bmatrix}$$

S is nonsingular for all finite values of k. The system is controllable.

State diagram:



Output equation: $y_2 = \mathbf{C}\mathbf{x}$ $\qquad \mathbf{C} = \begin{bmatrix} \dfrac{-1}{1+k} & \dfrac{1}{1+k} \end{bmatrix}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{D}' & \mathbf{A}'\mathbf{D}' \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{1+K} & \dfrac{3+2k}{(1+k)^2} \\ \dfrac{1}{1+k} & -\dfrac{3+2k}{(1+k)^2} \end{bmatrix}$$

V is singular for any *k*. The system with feedback is unobservable.

8-55 (a)

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -7 \end{bmatrix}$$

S is nonsingular. System is controllable.

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

V is nonsingular. System is observable.

(b) $u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix}\mathbf{x}$

$$\mathbf{A}_c = \mathbf{A} - \mathbf{BK} = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 2k_1 & 2k_2 \end{bmatrix} = \begin{bmatrix} -k_1 & 1-k_2 \\ -1-2k_1 & -3-2k_2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{A}_c\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & -k_1 - 2k_2 + 2 \\ 2 & -7 - 2k_1 - 4k_2 \end{bmatrix} \qquad |\mathbf{S}| = -11 - 2k_2 \neq 0$$

For controllabillity, $k_2 \neq -\dfrac{11}{2}$

$$\mathbf{V} = \begin{bmatrix} \mathbf{C}' & \mathbf{A}_c'\mathbf{C}' \end{bmatrix} = \begin{bmatrix} -1 & -1-3k_1 \\ 1 & -2-3k_2 \end{bmatrix}$$

For observability, $|\mathbf{V}| = -1 + 3k_1 - 3k_2 \neq 0$

8-56

Same as 8-21 (a)

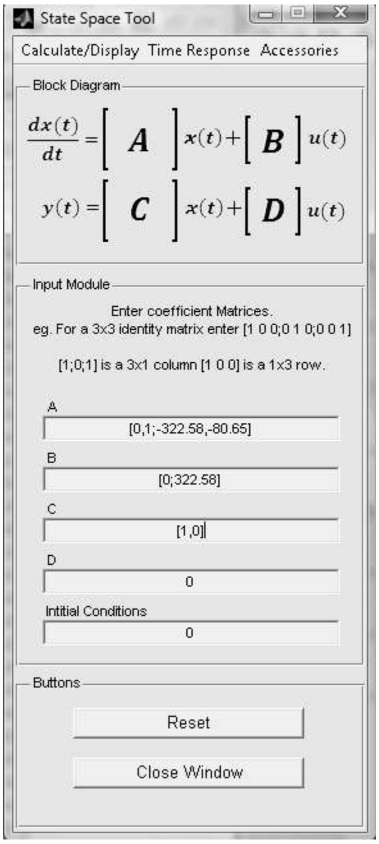
8-57

$$= \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix}\theta_r$$

From 8-22 $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 322.58 \end{bmatrix} \theta_r$

$$A = \begin{bmatrix} 0 & 1 \\ -322.58 & -80.65 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 322.58 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
$$D = 0$$

Use the state space tool of ACSYS



The A matrix is:

Amat =

0 1.0000

-322.5800 -80.6500

Characteristic Polynomial:

ans =

s^2+1613/20*s+16129/50

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-4.2206 0

0 -76.4294

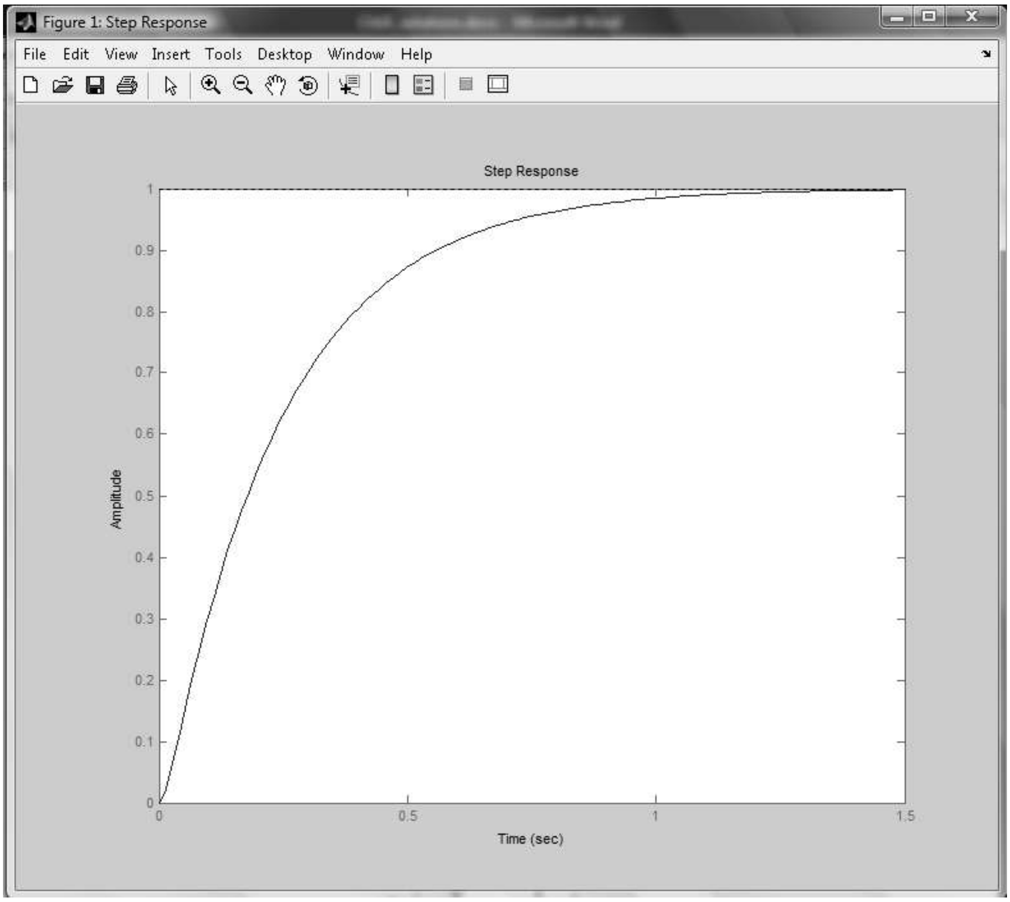
Eigen Vectors are

T =

0.2305 -0.0131

-0.9731 0.9999

```
State-Space Model is:

a =
        x1     x2
  x1      0      1
  x2  -322.6  -80.65

b =
       u1
  x1     0
  x2  322.6

c =
     x1  x2
  y1   1   0

d =
     u1
  y1   0

Continuous-time model.
 Characteristic Polynomial:

ans =
s^2+1613/20*s+16129/50

 Equivalent Transfer Function Model is:
Transfer function:
      322.6
---------------------
s^2 + 80.65 s + 322.6
 Pole, Zero Form:
Zero/pole/gain:
     322.58
-------------------
(s+76.43) (s+4.221)
```



8-58

Closed-loop System Transfer Function.

$$\frac{Y(s)}{R(s)} = \frac{1}{s^3 + (4+k_3)s^2 + (3+k_2+k_3)s + k_1}$$

For zero steady-state error to a step input, $k_1 = 1$. For the complex roots to be located at $-1 + j$ and $-1 - j$,

we divide the characteristic polynomial by $s^2 + 2s + 2$ and solve for zero remainder.

$$\begin{array}{r|l} & s + (2+k_2) \\ \hline s^2+2s+2 & s^3 + (4+k_3)s^2 + (3+k_2+k_3)s + 1 \\ & s^3 + \quad 2s^2 \quad\quad + 2s \\ \hline & \quad (2+k_3)s^2 + (1+k_2+k_3)s + 1 \end{array}$$

$$\frac{(2+k_3)s^2+(4+2k_3)s \quad +4+2k_3}{(\text{-3+k}_2-k_3)s-3-2k_3}$$

For zero remainder, $-3-2k_3=0$ Thus $k_3=-1.5$

$-3+k_2-k_3=0$ Thus $k_2=1.5$

The third root is at –0.5. Not all the roots can be arbitrarily assigned, due to the requirement on the steady-state error.

8-59 (a) Open-loop Transfer Function.

$$G(s)=\frac{X_1(s)}{E(s)}=\frac{k_3}{s\left[s^2+(4+k_2)s+3+k_1+k_2\right]}$$

Since the system is type 1, the steady-state error due to a step input is zero for all values of $k_1, k_2,$ and k_3 that correspond to a stable system. The characteristic equation of the closed-loop system is

$$s^3+(4+k_2)s^2+(3+k_1+k_2)s+k_3=0$$

For the roots to be at $-1+j$, $-1-j$, and -10, the equation should be:

$$s^3+12s^2+22s+20=0$$

Equating like coefficients of the last two equations, we have

$4+k_2=12$ Thus $k_2=8$

$3+k_1+k_2=22$ Thus $k_1=11$

$k_3=20$ Thus $k_3=20$

(b) Open-loop Transfer Function.

$$\frac{Y(s)}{E(s)} = \frac{G_c(s)}{(s+1)(s+3)} = \frac{20}{s\left(s^2+12s+22\right)} \qquad \text{Thus} \qquad G_c(s) = \frac{20(s+1)(s+3)}{s\left(s^2+12s+22\right)}$$

8-60 (a)

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.92 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.36 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B}^* = \begin{bmatrix} 0 \\ -0.0732 \\ 0 \\ 0.0976 \end{bmatrix}$$

The feedback gains, from k_1 to k_4:

–2.4071E+03 –4.3631E+02 –8.4852E+01 –1.0182E+02

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–1.5028E+02	–3.1938E+01	–6.2112E+00	–7.4534E+00
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
2/3258E+02	4.2584E+01	8.2816E+00	9.9379E+00

The **B** vector

0.0000E+00

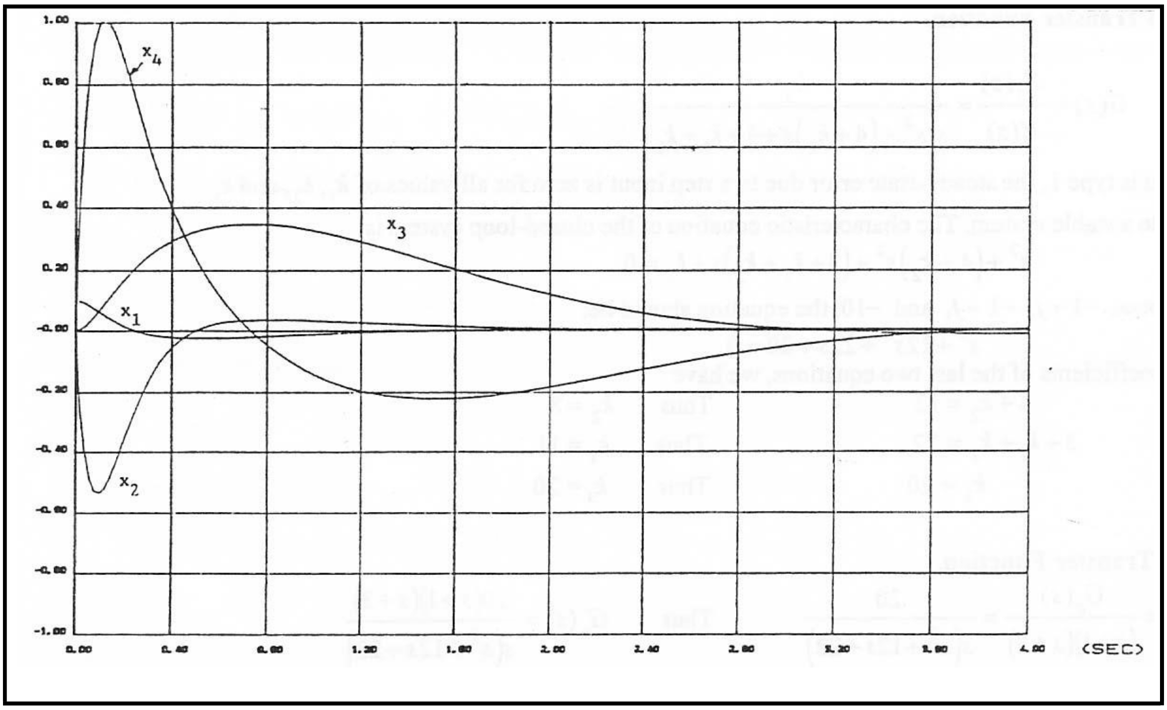
–7.3200E–02

0.0000E+00

9.7600E–02

Time Responses:

8-60
(b)



The feedback gains, from k_1 to k_2:

–9.9238E+03 –1.6872E+03 –1.3576E+03 –8.1458E+02

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–7.0051E+02	–1.2350E+02	–9.9379E+01	–5.9627E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
9.6621E+02	1.6467E+02	1.3251E+02	7.9503E+01

The **B** vector

0.0000E+00

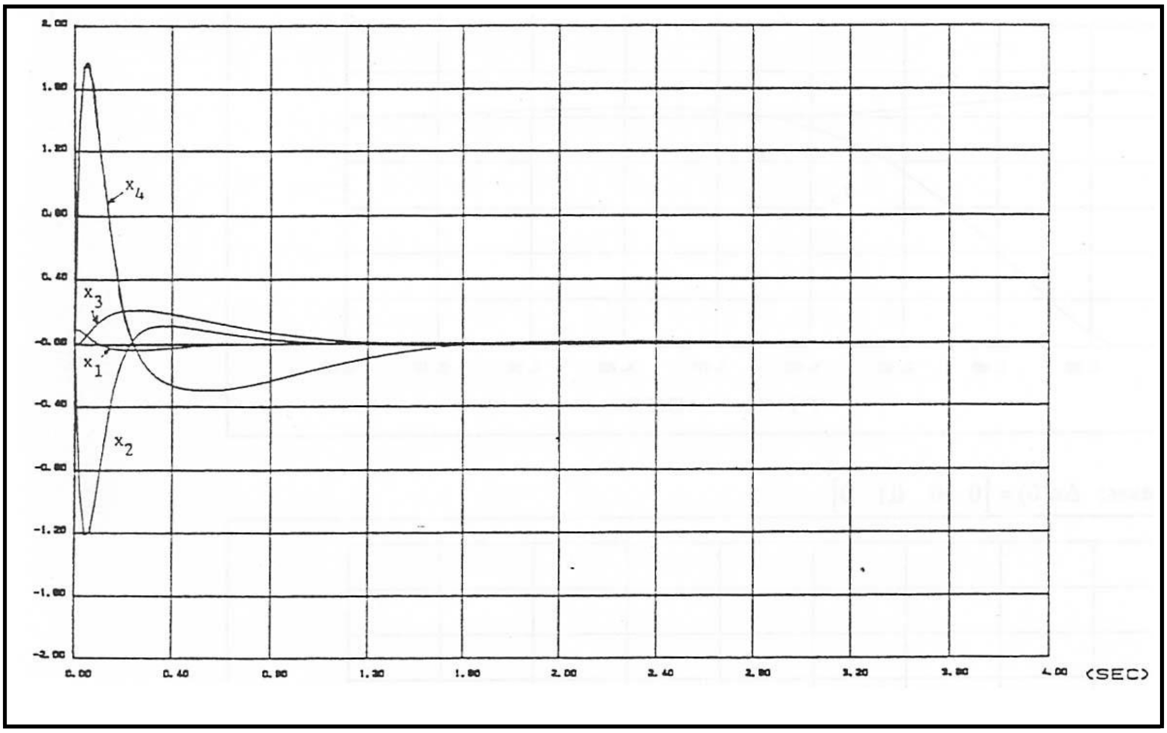
–7.3200E–02

0.0000E+00

9.7600E–02

Time Responses:

8-61
The



solutions using MATLAB

(a) The feedback gains, from k_1 to k_2:

–6.4840E+01 –5.6067E+00 2.0341E+01 2.2708E+00

The $\mathbf{A}^* - \mathbf{B}^*\mathbf{K}$ matrix of the closed-loop system

0.0000E+00	1.0000E+00	0.0000E+00	0.0000E+00
–3.0950E+02	–3.6774E+01	1.1463E+02	1.4874E+01
0.0000E+00	0.0000E+00	0.0000E+00	1.0000E+00
–4.6190E+02	–3.6724E+01	1.7043E+02	1.477eE+01

The **B** vector

0.0000E+00

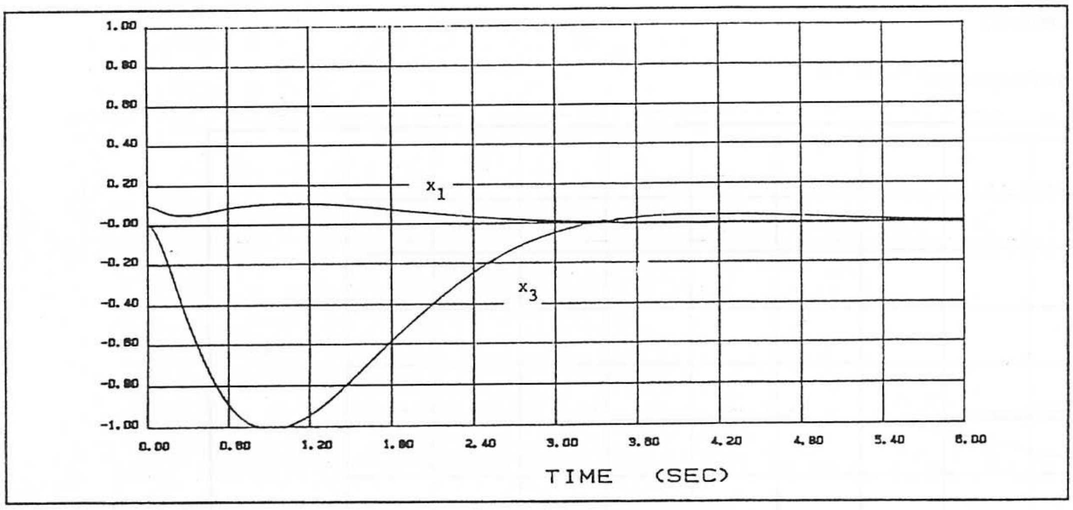
–6.5500E+00

0.0000E+00

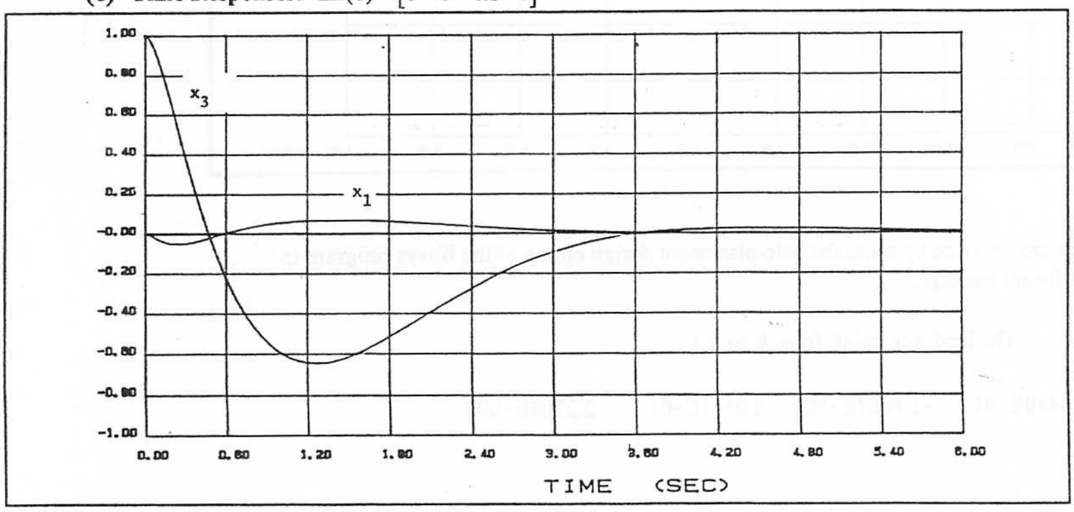
–6.5500E+00

(b) Time Responses: $\Delta\mathbf{x}(0) = \begin{bmatrix} 0.1 & 0 & 0 & 0 \end{bmatrix}'$

With the initial states



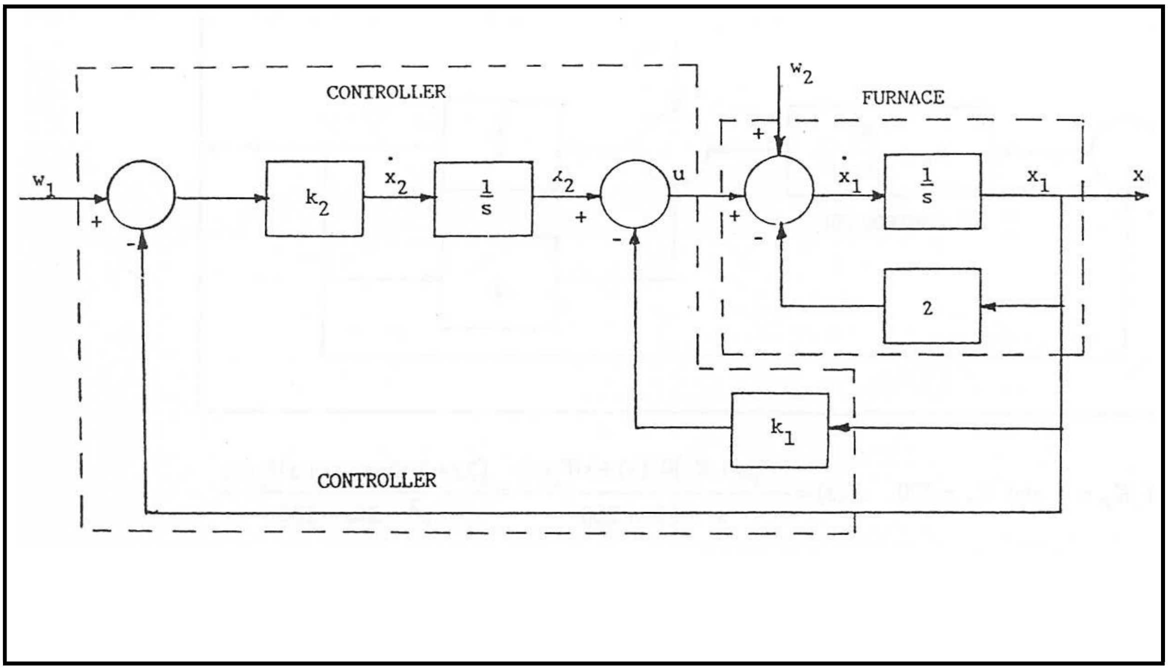
(c) Time Responses: $\Delta\mathbf{x}(0) = [0 \quad 0 \quad 0.1 \quad 0]'$



$\Delta\mathbf{x}(0) = [0.1 \quad 0 \quad 0 \quad 0]'$, the initial position of Δx_1 or Δy_1 is preturbed downward from its stable equilibrium position. The steel ball is initially pulled toward the magnet, so $\Delta x_3 = \Delta y_2$ is negative at first. Finally, the feedback control pulls both bodies back to the equilibrium position.

With the initial states $\Delta\mathbf{x}(0) = [0 \quad 0 \quad 0.1 \quad 0]'$, the initial position of Δx_3 or Δy_2 is preturbed downward from its stable equilibrium. For $t > 0$, the ball is going to be attracted up by the magnet toward the equilibrium position. The magnet will initially be attracted toward the fixed iron plate, and then settles to the stable equilibrium position. Since the steel ball has a small mass, it will move more actively.

8-62 (a) Block Diagram of System.



$u = -k_1 x_1 + k_2 \int \left(-x_1 + w_1\right) dt$

State Equations of Closed-loop System:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -2-k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ k_2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Characteristic Equation:

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+2+k_1 & -1 \\ k_2 & s \end{vmatrix} = s^2 + \left(2+k_1\right)s + k_2 = 0$$

For $s = -10, -10$, $|s\mathbf{I} - \mathbf{A}| = s^2 + 20s + 200 = 0$ Thus $k_1 = 18$ and $k_2 = 200$

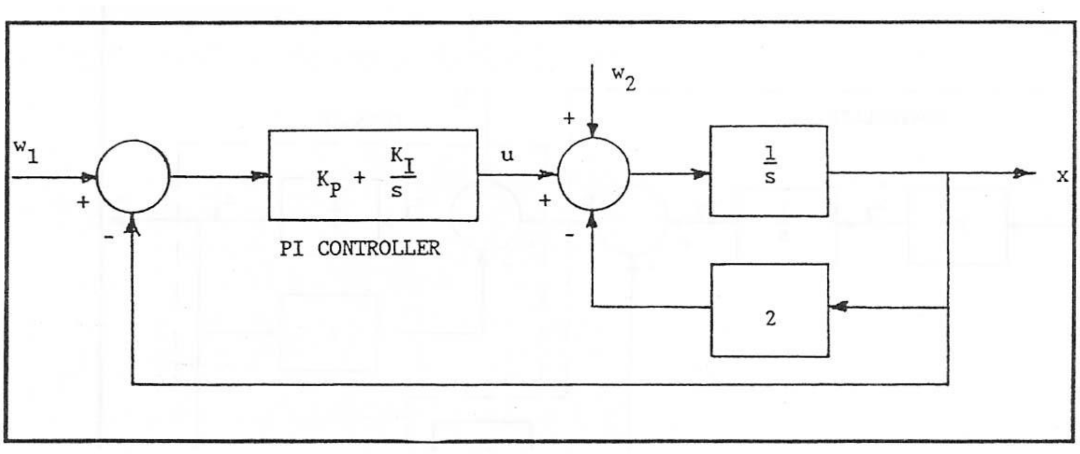
$$X(s) = X_1(s) = \frac{200W_1(s)s^{-2} + s^{-1}W_2(s)}{1 + 2s^{-1} + 18s^{-1} + 200s^{-2}} = \frac{200W_1(s) + sW_2(s)}{s^2 + 20s + 200}$$

$$W_1(s) = \frac{1}{s} \qquad W_2(s) = \frac{W_2}{s} \qquad W_2 = \text{ constant}$$

$$X(s) = \frac{200 + W_2 s}{s\left(s^2 + 20s + 200\right)} \qquad \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = 1$$

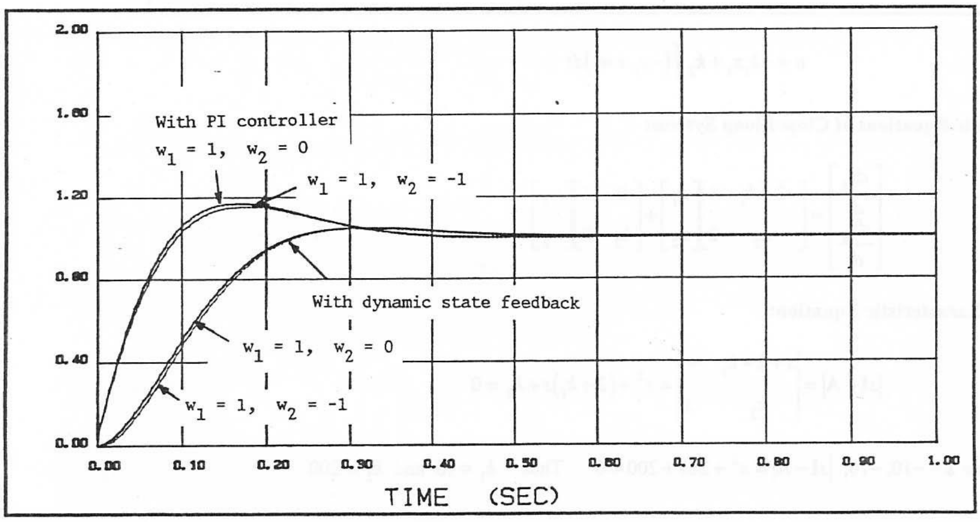
8-62 (b) **With PI Controller:**

Block Diagram of System:



Set $K_P = 2$ and $K_I = 200$. $X(s) = \dfrac{\left(K_P s + K_I\right)W_1(s) + sW_2(s)}{s^2 + 20s + 200} = \dfrac{\left(2s + 200\right)W_1(s) + sW_2(s)}{s^2 + 20s + 200}$

Time Responses:


2.00
1.60
1.20
0.80
0.40
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
TIME (SEC)
With PI controller
w1 = 1, w2 = 0
w1 = 1, w2 = -1
With dynamic state feedback
w1 = 1, w2 = 0
w1 = 1, w2 = -1

8-63)

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)} = \frac{10}{s^3+6s^2+11s+6}$$

Consider:

$$\begin{cases} Y(s) = s^{-3}X(s) \\ X(s) = 10U(s) - (6s^{-1} + 11s^{-2} + 6s^{-3})X(s) \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

As a result:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = [0]$$

Using MATLAB, we'll find:

$$K = \begin{bmatrix} 15.4 & 4.5 & 0.8 \end{bmatrix}$$

8-64)

Inverted Pendulum on a cart

The equations of motion from Problem 4-21 are obtained (by ignoring all the pendulum inertia term):

$$(M+m)\ddot{x} - ml\ddot{\theta}\cos\theta + ml\dot{\theta}^2\sin\theta = f$$
$$ml(-g\sin\theta - \ddot{x}\cos\theta + l\ddot{\theta}) = 0$$

These equations are nonlinear, but they can be linearized. Hence

$$\theta \approx 0$$
$$\cos\theta \approx 1$$
$$\sin\theta \approx \theta$$

$$(M+m)\ddot{x} + ml\ddot{\theta} = f$$
$$ml(-g\theta - \ddot{x} + l\ddot{\theta}) = 0$$

Or

$$\begin{bmatrix} (M+m) & ml \\ -ml & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} f \\ mlg\theta \end{bmatrix}$$

Pre-multiply by inverse of the coefficient matrix

inv([(M+m),m*l;-m*l,m*l^2])

ans =

[1/(M+2*m), -1/l/(M+2*m)]

[1/l/(M+2*m), (M+m)/m/l^2/(M+2*m)]

For values of M=2, m=0.5, l=1, g=9.8

ans =

0.3333 -0.3333

0.3333 1.6667

Hence

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 5/3 \end{bmatrix} \begin{bmatrix} f \\ 49/10\theta \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \text{1/3*f-49/30}\theta \\ \text{1/3*f+49/6}\theta \end{bmatrix}$$

The state space model is:

$$\begin{bmatrix} \dot{x}_4 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \text{1/3*f-49/30}x_1 \\ \text{1/3*f+49/6}x_1 \end{bmatrix}$$

Or:

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 1/3*\text{f}+49/6x_1 \\ 1/3*\text{f}-49/30x_1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix} f$$

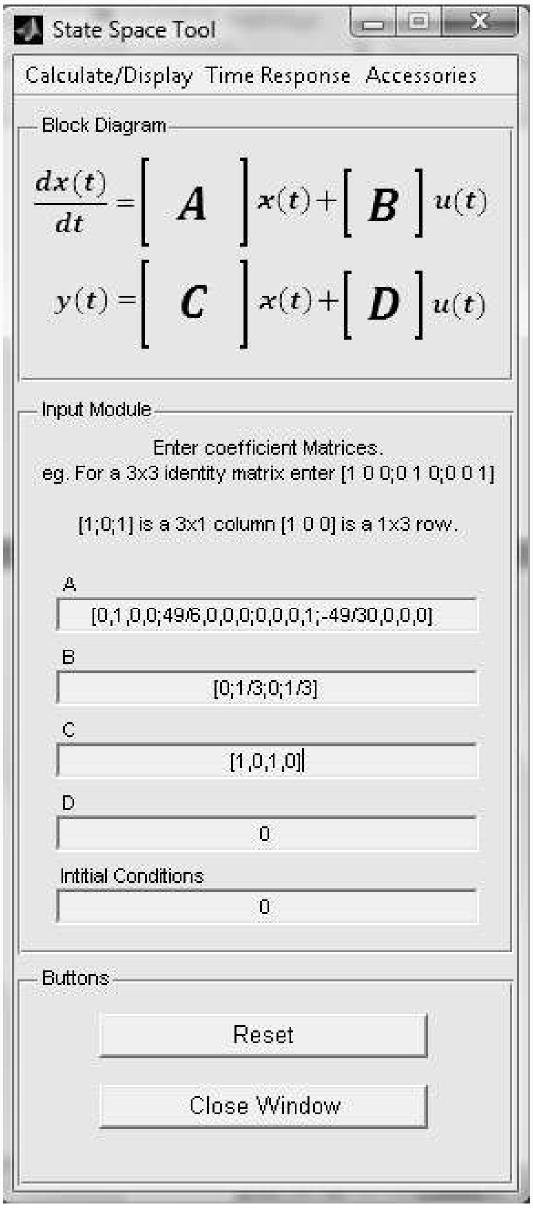
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 49/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -49/30 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1/3 \\ 0 \\ 1/3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

$$D = 0$$

Use ACSYS State tool and follow the design process stated in Example 8-17-1:



The A matrix is:

Amat =

```
       0    1.0000         0         0
   8.1667         0         0         0
       0         0         0    1.0000
  -1.6333         0         0         0
```

Characteristic Polynomial:

ans =

s^4-49/6*s^2

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

```
       0         0         0         0
       0         0         0         0
       0         0    2.8577         0
       0         0         0   -2.8577
```

Eigen Vectors are

T =

```
       0         0    0.3239   -0.3239
       0         0    0.9256    0.9256
   1.0000   -1.0000   -0.0648    0.0648
       0    0.0000   -0.1851   -0.1851
```

State-Space Model is:

a =

```
         x1      x2      x3      x4
  x1      0       1       0       0
  x2  8.167       0       0       0
  x3      0       0       0       1
  x4 -1.633       0       0       0
```

b =

```
         u1
  x1      0
  x2 0.3333
```

```
  x3       0
  x4  0.3333

c =
      x1  x2  x3  x4
  y1   1   0   1   0

d =
      u1
  y1   0
```

Continuous-time model.
Characteristic Polynomial:

ans =

s^4-49/6*s^2

Equivalent Transfer Function Model is:

Transfer function:

```
4.441e-016 s^3 + 0.6667 s^2 - 2.22e-016 s - 3.267
-------------------------------------------------
              s^4 - 8.167 s^2
```

Pole, Zero Form:

Zero/pole/gain:

```
4.4409e-016 (s+1.501e015) (s+2.214) (s-2.214)
---------------------------------------------
         s^2 (s-2.858) (s+2.858)
```

The Controllability Matrix [B AB A^2B ...] is =

Smat =

```
        0    0.3333         0    2.7222
   0.3333         0    2.7222         0
        0    0.3333         0   -0.5444
   0.3333         0   -0.5444         0
```

The system is therefore Not Controllable, rank of S Matrix is =

rankS =

4

Mmat =

```
        0   -8.1667         0    1.0000
  -8.1667         0    1.0000         0
        0    1.0000         0         0
   1.0000         0         0         0
```

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

```
        0         0    0.3333         0
        0         0         0    0.3333
  -3.2667         0    0.3333         0
        0   -3.2667         0    0.3333
```

The transformed matrices using CCF are:

Abar =

```
        0    1.0000         0         0
        0         0    1.0000         0
        0         0         0    1.0000
        0         0    8.1667         0
```

Bbar =

```
     0
     0
     0
     1
```

Cbar =

```
  -3.2667         0    0.6667         0
```

Dbar =

```
     0
```

Note incorporating –**K** in Abar:

Abar K=

```
 0    1.0000        0              0
 0         0   1.0000              0
 0         0        0         1.0000
-k1      -k2   8.1667-k3        -k4
```

System Characteristic equation is:
-k4*s^4+(8.1667-k3)*s^3-k2*s-k1=0

From desired poles we have:

```
>> collect((s-210)*(s-210)*(s+20)*(s-12))

 ans =

 -10584000+s^4-412*s^3+40500*s^2+453600*s
```

Hence: k1=10584000, k2=40500, k3=412+8.1667and k4=1

8-65) If $t_p = 3$ and $\xi = 0.707$, then $\omega_n = 1.414$. The 2nd order desired characteristic equation of the system is:

$$s^2 + 2s + 2 = 0 \qquad (1)$$

On the other hand:

$$\dot{X} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ -6 - K_1 & -5 - K_2 \end{bmatrix} x$$

where the characteristic equation would be:

$$s^2 + (5 + K_2)s + (6 + K_1) = 0 \qquad (2)$$

Comparing equation (1) and (2) gives:

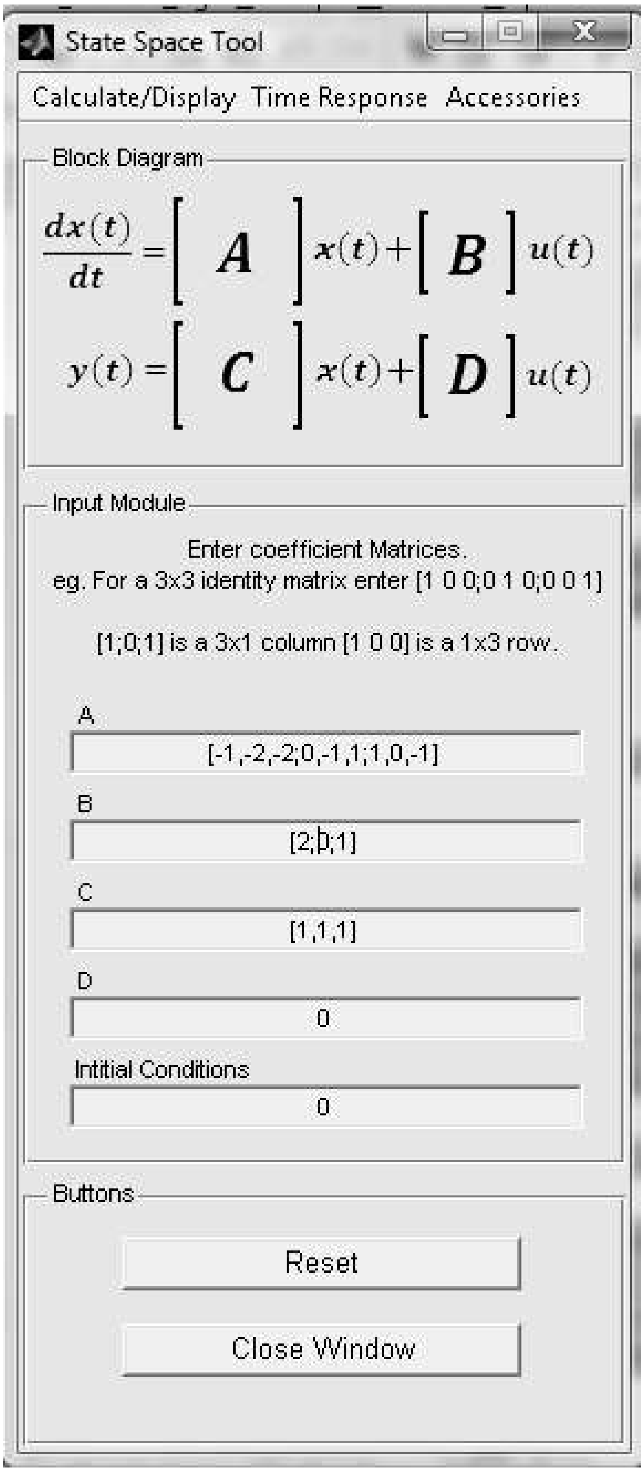
$$\begin{cases} 5 + K_2 = 2 \\ 6 + K_1 = 2 \end{cases}$$

which means $K_1 = -4$ and $K_2 = -3$

8-66) Using ACSYS we can convert the system into transfer function form.

USE ACSYS as illustrated in section 8-19-1

1) Activate MATLAB
2) Go to the folder containing ACSYS
3) Type in Acsys
4) Click the "State Space" pushbutton
5) Enter the A,B,C, and D values. Note C must be entered here and must have the same number of columns as A. We us [1,1] arbitrarily as it will not affect the eigenvalues.
6) Use the "Calculate/Display" menu and find the eigenvalues.
7) Next use the "Calculate/Display" menu and conduct State space calculations.
8) Next verify Controlability and find the $\overline{A}$ matrix
9) Follow the design procedures in section 8-17 (pole placement)


State Space Tool
Calculate/Display Time Response Accessories
Block Diagram
dx(t)/dt = [A] x(t) + [B] u(t)
y(t) = [C] x(t) + [D] u(t)
Input Module
Enter coefficient Matrices.
eg. For a 3x3 identity matrix enter [1 0 0;0 1 0;0 0 1]
[1;0;1] is a 3x1 column [1 0 0] is a 1x3 row.
A
[-1,-2,-2;0,-1,1;1,0,-1]
B
[2;0;1]
C
[1,1,1]
D
0
Intitial Conditions
0
Buttons
Reset
Close Window

The A matrix is:

Amat =

-1	-2	-2
0	-1	1
1	0	-1

Characteristic Polynomial:

ans =

s^3+3*s^2+5*s+5

Eigenvalues of A = Diagonal Canonical Form of A is:

Abar =

-0.6145 + 1.5639i	0	0
0	-0.6145 - 1.5639i	0
0	0	-1.7709

Eigen Vectors are

T =

-0.8074	-0.8074	-0.4259
0.2756 + 0.1446i	0.2756 - 0.1446i	-0.7166
-0.1200 + 0.4867i	-0.1200 - 0.4867i	0.5524

State-Space Model is:

a =

	x1	x2	x3
x1	-1	-2	-2
x2	0	-1	1
x3	1	0	-1

```
b =

      u1

   x1   2

   x2   0

   x3   1

c =

      x1  x2  x3

   y1   1   1   1

d =

      u1

   y1   0
```

Continuous-time model.

Characteristic Polynomial:

ans =

s^3+3*s^2+5*s+5

Equivalent Transfer Function Model is:

Transfer function:

```
  3 s^2 + 7 s + 4
---------------------
s^3 + 3 s^2 + 5 s + 5
```

Pole, Zero Form:

Zero/pole/gain:

```
       3 (s+1.333) (s+1)
---------------------------------
(s+1.771) (s^2  + 1.229s + 2.823)
```

The Controllability Matrix [B AB A^2B ...] is =

Smat =

2	-4	0
0	1	0
1	1	-5

The system is therefore Controllable, rank of S Matrix is =

rankS =

3

Mmat =

5	3	1
3	1	0
1	0	0

The Controllability Canonical Form (CCF) Transformation matrix is:

Ptran =

-2	2	2
3	1	0
3	4	1

The transformed matrices using CCF are:

Abar =

0	1.0000	0
0	0	1.0000
-5.0000	-5.0000	-3.0000

Bbar =

0

0

1

Cbar =

4 7 3

Dbar =

0

Using Equation (8-324) we get:

$$|sI-(A-BK)| = s^3+(3+k_3)s^2+(5+k_2)s+(5+k_1)=0$$

Using a 2nd order prototype system, for $t_s \le 5$, then $\xi\omega_n = 1$. For overshoot of 4.33%, $\xi = 0.707$. Then the desired 2nd order system will have a characteristic equation:

$$s^2+2\zeta\omega_n s+\omega_n{}^2 = s^2+2s+2=0$$

The above system poles are: $s_{1,2} = -1 \pm j$

One approach is to pick K=[k1 k2 k3] values so that two poles of the system are close to the desired second order poles and the third pole reduces the effect of the two system zeros that are at z=-1.333 and z=-1. Let's set the third pole at s=-1.333. Hence

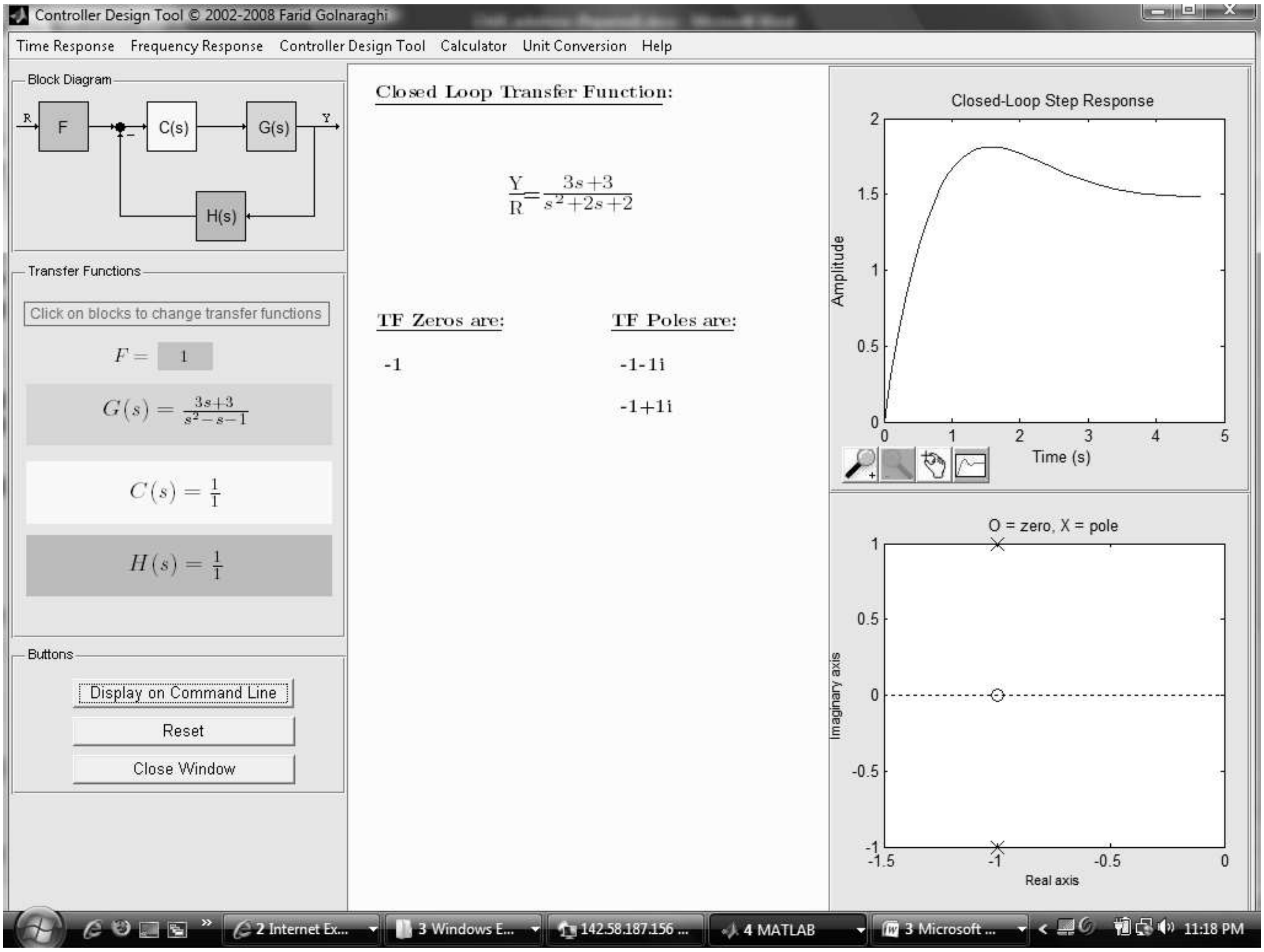
(s+1.333)*(s^2+2*s+2)= s^3+3.33*s^2+4.67*s+2.67

and K=[-2.37 -0.37 0.33].

$$\frac{Y}{R}=\frac{3(s+1)}{s^2+2s+2}$$

Use ACSYS control tool to find the time response. First convert the transfer function to a unity feedback system to make compatible to the format used in the Control toolbox.

$$G=\frac{3(s+1)}{s^2-s-1}$$



Overshoot is about 2%. You can adjust K values to obtain alternative results by repeating this process.

8-67) a) According to the circuit:

$$\frac{Ldi_2}{dt} = R_2 i_2 = v_c + R_1 C \frac{dv_c}{dt}$$

$$\frac{dv_c}{dt} = i(t) - i_2$$

$$y = (i(t) - i_2) R_2$$

If $i_2 = x_1, v_c = x_2 \; and \; i(t) = u$, then

$$\begin{cases} L\dot{x}_1 + R_2 x_1 = x_2 + R_1 C\dot{x}_2 \\ \dot{x}_2 = \dfrac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

or

$$\begin{cases} \dot{x}_1 = -\dfrac{2R_2}{L}x_1 + \dfrac{1}{L}x_2 + \dfrac{R_1}{L}(u - x_1) \\ \dot{x}_2 = \dfrac{1}{C}(u - x_1) \\ y = (u - x_1)R_2 \end{cases}$$

Therefore:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{2R_2}{L} & \dfrac{1}{L} \\ -\dfrac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{R_1}{L} \\ \dfrac{1}{C} \end{bmatrix} u$$

$$y = [-R_2 \quad 0]x + R_2 u$$

b) Uncontrollability condition is:

$$det[B \;\; AB] = det(C) = 0$$

According to the state-space of the system, C is calculated as:

$$C = \begin{bmatrix} \dfrac{R_1}{L} & -\dfrac{2R_1R_2}{L^2} + \dfrac{1}{LC} \\ \dfrac{1}{C} & -\dfrac{R_1}{LC} \end{bmatrix}$$

$$det(C) = \frac{R_1R_2}{L^2C} - \frac{1}{LC^2}$$

As $det(C) \neq 0$, because $R_1R_2 \neq RC$, then the system is controllable

c) Unobservability condition is:

$$det\begin{bmatrix} C \\ CA \end{bmatrix} = det(H) = 0$$

According to the state-space of the system, C is calculated as:

$$H = \begin{bmatrix} -R_2 & 0 \\ \frac{2R_2}{L} & -\frac{R_2}{L} \end{bmatrix}$$

$$det(H) = \frac{R_2^2}{L}$$

Since det(H) $\neq 0$, because R $\neq 0$ or L $\neq \infty$, then the system is observable.

d) The same as part (a)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1C} & 0\\ 0 & \frac{R_2}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\frac{1}{R_1C}\\ \frac{1}{L}\end{bmatrix}u$$

$$y=\begin{bmatrix}-\frac{1}{R_1} & 1\end{bmatrix}x+\frac{1}{R_1}u$$

For controllability, we define G as:

$$G=[B\ \ AB]=\begin{bmatrix}\frac{1}{R_1C}-\frac{1}{(R_1C)^2}\\ \frac{1}{L}-\frac{R_2}{L^2}\end{bmatrix}$$

$$det(G)=-\frac{R_2}{R_1CL^2}+\frac{1}{L(R_1C)^2}$$

If $R_1R_2C=L$,and then $det(G)=0$, which means the system is not controllable.

For observability, we define H as:

$$H=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1} & 1\\ \frac{1}{R_1^2C} & -\frac{R_2}{L}\end{bmatrix}$$

$$det(H)=\frac{R_2}{R_1L}-\frac{1}{R_1^2C}$$

If $R_1R_2C=L$, then $det(H)=0$, which means the system is not observable.