Estimation of Cosmological Parameters in ΛCDM- and DGP-Model Using Supernovae Type Ia Data

SUBMITTED BY

Danial Hagemann

Abschätzung kosmologischer Parameter im ΛCDM- und DGP-Modell anhand von Supernovae Typ Ia Daten

Bachelorarbeit

Universitätssternwarte München
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vorgelegt von

Danial Hagemann

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University Observatory Munich
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at the Ludwig-Maximilians-Universität
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Danial Hagemann

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First supervisor: Prof. Dr. Jochen Weller
Second supervisor: Dr. Steffen Hagstotz

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"The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy."

Steven Weinberg, [Wei93, Epilogue]

It was a long path to finally achieve this point in my life. And it was not an easy one. But no one said that life was going to be easy.
I was about 12 years old when I watched the sky the very first time with my telescope, looking out for the most beautiful planet of our solar system. Being able to admire the ring structure of Saturn opened my eyes and my heart to explore the beauty of nature, the beauty of space.

While growing up, we try, driven by curiosity and fascination, to understand the world around us. We may seek answers to "the big questions of life".

Where do we come from?
Why did the universe come to existence?
How old is the universe?
Which mechanisms in nature guide the course of all events?
What will happen in the future?
Which role do I play in this unimaginable big universe?
What is the purpose of my life?
And, where do I want to go?

Those are difficult questions. And that is why those questions are probably the oldest ones of humanity.
Throughout my "journey of life", I tried to find answers to those questions. Maybe I could find some attempts to answer a few of those in this thesis. Maybe I will never find an answer to some of these questions.

At least I found an answer what the purpose of *my* life is: to discover the beauty of nature. Going on with my journey, trying to find answers. And never giving up during this adventure – no matter, how hard and how frustrating life can be sometimes. No matter how often I have fallen or I will fall down to the harsh ground of reality. Because it is my curiosity, my ambition and my passion that makes me standing up again to turn my eyes, my gaze into the sky.

That is the most important lesson I have learned in the last years, studying physics.

The fact that I made it so far is not only through reading textbooks, doing my exercise sheets or preparing for exams. Because it is not only the effort I brought up.

I am where I am thanks to the effort of several people that came into my life. I am very glad that I had the luck to get to know these people. And therefore, this is the best occasion for me to thank them.

First of all, I want to thank my teachers, who not only taught me a lot, but also did their best to answer my difficult questions and quench my thirst for knowledge, which was probably not always easy and took a lot of patience,

Pierette Al-Korey, Ulrike Blattert, Peter Bodden, Nicole Bömecke, Jörg Bühler, Reiner Büter, Rolf Heckmann, Petra Jähnigen, Thomas Schröder Klementa, Axel Knuth, Eva Krüger, René Meyer-Brede, Angelika Nieboer, Roland Petereit, David Stephan, Volkhard Stierhof, Stefan Usée, Selma Weiß-Tümmers, Wolf Wingenfeld, Sema Yilmaz, Mike Ziegner.

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And last but not least, I would like to express my greatest gratitude to my family – especially **my father**, **my mother** and **my sister**, who raised me and cared for me since the day I was born, who provided me everything that was needed to enjoy an excellent education, who opened the path on my way to a self-determined life and contributed to becoming the person I am today.

From the bottom of my heart – thank you.

In loving memory of Simon Jacob Gallo and Gamal Tadros Mishreky

NOTATION AND CONVENTIONS

Throughout this thesis, we choose the signature for the metric of spacetime to be $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$. This is a list of books on astrophysics, cosmology and general relativity I have came across while working on this thesis, divided into the different sign convention for the metric of spacetime.

(+) $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$

- Matthias Bartelmann. *Das kosmologische Standardmodell* [Bar19]
- Sean M. Carroll. *Spacetime and Geometry* [Car19]
- Scott Dodelson, Fabian Schmidt. *Modern Cosmology* [DS20]
- C.W. Misner, K.S. Thorne, J.A. Wheeler. *Gravitation* [MTW17]
- Bernard Schutz. *A First Course in General Relativity* [Sch09]
- Karl-Heinz Spatscheck. *Astrophysik* [Spa17]
- Steven Weinberg. *Cosmology* [Wei08]
- A. Zee. *Einstein Gravity in a Nutshell* [Zee13]

(−) $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$

- Bradley W. Carroll, Dale A. Ostlie. *An Introduction to Modern Astrophysics* [CO07]
- Viatcheslav Mukhanov. *Physical Foundations of Cosmology* [Muk05]
- P.J.E. Peebles. *Principles of Physical Cosmology* [Pee93]

In order not to lose the connection between empirically measured quantities we can observe and theoretical quantities introduced by the cosmological models, we choose to let $c = 299\,792\,458\,\mathrm{m/s}$ and $G = 6.674 \times 10^{-11}\,\mathrm{m^3/(kg\ s^2)}$ in SI-units.[1]

[1] ... although the convenience for mathematical manipulations in the case of natural units where $c = G = 1$ cannot be denied.

ABSTRACT

In this thesis, we consider measurements of type Ia supernovae by the *Supernova Cosmology Project*, the dataset “Union2.1”,[2] to obtain best-fit values to the free parameters of the ΛCDM-model, known as the “standard model of cosmology”, and the DGP-model proposed by Gia Dvali, Gregory Gabadadze and Massimo Porrati.
While the ΛCDM-model describes the expansion of the universe by the existence of a cosmological constant Λ, the DGP-model introduces an additional spatial dimension to spacetime, in which gravity mimics the expansion of the universe by reaching the introduced crossover scale r_{C}.
After introducing theoretical backgrounds on how to relate physical quantities we observe like magnitude and redshift with the cosmological parameters that both cosmological models contain, we estimate best-fit values for the parameters $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0})$ and $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \alpha)$, where α is an interpolation parameter for which $\alpha = 0$ describes the ΛCDM-model and $\alpha = 1$ the DGP-model, by minimizing the χ^2-distribution using the “Union2.1” SN Ia dataset.

In the case of estimating $\boldsymbol{\theta}_{\mathrm{best}} = (\Omega_{\mathrm{m},0,\mathrm{best}}, \Omega_{\Lambda,0,\mathrm{best}})$, we obtain $\boldsymbol{\theta}_{\mathrm{best}} = (0.28, 0.72) \pm (0.07, 0.12)$, which is consistent with the results provided by the *Supernova Cosmology Project* (see [Suz+12]).
In the case of estimating $\boldsymbol{\theta}_{\mathrm{best}} = (\Omega_{\mathrm{m},0,\mathrm{best}}, \alpha_{\mathrm{best}})$, we obtain $(\Omega_{\mathrm{m},0,\mathrm{best}}, \alpha_{\mathrm{best}}) = (0.28, -0.05)$.

While an adequate estimation of σ-values for the individual parameters in the case of $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \alpha)$ is not possible by minimizing the χ^2-distribution using the “Union2.1” dataset and we therefore cannot make any final judgement in favor of any model, our results are indicative towards the ΛCDM-model.

[2] Website on the “Union2.1” dataset by the *Supernova Cosmology Project*: https://supernova.lbl.gov/Union/

CONTENTS

CHAPTER 1

INTRODUCTORY CONCEPTS OF ASTROPHYSICS

In 2011, Saul Perlmutter, Adam G. Riess, and Brian P. Schmidt received the Nobel Prize in Physics *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"*. [1]
Since then, we know that the expansion of the universe is accelerating. The cause of this accelerated expansion is still unknown, but there are some theoretical models which attempt to explain this phenomenon.

The established model in recent cosmology is the **ΛCDM-model**, sometimes referred as the *standard model of cosmology.* In this model, the cause of the accelerated expansion is due to the so called *cosmological constant* Λ, which appears as a physical constant in Einstein's field equations of general relativity like the Newtonian gravitational constant G.

Another model, published in April 2000 by Gia Dvali, Gregory Gabadadze and Massimo Porrati, – the **DGP-model** – proposes a modification of Einstein's field equations by introducing a fifth dimension to the four-dimensional spacetime, so that gravity behaves equivalently to Newtonian gravity on small distances, but weakens on large scales.

Before going into details about both cosmological models, let me introduce relevant concepts in physics and astronomy from which we can relate measurable physical quantities, like the brightness of stars or their redshift, to more abstract properties of the universe like its scale factor. Based on certain assumptions of a theory, it is possible to develop models that make predictions about properties of the universe, for example, how the expansion and its evolution in time influences the relation of physical quantities.

1.1 Distance Measurement in Astronomy

Essential to astronomical observations and measurements is to determine the distance to objects in space like stars, star clusters, galaxies or even clusters of galaxies.
By looking at night into the sky, the only information perceived by our human eye are the brightness and some color in which objects (mostly stars) appear. How do we determine the distance to those objects?

1.1.1 The Parallax Method

One method to determine the distance to an object is by using the *parallax effect.*
For this method, we assume that the observed object is almost stationary relative to earth. First, we detect the position of the measured object in the sky. After a while (for example, after a half year), the object appears at a slightly different position in the sky, since earth moved on its elliptical orbit which leads to another point of view for the observation.

[1] Press release: https://www.nobelprize.org/prizes/physics/2011/press-release/

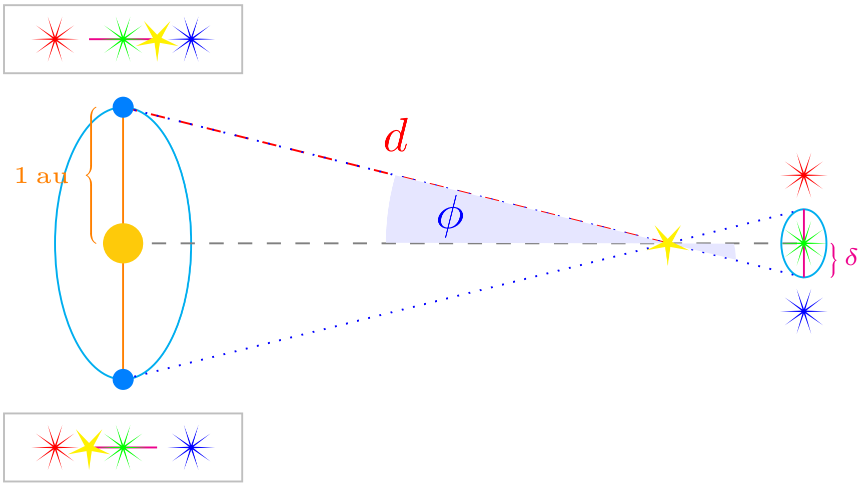


Figure 1.1: To determine the distance d between earth and the yellow star, we consider the small change of the position δ around the green star under which the yellow star appears. The difference δ relates to the angle ϕ.

By measuring the appeared difference δ of the object's position and relating this difference to the angle ϕ under which the object changed its position in the sky, we can calculate the distance d by

$$d = \frac{1\,\mathrm{au}}{\sin(\phi)} \approx \frac{1\,\mathrm{au}}{\phi}, \tag{1.1}$$

where $1\,\mathrm{au} \approx 149.6 \times 10^9\,\mathrm{m}$ ("astronomical unit"[2]) is defined as the average distance $d_{♁\odot}$ of earth to the sun.
The small-angle approximation can be considered justified, since the distance to earth nearest star (beyond of our solar system), Proxima-Centauri, is about $d_{\mathrm{P.C.}} \approx 4.2\,\mathrm{ly}$ and therefore $\phi_{\mathrm{P.C.}} \approx \arcsin\left(\frac{1\,\mathrm{au}}{4.2\,\mathrm{ly}}\right) \approx 3.723 \times 10^{-6} \ll 1$. The larger the distance to the measured object, the smaller the angle is.
Since the parallax method is one of the most important ways to measure the distance of far away objects, we define the distance of a *parsec* ("parallax second") as the distance to an object, under which the parallax is $\phi = 1\,\mathrm{arcsec} = \frac{1}{3600}{}^\circ$:

$$1\,\mathrm{pc} := \frac{1\,\mathrm{au}}{\sin\left(\frac{1}{3600}{}^\circ\right)} \approx 3.26\,\mathrm{ly} \approx 3.086 \times 10^{16}\,\mathrm{m}. \tag{1.2}$$

This is the usual unit by which cosmological distances or parameters are expressed, for example the *Hubble constant* $H_0 \approx 67.66\,\frac{\mathrm{km}}{\mathrm{s}}\mathrm{Mpc}^{-1}$.[3]

We should keep in mind that this method has of course boundaries and is practically useful for the local region of our galaxy, since the larger the distance to an observed object is, the apparent change in position in the sky gets smaller and is almost unnoticable for objects with a distance on cosmological scale ($d \gtrsim 300\,\mathrm{Mpc}$).

[2] Since 2012, the astronomical unit was redefined by the IAU (International Astronomical Union) to be exactly $1\,\mathrm{au} := 149\,597\,870\,700\,\mathrm{m}$, see *Resolution B2* at the XXVIII General Assembly of IAU: https://www.iau.org/static/resolutions/IAU2012_English.pdf

[3] The value of H_0 according to the results of the *Planck Collaboration 2018*, [Col+20, Table 7]

1.1.2 The Distance Modulus

Another method to determine the distance to an object with a certain luminosity L is to measure its radiant flux F at a distance d by

$$F = \frac{L}{4\pi d^2}. \tag{1.3}$$

In general astronomical observations, it is not the radiant flux of a star that is measured. Rather, we observe *differences* of brightness or magnitude m between two objects. We define a difference in magnitude between two objects $m_1 - m_2 =: \Delta m$ so that the radiant flux F_2 of object 2 is 100 times higher than the radiant flux F_1 of object 1 when their difference in magnitude is $\Delta m = 5$, so

$$\frac{F_2}{F_1} = 100 \Leftrightarrow m_1 - m_2 = \Delta m = 5. \tag{1.4}$$

This leads us to the relation between the difference of magnitude Δm and the relation of the radiant flux of two objects

$$\frac{F_2}{F_1} = 100^{\frac{m_1 - m_2}{5}} \tag{1.5}$$

and therefore with Equation (1.3)

$$m_1 - m_2 = \frac{5}{2}\log_{10}\left(\frac{F_2}{F_1}\right) = \frac{5}{2}\log_{10}\left(\frac{L_2}{4\pi d_2^2}\frac{4\pi d_1^2}{L_1}\right) = \frac{5}{2}\log_{10}\left(\frac{L_2}{L_1}\right) + 5\log_{10}\left(\frac{d_1}{d_2}\right). \tag{1.6}$$

So, calculating the distance to one of both objects, for example object 2, would require us to know their *relative* magnitudes m_1 and m_2, their luminosities L_1 and L_2 and the distance d_1 to object 1.
To eliminate the need of knowing five quantities, we can define an *absolute* magnitude M as the magnitude an object would have at a distance of $d = 10\,\mathrm{pc}$. If we consider only one object, we therefore can calculate the distance to this object by

$$m - M = \underbrace{\frac{5}{2}\log_{10}\left(\frac{L}{L}\right)}_{=0} + 5\log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right) = 5\log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right). \tag{1.7}$$

We call the difference between the relative magnitude m and the absolute magnitude M of one object its *distance modulus*.
The benefit of the distance modulus is that we only have to know two quantities, m and M, to calculate the distance to an object.
Unfortunately, we cannot measure the absolute magnitude M of an object that easily (since we cannot fly to far away stars and measure the observed magnitude or radiant flux at a distance of $10\,\mathrm{pc}$ to them), which is why we rely on Equation (1.6). However, we could reduce the amount of unknown variables by calibrating the relation (1.6) with an object which luminosity and distance is known.

One could choose to calibrate Equation (1.6) by measuring the properties of the nearest star to earth – our sun. Given the magnitude $m_\odot$, the luminosity $L_\odot$ and the distance to sun $d_{♁\odot} = 1\,\mathrm{au}$, we obtain

$$m = m_\odot - \frac{5}{2}\log_{10}\left(\frac{L}{L_\odot}\right) + 5\log_{10}\left(\frac{d}{1\,\mathrm{au}}\right). \tag{1.8}$$

By this calibration, we could determine the distance to an observed object, if we knew its luminosity L and measured its relative magnitude m. Generally, the luminosity of objects like stars could varying arbitrarily. To obtain a reliable distance measurement, we are looking for objects whose luminosity can be predicted very precisely – so called *standard candles*.

1.1.3 Possible Candidates for Standard Candles

Generally, there are two established candidates for standard candles in astronomy. Since we deal in this thesis with data of type Ia supernovae, the focus lies on the second paragraph of this subsection. For the sake of completeness, however, the cepheids as possible standard candles should not be unmentioned – also to emphasize why we rely on supernovae of type Ia to determinine cosmological distances.

1.1.3.1 Cepheids

There are several types and classes of cepheids, but they all have in common that those stars obey a certain, periodic relation of luminosity and time.

Without going into details[4] the periodicity of luminosity is caused by fluctuations of temperature dependent opacity in the stars photosphere due to transitions between single- and dual-ionized Helium inside the star.
It is important to identify cepheids of the same type – cepheids that share the same physical properties, like their metalicity or the same periodicity pattern in their luminosity, to ensure that the same physical process is occuring in all cepheids of a certain type. From then on, it is possible to determine the luminosity of all cepheids of the same type by observing one cepheid, measuring its relative magnitude m and its distance d by the parallax method.

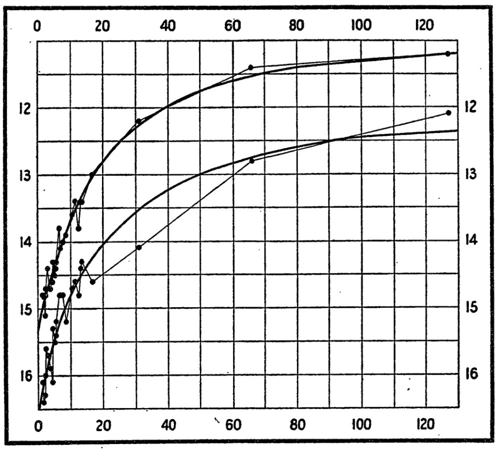


FIG. 1.

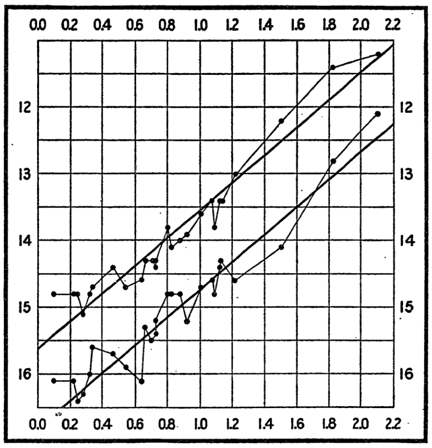


FIG. 2.

Figure 1.2: First observed direct, logarithmic periodicity between time (x-axis in days) and magnitude (y-axis) of 25 stars in the Small Magellanic Cloud by Henrietta Leavitt, published 1912. Source: [LP12].

[4] For further information and more detailed explanation how stellar pulsation and its underlying κ-mechanism can be calculated, I recommend chapter 14 "*Stellar Pulsation*" in [CO07].

However, in order to use cepheids for distance determination, they must first be observed and resolved. Even with our most powerful telescopes that we have today, resolution of cepheids is only possible in our local, galactic neighborhood, for example in the Large Magellanic Cloud or the Andromeda Galaxy. Therefore, the boundary to resolute cepheids in other galaxies is currently about $\sim 30\,\text{Mpc}$ ([Bar19, p. 47] and [Eng13, p. 3]).
For larger scales, we need much brighter standard candles.

1.1.3.2 Supernovae of Type Ia

In general, supernovae are abrupt bursts of luminosity from massive stars, often accompanied by explosive thermonuclear reactions. While supernovae of type Ib, Ic and II occur due to an imbalance between the star's gravity, which causes its nucleus to collapse, and radiation emitted by nuclear fusion inside the star, which pushes against its photosphere, supernovae of type Ia are caused by a white dwarf accreting a companion star and therefore increasing in mass.
If, due to accreating a companion star, the white dwarf reaches a mass limit over $\sim 1.4M_\odot$ (where $M_\odot \approx 1.98 \times 10^{30}\,\text{kg}$ is the mass of the sun, [Bar19, p. 48]), which is called the *Chandrasekhar limit* M_{Ch}, the nucleus of the white dwarf beocmes unstable, since the degeneracy pressure of the electrons (Pauli exclusion principle) cannot resist the gravitational forces any more.
Inside the white dwarf's nucleus, a fusion of carbon and oxygen is triggered, which leads to further nuclear processes and an amount of energy release up to $\sim 10^{44}\,\text{J}$ ([Mag17, p. 295], [Spa17, p. 321]). This makes supernovae of type Ia one of the most energetic – and therefore also luminous – phenomena known in nature. Supernovae of type Ia are almost as bright as their host galaxy. Since they are not only extremly luminous, but the process of star collapse is triggered after crossing a fixed limit of mass (the Chandrasekhar limit), this makes them also very unique and therefore good candidates for standard candles.
We can distinguish between supernovae of type Ia from other supernovae types by observing and analizing their spectrum.

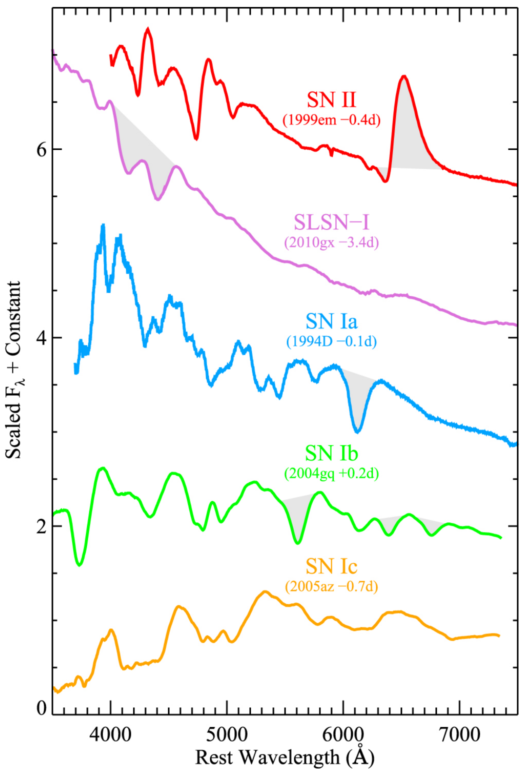


Figure 1.3: Key features in the spectra of different supernovae types are shaded in grey. While supernovae of type II have significant hydrogen lines, supernovae of type Ia and Ib are lacking of hydrogen lines. In type Ia supernovae, the Si II line (at $\sim 6150\,\text{Å}$) is very significant, while it is weak in type Ib supernovae. In type Ib supernovae, the helium lines seems to be very strong. Supernovae of type Ic do have non of the mentioned features.
Source: [Qui+18, Figure 1]

Yet there are small problems that occur here.
First, the mechanism which leads to the supernovae explosion is somewhat controversial, since there are two possible scenarios: the "single-degenerate"-model, in which the companion star is a star of the main sequence or a giant star, and the "double-degenerate"-model, in which the companion star is also a white dwarf.

In the "single-degenerate"-model, the accretion must not be too slow, since the hydrogen-rich material of the companion star could be burnt at the same rate as it is accreted, which results in no growth of mass for the white dwarf. On the other hand, the accretion must not be too high, since the accretion might stop due to the companion star's loss of mass at a high rate or being engulfed by the accreted material ([Mag17, p. 308]).

In the "double-degenrate"-model, it is assumed that one of both white dwarfs reaches the Chandrasekhar limit if they get too close ([Bar19, p. 48]), but there are also other possible scenarios that could occur (see [Mag17, p. 308/309]).
This could lead to slight variations in the luminosity behavior.

Other variations in luminosity behavior could occur due the amount of ^{56}Ni in the thermonuclear process whose decay influences the peak of the supernovae type Ia light curve ([Mag17, p. 295]). However, those variations can be compensated very well since a relation between the lumninosity's decline and its peak is found so that it is possible to "normalize" or "stretch" the light curves so they obey a uniform distribution ([PS03, p. 4], [Phi93]).
The fact that the light curves of type Ia supernovae are distributed exactly the same way after some "normalization" or "stretching" shows us that the same physical process underlies all light curves of type Ia supernovae, even if they seem to be stretched, not only due to the relation of luminositys decline and luminositys peak. We will mention later why this property of supernovae type Ia light curves verify the expansion of the universe.

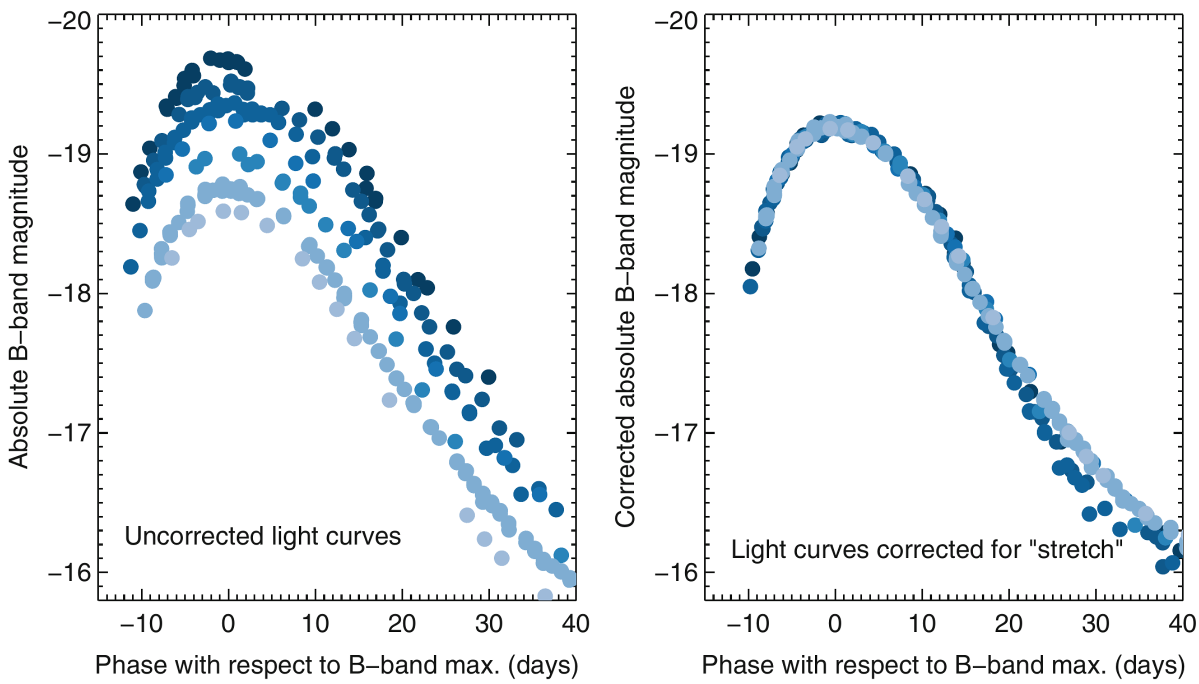


Figure 1.4: Uncorrected vs. corrected sample of supernovae type Ia light curves.
Source: [Mag17, p. 298, Figure 2]

Despite the mentioned slight variations in the luminosity spectrum, supernovae of type Ia can be considered as the best candidates for standard candles at the current state of research, not only because their light curves are very homogeneous, but they are also much brighter than cepheids and therefore also visible at large distances, which is essential for cosmological research.

1.1.4 Redshift and Hubble's Observation

The most important tool to determine distances on a cosmological scale is by observing the redshift of extragalactic objects. The redshift is a shift in the observed spectrum of light emmitted by a source with wavelength λ_{e} to an observed wavelength λ_{o} and defined as

$$z := \frac{\lambda_{\mathrm{o}} - \lambda_{\mathrm{e}}}{\lambda_{\mathrm{e}}} = \frac{\lambda_{\mathrm{o}}}{\lambda_{\mathrm{e}}} - 1. \tag{1.9}$$

The term "red" in "redshift" might be misleading, since z could take also values < 0 and therefore $\lambda_{\mathrm{o}} < \lambda_{\mathrm{e}}$, which implicates a shift of the light's spectrum to the blue.
This term has been established because observations show that the spectrum of most galaxies and extragalactic objects is shifted towards red.

Towards the end of the 1920s, Edwin Hubble observed a certain relation between the measured redshift of the galaxies and extragalactic objects and their distance to us. If one assumes the redshift due to (relativistic) doppler effect,[5] one could calculate the galaxies' velocity component towards the direction of observation by

$$z = \gamma\left(1 + \frac{v}{c}\right) = \sqrt{\frac{c+v}{c-v}} - 1. \tag{1.10}$$

An observed shift of the spectrum to the red would imply that the observed object moves away from the observer, and a shift of the spectrum to the blue implies a motion towards the observer.
Edwin Hubble applied a linear relation (see Figure 1.6) between the measured distance d to the observed objects and the calculated velocity v due to the doppler effect

$$v = H_0 d, \tag{1.11}$$

which is known as the *Hubble law*. The proportionality constant H_0 has the dimension of an inverse time. It is one of the most important parameters in cosmology and its precise determination a challenge of active research (often called "Hubble tension", [Val+21]).

For small velocities ($v \ll c$), we can approximate Equation (1.10) (first order Taylor series) as $z \approx \frac{v}{c}$, which leads with Equation (1.11) to

$$d \approx \frac{c}{H_0} z. \tag{1.12}$$

[5] Some raised objections against the interpretation that the observed redshift is a result of the doppler effect, but claimed that it is caused by energy loss of the photons traveling through space ("tired light"-hypothesis). We will address later why this hypothesis cannot be hold anymore in the light of tremendous evidence for the expansion model.

We will see later that this relation is actually a first order approximation between the *luminosity distance* d_L and the redshift z.
The correct relation between the luminosity distance d_L and the redshift z will play an essential role when estimating parameters of cosmological models.
From Hubble's law follows that the further the distance to a galaxy (or other cosmological object) is, the higher the redshift and therefore, the faster it seems to move apart from us. This observation was one of the milestones in the history of cosmology and was the first indicator that our universe is truly expanding.

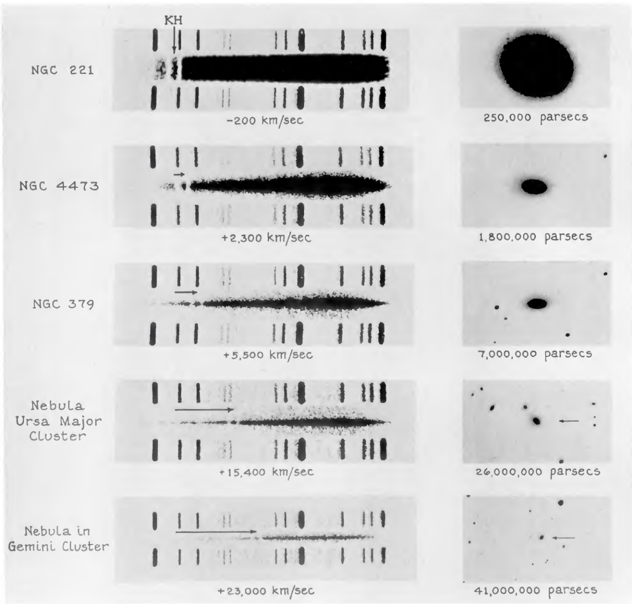


Figure 1.5: Observed redshift of H and K lines of calcium, shifted to the red, by Milton L. Humason.
Source: [Hum36, Figure "Red-shifts in the Spectra of Extra-galactic Nebulae"]

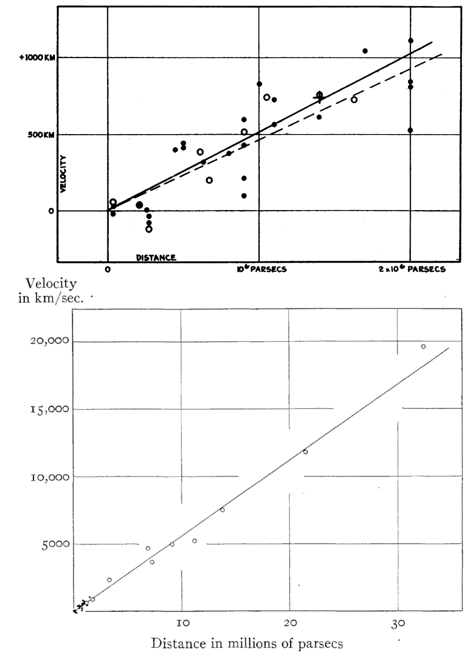


Figure 1.6: Linear regression between the distance of observed objects and the calculated velocity due to redshift.
Source: [Hub29] and [HH31, Figure 5]

1.2 The Theory of Gravity and Spacetime

How can one speak about the nature on large scales without mentioning the force that dominates in this regime? Beginning with Newton's ideas about gravity, the modern formulation about gravity is described in the framework of Einstein's general theory of relativity, in which gravity is, strictly speaking, not anymore a force in the Newtonian sense, but rather a property of the four dimensional spacetime that interacts with matter.

For a deep understanding of general relativity, it is required to have knowledge on the mathematics of differential geometry.
Besides the fact that, as an undergraduate student, I do not (yet) possess this knowledge, this would be beyond the scope of this bachelor thesis.
However, to motivate the basic equations of the ΛCDM-model, which are derived from Einstein's field equations under certain assumptions that we are going to formulate in the next chapter, I would like to mention the concept of a *metric* and give brief view on Einstein's field equations.

1.2.1 The Metric of Spacetime

Generally speaking, a metric is a function that takes two points in space and returns a distance. For example, let $\mathbb{E}^2$ be the Euclidean, two dimensional space, then

$$d(\cdot,\cdot) : \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}, (\boldsymbol{p}_1, \boldsymbol{p}_2) \mapsto d(\boldsymbol{p}_1, \boldsymbol{p}_2) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{1.13}$$

would be a function that gives us the distance between two points $\boldsymbol{p}_1 = (x_1, y_1)^\intercal$ and $\boldsymbol{p}_2 = (x_2, y_2)^\intercal$. This distance function d would return the Pythagorean distance in Cartesian coordinates that we are familiar with.
Let us define $x := x_1 - x_2$, $y := y_1 - y_2$ and $s := d(\boldsymbol{p}_1, \boldsymbol{p}_2)$ so we could write for the (infinitesimal) distance

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \,. \tag{1.14}$$

Now, let us switch to polar coordinates so that $\boldsymbol{p}_1 := (r_1, \phi_1)^\intercal$ and $\boldsymbol{p}_2 := (r_2, \phi_2)^\intercal$. With the given distance function d, we would obtain

$$d(\boldsymbol{p}_1, \boldsymbol{p}_2) = \sqrt{(r_1 - r_2)^2 + (\phi_1 - \phi_2)^2}, \tag{1.15}$$

but this is *not* the same distance as in Cartesian coordinates. The distance that would correspond to the same distance as in Cartesian coordinates (see Equation (1.14)), is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 \tag{1.16}$$

with $r := r_1 - r_2$, $\phi := \phi_1 - \phi_2$.
We have to introduce the *metric tensor* (in most applications of physics a 3×3- or 4×4-matrix) that guarantees the invariance of the distance function d under coordinate transformation.
We define

$$g_{ij} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{in cart.} \\ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} & \text{in polar} \end{cases} \qquad x^1 := \begin{cases} x & \text{in cart.} \\ r & \text{in polar} \end{cases} \qquad x^2 := \begin{cases} y & \text{in cart.} \\ \phi & \text{in polar} \end{cases} \tag{1.17}$$

so that we can express the invariant distance between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ through

$$\mathrm{d}s^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{11}\Big(\mathrm{d}x^1\Big)^2 + \underbrace{g_{12}}_{=0}\,\mathrm{d}x^1\,\mathrm{d}x^2 + \underbrace{g_{21}}_{=0}\,\mathrm{d}x^2\,\mathrm{d}x^1 + g_{22}\Big(\mathrm{d}x^2\Big)^2 \tag{1.18}$$

$$= g_{11}\Big(\mathrm{d}x^1\Big)^2 + g_{22}\Big(\mathrm{d}x^2\Big)^2 \overset{\text{cart.}}{=} \mathrm{d}x^2 + \mathrm{d}y^2 \overset{\text{polar}}{=} \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 \,. \tag{1.19}$$

In the framework of relativity, we express the distance (called "world line") between to *events* $\boldsymbol{p}_1 := (ct_1, x_1, y_1, z_1)^\intercal$ and $\boldsymbol{p}_2 := (ct_2, x_2, y_2, z_2)^\intercal$ in four dimensional spacetime as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{1.20}$$

with the implicit sum over double occuring indices $\mu, \nu \in \{0, 1, 2, 3\}$. In flat Minkowski-spacetime of special relativity for example, we have

$$g_{\mu\nu} = \eta_{\mu\nu} := \mathrm{diag}(-1, 1, 1, 1) \tag{1.21}$$

and therefore for distances in spacetime

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(c\,\mathrm{d}t)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1.22}$$

1.2.2 Einstein's Field Equations

Similar to the derivation of the Euler–Lagrange equations in classical mechanics (or classical field theory) by formulating an action $S[\boldsymbol{q}(t)]$ (or $S[\phi(x)]$) and find the path $\boldsymbol{q}(t)$ (or field $\phi(x)$) that extremizes the action ($\delta S = 0$), Einstein's field equations can be derived[6] from the Einstein–Hilbert action given by

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \int \mathrm{d}^4x \sqrt{-\det(g_{\mu\nu})}\left[\frac{c^4}{16\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right], \tag{1.23}$$

where d^4x is the four dimensional spacetime-volume element, $g_{\mu\nu}$ the metric tensor, c the speed of light (in vacuum), G the Newtonian gravitational constant, R the Ricci scalar, Λ the cosmological constant, and $\mathcal{L}_{\mathrm{M}}$ the Lagrange density of matter fields.
With the action principle, the variation $\delta S[g_{\mu\nu}]$ of the Einstein–Hilbert action with respect to the metric leads to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.24}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor and $T_{\mu\nu}$ the energy-momentum tensor.
Without getting too much into details about the single components of this equation, we keep in mind that the left-hand side of Equation (1.24) describes how spacetime behaves, while the right-hand side of Equation (1.24) describes how matter behaves.

1.2.2.1 The Cosmological Constant Λ

Originally, Einstein formulated his field equations without the cosmological constant Λ. Since Einstein believed in a static universe and found that his equations cannot hold for a static universe (it would have collapsed due to the gravity of matter), he introduced the Λ term that acts repulsive towards the attraction of gravity, so that a static universe as he proposed would be possible.

[6] For a proper and detailed derivation, see subsection 4.3 "*Lagrangian Formulation*" in [Car19, p. 159].

CHAPTER 2

THE STANDARD MODEL OF COSMOLOGY

The standard model of cosmology describes an expanding universe. Its basic assumptions are that the cosmological principle is valid and that general relativity can describe nature on cosmological scales.

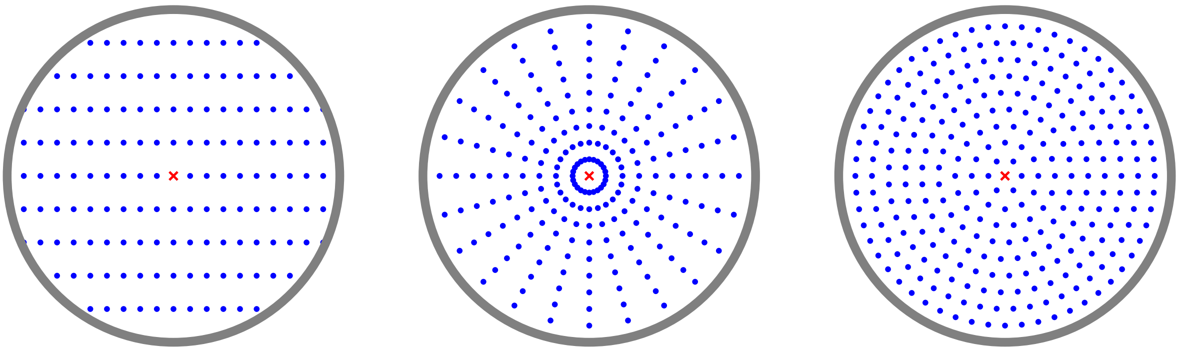
The dynamics of the expansion is guided by the cosmological constant Λ in Einstein's field equations, since this term was initally introduced to act repulsive towards gravity, so that the universe does not collapse due to the gravity of matter it contains. The drive behind the *accelerated* expansion was given the name "dark energy", which refers in the ΛCDM-model to the cosmological constant Λ.
Nevertheless, it should be mentioned that the term "dark energy" used in cosmology is not restricted to the cosmological constant. Its meaning depends on the particular cosmological model that is considered (see subsection 2.4.6 "*Dark energy*" in [DS20, p. 50] and [FTH08]).

The "CDM“ in "ΛCDM" stands for "cold dark matter". There are several indications that lead to the assumption of the existence of dark matter. The first and probably most dominant indication is by observing the velocity of stars while surrounding their galactic center. While, according to classical mechanics, the rotation velocity of an object in a gravitational field should decrease the further the object is to the center of mass, it is observed that the rotation curve (the distance-velocity-relation, see [Sch06, p. 64]) in spiral galaxies does not decrease, which leads to the assumption that there has to be some mass in galactic halos that holds the galactic disk together due to its gravity (otherwise, galaxies should fall apart due to the high velocity of the stars that surround the galactic nucleus).
Apart from the absence of other interactions than gravitational, dark matter behaves like ordinary (baryonic) matter according to the ΛCDM-model. While theories on "hot" dark matter assume small masses for the particles that dark matter consists of, so that they behave more like relativistic hot gas, the ΛCDM-model assumes "cold" dark matter, which means in this context heavy particles that behave more classically.

For the purpose of this thesis, the distinction between ordinary, baryonic matter and dark matter can be neglected.

2.1 The Cosmological Principle

The cosmological principle states that, assuming the universe does not prefer any direction in space, i.e. the universe is isotropic for *every* observer, the larger scales we consider, the smaller the variance and thus the more homogeneous the matter distribution appears.
A more mathematically rigorous definition of the cosmological principle can be found in [Bar19, p. 5] and [MTW17, p. 713/714].



(a) (b) (c)

Figure 2.1: The cosmological principle visualized.
The matter distribution (a) is homogeneous, but not isotropic.
The matter distribution (b) is for an observer at the center isotropic, but not homogeneous.
The matter distribution (c) is for an observer at the center isotropic *and* homogeneous.

Obviously, the universe is not perfectly homogeneous (especially not on small scales, for example our galactic neighborhood), but keep in mind that the cosmological principle does *not* postulate a perfect homogeneous and isotropic universe.
Instead, the cosmological principle is a statement about the *statistics* of matter distribution. The *larger* the scales an observer considers at *any* point in the universe, the *more* homogeneous the matter distribution appears.

The strongest evidence at current state that supports the cosmological principle is the measurement of the cosmic microwave background, whose radiation spectrum follows the Planck distribution of a black body radiator with $T \approx 2.736\,\mathrm{K}$ more accurately than any other measurement in nature ([Pee93, p. 131/132] and [Whi99]).

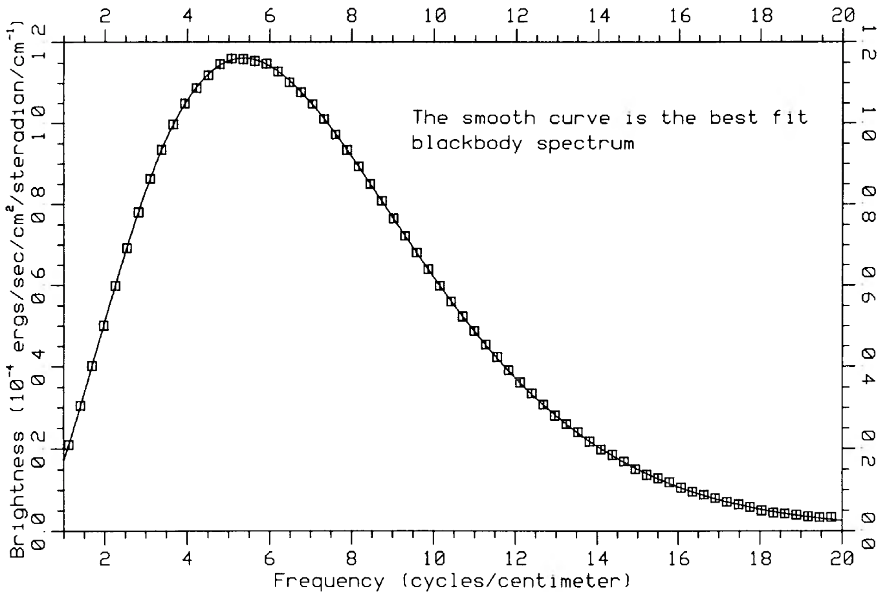


Figure 2.2: The spectrum of the microwave background radiation, measured by “COBE” (Cosmic Background Explorer) in 1990. For this exploration, John C. Mather and George F. Smoot were awarded the Nobel Prize in Physics in 2006.
Source: [Mat+90, Figure 2]

2.1.1 FLRW-Metric

Motivated by the cosmological principle, there are mathematically three possible geometries for the four-dimensional spacetime that satisfy spatial homogeneity and isotropy of the metric:

- a four-dimensional sphere $\mathcal{S}^3$ with positive curvature,
- a four-dimensional flat space $\mathcal{F}^3$ with zero curvature,
- a four-dimensional hyperboloid $\mathcal{H}^3$ with negative curvature.

We can parameterize these geometries by

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} &\overset{\mathcal{S}^3}{=} \begin{pmatrix} \sin(\psi)\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta)\sin(\phi) \\ \sin(\psi)\cos(\theta) \\ \cos(\psi) \end{pmatrix} \overset{\mathcal{F}^3}{=} \begin{pmatrix} \psi\sin(\theta)\cos(\phi) \\ \psi\sin(\theta)\sin(\phi) \\ \psi\cos(\theta) \\ 1 \end{pmatrix} \overset{\mathcal{H}^3}{=} \begin{pmatrix} \sinh(\psi)\sin(\theta)\cos(\phi) \\ \sinh(\psi)\sin(\theta)\sin(\phi) \\ \sinh(\psi)\cos(\theta) \\ \cosh(\psi) \end{pmatrix} \\ &= \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \\ \frac{\partial r}{\partial \psi} \end{pmatrix} \quad \text{with} \quad r = \begin{cases} \sin(\psi) & \text{for} \quad \mathcal{S}^3 \\ \psi & \text{for} \quad \mathcal{F}^3 \\ \sinh(\psi) & \text{for} \quad \mathcal{H}^3 \end{cases}. \end{aligned} \tag{2.1}$$

The metric for the spatial line element $\mathrm{d}\ell$ is

$$\mathrm{d}\ell^2 = \begin{cases} \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 & \text{for} \quad \mathcal{S}^3 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & \text{for} \quad \mathcal{F}^3 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}w^2 & \text{for} \quad \mathcal{H}^3 \end{cases}. \tag{2.2}$$

With the parameterization in Equation (2.1), we can write for the infinitesimals

$$\begin{aligned} \mathrm{d}x &= \sin(\theta)\cos(\phi)\,\mathrm{d}r + r\cos(\theta)\cos(\phi)\,\mathrm{d}\theta - r\sin(\theta)\sin(\phi)\,\mathrm{d}\phi\,, \\ \mathrm{d}y &= \sin(\theta)\sin(\phi)\,\mathrm{d}r + r\cos(\theta)\sin(\phi)\,\mathrm{d}\theta + r\sin(\theta)\cos(\phi)\,\mathrm{d}\phi\,, \\ \mathrm{d}z &= \cos(\theta)\,\mathrm{d}r - r\sin(\theta)\,\mathrm{d}\theta\,, \\ \mathrm{d}w &= \frac{\partial w}{\partial r}\,\mathrm{d}r = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial \psi}\right)\mathrm{d}r\,. \end{aligned}$$

The components $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ lead to

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\underbrace{\left[\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right]}_{=:\mathrm{d}\Omega^2} = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,,$$

where $\mathrm{d}\Omega$ is the infinitesimal spatial angle.
Let us have a closer look on $\mathrm{d}w$. We obtain

$$\mathrm{d}w = \mathrm{d}r \begin{cases} \frac{\partial}{\partial r}\cos(\psi) = \frac{\partial}{\partial r}\cos(\arcsin(r)) = -\frac{r}{\sqrt{1-r^2}} & \text{for} \quad \mathcal{S}^3 \\ \frac{\partial}{\partial r}1 = 0 & \text{for} \quad \mathcal{F}^3 \\ \frac{\partial}{\partial r}\cosh(\psi) = \frac{\partial}{\partial r}\cosh(\operatorname{arsinh}(r)) = \frac{r}{\sqrt{1+r^2}} & \text{for} \quad \mathcal{H}^3 \end{cases}, \tag{2.3}$$

and hence for the spatial component of the metric (2.2)

$$\mathrm{d}\ell^2 = \begin{cases} \frac{1}{1-r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{S}^3 \\ \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{F}^3 \\ \frac{1}{1+r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{H}^3 \end{cases}. \tag{2.4}$$

If we define

$$k := \begin{cases} 1 & \text{for} \quad \mathcal{S}^3 \\ 0 & \text{for} \quad \mathcal{F}^3 \\ -1 & \text{for} \quad \mathcal{H}^3 \end{cases}, \tag{2.5}$$

we can write Equation (2.4) in compact notation as

$$\mathrm{d}\ell^2 = \frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.6}$$

Since the cosmological model describes an expanding universe, we introduce the scale factor $a(t)$, by which the spatial line element $\mathrm{d}\ell$ is scaled at time t.
Therefore, we finally obtain the Friedmann–Lemaître–Robertson–Walker metric of spacetime

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\ell^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right]. \tag{2.7}$$

2.2 Friedmann Equations

Applying the FLRW-metric to Einstein's field equations leads to the fundamental equations – the Friedmann equations – that describe the dynamics of the expansion of the universe by the scale factor $a(t)$ with ([Bar19, p. 11])

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{2.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{2.9}$$

Those equations contain the energy density $\rho(t) = \rho_{\mathrm{m}}(t) + \rho_{\mathrm{r}}(t)$ of matter and radiation, the curvature parameter k of the FLRW-metric, the pressure p that matter and radiation exerts as a perfect fluid and the cosmological constant Λ.

Taking the derivative of Equation (2.8) with respect to time and inserting into Equation (2.9) leads to the differential equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{2.10}$$

We define the Hubble parameter $H(t) := \frac{\dot{a}}{a}$ and $w := \frac{p}{\rho c^2}$, which leads with Equation (2.10) to

$$\dot{\rho} + 3H\rho(1+w) = 0. \tag{2.11}$$

This Equation (2.11) is sometimes called “the continuity equation of cosmology”. We obtain by integration the relation between density $\rho(t)$ and scale factor $a(t)$

$$\rho(a) = \rho_0 a^{-3(1+w)}(t) \tag{2.12}$$

with $\rho_0 := \rho(a(t_0))$.

2.2.1 Density Parameters

We are now interested in how the density components of the universe change as a function of the expansion of the universe scaled by $a(t)$.

In the case of matter at rest, the density of matter ρ_m shrinks by $a^3(t)$ when the volume that contains the matter is scaled by $a(t)$, so

$$\rho_\mathrm{m}(a) = \frac{E}{V(a)} = \frac{E}{V_0 a^3(t)} = \rho_{\mathrm{m},0} a^{-3}(t). \tag{2.13}$$

With comparison to Equation (2.12) follows $w_\mathrm{m} = 0$.
In the case of radiation, we have to account that its wavelength λ is scaled by the factor $a(t)$, so $\lambda(a) = \lambda_0 a(t)$, which leads with $E(\lambda) = hc/\lambda$ to the energy density

$$\rho_\mathrm{r}(a) = \frac{E(a)}{V(a)} = \frac{1}{V_0 a^3(t)} \frac{hc}{\lambda_0 a(t)} = \frac{hc}{V_0 \lambda_0} \frac{1}{a^4(t)} = \rho_{\mathrm{r},0} a^{-4}(t). \tag{2.14}$$

With comparison to Equation (2.12) follows $w_\mathrm{r} = \frac{1}{3}$.
Now, let us rewrite the cosmological constant Λ by defining

$$\rho_\Lambda := \frac{\Lambda c^2}{8\pi G} \tag{2.15}$$

as the density that relates to the cosmological constant. As the name might imply, this density remains constant as the universe expands and is therefore independent of the scale factor $a(t)$. This interpretation of Λ leads to the assumption, that ρ_Λ can be viewed as the energy density of vacuum. As vacuum increases exact the same way as space expands, its density remains constant, so

$$\rho_\Lambda(a) = \rho_{\Lambda,0}. \tag{2.16}$$

With comparison to Equation (2.12) follows $w_\Lambda = -1$.
Our aim is now to write Equation (2.8) as a sum of densities. We therefore define a "density" that would relate to the spatial curvature k,

$$\rho_k := -\frac{3}{8\pi G} k c^2 a^{-2}(t). \tag{2.17}$$

Further, we define the critical density ρ_cr as the density the universe would have if it was flat ($k = 0$) and had no component of Λ at the time t_0, so with $H_0 := H(t_0)$

$$\rho_\mathrm{cr} := \frac{3H_0^2}{8\pi G}. \tag{2.18}$$

Finally, we define

$$\Omega_i := \frac{\rho_i}{\rho_\mathrm{cr}} \tag{2.19}$$

where i is the label for the density type. We can then write Equation (2.8) as

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}\rho_{\mathrm{r+m}} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ &= \frac{8\pi G}{3}\Big(\rho_\mathrm{r} + \rho_\mathrm{m} + \rho_k + \rho_\Lambda\Big) \\ &= H_0^2 \frac{1}{\rho_\mathrm{cr}}\Big(\rho_\mathrm{r} + \rho_\mathrm{m} + \rho_k + \rho_\Lambda\Big) \\ &= H_0^2 \sum_{i\in\{\mathrm{r},\mathrm{m},k,\Lambda\}} \Omega_i. \end{aligned} \tag{2.20}$$

We clearly see that the sum of all normalized density parameters $\Omega_{i,0}$ at time t_0 lead to

$$1 = \Omega_{\mathrm{r},0} + \Omega_{\mathrm{m},0} + \Omega_{k,0} + \Omega_{\Lambda,0}. \tag{2.21}$$

With the relations (2.13), (2.14), (2.16), (2.17) between the densities and the scale factor, we obtain

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}\Big(\rho_{\mathrm{r}} + \rho_{\mathrm{m}} + \rho_k + \rho_\Lambda\Big) \\ &= H_0^2 \frac{1}{\rho_{\mathrm{cr}}}\Big(\rho_{\mathrm{r},0}a^{-4}(t) + \rho_{\mathrm{m},0}a^{-3}(t) + \rho_{\mathrm{k},0}a^{-2}(t) + \rho_{\Lambda,0}\Big) \\ &= H_0^2\Big(\Omega_{\mathrm{r},0}a^{-4}(t) + \Omega_{\mathrm{m},0}a^{-3}(t) + \Omega_{k,0}a^{-2}(t) + \Omega_{\Lambda,0}\Big) \\ &=: H_0^2 E^2(a), \end{aligned}$$

and we call

$$E(a) := \sqrt{\Omega_{\mathrm{r},0}a^{-4}(t) + \Omega_{\mathrm{m},0}a^{-3}(t) + \Omega_{k,0}a^{-2}(t) + \Omega_{\Lambda,0}} \tag{2.22}$$

the expansion function.

With the obtained values by the *Planck Collaboration* (see [Col+20, Table 7]), $\Omega_{\mathrm{m},0} \approx 0.3111 \pm 0.0056$, $\Omega_{\mathrm{r},0} = \frac{1}{1+z_{\mathrm{eq}}}\Omega_{\mathrm{m},0} \approx 9.812 \times 10^{-5}$ (at $z_{\mathrm{eq}} = 3387 \pm 21$) and $\Omega_{\Lambda,0} \approx 0.6889 \pm 0.0056$, we see that the radiation component was only relevant in the first $\sim 380\,000\,\mathrm{yr}$ of the universe until matter and radiation decoupled.

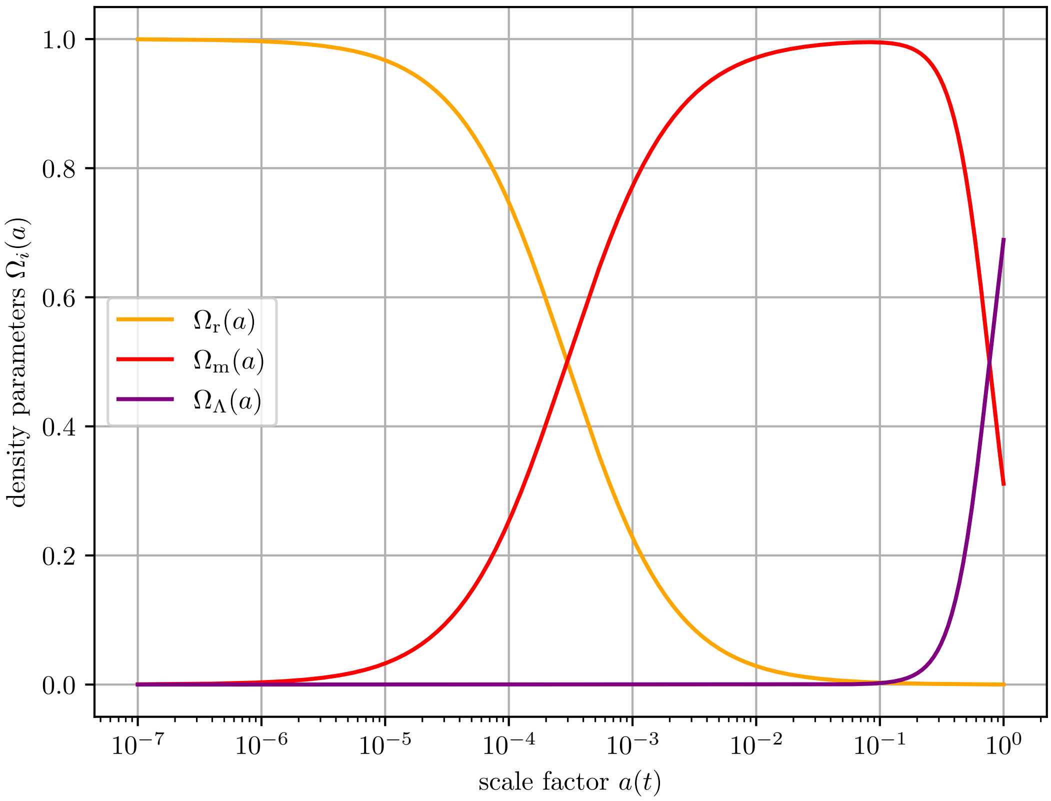


Figure 2.3: The normalized density parameters of radiation, matter and Λ in dependence of the scale factor $a(t)$ on logarithmic axis.

Since the radiation does not contribute significantly to the expansion of the universe, we assume for the rest of this thesis $\Omega_{\mathrm{r},0} = 0$.
Further, the universe appears to be almost flat ([Col+20, Table 7]) with a measured value of $\Omega_{k,0} \approx 0.0007 \pm 0.0019$.
Let us have a look how $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ contribute to the expansion of a flat universe.

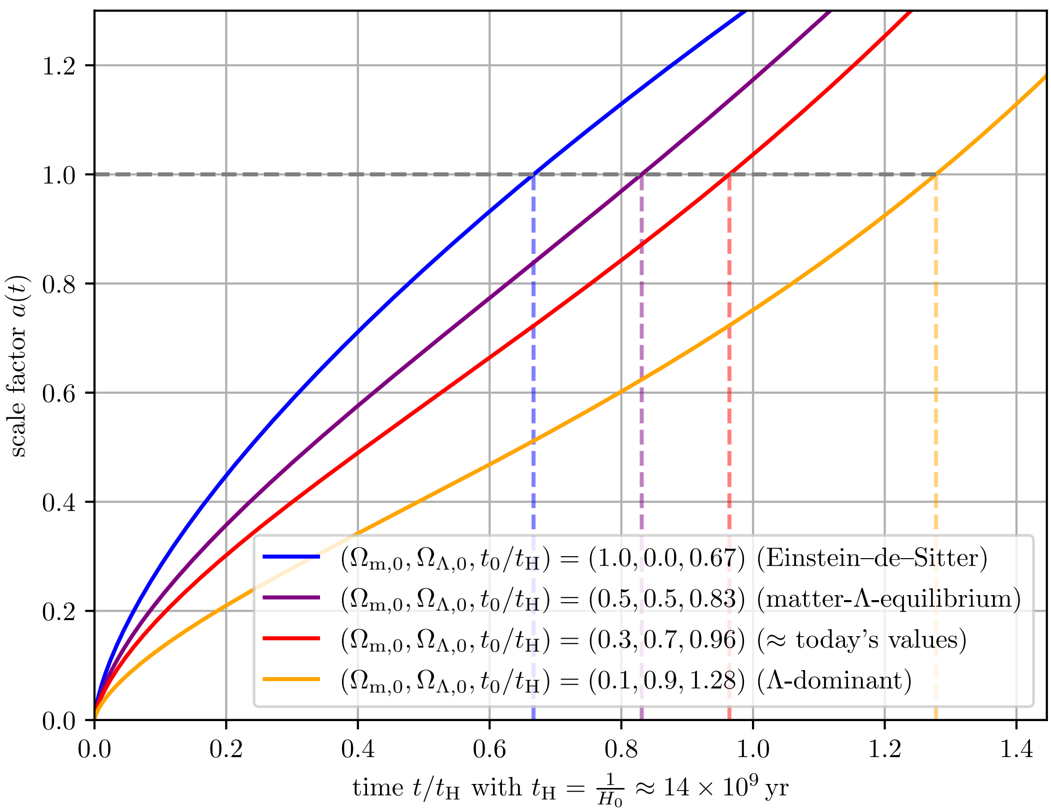


Figure 2.4: Contribution of $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ to the expansion in a flat universe ($k = 0$) without radiation $\Omega_{\mathrm{r},0} = 0$ for several cosmologies. $t_\mathrm{H} = \frac{1}{H_0} \approx 14 \times 10^9\,\mathrm{yr}$ is the Hubble time.

We notice from Figure 2.4 that the higher the value of $\Omega_{\Lambda,0}$, the more it contributes to an accelerated expansion.

We define $a(t_0) := 1$ as the scale factor at the time $t_0 =: t_\mathrm{today}$.
With $\dfrac{\mathrm{d}a}{\mathrm{d}t} = H_0 a E(a)$, the age of the universe can be calculated by

$$t_0 = \frac{1}{H_0} \int_0^{a(t_0)} \mathrm{d}a\, \frac{1}{aE(a)} = \frac{1}{H_0} \int_0^1 \mathrm{d}a\, \frac{1}{\sqrt{\Omega_{\mathrm{r},0} a^{-2} + \Omega_{\mathrm{m},0} a^{-1} + \Omega_{k,0} + \Omega_{\Lambda,0} a^2}}. \tag{2.23}$$

For a flat universe without considering radiation ($\Omega_{k,0} = 0$, $\Omega_{\mathrm{r},0} = 0$), the expression (2.23) can be calculated analytically. With $\Omega_{\mathrm{m},0} = 0.3$ and therefore $\Omega_{\Lambda,0} = 1 - \Omega_{\mathrm{m},0} = 0.7$, we obtain

$$t_0 = \frac{2}{3H_0\sqrt{1 - \Omega_{\mathrm{m},0}}} \operatorname{arsinh}\left(\sqrt{\frac{1 - \Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}}} a^{\frac{3}{2}}(t_0)\right) \approx 13.8 \times 10^9\,\mathrm{yr}. \tag{2.24}$$

2.3 Measures of Distance

Given the equations the standard model provides, we can derive measures of cosmological distances.
As already mentioned, the wavelength of light that travels to an observer is being redshifted due to the expansion of the universe. The relation between the scale factor and the redshift is given by (see [Bar19, p. 9])

$$a(z) = \frac{1}{1+z}. \tag{2.25}$$

For the path that light travels between two points (events) in spacetime (also called "lightlike geodesics"), we have $\mathrm{d}s = 0$, which leads with the FLRW-metric (see Equation (2.7)) and isotropy ($\mathrm{d}\Omega = 0$) to

$$-\frac{c}{a(t)}\,\mathrm{d}t = \frac{1}{\sqrt{1-kr^2}}\,\mathrm{d}r\,. \tag{2.26}$$

For the left-hand side of Equation (2.26), we can write

$$-\frac{c}{a(t)}\,\mathrm{d}t = -\frac{c}{a}\frac{1}{\dot{a}}\,\mathrm{d}a = -\frac{c}{H_0}\frac{1}{a^2E(a)}\,\mathrm{d}a \tag{2.27}$$

and with $\mathrm{d}z = -\dfrac{1}{a^2}\,\mathrm{d}a$,

$$-\frac{c}{a(t)}\,\mathrm{d}t = -\frac{c}{H_0}\frac{1}{a^2E(a)}\,\mathrm{d}a = \frac{c}{H_0}\frac{1}{E(z)}\,\mathrm{d}z\,. \tag{2.28}$$

We define $d_\mathrm{H} := \dfrac{c}{H_0}$ as the Hubble distance.
For the right-hand side of Equation (2.26), we obtain for the integration

$$\int_0^{d_\mathrm{C}} \frac{1}{\sqrt{1-kr^2}}\,\mathrm{d}r =: S_k^{-1}(d_\mathrm{C}) = \begin{cases} \frac{1}{\sqrt{k}}\operatorname{arcsin}\left(\sqrt{k}d_\mathrm{C}\right) & \text{for} \quad k>0 \\ d_\mathrm{C} & \text{for} \quad k=0 \\ \frac{1}{\sqrt{|k|}}\operatorname{arsinh}\left(\sqrt{|k|}d_\mathrm{C}\right) & \text{for} \quad k<0 \end{cases}. \tag{2.29}$$

The substitution $-k = \dfrac{1}{d_\mathrm{H}^2}\Omega_{k,0}$ (see Equations (2.17) and (2.18)) leads to

$$S_k^{-1}\left(\frac{d_\mathrm{C}}{d_\mathrm{H}}\right) = d_\mathrm{H}\cdot \begin{cases} \frac{1}{\sqrt{|\Omega_{k,0}|}}\operatorname{arcsin}\left(\sqrt{|\Omega_{k,0}|}\frac{d_\mathrm{C}}{d_\mathrm{H}}\right) & \text{for} \quad \Omega_{k,0}<0 \\ \frac{d_\mathrm{C}}{d_\mathrm{H}} & \text{for} \quad \Omega_{k,0}=0 \\ \frac{1}{\sqrt{\Omega_{k,0}}}\operatorname{arsinh}\left(\sqrt{\Omega_{k,0}}\frac{d_\mathrm{C}}{d_\mathrm{H}}\right) & \text{for} \quad \Omega_{k,0}>0 \end{cases}. \tag{2.30}$$

By integration of Equation (2.28) follows finally[1]

$$d_\mathrm{C} = S_k(I) = d_\mathrm{H}\cdot \begin{cases} \frac{1}{\sqrt{|\Omega_{k,0}|}}\sin\left(\sqrt{|\Omega_{k,0}|}I\right) & \text{for} \quad \Omega_{k,0}<0 \\ I & \text{for} \quad \Omega_{k,0}=0 \\ \frac{1}{\sqrt{\Omega_{k,0}}}\sinh\left(\sqrt{\Omega_{k,0}}I\right) & \text{for} \quad \Omega_{k,0}>0 \end{cases} \tag{2.31}$$

[1] I have explicitly performed this derivation, since Equations (2.31) and (2.32) are essential for the evaluation of supernovae data and implementation in the source codes in this thesis.

with

$$I := \int_0^z \mathrm{d}z' \, \frac{1}{E(z')}. \tag{2.32}$$

We call d_C the *comoving distance*. We define

$$d_\mathrm{A} := \frac{1}{1+z} d_\mathrm{C} \tag{2.33}$$

as the *angular diameter distance* and

$$d_\mathrm{L} := (1+z) d_\mathrm{C} \tag{2.34}$$

as the *luminosity distance*.

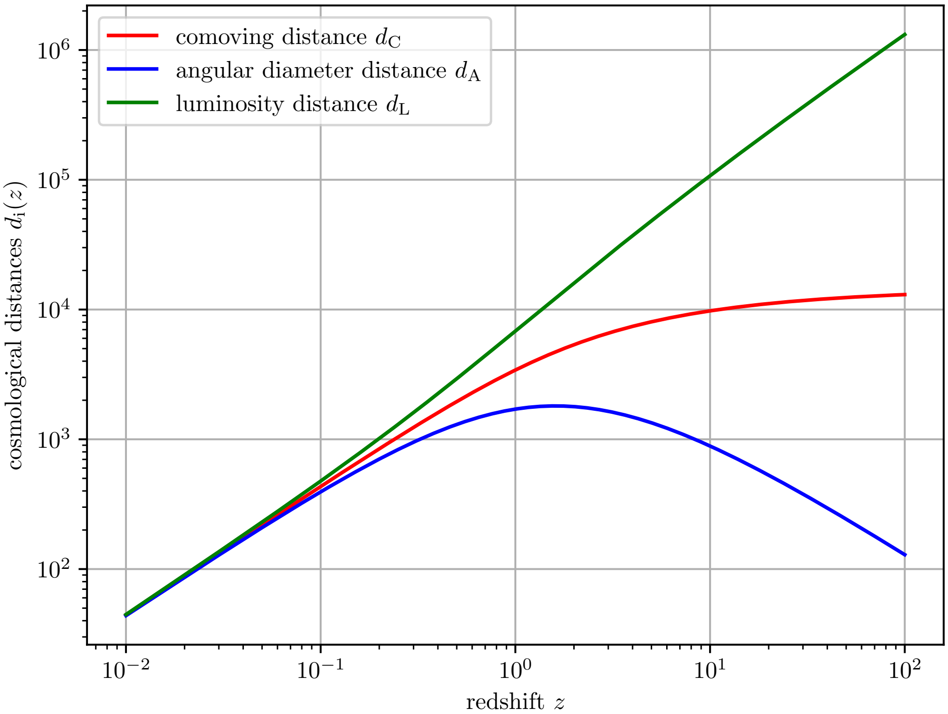


Figure 2.5: Comoving distance d_C, angular diameter distance d_A and luminosity distance d_L with respect to redshift z (logarithmic axis) for the ΛCDM-model with $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ and $H_0 = 67.66\,\frac{\mathrm{km}}{\mathrm{s}}\mathrm{Mpc}^{-1}$.

We can clearly see in Figure 2.5 the linear approximation for low redshifts, which explains the linear relation (1.12) that Edwin Hubble observed. Further, we see that those distances require observations at high redshift to distinguish them from each other.
The fact of the observed relation $d_\mathrm{L}/d_\mathrm{A} = (1+z)^2$ ([Wei08, p. 58]) and the mentioned slightly broadened light curve of supernovae type Ia due to the slowed supernovae explosion by the factor of $1+z$ ([Gol+01, p. 10/11]) shows that the observed redshift is due to an expansion of space,[2] which supports the ΛCDM-model.
For this thesis, the most relevant distance is the luminosity distance d_L.

[2] Therefore, interpretations of observed redshift due to loss of energy by the received light in a static universe ("tired light"-theory) can be considered as ruled out.

CHAPTER 3

THE DVALI–GABADADZE–PORRATI-MODEL

The model proposed by Gia Dvali, Gregory Gabadadze and Massimo Porrati ([DGP00]) is an alternative cosmological model, in which our four-dimensional universe is embedded as a brane in a five-dimensional Minkowski spacetime.
The authors introduce a fourth spatial dimension y. While the electromagnetic, the weak and the strong nuclear force are limited to our four-dimensional world, gravity also acts onto the postulated additional spatial dimension.
The scale to which gravity acts familiar as the theory of general relativity predicts, but mimics dark energy as it acts onto the additional dimension, is introduced as the crossover scale r_c.
From the modified Einstein–Hilbert action given by (see [DT03])

$$S_\mathrm{DGP} = \frac{1}{r_\mathrm{c}} \frac{c^4}{8\pi G} \int \mathrm{d}^4x\,\mathrm{d}y \sqrt{-\det\left(g^{(5)}_{AB}\right)}\mathcal{R} + \int \mathrm{d}^4x \sqrt{-\det(g_{\mu\nu})}\left(\frac{c^4}{8\pi G}R + \mathcal{L}_\mathrm{SM}\right), \tag{3.1}$$

where $g^{(5)}_{AB}$ is the metric with $A, B \in \{0, 1, 2, 3, 4\}$ and $\mathcal{R}$ the Ricci scalar on five-dimensional spacetime, follow the modified Einstein field equations

$$\frac{1}{r_\mathrm{c}}\mathcal{G}_{AB} + \delta(y)\delta^\mu_A\delta^\nu_B\left(G_{\mu\nu} - \frac{8\pi G}{c^4}T_{\mu\nu}\right) = 0, \tag{3.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor (left-hand side of Equation (1.24) without cosmological constant Λ) and $\mathcal{G}_{AB}$ its five-dimensional analogon. An ansatz as proposed in [DT03], the metric

$$\mathrm{d}s^2_{(5)} = f(y, H)\,\mathrm{d}s^2 - \mathrm{d}y^2\,, \tag{3.3}$$

where $\mathrm{d}s^2$ is the four-dimensional maximally-symmetric FLRW-metric, H the four-dimensional Hubble parameter and $f(y, H)$ the so called warp-factor, leads to the modified Friedmann equation

$$H^2 \pm \frac{c}{r_\mathrm{c}}H = \frac{8\pi G}{3}\rho. \tag{3.4}$$

The positive sign accounts to a decelerated expansion, while a negative sign corresponds to an accelerated expansion. Thus, we consider in the following the case in which the sign in Equation (3.4) is negative.

We are now introducing the parameter $\alpha \in \mathbb{R}$ which should interpolate between the ΛCDM-model in the case of neglecting radiation in a flat universe ($\Omega_{\mathrm{r},0} = 0, \Omega_{k,0} = 0$) for $\alpha = 0$ and the DGP-model for $\alpha = 1$. Therefore, we can rewrite Equation (3.4) in the accelerated case as

$$H^2 - \left(\frac{c}{r_\mathrm{c}}\right)^{2-\alpha} H^\alpha = \frac{8\pi G}{3}\rho \tag{3.5}$$

and can conclude with Equation (2.13), Equation (2.18), and the crossover scale expressed by ([DT03, p. 3])

$$r_{\mathrm{c}} = \frac{c}{H_0}(1 - \Omega_{\mathrm{m},0})^{\frac{1}{\alpha-2}}, \tag{3.6}$$

the equation

$$E^2(z) - (1 - \Omega_{\mathrm{m},0})E^{\alpha}(z) - \Omega_{\mathrm{m},0}(1+z)^3 = 0, \tag{3.7}$$

where $E(z) = \dfrac{H(z)}{H_0}$ is the expansion function.

The DGP-model refrains from introducing a cosmological constant. The free parameters of the DGP-model are therefore $\Omega_{\mathrm{m},0}$ and α. We consider in this thesis the DGP-model of a flat universe. The definition of the luminosity distance d_{L} remains the same as in the ΛCDM-model (see Equation (2.34)).
Since a deeper understanding of the mathematical formalism of the DGP-model and a detailed description of its phenomenological implications would go beyond the scope of this thesis, I recommend for further readings [Lue06].

CHAPTER 4

PARAMETER ESTIMATION

Let us now compare the ΛCDM- and DGP-model. To do this, we will consider the "Union2.1" SN Ia compilation and determine best-fit values for the parameter pair $(\Omega_{\mathrm{m},0},\ \Omega_{\Lambda,0})$ in ΛCDM- and $(\Omega_{\mathrm{m},0},\ \alpha)$ in DGP-model by computing the χ^2-distribution, which is a measure for the likelihood.

4.1 Supernovae Type Ia "Union2.1" Dataset

The SN Ia "Union2.1" dataset (see appendix A) used in this thesis contains the name, the redshift, the distance modulus, and the distance modulus error of 580 supernovae, measured by the Hubble Space Telescope.

First, let us have a look at the dataset. We plot the distance modulus (see Equation (1.7)) $m - M$ against the redshift z for the predicted luminosity distance d_L by the ΛCDM-model with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0.7)$ and DGP-model with $(\Omega_\mathrm{m}, \alpha) = (0.3, 1.0)$.

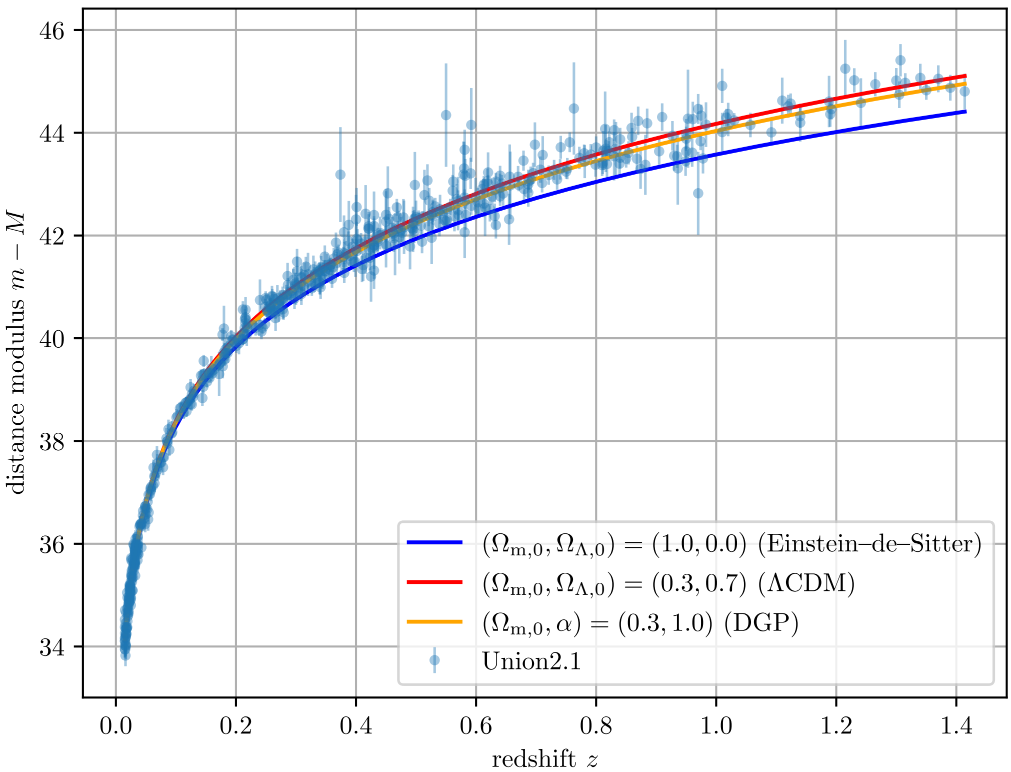


Figure 4.1: Distance modulus $m - M$ against redshift z for Einstein–de–Sitter-model $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (1.0, 0.0)$, the ΛCDM-model with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0.7)$ and DGP-model with $(\Omega_{\mathrm{m},0}, \alpha) = (0.3, 1.0)$.

At first sight, we see that a universe that contains only matter (Einstein–de–Sitter-model) cannot match the data, especially for high redshifts.
Further, we can conclude that the data points at high redshift ($z \gtrsim 0.8$) have a greater influence on the fit for the estimated parameters, since there is no significant fluctuation of data points at low redshift.
The apparently large errors of some data points in the range of $0.35 \leq z \leq 1.0$ should not cause any concerns whether this dataset is suitable for an adequate parameter estimation, since the error of low-redshift data points will impact the fit to a lesser degree than errors in high-redshift data points.

4.2 Statistical Analysis

To obtain values for the best-fit parameters, we consider the distance of every data point between its *measured* relative magnitude to the *theoretical* value the relative magnitude would have dependent on the parameters of the model.
From now on, we denote $\boldsymbol{\theta}$ as the parameter pairs of the model we want to estimate, so

$$\boldsymbol{\theta} = \begin{cases} (\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) & \text{for } \Lambda\text{CDM-model} \\ (\Omega_{\mathrm{m},0}, \alpha) & \text{for DGP-model} \end{cases}. \tag{4.1}$$

The given dataset is a sample of $N = 580$ datapoints which contain for datapoint $i \in [1, N]$ the redshift z_i, the distance modulus and therefore implicitly[1] the relative magnitude m_i and its error σ_{m_i}.
Our goal is to express the quantities given *only* through the free parameters $\boldsymbol{\theta}$.
Now, let us consider the relative magnitude m, which is given by the distance modulus (see Equation (1.7)). First, we want to express the luminosity distance d_L in multiples of $1\,\mathrm{Mpc}$, so we obtain

$$\begin{aligned} m &= M + 5\log_{10}\left(\frac{d_\mathrm{L}}{10\,\mathrm{pc}}\right) = M + 5\log_{10}\left(10^5\frac{d_\mathrm{L}}{1\,\mathrm{Mpc}}\right) \\ &= M + 25 + 5\log_{10}\left(\frac{d_\mathrm{L}}{1\,\mathrm{Mpc}}\right). \end{aligned}$$

Since the luminosity distance d_L depends on the Hubble distance d_H (see Equation (2.31) and (2.34)) and is therefore proportional to $d_\mathrm{L} \propto 1/H_0$, we redefine the luminosity distance so that it is independent of the Hubble constant. Hence, we go on with

$$\begin{aligned} m &= M + 25 + 5\log_{10}\left(\frac{1}{H_0}H_0 d_\mathrm{L}(z, H_0, \boldsymbol{\theta})\mathrm{Mpc}^{-1}\right) \\ &= \underbrace{M + 25 - 5\log_{10}(H_0)}_{=:\mathcal{M}(H_0)} + 5\log_{10}\left(\underbrace{H_0 d_\mathrm{L}(z, H_0, \boldsymbol{\theta})}_{=:\mathcal{D}_\mathrm{L}(z,\boldsymbol{\theta})}\mathrm{Mpc}^{-1}\right) \\ &= \mathcal{M}(H_0) + 5\log_{10}(\mathcal{D}_\mathrm{L}(z, \boldsymbol{\theta})\mathrm{Mpc}^{-1}). \end{aligned} \tag{4.2}$$

With Equation (4.2), the dependency on the Hubble constant is now in an additive constant $\mathcal{M}(H_0)$, which will be useful as we see later.

[1] Although the "Union2.1" SN Ia compilation contains the distance modulus $m - M$, we will handle it throughout the parameter estimation as if it only contains the relative magnitude m. Since we are going to marginalize the parameter M, it has no influence on the parameter estimation.

From now on, we are going to call the relative magnitude in Equation (4.2) the *theoretical* relative magnitude m_{th}, since it contains the cosmological parameters $\boldsymbol{\theta}$.

Given the dataset $D := \{z_i, m_i, \sigma_{m_i}\}$, where z_i is the measured redshift, m_i the measured relative magnitude and σ_{m_i} the error of the relative magnitude m_i, we define

$$\chi^2(\mathcal{M}, \boldsymbol{\theta}|D) := \sum_{i=1}^{N} \left(\frac{m_{\mathrm{th}}(z_i, \mathcal{M}, \boldsymbol{\theta}) - m_i}{\sigma_{m_i}} \right)^2 \tag{4.3}$$

as the χ^2-distribution of $\mathcal{M}$ and $\boldsymbol{\theta}$.
The likelihood, which is a probability density, is given by

$$L(\mathcal{M}, \boldsymbol{\theta}|D) := L_0 \exp\left(-\frac{1}{2} \chi^2(\mathcal{M}, \boldsymbol{\theta}|D) \right), \tag{4.4}$$

where L_0 is a normalization factor. With the likelihood L, it is possible to calculate the probability P to find $\mathcal{M} \in \mathcal{I}_{\mathcal{M}}$ and $\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}$ in a parameter interval $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}_{\boldsymbol{\theta}}$ with

$$P(\mathcal{M} \in \mathcal{I}_{\mathcal{M}}, \boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D) = \int\limits_{\mathcal{I}_{\mathcal{M}}} \mathrm{d}\mathcal{M} \int\limits_{\mathcal{I}_{\boldsymbol{\theta}}} \mathrm{d}\boldsymbol{\theta}\, L(\mathcal{M}, \boldsymbol{\theta}|D). \tag{4.5}$$

However, this probability P still depends on $\mathcal{M}$, which is not measured. Since our goal is to find the best-fit values for $\boldsymbol{\theta}$ and therefore express the probability only through $\boldsymbol{\theta}$, we are going to *marginalize* over $\mathcal{M}$. We obtain the *marginalized* likelihood L_{M} by integrating the likelihood L over all possible values that $\mathcal{M}$ could take. Assuming $\mathcal{M} \in (-\infty, \infty)$, it follows

$$L_{\mathrm{M}}(\boldsymbol{\theta}|D) = \int\limits_{-\infty}^{\infty} \mathrm{d}\mathcal{M}\, L(\mathcal{M}, \boldsymbol{\theta}|D). \tag{4.6}$$

Thereby, we can calculate the probability $P(\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D)$ to find values for $\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}$, given the dataset D, but without any information on $\mathcal{M}$, which implicitly contains the Hubble constant H_0, so

$$P(\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D) = \int\limits_{\mathcal{I}_{\boldsymbol{\theta}}} \mathrm{d}\boldsymbol{\theta}\, L_{\mathrm{M}}(\boldsymbol{\theta}|D). \tag{4.7}$$

Since $\mathcal{M}$ is only an additive constant (see Equation (4.2)), this can be done analytically. To do so, we set $\mathcal{M} = 0$ and introduce the terms

$$c_1(\{\sigma_{m_i}\}) := \sum_{i=1}^{N} \frac{1}{\sigma_{m_i}^2}, \tag{4.8}$$

$$b_0(\boldsymbol{\theta}|D) := \sum_{i=1}^{N} \frac{m_{\mathrm{th}}(z_i, \boldsymbol{\theta}) - m_i}{\sigma_{m_i}^2}, \tag{4.9}$$

$$b_1(\boldsymbol{\theta}|D) := \sum_{i=1}^{N} \left(\frac{m_{\mathrm{th}}(z_i, \boldsymbol{\theta}) - m_i}{\sigma_{m_i}} \right)^2, \tag{4.10}$$

by which we obtain

$$\chi^2(0, \boldsymbol{\theta}|D) = b_1(\boldsymbol{\theta}|D) - \frac{b_0^2(\boldsymbol{\theta}|D)}{c_1(\{\sigma_{m_i}\})} =: \chi_{\mathrm{A}}^2(\boldsymbol{\theta}|D) \tag{4.11}$$

and thus for the likelihood

$$L(0,\boldsymbol{\theta}|D) = L_0 \exp\left(-\frac{1}{2}\chi^2_{\mathrm{A}}(\boldsymbol{\theta}|D)\right) =: L_{\mathrm{A}}(\boldsymbol{\theta}|D). \tag{4.12}$$

The expressions L_{M} (see Equation (4.6)) and L_{A} (see Equation (4.12)) lead to the same results, which we will check later for a minimal working example in the case of the ΛCDM-model. In the following, we denote the likelihood and the χ^2-distribution that are independent of $\mathcal{M}$ simply as $L(\boldsymbol{\theta}|D)$ and $\chi^2(\boldsymbol{\theta}|D)$.
The best-fit parameters $\boldsymbol{\theta}_{\mathrm{best}}$ are located at the maximum of the likelihood, so

$$\left.\frac{\partial L(\boldsymbol{\theta}|D)}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}_{\mathrm{best}}} = 0. \tag{4.13}$$

Since $L(\boldsymbol{\theta}|D)$ is a concave function of $\chi^2(\boldsymbol{\theta}|D)$, we can find the best-fit parameters $\boldsymbol{\theta}_{\mathrm{best}}$ at the minimum of $\chi^2(\boldsymbol{\theta}|D)$. The χ^2-distribution is a measure for the so called *log-likelihood*, which is sometimes considered rather than the likelihood for practical purposes and convenience of computational work. However, this has no influence on the outcome of the parameter estimation.

After we obtain $\boldsymbol{\theta}_{\mathrm{best}}$, it is important to find the $\sigma_{\boldsymbol{\theta}}$-regions, which make a statement about the probability to find $\boldsymbol{\theta}$ in a certain parameter interval[2] $\mathcal{I}_{n\sigma_{\boldsymbol{\theta}}} \subset \mathcal{I}_{\boldsymbol{\theta}}$. Their construction is based on the standard deviation σ_x of a Gaussian distribution

$$g(x|\mu_x,\sigma_x) = \frac{1}{\sqrt{2\pi}\sigma_x}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right], \tag{4.14}$$

where μ_x is the mean of x and σ_x^2 the variance of x, so that the probability of finding $x \in [\mu_x - n\sigma_x, \mu_x + n\sigma_x] =: \mathcal{I}_{n\sigma_x}, n \in \mathbb{N}$ is given by

$$P(x \in \mathcal{I}_{n\sigma_x}|\mu_x,\sigma_x) = \int_{\mathcal{I}_{n\sigma_x}} \mathrm{d}x\, g(x|\mu_x,\sigma_x) =: P_{n\sigma_x} \tag{4.15}$$

and therefore

$$P(\boldsymbol{\theta} \in \mathcal{I}_{n\sigma_{\boldsymbol{\theta}}}|D) = \int_{\mathcal{I}_{n\sigma_{\boldsymbol{\theta}}}} \mathrm{d}\boldsymbol{\theta}\, L(\boldsymbol{\theta}|D) = P_{n\sigma_x}. \tag{4.16}$$

It is common to express statistical accuracies in multiples of the standard deviation. To calculate $P_{n\sigma_x}$, let us take a step back to the meaning of the χ^2-distribution. In general, the χ^2_k-distribution is defined to be proportional to the sum of the squares of k statistically independent, standard normal distributed random variables X_i $(i \in [1,k])$, so

$$P(X_1,...,X_k) = \prod_{i=1}^{k} P(X_i) \wedge X_i \sim g(x|0,1) \Rightarrow \sum_{i=1}^{k} X_i^2 \sim \chi^2_k. \tag{4.17}$$

The χ^2_k-distribution depends on the amount of statistically independent random variables k, which is often called the *degree of freedom*.
The probability density function ("PDF") of the χ^2_k-distribution is given by

$$f_k(x) = \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} \mathrm{e}^{-\frac{1}{2}x} \quad \text{for} \quad x > 0, \tag{4.18}$$

[2] The interval $\mathcal{I}_{n\sigma_{\boldsymbol{\theta}}}$ is sometimes called the "confidence interval".

where

$$\Gamma(s) := \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-t} \tag{4.19}$$

is the gamma function.
The cumulative density function ("CDF") of the χ_k^2-distribution is given by

$$F_k(x) = \frac{1}{\Gamma(\frac{k}{2})} \gamma(\tfrac{k}{2}, \tfrac{x}{2}) = \frac{1}{\Gamma(\frac{k}{2})} \int_0^{\frac{x}{2}} \mathrm{d}t\, t^{\frac{k}{2}-1} \mathrm{e}^{-t} \quad \text{for} \quad x > 0, \tag{4.20}$$

where $\gamma(\frac{k}{2}, \frac{x}{2})$ is the lower incomplete gamma function.
The values of $P_{n\sigma_x}$ can be computed by

$$P_{n\sigma_x} = F_1(n^2) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^{\frac{n^2}{2}} \mathrm{d}t\, t^{-\frac{1}{2}} \mathrm{e}^{-t}. \tag{4.21}$$

Since $\boldsymbol{\theta}$ contains for both cosmological models two parameters (see Equation (4.1)), the χ^2-distribution that only depends on $\boldsymbol{\theta}$ has $k = 2$ degrees of freedom, which leads to the probability density function

$$f_2(x) = \frac{1}{2} \mathrm{e}^{-\frac{1}{2}x} \quad \text{for} \quad x > 0 \tag{4.22}$$

and the cumulative density function

$$F_2(x) = 1 - \mathrm{e}^{-\frac{1}{2}x} \quad \text{for} \quad x > 0. \tag{4.23}$$

The quantile function, sometimes called "percent-point function" ("PPF") or "inverse cumulative distribution", $Q(P)$ returns the value x of a random variable X for the probability P to find $X \leq x$. With the cumulative distribution function $F_2(x)$, the quantile function $Q(P)$ is

$$Q(P) = -2 \ln(1 - P). \tag{4.24}$$

For $P_{n\sigma_x}$, we define

$$Q_n := -2 \ln(1 - P_{n\sigma_x}). \tag{4.25}$$

The $\sigma_{\boldsymbol{\theta}}$-regions satisfy the condition

$$\forall \boldsymbol{\theta} \in \mathcal{I}_{n\sigma_{\boldsymbol{\theta}}} : L(\boldsymbol{\theta}|D) = L_0 \exp\left[-\frac{1}{2}\left(\chi^2(\boldsymbol{\theta}_{\text{best}}|D) + Q_n\right)\right]. \tag{4.26}$$

4.3 Computational Implementation and Results

The source codes which are subject to this analysis are written in Python.
The main computational implementations that lead to the following results are only outlined for a minimal working example in the case of the ΛCDM-model. For a full insight to the entire source codes that are used to produce the plots and the results in this thesis, see appendix A.

4.3.1 ΛCDM-Model

4.3.1.1 Minimal Working Example – ΛCDM-Model for a Flat Universe ($\Omega_{k,0} = 0$)

Before writing huge and complex code, it is always a good idea to reduce the complexity of the cosmological model by creating a simple, minimalistic "toy model". To do so for the ΛCDM-model, we assume a flat universe (and, as mentioned on page 17, no radiation) – so $\Omega_{k,0} = 0$ (and $\Omega_{\mathrm{r},0} = 0$).
With Equation (2.21) follows $\Omega_{\Lambda,0} = 1 - \Omega_{\mathrm{m},0}$ and therefore with Equation (2.25) for the expansion function (see Equation (2.22))

$$E(z) = \sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + 1 - \Omega_{\mathrm{m},0}}. \tag{4.27}$$

In the case of a flat universe, the comoving distance is simply given by $d_{\mathrm{C}} = d_{\mathrm{H}} I$, where I is the expression (2.32) with the expansion function in Equation (4.27). With Equation (2.34) and (4.2), the (modified) luminosity distance $\mathcal{D}_{\mathrm{L}}$ for our simplified model is

$$\mathcal{D}_{\mathrm{L}}(z, \Omega_{\mathrm{m},0}) = (1+z)c \int_0^z \mathrm{d}z' \, \frac{1}{\sqrt{\Omega_{\mathrm{m},0}(1+z')^3 + 1 - \Omega_{\mathrm{m},0}}}. \tag{4.28}$$

A computational implementation of those functions could look as in Listing 4.1.

```
def expansion_function(z, Omega_m0):
    return np.sqrt(Omega_m0 * np.power(1.0 + z, 3) + 1.0 - Omega_m0)


def integrand(z, Omega_m0):
    if z == 0.0:
        return 1.0
    E = expansion_function(z, Omega_m0)
    return 1.0/E


def integral(z, Omega_m0):
    # d_C/d_H = Integrate[1/E(z'), {z',0,z}]
    return quad(integrand, 0.0, z, args=(Omega_m0,))[0]


def mod_luminosity_distance(z, Omega_m0):
    # Cosmological Parameters
    # ======================
    c = 299792.458          # speed of light in vacuum in km/s
    H_0 = 1.0               # dependence on hubble constant is set into
  mod_absolute_magnitude
    d_H = c/H_0             # hubble distance
    # ======================

    I = np.array([integral(zi, Omega_m0)  for zi in z])

    return (1.0 + z) * d_H * I
```

Listing 4.1: Function for $E(z)$ and $\mathcal{D}_{\mathrm{L}}(z, \Omega_{\mathrm{m},0})$.

The function `quad`, which is returned by `integral`, is imported from `scipy.integrate`.[3] It returns the value of the calculated integration and its error. This is especially important when

[3] Documentation of `scipy.integrate`-package: https://docs.scipy.org/doc/scipy/reference/integrate.html.

we compute the marginalized likelihood L_M. If the integration error is of the same order of magnitude as the computed value of the integration, this can lead to erroneous parameter estimation of $\boldsymbol{\theta}_\mathrm{best}$. The `scipy.integrate`-package uses integration techniques from the Fortran library `QUADPACK`.[4] For our purposes, the integrator `qagse` and subroutine `qk21` are called, which use the Gauss-Kronrod quadrature with 21 quadrature points, where the function that should be integrated is evaluated at the quadrature points with corresponding weighting.

So, let us first compute the likelihood $L_\mathrm{A}(\Omega_{\mathrm{m},0}|D)$ by calculating χ^2 analytically. Therefore, we implement the theoretical magnitude m_th as in Equation (4.2), χ^2_A as in Equation (4.11) and L_A as in Equation (4.12).

```
@njit
def theoretical_magnitude(mod_absolute_magnitude, mod_luminosity_distance):
    # mod_luminosity_distance := H_0 * luminosity_distance
    # mod_absolute_magnitude := absolute_magnitude -  5.0 * np.log10(H_0) + 25.0
    return mod_absolute_magnitude + 5.0 * np.log10(mod_luminosity_distance)

@njit
def analytic_chi_square(magnitudes, error_magnitudes, theoretical_magnitudes):
    c1 = 0.0
    b0 = 0.0
    b1 = 0.0
    for m, sigma, m_th in zip(magnitudes, error_magnitudes, theoretical_magnitudes):
        c1 += 1.0/(sigma * sigma)
        b0 += (m_th - m)/(sigma * sigma)
        b1 += ((m_th - m)/sigma) * ((m_th - m)/sigma)
    return b1 - b0 * b0/c1

def analytic_likelihood(Omega_m0, redshifts, magnitudes, error_magnitudes, L0):
    mod_absolute_magnitude = 0.0
    D_L = mod_luminosity_distance(redshifts, Omega_m0)
    m_th = theoretical_magnitude(mod_absolute_magnitude, D_L)
    chi_2 = analytic_chi_square(magnitudes, error_magnitudes, m_th)
    return L0 * np.exp(-0.5 * chi_2)
```

Listing 4.2: Function for $m_\mathrm{th}(z, \mathcal{M}, \Omega_{\mathrm{m},0})$, $\chi^2_\mathrm{A}(\Omega_{\mathrm{m},0}|D)$ and $L_\mathrm{A}(\Omega_{\mathrm{m},0}|D)$.

The normalization factor L_0 does not impact the parameter estimation in the case of computing χ^2 analytically, so we could omit the multiplication with `L0` in this case.
As result, we obtain values for the best-fit parameters rounded to two decimal places $\boldsymbol{\theta}_\mathrm{best} = (\Omega_{\mathrm{m},0,\mathrm{best}}, \Omega_{\Lambda,0,\mathrm{best}}) = (0.28, 0.72)$.
By plotting the likelihood L_A with $L_0 = 1$, we see that its magnitude is in the range of $\sim 10^{-123}$ (see Figure 4.3). If we had a χ^2-distribution that depends on $\mathcal{M}$, implement its likelihood as in Listing 4.2 with $L_0 = 1$ and integrate over it by using `quad` to calculate L_M, we would obtain an integration error in the same order of magnitude and therefore erroneous parameter estimation.
To avoid this, we are going to normalize the likelihood before we integrate over $\mathcal{M}$ to compute the marginalized likelihood L_M for this simplified model.
Further, it should be mentioned that it is recommended to speed-up the computation with the

[4] Fortran codes that are implemented in `QUADPACK`: https://netlib.org/quadpack/

Python-package `numba` [5] by importing `njit` and writing `@njit` before defining the function that should be improved. Even if this is not possible for all functions, for example functions that contain or execute `quad`, this technique saves a lot of computation time.

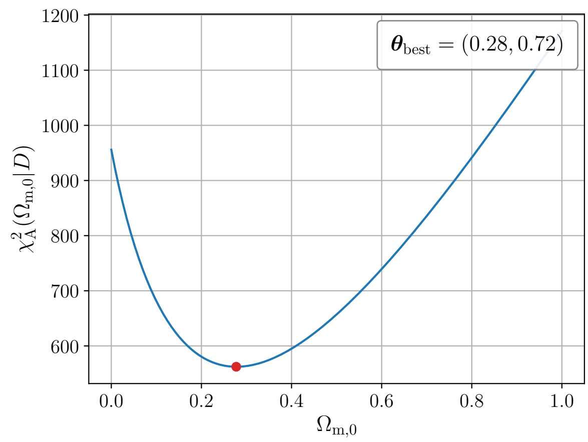


Figure 4.2: Density parameter $\Omega_{\mathrm{m},0}$ vs. analytically computed $\chi^2_{\mathrm{A}}(\Omega_{\mathrm{m},0}|D)$.

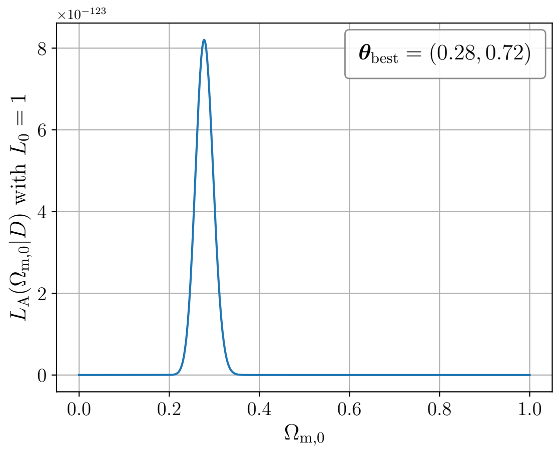


Figure 4.3: Density parameter $\Omega_{\mathrm{m},0}$ vs. likelihood $L_{\mathrm{A}}(\Omega_{\mathrm{m},0}|D)$ with $L_0 = 1$.

Before calculating the marginalized likelihood $L_{\mathrm{M}}(\Omega_{\mathrm{m},0}|D)$, let us compute the likelihood $L(\mathcal{M}, \Omega_{\mathrm{m},0}|D)$ (see Equation 4.4). To do so, we implement the χ^2-distribution according to Equation 4.3 as in Listing 4.3. Since we call `mod_luminosity_distance` which uses `integral` and therefore `quad`, we split the computation of the likelihood in a function `l` which we can provide with `@njit` and another function `likelihood` in which we compute the luminosity distance $\mathcal{D}_{\mathrm{L}}$.

```
@njit
def chi_square(magnitudes, error_magnitudes, theoretical_magnitudes):
    return np.sum(np.square((theoretical_magnitudes - magnitudes) /
   error_magnitudes))
```

Listing 4.3: Function for $\chi^2(\mathcal{M}, \Omega_{\mathrm{m},0}|D)$.

```
@njit
def l(mod_absolute_magnitude, mod_luminosity_distance, magnitudes,
   error_magnitudes, L0):
    m_th = theoretical_magnitude(mod_absolute_magnitude,
   mod_luminosity_distance)
    chi_2 = chi_square(magnitudes, error_magnitudes, m_th)
    return L0 * np.exp(-0.5 * chi_2)


def likelihood(mod_absolute_magnitude, Omega_m0, redshifts, magnitudes,
   error_magnitudes, L0):
    D_L = mod_luminosity_distance(redshifts, Omega_m0)
    L = l(mod_absolute_magnitude, D_L, magnitudes, error_magnitudes, L0)
    return L
```

Listing 4.4: Functions for the likelihood $L(\mathcal{M}, \Omega_{\mathrm{m},0}|D)$.

For calculating the marginalized likelihood L_{M}, we integrate over all theoretically possible values of $\mathcal{M} \in (-\infty, \infty)$ (see Equation 4.6). Numerically, we integrate over an interval in

[5] Documentation of `numba`-package: https://numba.readthedocs.io/en/stable/user/5minguide.html

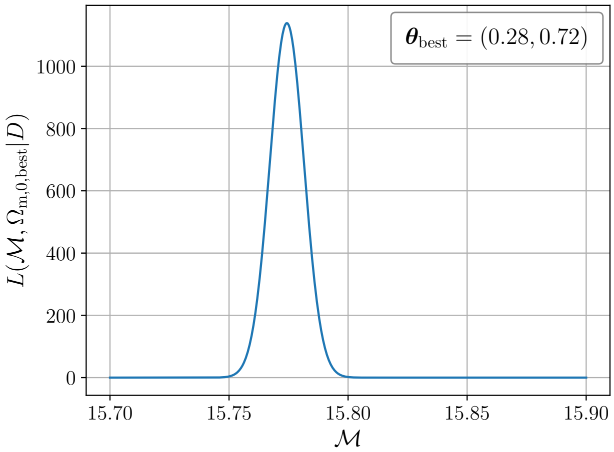


Figure 4.4: Modified magnitude $\mathcal{M}$ vs. likelihood $L(\mathcal{M}, \Omega_{\text{m,0,best}}|D)$ at $\Omega_{\text{m,0,best}} = 0.28$.

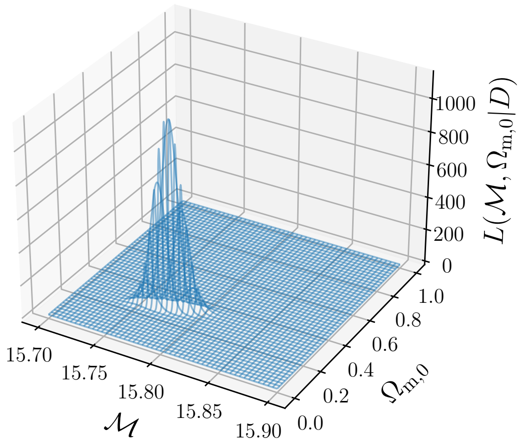


Figure 4.5: Modified magnitude $\mathcal{M}$ vs. density parameter $\Omega_{\text{m,0}}$ vs. likelihood $L(\mathcal{M}, \Omega_{\text{m,0}}|D)$.

which $L(\mathcal{M}, \Omega_{\text{m,0}}|D)$ does no longer change significantly as a function of $\mathcal{M}$. As we see from the computation of the likelihood $L(\mathcal{M}, \Omega_{\text{m,0}}|D)$ (see Figure 4.4 and 4.5), the relevant interval can be chosen as $\mathcal{I}_{\mathcal{M}} = [15.7, 15.9]$.

The marginalized likelihood $L_{\text{M}}(\Omega_{\text{m,0}}|D)$ can then be computed as in Listing 4.5.

```
def marginalized_likelihood(Omega_m0, redshifts, magnitudes,
   error_magnitudes, L0):
    D_L = mod_luminosity_distance(redshifts, Omega_m0)
    min_mod_absolute_magnitude = 15.7
    max_mod_absolute_magnitude = 15.9
    margin_L = quad(l, min_mod_absolute_magnitude,
   max_mod_absolute_magnitude, args=(D_L, magnitudes, error_magnitudes, L0))
   [0]
    return margin_L
```

Listing 4.5: Function for the marginalized likelihood $L_{\text{M}}(\Omega_{\text{m,0}}|D)$.

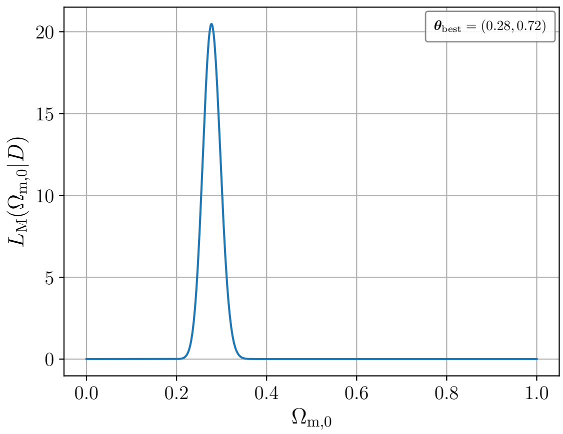


Figure 4.6: Density parameter $\Omega_{\text{m,0}}$ vs. marginalized likelihood $L_{\text{M}}(\Omega_{\text{m,0}}|D)$.

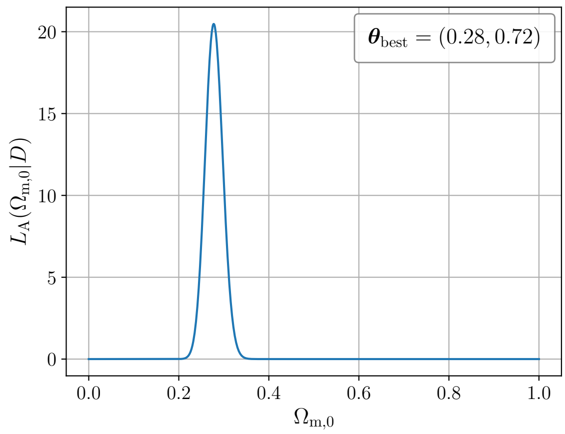


Figure 4.7: Density parameter $\Omega_{\text{m,0}}$ vs. analytic likelihood $L_{\text{A}}(\Omega_{\text{m,0}}|D)$.

As we compare the results for the marginalized likelihood $L_{\text{M}}(\Omega_{\text{m,0}}|D)$ and the analytic like-

lihood $L_{\mathrm{A}}(\Omega_{\mathrm{m},0}|D)$ in the Figures 4.6 and 4.7, we see that both likelihoods are the same and lead to the same values for the parameter estimation $\boldsymbol{\theta}_{\mathrm{best}} = (\Omega_{\mathrm{m},0,\mathrm{best}}, \Omega_{\Lambda,0,\mathrm{best}}) = (0.28, 0.72)$ (rounded to two decimal places), as we have expected.
Nevertheless, the sharp pulse shape of the likelihood could lead to incorrect integration as mentioned in the `quad`-documentation, since the size of the function that is integrated compared to the integration interval is important to compute correct values. Therefore, we are going to calculate the χ^2-distribution analytically for the evaluation of the ΛCDM-model in the case of an arbitrary curvature parameter $\Omega_{k,0}$ and for the valuation of the DGP-model.

4.3.1.2 ΛCDM-Model with an Arbitrary Curvature Parameter $\Omega_{k,0}$

By allowing an arbitrary curvature parameter $\Omega_{k,0} = 1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0}$, the expansion function is given by

$$E(z) = \sqrt{\Omega_{\mathrm{m},0}(1+z)^3 + (1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0})(1+z)^2 + \Omega_{\Lambda,0}}. \tag{4.29}$$

Further, we need to take the cases of $\Omega_{k,0} < 0$ and $\Omega_{k,0} > 0$ into account when computing the (modified) luminosity distance $\mathcal{D}_{\mathrm{L}}(z, \Omega_{\mathrm{m},0}, \Omega_{\Lambda,0})$ (see Equations (2.31), (2.32), (2.34) and (4.2)). First, let us have a look at the computed results for the analytically calculated χ^2-distribution (see Listing 4.2).

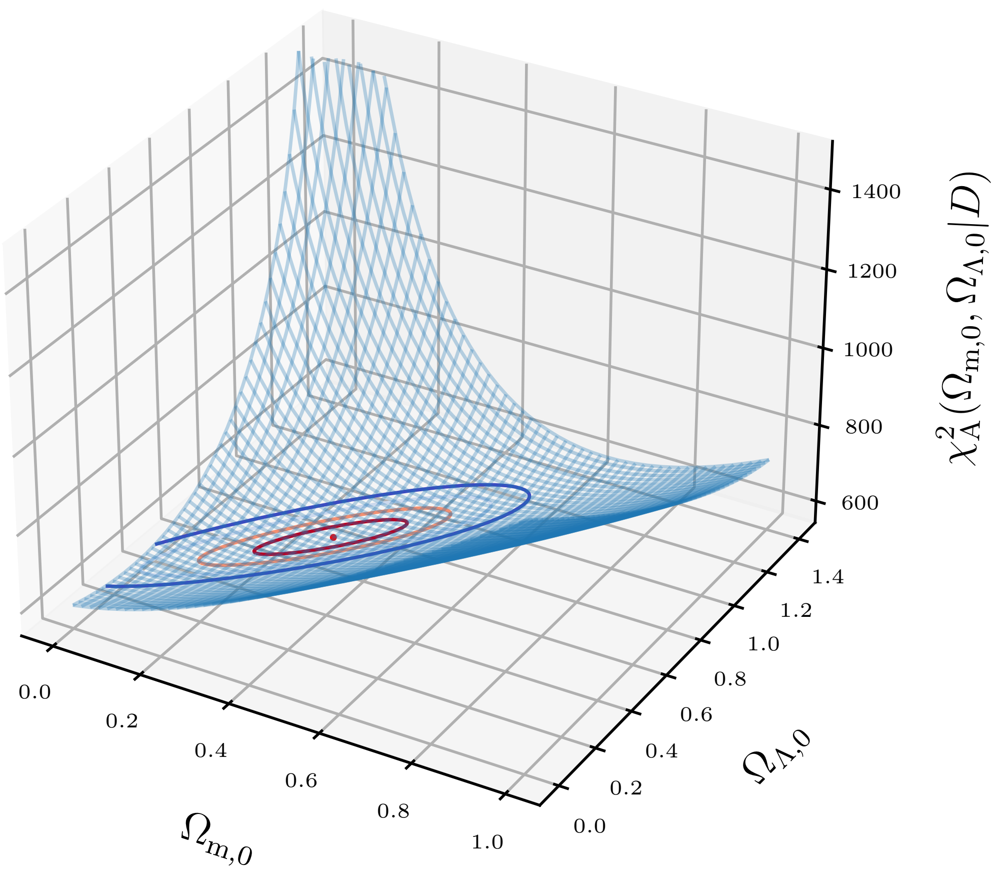


Figure 4.8: Density parameters $\Omega_{\mathrm{m},0}$ vs. $\Omega_{\Lambda,0}$ vs. analytic $\chi^2_{\mathrm{A}}(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}|D)$.

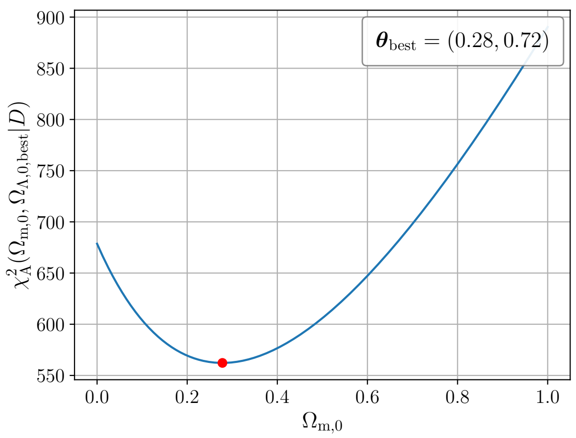


Figure 4.9: Density parameter $\Omega_{m,0}$ vs. analytic $\chi^2_A(\Omega_{m,0}, \Omega_{\Lambda,0,\text{best}}|D)$ at $\Omega_{\Lambda,0,\text{best}} = 0.72$.

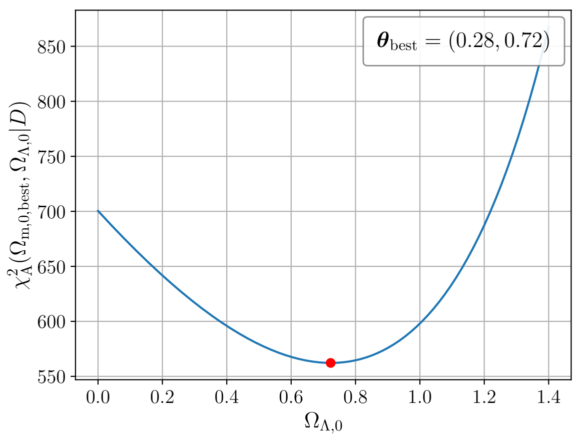


Figure 4.10: Density parameter $\Omega_{\Lambda,0}$ vs. analytic $\chi^2_A(\Omega_{m,0,\text{best}}, \Omega_{\Lambda,0}|D)$ at $\Omega_{m,0,\text{best}} = 0.28$.

We obtain for the best-fit values of the ΛCDM-model with an arbitrary curvature $\boldsymbol{\theta}_{\text{best}} = (\Omega_{m,0,\text{best}}, \Omega_{\Lambda,0,\text{best}}) = (0.28, 0.72)$.

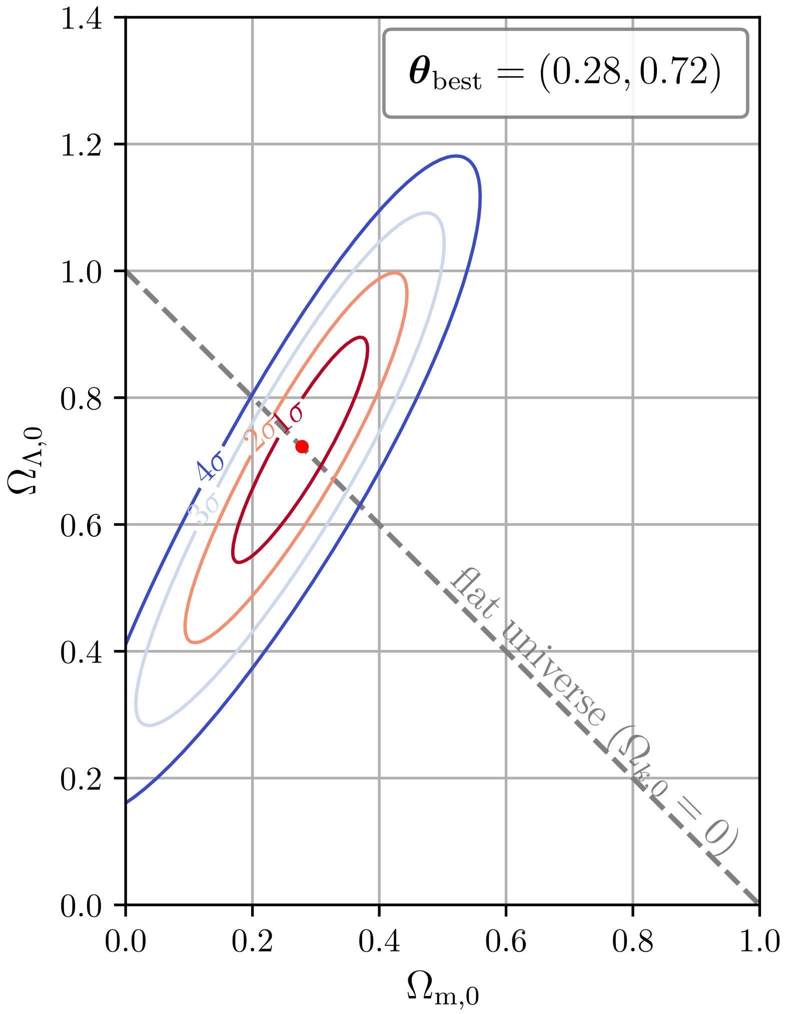


Figure 4.11: Density parameters $\Omega_{m,0}$ vs. $\Omega_{\Lambda,0}$ with calculated $\sigma_{\boldsymbol{\theta}}$-regions by computing $\chi^2_A(\Omega_{m,0}, \Omega_{\Lambda,0}|D)$.

To calculate σ-values for the individual parameters, we sum the (analytic) likelihood calculated for discrete values of $\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta},n} = [\boldsymbol{\theta}_{\min}, \boldsymbol{\theta}_{\max}, n]$ where n is the amount of linear spaced points between $\boldsymbol{\theta}_{\min}$ and $\boldsymbol{\theta}_{\max}$ such that the dependency of the likelihood $L_{\mathrm{A}}(\boldsymbol{\theta}|D)$ is reduced to one parameter. We denote

$$L_{\mathrm{A},\sum\Omega_{\Lambda,0}}(\Omega_{\mathrm{m},0}|D) := \sum_{\Omega_{\Lambda,0}\in\mathcal{I}_{\Omega_{\Lambda,0},n}} L_{\mathrm{A}}(\Omega_{\mathrm{m},0},\Omega_{\Lambda,0}|D), \tag{4.30}$$

$$L_{\mathrm{A},\sum\Omega_{\mathrm{m},0}}(\Omega_{\Lambda,0}|D) := \sum_{\Omega_{\mathrm{m},0}\in\mathcal{I}_{\Omega_{\mathrm{m},0},n}} L_{\mathrm{A}}(\Omega_{\mathrm{m},0},\Omega_{\Lambda,0}|D) \tag{4.31}$$

This to one parameter reduced likelihood has the shape of a (one dimensional) Gaussian curve. Its standard deviation is considered as the σ-value for the individual parameter.
We obtain this σ-value by fitting a Gaussian curve with `scipy.optimize.curve_fit` [6] for the computed, reduced likelihood.

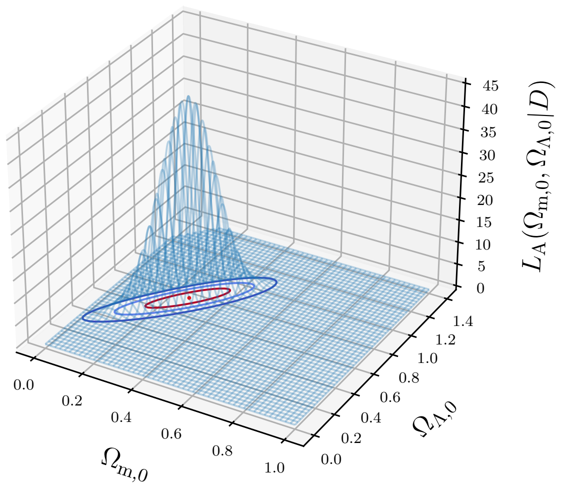


Figure 4.12: Density parameters $\Omega_{\mathrm{m},0}$ vs. $\Omega_{\Lambda,0}$ vs. analytic likelihood $L_{\mathrm{A}}(\Omega_{\mathrm{m},0},\Omega_{\Lambda,0}|D)$.

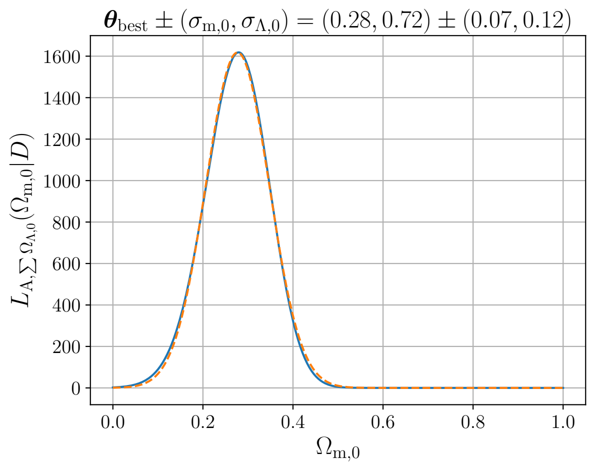


Figure 4.13: Density parameters $\Omega_{\mathrm{m},0}$ vs. reduced, analytic likelihood $L_{\mathrm{A},\sum\Omega_{\Lambda,0}}(\Omega_{\mathrm{m},0}|D)$ and its fit.

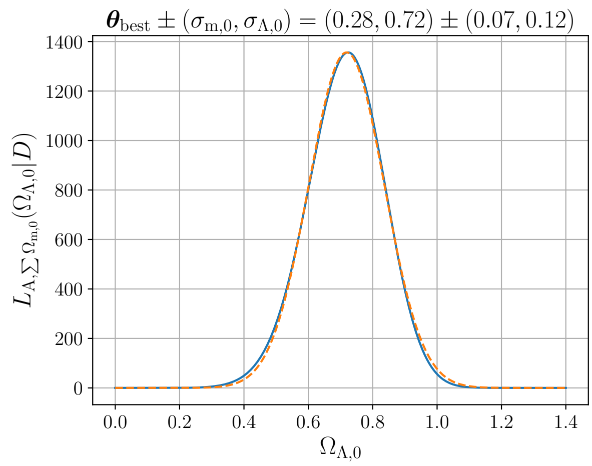


Figure 4.14: Density parameters $\Omega_{\Lambda,0}$ vs. reduced, analytic likelihood $L_{\mathrm{A},\sum\Omega_{\mathrm{m},0}}(\Omega_{\Lambda,0}|D)$ and its fit.

[6] Documentation of `scipy.optimize`-package: https://docs.scipy.org/doc/scipy/reference/optimize.html

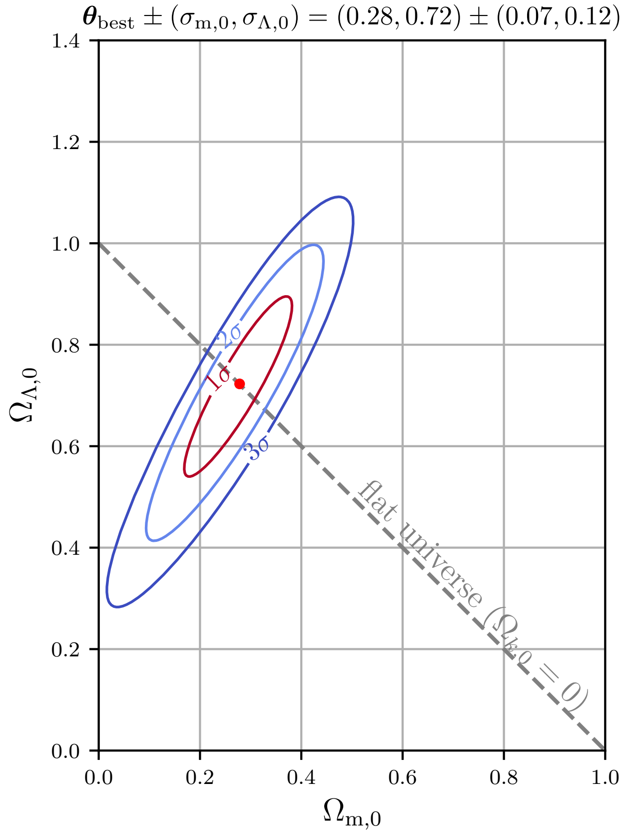


Figure 4.15: Density parameters $\Omega_{\text{m},0}$ vs. $\Omega_{\Lambda,0}$ with calculated $\sigma_{\boldsymbol{\theta}}$-regions by computing the analytic likelihood $L_{\text{A}}(\Omega_{\text{m},0}, \Omega_{\Lambda,0}|D)$.

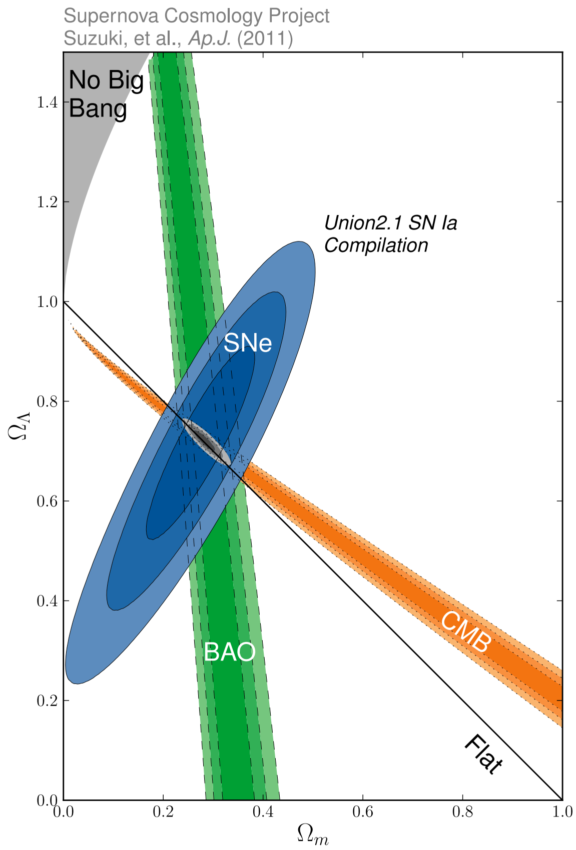


Figure 4.16: Density parameters $\Omega_{\text{m},0}$ vs. $\Omega_{\Lambda,0}$ with calculated $\sigma_{\boldsymbol{\theta}}$-regions by the *Supernova Cosmology Project*.
Source: [Suz+12, Figure 5]

Finally, we can specify the best-fit values with errors as $\boldsymbol{\theta}_{\text{best}} \pm (\sigma_{\text{m},0}, \sigma_{\Lambda,0}) = (0.28, 0.72) \pm (0.07, 0.12)$. As we compare Figures 4.15 and 4.16, we can conclude that our method and code (see appendix A) can reproduce the results by the *Supernovae Cosmology Project*, though the calculated errors for the individual parameters are slightly larger in our case.
By considering further constraints like measurements of the cosmic microwave background (CMB) or baryonic acoustic oscillations (BAO), it is possible to specify the best-fit values more precisely and reduce their uncertainty significantly (see [Suz+12, Table 7] and [Col+20, Table 6 and 7]).

4.3.2 DGP-Model

For the DGP-model, we implement the modified Friedmann Equations (see Equation 3.7) and its derivative with respect to z assuming $z = \text{const.}$ as in Listing 4.6.

```
@njit
def mod_friedmann(E, z, Omega_m0, alpha):
    return E * E - (1.0 - Omega_m0) * np.power(E, alpha) - Omega_m0 * np.
   power(1.0 + z, 3)

@njit
def deriv_mod_friedmann(E, _, Omega_m0, alpha):
    return 2.0 * E - alpha * (1.0 - Omega_m0) * np.power(E, alpha - 1.0)
```

Listing 4.6: Function for modified Friedmann Equation and its derivative assuming $z = \text{const.}$

To compute the solution of the modified Friedmann Equation for $E(z)$, we use the `root`-function provided by `scipy.optimize` and implement the solution as in Listing 4.7.

```
def sol_friedmann(z, Omega_m0, alpha, mod_friedmann, mod_deriv_friedmann):
    # Solves the modified friedmann equation f(z) = 0 for z with exterior derivative mod_deriv_friedmann
    return opt.root(mod_friedmann, 1.0, args=(z, Omega_m0, alpha), jac=mod_deriv_friedmann).x[0]
```

Listing 4.7: Function to find the solution of the modified Friedmann Equation.

As in the ΛCDM-model, it is demanded to calculate the (modified) luminosity distance $\mathcal{D}_{\mathrm{L}}$. Since the implemented computation of $E(z)$ as in Listing 4.7 is not trivial for `quad` to integrate over, we interpolate the integrand by a sample of redshifts z' and the expansion function $E(z')$ that solves Equation 3.7 for this sample.

```
@njit
def interp_integrand(z, sample_redshifts, sample_E):
    if z == 0.0:
        return 1.0

    E = np.interp(z, sample_redshifts, sample_E)
    return 1.0/E

def interp_integral(z, sample_redshifts, sample_E):
    # d_C/d_H = Integrate[1/E(z'), {z', 0, z}]
    return quad(interp_integrand, 0.0, z, args=(sample_redshifts, sample_E))[0]

def mod_luminosity_distance(z, Omega_m0, alpha):
    # Cosmological Parameters
    # ======================
    c = 299792.458            # speed of light in vacuum in km/s
    H_0 = 1.0                 # dependence on hubble constant is set into the mod_absolute_magnitude, see theoretical_magnitude
    d_H = c/H_0               # hubble distance
    # ======================

    sample_redshifts = np.linspace(0.0, max(z), 1000)
    sample_E = np.array([sol_friedmann(zi, Omega_m0, alpha, mod_friedmann, deriv_mod_friedmann) for zi in sample_redshifts])

    I = np.array([interp_integral(zi, sample_redshifts, sample_E) for zi in z])

    return (1.0 + z) * d_H * I
```

Listing 4.8: Functions for computation of the (modified) luminosity distance $\mathcal{D}_{\mathrm{L}}$ in the DGP-model.

Further, we compute the χ^2-distribution analytically as in Listing 4.2 and 4.9. In the following, we only consider $\chi^2_{\mathrm{A}}(\Omega_{\mathrm{m},0}, \alpha|D)$, which is sufficient to estimate best-fit values for $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \alpha)$.

```
def chi2(Omega_m0, alpha, redshifts, magnitudes, error_magnitudes):
    mod_absolute_magnitude = 0.0
    D_L = mod_luminosity_distance(redshifts, Omega_m0, alpha)
    m_th = theoretical_magnitude(mod_absolute_magnitude, D_L)
    chi_2 = analytic_chi_square(magnitudes, error_magnitudes, m_th)
    return chi_2
```

Listing 4.9: Function for analytic $\chi^2_{\mathrm{A}}(\Omega_{\mathrm{m},0}, \alpha|D)$.

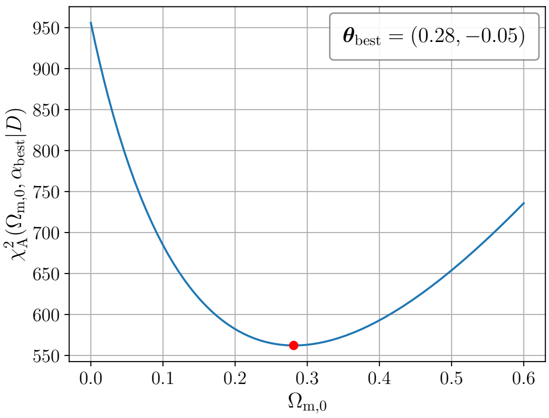


Figure 4.17: Density parameter $\Omega_{\text{m,0}}$ vs. analytic $\chi^2_{\text{A}}(\Omega_{\text{m,0}}, \alpha_{\text{best}}|D)$ at $\alpha_{\text{best}} = -0.05$.

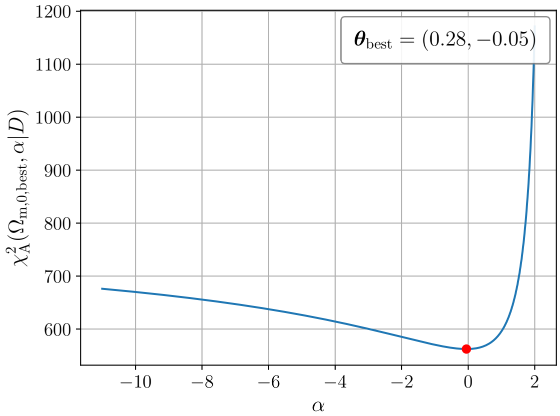


Figure 4.18: Interpolation parameter α vs. analytic $\chi^2_{\text{A}}(\Omega_{\text{m,0,best}}, \alpha|D)$ at $\Omega_{\text{m,0,best}} = 0.28$.

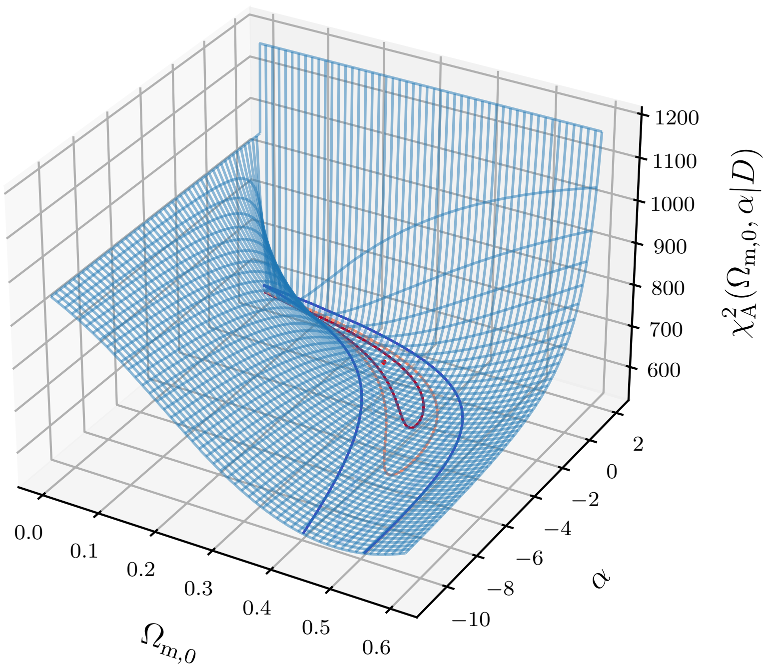


Figure 4.19: Density parameter $\Omega_{\text{m,0}}$ vs. interpolation parameter α vs. analytic $\chi^2_{\text{A}}(\Omega_{\text{m,0}}, \alpha|D)$.

Finally, we obtain for the best-fit values $\boldsymbol{\theta}_{\text{best}} = (0.28, -0.05)$. As we see in Figure 4.19 and Figure 4.20, the $\sigma_{\boldsymbol{\theta}}$-regions spread out broadly in parameter space as the χ^2-distribution flattens. This makes it difficult to give an adequate estimate of the σ-uncertainties for the individual parameters. Nevertheless, we can conclude that this parameter estimation of $\boldsymbol{\theta} = (\Omega_{\text{m,0}}, \alpha)$ tends to prefer the ΛCDM-model (for which $\alpha = 0$) over the DGP-model (for which $\alpha = 1$). As shown in Figure 4.22 (see [LX13, Figure 3]), accounting multiple datasets such as measurements of baryonic acoustic oscillations (BAO) or cosmic microwave background (Planck), but also measurements of weak gravitational lensing (see [TAW09, Figure 7]) allow constraints that disfavour the DGP-model.

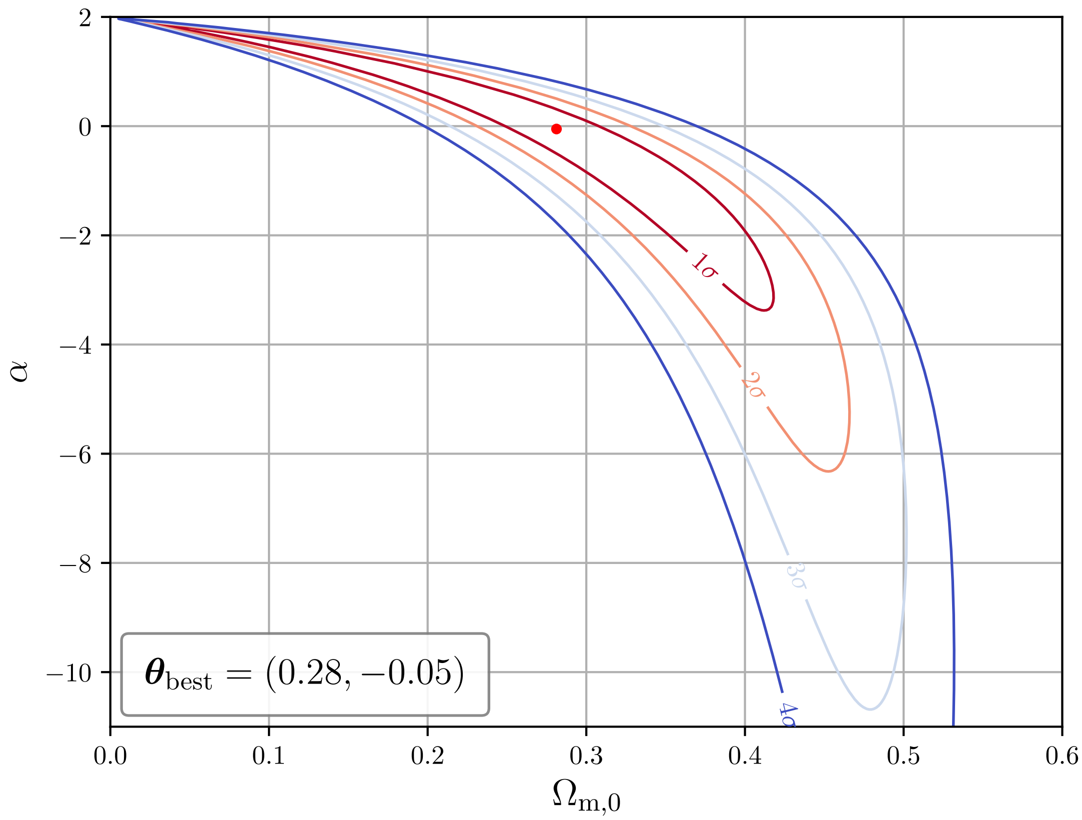


Figure 4.20: Density parameter $\Omega_{\text{m},0}$ vs. interpolation parameter α with calculated $\sigma_{\boldsymbol{\theta}}$-regions by computing $\chi^2_{\text{A}}(\Omega_{\text{m},0}, \alpha|D)$.

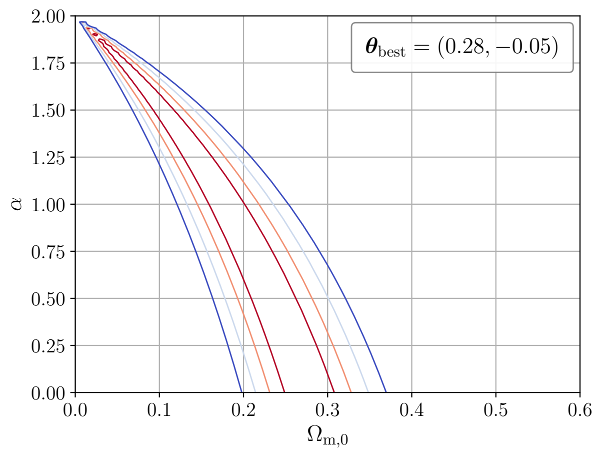


Figure 4.21: Density parameter $\Omega_{\text{m},0}$ vs. interpolation parameter α with calculated $\sigma_{\boldsymbol{\theta}}$-regions for $\alpha \in [0.0, 2.0]$.

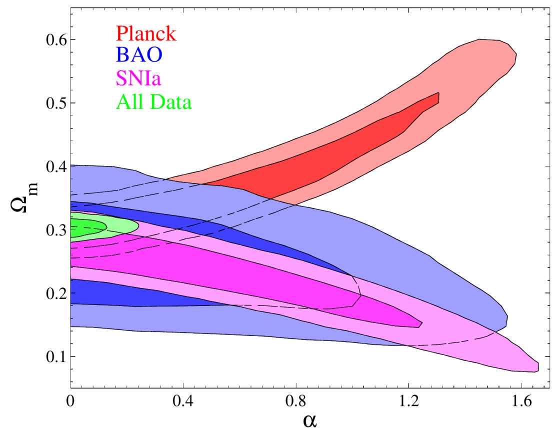


Figure 4.22: Density parameter $\Omega_{\text{m},0}$ vs. interpolation parameter α considering several cosmological constraints.
Source: [LX13, Figure 3]

CHAPTER 5

CONCLUSION

First of all, we see that the computational implementation carried out in this thesis are able to reproduce the results of the *Supernovae Cosmology Project* using the "Union2.1" SN Ia dataset. Further, the result of $\Omega_{\mathrm{m,0,best}} = 0.28$ in both cases, $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0})$ and $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \alpha)$, shows consistency in the computational implementation of our parameter estimation.

The obtained values of $(\Omega_{\mathrm{m,0,best}}, \Omega_{\Lambda,\mathrm{0,best}}) \pm (\sigma_{\mathrm{m},0}, \sigma_{\Lambda,0}) = (0.28, 0.72) \pm (0.07, 0.12)$ and $(\Omega_{\mathrm{m,0,best}}, \alpha_{\mathrm{best}}) = (0.28, -0.05)$ prefer the ΛCDM-model for a flat universe. Yet, the large errors, especially in the case of analyzing the parameter pair $\boldsymbol{\theta} = (\Omega_{\mathrm{m},0}, \alpha)$, do not allow a final judgement on the cosmological models by using the "Union2.1" SN Ia dataset alone (as mentioned in [TAW09]). This can also be seen in Figure 4.1, where the predicted relation between distance modulus and redshift in both the ΛCDM-model and the DGP-model match even for high-redshift datapoints and therefore do not allow a clear distinction. Only under consideration of multiple measurements as the cosmic microwave background (CMB), baryonic acoustic oscillations (BAO) and weak gravitational lensing, it is possible to provide constraints that favour the ΛCDM-model and disfavour the DGP-model.

Under the assumptions of homogeneity and isotropy on large scales, correct description of gravity by general relativity and supernovae type Ia as standard candles, we can conclude that the ΛCDM-model is at the current state of research in physical cosmology one of the best theoretical models we can provide that matches with multiple data and observations.

Nevertheless, the physical interpretation of the cosmological constant Λ as corresponding to vacuum energy density is questionable, since the obtained value for vacuum energy density by cosmological observations differ highly from predictions made by quantum field theory. This problem is known as the "cosmological constant problem" (see [Wei89]). As long as this problem is not considered to be solved, there is at least one legitimate concern for developing cosmological models that refrain from introducing a cosmological constant as provided by the DGP-model.
Furthermore it must be ensured that supernovae of type Ia can be seen as good candidates for standard candles.

Finally, we can conclude that cosmological parameter estimation should always account for multiple datasets, observations and methods to provide constraints and therefore good predictions on which cosmological model is most suitable for describing the universe we observe.

APPENDIX A

DATASET AND SOURCE CODES

The dataset used in this thesis is a reduced version of the full SN Ia "Union2.1" dataset.[1] The full table of all supernovae contains additional data like B-band magnitude, stretch or color.[2][3]

All source codes that I wrote to produce the plots are open-source and available at the GitHub repository of this thesis.[4]
Here is the list of the source codes that correspond to following Figures:

- Figure 1.1: `parallax.tex`
- Figure 2.1: `homogeneous-isotropic.tex`
- Figure 2.3: `scale-factor-vs-density-parameters.py`
- Figure 2.4: `time-vs-scale-factor.py`
- Figure 2.5: `redshift-vs-cosmological-distances.py`
- Figure 4.1: `redshift-vs-distance-modulus.py`
- Figure 4.2: `MWE-analytic-chi2.py`
- Figure 4.3, Figure 4.7: `MWE-analytic-likelihood.py`
- Figure 4.4, Figure 4.5: `MWE-likelihood.py`
- Figure 4.6: `MWE-marginalized-likelihood.py`
- Figure 4.8, Figure 4.9, Figure 4.10, Figure 4.11: `Lambda-CDM-analytic-chi2.py`
- Figure 4.12, Figure 4.13, Figure 4.14, Figure 4.15: `Lambda-CDM-analytic-likelihood.py`
- Figure 4.17, Figure 4.18, Figure 4.19, Figure 4.20, Figure 4.21: `DGP-analytic-chi2.py`

[1] Download the dataset used in this thesis: https://supernova.lbl.gov/Union/figures/SCPUnion2.1_mu_vs_z.txt
[2] Description of the full table: https://supernova.lbl.gov/Union/descriptions.html#FullTable
[3] Download full table as `.tex`-file: https://supernova.lbl.gov/Union/figures/SCPUnion2.1_AllSNe.tex
[4] GitHub repository of this thesis: https://github.com/DaHaCoder/bachelor-thesis

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SELBSTSTÄNDIGKEITSERKLÄRUNG

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst und keine anderen als die in der Arbeit angegebenen Quellen und Hilfsmittel verwendet zu haben.

München, den 01.06.2023.

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Danial Hagemann

DECLARATION OF AUTHORSHIP

Hereby, I declare that I have written this thesis by myself and that I have not used any sources or aids other than those indicated in this thesis.

Munich, the 01.06.2023.

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Danial Hagemann