Chapter 9

9-1 (a) $P(s) = s^4 + 4s^3 + 4s^2 + 8s \qquad Q(s) = s + 1$

Finite zeros of *P*(*s*): $0, \ -3.5098, \ -0.24512 \pm j1.4897$

Finite zeros of *Q*(*s*): -1

Asymptotes: $\boldsymbol{K > 0}$: $60^o, \ 180^o, \ 300^o$ $\qquad$ $\boldsymbol{K < 0}$: $0^o, \ 120^o, \ 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-3.5 - 0.24512 - 0.24512 - (-1)}{4 - 1} = -1$$

(b) $P(s) = s^3 + 5s^2 + s \qquad Q(s) = s + 1$

Finite zeros of *P*(*s*) : $0, \ -4.7912, \ -0.20871$

Finite zeros of *Q*(*s*): -1

Asymptotes: $\boldsymbol{K > 0}$: $90^o, \ 270^o$ $\qquad$ $\boldsymbol{K < 0}$: $0^o, \ 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-4.7913 - 0.2087 - (-1)}{3 - 1} = -2$$

(c) $P(s) = s^2 \qquad Q(s) = s^3 + 3s^2 + 2s + 8$

Finite zeros of *P*(*s*): $0, \ 0$

Finite zeros of *Q*(*s*): $-3.156, \quad 0.083156 \pm j1.5874$

Asymptotes: $\boldsymbol{K > 0}$: 180^o $\qquad$ $\boldsymbol{K < 0}$: 0^o

(d) $P(s) = s^3 + 2s^2 + 3s \qquad Q(s) = \left(s^2 - 1\right)\left(s + 3\right)$

Finite zeros of *P*(*s*): $0, \quad -1 \pm j1.414$

Finite zeros of *Q*(*s*): $1, -1, -3$

Asymptotes: **There are no asymptotes, since the number of zeros of *P*(*s*) and *Q*(*s*) are equal.**

(e) $P(s) = s^5 + 2s^4 + 3s^3 \qquad Q(s) = s^2 + 3s + 5$

Finite zeros of *P*(*s*): $0, \quad 0, \quad 0, \quad -1 \pm j1.414$

Finite zeros of *Q*(*s*): $-1.5 \pm j1.6583$

Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1 - 1 - (-1.5) - (-1.5)}{5 - 2} = \frac{1}{3}$$

(f) $P(s) = s^4 + 2s^2 + 10 \qquad Q(s) = s + 5$

Finite zeros of *P*(*s*): $-1.0398 \pm j1.4426, \quad 1.0398 \pm j1.4426$

Finite zeros of *Q*(*s*): -5

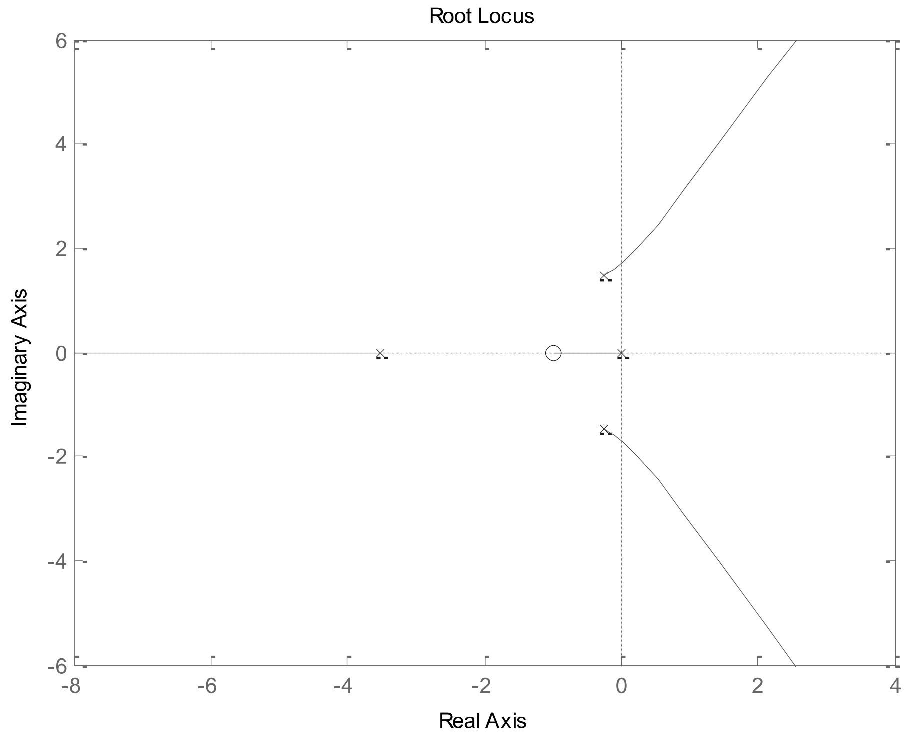
Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1.0398 - 1.0398 + 1.0398 + 1.0398 - (-5)}{4 - 1} = \frac{-5}{3}$$

9-2(a) MATLAB code:

```
s = tf('s')
num_GH=(s+1);
den_GH=(s^4+4*s^3+4*s^2+8*s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

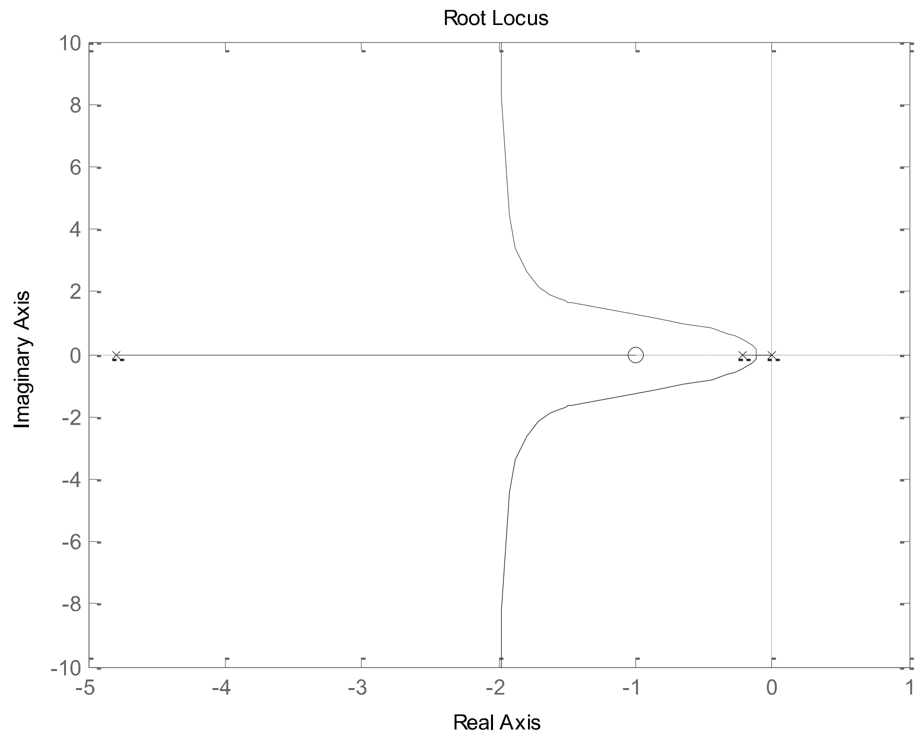
Assymp2_angle = -60

Assymp3_angle = 180

sigma = -1.0000 (intersect of asymptotes)

9-2(b) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+1);
den_GH=(s^3+5*s^2+s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



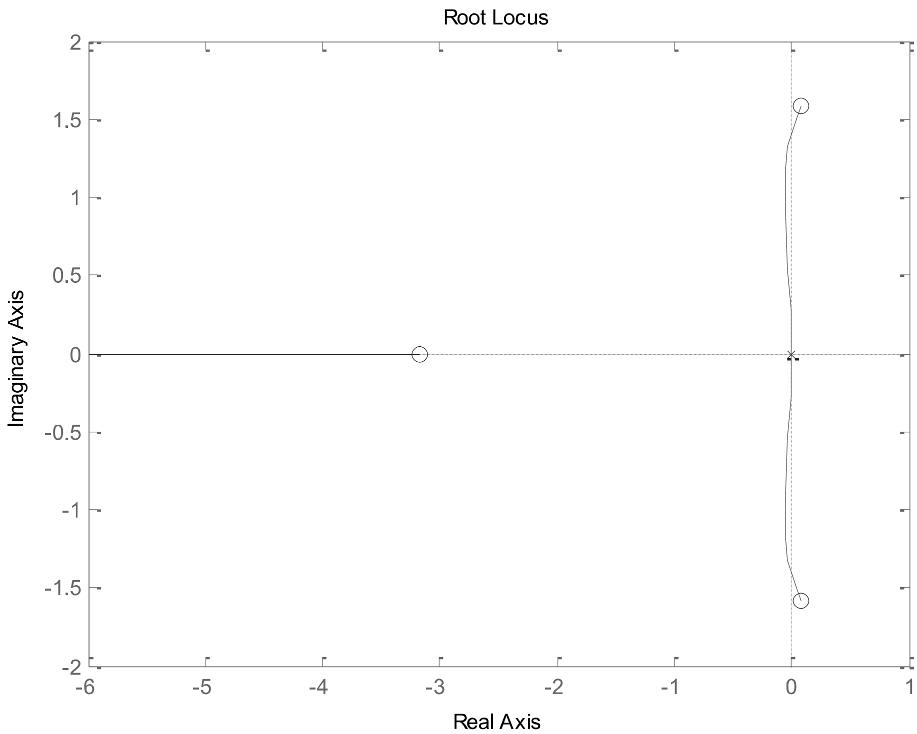
Assymp1_angle = 90

Assymp2_angle = -90

sigma = -2 (intersect of asymptotes)

9-2(c) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s^3+3*s^2+2*s+8);
den_GH=(s^2);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)   %number of poles of G(s)H(s)
m=length(GH_z)   %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```

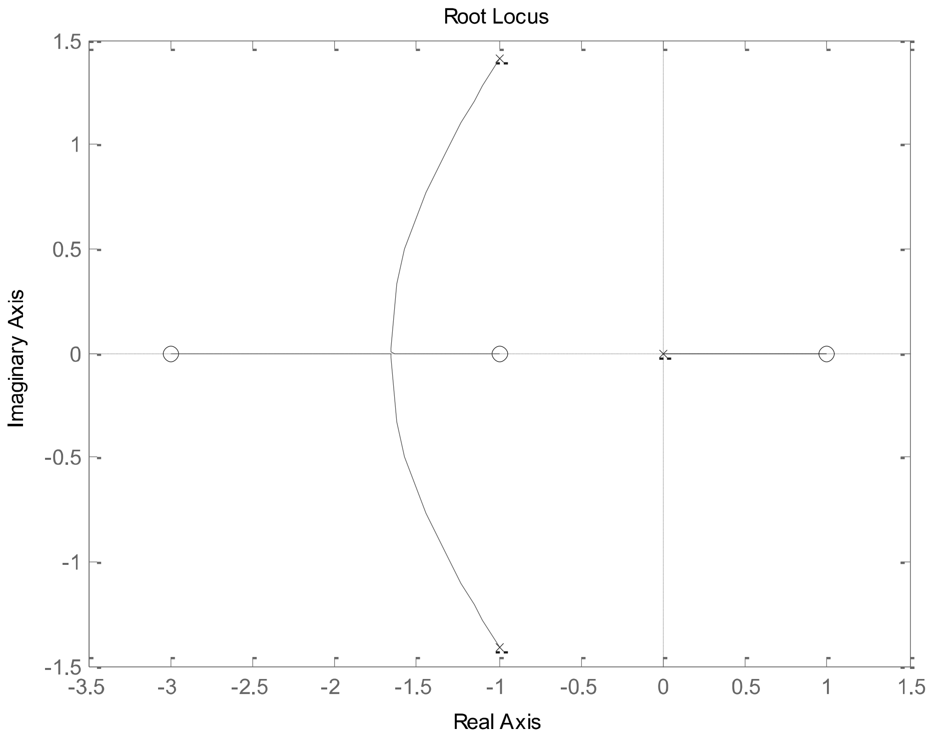


Assymp1_angle = 180

sigma = -3.0000 (intersect of asymptotes)

9-2(d) MATLAB code:

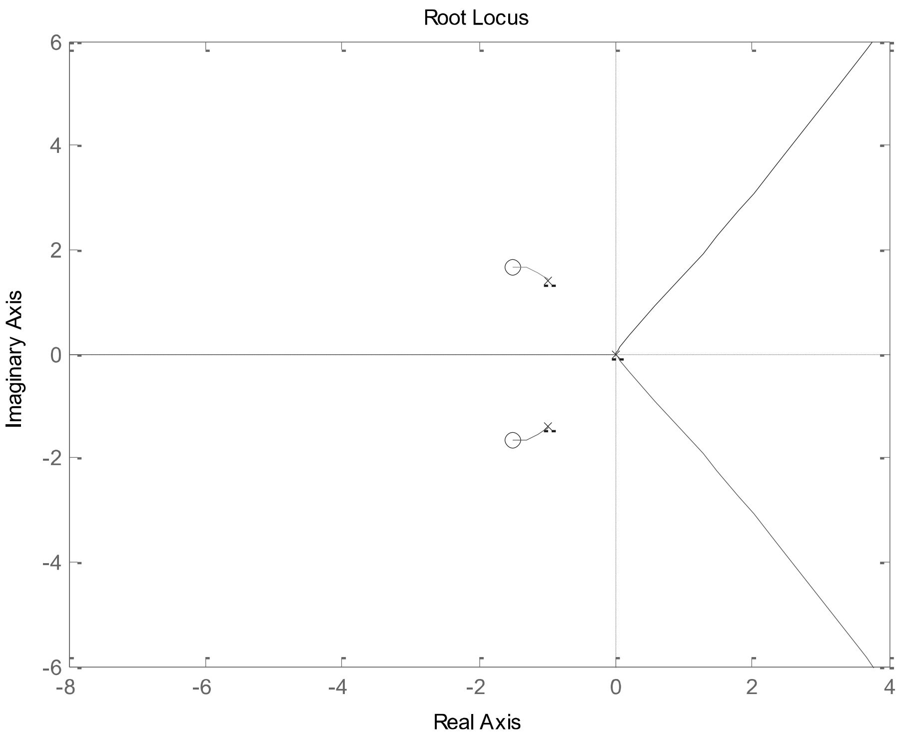
```
s = tf('s')
'Generating the transfer function:'
num_GH=((s^2-1)*(s+3));
den_GH=(s^3+2*s^2+3*s);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
```



No asymptotes

9-2(e) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s^2+3*s+5);
den_GH=(s^5+2*s^4+3*s^3);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

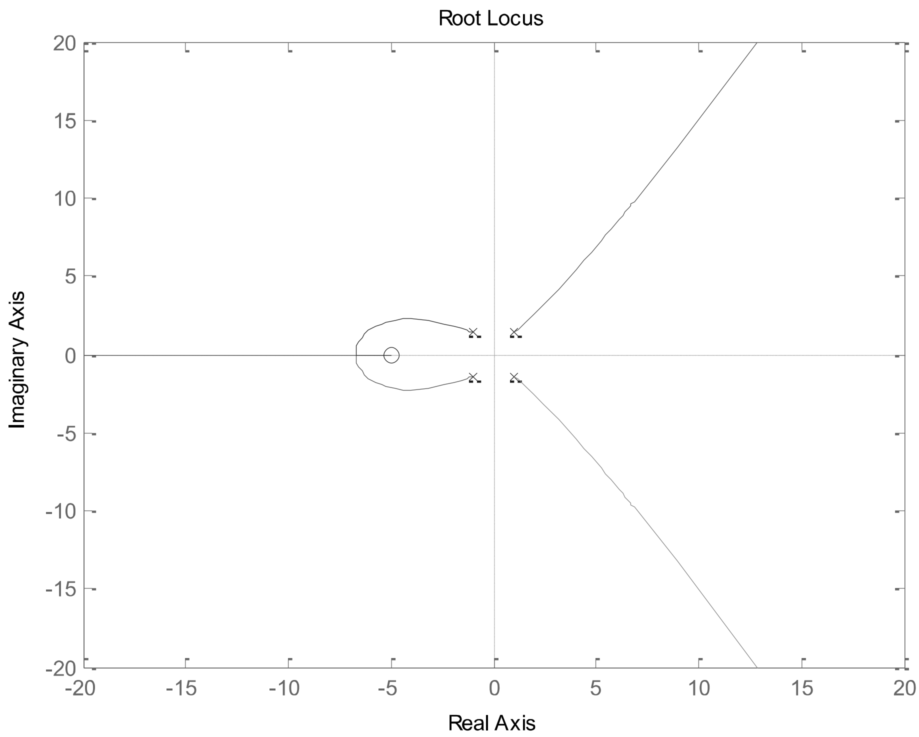
Assymp2_angle = -60

Assymp3_angle = 180

sigma = 0.3333 (intersect of asymptotes)

9-2(f) MATLAB code:

```
s = tf('s')
'Generating the transfer function:'
num_GH=(s+5);
den_GH=(s^4+2*s^2+10);
GH_a=num_GH/den_GH;
figure(1);
rlocus(GH_a)
xlim([-20 20])
ylim([-20 20])
GH_p=pole(GH_a)
GH_z=zero(GH_a)
n=length(GH_p)    %number of poles of G(s)H(s)
m=length(GH_z)    %number of zeros of G(s)H(s)
%Asymptotes angles:
k=0;
Assymp1_angle=+180*(2*k+1)/(n-m)
Assymp2_angle=-180*(2*k+1)/(n-m)
k=1;
Assymp3_angle=+180*(2*k+1)/(n-m)
%Asymptotes intersection point on real axis:
sigma=(sum(GH_p)-sum(GH_z))/(n-m)
```



Assymp1_angle = 60

Assymp2_angle = -60

Assymp3_angle = 180

sigma = 1.6667 (intersect of asymptotes)

9-3) Consider

$$G(s)H(s) = K\frac{Q(s)}{P(s)} = K\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

As the asymptotes are the behavior of G(s)H(s) when |s|→∞ , then

$|s| > |z_i|$ for i = 1,2,…,m and $|s| > |p_i|$ for i = 1,2,…,n

therefore $\angle G(s)H(s) = m\arg(s) - n\arg(s) = -(n-m)\arg(s)$

According to the condition on angles:

$$\angle G(s)H(s) = \begin{cases} 2(i+1)\pi & K \geq 0 \\ 2i\,\pi & K \leq 0 \end{cases}$$

If we consider $\arg(s) = \theta_i$, then:

$$\angle G(s)H(s) = \begin{cases} -(n-m)\theta_i = (2i+1)\pi & K \geq 0 \\ -(n-m)\theta_i = 2i\,\pi & K \leq 0 \end{cases}$$

or

$$\begin{cases} \theta_i = \dfrac{2i+1}{|n-m|}\pi & K \geq 0 \\ \theta_i = \dfrac{2i}{|n-m|}\pi & K \leq 0 \end{cases}$$

9-4) If $G(s)H(s) = K\frac{Q(s)}{P(s)}$, then each point on root locus must satisfy the characteristic equation of $P(s) + KQ(s) = 0$

If $P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ and $Q(s) = s^m + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$, then

$$s^n + a_{n-1}s^{n-1} + \cdots + a_0 + K(s^m - b_{m-1}s^{m-1} + \cdots + b_0) = 0$$

or

$$s^{n-m} + (a_{n-1} - b_{m-1})s^{n-m-1} + \cdots + K = 0$$

If the roots of above expression is considered as s_i for i = 1,2,…,(n-m), then

$$a_{n-1} - b_{m-1} = -\sum_{i=1}^{n-m} s_i = -\sum_{i=1}^{n} s_i - \sum_{i=1}^{m} s_i = -\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i$$

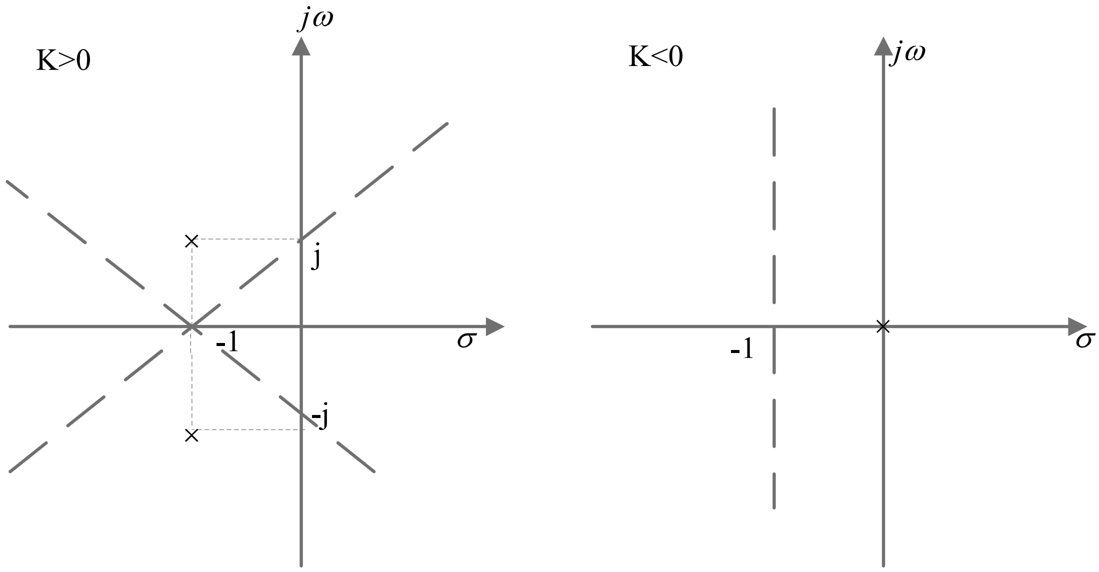
since the intersect of (n-m) asymptotes lies on the real axis of the s-plane and $-(a_{n-1} - b_{m-1})$ is real, therefore

$$\sigma_1 = -\frac{a_{n-1} - b_{m-1}}{n-m} = -\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n-m}$$

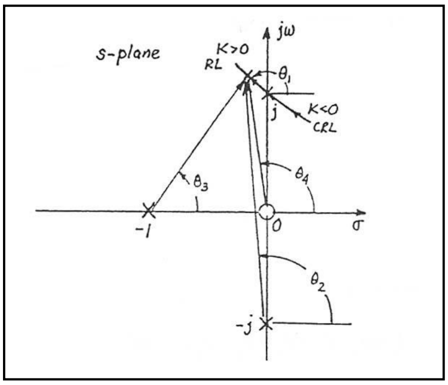
9-5) Poles of GH is s = 0, -2, -1 + j, -1 – j, therefore the center of asymptotes:

$$\sigma_1 = \frac{\sum p_i - \sum z_i}{n-m} = -1$$

The angles of asymptotes: $\begin{cases} \theta_i = 45^o, 135^o, 225^o, 315^o & K > 0 \\ \theta_i = 0^o, 90^o, 180^o, 270^o & K < 0 \end{cases}$


K>0
jω
j
-1
σ
-j
K<0
jω
-1
σ

9-6 (a) Angles of departure and arrival.



K > 0: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

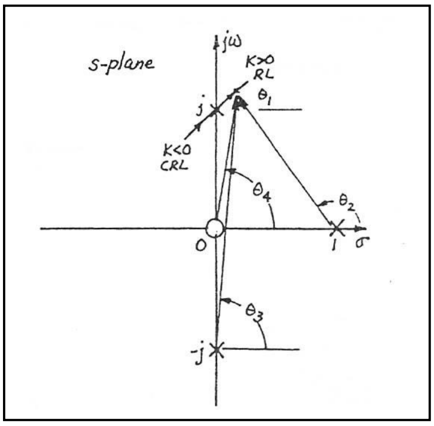
$-\theta_1 - 90^o - 45^o + 90^o = -180^o$

$\theta_1 = 135^o$

K < 0: $-\theta_1 - 90^o - 45^o + 90^o = 0^o$

$\theta_1 = -45^o$

(b) Angles of departure and arrival.



K > 0: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

K < 0: $-\theta_1 - 135^o - 90^o + 90^o = 0^o$

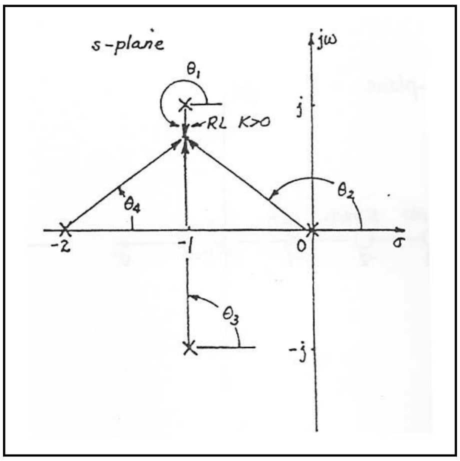
$\theta_1 = -135^o$

(c) Angle of departure:

$K > 0$: $-\theta_1 - \theta_2 - \theta_3 + \theta_4 = -180^o$

$-\theta_1 - 135^o - 90^o - 45^o = -180^o$

$\theta_1 = -90^o$

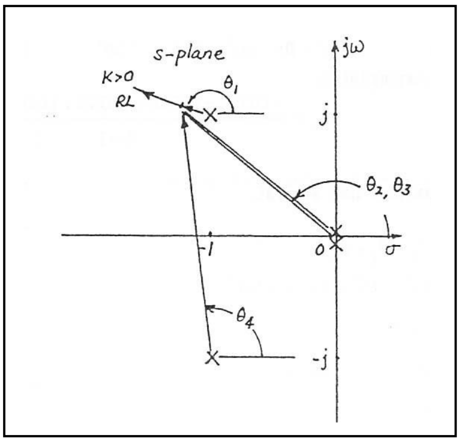


(d) Angle of departure

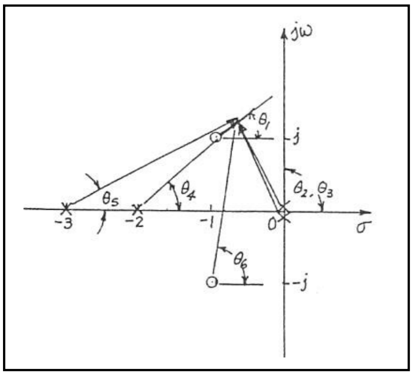
$K > 0$: $-\theta_1 - \theta_2 - \theta_3 - \theta_4 = -180^o$

$-\theta_1 - 135^o - 135^o - 90^o = -180^o$

$\theta_1 = -180^o$



(e) Angle of arrival



***K* < 0:** $\theta_1 + \theta_6 - \theta_2 - \theta_3 - \theta_4 - \theta_5 = -360^o$

$\theta_1 + 90^o - 135^o - 135^o - 45^o - 26.565^o = -360^o$

$\theta_1 = -108.435^o$

9-7) a) $\angle$ G(s)H(s) = $\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$

$= \sum_{i=1}^{m} \angle(s+z_i) + \sum_{i=1, i\neq j}^{n} \angle(s+p_i) - \angle(s+p_j)$

$= \angle G(s)H'(s) - \angle(s+p_j)$

$= \angle G(s)H'(s) - \theta_D$

we know that $\angle G(s)H(s) = \begin{cases} (2i+1)\times 180 & K\geq 0 \quad i=0,\pm 1,\ldots \\ (2i)\times 180 & K\leq 0 \quad i=0,\pm 1,\ldots \end{cases}$

therefore

$$\begin{cases} \angle G(s)H'(s) - \theta_D = 180 & K\geq 0 \\ \angle G(s)H'(s) - \theta_D = 0 & K\leq 0 \end{cases}$$

As a result, θ_D = $\angle$ G(s)H'(s) – 180° = 180 + $\angle G(s)H'(s)$, when -180° = 180°

b) Similarly:

$\angle$ G(s)H(s) = $\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i)$

$= \sum_{i=1, i\neq j}^{m} \angle(s+z_i) + \sum_{i=1}^{n} \angle(s-p_i) + \angle(s+z_j)$

$= \angle G(s)H''(s) + \angle(s+z_j)$

$= \angle G(s)H''(s) + \theta$

Therefore:

$$\begin{cases} \angle G(s)H''(s) + \theta = 180 & K\geq 0 \\ \angle G(s)H''(s) + \theta = 0 & K\leq 0 \end{cases}$$

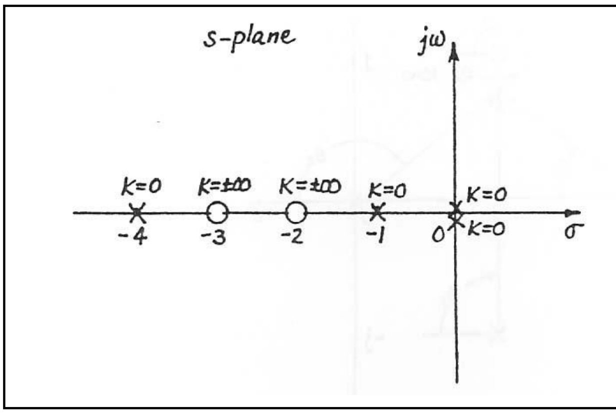
As a result, θ =180 – $\angle G(s)H''(s)$

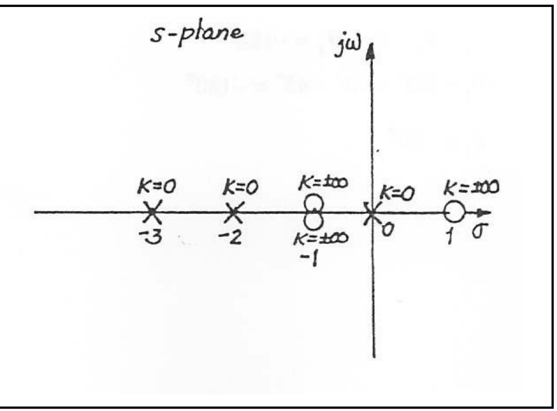
9-8) zeros: $s=-1-j, -1+j$ and poles: $s = 0, -2j, +2j$

Departure angles from: $\begin{cases} s=2j & : \quad \theta = 180-63.4 = 116.6 \\ s=-2j & : \quad \theta = -198.4 \end{cases}$

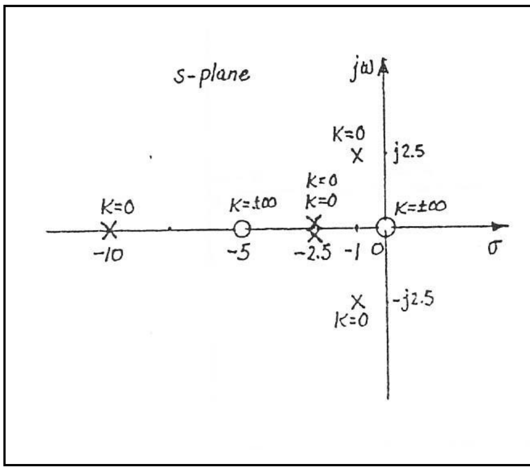
Arrival angles at $\begin{cases} s=-1+j & : \quad \theta = 180-(-18.4) = 198.4 \\ s=-1-j & : \quad \theta = -198.4 \end{cases}$

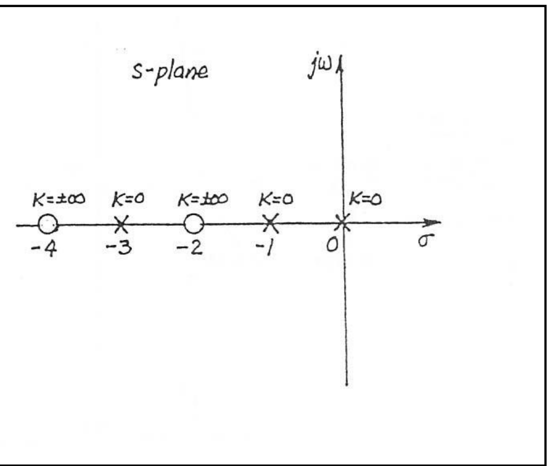
9-9) (a) **(b)**


s-plane
jω
K=0
K=±∞
K=±∞
K=0
K=0
K=0
-4
-3
-2
-1
0
σ


s-plane
jω
K=0
K=0
K=±∞
K=±∞
K=0
K=±∞
-3
-2
-1
0
1
σ

(c) **(d)**


s-plane
jω
K=0
j2.5
K=0
K=0
K=0
K=±∞
K=±∞
-10
-5
-2.5
-1
0
σ
-j2.5
K=0


s-plane
jω
K=±∞
K=0
K=±∞
K=0
K=0
-4
-3
-2
-1
0
σ

9-10) The breaking points are on the real axis of $1 + KG(s)H(s) = 0$ and must satisfy

$$\frac{dG(s)H(s)}{ds} = 0$$

If $G(s)H(s) = \frac{Q(s)}{P(s)}$ and α is a breakaway point, then

$$1 + K\frac{Q(\alpha)}{P(\alpha)} = 0 \Rightarrow K = -\frac{P(\alpha)}{Q(\alpha)}$$

Finding α where K is maximum or minimum $\frac{dK}{d\,\alpha} = 0$, therefore

$$\frac{d}{d\,\alpha}\left[\frac{P(\alpha)}{Q(\alpha)}\right] = 0$$

or

$$\frac{d}{d\alpha}\left[\frac{(\alpha + p_1)(\alpha + p_2)\dots(\alpha + p_n)}{(\alpha + z_1)(\alpha + z_2)\dots(\alpha + z_m)}\right] = 0$$

$$\sum_{i=1}^{n}\frac{1}{\alpha + p_1}\left[\frac{P(\alpha)}{Q(\alpha)}\right] - \sum_{i=1}^{m}\frac{1}{\alpha + z_i}\left[\frac{P(\alpha)}{Q(\alpha)}\right] = 0$$

$$\sum_{i=1}^{n}\frac{1}{\alpha + p_i} = \sum_{i=1}^{m}\frac{1}{\alpha + z_i}$$

9-11) (a) Breakaway-point Equation: $2s^5 + 20s^4 + 74s^3 + 110s^2 + 48s = 0$

Breakaway Points: $-0.7275, \quad -2.3887$

(b) Breakaway-point Equation: $3s^6 + 22s^5 + 65s^4 + 100s^3 + 86s^2 + 44s + 12 = 0$

Breakaway Points: $-1, \quad -2.5$

(c) Breakaway-point Equation: $3s^6 + 54s^5 + 347.5s^4 + 925s^3 + 867.2s^2 - 781.25s - 1953 = 0$

Breakaway Points: $-2.5, \quad 1.09$

(d) Breakaway-point Equation: $-s^6 - 8s^5 - 19s^4 + 8s^3 + 94s^2 + 120s + 48 = 0$

Breakaway Points: $-0.6428, \quad 2.1208$

9-12) (a)

$$G(s)H(s) = \frac{K(s+8)}{s(s+5)(s+6)}$$

Asymptotes: *K* > 0: 90^o and 270^o **K < 0:** 0^o and 180^o

Intersect of Asymptotes:

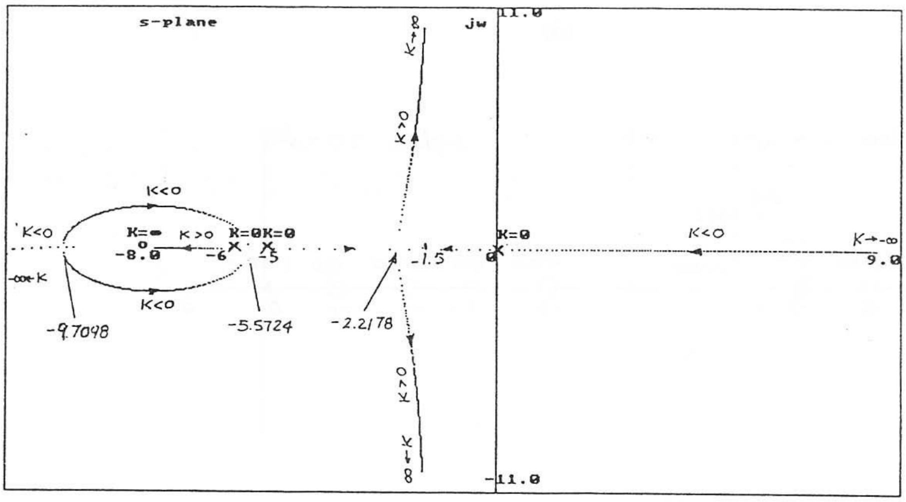
$$\sigma_1 = \frac{0-5-6-(-8)}{3-1} = -1.5$$

Breakaway-point Equation:

$$2s^3 + 35s^2 + 176s + 240 = 0$$

Breakaway Points: $-2.2178, \quad -5.5724, \quad -9.7098$

Root Locus Diagram:



9-12 (b)

$$G(s)H(s) = \frac{K}{s(s+1)(s+3)(s+4)}$$

Asymptotes: *K* > 0: $45^o,\ 135^o,\ 225^o,\ 315^o$ ***K* < 0:** $0^o,\ 90^o,\ 180^o,\ 270^o$

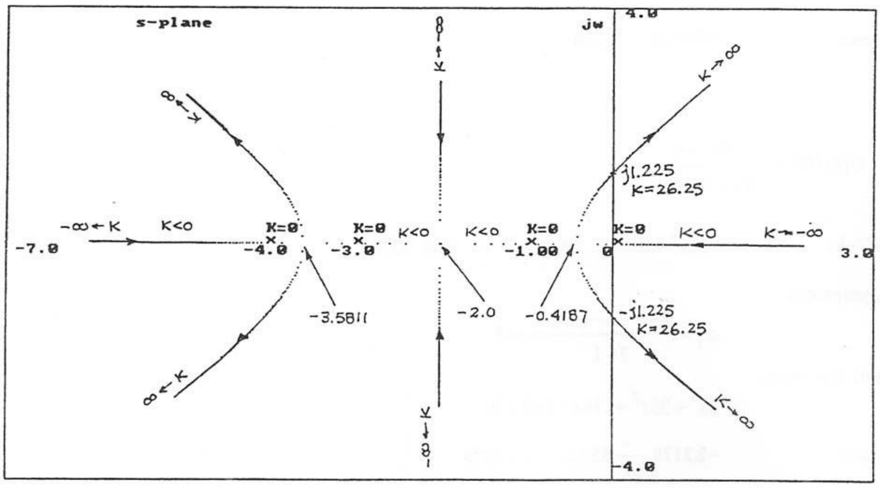
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-4}{4} = -2$$

Breakaway-point Equation: $4s^3 + 24s^2 + 38s + 12 = 0$

Breakaway Points: $-0.4189,\ -2,\ -3.5811$

Root Locus Diagram:



9-12 (c)

$$G(s)H(s) = \frac{K(s+4)}{s^2(s+2)^2}$$

Asymptotes: ***K* > 0:** $60^o, \quad 180^o, \quad 300^o$ ***K* < 0:** $0^o, \quad 120^o, \quad 240^o$

Intersect of Asymptotes:

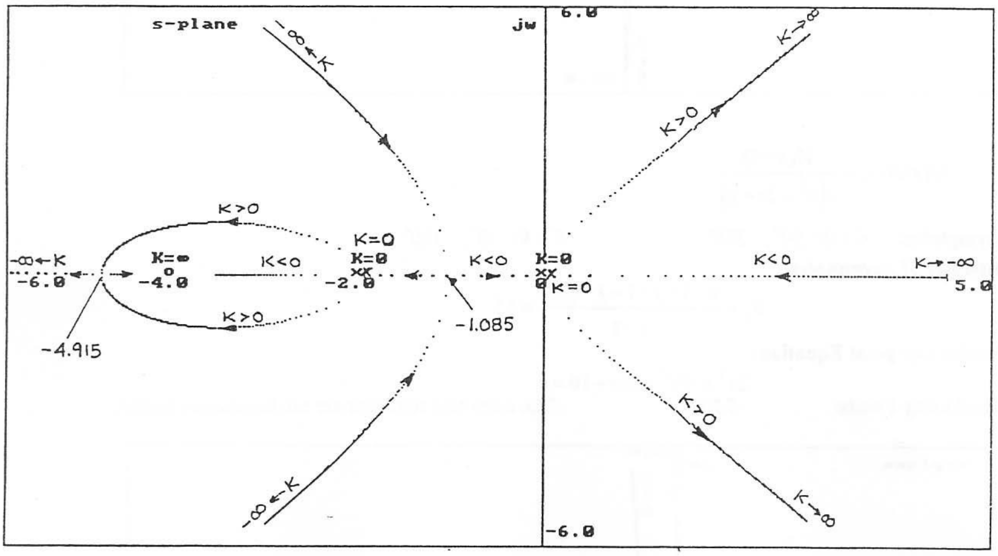
$$\sigma_1 = \frac{0+0-2-2-(-4)}{4-1} = 0$$

Breakaway-point Equation:

$$3s^4 + 24s^3 + 52s^2 + 32s = 0$$

Breakaway Points: $0, \quad -1.085, \quad -2, \quad -4.915$

Root Locus Diagram:



9-12 (d)

$$G(s)H(s) = \frac{K(s+2)}{s(s^2+2s+2)}$$

Asymptotes: ***K* > 0:** 90^o, 270^o ***K* < 0:** 0^o, 180^o

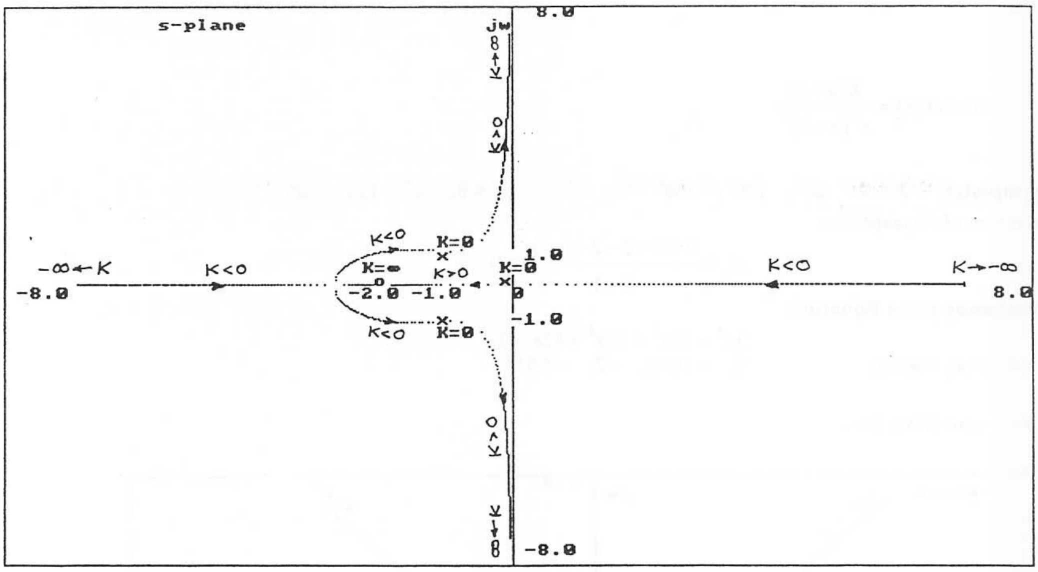
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-j-1-j-(-2)}{3-1} = 0$$

Breakaway-point Equation: $2s^3 + 8s^2 + 8s + 4 = 0$

Breakaway Points: -2.8393 The other two solutions are not breakaway points.

Root Locus Diagram



9-12 (e)

$$G(s)H(s) = \frac{K(s+5)}{s\left(s^2+2s+2\right)}$$

Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

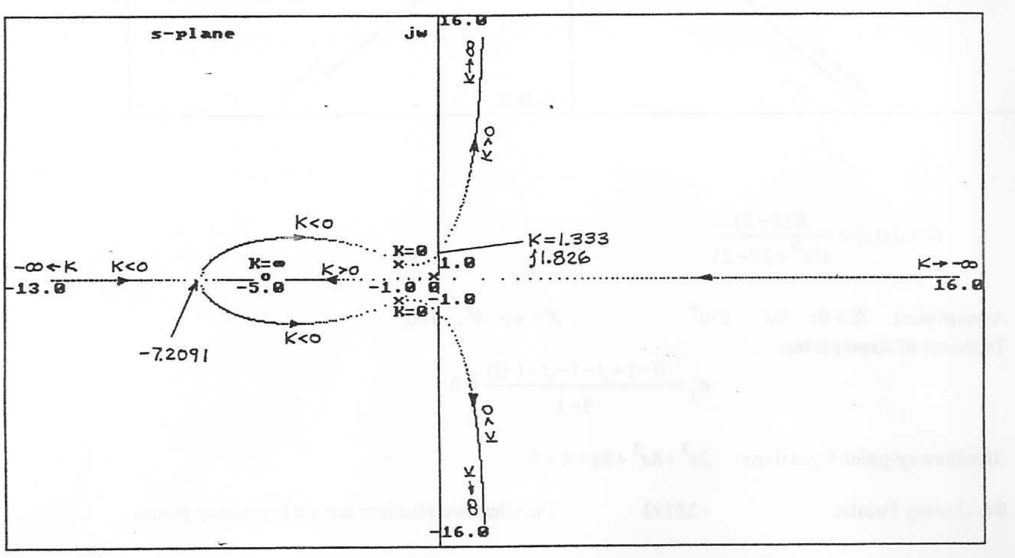
Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-j-1-j-(-5)}{3-1} = 1.5$$

Breakaway-point Equation:

$$2s^3 + 17s^2 + 20s + 10 = 0$$

Breakaway Points: -7.2091 The other two solutions are not breakaway points.



9-12 (f)

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+2s+2)}$$

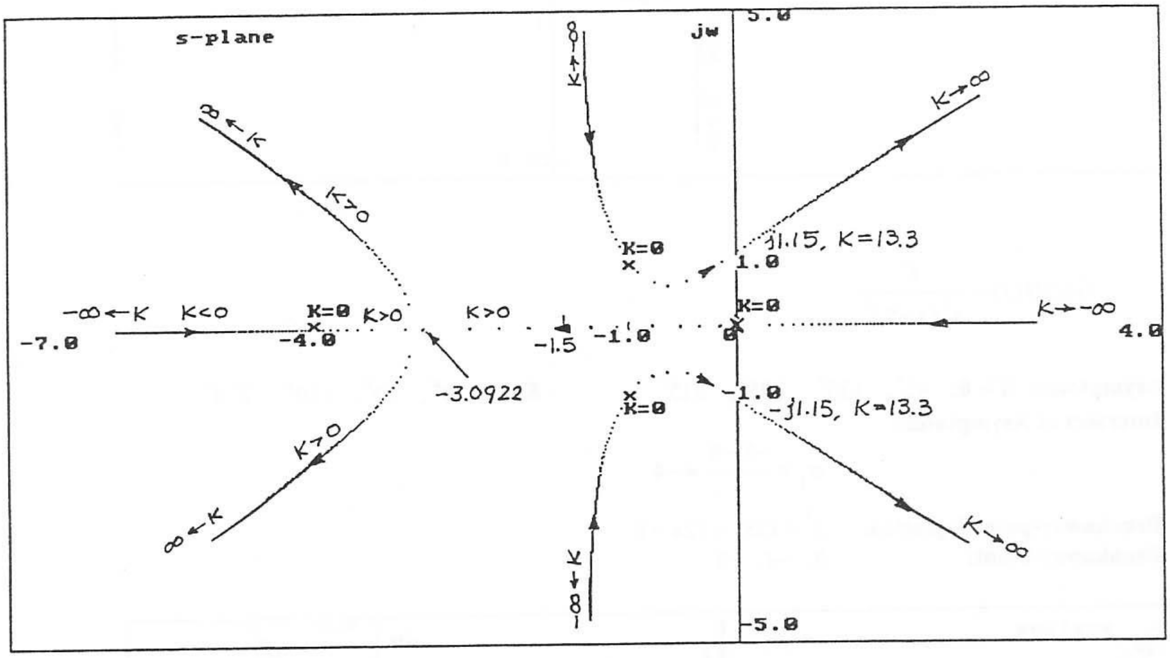
Asymptotes: ***K* > 0:** $45^o, \ 135^o, \ 225^o, \ 315^o$ ***K* < 0:** $0^o, \ 90^o, \ 180^o, \ 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-j-1+j-4}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 20s + 8 = 0$

Breakaway Point: -3.0922 The other solutons are not breakaway points.



9-12 (g)

$$G(s)H(s)=\frac{K(s+4)^2}{s^2(s+8)^2}$$

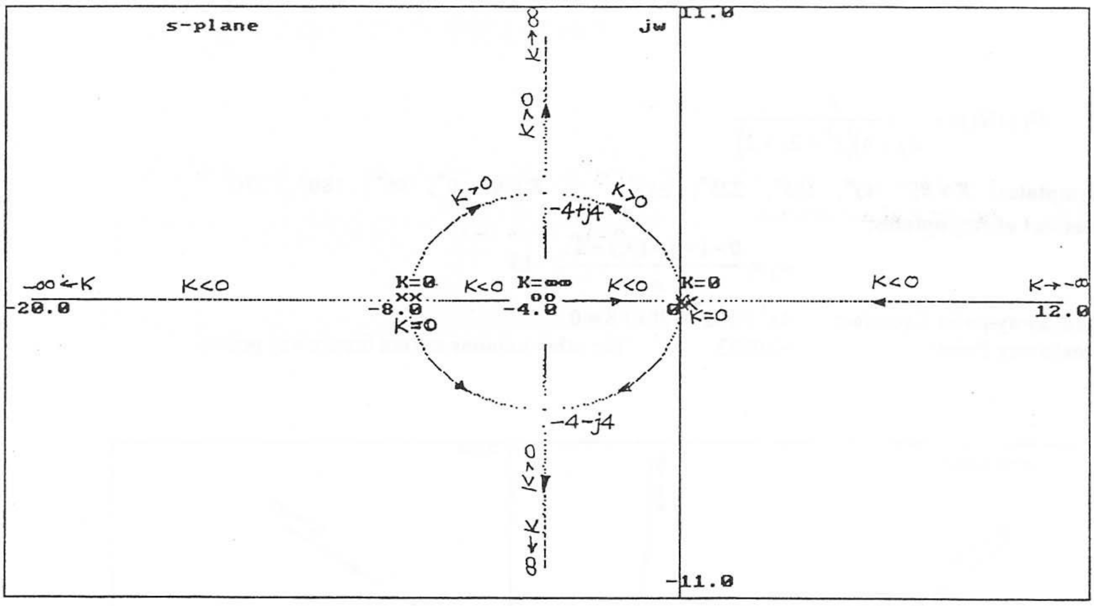
Asymptotes: ***K* > 0:** $90^o,\ \ 270^o$ ***K* < 0:** $0^o,\ \ 180^o$

Intesect of Asymptotes:

$$\sigma_1=\frac{0+0-8-8-(-4)-(-4)}{4-2}$$

Breakaway-point Equation: $s^5+20s^4+160s^3+640s^2+1040s=0$

Breakaway Points: 0, –4, –8, $-4-j4$, $-4+j4$



9-12 (h)

$$G(s)H(s) = \frac{K}{s^2(s+8)^2}$$

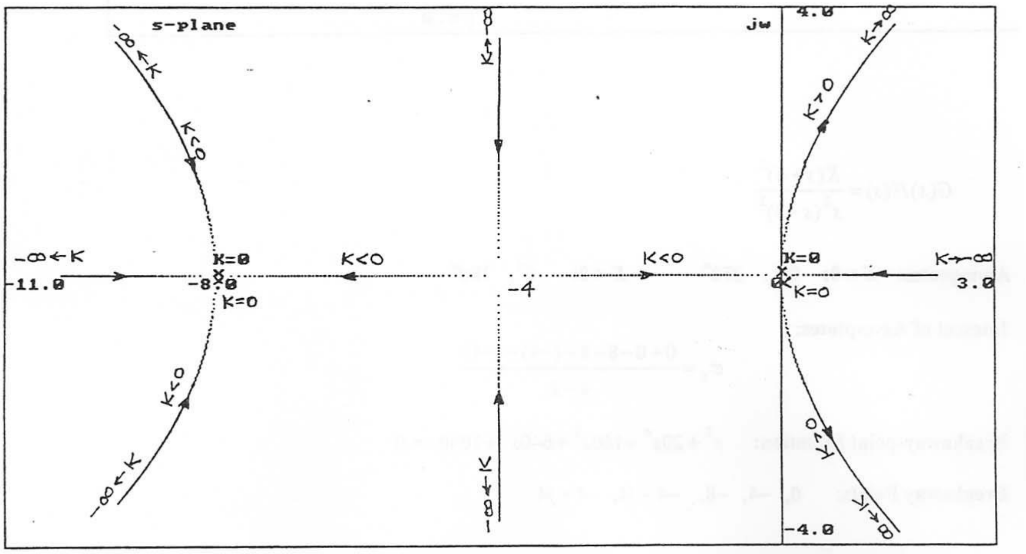
Asymptotes: ***K* > 0:** $45^o, \ \ 135^o, \ \ 225^o, \ \ 315^o$ ***K* < 0:** $0^o, \ \ 90^o, \ \ 180^o, \ \ 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-8}{4} = -4$$

Breakaway-point Equation: $s^3 + 12s^2 + 32s = 0$

Breakaway Point: 0, −4, −8



9-12 (i)

$$G(s)H(s) = \frac{K\left(s^2 + 8s + 20\right)}{s^2(s+8)^2}$$

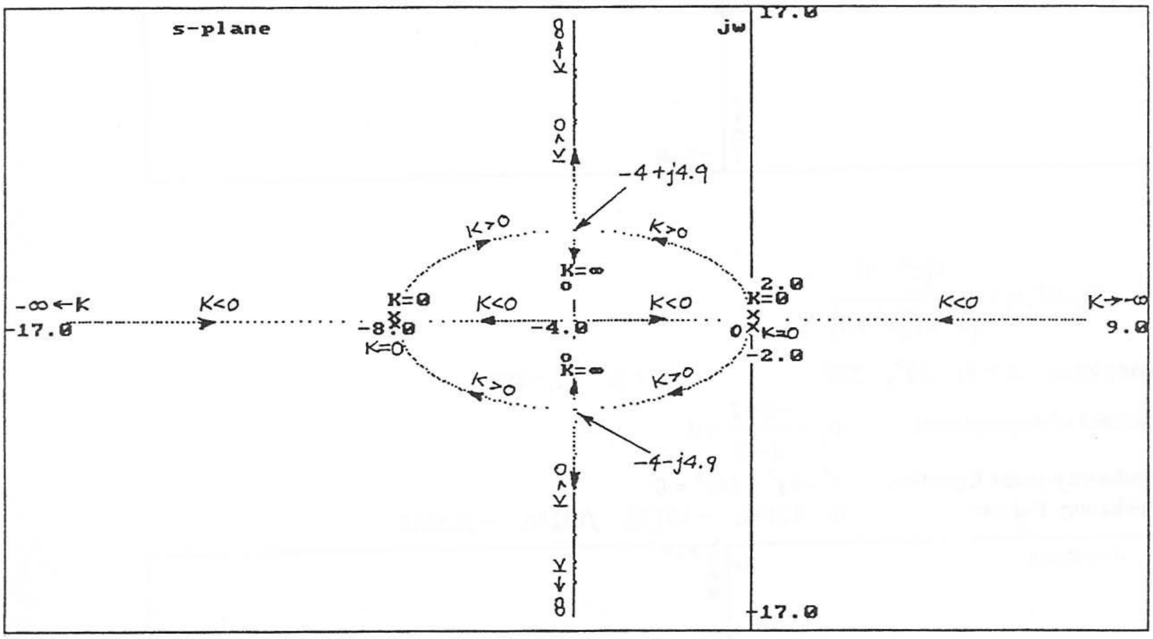
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-8-(-4)-(-4)}{4-2} = -4$$

Breakaway-point Equation: $s^5 + 20s^4 + 128s^3 + 736s^2 + 1280s = 0$

Breakaway Points: $-4, \quad -8, \quad -4+j4.9, \quad -4-j4.9$



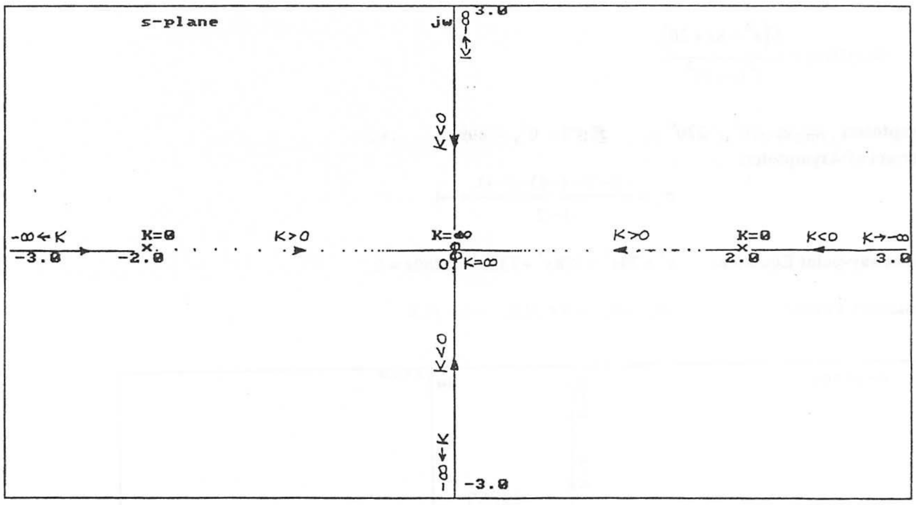
(j)

$$G(s)H(s) = \frac{Ks^2}{\left(s^2 - 4\right)}$$

Since the number of finite poles and zeros of $G(s)H(s)$ are the same, there are no asymptotes.

Breakaway-point Equation: $8s = 0$

Breakaway Points: $s = 0$



9-12 (k)

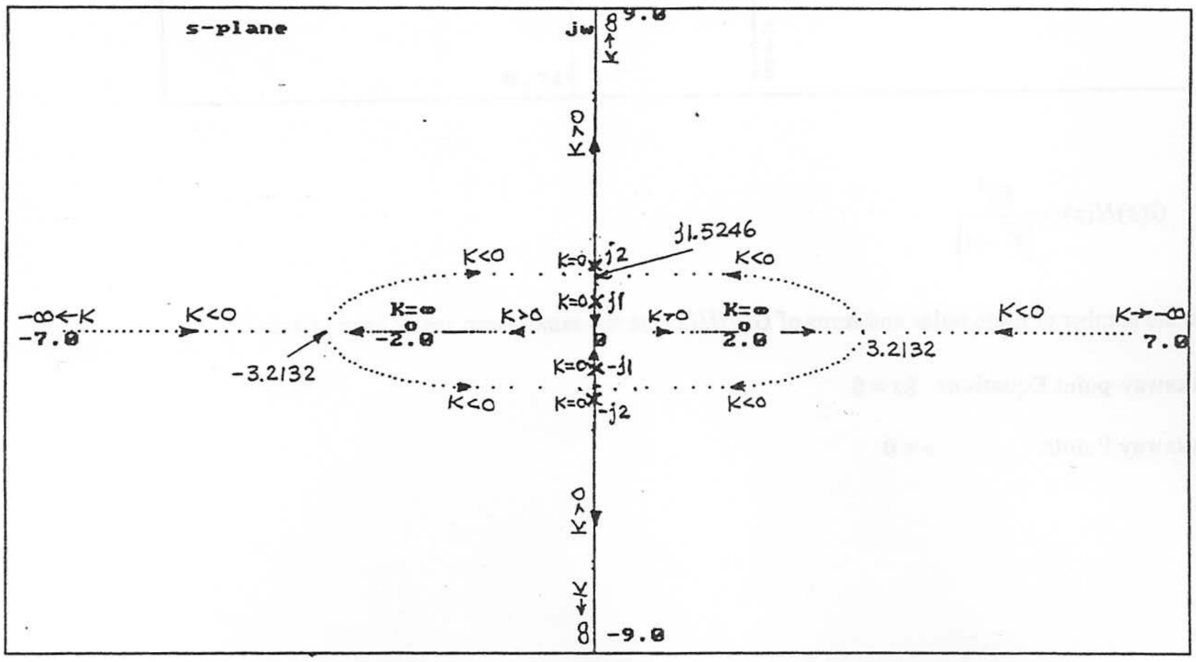
$$G(s)H(s) = \frac{K\left(s^2-4\right)}{\left(s^2+1\right)\left(s^2+4\right)}$$

Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes: $\sigma_1 = \dfrac{-2+2}{4-2} = 0$

Breakaway-point Equation: $s^6 - 8s^4 - 24s^2 = 0$

Breakaway Points: $0, \quad 3.2132, \quad -3.2132, \quad j1.5246, \quad -j1.5246$



9-12 (l)

$$G(s)H(s) = \frac{K(s^2 - 1)}{\left(s^2 + 1\right)\left(s^2 + 4\right)}$$

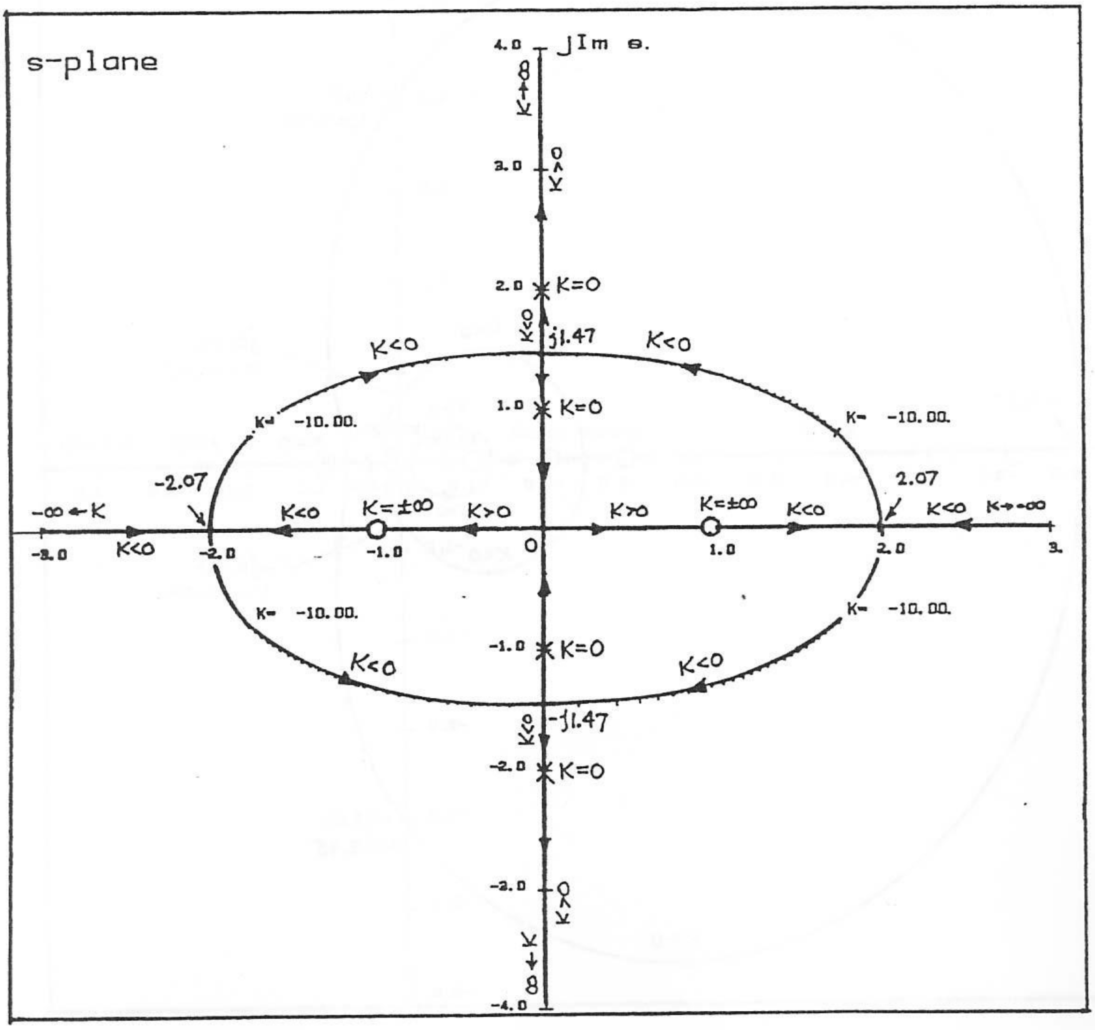
Asymptotes: *K* > 0: $90^o, \quad 270^o$ **_K_ < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1+1}{4-2} = 0$$

Breakaway-point Equation: $s^5 - 2s^3 - 9s = 0$

Breakaway Points: $-2.07, \quad 2.07, \quad -j1.47, \quad j1.47$



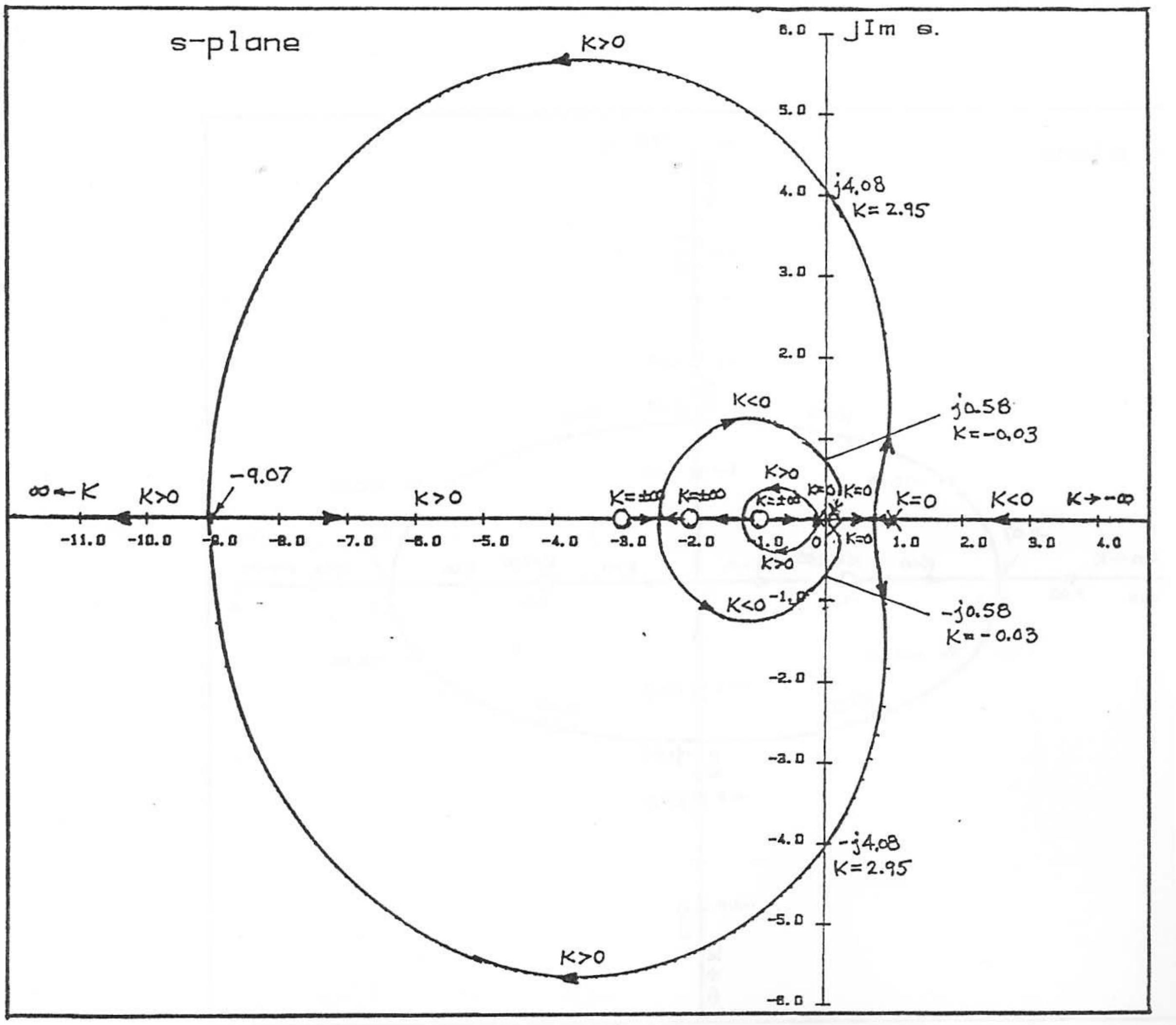
(m)

$$G(s)H(s) = \frac{K(s+1)(s+2)(s+3)}{s^3(s-1)}$$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^6 + 12s^5 + 27s^4 + 2s^3 - 18s^2 = 0$

Breakaway Points: –1.21, –2.4, –9.07, 0.683, 0, 0



(n)

$$G(s)H(s) = \frac{K(s+5)(s+40)}{s^3(s+250)(s+1000)}$$

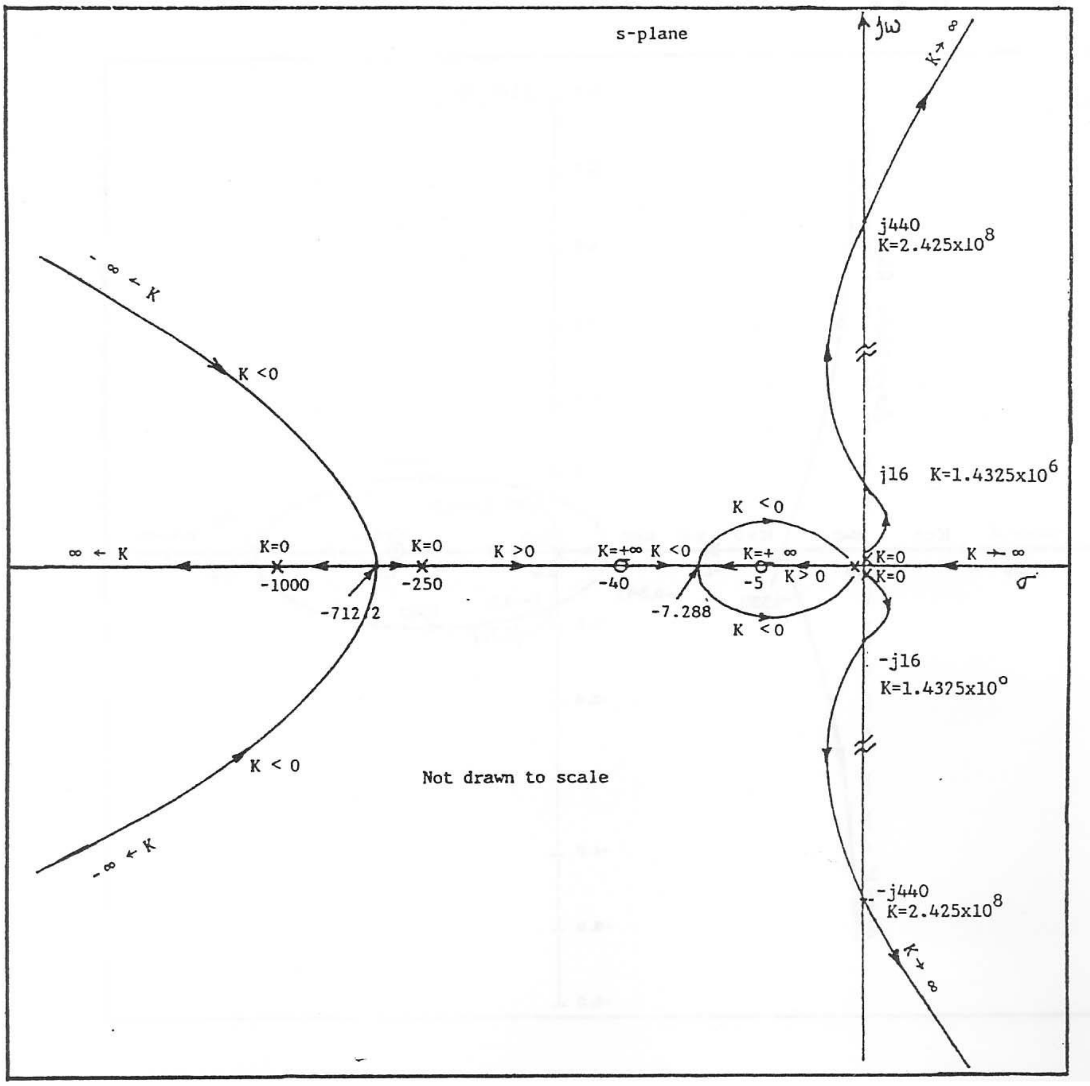
Asymptotes: ***K* > 0:** $60^o,\quad 180^o,\quad 300^o$ ***K* < 0:** $0^o,\quad 120^o,\quad 240^o$

Intersect of asymptotes:

$$\sigma_1 = \frac{0+0+0-250-1000-(-5)-(-40)}{5-2} = -401.67$$

Breakaway-point Equation: $3750s^6 + 335000s^5 + 5.247\times10^8 s^4 + 2.9375\times10^{10} s^3 + 1.875\times10^{11} s^2 = 0$

Breakaway Points: –7.288, –712.2, 0, 0



9-12 (o)

$$G(s)H(s) = \frac{K(s-1)}{s(s+1)(s+2)}$$

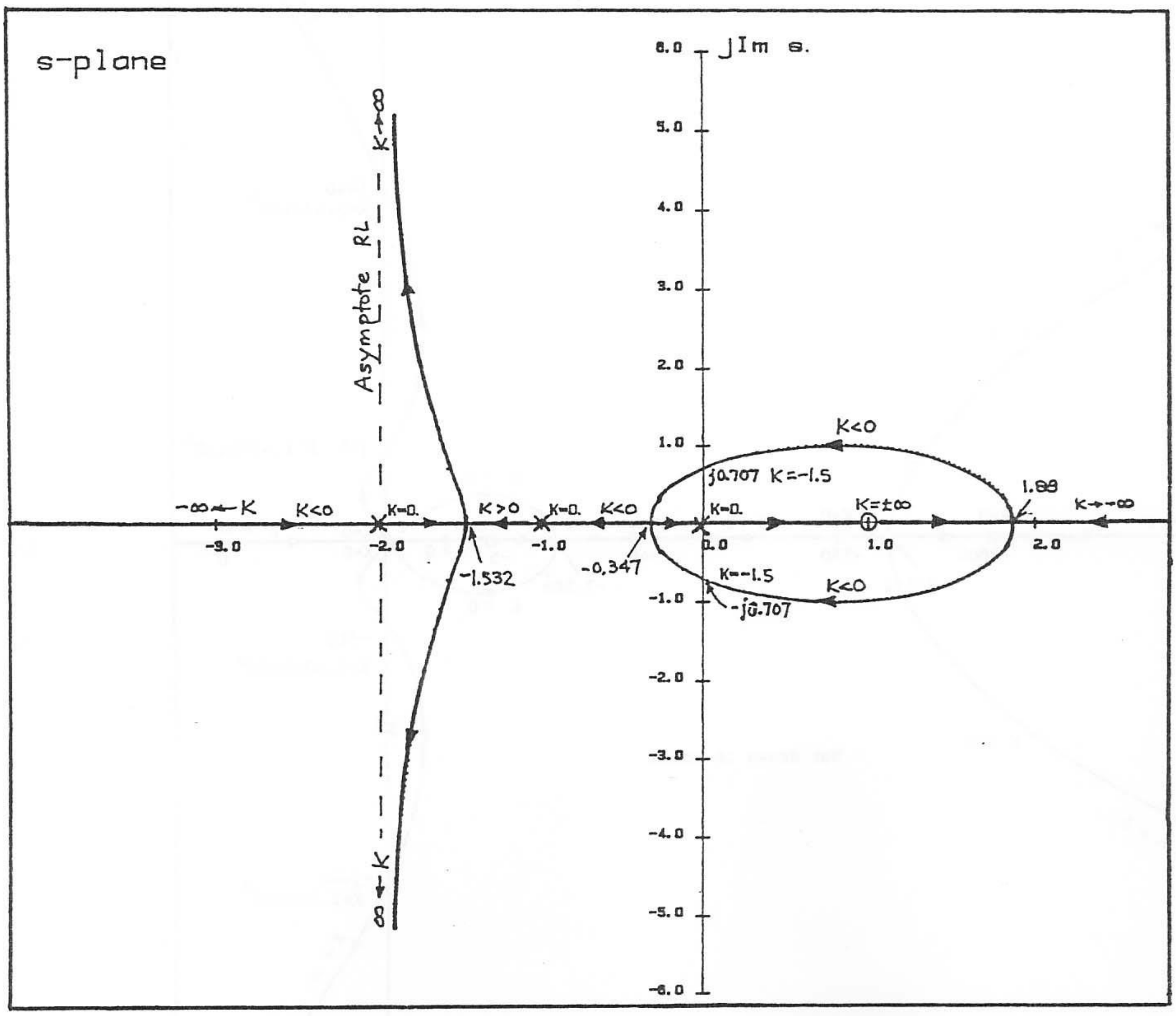
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-2-1}{3-1} = -2$$

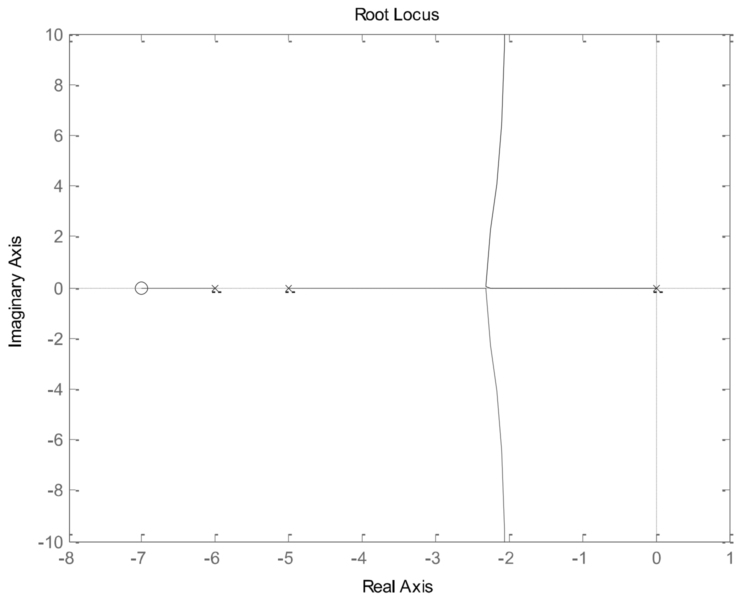
Breakaway-point Equation: $s^3 - 3s - 1 = 0$

Breakaway Points; –0.3473, –1.532, 1.879



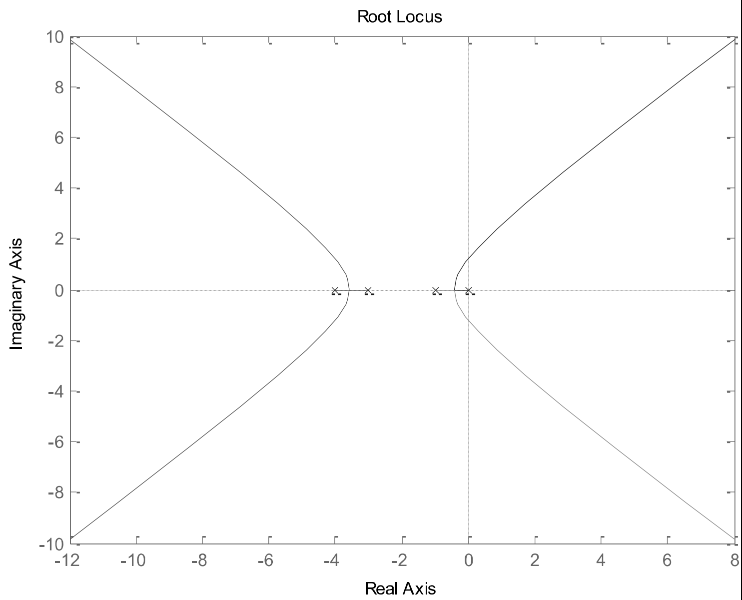
9-13(a) MATLAB code:

```
num=[1 7];
den=conv([1 0],[1 5]);
den=conv(den,[1 6]);
mysys=tf(num,den)
rlocus(mysys);
```



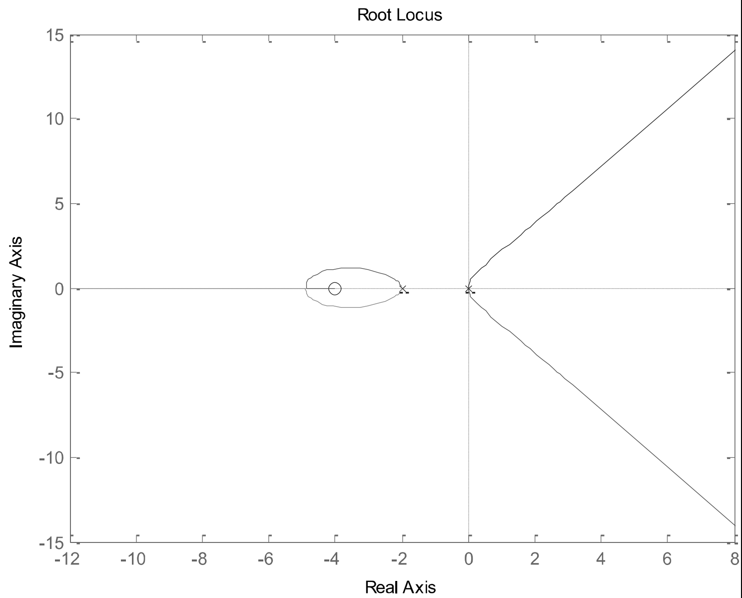
9-13(b) MATLAB code:

```
num=[0 1];
den=conv([1 0],[1 1]);
den=conv(den,[1 3]);
den=conv(den,[1 4]);
mysys=tf(num,den)
rlocus(mysys);
```



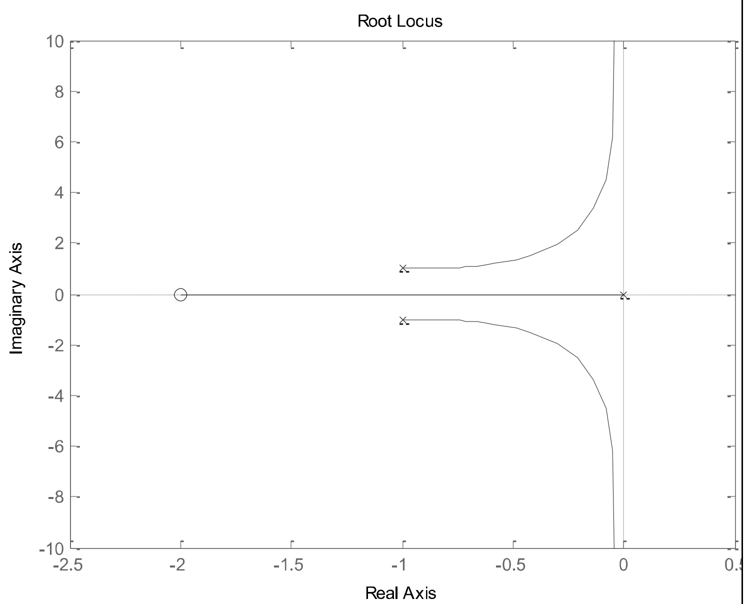
9-13(c) MATLAB code:

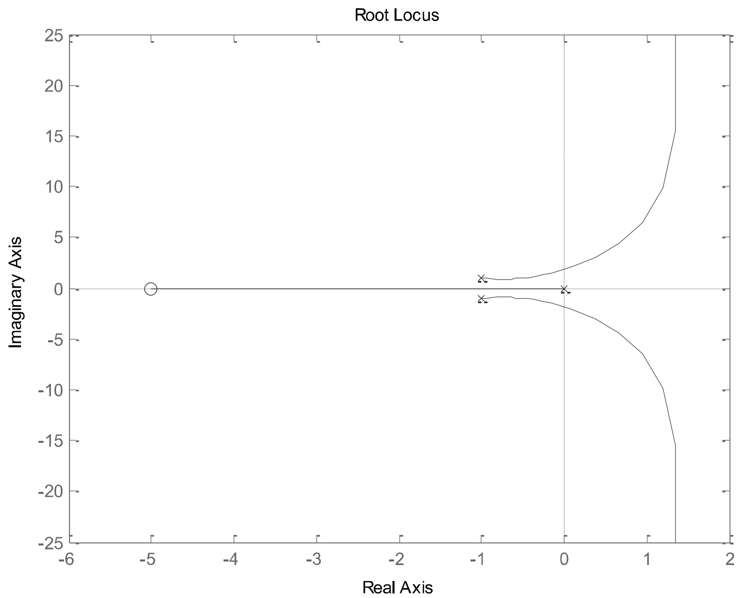
```
num=[1 4];
den=conv([1 0],[1 0]);
den=conv(den,[1 2]);
den=conv(den,[1 2]);
mysys=tf(num,den)
rlocus(mysys);
```



9-13(d) MATLAB code:

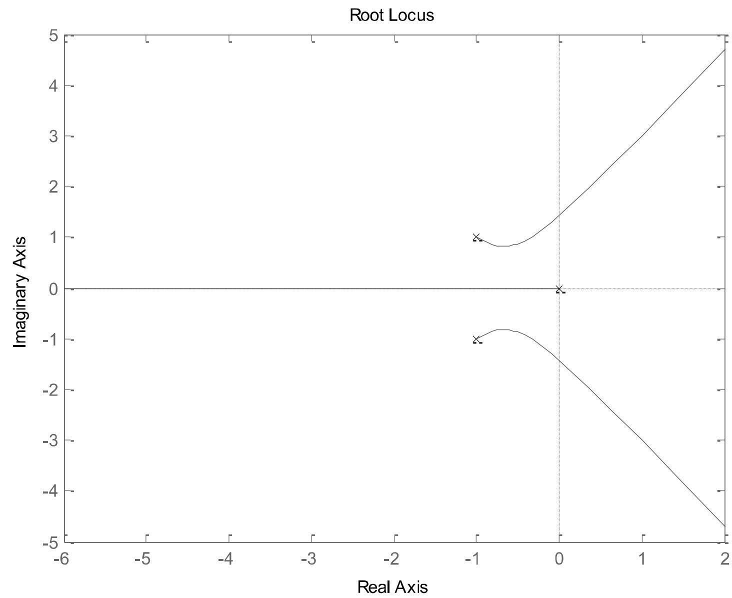
```
num=[1 2];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);
```





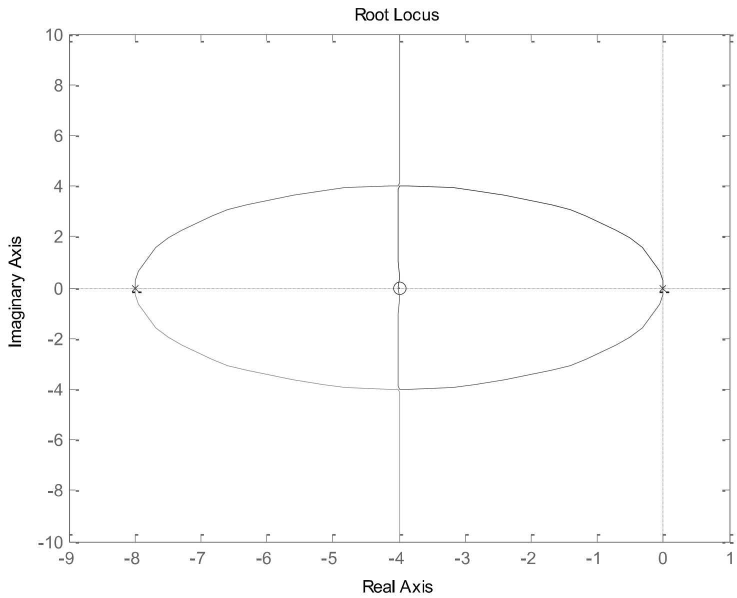
9-13(e) MATLAB code:

```
num=[1 5];
den=conv([1 0],[1
(1+j)]);
den=conv(den,[1 (1-
j)]);
mysys=tf(num,den)
rlocus(mysys);
```



9-13(f) MATLAB code:

```
num=conv([1 4],[1 4]);
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```

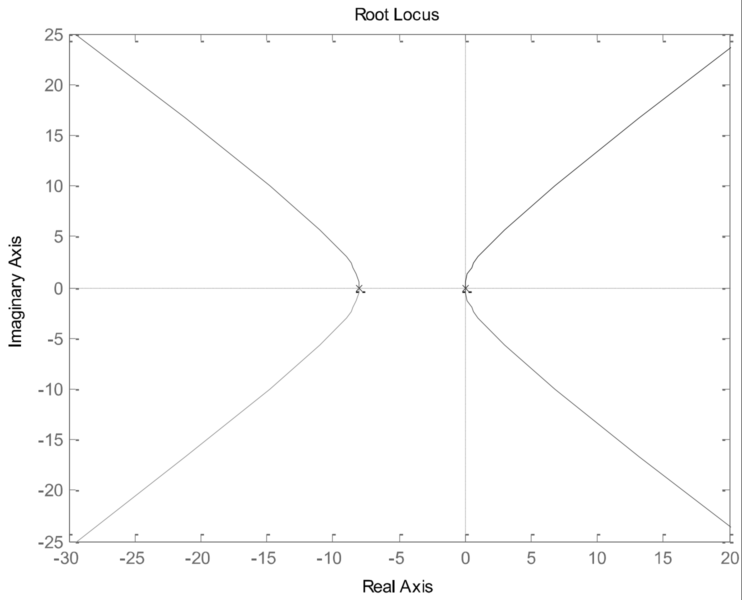


9-13(g) MATLAB code:

```
num=conv([1 4],[1 4]);
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```

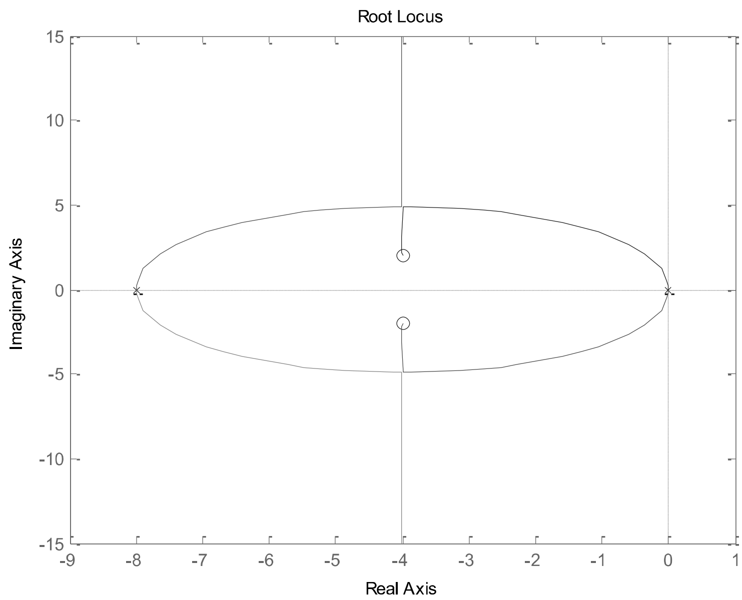
9-13(h) MATLAB code:

```
num=[0 1];
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



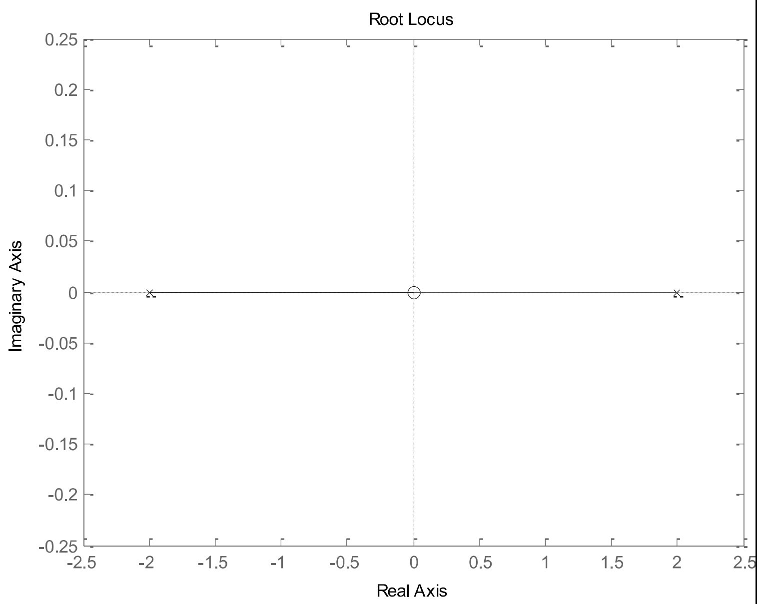
9-13(i) MATLAB code:

```
num=conv([1 4-2j],[1
4+2j])
den=conv([1 0],[1 0]);
den=conv(den,[1 8]);
den=conv(den,[1 8]);
mysys=tf(num,den)
rlocus(mysys);
```



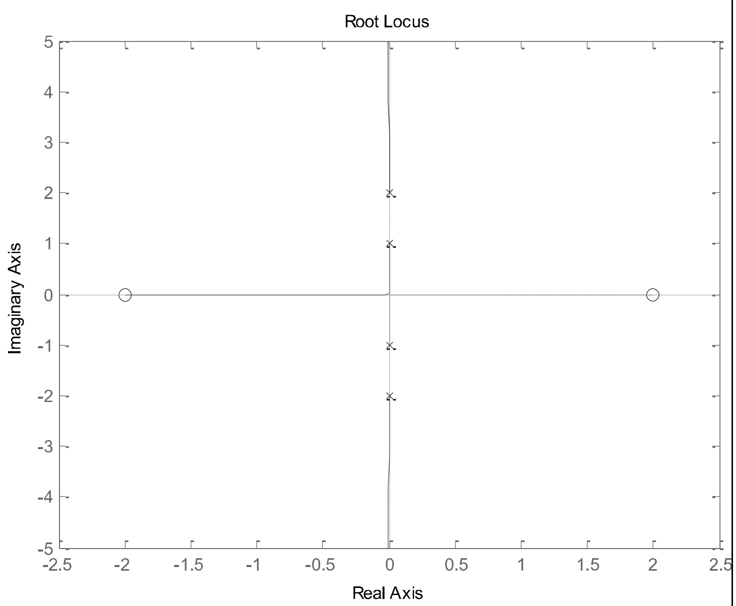
9-13(j) MATLAB code:

```
num=conv([1 0],[1 0]);
den=conv([1 2],[1 -2]);
mysys=tf(num,den)
rlocus(mysys);
```



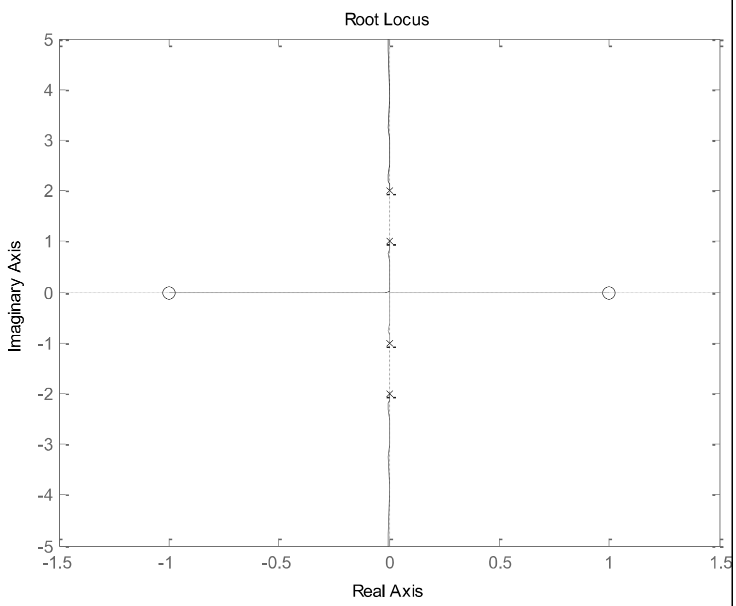
9-13(k) MATLAB code:

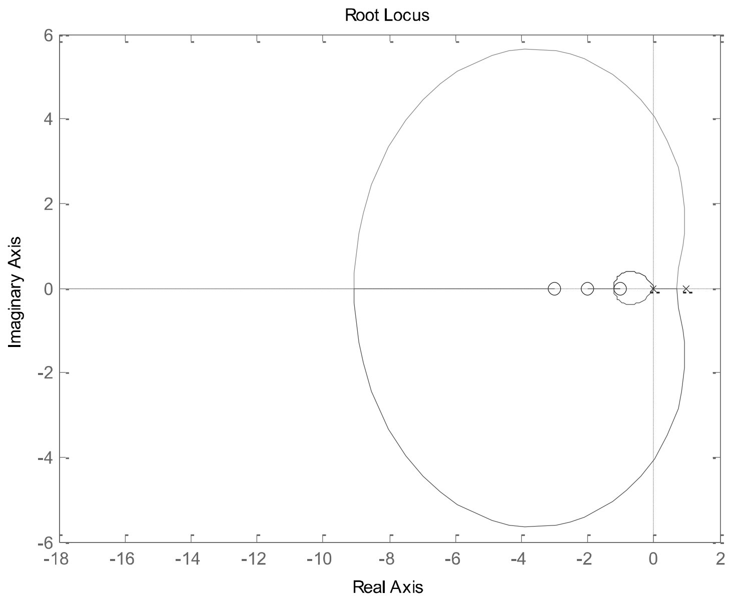
```
num=conv([1 2],[1 -2]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
rlocus(mysys);
```

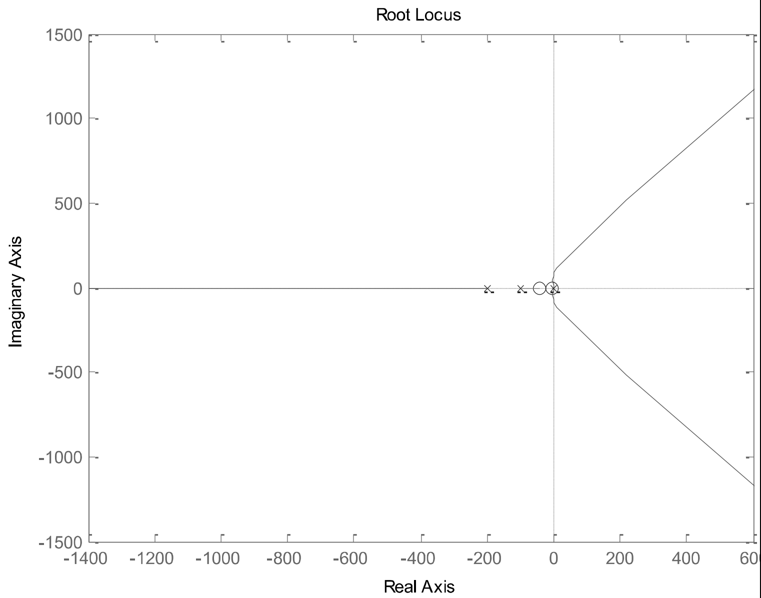


9-13(l) MATLAB code:

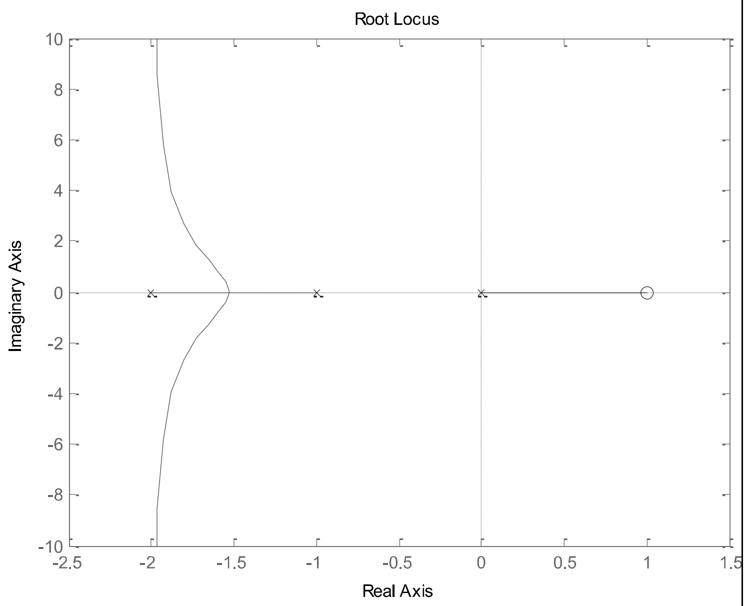
```
num=conv([1 1],[1 -1]);
den=conv([1 -j],[1 j]);
den=conv(den,[1 -2j]);
den=conv(den,[1 2j]);
mysys=tf(num,den)
rlocus(mysys);
```





9-13(m) MATLAB code:
`num=conv([1 1],[1 2]);`
`num=conv(num,[1 3]);`
`den=conv([1 0],[1 0]);`
`den=conv(den,[1 0]);`
`den=conv(den,[1 -1]);`
`mysys=tf(num,den)`
`rlocus(mysys);`
	

9-13(n) MATLAB code:
`num=conv([1 5],[1 40]);`
`den=conv([1 0],[1 0]);`
`den=conv(den,[1 0]);`
`den=conv(den,[1 100]);`
`den=conv(den,[1 200]);`
`mysys=tf(num,den)`
`rlocus(mysys);`	

9-13(o) MATLAB code:
`num=conv([1 5],[1 40]);`
`den=conv([1 0],[1 0]);`
`den=conv(den,[1 0]);`
`den=conv(den,[1 100]);`
`den=conv(den,[1 200]);`
`mysys=tf(num,den)`
`rlocus(mysys);`
	

9-14) (a) $Q(s) = s + 5 \qquad P(s) = s\left(s^2 + 3s + 2\right) = s(s+1)(s+2)$

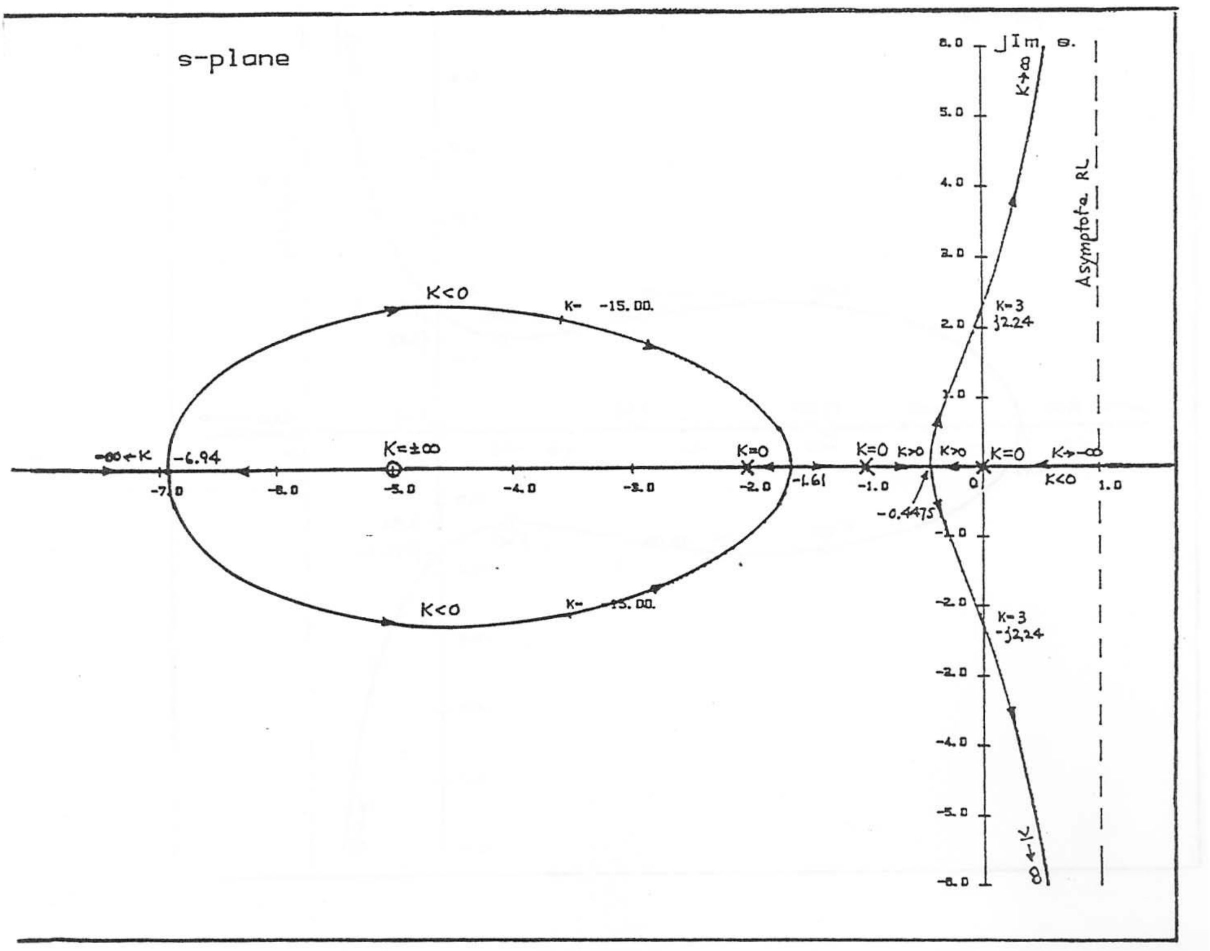
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-2-(-5)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 9s^2 + 15s + 5 = 0$

Breakaway Points: $-0.4475, \quad -1.609, \quad -6.9434$



9-14 (b) $Q(s) = s + 3 \qquad P(s) = s(s^2 + s + 2)$

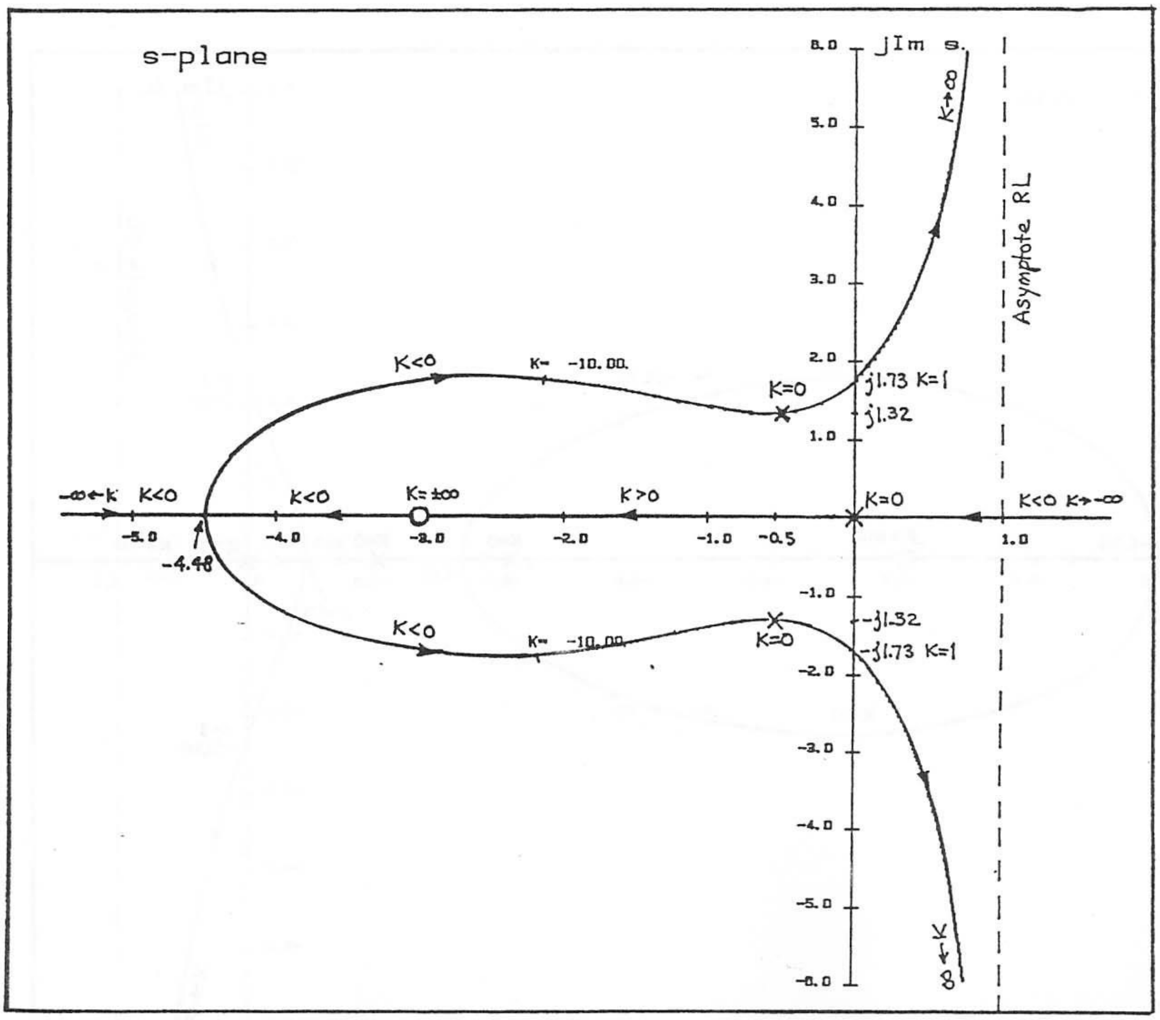
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-1-(-3)}{3-1} = 1$$

Breakaway-point Equation: $s^3 + 5s^2 + 3s + 3 = 0$

Breakaway Points: –4.4798 The other solutions are not breakaway points.

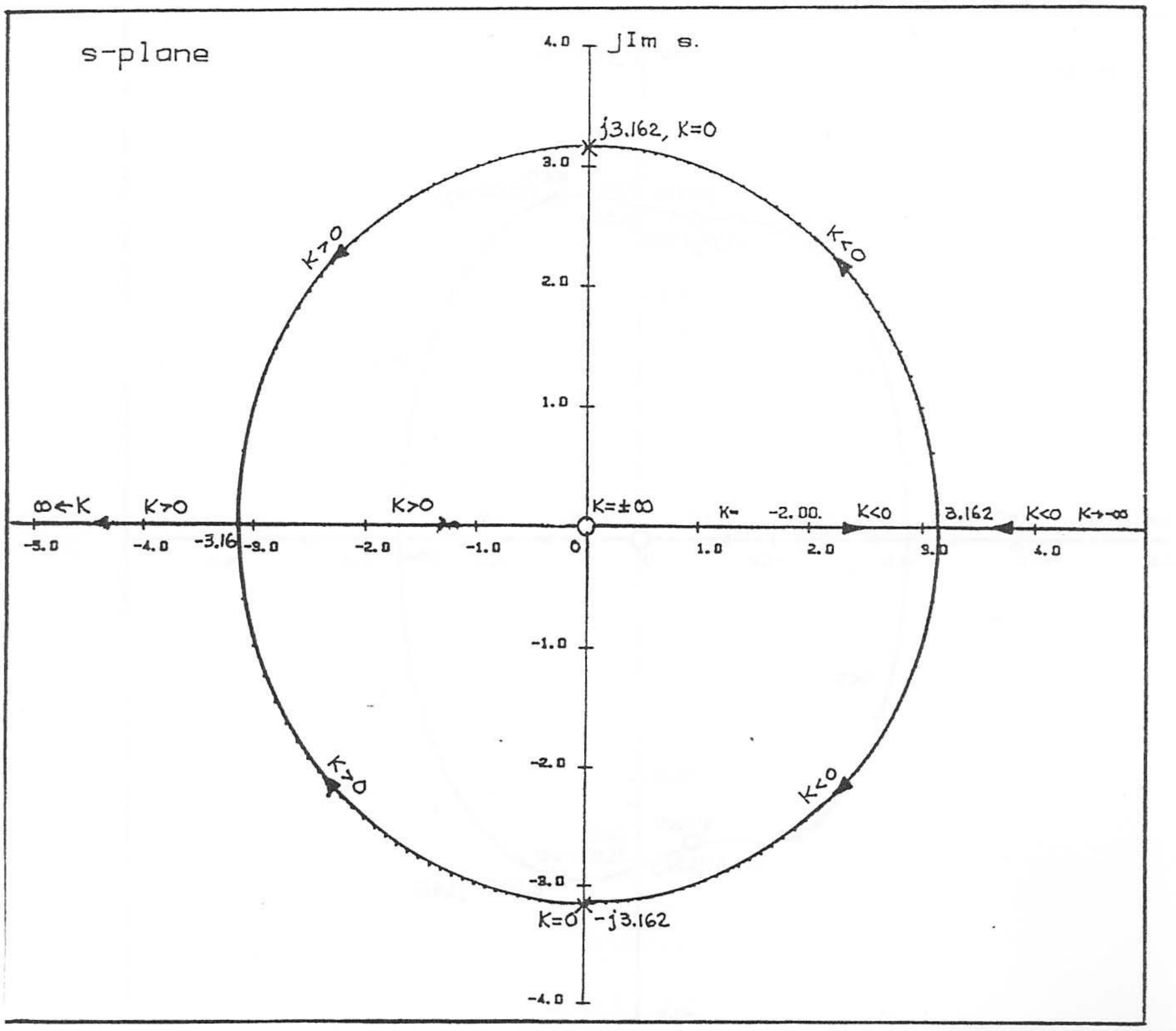


9-14 (c) $Q(s) = 5s \qquad P(s) = s^2 + 10$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $5s^2 - 50 = 0$

Breakaway Points: –3.162, 3.162

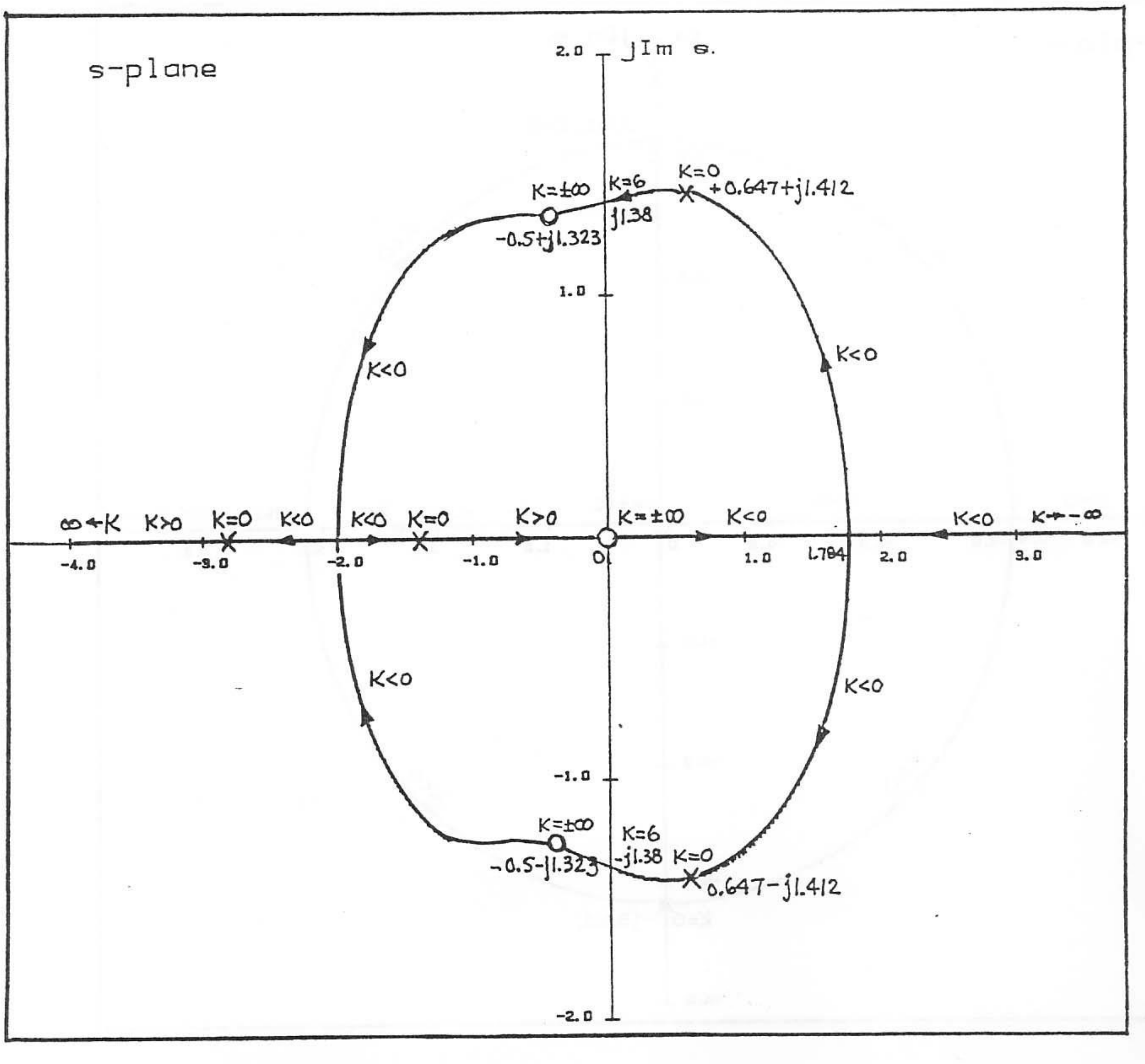


9-14 (d) $Q(s) = s\left(s^2 + s + 2\right)$ $P(s) = s^4 + 3s^3 + s^2 + 5s + 10$

Asymptotes: *K* > 0: 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^6 + 2s^5 + 8s^4 + 2s^3 - 33s^2 - 20s - 20 = 0$

Breakaway Points: -2, 1.784. The other solutions are not breakaway points.

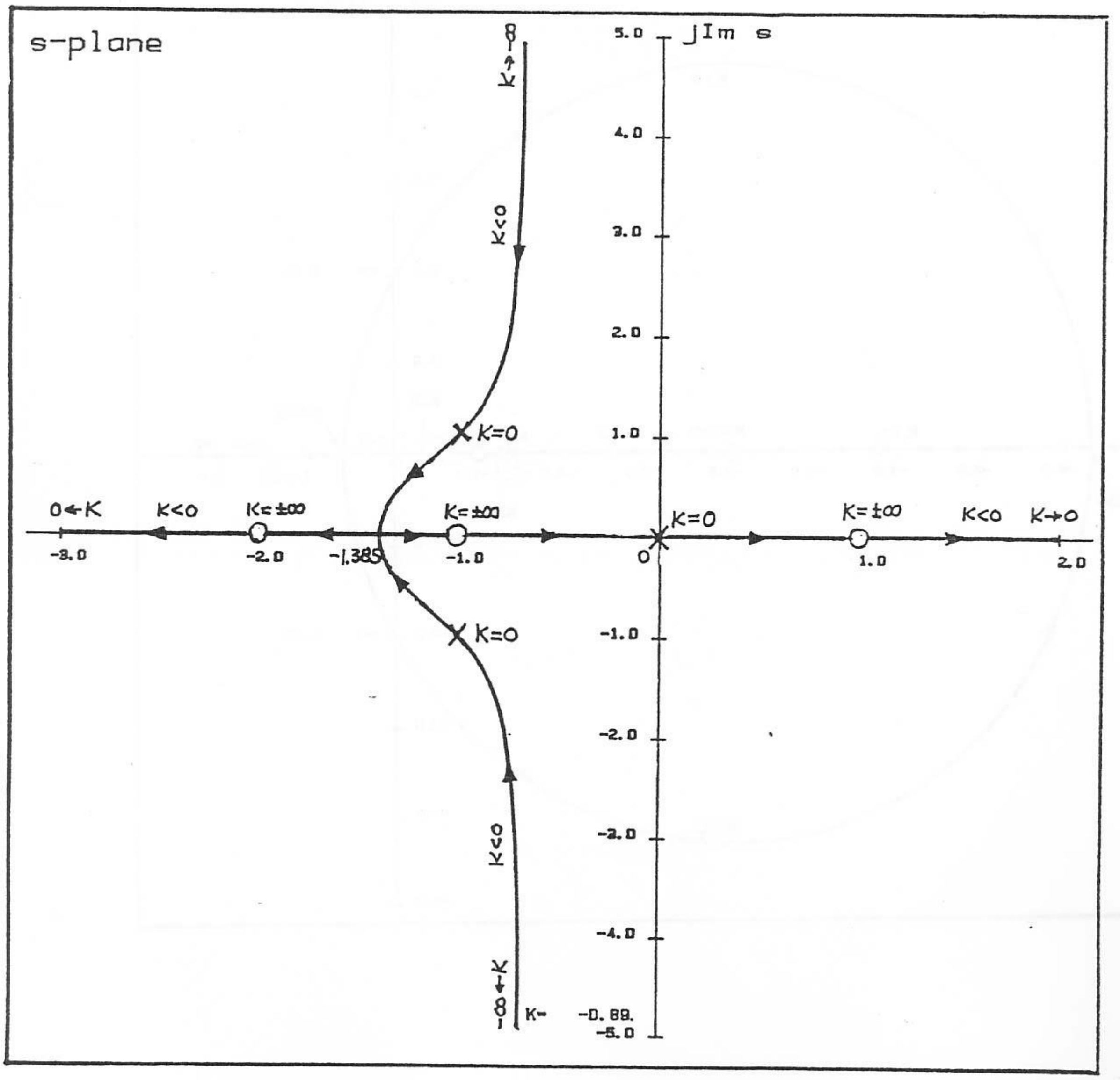


9-14 (e) $Q(s) = \left(s^2 - 1\right)\left(s + 2\right)$ $P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway-point Equation: $6s^3 + 12s^2 + 8s + 4 = 0$

Breakaway Points: –1.3848

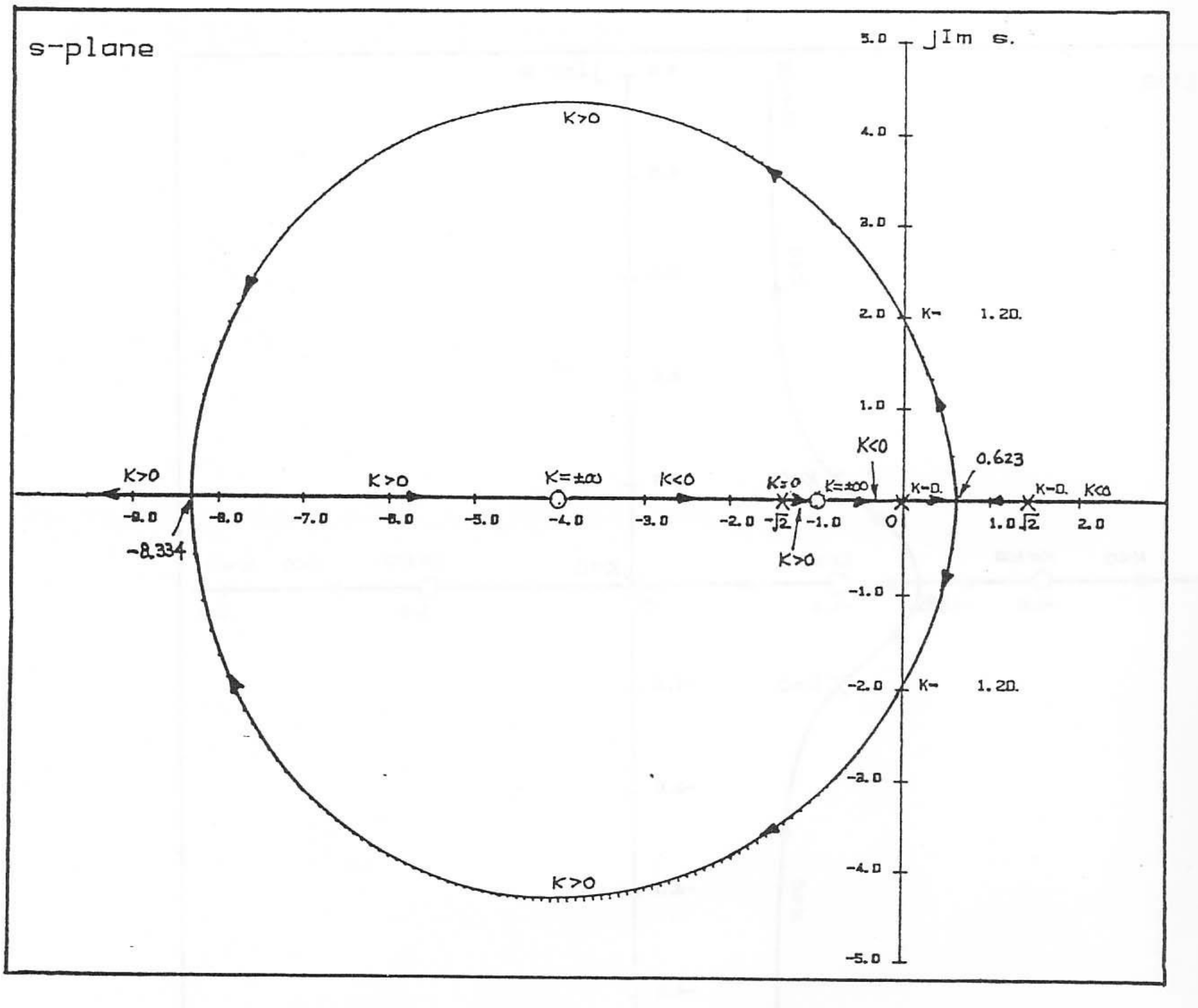


9-14 (f) $Q(s) = (s+1)(s+4) \qquad P(s) = s\left(s^2 - 2\right)$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point equations: $s^4 + 10s^3 + 14s^2 - 8 = 0$

Breakaway Points: –8.334, 0.623



9-14 (g) $Q(s) = s^2 + 4s + 5 \qquad P(s) = s^2\left(s^2 + 8s + 16\right)$

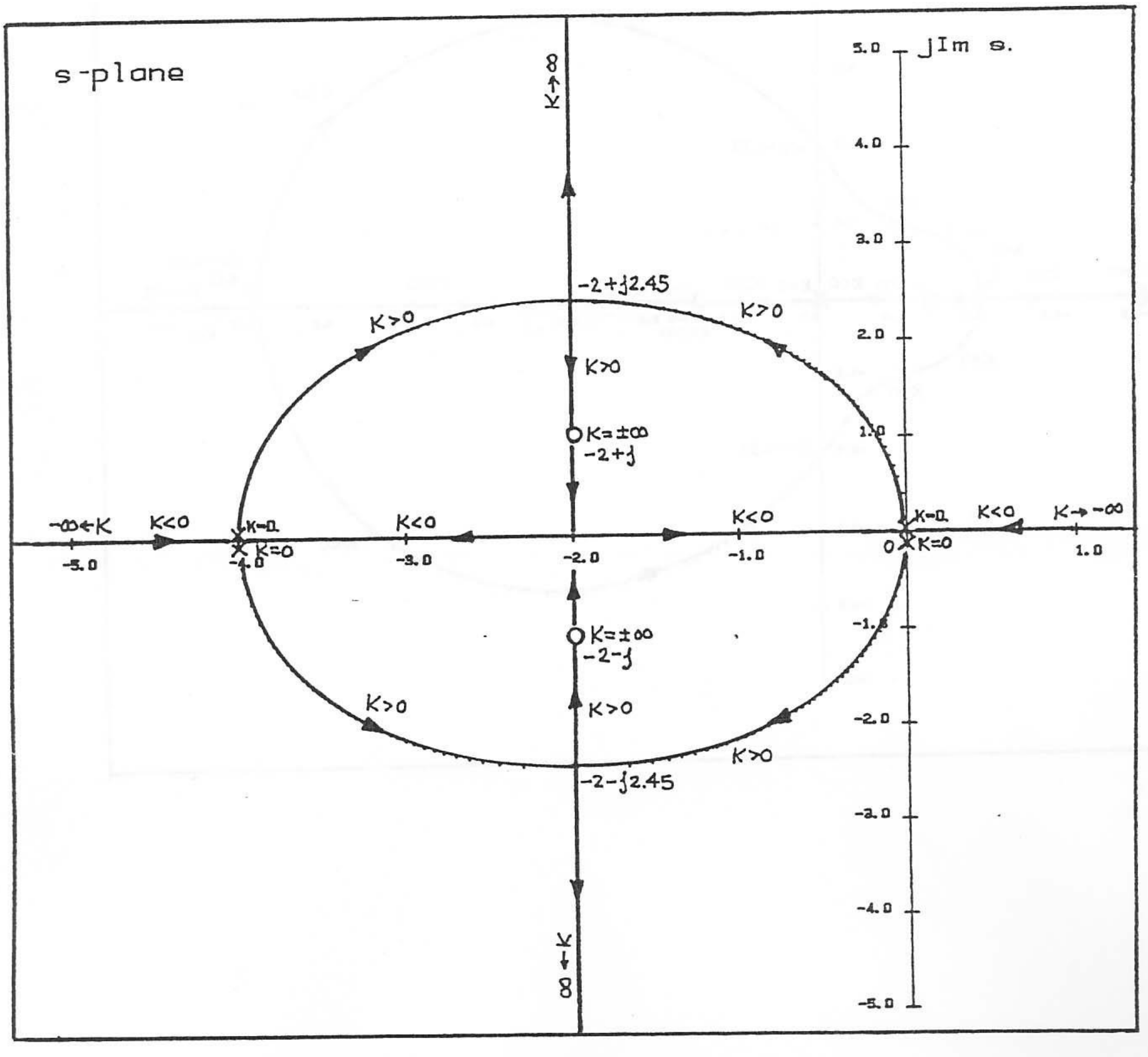
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$ ***K* < 0:** $0^o, \quad 180^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-8-(-4)}{4-2} = -2$$

Breakaway-point Equation: $s^5 + 10s^4 + 42s^3 + 92s^2 + 80s = 0$

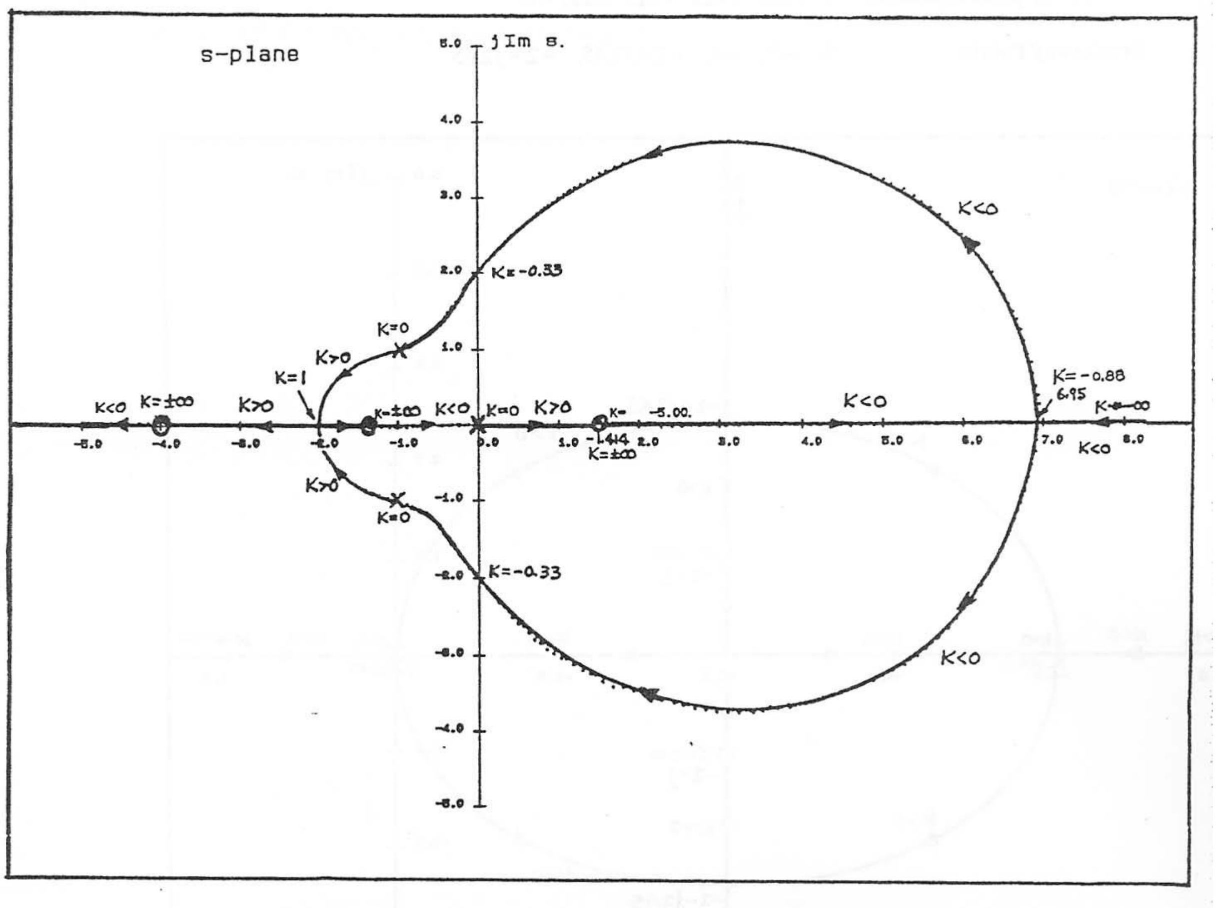
Breakaway Points: $0, \quad -2, \quad -4, \quad -2+j2.45, \quad -2-j2.45$



9-14 (h) $Q(s) = \left(s^2 - 2\right)\left(s + 4\right)$ $P(s) = s\left(s^2 + 2s + 2\right)$

Since $Q(s)$ and $P(s)$ are of the same order, there are no asymptotes.

Breakaway Points: -2, 6.95

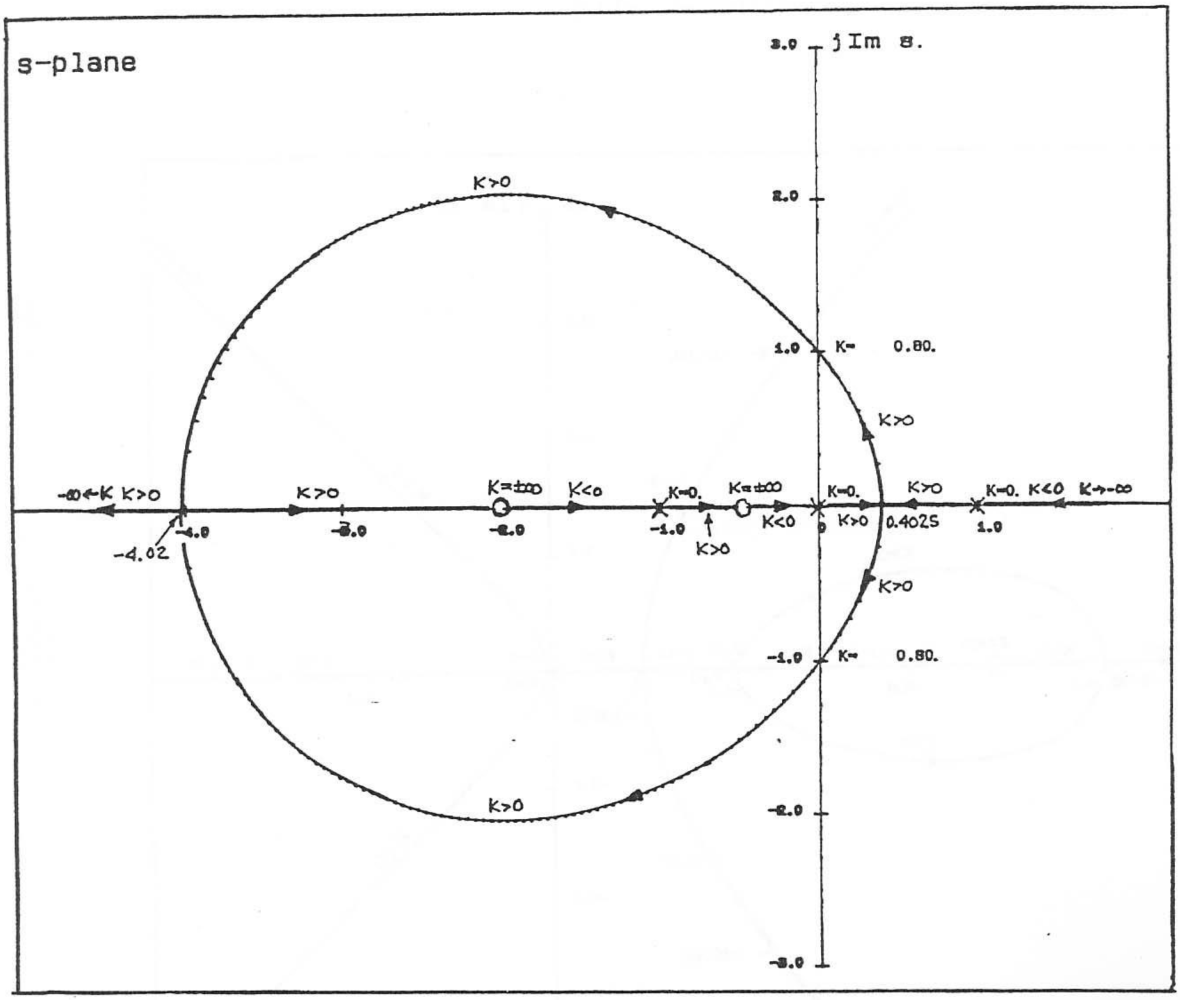


9-14 (i) $Q(s) = (s+2)(s+0.5) \qquad P(s) = s(s^2-1)$

Asymptotes: ***K* > 0:** 180^o ***K* < 0:** 0^o

Breakaway-point Equation: $s^4 + 5s^3 + 4s^2 - 1 = 0$

Breakaway Points: –4.0205, 0.40245 The other solutions are not breakaway points.


s-plane
jIm s.
3.0
2.0
1.0
-1.0
-2.0
-3.0
K>0
K= 0.80.
K>0
-∞←K K>0
K>0
K=±∞
K<o
K=0.
K=±∞
K=0.
K>0
K=0. K<0 K→-∞
-4.0
-3.0
-2.0
-1.0
K<0
0
K>0
0.4025
1.0
-4.02
K>0
K>0
K= 0.80.
K>0

9-14 (j) $Q(s) = 2s + 5 \qquad P(s) = s^2\left(s^2 + 2s + 1\right) = s^2\left(s+1\right)^2$

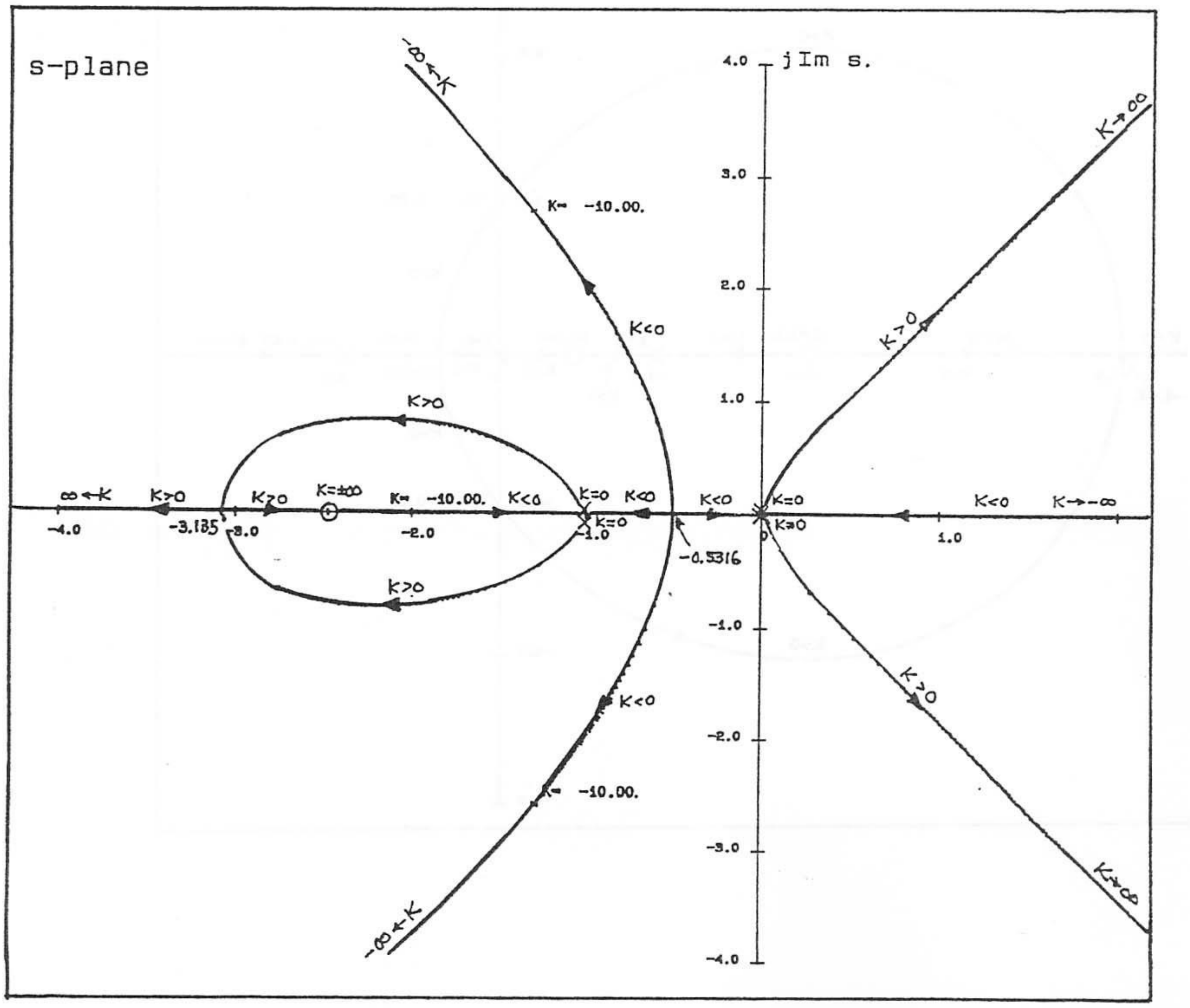
Asymptotes: ***K* > 0:** $60^o,\ 180^o,\ 300^o$ ***K* < 0:** $0^o,\ 120^o,\ 240^o$

Intersect of Asymptotes;

$$\sigma_1 = \frac{0+0-1-1-(-2.5)}{4-1} = \frac{0.5}{3} = 0.167$$

Breakaway-point Equation: $6s^4 + 28s^3 + 32s^2 + 10s = 0$

Breakaway Points: 0, –0.5316, –1, –3.135



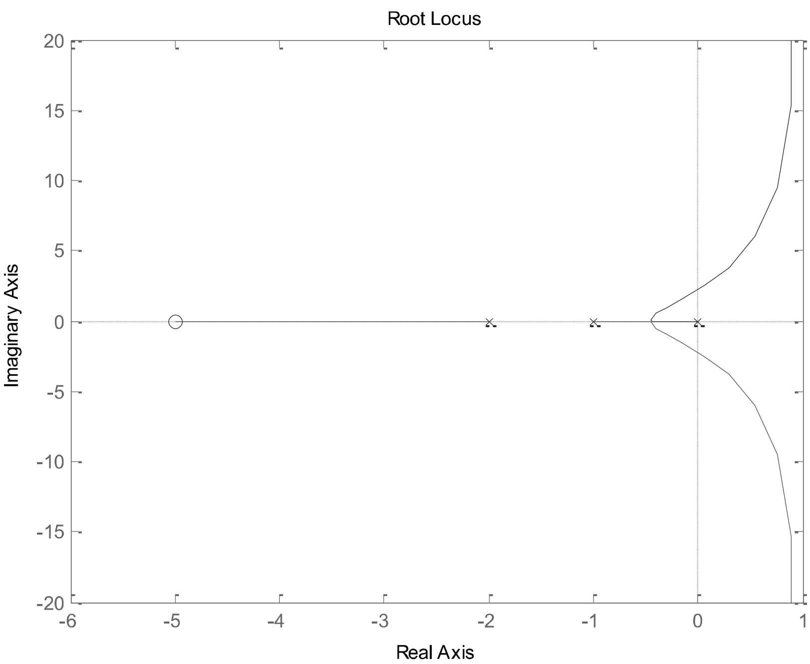
9-15) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
num_GH_a=(s+5);
den_GH_a=(s^3+3*s^2+2*s);
GH_a=num_GH_a/den_GH_a;
figure(1);
rlocus(GH_a)
```

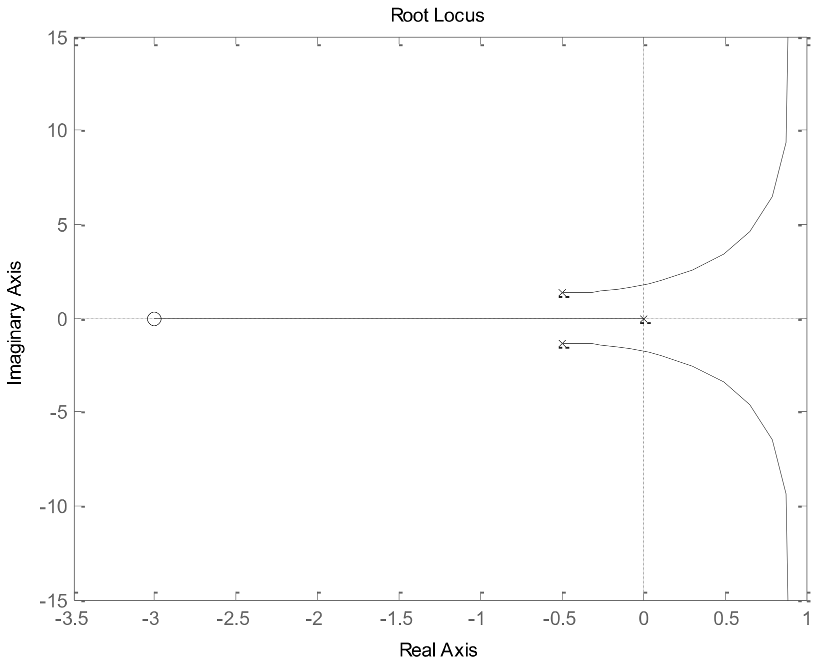
```
%b)
num_GH_b=(s+3);
den_GH_b=(s^3+s^2+2*s);
GH_b=num_GH_b/den_GH_b;
figure(2);
rlocus(GH_b)

%c)
num_GH_c= 5*s^2;
den_GH_c=(s^3+10);
GH_c=num_GH_c/den_GH_c;
figure(3);
rlocus(GH_c)

%d)
num_GH_d=(s^3+s^2+2);
den_GH_d=(s^4+3*s^3+s^2+15);
GH_d=num_GH_d/den_GH_d;
figure(4);
rlocus(GH_d)

%e)
num_GH_e=(s^2-1)*(s+2);
den_GH_e=(s^3+2*s^2+2*s);
GH_e=num_GH_e/den_GH_e;
figure(5);
rlocus(GH_e)

%f)
num_GH_f=(s+4)*(s+1);
den_GH_f=(s^3-2*s);
GH_f=num_GH_f/den_GH_f;
figure(6);
rlocus(GH_f)

%g)
num_GH_g=(s^2+4*s+5);
den_GH_g=(s^4+6*s^3+9*s^2);
GH_g=num_GH_g/den_GH_g;
figure(7);
rlocus(GH_g)


%h)
num_GH_h=(s^2-2)*(s+4);
den_GH_h=(s^3+2*s^2+2*s);
GH_h=num_GH_h/den_GH_h;
```

```
figure(8);
rlocus(GH_h)

%i)
num_GH_i=(s+2)*(s+0.5);
den_GH_i=(s^3-s);
GH_i=num_GH_i/den_GH_i;
figure(9);
rlocus(GH_i)

%j)
num_GH_j=(2*s+5);
den_GH_j=(s^4+2*s^3+2*s^2);
GH_j=num_GH_j/den_GH_j;
figure(10);
rlocus(GH_j)

%k)
num_GH_k=1;
den_GH_k=(s^5+2*s^4+3*s^3+2*s^2+s);
GH_k=num_GH_k/den_GH_k;
figure(11);
rlocus(GH_k)
```

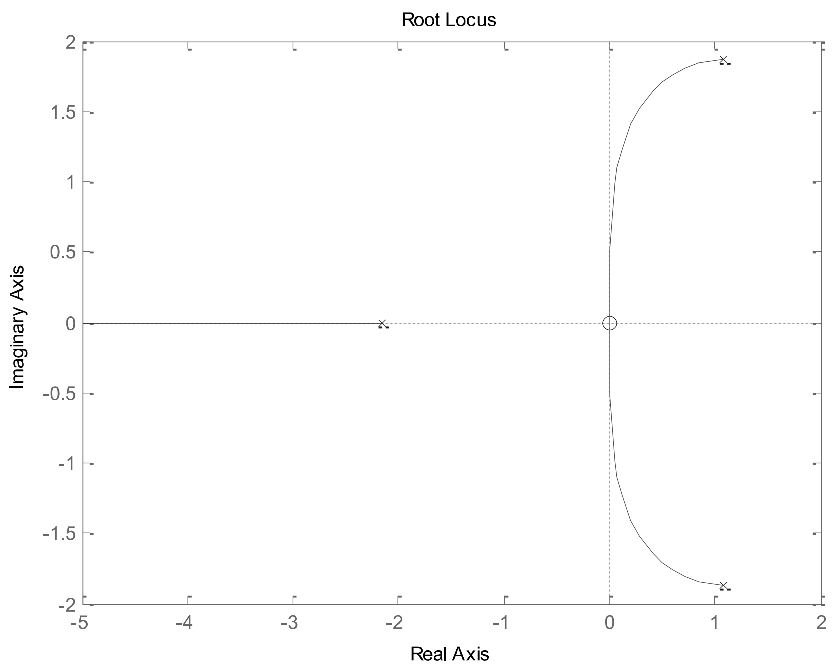
Root Locus diagram – 9-15(a):



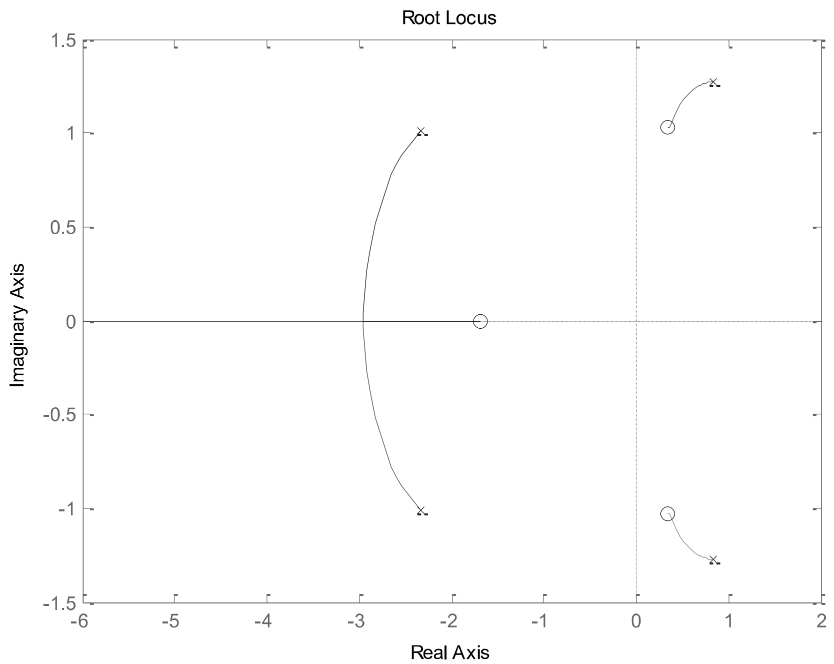
Root Locus diagram – 9-15(b):



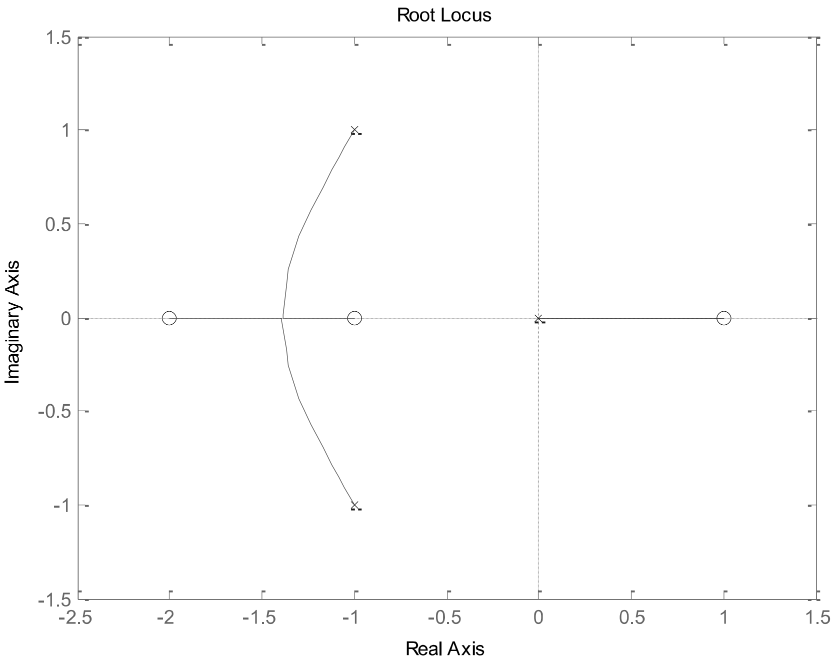
Root Locus diagram – 9-15(c):



Root Locus diagram – 9-15(d):



Root Locus diagram – 9-15(e):



Root Locus diagram – 9-15(f):

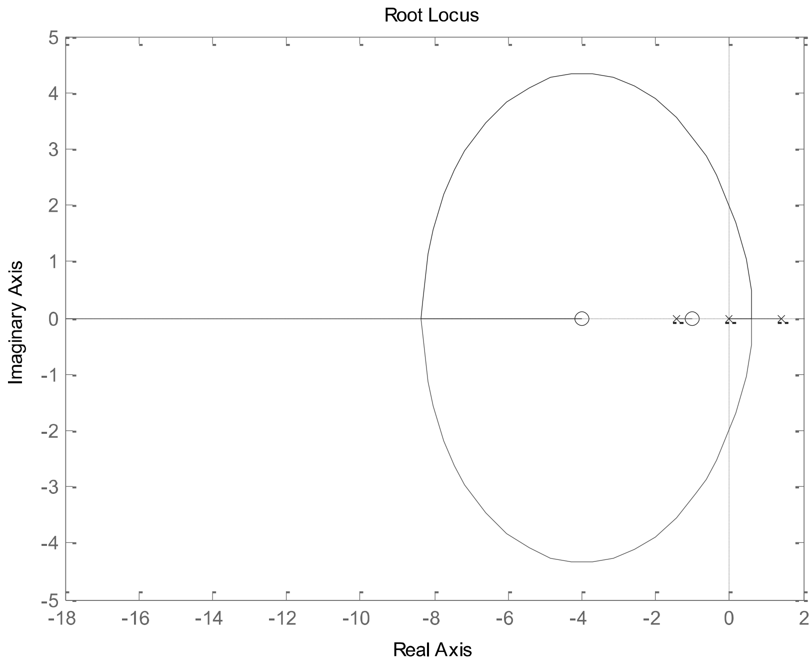
poles: $s = -1, -2 + j, -2 - j$

$\sigma_1 = \frac{-1-2-j-2+j}{3} = -1.67$

Asymptotes angle: $\theta_i = \frac{2i+1}{|n-m|} \times 180 = \frac{2i+1}{3} \times 180$

Therefore, $\theta_i = 60,180,300$

Departure angle from: $\begin{cases} s = -2 - j & : \quad \theta = 45 \\ s = -2 + j & : \quad \theta = -45 \end{cases}$



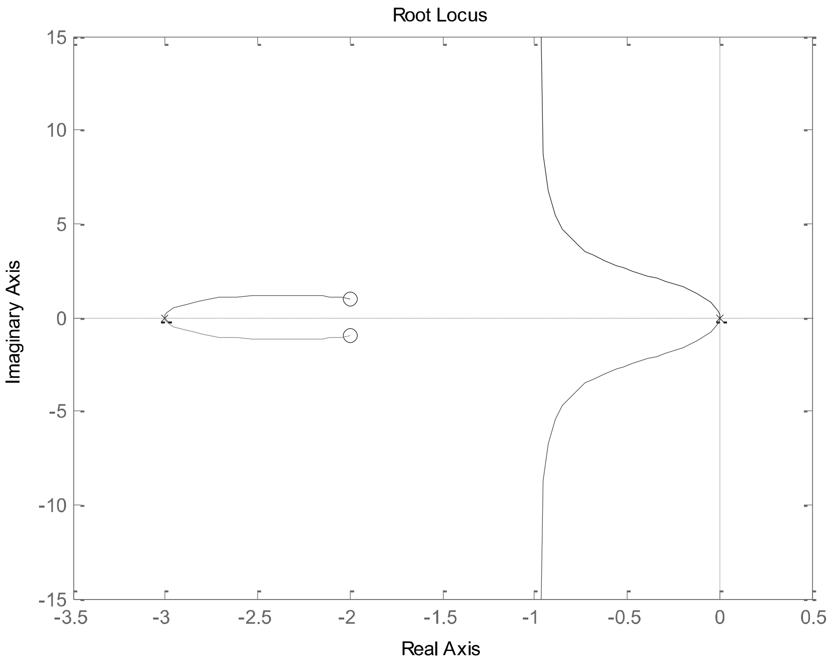
Root Locus diagram – 9-15(g):

Poles: $s = -1, -5 - j, 3 + j$ and zeroes: $s = -2$

$$\sigma_1 = \frac{-1-3-j-3+j+2}{2} = -2.5$$

Asymptotes angles: $\begin{cases} \theta_i = \frac{2i+1}{n-m} 180 = \frac{2i-1}{3-1} 180 \\ \theta_i = 90,270 \end{cases}$

Departure angles from: $\begin{cases} s = -3 - j \quad : \quad \theta = -72^o \\ s = -2 + j \quad : \quad \theta = 72^o \end{cases}$



Root Locus diagram – 9-15(h):

Poles: $s = 0, -1$ and zeros: $s = -2, -3$

The break away points:

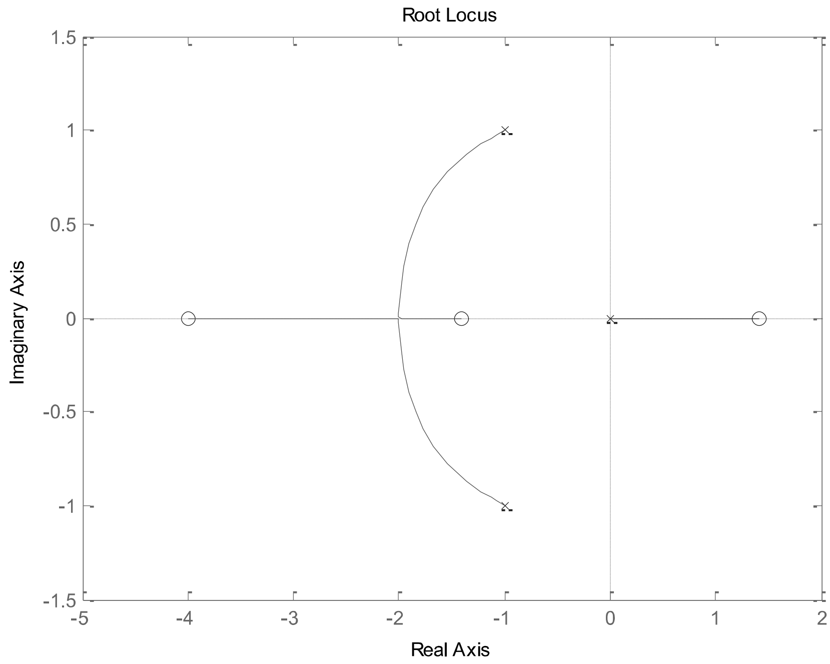
$$-\frac{d}{ds}\left[\frac{P(s)}{Q(s)}\right] = 0$$

which means:

$$-\frac{d}{ds}\left[\frac{s(s+1)}{(s+2)(s+3)}\right] = 0$$

or

$$\frac{1}{s+1} + \frac{1}{s} = \frac{1}{s+2} + \frac{1}{s+3}$$
$$(2s+1)(s^2+5s+6) - (2s+5)(s^2+s) = 0$$
$$4s^2 + 12s + 6 = 0$$
$$\begin{cases} s = -0.634 \\ s = -2.366 \end{cases}$$



Root Locus diagram – 9-15(i):

Poles: $s = 0, -2-j, -2+j$

breaking points: $-\frac{d}{ds}(s^3 + 4s^2 + 5s) = 0$

which means :

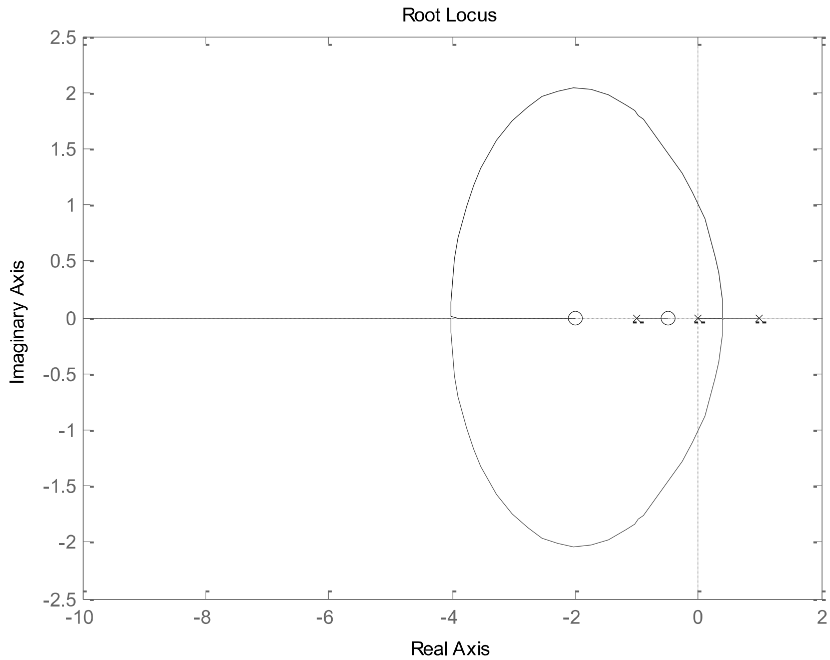
$$\begin{cases} s = -1 \\ s = -1.67 \end{cases}$$

Departure angles from: $\begin{cases} s = -2j & : \quad \theta = -63.43 \\ s = -2+j & : \quad \theta = 63.43 \end{cases}$

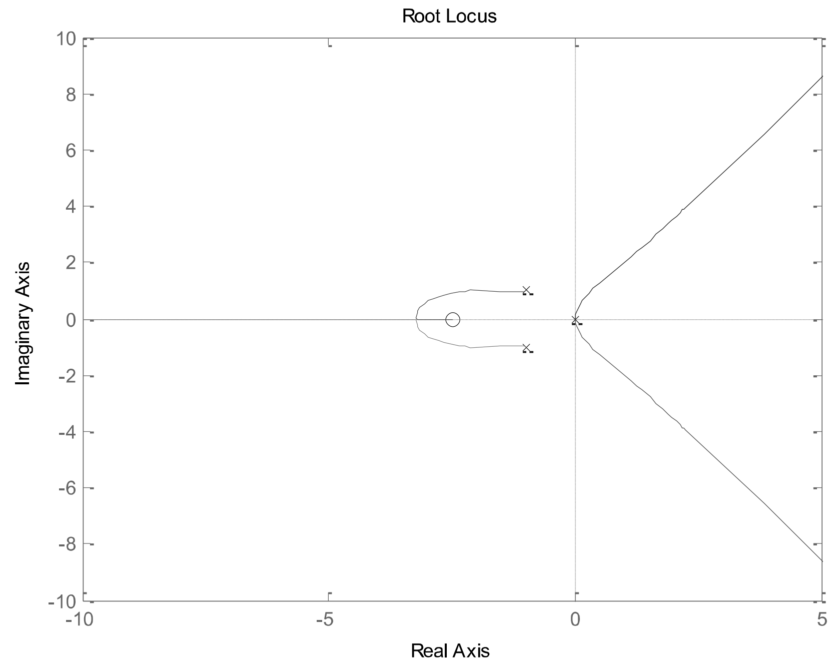
Asymptotes angles: $\theta_i = \frac{2i+1}{n-m} \times 180 = \frac{2i+1}{3} \times 180$

or θ = $60^o, 180^o, 300^o$

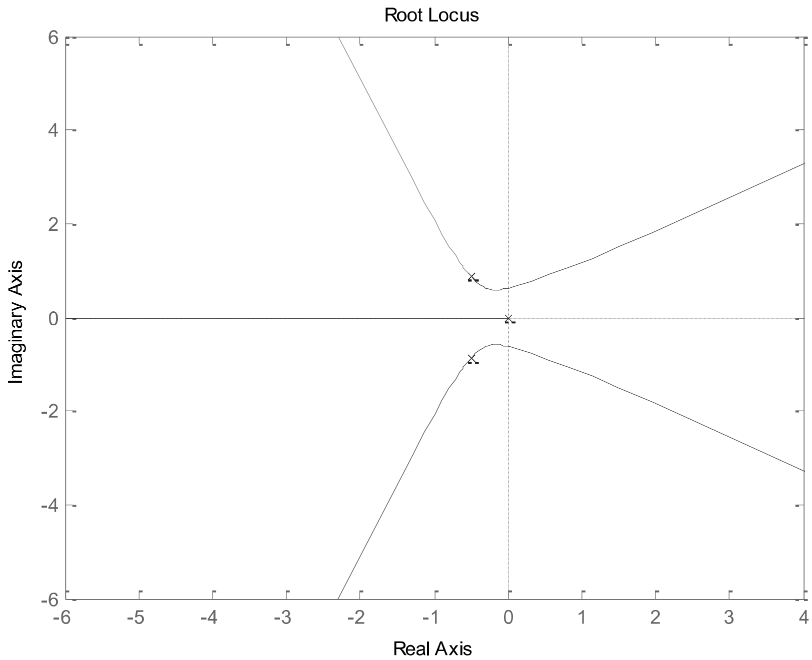
$$v_1 = \frac{-2-j-2+j}{3} = -\frac{4}{3}$$



Root Locus diagram – 9-15(j):



Root Locus diagram – 9-15(k):



9-16) (a) **Asymptotes: *K* > 0:** $45^o, \quad 135^o, \quad 225^o, \quad 315^o$

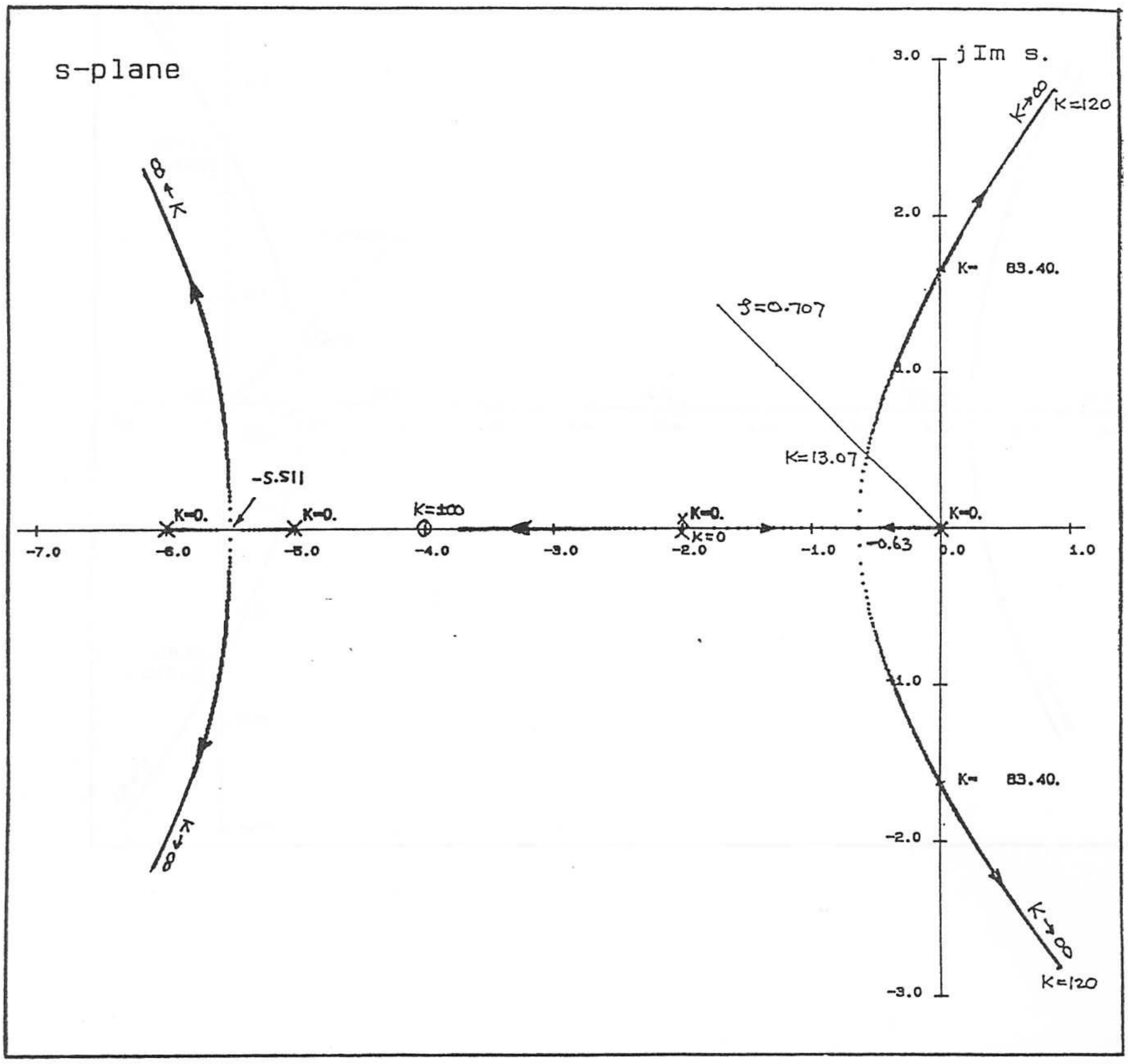
Intersect of Asymptotes:

$$\sigma_1 = \frac{-2-2-5-6-(-4)}{5-1} = -2.75$$

Breakaway-point Equation: $4s^5 + 65s^4 + 396s^3 + 1100s^2 + 1312s + 480 = 0$

Breakaway Points: –0.6325, –5.511 (on the RL)

When $\zeta = 0.707$, ***K* = 13.07**



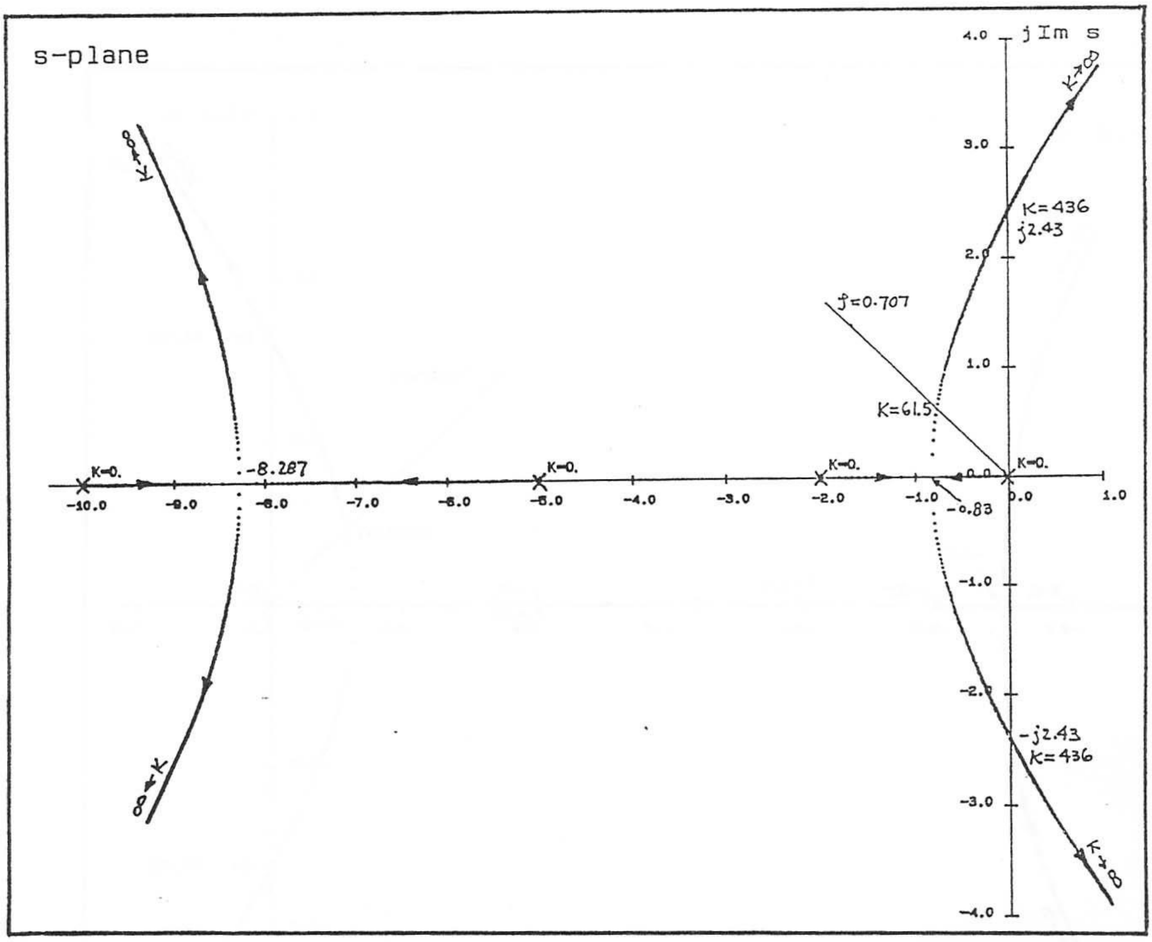
9-16 (b) **Asymptotes: *K* > 0:** $45^o,\ 135^o,\ 225^o,\ 315^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-2-5-10}{4} = -4.25$$

Breakaway-point Equation: $4s^3 + 51s^2 + 160s + 100 = 0$

When $\zeta = 0.707,$ ***K* = 61.5**

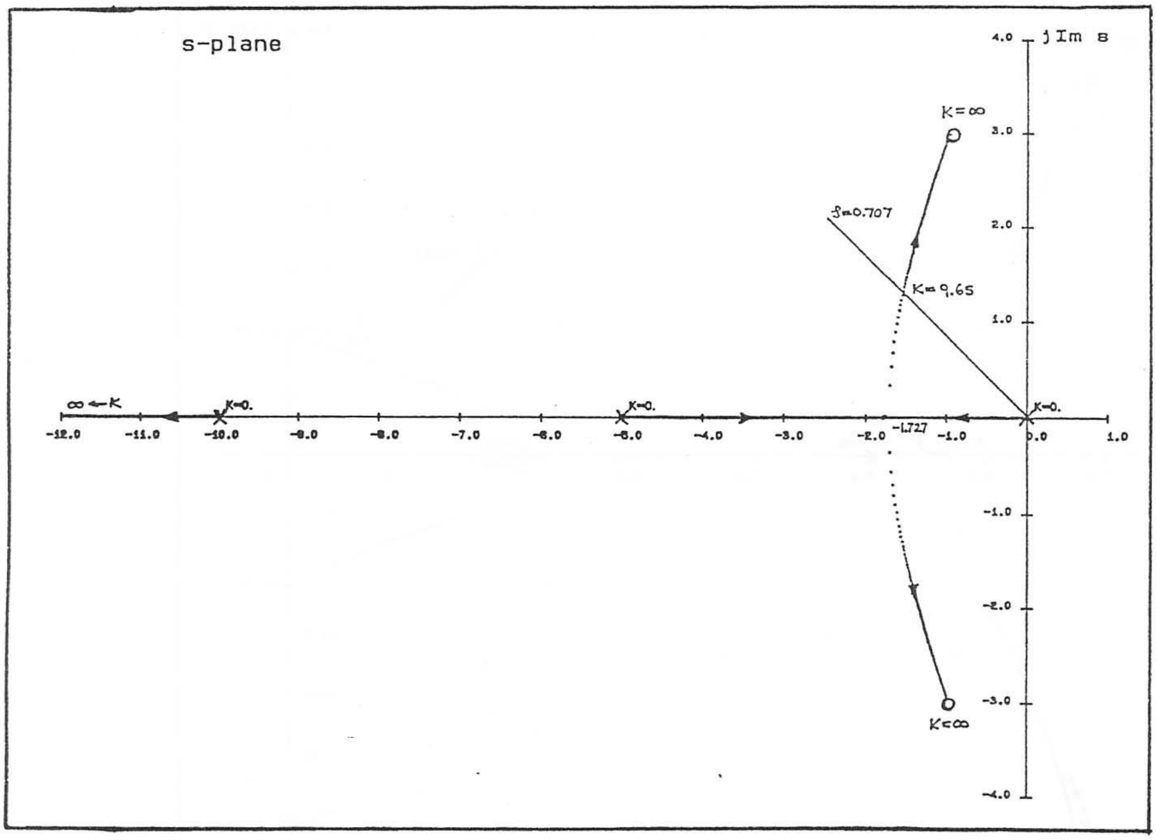


9-16 (c) **Asymptotes:** ***K* > 0:** 180^o

Breakaway-point Equation: $s^4 + 4s^3 + 10s^2 + 300s + 500 = 0$

Breakaway Points: –1.727 (on the RL)

When $\zeta = 0.707$, ***K* = 9.65**

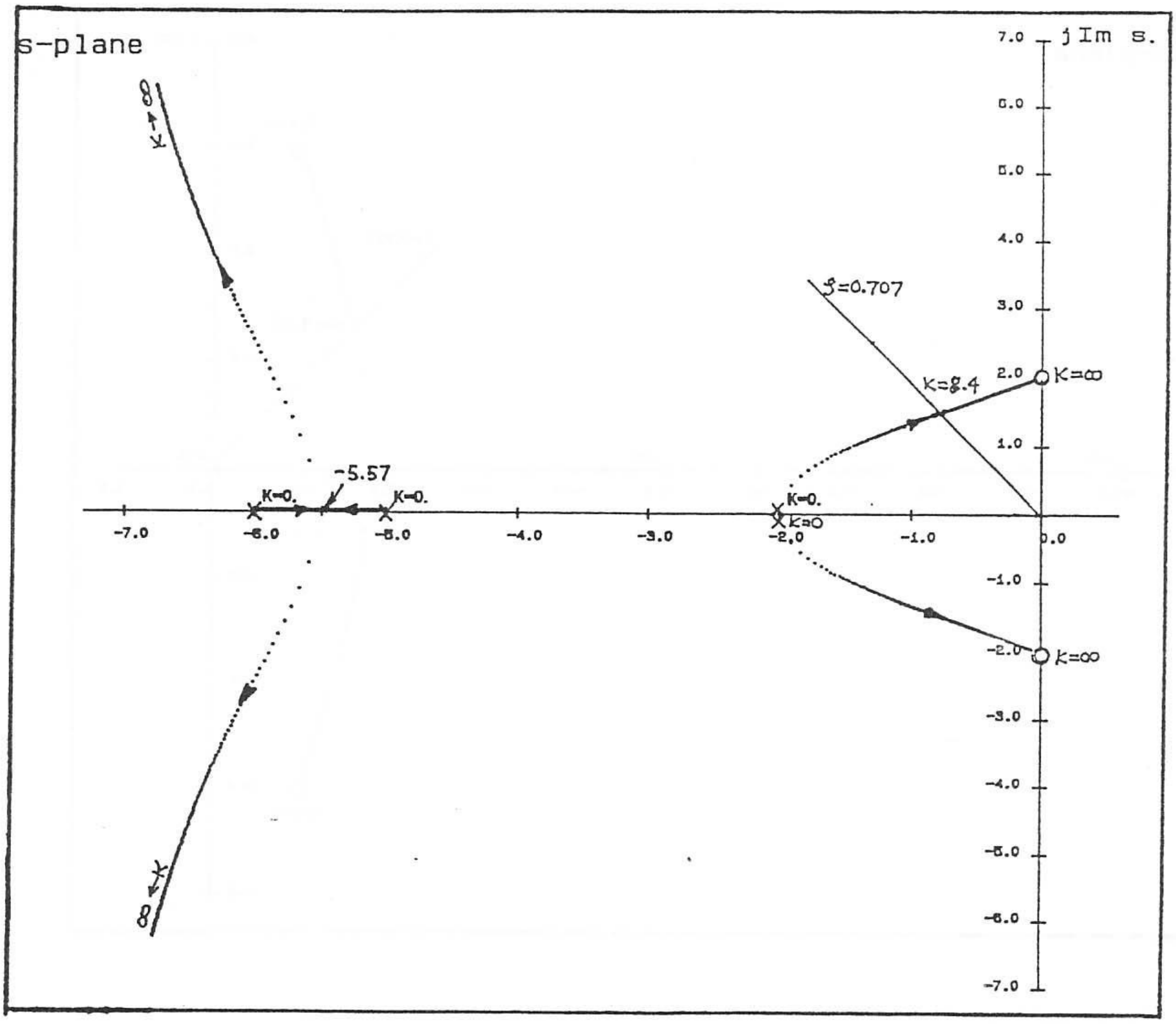

s-plane
jIm s
K=∞
K=∞
ζ=0.707
K=9.65
∞ ← K
K=0.
K=0.
K=0.
-1.727
-12.0
-11.0
-10.0
-9.0
-8.0
-7.0
-6.0
-5.0
-4.0
-3.0
-2.0
-1.0
0.0
1.0
4.0
3.0
2.0
1.0
-1.0
-2.0
-3.0
-4.0

9-16 (d) ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-2-2-5-6}{4-2} = -7.5$$

When $\zeta = 0.707,$ ***K* = 8.4**



9-17) MATLAB code:

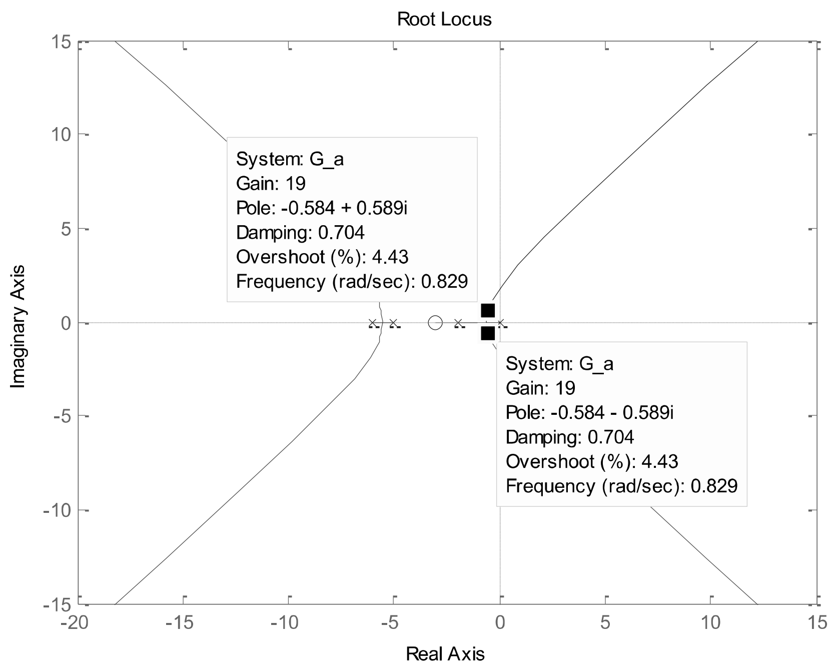
```
clear all;
close all;
s = tf('s')

%a)
num_G_a=(s+3);
den_G_a=s*(s^2+4*s+4)*(s+5)*(s+6);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
```

```
num_G_b= 1;
den_G_b=s*(s+2)*(s+4)*(s+10);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
num_G_c=(s^2+2*s+8);
den_G_c=s*(s+5)*(s+10);
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)

%d)
num_G_d=(s^2+4);
den_G_d=(s+2)^2*(s+5)*(s+6);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)

%e)
num_G_e=(s+10);
den_G_e=s^2*(s+2.5)*(s^2+2*s+2);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

%f)
num_G_f=1;
den_G_f=(s+1)*(s^2+4*s+5);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)

%g)
num_G_g=(s+2);
den_G_g=(s+1)*(s^2+6*s+10);
G_g=num_G_g/den_G_g;
figure(7);
rlocus(G_g)

%h)
num_G_h=(s+3)*(s+2);
den_G_h=s*(s+1);
G_h=num_G_h/den_G_h;
figure(8);
rlocus(G_h)
```

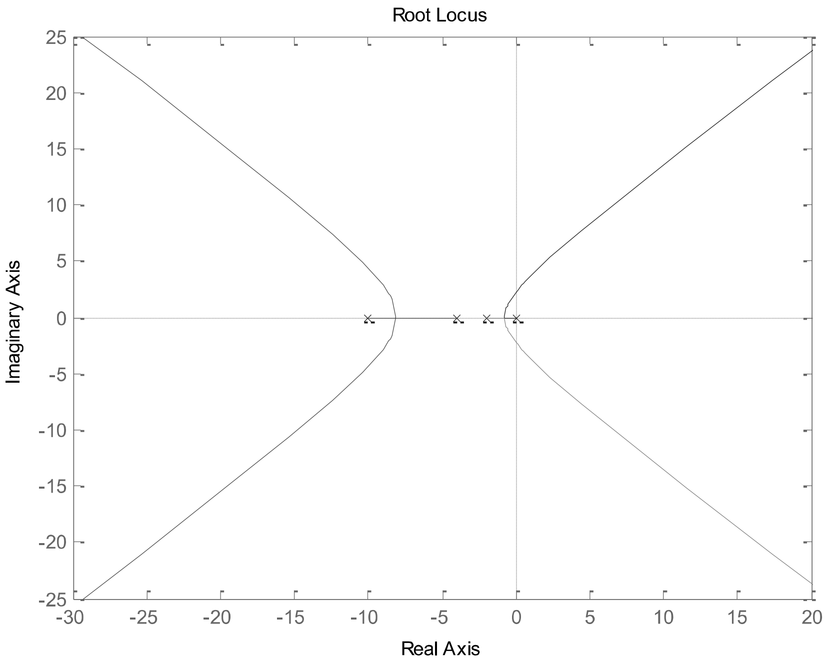
```
%i)
num_G_i=1;
den_G_i=s*(s^2+4*s+5);
G_i=num_G_i/den_G_i;
figure(9);
rlocus(G_i)
```

Root Locus diagram – 9-17(a):

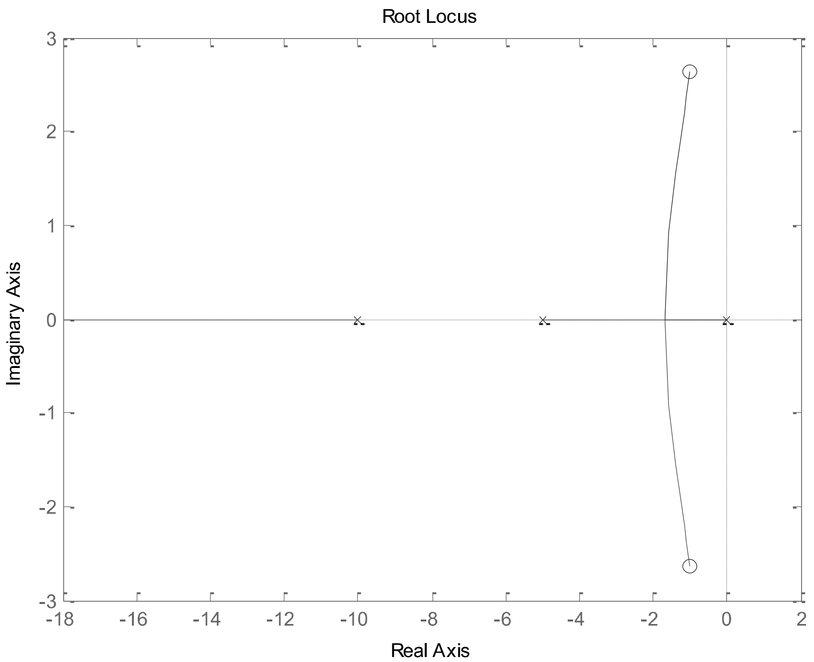
By using "Data Cursor" tab on the figure window and clicking on the root locus diagram, gain and damping values can be observed. Damping of ~0.707 can be observed on intersection of the root locus diagram with two lines originating from (0,0) by angles of ArcCos(0.707) from the real axis. These intersection points are shown for part (a) where the corresponding gain is 19. In the other figures for section (b) to (i), similar points have been picked by the "Data Cursor", and the gains are reported here.



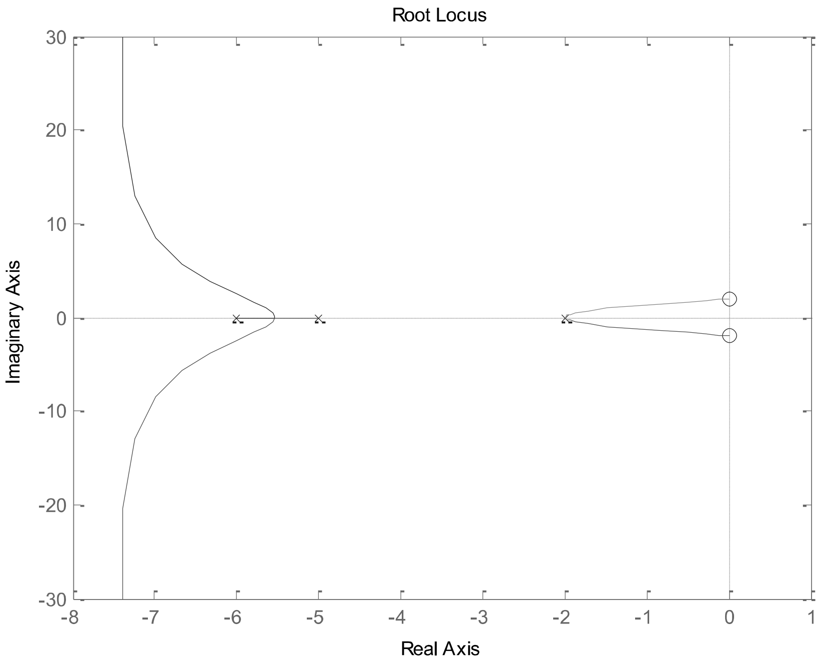
Root Locus diagram – 9-17(b): (K = 45.5 @ damping = ~0.0707)



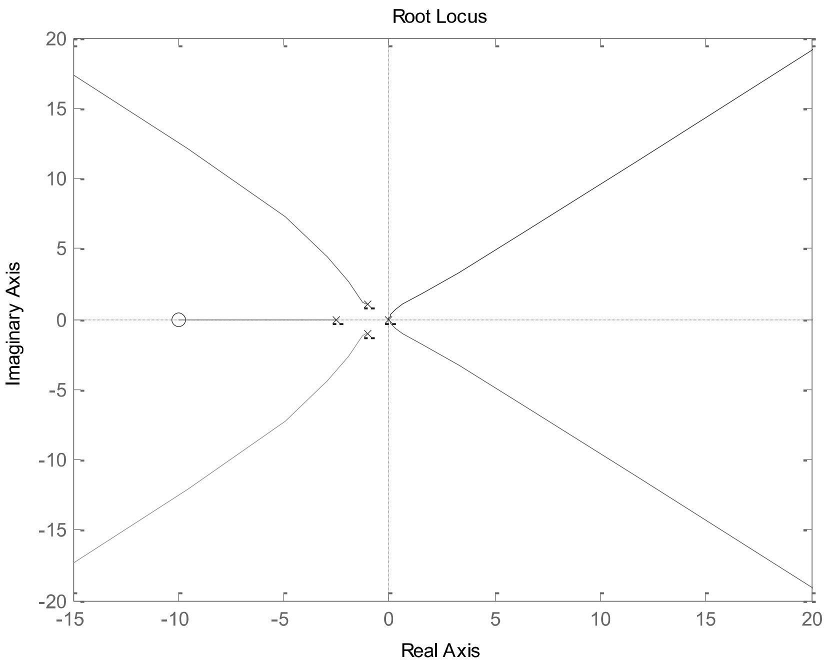
Root Locus diagram – 9-17(c): (K = 12.8 @ damping = ~0.0707)



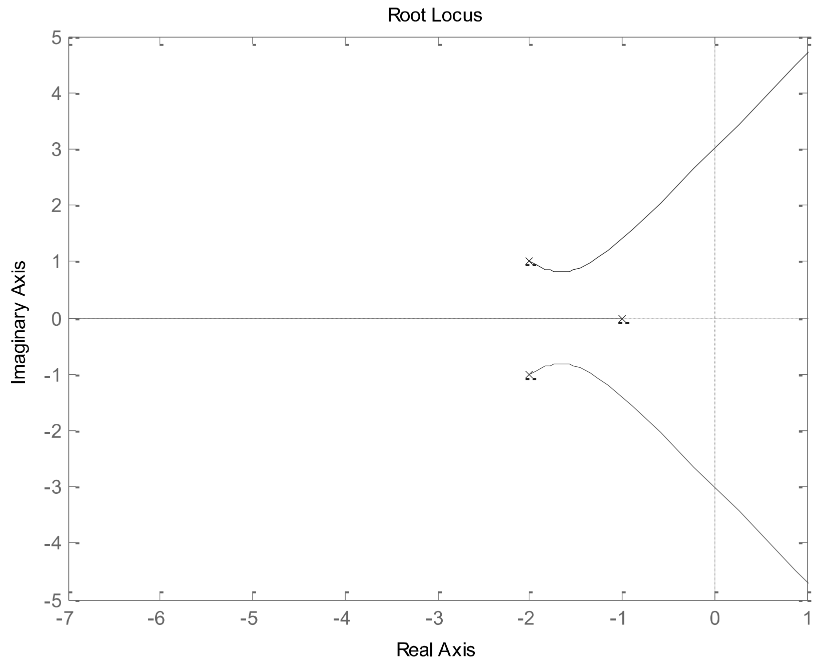
Root Locus diagram – 9-17(d): (K = 8.3 @ damping = ~0.0707)



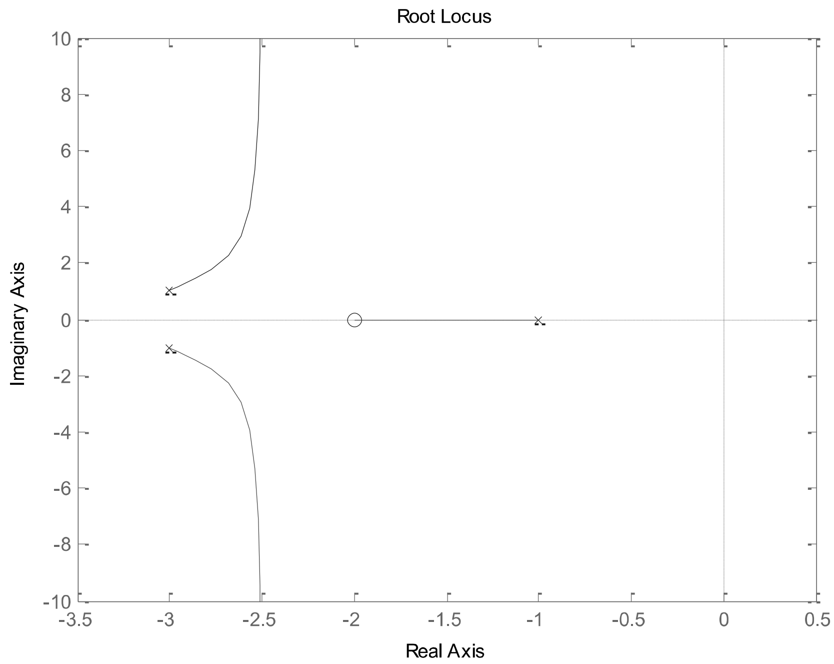
Root Locus diagram – 9-17(e): (K = 0 @ damping = 0.0707)



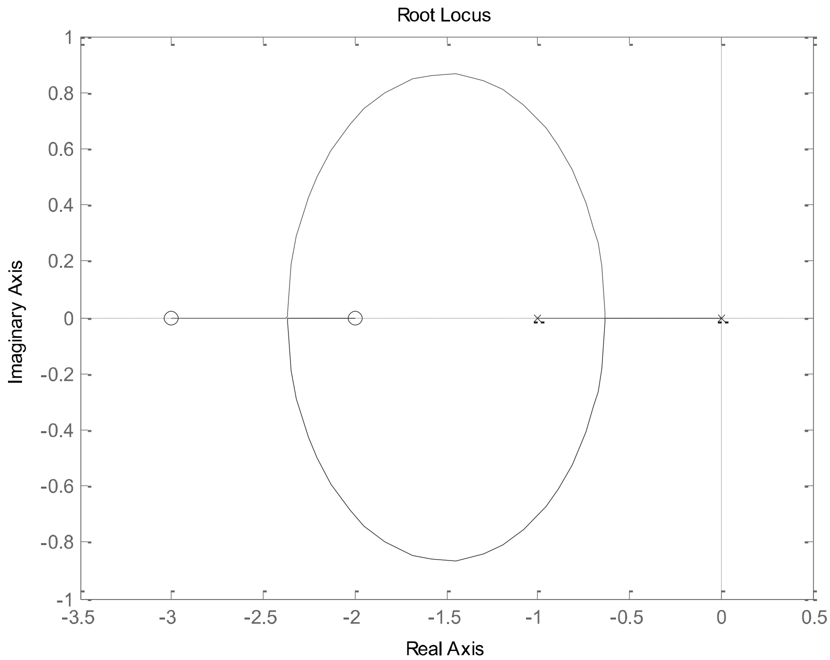
Root Locus diagram – 9-17(f): (K = 2.33 @ damping = ~0.0707)



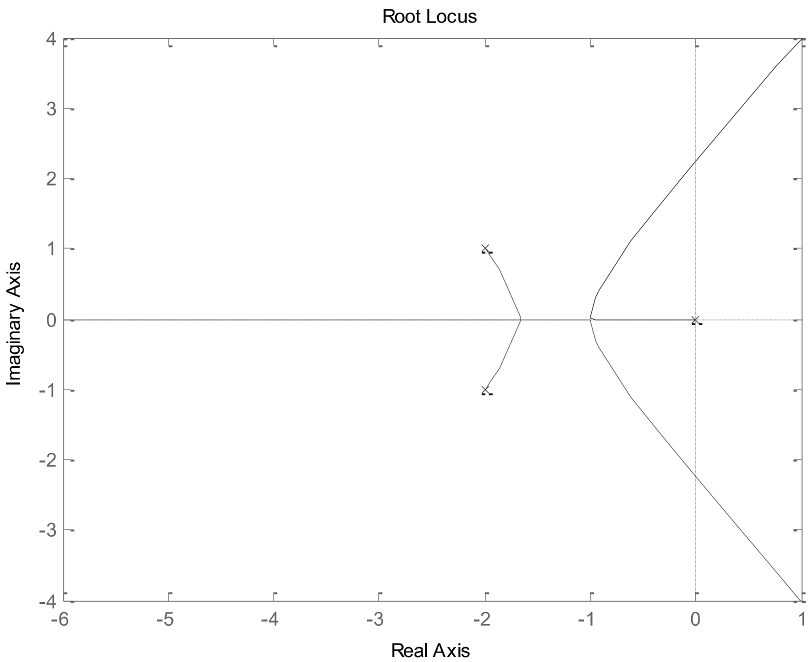
Root Locus diagram – 9-17(g): (K = 7.03 @ damping = ~0.0707)



Root Locus diagram – 9-17(h): (no solution exists for damping =0.0707)



Root Locus diagram – 9-17(i): (K = 2.93 @ damping = ~0.0707)

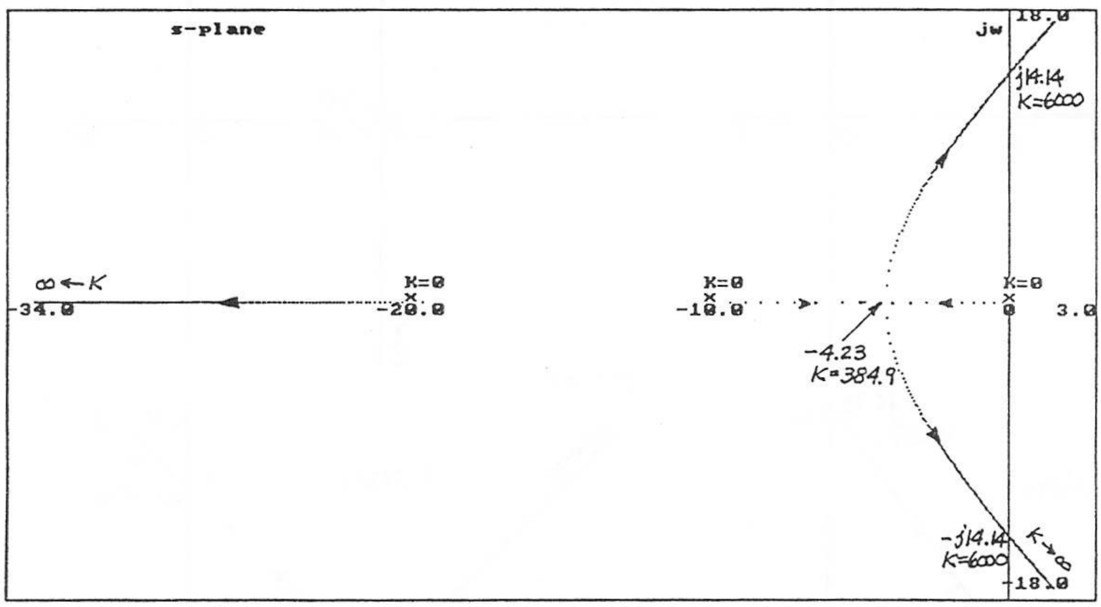


9-18) **(a)** **Asymptotes:** ***K* > 0:** 60^o, 180^o, 300^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-10-20}{3} = -10$$

Breakaway-point Equation: $3s^2 + 60s + 200 = 0$ **Breakaway Point: (RL)** −4.2265, ***K* = 384.9**



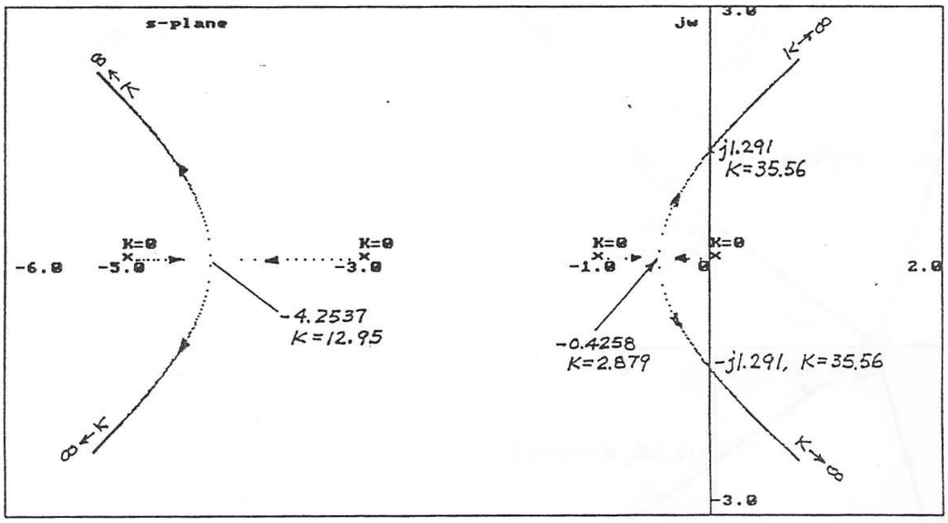
(b) Asymptotes: ***K* > 0:** 45^o, 135^o, 225^o, 315^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-3-5}{4} = -2.25$$

Breakaway-point Equation: $4s^3 + 27s^2 + 46s + 15 = 0$

Breakaway Points: (RL) −0.4258 ***K* = 2.879**, −4.2537 ***K* = 12.95**



c) Zeros: $s = 0.5$ and poles: $s = 1$

Angle of asymptotes: $\theta = (2i + 1)180 = 180$

The breakaway points: $\frac{1}{(s+1)^2} = \frac{1}{s+0.5}$ ➔ $s^2 + s + 0.5 = 0$

Then $s = -0.5 - 0.5j, -0.5 + 0.5j$ and $\sigma_1 = \frac{+1-0.5}{1} = 0.5$

d) Poles: $s = -0.5, 4.5$

Angle of asymptotes: $\theta_i = \frac{2i+1}{2} \times 180 = 90, 270$

breakaway points:

$s^2 + s + 0.75 = 0$ ➔ $s = -1 - \sqrt{2}j, -1 + \sqrt{2}j$

$\sigma_1 = \frac{-0.5+1.5}{2} = 0.5$

e) Zeros: $s = -\frac{1}{3}, -1$ and poles: $s = 0, 0.5, 1$

Angle of asymptotes: $\theta_i = \frac{2i+1}{3-2} 180 = 180$

breakaway points: $\frac{1}{s} + \frac{1}{s+\frac{1}{2}} + \frac{1}{s-1} = \frac{1}{s+\frac{1}{3}} + \frac{1}{s+1}$ ➔ $s = 0.383, -2.22$

$$\sigma = -\frac{1-0.5+\frac{1}{3}+1}{1} = -\frac{11}{6}$$

f) Poles: $s = 0, -3+4j, -3-4j$

Angles of asymptotes: $\theta_i = \frac{2i+1}{3} \times 180 = 60,180,300$

$\sigma_1 = -\frac{0+3-4j+3+4j}{3} = 2$

breakaway point: $-\frac{d}{ds}[s(s^2+6s+25)] = 0$

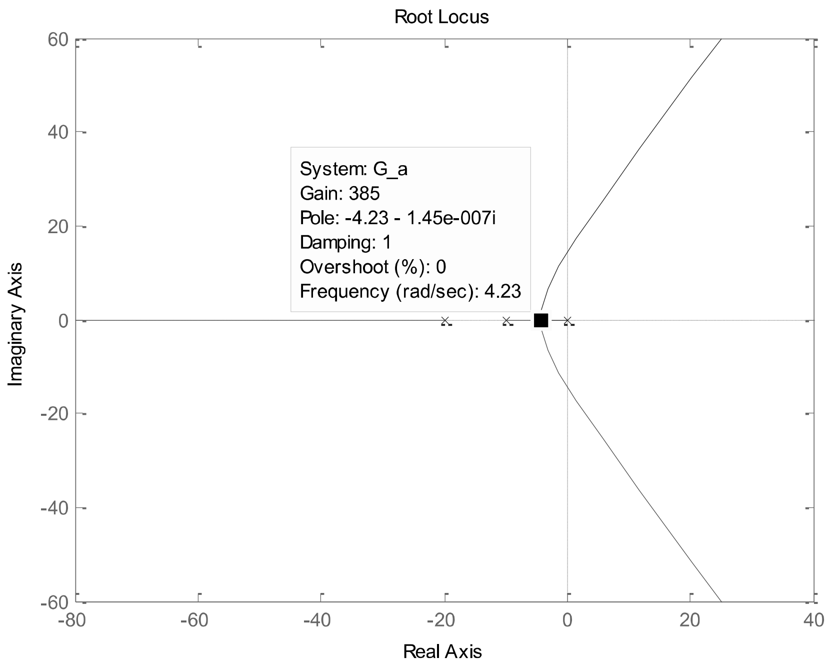
$3s^2+12s+25 = 0$ ➔ $s \approx -2+2.1j, -2-2.1j$

9-19) MATLAB code:

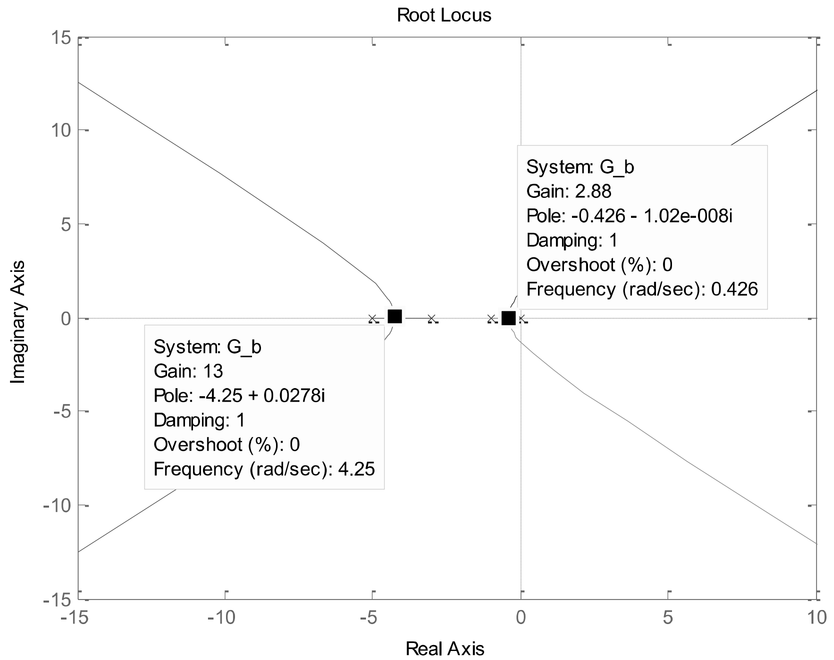
```
clear all;
close all;
s = tf('s')

%a)
num_G_a=1;
den_G_a=s*(s+10)*(s+20);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
num_G_b= 1;
den_G_b=s*(s+1)*(s+3)*(s+5);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
num_G_c=(s-0.5);
den_G_c=(s-1)^2;
G_c=num_G_c/den_G_c;
figure(3);
rlocus(G_c)

%d)
num_G_d=1;
den_G_d=(s+0.5)*(s-1.5);
G_d=num_G_d/den_G_d;
figure(4);
rlocus(G_d)
```

```
%e)
num_G_e=(s+1/3)*(s+1);
den_G_e=s*(s+1/2)*(s-1);
G_e=num_G_e/den_G_e;
figure(5);
rlocus(G_e)

%f)
num_G_f=1;
den_G_f=s*(s^2+6*s+25);
G_f=num_G_f/den_G_f;
figure(6);
rlocus(G_f)
```

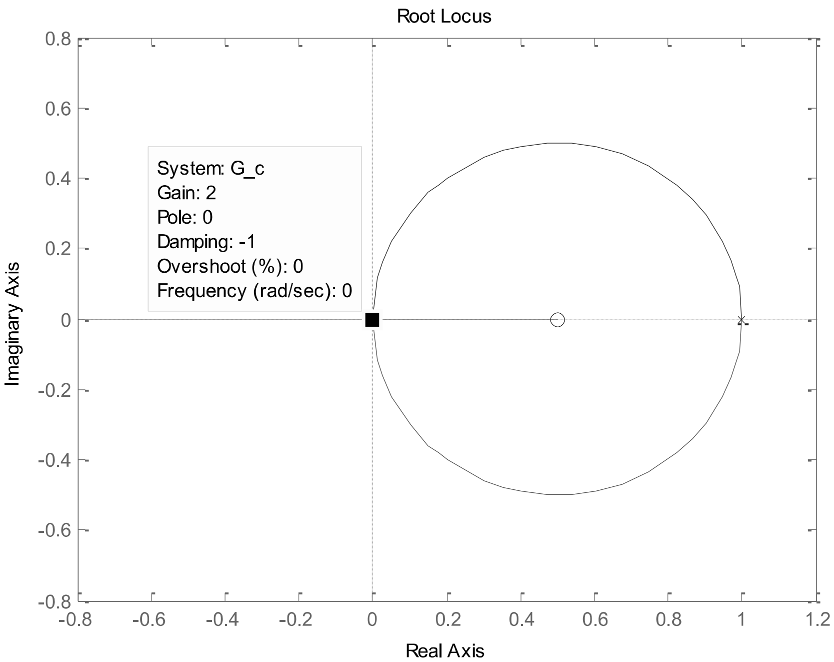
Root Locus diagram – 9-19(a):



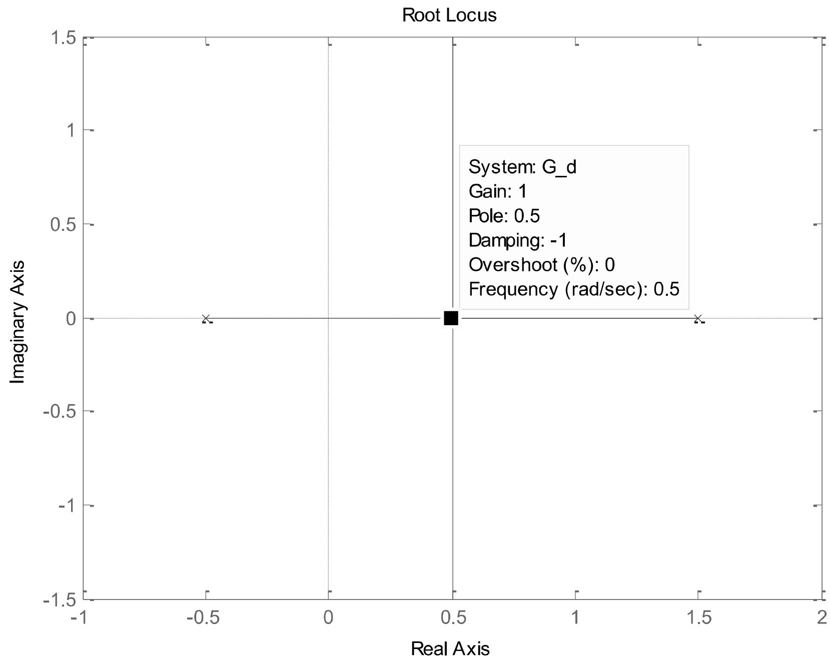
Root Locus diagram – 9-19(b):



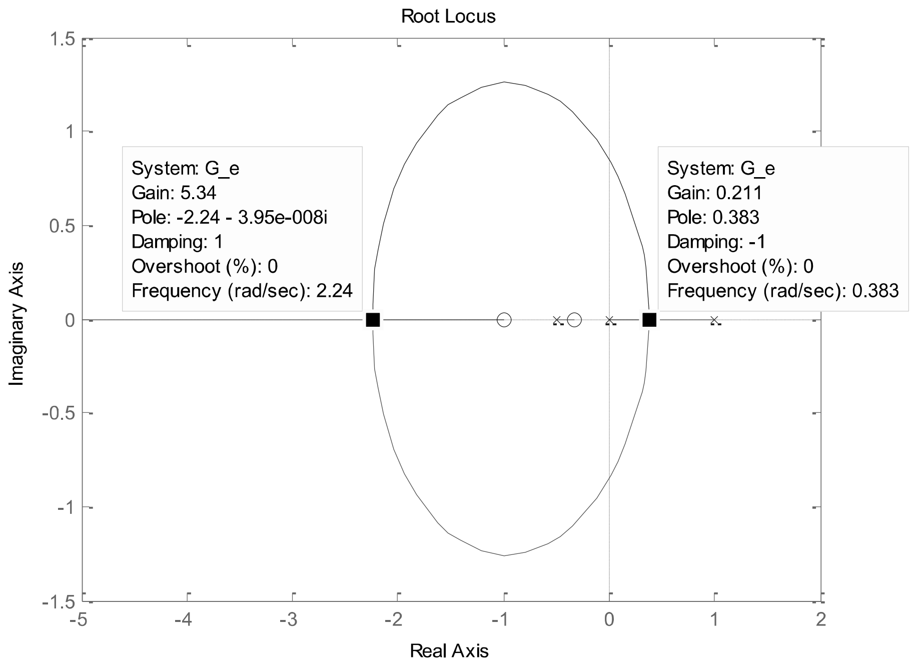
Root Locus diagram – 9-19(c):



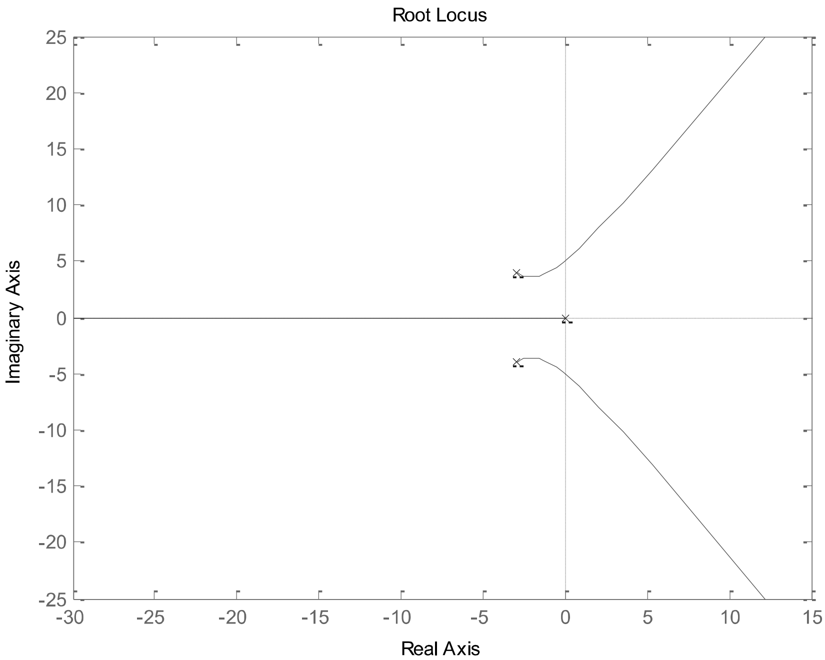
Root Locus diagram – 9-19(d):



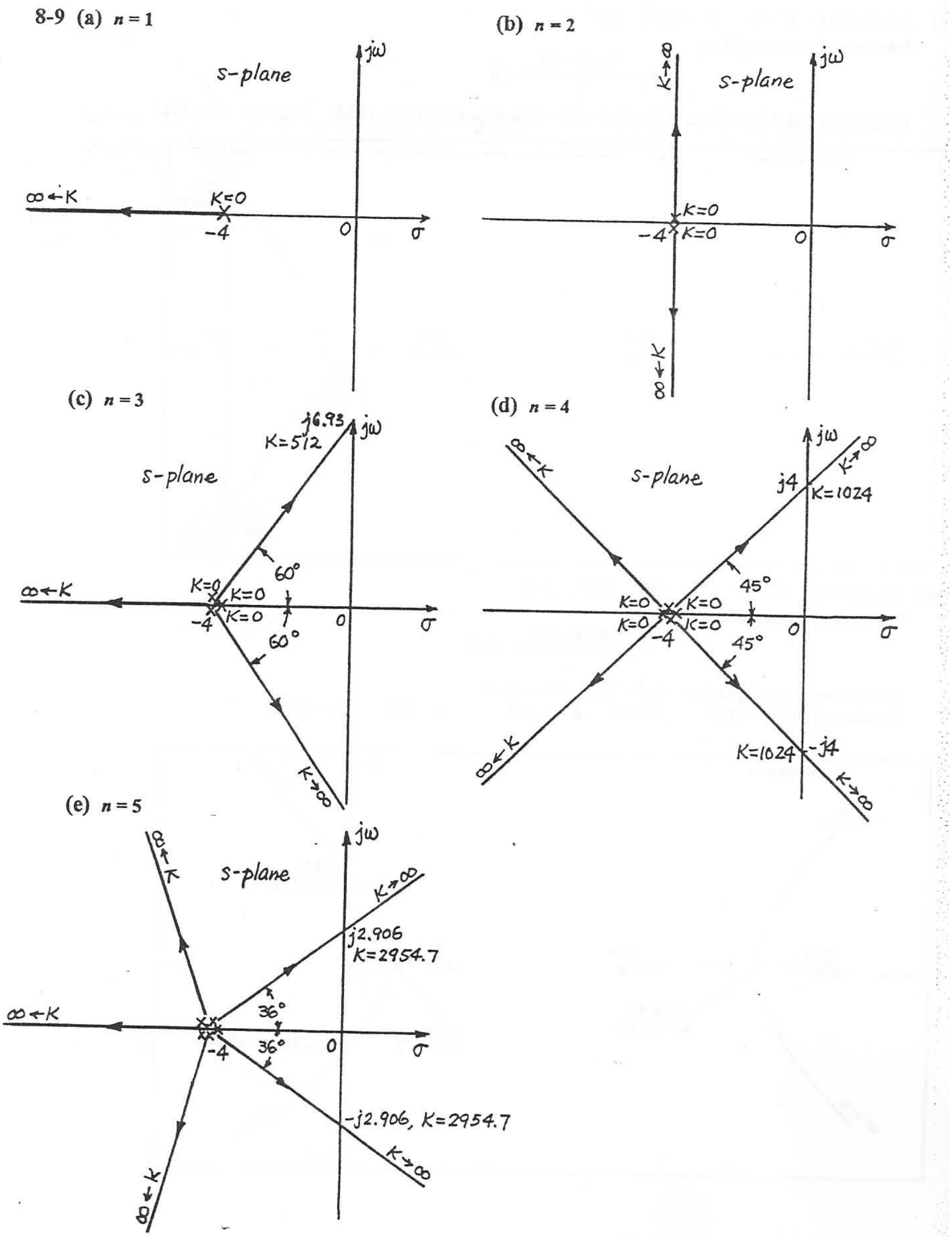
Root Locus diagram – 9-19(e):



Root Locus diagram – 9-19(f): (No breakaway points)


Root Locus
Imaginary Axis
Real Axis
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-25
-20
-15
-10
-5
0
5
10
15

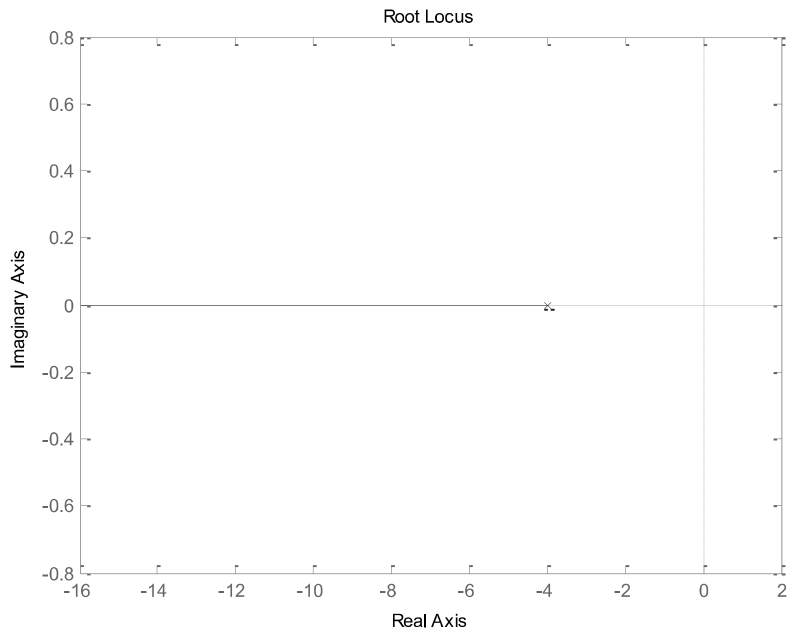
9-20)


8-9 (a) n = 1
s-plane
jω
∞←K
K=0
-4
0
σ
(b) n = 2
K→∞
K=0
-4
K=0
(c) n = 3
j6.93
K=512
60°
K=0
K→∞
(d) n = 4
j4
K=1024
45°
-j4
K=1024
K→∞
(e) n = 5
j2.906
K=2954.7
36°
-j2.906, K=2954.7
K→∞

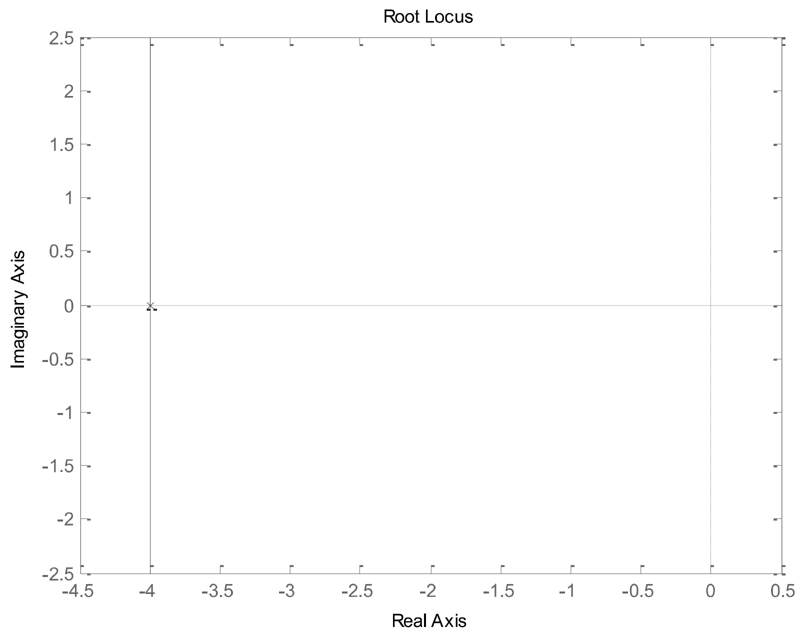
9-21) MATLAB code:

```
clear all;
close all;
s = tf('s')

%a)
n=1;
num_G_a= 1;
den_G_a=(s+4)^n;
G_a=num_G_a/den_G_a;
figure(n);
rlocus(G_a)

%b)
n=2;
num_G_b= 1;
den_G_b=(s+4)^n;
G_b=num_G_b/den_G_b;
figure(n);
rlocus(G_b)

%c)
n=3;
num_G_c= 1;
den_G_c=(s+4)^n;
G_c=num_G_c/den_G_c;
figure(n);
rlocus(G_c)

%d)
n=4;
num_G_d= 1;
den_G_d=(s+4)^n;
G_d=num_G_d/den_G_d;
figure(n);
rlocus(G_d)

%e)
n=5;
num_G_e= 1;
den_G_e=(s+4)^n;
G_e=num_G_e/den_G_e;
figure(n);
rlocus(G_e)
```

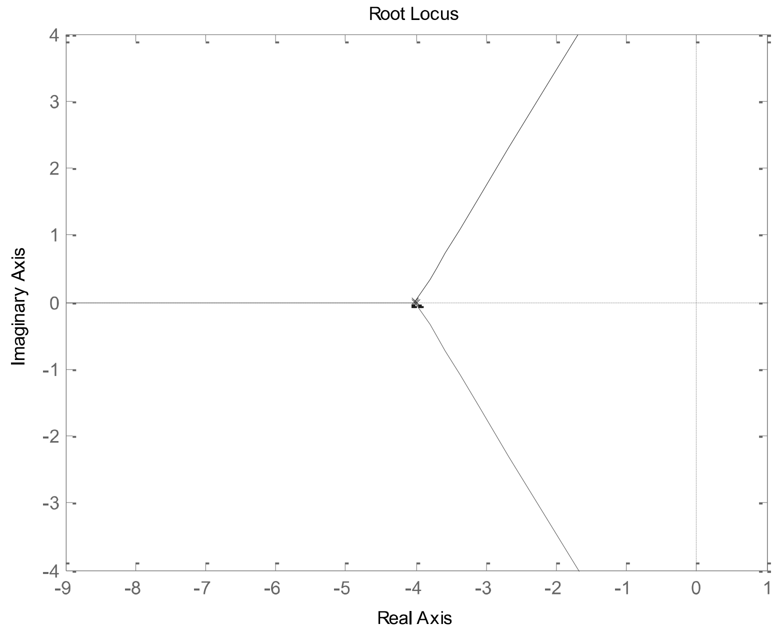
Root Locus diagram – 9-21(a):



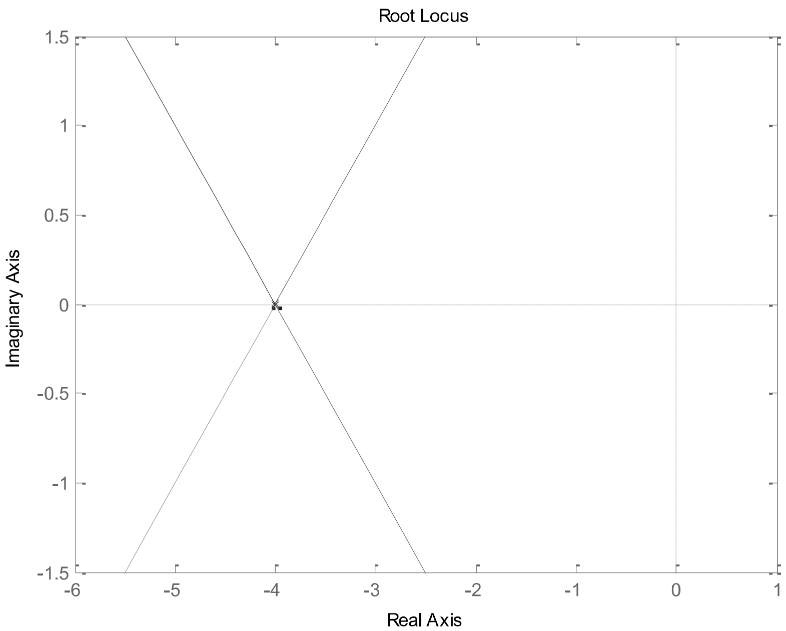
Root Locus diagram – 9-21(b):



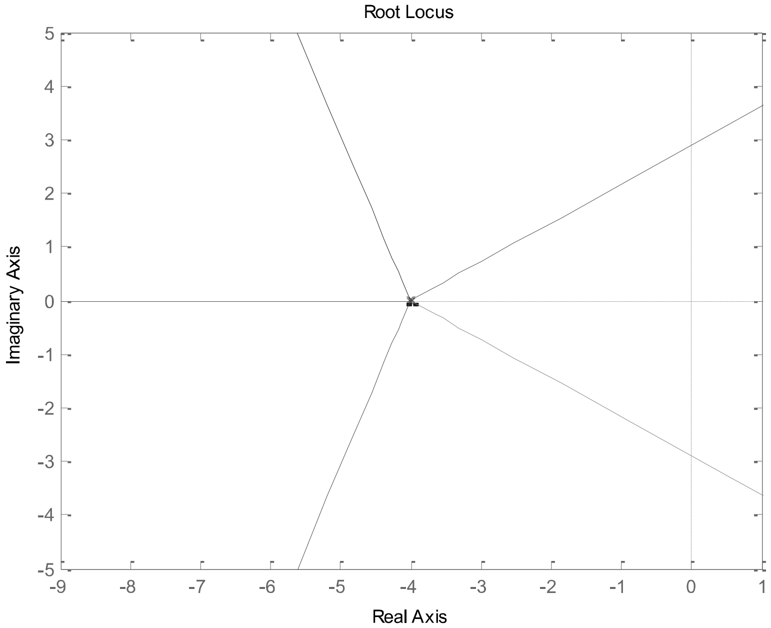
Root Locus diagram – 9-21(c):



Root Locus diagram – 9-21(d):



Root Locus diagram – 9-21(e):



9-22) $P(s) = s^3 + 25s^2 + 2s + 100 \qquad Q(s) = 100s$

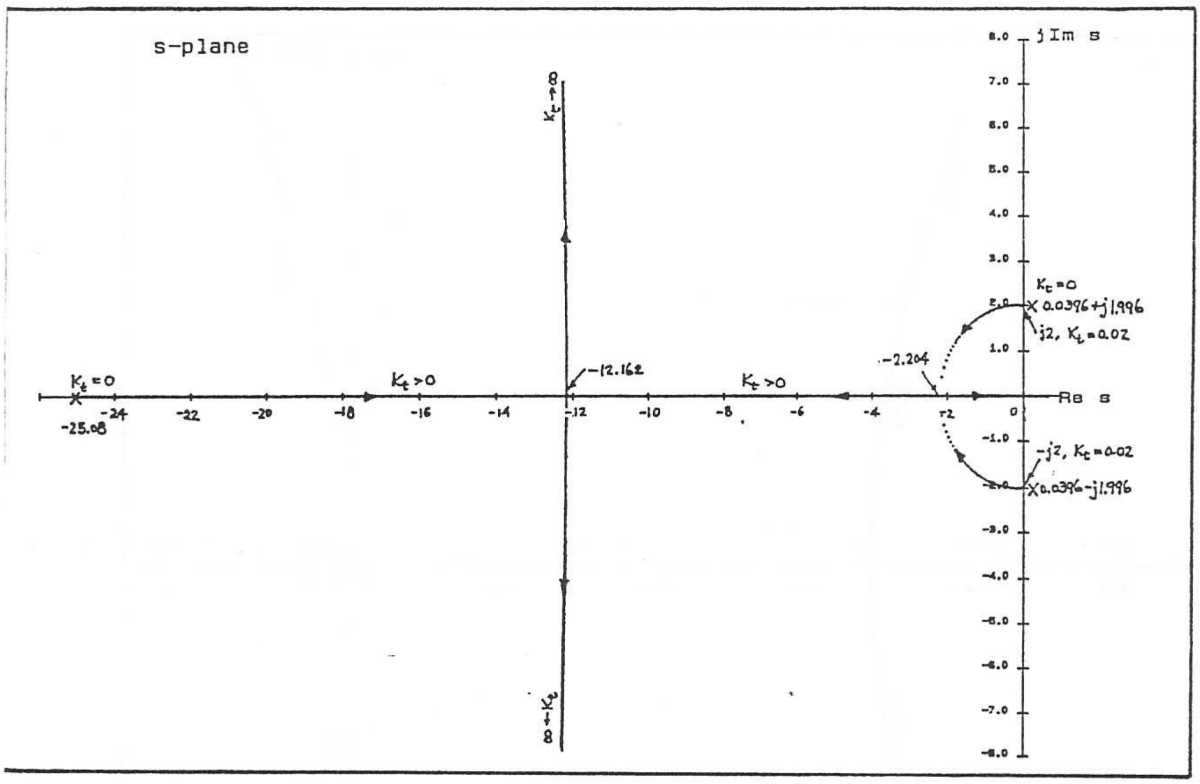
Asymptotes: $K_t > 0$: 90^o, 270^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{-25-0}{3-1} = -12.5$$

Breakaway-point Equation: $s^3 + 12.5s^2 - 50 = 0$

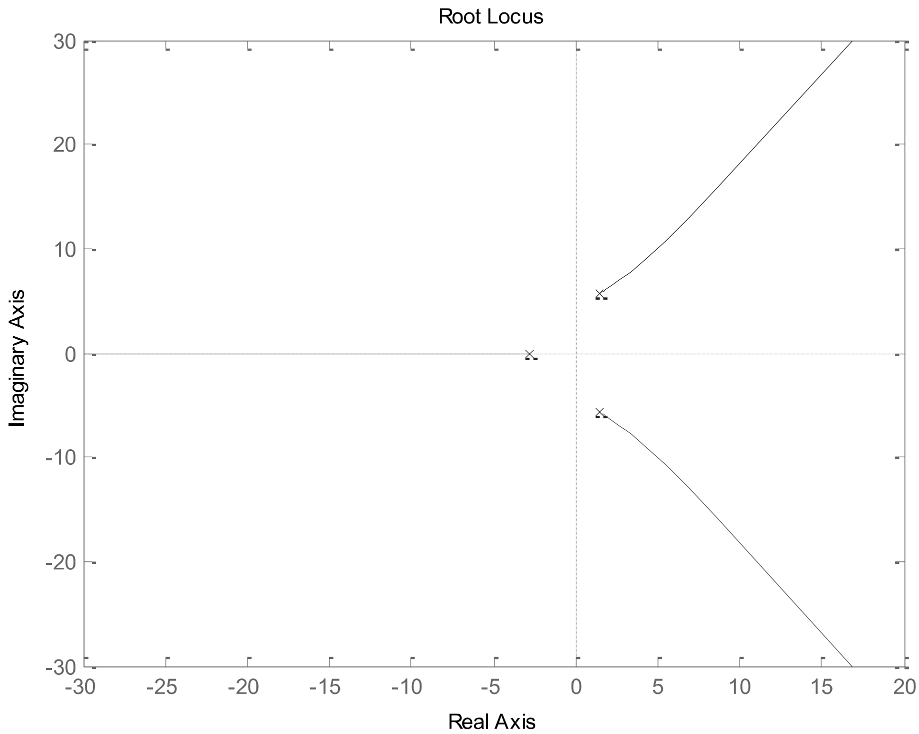
Breakaway Points: **(RL)** -2.2037, -12.162



9-23) MATLAB code:

```
s = tf('s')
num_G= 100;
den_G=s^3+25*s+2*s+100;
G=num_G/den_G;
figure(1);
rlocus(G)
```

Root Locus diagram – 9-23:



9-24) **Characteristic equation:** $s^3+5s^2+K_t s+K=0$

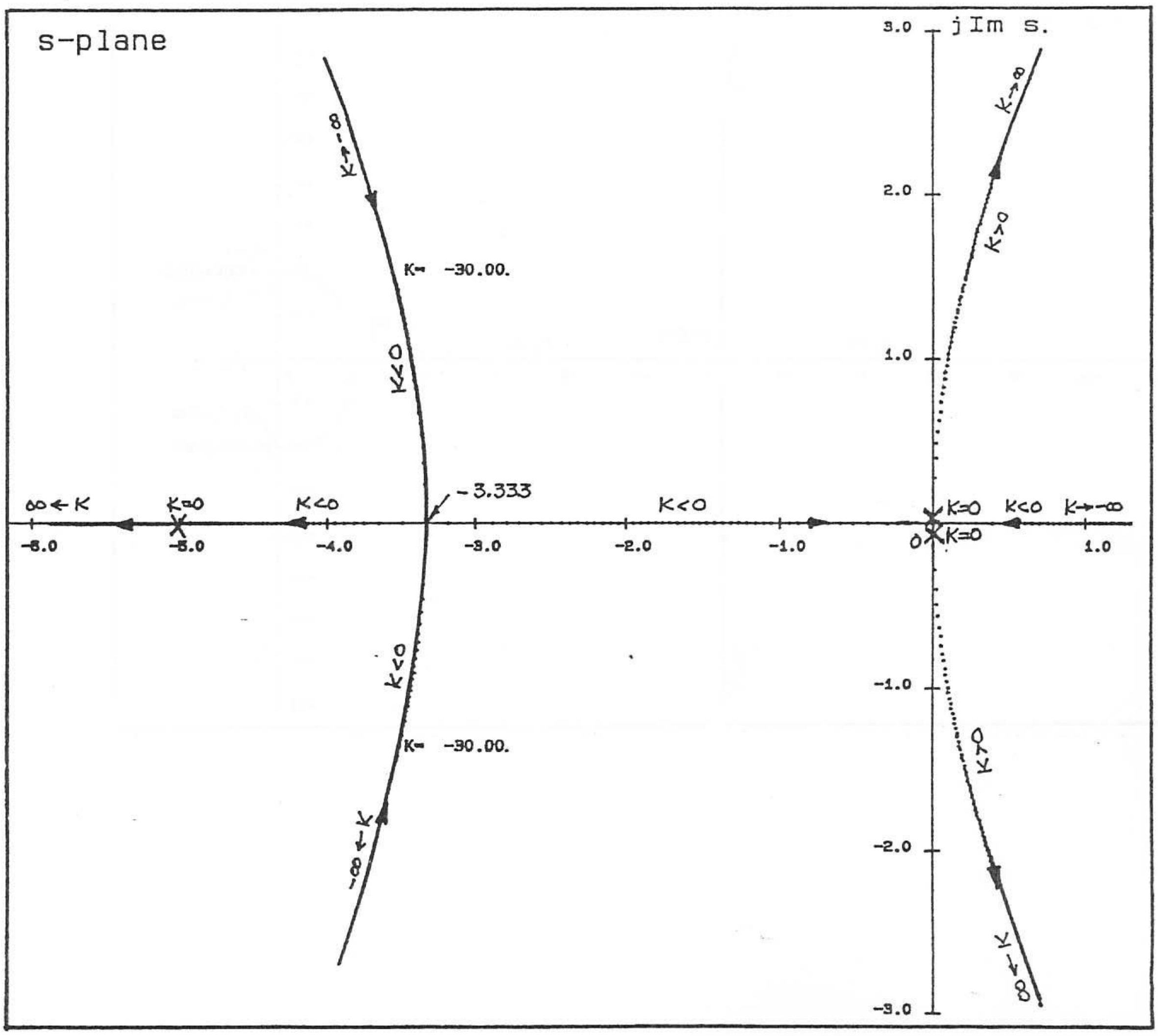
(a) $K_t=0$: $P(s)=s^2\left(s+5\right)$ $Q(s)=1$

Asymptotes: ***K* > 0:** 60^o, 180^o, 300^o

Intersect of Asymptotes:

$$\sigma_1=\frac{-5-0}{3}=-1.667$$

Breakaway-point Equation: $3s^2 + 10s = 0$ **Breakaway Points:** 0, -3.333



9-24 (b) $P(s) = s^3 + 5s^2 + 10 = 0 \qquad Q(s) = s$

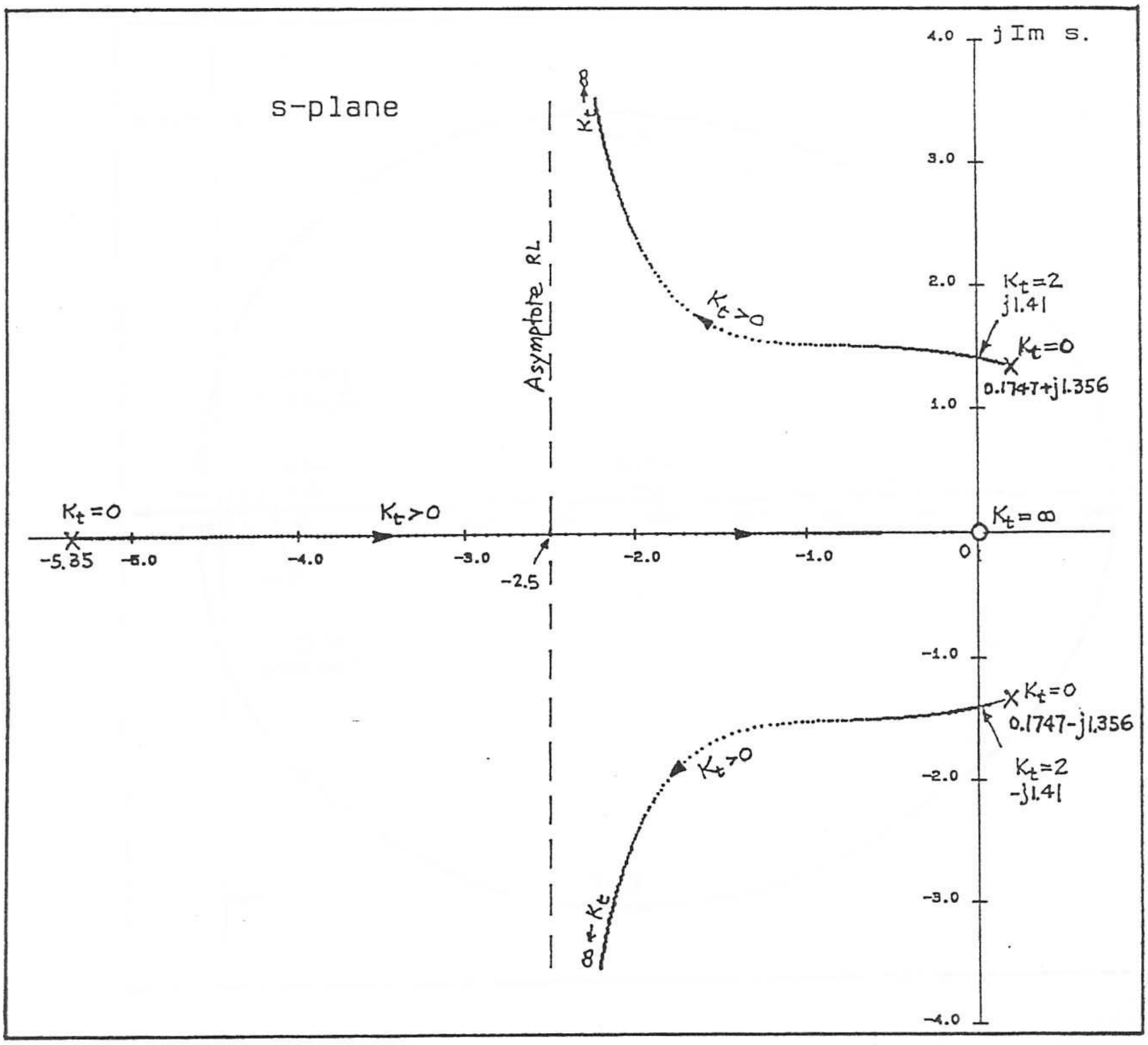
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-5-0}{2-1} = 0$$

Breakaway-point Equation: $2s^3 + 5s - 10 = 0$

There are no breakaway points on RL.



9-25)

By collapsing the two loops, and finding the overall close loop transfer function, the characteristic equation (denominator of closed loop transfer function) can be found as:

$$1+GH=\frac{s^3+5s^2+K_t s+K}{s^2(s+5)+K_t s}$$

For part (a):

K_t =0. Therefore, assuming

Den(GH)= s^3+5s^2 and

Num(GH) =1, we can use rlocus command to construct the root locus diagram.

For part (b):

K =10. Therefore, assuming

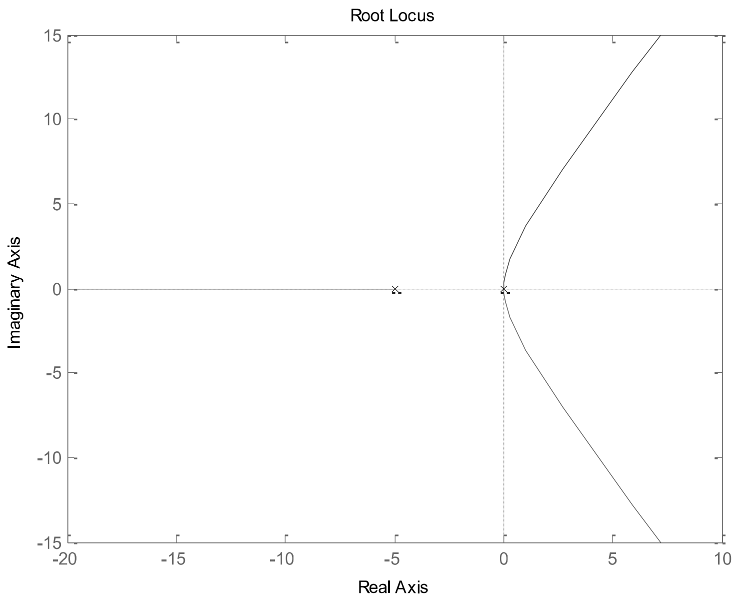
Den(GH)= s^3+5s^2+10 and

Num(GH) = s , we can use rlocus command to construct the root locus diagram.

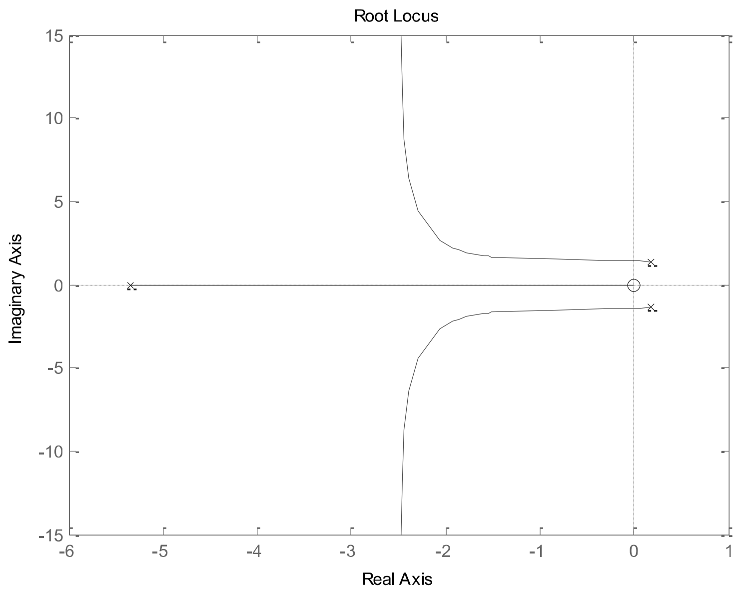
MATLAB code (9-25):

```
s = tf('s')
%a)
num_G_a= 1;
den_G_a=s^3+5*s^2;
GH_a=num_G_a/den_G_a;
figure(1);
rlocus(GH_a)

%b)
num_G_b= s;
den_G_b=s^3+5*s^2+10;
GH_b=num_G_b/den_G_b;
figure(2);
rlocus(GH_b)
```

Root locus diagram, part (a):



Root locus diagram, part (b):

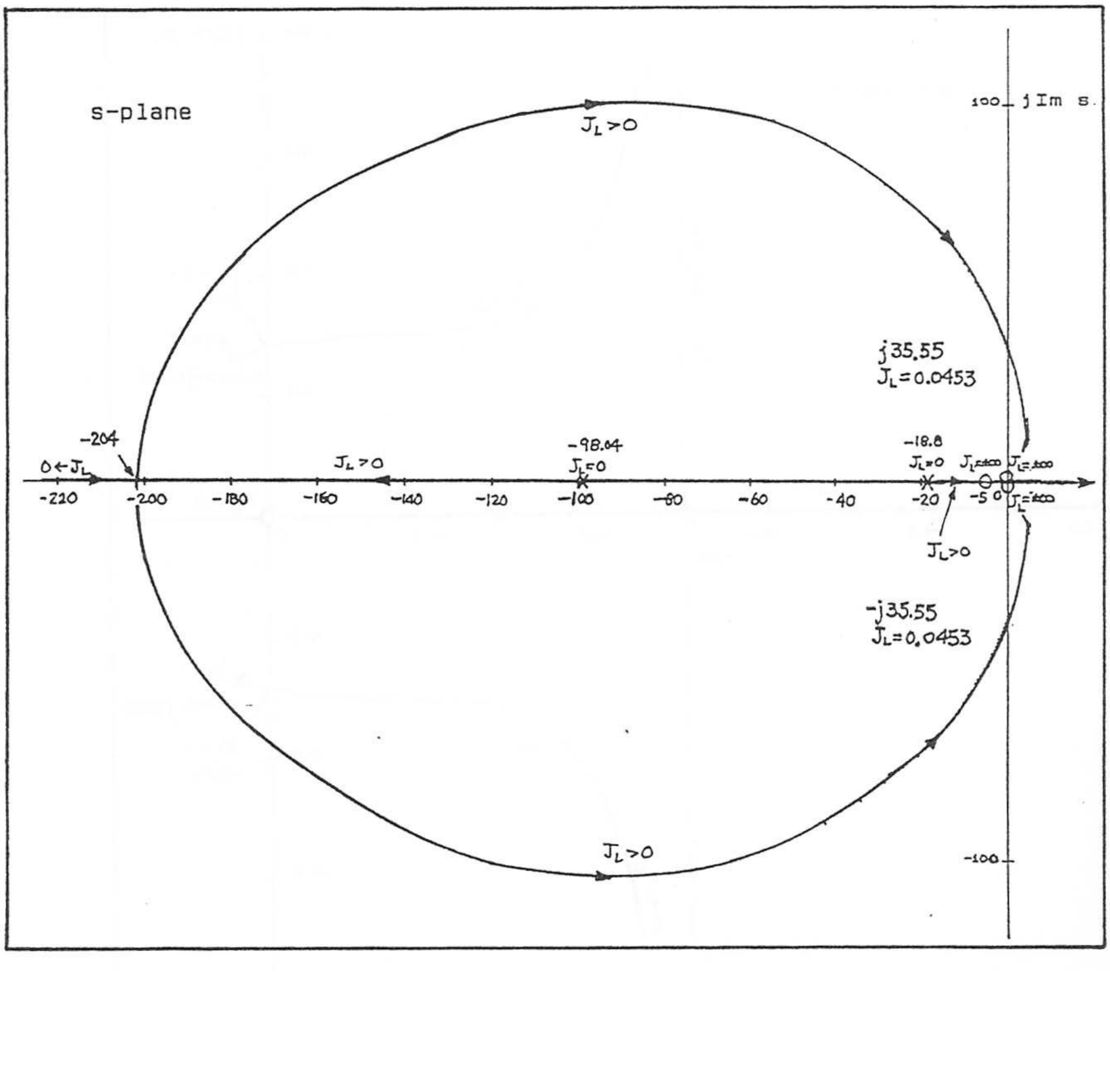


9-26) $P(s) = s^2 + 116.84s + 1843 \qquad Q(s) = 2.05s^2(s+5)$

Asymptotes: $J_L = 0$: 180^o

Breakaway-point Equation: $-2.05s^4 - 479s^3 - 12532s^2 - 37782s = 0$

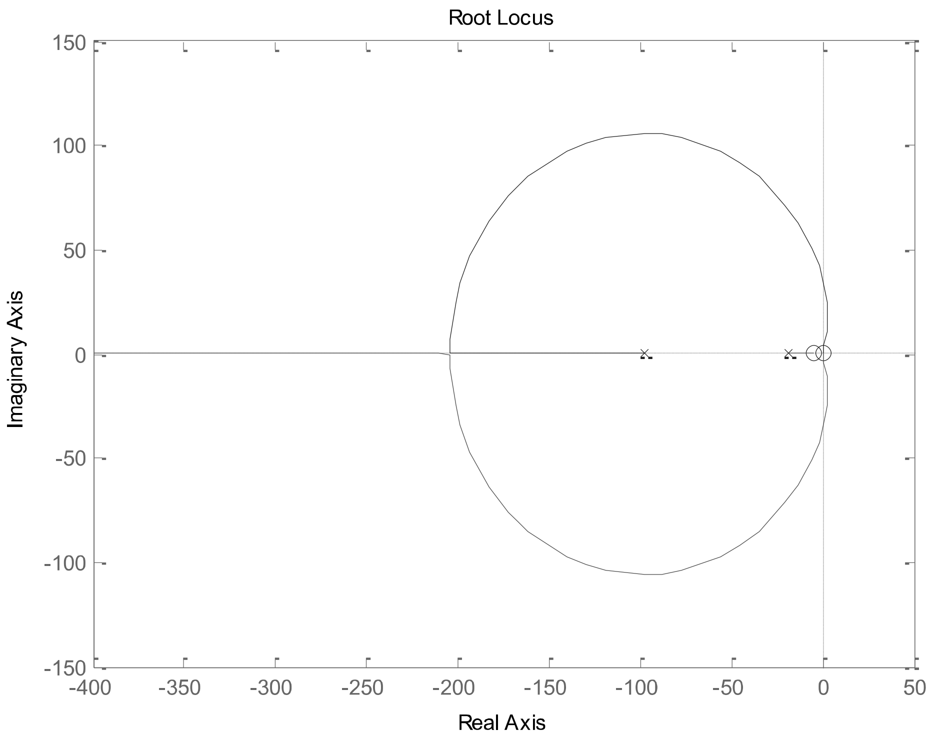
Breakaway Points: **(RL)** 0, −204.18



9-27) MATLAB code:

```
s = tf('s')
num_G = (2.05*s^3 + 10.25*s^2);
den_G = (s^2 + 116.84*s + 1843);
G = num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:

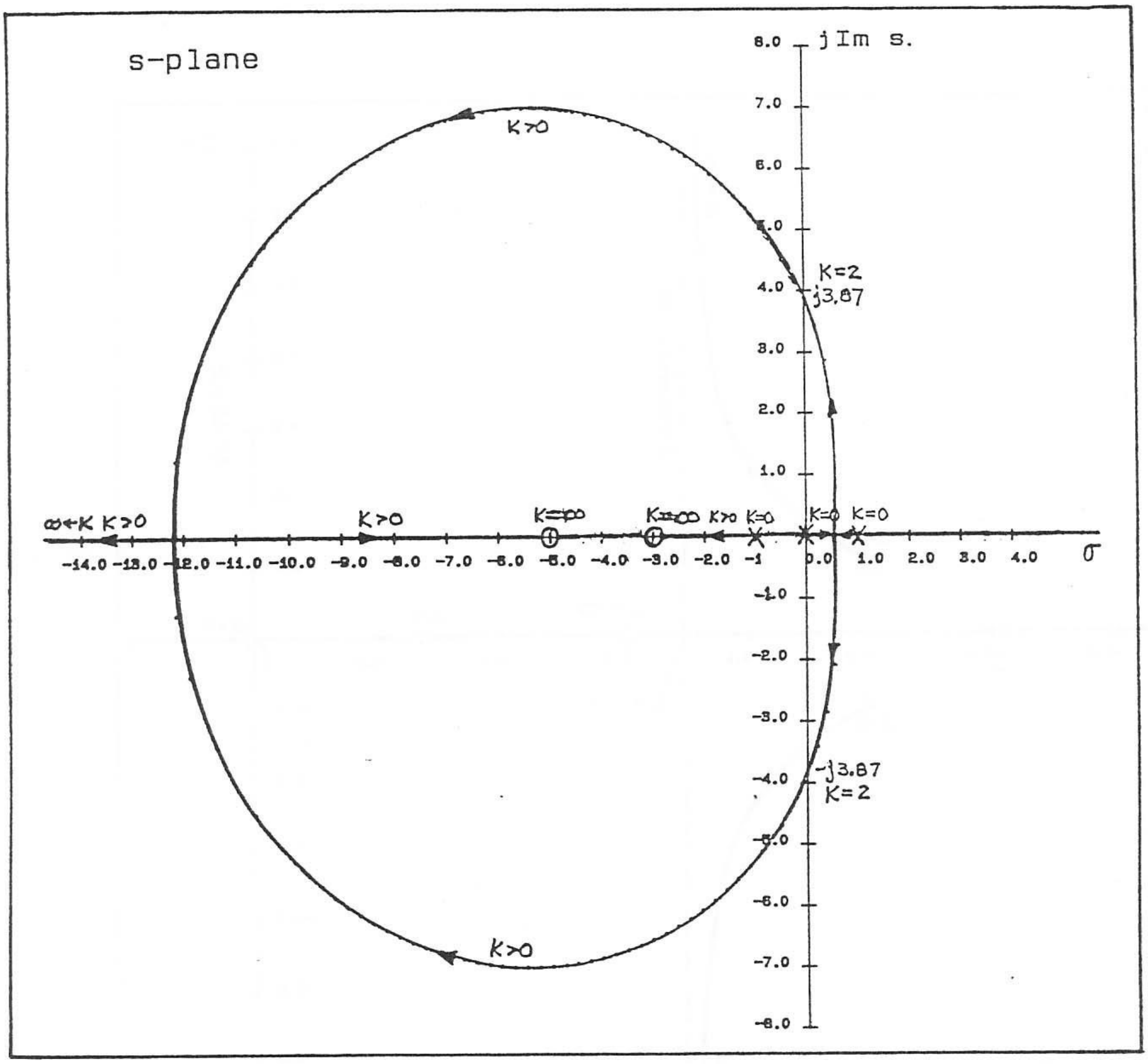


9-28) (a) $P(s) = s\left(s^2 - 1\right) \qquad Q(s) = (s+5)(s+3)$

Asymptotes: ***K* > 0:** 180^o

Breakaway-point Equation: $s^4 + 16s^3 + 46s^2 - 15 = 0$

Breakaway Points: (RL) 0.5239, −12.254



9-28 (b) $P(s) = s\left(s^2 + 10s + 29\right) \qquad Q(s) = 10(s+3)$

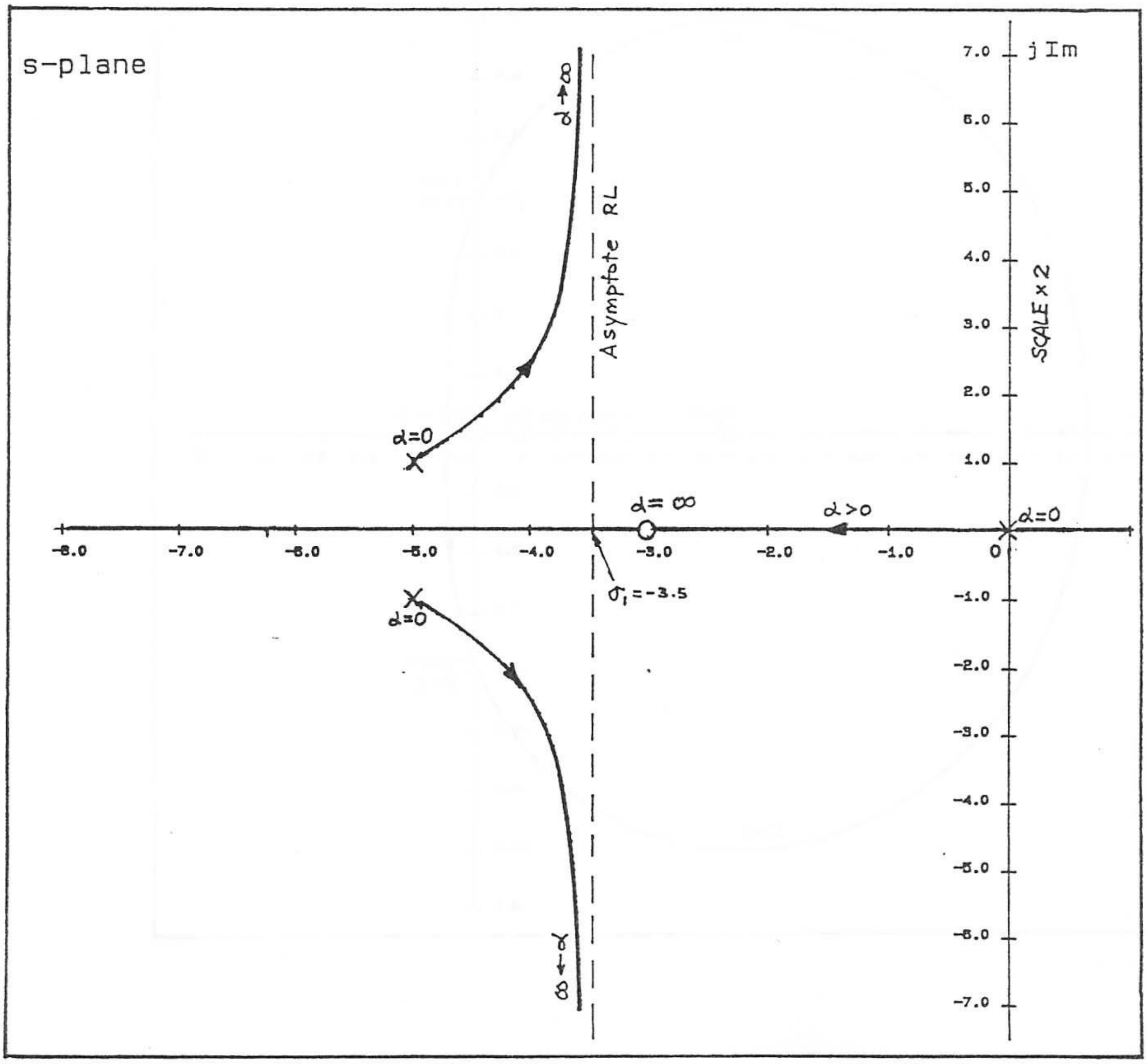
Asymptotes: ***K* > 0:** $90^o, \quad 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0 - 10 - (-3)}{3 - 1} = -3.5$$

Breakaway-point Equation: $20s^3 + 190s^2 + 600s + 870 = 0$

There are no breakaway points on the RL.

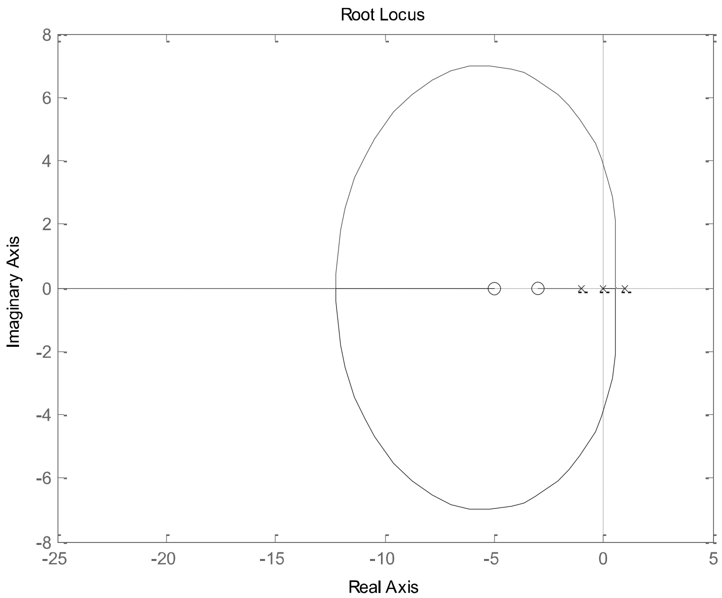


9-29)

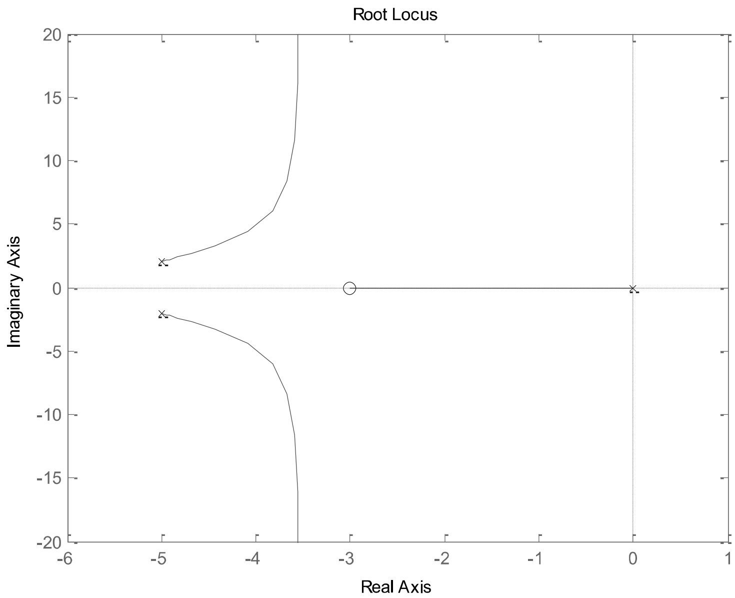
MATLAB code (9-29):	**Root locus diagram, part (a):**

`s = tf('s')`
`%a)`
`num_G_a = (s+5)*(s+3);`
`den_G_a = s*(s^2 - 1);`
`G_a = num_G_a/den_G_a;`
`figure(1);`
`rlocus(G_a)`	

```
K=10;
%b)
num_G_b  = (3*K+K*s);
den_G_b =
(s^3+K*s^2+K*3*s-s);
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)
```



Root locus diagram, part (b):



9-30) Poles: $s = 0, -3.6$ zeros: $s = -0.4$

Angles of asymptotes: $\theta_i = \frac{2i+1}{3-1} \times 180 = 90^o, 270^o$

$\sigma = -\frac{3.6-0.4}{3-1} = -1.6$

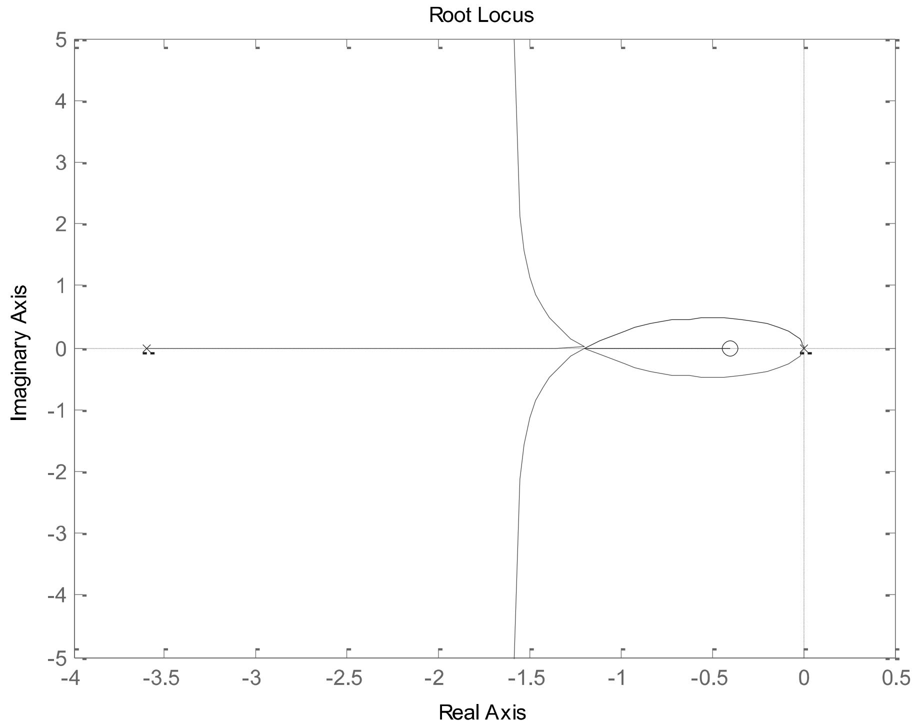
breakaway points: $\frac{1}{s^2} + \frac{1}{s+3.6} = \frac{1}{s+0.4}$

⇨ $s^3 + 2.4s^2 + 1.44s = 0$ ➔ $s = 0, -1.2$

MATLAB code:

```
s = tf('s')
num_G=(s+0.4);
den_G=s^2*(s+3.6);
G=num_G/den_G;
figure(1);
rlocus(G)
```

Root locus diagram:



9-31 (a) $P(s) = s(s+12.5)(s+1)$ $Q(s) = 83.333$

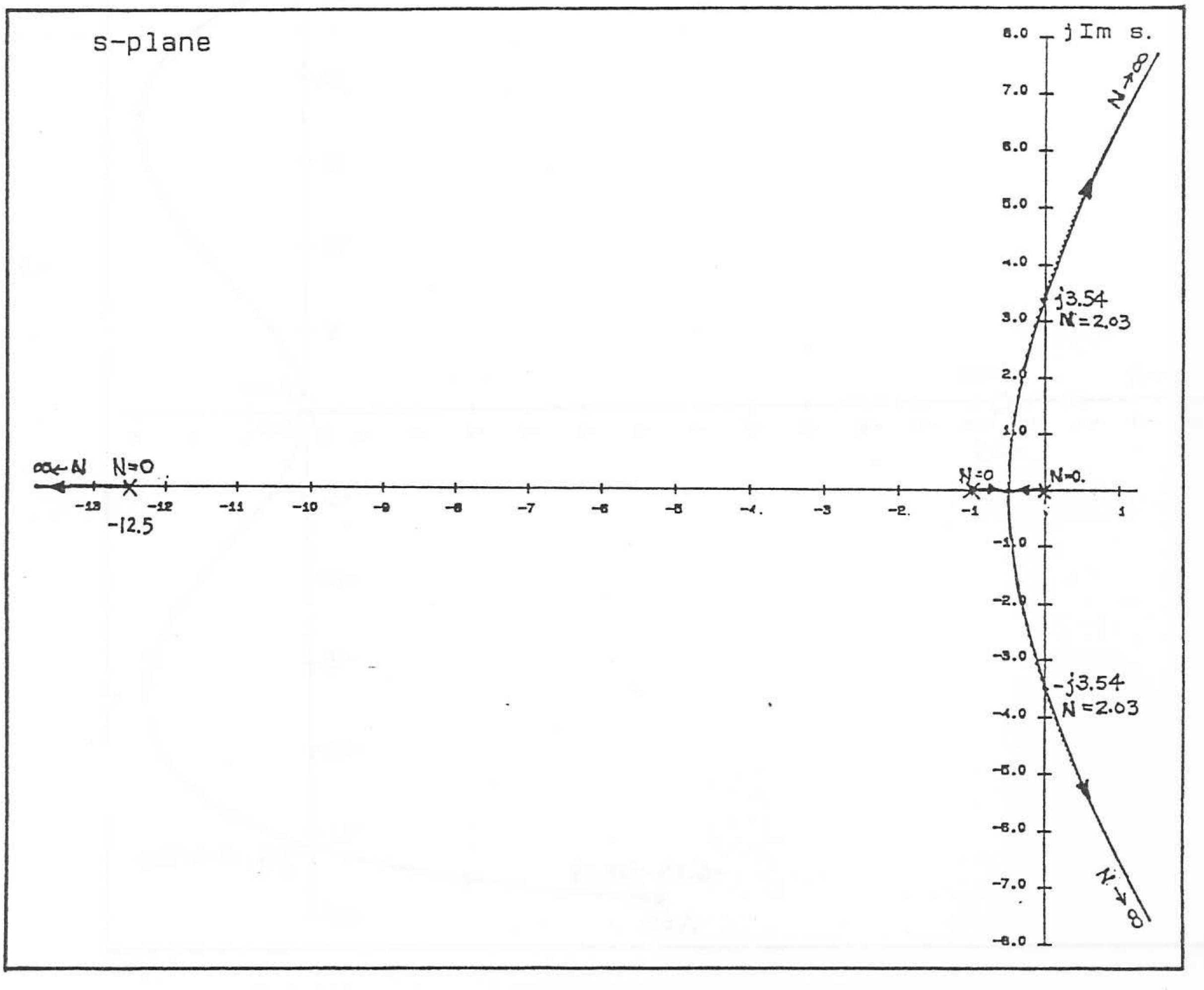
Asymptotes: ***N* > 0:** $60^o, \quad 180^o, \quad 300^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-12.5-1}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s\_12.5 = 0$

Breakaway Point: (RL) -0.4896

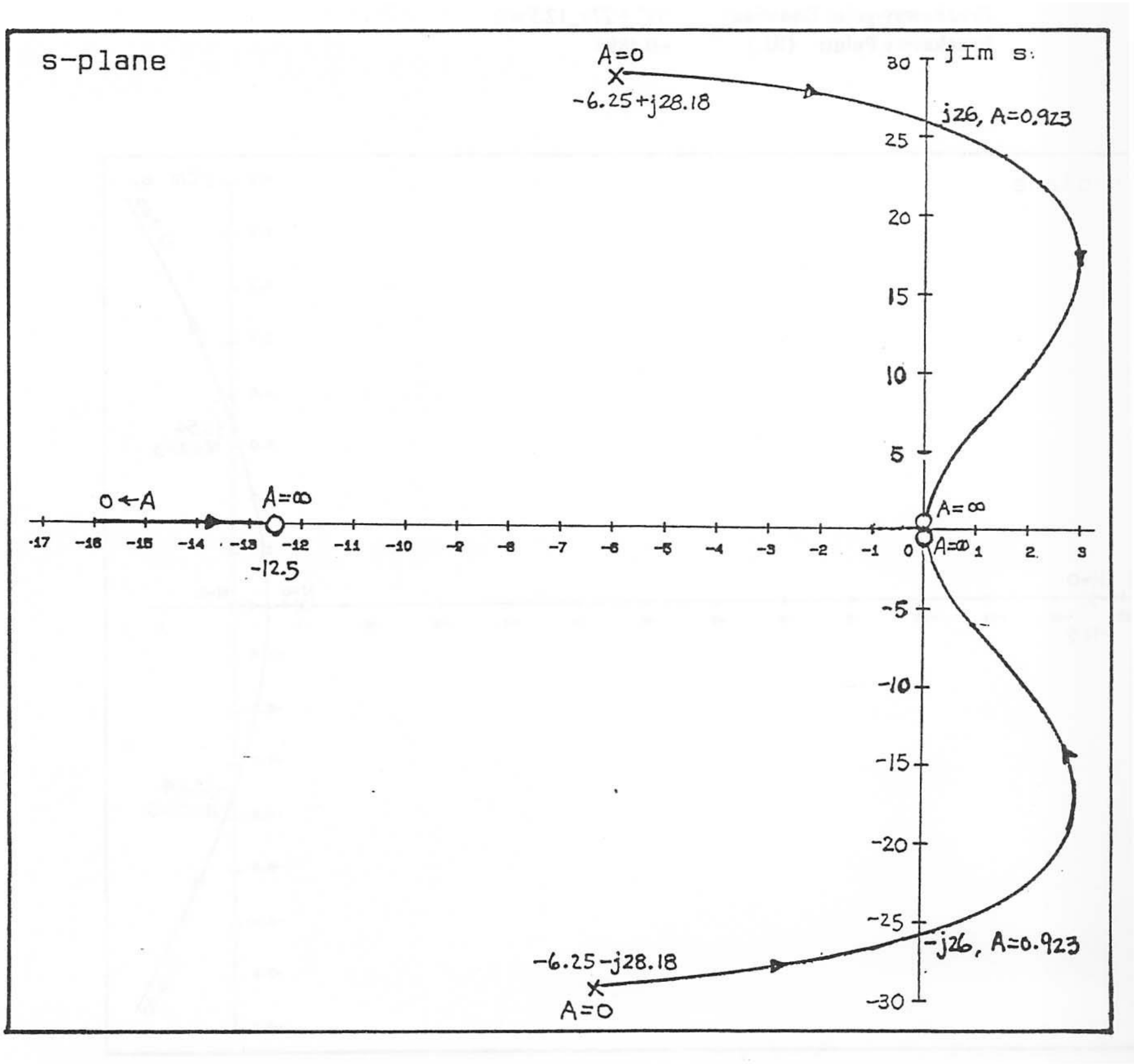


9-31 (b) $P(s) = s^2 + 12.5s + 833.333 \qquad Q(s) = 0.02s^2(s+12.5)$

A > 0: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 53.125s^2 + 416.67s = 0$

Breakaway Points: **(RL)** 0



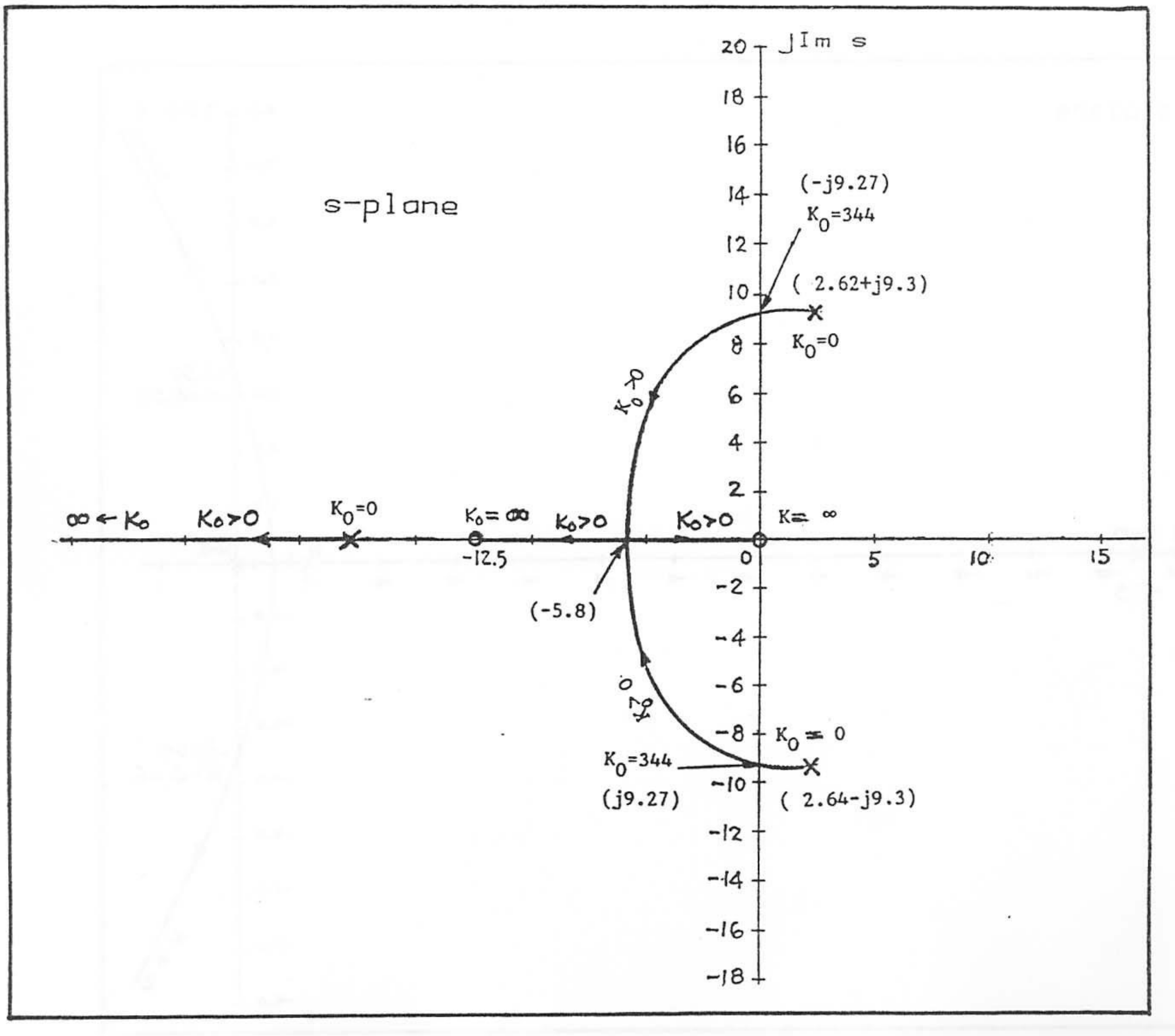
9-31 c) $P(s) = s^3 + 12.5s^2 + 1666.67 = (s + 17.78)(s - 2.64 + j9.3)(s - 2.64 - j9.3)$

$Q(s) = 0.02s(s + 12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 3.125s^2 - 66.67s - 416.67 = 0$

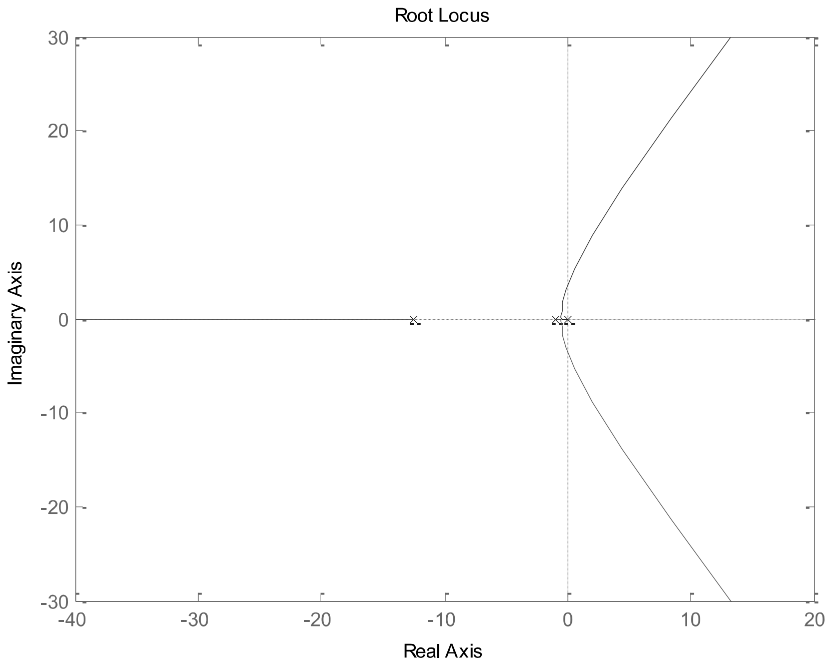
Breakaway Point: (RL) -5.797



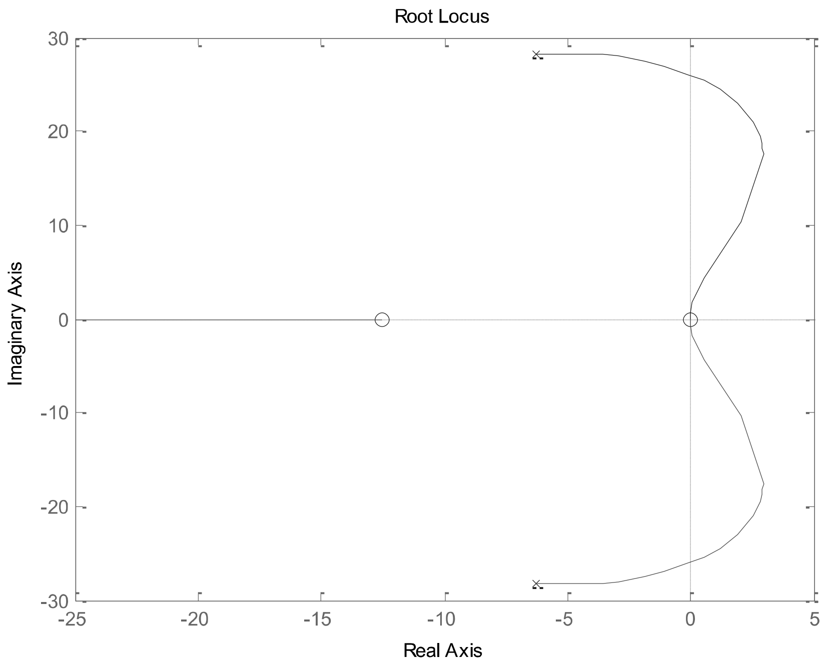
9-32) MATLAB code:

```
s = tf('s')
%a)
A=50;
K0=50;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
N=10;
K0=50;
num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
A=50;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

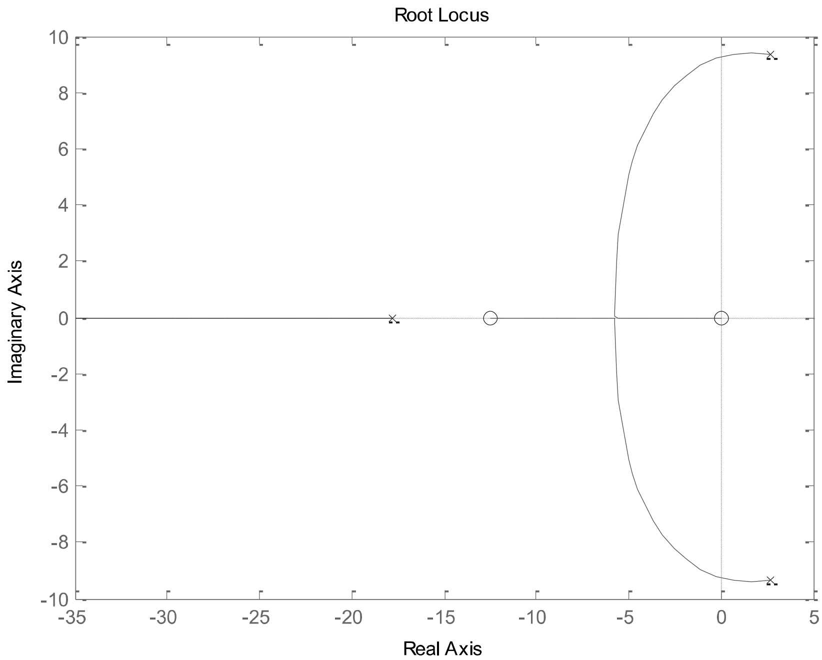
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c):



9-33) (a) $A = K_o = 100$: $P(s) = s(s+12.5)(s+1)$ $Q(s) = 41.67$

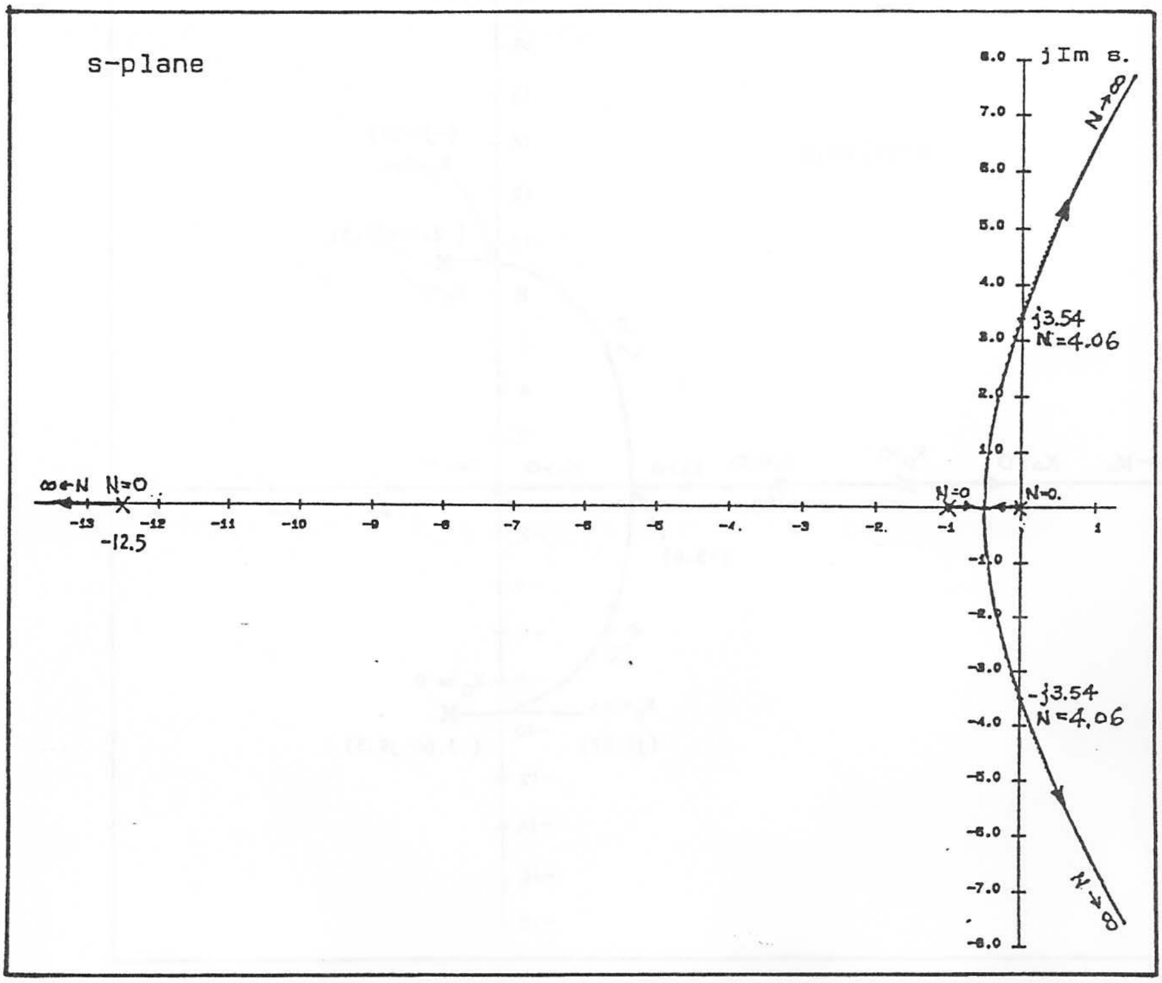
Asymptotes: ***N* > 0:** 60^o 180^o 300^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0-1-12.5}{3} = -4.5$$

Breakaway-point Equation: $3s^2 + 27s + 12.5 = 0$

Breakaway Points: (RL) -0.4896



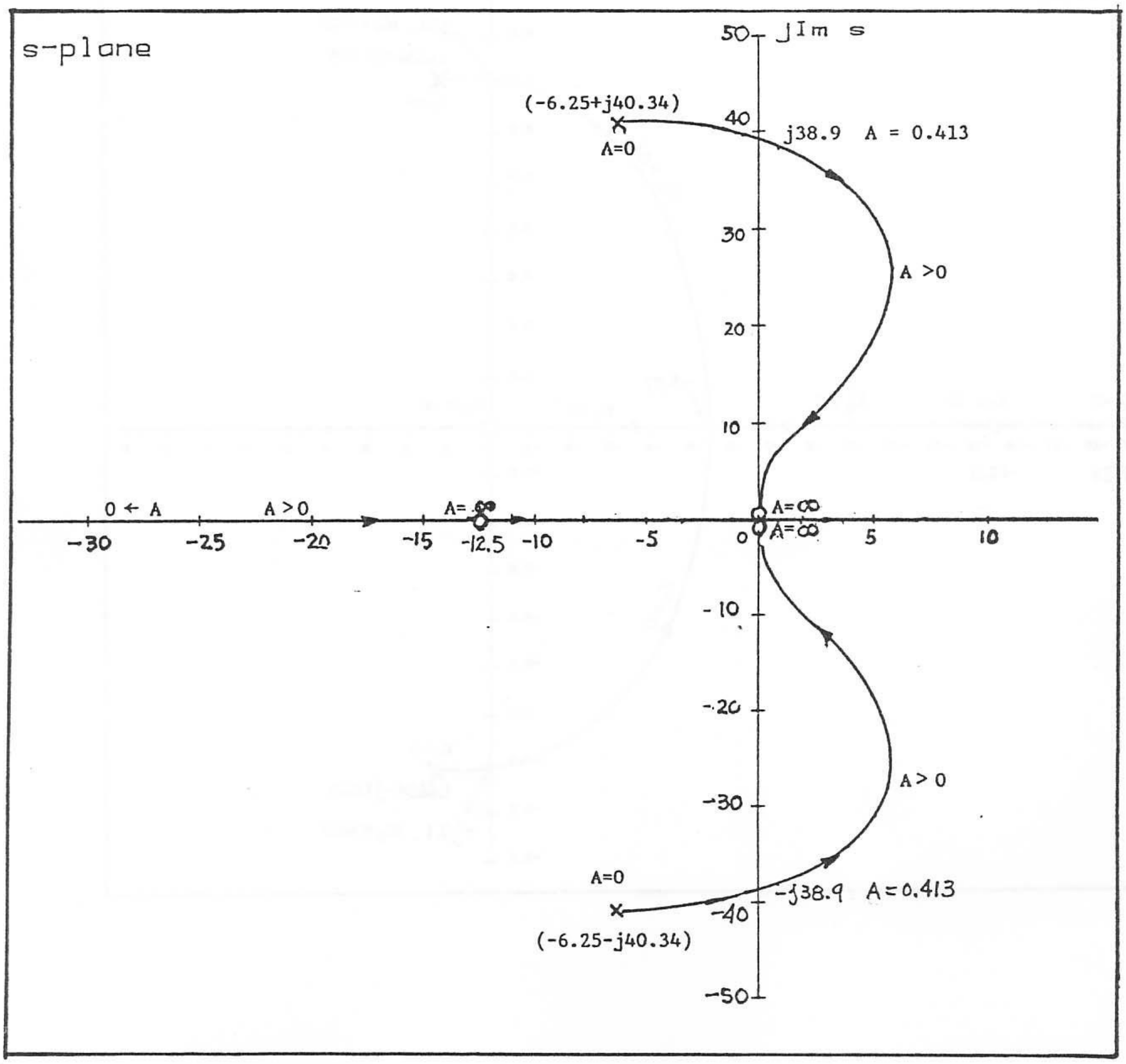
9-33 (b) $P(s)=s^2+12.5+1666.67=(s+6.25+j40.34)(s+6.25-j40.34)$

$Q(s)=0.02s^2(s+12.5)$

Asymptotes: ***A* > 0:** 180^o

Breakaway-point Equation: $0.02s^4 + 0.5s^3 + 103.13s^2 + 833.33s = 0$

Breakaway Points: **(RL)** 0



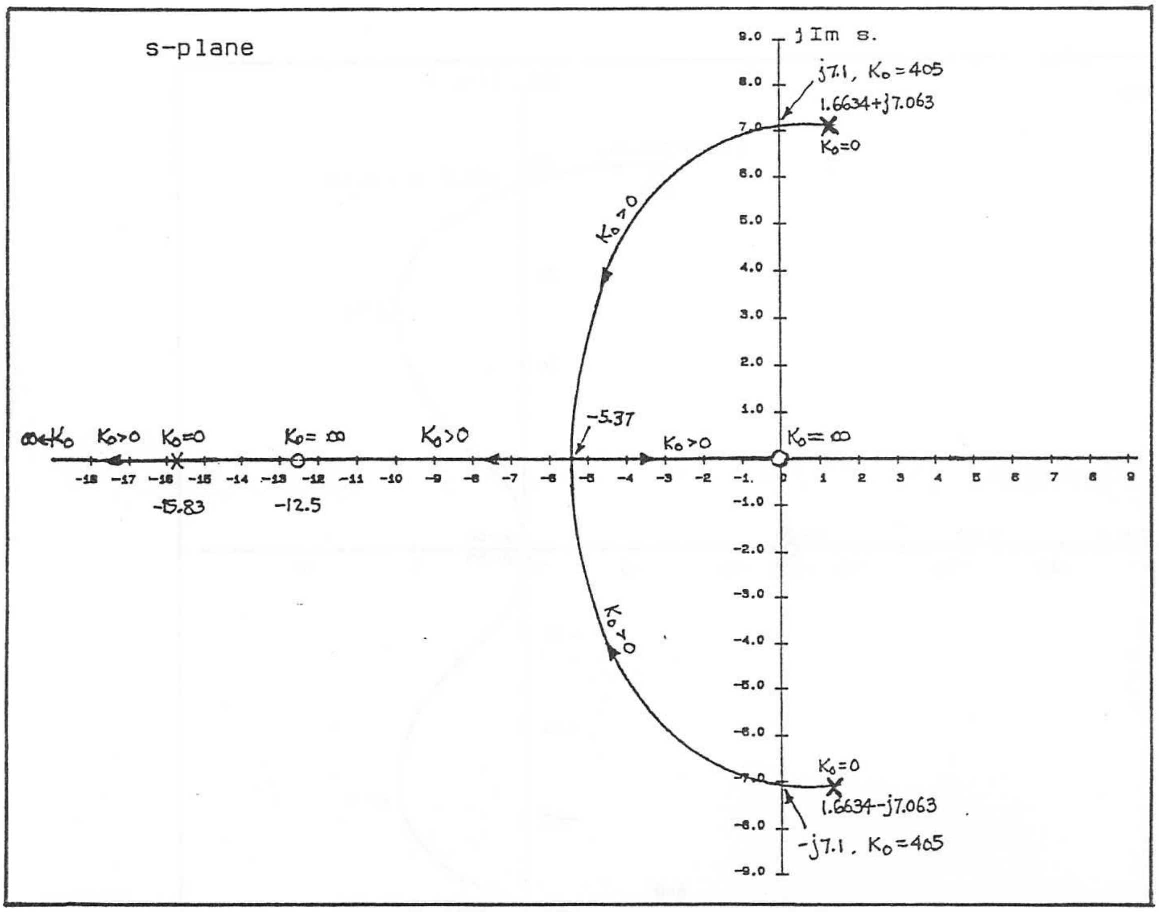
9-33 (c) $P(s) = s^3 + 12.5s^2 + 833.33 = (s+15.83)(s-1.663+j7.063)(s-1.663-j7.063)$

$Q(s) = 0.01s(s+12.5)$

Asymptotes: $K_o > 0$: 180^o

Breakaway-point Equation: $0.01s^4 + 0.15s^3 + 1.5625s^2 - 16.67s - 104.17 = 0$

Breakaway Point: **(RL)** -5.37

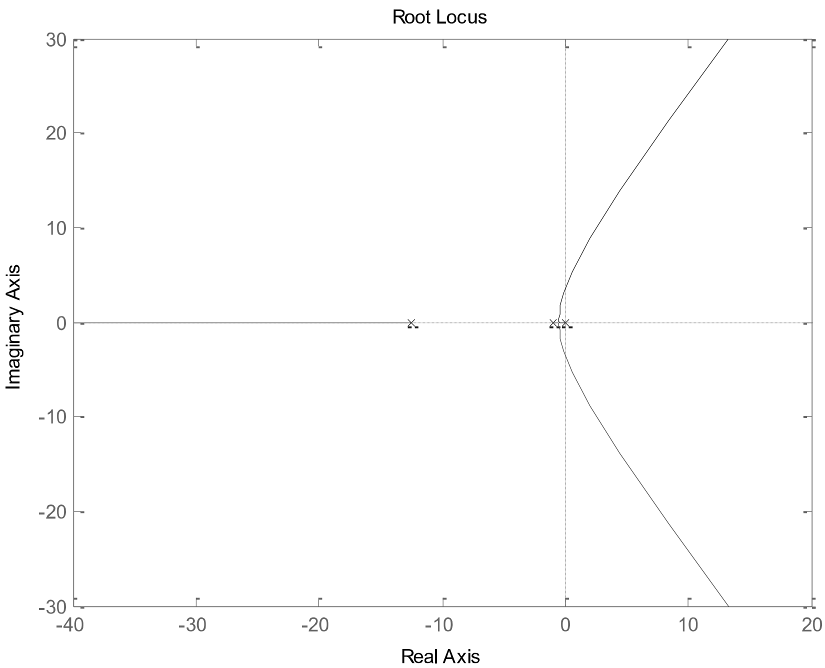


9-34) MATLAB code:

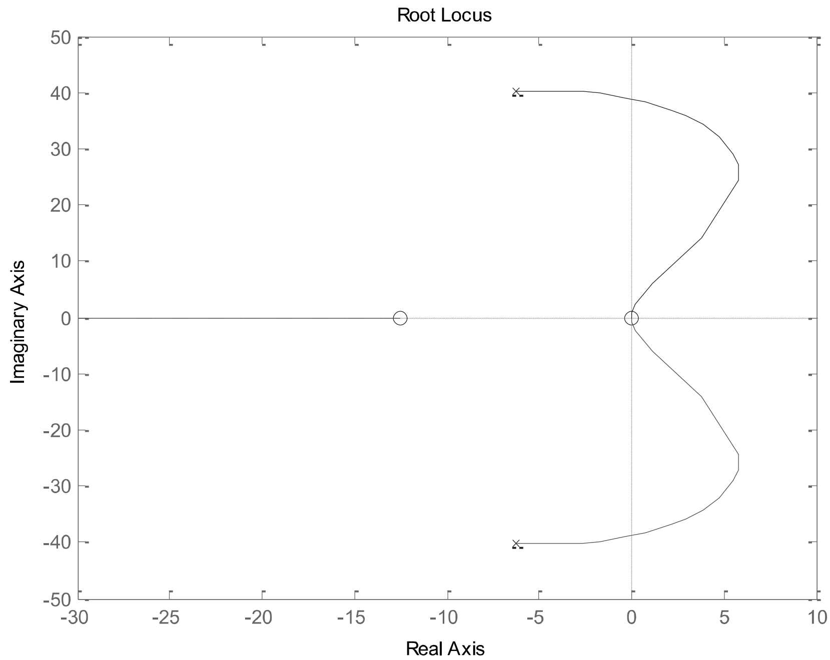
```
s = tf('s')
%a)
A=100;
K0=100;
num_G_a = 250;
den_G_a = 0.06*s*(s + 12.5)*(A*s+K0);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)

%b)
N=20;
K0=50;
```

```
num_G_b = 0.06*s*(s+12.5)*s
den_G_b = K0*(0.06*s*(s+12.5))+250*N;
G_b = num_G_b/den_G_b;
figure(2);
rlocus(G_b)

%c)
A=100;
N=20;
num_G_c = 0.06*s*(s+12.5);
den_G_c = 0.06*s*(s+12.5)*A*s+250*N;
G_c = num_G_c/den_G_c;
figure(3);
rlocus(G_c)
```

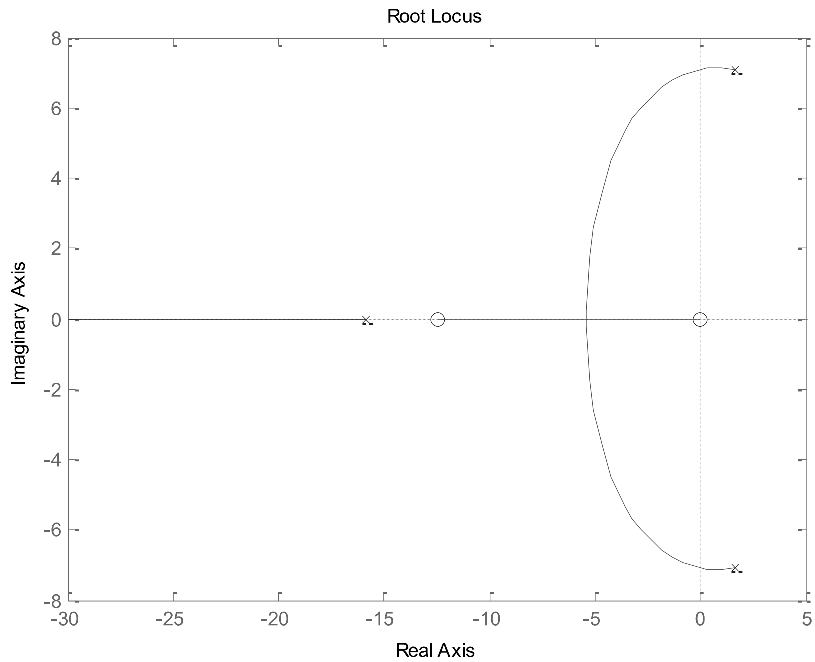
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c):



9-35) a) zeros: $s = -2$, poles: $s = -2j, +2j, -5$

Angle of asymptotes: $\theta_i = \frac{2i+1}{4-2} \times 180 = 90,270$

$\sigma = -3$

Breakaway points: $\frac{1}{s^2+4} + \frac{1}{(s+5)^2} = \frac{1}{(s+2)^2}$

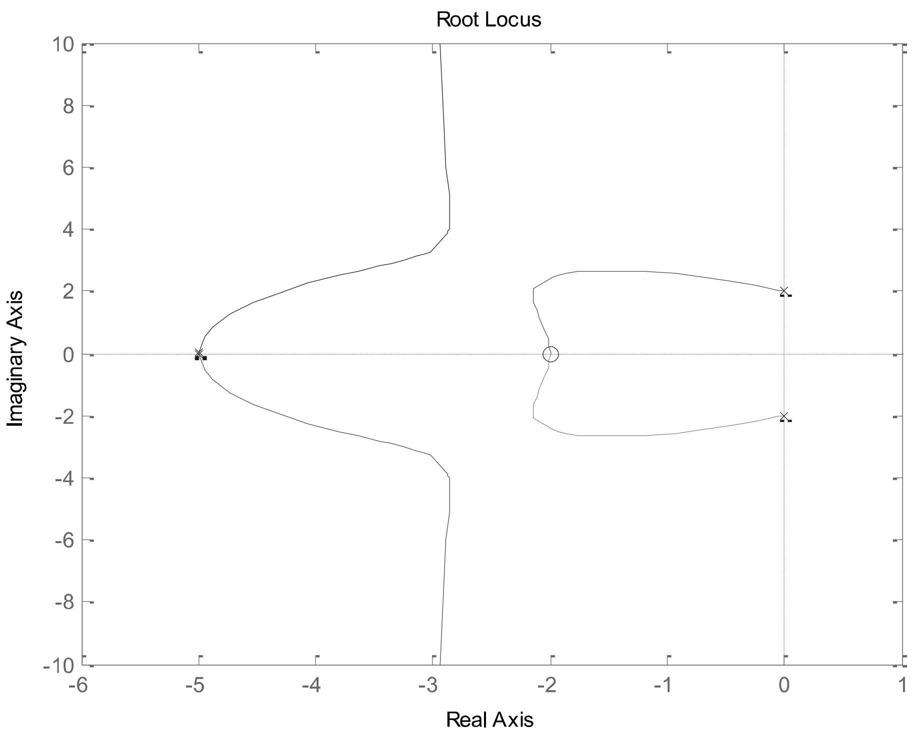
⇨ $s^2 + 6s + 25 = 0$ ➔ $s = -1.5 + 2j, 1.5 - 2j$

b) There is no closed loop pole in the right half s-plane; therefore the system is stable for all K>0

c) MATLAB code:

```
num_G=25*(s+2)^2;
den_G=(s^2+4)*(s+5)^2;
G_a=num_G/den_G;
figure(1);
rlocus(G_a)
```

Root locus diagram:



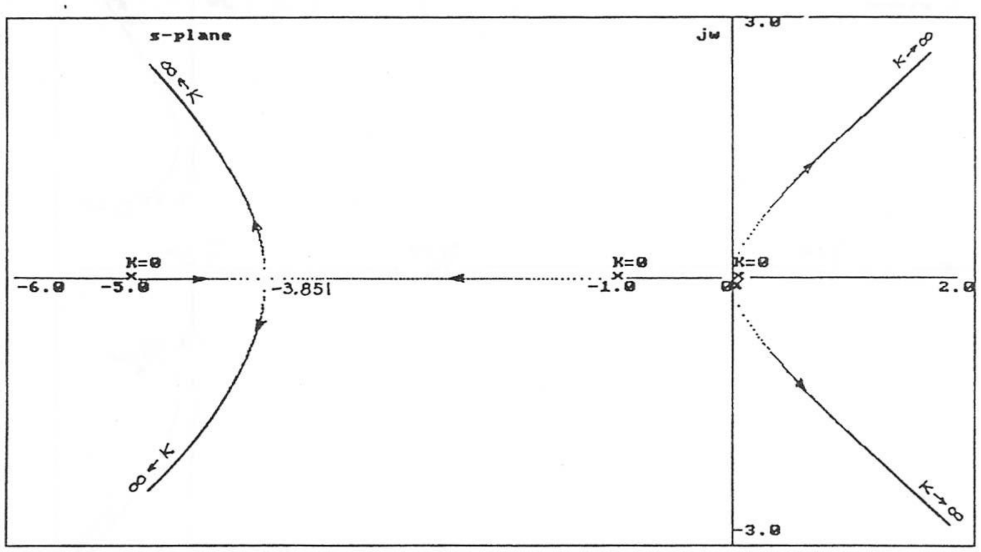
9‑36) (a) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 1$

Asymptotes: *K* > 0: $45^o,\ 135^o,\ 225^o,\ 315^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-1-5}{4} = -1.5$$

Breakaway-point Equation: $4s^3 + 18s^2 + 10s = 0$ **Breakaway point: (RL)** 0, –3.851



(b) $P(s) = s^2(s+1)(s+5) \qquad Q(s) = 5s+1$

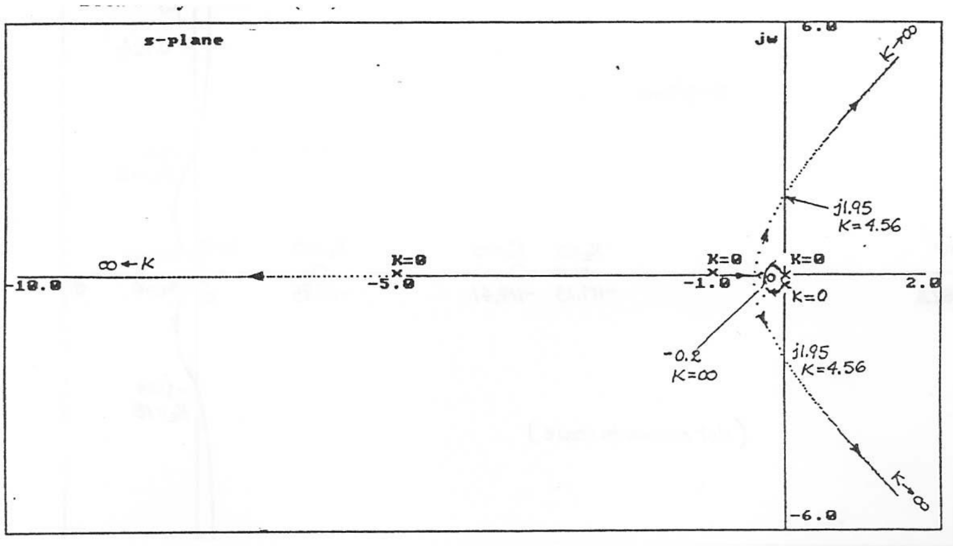
Asymptotes: *K* > 0: $60^o,\ 180^o,\ 300^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-1-5-(-0.2)}{4-1} = -\frac{5.8}{3} = -1.93$$

Breakaway-point Equation: $15s^4 + 64s^3 + 43s^2 + 10s = 0$

Breakaway Points: (RL) –3.5026

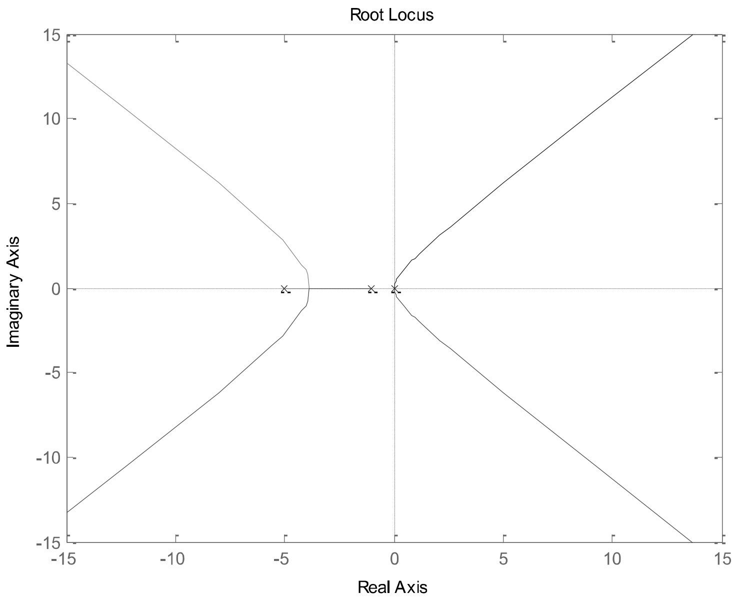


9-37)

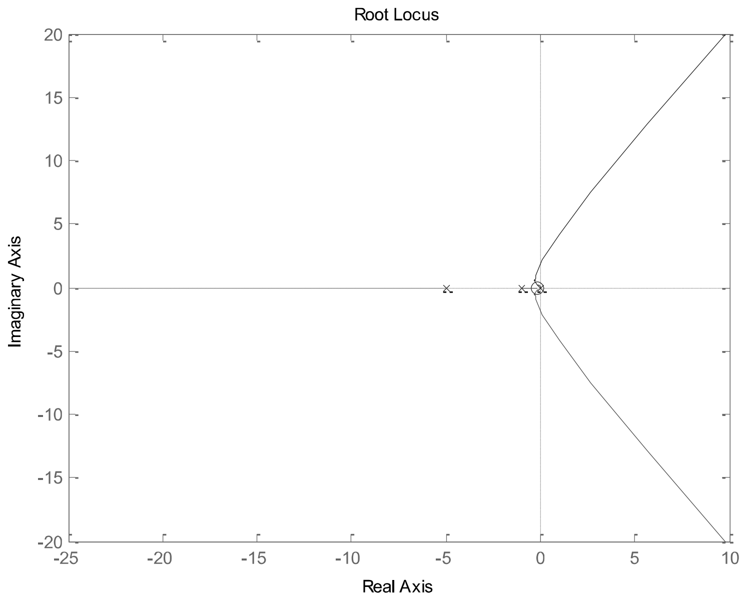
MATLAB code (9-37):

```
s = tf('s')
%a)
num_GH_a= 1;
den_GH_a=s^2*(s+1)*(s+
5);
GH_a=num_GH_a/den_GH_a
;
figure(1);
rlocus(GH_a)

%b)
num_GH_b= (5*s+1);
den_GH_b=s^2*(s+1)*(s+
5);
GH_b=num_GH_b/den_GH_b
;
figure(2);
rlocus(GH_b)
```

Root locus diagram, part (a):



Root locus diagram, part (b):



9-38) a) e^{-s} can be approximated by (easy way to verify is to compare both funtions' Taylor series expansions)

$$e^{-s} \approx \frac{2-s}{2+s}$$

Therefore:

$$G(s) = -\frac{K(s-2)}{(s+1)(s+2)}$$

Zeros: $s = 2$ and poles: $s = -1, -2$

Angle of asymptotes : $\theta_i = (2i+1)180 = 180$

$$\sigma_c = -(1+2-2) = -1$$

Breakaway points: $\frac{1}{s+1} + \frac{1}{s+2} = \frac{1}{2-s}$

Which means: $s^2 + 4s = 0$ ➔ $s = 0, s = \pm 2$

b) $s + 1 + K\frac{2-s}{s+2} = 0$ ➔ $s^2 + 3s + 2 - Ks + 2K = 0$

S^2	1	2+2k
S	3-k	0

S^0	(3-k)(2+2k)

As a result:

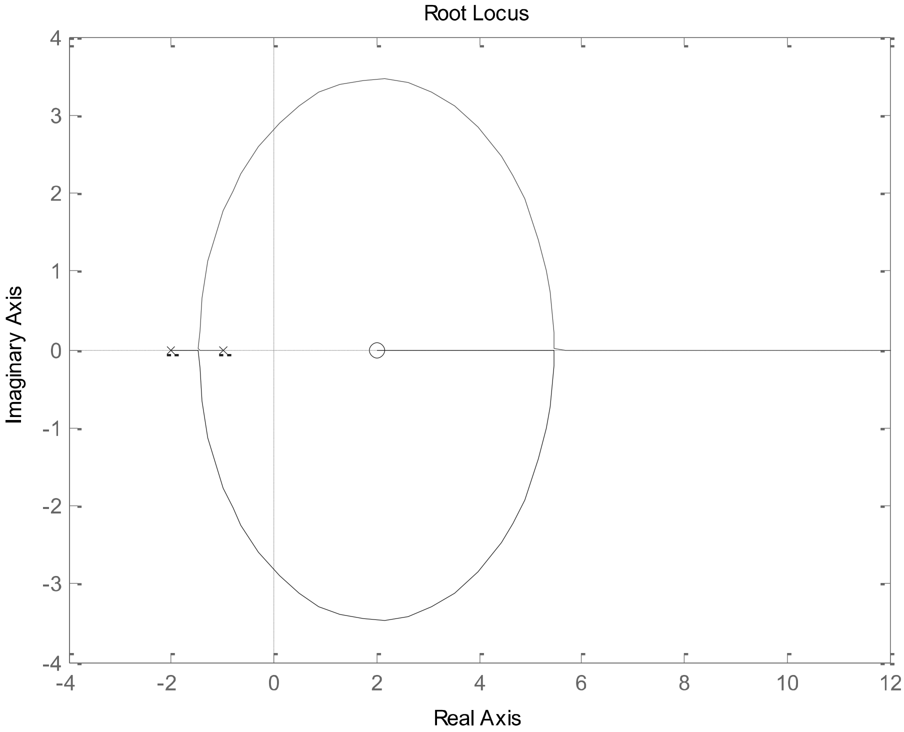
$$\begin{cases} 3-K>0 \ \rightarrow \ K<3 \\ (3-K)(2+2K)>0 \ \rightarrow \ 2+2k>0 \rightarrow \ K>-1 \end{cases}$$

Since K must be positive, the range of stability is then $0<k<3$

c) In this problem, e^{-Ts} term is a time delay. Therefore, MATLB PADE command is used for pade approximation, where brings e^{-Ts} term to the polynomial form of degree N.

```
s = tf('s')
T=1
N=1;
num_GH= pade(exp(-1*T*s),N);
den_GH=(s+1);
GH=num_GH/den_GH;
figure(5);
rlocus(GH)
```

Root locus diagram:



9-39)

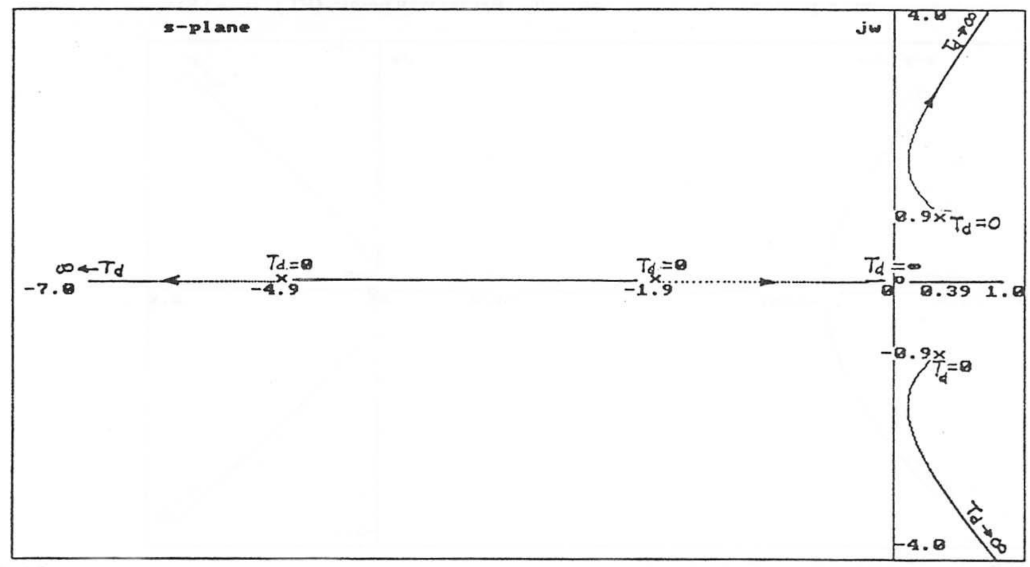
(a) $P(s) = s^2(s+1)(s+5)+10 = (s+4.893)(s+1.896)(s-0.394+j0.96)(s-0.394+j0.96)$

$Q(s) = 10s$

Asymptotes: $T_d > 0$: 60^o, 180^o, 300^o

Intersection of Asymptotes: $\sigma_1 = \dfrac{-4.893-1.896+0.3944+0.3944}{4-1} = -2$

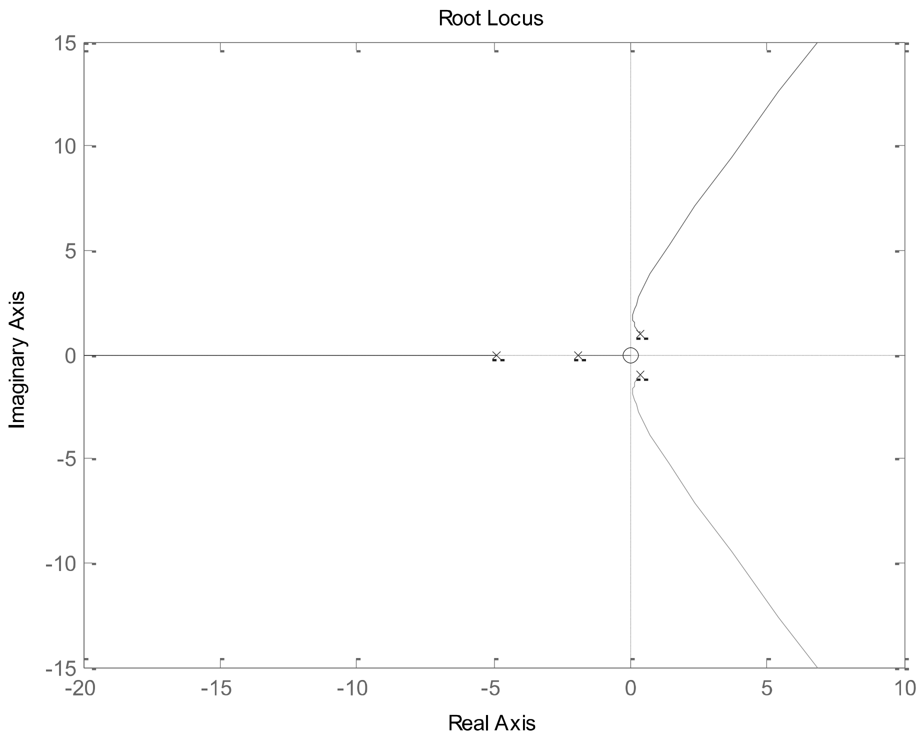
There are no breakaway points on the RL.



(b) MATLAB code:

```
s = tf('s')
num_GH= 10*s;
den_GH=s^2*(s+1)*(s+5)+10;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)
```

Root locus diagram:



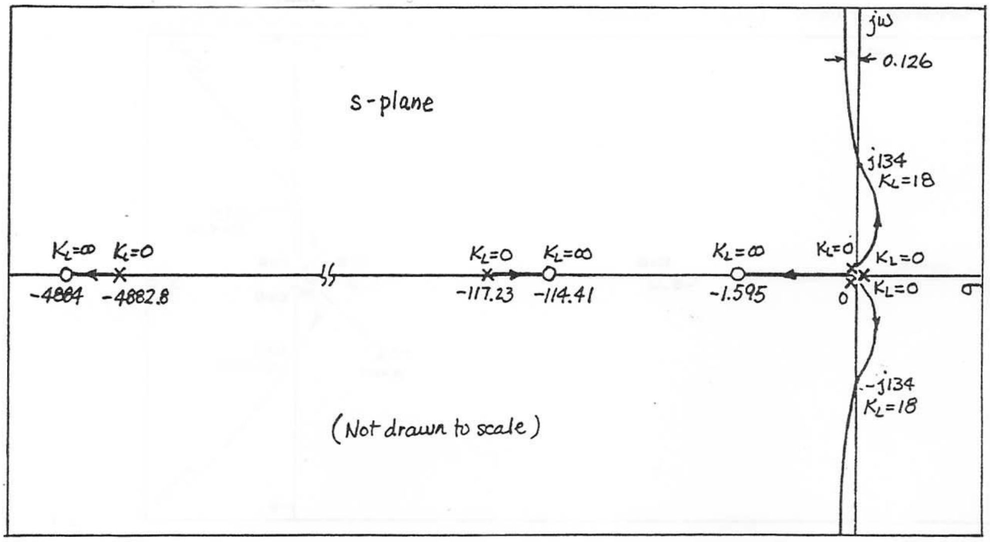
9-40) (a) ***K* = 1:** $P(s) = s^3(s+117.23)(s+4882.8)$ $\qquad Q(s) = 1010(s+1.5948)(s+114.41)(s+4884)$

Asymptotes: $K_L > 0$: $\quad 90^o, \ 270^o$

Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 1.5948 + 114.41 + 4884}{5-3} = -0.126$$

Breakaway Point: (RL) 0



9-40 (b) ***K* = 1000:** $P(s) = s^3(s+117.23)(s+4882.8)$

$$Q(s) = 1010(s^3 + 5000s^2 + 5.6673\times 10^5 s + 891089110)$$
$$= 1010(s+4921.6)(s+39.18+j423.7)(s+39.18-423.7)$$

Asymptotes: $K_L > 0$: 90^o, 270^o

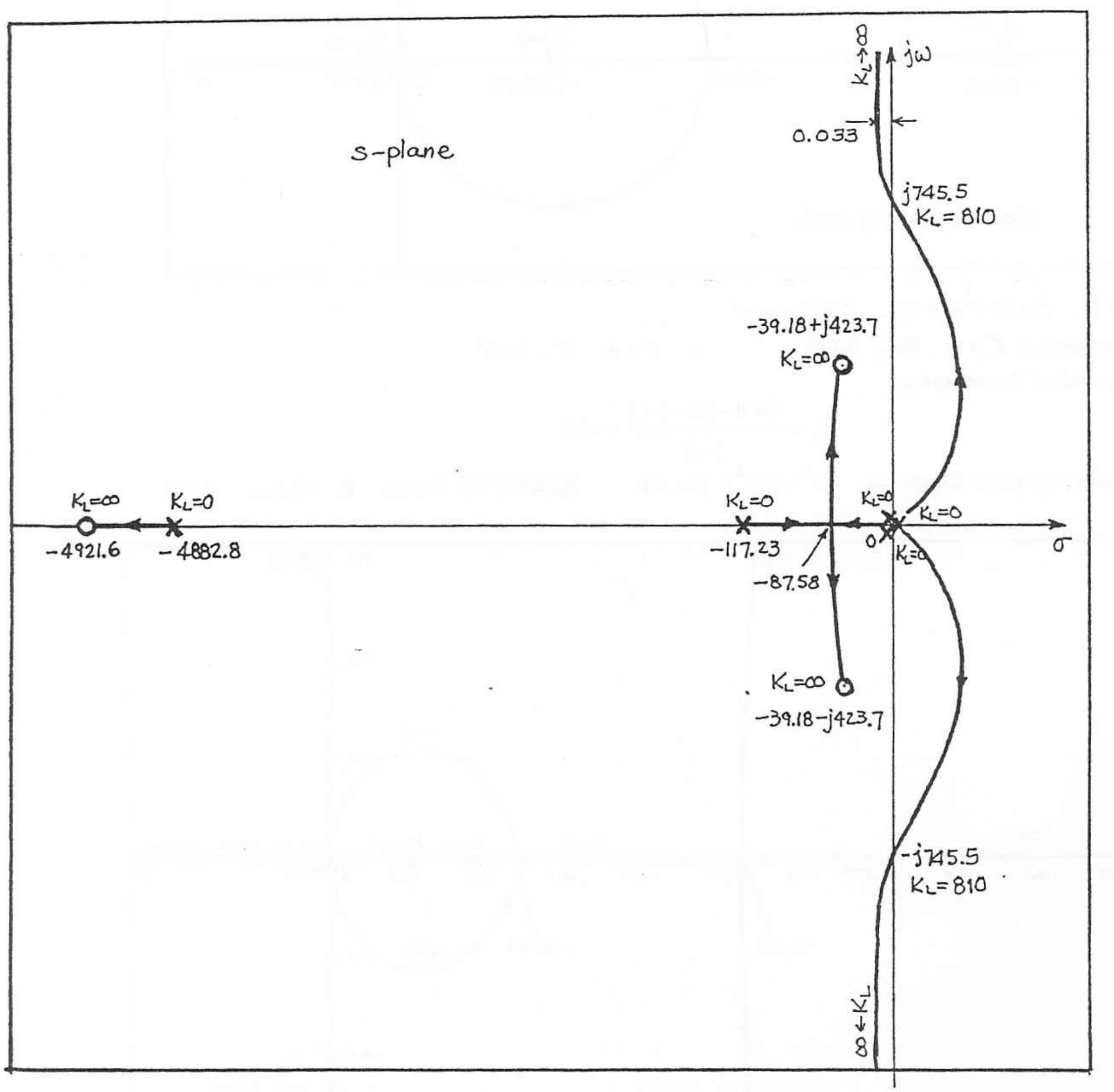
Intersect of Asymptotes:

$$\sigma_1 = \frac{-117.23 - 4882.8 + 4921.6 + 39.18 + 39.18}{5-3} = -0.033$$

Breakaway-point Equation:

$$2020s^7 + 2.02\times 10^7 s^6 + 5.279\times 10^{10} s^5 + 1.5977\times 10^{13} s^4 + 1.8655\times 10^{16} s^3 + 1.54455\times 10^{18} s^2 = 0$$

Breakaway points: (RL) 0, –87.576

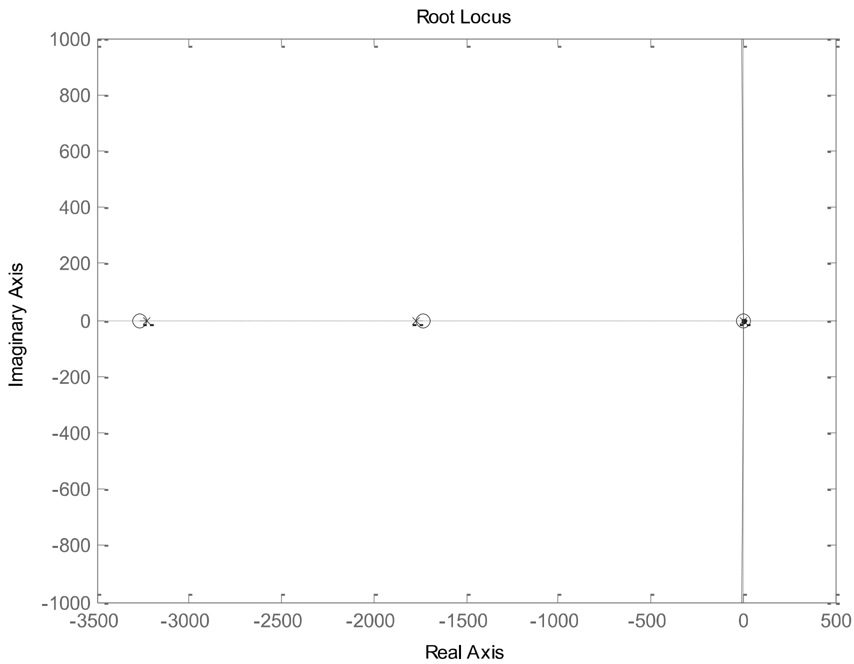


9-41) MATLAB code:

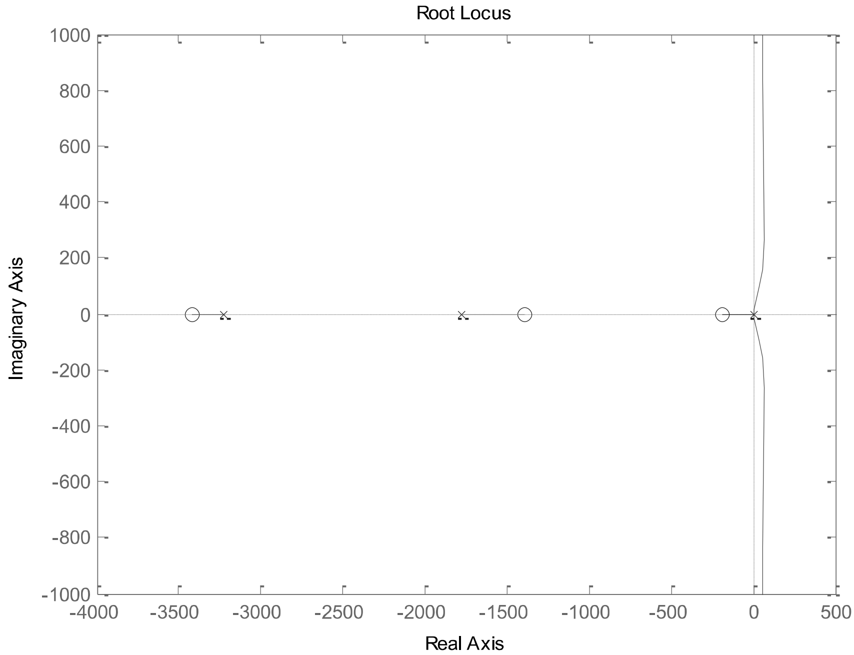
```
s = tf('s')
Ki=9;
Kb=0.636;
Ra=5;
La=.001;
Ks=1;
n=.1;
Jm=0.001;
Jl=0.001;
Bm=0;
%a)
```

```
K=1;
num_G_a=((n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_a=((La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_a=num_G_a/den_G_a;
figure(1);
rlocus(G_a)
%b)
K=1000;
num_G_b=((n^2*La*Jl+La*Jm)*s^3+(n^2*Ra*Jl+Ra*Jm+Bm*La)*s^2+Ra*Bm*s+Ki
*Kb*s+n*Ks*K*Ki);
den_G_b=((La*Jm*Jl)*s^5+(Jl*Ra*Jm+Jl*Bm*La)*s^4+(Ki*Kb*Jl+Ra*Bm*Jl)*s
^3);
G_b=num_G_b/den_G_b;
figure(2);
rlocus(G_b)
```

Root locus diagram, part (a):



Root locus diagram, part (b):



9-42

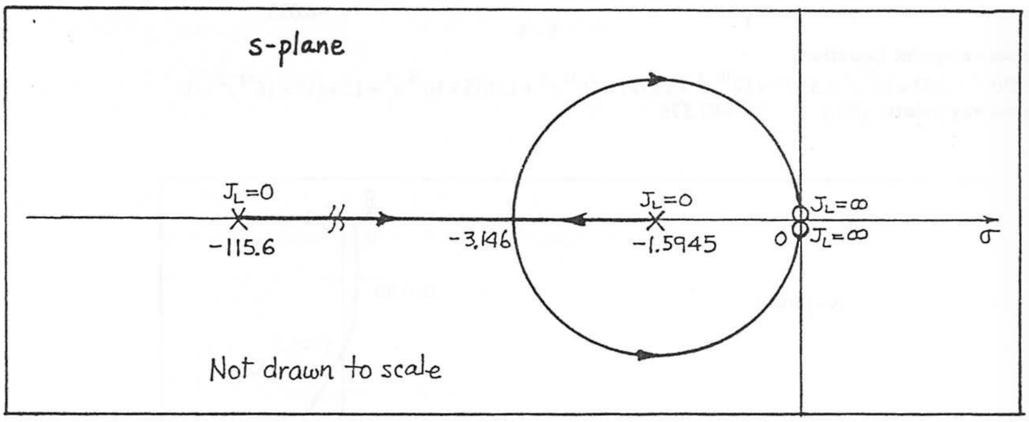
(a) Characteristic Equation: $s^3 + 5000s^2 + 572{,}400s + 900{,}000 + J_L\left(10s^3 + 50{,}000s^2\right) = 0$

$P(s) = s^3 + 5000s^2 + 572{,}400s + 900{,}000 = (s + 1.5945)(s + 115.6)(s + 4882.8)$ $\qquad Q(s) = 10s^2(s + 5000)$

Since the pole at –5000 is very close to the zero at –4882.8, $P(s)$ and $Q(s)$ can be approximated as:

$P(s) \cong (s + 1.5945)(s + 115.6) \qquad Q(s) \cong 10.24s^2$

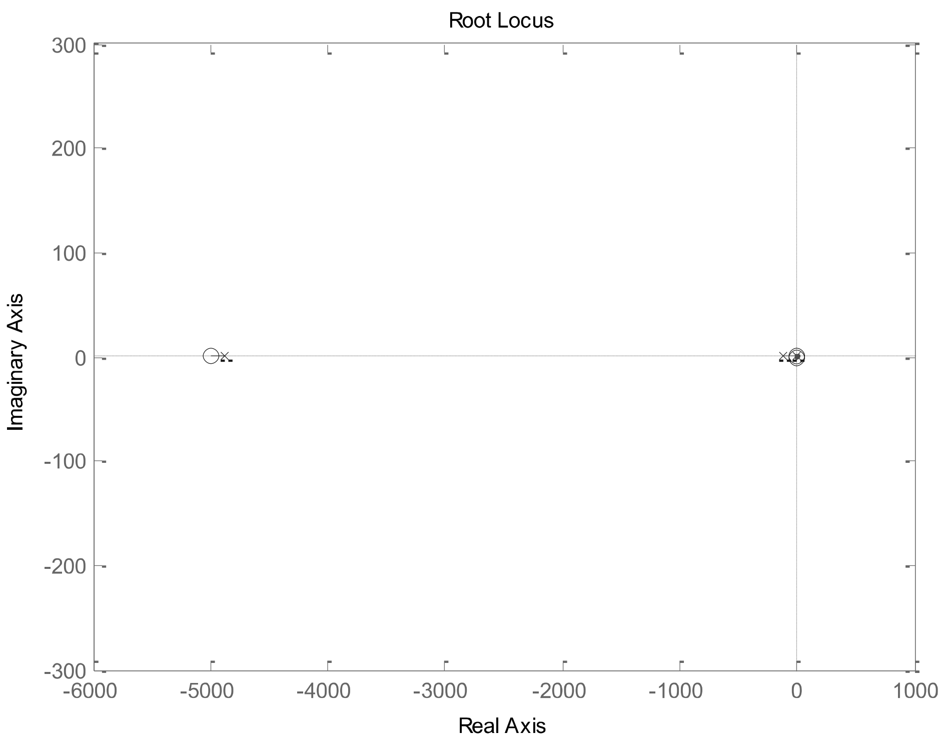
Breakaway-point Equation: $1200s^2 + 3775s = 0$ **Breakaway Points: (RL):** 0, –3.146



(b) MATLAB code:

```
s = tf('s')
K=1;
Jm=0.001;
La=0.001;
n=0.1;
Ra=5;
Ki=9;
Bm=0;
Kb=0.0636;
Ks=1;

num_G_a = (n^2*La*s^3+n^2*Ra*s^2);
den_G_a = (La*Jm*s^3+(Ra*Jm+Bm*La)*s^2+(Ra*Bm+Ki*Kb)*s+n*Ki*Ks*K);
G_a = num_G_a/den_G_a;
figure(1);
rlocus(G_a)
```

Root locus diagram:



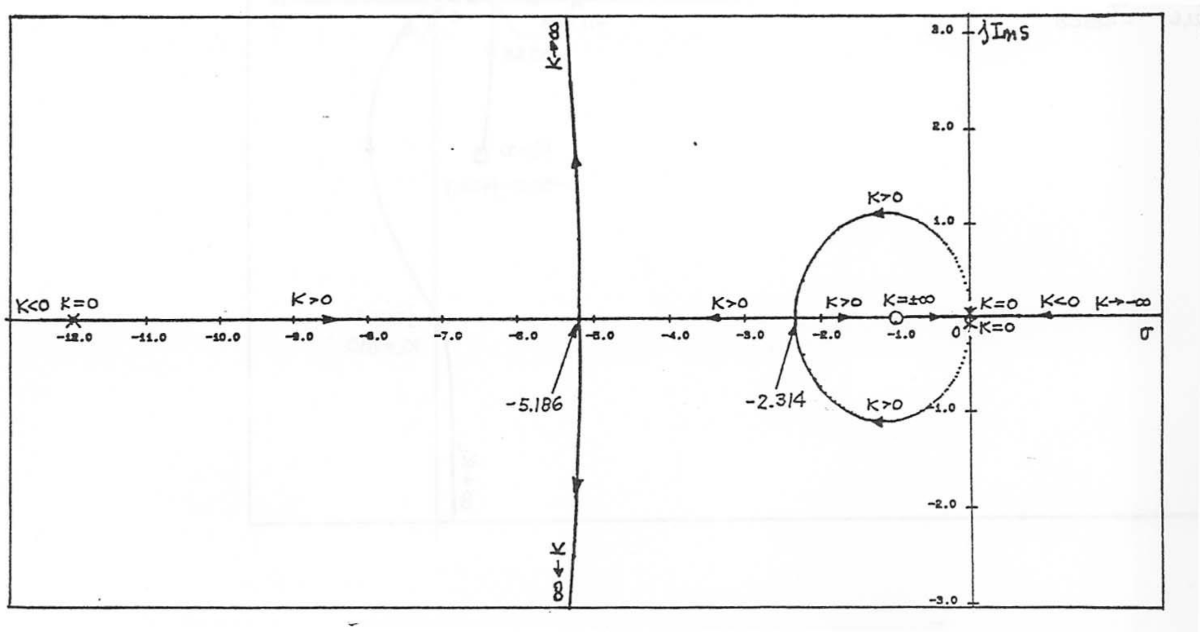
9-43) (a) $\alpha = 12$: $P(s) = s^2(s+12)$ $\quad Q(s) = s+1$

Asymptotes: ***K* > 0:** 90^o, 270^o ***K* < 0:** 0^o, 180^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-12-(-1)}{3-1} = -5.5$$

Breakaway-point Equation: $2s^3 + 15s^2 + 24s = 0$ **Breakaway Points:** 0, −2.314, −5.186



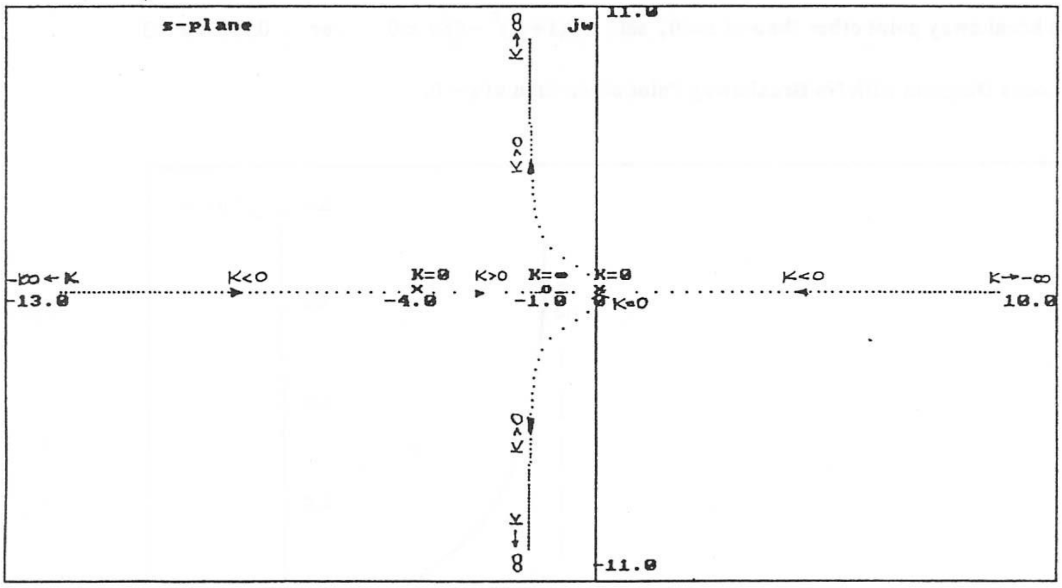
9-43 (b) $\alpha = 4$: $P(s) = s^2(s+4)$ $\quad Q(s) = s+1$

Asymptotes: ***K* > 0:** 90^o, 270^o ***K* < 0:** 0^o, 180^o

Intersect of Asymptotes:

$$\sigma_1 = \frac{0+0-4-(-10}{3-1} = -1.5$$

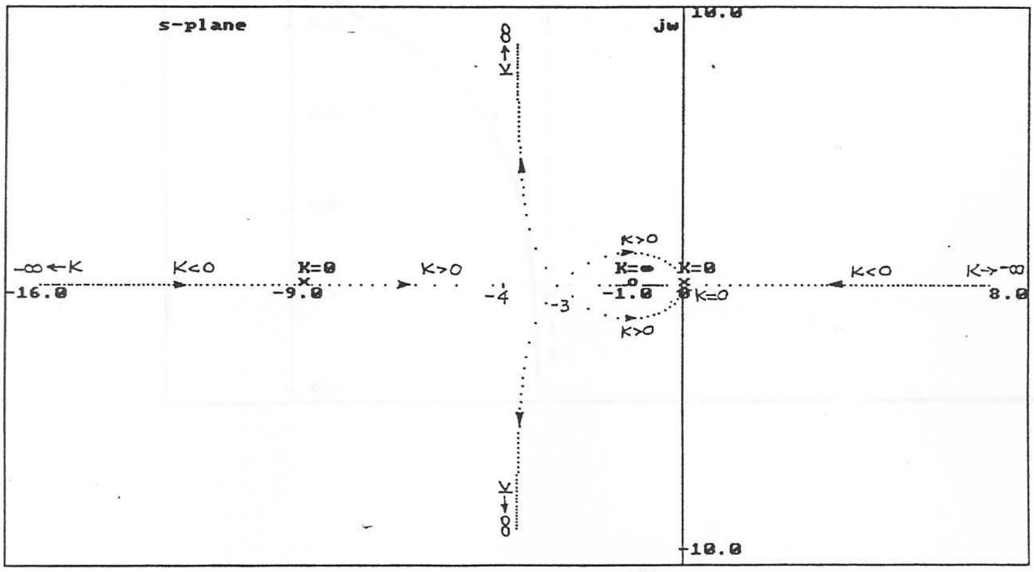
Breakaway-point Equation: $2s^3 + 7s^2 + 8s = 0$ **Breakaway Points:** ***K* > 0** 0. None for ***K* < 0.**



(c) **Breakaway-point Equation:** $2s^2 + (\alpha+3)s + 2s = 0$ **Solutions:** $s = -\dfrac{\alpha+3}{4} \pm \dfrac{\sqrt{(\alpha+3)^2 - 16\alpha}}{4}$, $s = 0$

For one nonzero breakaway point, the quantity under the square-root sign must equal zero.

Thus, $\alpha^2 - 10\alpha + 9 = 0$, $\alpha = 1$ or $\alpha = 9$. The answer is $\alpha = 9$. The $\alpha = 1$ solution represents pole-zero cancellation in the equivalent $G(s)$. When $\alpha = 9$, the nonzero breakaway point is at $s = -3$. $\sigma_1 = -4$.



9-44)

For part (c), after finding the expression for:

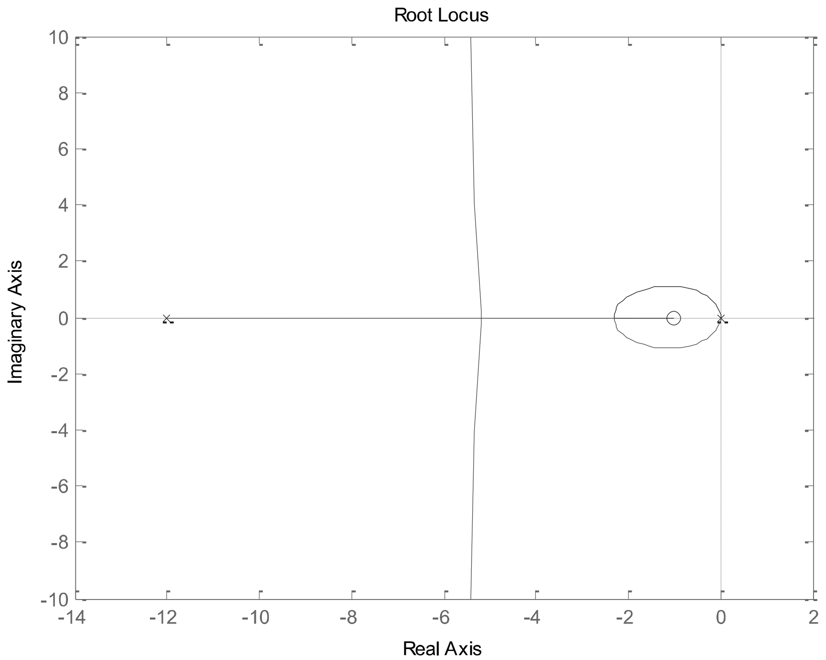
$$\frac{dk}{ds} = \frac{-3-\alpha \pm \sqrt{(\alpha-1)(\alpha-9)}}{4},$$

there is one acceptable value of alpha that makes the square root zero ($\alpha = 9$). Zero square root means one answer to the breakaway point instead of 2 answers as a result of $\pm$ sign. $\alpha = 1$ is not acceptable since it results in $s = -1 @ \frac{dk}{ds} = 0$ and then $k = \frac{0}{0}$.

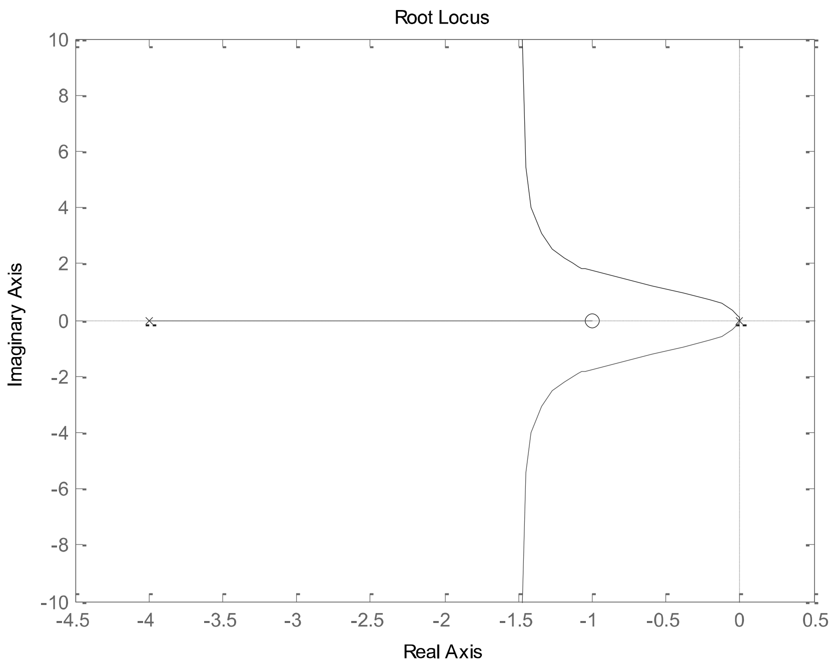
MATLAB code:

```
s = tf('s')

%(a)
alpha=12
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(1);
rlocus(GH)

%(b)
alpha=4
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(2);
rlocus(GH)

%(c)
alpha=9
num_GH= s+1;
den_GH=s^3+alpha*s^2;
GH=num_GH/den_GH;
figure(3);
rlocus(GH)
```

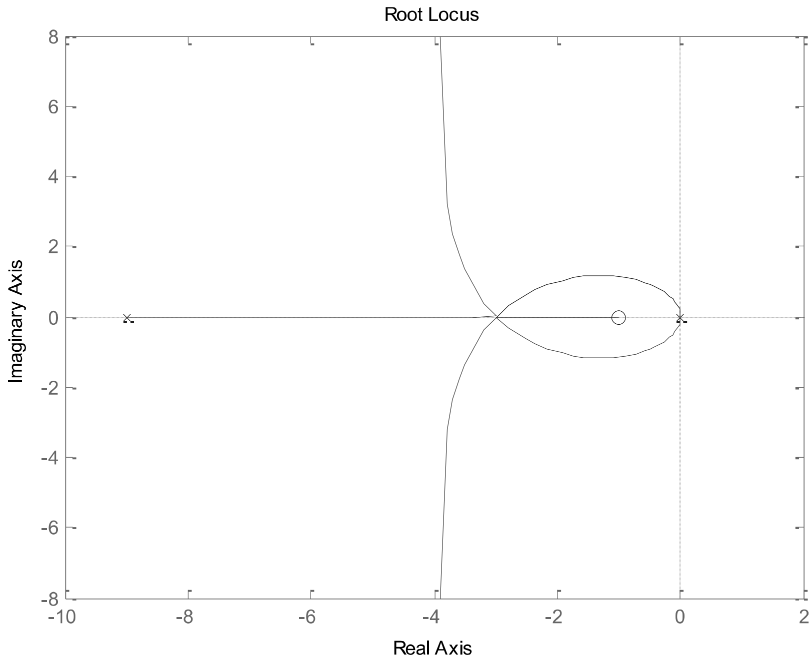
Root locus diagram, part (a):



Root locus diagram, part (b):



Root locus diagram, part (c): (`alpha=9 resulting in 1 breakaway point`)



9-45) (a) $P(s) = s^2(s+3) \qquad Q(s) = s + \alpha$

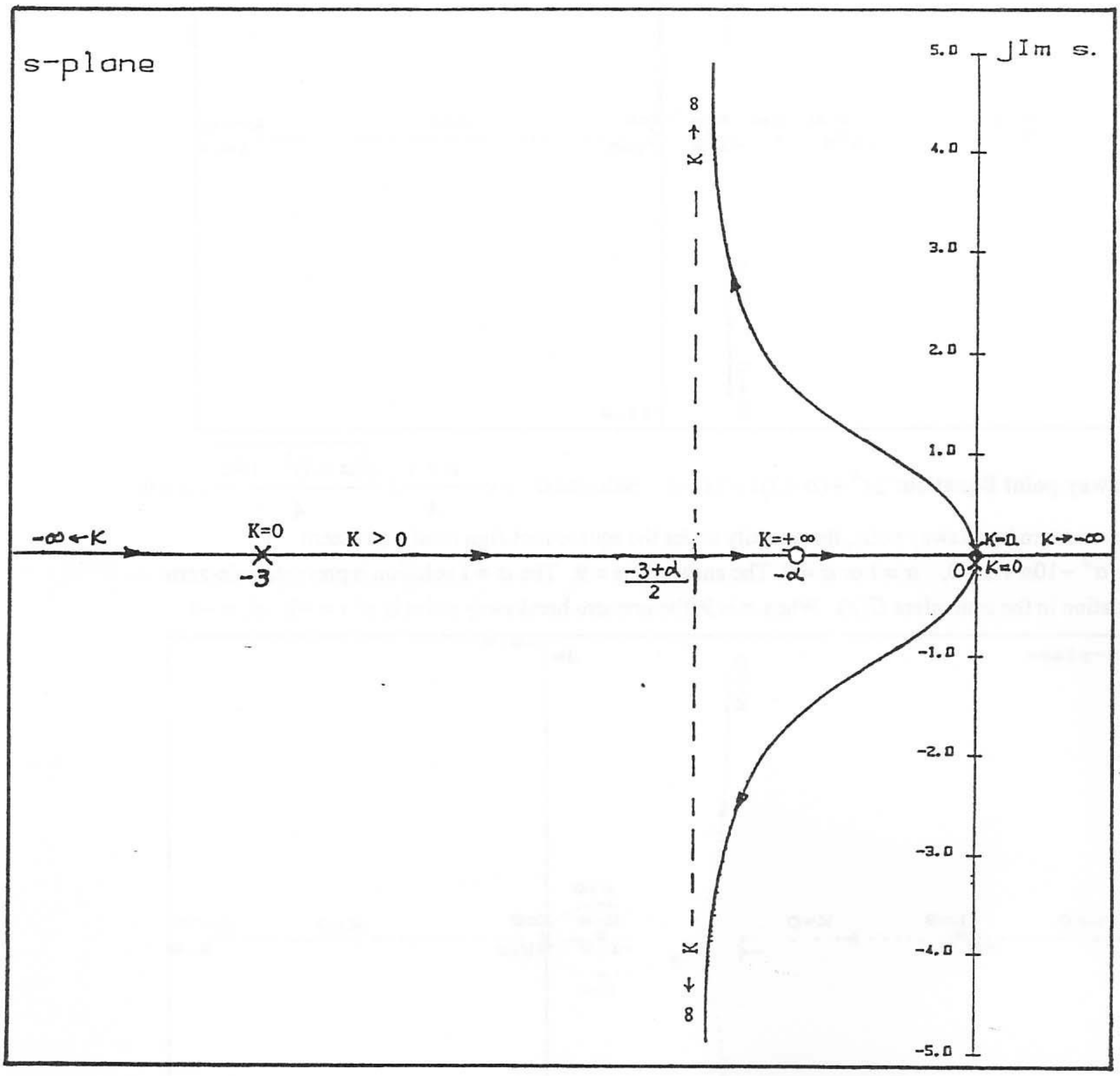
Breakaway-point Equation: $2s^3 + 3(1+\alpha)s + 6\alpha = 0$

The roots of the breakaway-point equation are:

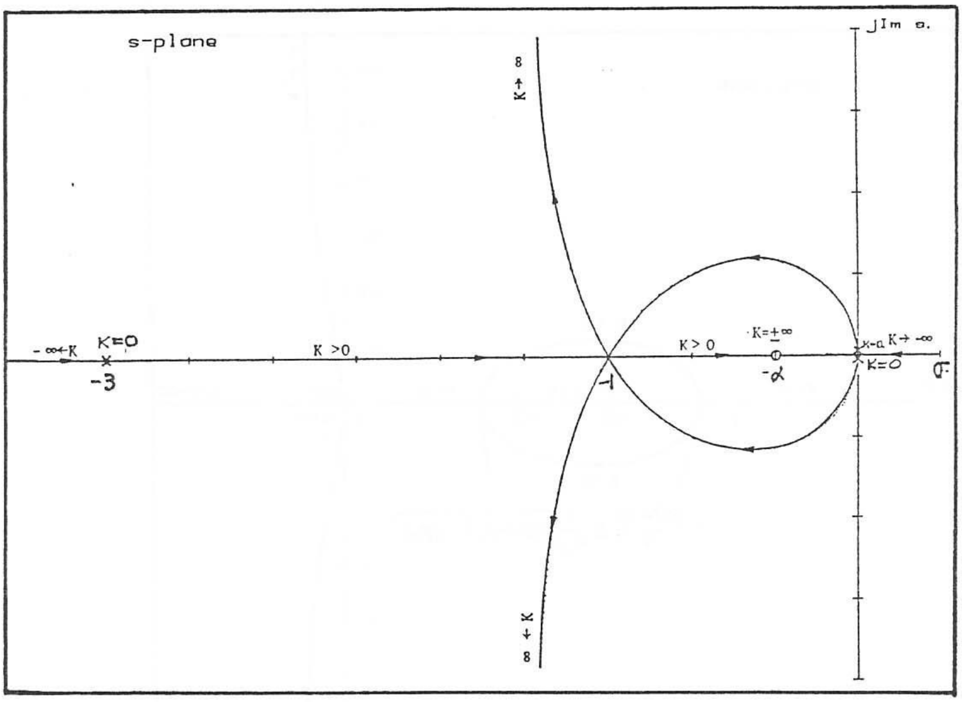
$$s = \frac{-3(1+\alpha)}{4} \pm \frac{\sqrt{9(1+\alpha)^2 - 48\alpha}}{4}$$

For no breakaway point other than at $s = 0$**, set** $9(1+\alpha)^2 - 48\alpha < 0$ **or** $-0.333 < \alpha < 3$

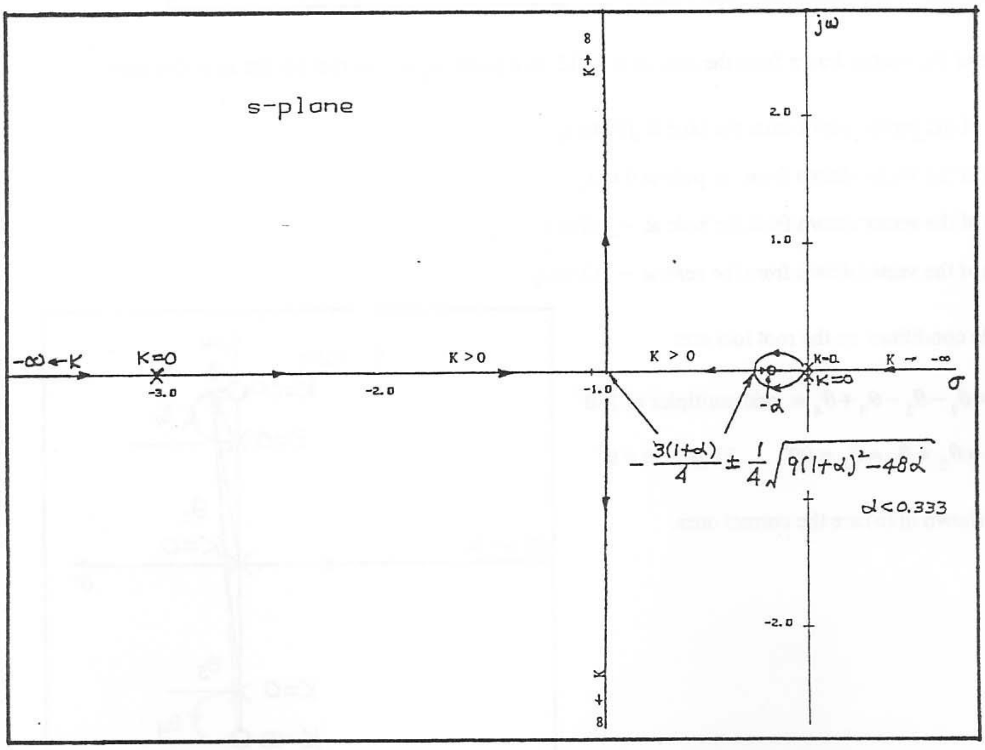
Root Locus Diagram with No Breakaway Point other than at *s* = 0.



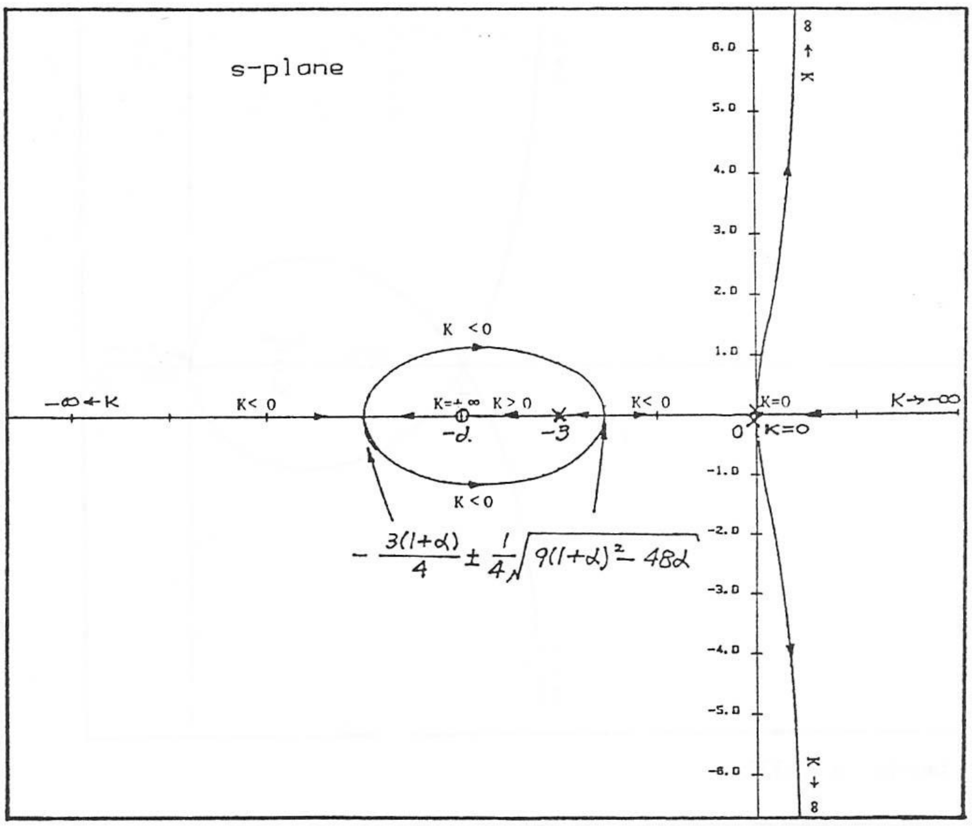
9-45 (b) **One breakaway point other than at *s* = 0:** $\alpha = 0.333$, Breakaway point at $s = -1$.



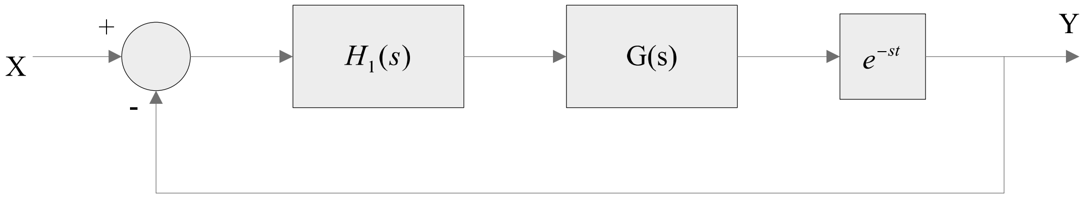
(c) Two breakaway points: $\alpha < 0.333$:



9-45 (d) Two breakaway points: $\alpha > 3$:



9-46) First we can rearrange the system as:



where

$$H_1(s) = \frac{H}{1 + (1 - e^{-st})GH}$$

Now designing a controller is similar to the designing a controller for any unity feedback system.

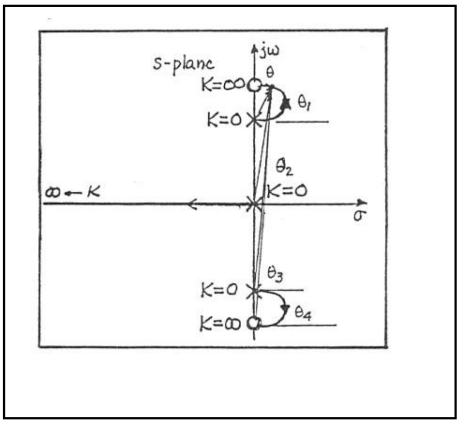
9-47) Let the angle of the vector drawn from the zero at $s = j12$ to a point s_1 on the root locuss near the zero be θ. Let

$\theta_1 =$ angle of the vector drawn from the pole at $j10$ to s_1.

$\theta_2 =$ angle of the vector drawn from the pole at 0 to s_1.

$\theta_3 =$ angle of the vector drawn from the pole at $-j10$ to s_1.

$\theta_4 =$ angle of the vector drawn from the zero at $-j12$ to s_1.



Then the angle conditions on the root loci are:

$$\theta = \theta_1 - \theta_2 - \theta_3 + \theta_4 = \text{ odd multiples of } 180^o$$

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = 90^o \qquad \text{Thus,} \quad \theta = 0^o$$

The root loci shown in (b) are the correct ones.

Answers to True and False Review Questions:

6. (F) 7. (T) 8. (T) 9. (F) 10. (F) 11. (T) 12. (T) 13. (T) 14. (T)