Estimation of Cosmological Parameters in ΛCDM- and DGP-Model using Supernovae Type Ia Data

SUBMITTED BY

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Abschätzung kosmologischer Parameter im ΛCDM- und DGP-Modell anhand von Supernovae Typ Ia Daten

Bachelorarbeit

Universitätssternwarte München
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It was a long path to achieve finally this point in my life. And it was not an easy one. But no one said, that life is going to be easy.
I was about 12 years old, when I watched the sky the very first time with my a telescope, looking out for the most beautiful planet of our solar system. Being able to admire the ring structure of Saturn opened my eyes and my heart to explore the beauty of nature, the beauty of space.

While growing up, we try, driven by curiosity and fascination, to understand the world around us. We may seek answers to “the big questions of life”.

Where do we come from ?
Why did the universe come to existence ?
How old is the universe ?
Which mechanisms in nature guide the course of all events ?
What will happen in the future ?
Which role do I play in this unimaginable big universe ?
What is the purpose of my life ?
And, where do I want to go ?

Those are not easy questions. And that is why those questions are probably the oldest ones of humanity.
Throughout my “journey of life”, I tried to find answers to those questions. Maybe, I could find some attempts to answer a few of those in this thesis. Maybe, I will never find an answer to some of these questions.

But at least, I found an answer, what the purpose of *my* life is: to discover the beauty of nature. Going on with my journey, trying to find answers. And never giving up during this adventure - no matter, how hard and how frustrating life sometimes can be. No matter, how often I have fallen or I will fall down to the harsh ground of reality. Because it is my curiosity, my ambition and my passion that makes me standing up again to turn my eyes, my gaze into the sky.

That is the most important lesson I learned in the last years, studying physics.

The fact that I made it so far is not only through reading textbooks, doing my exercise sheets or preparing for exames. Because it is not only the effort I brought up.
I am where I am thanks to the effort of several people that came into my life. I am very glad that I had the luck to get to know these people. And therefore, this is the best occasion for me to thank them.

First of all, I want to thank my teachers, who not only taught me a lot, but also did their best to answer my difficult questions and quench my thirst for knowledge, which was probably not always easy and took a lot of patience,

Pierette Al-Korey, Ulrike Blattert, Peter Bodden, Nicole Bömecke, Jörg Bühler, Reiner Büter, Rolf Heckmann, Petra Jähnigen, Thomas Schröder Klementa, Axel Knuth, Eva Krüger, René Meyer-Brede, Angelika Nieboer, Roland Petereit, David Stephan, Volkhard Stierhof, Stefan Usée, Selma Weiß-Tümmers, Wolf Wingenfeld, Sema Yilmaz, Mike Ziegner.

In loving memory of Simon Jacob Gallo and Gamal Tadros Mishreky

NOTATION AND CONVENTIONS

List of Books on Astrophysics, Cosmology and/or General Relativity with

(+) $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$

- Scott Dodelson, Fabian Schmidt. *Modern Cosmology* [DS20]
- Matthias Bartelmann. *Das kosmologische Standardmodell* [Bar19]

(−) $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$

- Bradley W. Carroll, Dale A. Ostlie. *An Introduction to Modern Astrophysics* [CO07]
- Viatcheslav Mukhanov. *Physical Foundations of Cosmology*

ABSTRACT

In this thesis, we consider measurements of type Ia Supernovae by the *Supernova Cosmology Project* [1] (dataset “Union2.1”) to obtain best-fit values to the free parameters and constraints to both cosmological models.

[1] Supernova Cosmology Project: https://supernova.lbl.gov/Union/

CONTENTS

CHAPTER 1

INTRODUCTORY CONCEPTS OF ASTROPHYSICS

In 2011, Saul Perlmutter, Adam G. Riess and Brian P. Schmidt received the Nobel Prize in Physics *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"*. [1]
Since then, we know that the expansion of the universe is accelerating. The cause of this accelerated expansion is still unknown, but there are some theoretical models which attempt to explain this phenomenon.

The established model in recent cosmology is the **ΛCDM-model**, sometimes referred as the *standard model of cosmology.* In this model, the cause of the accelerated expansion is due to the so called *cosmological constant* Λ, which appears as a physical constant in Einstein's field equations of general relativity like the Newtonian gravitational constant G.

Another model, published in April 2000 by Gia Dvali, Gregory Gabadadze and Massimo Porrati, – the **DGP-model** – proposes a modification of Einstein's field equations by introducing a fifth dimension to the four-dimensional spacetime, so that gravity behaves equivalently to Newtonian gravity on small distances, but weakens on large scales.

Before going into details about both cosmological models, let me introduce relevant concepts in physics and astronomy from which we can relate measurable physical quantities, like the brightness of stars or their redshift, to more abstract properties of the universe like its scale factor. Based on certain assumptions of a theory, it is possible to develope models that make predictions about properties of the universe, for example, how the expansion and its evolution in time influences the relation of physical quantities.

1.1 Distance Measurement in Astronomy

Essential to astronomical observations and measurements is to determine the distance to objects in space like stars, star clusters, galaxies or even clusters of galaxies.
By looking at night into the sky, the only information perceived by our human eye are the brightness and some color in which objects (mostly stars) appear. How do we determine the distance to those objects ?

1.1.1 The Parallax Method

One method to determine the distance to an object is by using the *parallax effect.*
For this method, we assume that the observed object is almost stationary relative to earth. First, we detect the position of the measured object in the sky. After a while (for example, after a half year), the object appears at a slightly different position in the sky, since earth moved on its elliptical orbit which leads to another point of view for the observation.

[1] Press release: https://www.nobelprize.org/prizes/physics/2011/press-release/

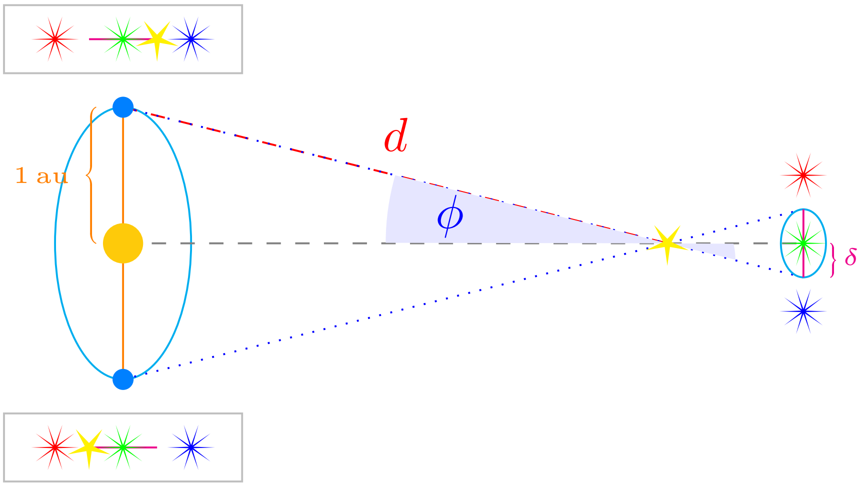


Figure 1.1: To determine the distance d between earth and the yellow star, we consider the small change of the position δ around the green star under which the yellow star appears. The difference δ relates to the angle ϕ.

By measuring the appeared difference δ of the object's position and relating this difference to the angle ϕ under which the object changed its position in the sky, we can calculate the distance d by

$$d = \frac{1\,\mathrm{au}}{\sin(\phi)} \approx \frac{1\,\mathrm{au}}{\phi}, \tag{1.1}$$

where $1\,\mathrm{au} \approx 149.6 \times 10^9\,\mathrm{m}$ ("astronomical unit"[2]) is defined as the average distance $d_{♁\odot}$ of earth to the sun.
The small-angle approximation can be considered justified, since the distance to earth nearest star (beyond of our solar system), Proxima-Centauri, is about $d_{\mathrm{P.C.}} \approx 4.2\,\mathrm{ly}$ and therefore $\phi_{\mathrm{P.C.}} \approx \arcsin\left(\frac{1\,\mathrm{au}}{4.2\,\mathrm{ly}}\right) \approx 3.723 \times 10^{-6} \ll 1$. The larger the distance to the measured object, the smaller the angle is.
Since the parallax method is one of the most important ways to measure the distance far away objects, we define the distance of a *parsec* ("parallax second") as the distance to an object, under which the parallax is $\phi = 1\,\mathrm{arcsec} = \frac{1}{3600}^\circ$:

$$1\,\mathrm{pc} := \frac{1\,\mathrm{au}}{\sin\left(\frac{1}{3600}^\circ\right)} \approx 3.26\,\mathrm{ly} \approx 3.086 \times 10^{16}\,\mathrm{m}. \tag{1.2}$$

This is the usual unit by which cosmological distances or parameters are expressed, for example the *Hubble constant* $H_0 \approx 67.66\,\frac{\mathrm{km}}{\mathrm{s}}\mathrm{Mpc}^{-1}$[3].

We should keep in mind that this method has of course boundaries and is practically useful for the local region of our galaxy, since the larger the distance to an observerd object is, the appeared change in position at sky gets smaller and is for objects with a distance on cosmological scale ($d \gtrsim 300\,\mathrm{Mpc}$) almost unnoticable.

[2] Since 2012, the astronomical unit was redefined by the IAU (International Astronomical Union) to be exactly $1\,\mathrm{au} := 149\,597\,870\,700\,\mathrm{m}$, see *Resolution B2* at the XXVIII General Assembly of IAU: https://www.iau.org/static/resolutions/IAU2012_English.pdf
[3] The value of H_0 according to the results of the *Planck Collaboration 2018*, [Col+20, Table 7]

1.1.2 The Distance Modulus

Another method to determine the distance to an object with a certain luminosity L is to measure its radiant flux F at a distance d by

$$F = \frac{L}{4\pi d^2}. \tag{1.3}$$

In general astronomical observations, it is not the radiant flux of a star that is measured. Rather, we observe *differences* of brightness or magnitude m between two objects. We define a difference in magnitude between two objects $m_1 - m_2 =: \Delta m$ so that the radiant flux F_2 of object 2 is 100 times higher than the radiant flux F_1 of object 1, when their difference in magnitude is $\Delta m = 5$, so

$$\frac{F_2}{F_1} = 100 \Leftrightarrow m_1 - m_2 = \Delta m = 5. \tag{1.4}$$

This leads us to the relation between the difference of magnitude Δm and the relation of the radiant flux of two objects

$$\frac{F_2}{F_1} = 100^{\frac{m_1 - m_2}{5}} \tag{1.5}$$

and therefore with (1.3)

$$m_1 - m_2 = \frac{5}{2}\log_{10}\left(\frac{F_2}{F_1}\right) = \frac{5}{2}\log_{10}\left(\frac{L_2}{4\pi d_2^2}\frac{4\pi d_1^2}{L_1}\right) = \frac{5}{2}\log_{10}\left(\frac{L_2}{L_1}\right) + 5\log_{10}\left(\frac{d_1}{d_2}\right). \tag{1.6}$$

So, to calculate the distance to one of both objects, for example object 2, would require us to know their *relative* magnitudes m_1 and m_2, their luminosities L_1 and L_2 and the distance d_1 to object 1.
To eliminate the need of knowing five quantities, we can define an *absolute* magnitude M as the magnitude an object would have at a distance of $d = 10\,\mathrm{pc}$. If we consider only one object, we therefore can calculate the distance to this object by

$$m - M = \underbrace{\frac{5}{2}\log_{10}\left(\frac{L}{L}\right)}_{=0} + 5\log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right) = 5\log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right). \tag{1.7}$$

We call the difference between the relative magnitude m and the absolute magnitude M of one object its *distance modulus*.
The benefit of the distance modulus is that we only have to know two quantities, m and M, to calculate the distance to an object.
Unfortunately, we can not measure the absolute magnitude M of an object that easily (since we can not fly to far away stars and measure the observed magnitude or radiant flux at a distance of $10\,\mathrm{pc}$ to them). That is why we rely on (1.6). But at least, we could reduce the amount of unknown variables by calibrating the relation (1.6) with an object which luminosity and distance is known.

One could choose to calibrate equation (1.6) by measuring the properties of the nearest star to earth – our sun. Given the magnitude $m_\odot$, the luminosity $L_\odot$ and the distance to sun $d_{♁\odot} = 1\,\mathrm{au}$, we obtain

$$m = m_\odot - \frac{5}{2}\log_{10}\left(\frac{L}{L_\odot}\right) + 5\log_{10}\left(\frac{d}{1\,\mathrm{au}}\right). \tag{1.8}$$

By this calibration, we can determine the distance to an observed object, if we could know its luminosity L and measure its relative magnitude m. Generally, the luminosity of objects like stars could varying arbitrarly. To obtain a reliable distance measurement, we are looking for objects which luminosity can be predicted very precisely – so called *standard candles*.

1.1.3 Possible Candidates for Standard Candles

Generally, there are two established candidates for standard candles in astronomy. Since we deal in this thesis with data of type Ia supernovae, the focus lies on the second paragraph of this subsection. For the sake of completeness, however, the cepheids as possible standard candles should not be unmentioned – also to emphasize, why we rely on supernovae of type Ia to determinine cosmological distances.

1.1.3.1 Cepheids

There are several types and classes of cepheids, but they all have in common that those stars obey a certain, periodic relation of luminosity and time.

Without going into details[4], the periodicity of luminosity is caused by fluctuations of temperature dependent opacity in the stars photosphere due to transitions between single- and dual-ionized Helium inside the star.
It is important to identify cepheids of the same type – cepheids that share the same physical properties, like their metalicity or the same periodicity pattern in their luminosity, to ensure that the same physical process is occuring in all cepheids of a certain type. From then on, it is possible to determine the luminosity of all cepheids of the same type by observing one cepheid, measuring its relative magnitude m and its distance d by the parallax method.

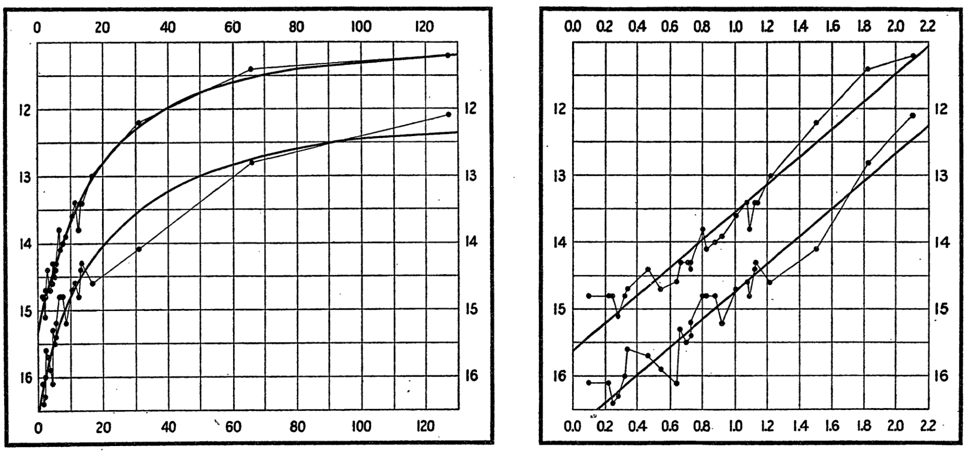


FIG. 1. FIG. 2.

Figure 1.2: First observed direct, logarithmic periodicity between time (x-axis in days) and magnitude (y-axis) of 25 stars in the Small Magellanic Cloud by Henrietta Leavitt, published 1912. Source: [LP12].

[4] For further information and more detailed explanation how stellar pulsation and its underlying κ-mechanism can be calculated, I recommend chapter 14 "*Stellar Pulsation*" in [CO07]

However, in order to use cepheids for distance determination, they must first be observed and resolved. Even with our most powerful telescopes that we have today, resolution of cepheids is only possible in our local, galactic neighborhood, for example in the Large Magellanic Cloud or the Andromeda galaxy. Therefore, the boundary to resolute cepheids in other galaxies is currently about $\sim 30\,\mathrm{Mpc}$ ([Bar19, p. 47] and [Eng13, p. 3]).
For larger scales, we need much brighter standard candles.

1.1.3.2 Supernovae of Type Ia

In general, supernovae are abrupt bursts of luminosity from massive stars, often accompanied by explosive thermonuclear reactions. While supernovae of type Ib, Ic and II occur due to an imbalance between the star's gravity, which causes its nucleus to collapse, and radiation emitted by nuclear fusion inside the star, which pushes against its photosphere, supernovae of type Ia are caused by a white dwarf, accreting a companion star and therefore increasing in mass.
If the white dwarf reaches due to accreting a companion star a mass limit over $\sim 1.4 M_\odot$ (where $M_\odot \approx 1.98 \times 10^{30}\,\mathrm{kg}$ is the mass of the sun, [Bar19, p. 48]), which is called the *Chandrasekhar limit* M_{Ch}, the nucleus of the white dwarf beocmes unstable, since the degeneracy pressure of the electrons (Pauli exclusion principle) can not resist the gravitational forces any more.
Inside the white dwarf's nucleus, a fusion of carbon and oxygen is triggered, which leads to further nuclear processes and an amount of energy release up to $\sim 10^{44}\,\mathrm{J}$ ([Mag17, p. 295], [Spa17, p. 321]). This makes supernovae of type Ia one of the most energetic – and therefore also luminous – phenomena known in nature. Supernovae of type Ia are almost as bright as their host galaxy. Since they are not only extremly luminous, but the process of star collapse is triggered after crossing a fixed limit of mass (the Chandrasekhar limit), this makes them also very unique and therefore good candidates for standard candles.
We can distinguish between supernovae of type Ia from other supernovae types by observing and analizing their spectrum.

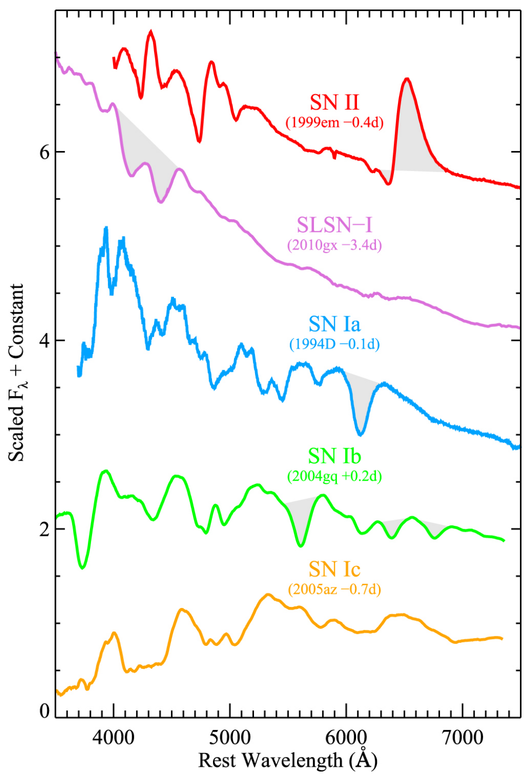


Figure 1.3: Key features in the spectra of different supernovae types are shaded in grey. While supernovae of type II have significant hydrogen lines, supernovae of type Ia and Ib are lacking of hydrogen lines. In type Ia supernovae, the Si II line (at $\sim 6150\,\text{Å}$) is very significant, while it is weak in type Ib supernovae. In type Ib supernovae, the helium lines seems to be very strong. Supernovae of type Ic do have non of the mentioned features.
Source: [Qui+18, Figure 1]

Yet there are small problems that occur here.
First, the mechanism which leads to the supernovae explosion is somewhat controversial, since there are two possible scenarios: the "single-degenerate"-model, in which the companion star is a star of the main sequence or a giant star, and the "double-degenerate"-model, in which the companion star is also a white dwarf.

In the "single-degenerate"-model, the accretion must not be too slow, since the hydrogen-rich material of the companion star could be burnt at the same rate as it is accreted, which results in no growth of mass for the white dwarf. On the other hand, the accretion must not be too high, since the accretion might stop due to the companion star's loss of mass at a high rate or being engulfed by the accreted material ([Mag17, p. 308]).

In the "double-degenrate"-model, it is assumed that one of both white dwarfs reaches the Chandrasekhar limit if they get too close ([Bar19, p. 48]), but there are also other possible scenarios that could occur (see [Mag17, p. 308/309]).
This could lead to slight variations in the luminosity behavior.

Other variations in luminosity behavior could occur due the amount of ^{56}Ni in the thermonuclear process, which decay influences the peak of the supernovae type Ia light curve ([Mag17, p. 295]). But those variations can be compensated very well since a relation between the lumninosity's decline and its peak is found so that it is possible to "normalize" or "stretch" the light curves so they obey a uniform distribution ([PS03, p. 4], [Phi93]).
The fact that the light curves of type Ia supernovae are distributed exactly the same way after some "normalization" or "stretching" shows us, that the same physical process underlies all light curves of type Ia supernovae, even if they seem to be stretched, not only due to the relation of luminositys decline and luminosity peak. We will mention later, why this property of supernovae type Ia light curves verify the expansion of the universe.

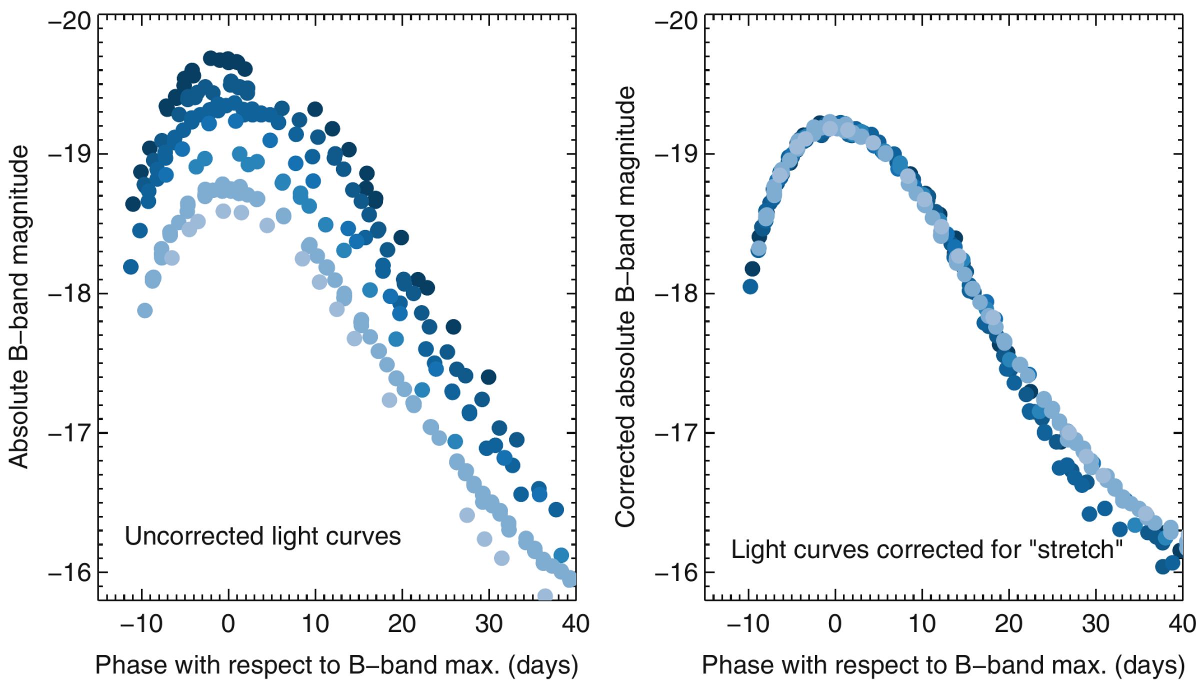


Figure 1.4: Uncorrected vs. corrected sample of supernovae type Ia light curves.
Source: [Mag17, p. 298, Figure 2]

Despite the mentioned slight variations in the luminosity spectrum, supernovae of type Ia can be considered as the best candidates for standard candles at the current state of research, not only because their light curves are very homogeneous, but they are also much brighter than cepheids and therefore also at large distances visible, which is essential for cosmological research.

1.1.4 Redshift and Hubble's Observation

The most important tool to determine distances on a cosmological scale is by observing the redshift of extragalactic objects. The redshift is a shift in the observed spectrum of light emmited by a source with wavelength λ_e to an observed wavelength λ_o and defined as

$$z := \frac{\lambda_\mathrm{o} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}} = \frac{\lambda_\mathrm{o}}{\lambda_\mathrm{e}} - 1. \tag{1.9}$$

The term "red" in "redshift" might be misleading, since z could take also values < 0 and therefore $\lambda_\mathrm{o} < \lambda_\mathrm{e}$, which implicates a shift of the light's spectrum to the blue.
This term has been established, because observations show that the spectrum of most galaxies and extragalactic objects is shifted towards red.

In the end of the 1920s, Edwin Hubble observed a certain relation between the measured redshift of the galaxies and extragalactic objects and their distance to us. If one assumes the redshift due to (relativistic) doppler effect[5], one could calculate the galaxys velocity component towards the direction of observation by

$$z = \gamma\left(1 + \frac{v}{c}\right) = \sqrt{\frac{c+v}{c-v}} - 1. \tag{1.10}$$

An observed shift of the spectrum to the red would imply, that the observed object moves away from the observer, and a shift of the spectrum to the blue implies a motion towards the observer.
Edwin Hubble applied a linear relation (see figure 1.6) between the measured distance d to the observed objects and the calculated velocity v due to the doppler effect

$$v = H_0 d, \tag{1.11}$$

which is known as the *Hubble law*. The proportionality constant H_0 has the dimension of an inverse time. It is one of the most important parameters in cosmology and its precise determination a challenge of active research (often called "Hubble tension", [Val+21]).

For small velocities ($v \ll c$), we can approximate (1.10) (first order taylor series) as $z \approx \frac{v}{c}$, which leads with (1.11) to

$$d \approx \frac{c}{H_0} z. \tag{1.12}$$

We will see later that this relation is actually a first order approximation between the *luminosity distance* d_L and the redshift z.

[5] Some raised objections against the interpretation that the observed redshift is a result of the doppler effect, but claimed that it is caused by energy loss of the photons traveling through space ("tired light"-hypothesis). We will address later, why this hypothesis can not be hold anymore in the light of tremendous evidence for the expansion model.

The correct relation between the luminosity distance d_L and the redshift z will play an essential role when estimating parameters of cosmological models.

From Hubble's law follows that the further the distance to a galaxy (or other cosmological object) is, the higher the redshift and therefore, the faster it seems to move apart from us. This observation was one of the milestones in the history of cosmology and was the first indicator, that our universe is truly expanding.

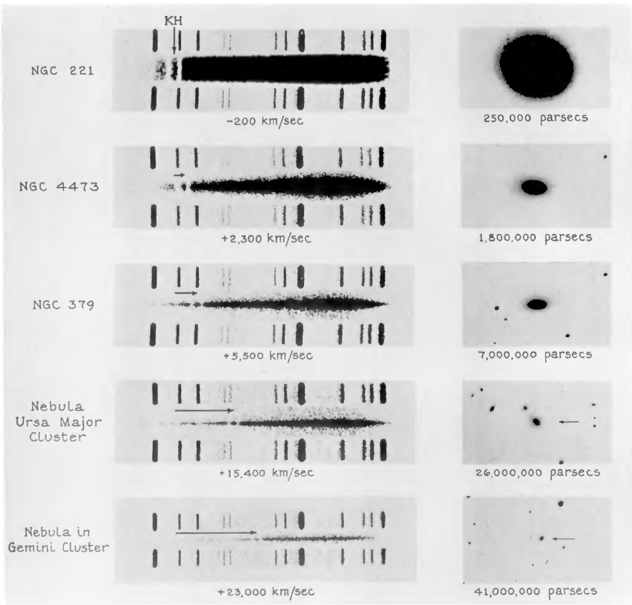


Figure 1.5: Observed redshift of H and K lines of calcium, shifted to the red, by Milton L. Humason.
Source: [Hum36, Figure "Red-shifts in the Spectra of Extra-galactic Nebulae"]

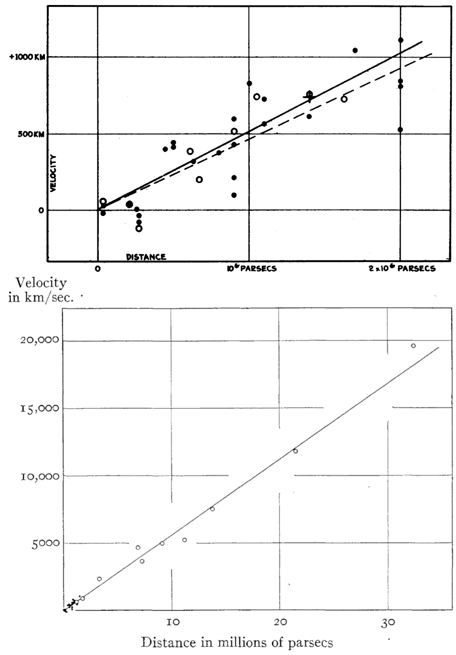


Figure 1.6: Linear regression between the distance of observed objects and the calculated velocity due to redshift.
Source: [Hub29] and [HH31, Figure 5]

1.2 The Theory of Gravity and Spacetime

How can one speak about the nature on large scales without mentioning the force that dominates in this regime ? Beginning with Newton's ideas about gravity, the modern formulation about gravity is described in the framework of Einstein's general theory of relativity, in which gravity is, strictly speaking, not anymore a force in the Newtonian sense, but rather a property of the four dimensional spacetime that interacts with matter.

For a deep understanding of general relativity, it is required to have knowledge on the mathematics of differential geometry.
Despite the fact that, as an undergraduate student, I do not have this knowledge (yet, this would be beyond the scope of this bachelor thesis.
But to motivate the basic equations of the ΛCDM-Model, which are derived from Einstein's field equations under certain assumptions that we formulate in the next chapter, I would like to mention the concept of a *metric* and give brief view on Einstein's field equations.

1.2.1 The Metric of Spacetime

Generally speaking, a metric is a function that takes two points in space and returns a distance. For example, let $\mathbb{E}^2$ be the Euclidean, two dimensional space, then

$$d(\cdot,\cdot) : \mathbb{E}^2 \times \mathbb{E}^2 \to \mathbb{R}, (\boldsymbol{p}_1, \boldsymbol{p}_2) \mapsto d(\boldsymbol{p}_1, \boldsymbol{p}_2) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{1.13}$$

would be a function that gives us the distance between two points $\boldsymbol{p}_1 = (x_1, y_1)^\intercal$ and $\boldsymbol{p}_2 = (x_2, y_2)^\intercal$. This distance function d would return the pythagorean distance in cartesian coordinates that we are familiar with.
Let us define $x := x_1 - x_2$, $y := y_2 - y_2$ and $s := d(\boldsymbol{p}_1, \boldsymbol{p}_2)$ so we could write for the (infinitesimal) distance

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \,. \tag{1.14}$$

Now, let us switch to polar coordinates so that $\boldsymbol{p}_1 := (r_1, \phi_1)^\intercal$ and $\boldsymbol{p}_2 := (r_2, \phi_2)^\intercal$. With the given distance function d, we would obtain

$$d(\boldsymbol{p}_1, \boldsymbol{p}_2) = \sqrt{(r_1 - r_2)^2 + (\phi_1 - \phi_2)^2}. \tag{1.15}$$

But this is *not* the same distance as in cartesian coordinates. The distance that would correspond to the same distance as in cartesian coordinates (1.14), is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 \tag{1.16}$$

with $r := r_1 - r_2$, $\phi := \phi_1 - \phi_2$.
We have to introduce the *metric tensor* (in most applications of physics a 3×3- or 4×4-matrix) that garantuees as the invariance of the distance function d under coordinate transformation. We define

$$g_{ij} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{in cart.} \\ \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} & \text{in polar} \end{cases} \qquad x^1 := \begin{cases} x & \text{in cart.} \\ r & \text{in polar} \end{cases} \qquad x^2 := \begin{cases} y & \text{in cart.} \\ \phi & \text{in polar} \end{cases} \tag{1.17}$$

so that we can express the invariant distance between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ through

$$\mathrm{d}s^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{11}\left(\mathrm{d}x^1\right)^2 + \underbrace{g_{12}}_{=0}\,\mathrm{d}x^1\,\mathrm{d}x^2 + \underbrace{g_{21}}_{=0}\,\mathrm{d}x^2\,\mathrm{d}x^1 + g_{22}\left(\mathrm{d}x^2\right)^2 \tag{1.18}$$

$$= g_{11}\left(\mathrm{d}x^1\right)^2 + g_{22}\left(\mathrm{d}x^2\right)^2 \overset{\text{cart.}}{=} \mathrm{d}x^2 + \mathrm{d}y^2 \overset{\text{polar}}{=} \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 \,. \tag{1.19}$$

In the framework of relativity, we express the distance (called "world line") between to *events* $\boldsymbol{p}_1 := (ct_1, x_1, y_1, z_1)^\intercal$ and $\boldsymbol{p}_2 := (ct_2, x_2, y_2, z_2)^\intercal$ in four dimensional spacetime as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{1.20}$$

with the implicit sum over double occuring indices $\mu, \nu \in \{0, 1, 2, 3\}$. In flat Minkowski-spacetime of special relativity for example, we have

$$g_{\mu\nu} = \eta_{\mu\nu} := \mathrm{diag}(-1, 1, 1, 1) \tag{1.21}$$

and therefore for distances in spacetime

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -(c\,\mathrm{d}t)^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1.22}$$

1.2.2 Einstein's Field Equations

Similar to the derivation of the Euler-Lagrange equations in classical mechanics (or classical field theory) by formulating an action $S[\boldsymbol{q}(t)]$ (or $S[\phi(x)]$) and find the path $\boldsymbol{q}(t)$ (or field $\phi(x)$) that extremizes the action ($\delta S = 0$), Einstein's field equations can be derived[6] from the Einstein-Hilbert action given by

$$S_{\mathrm{EH}}[g_{\mu\nu}] = \int \mathrm{d}^4x \sqrt{-\det(g_{\mu\nu})}\left[\frac{c^4}{16\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}}\right], \tag{1.23}$$

where d^4x is the four dimensional spacetime-volume element, $g_{\mu\nu}$ the metric tensor, c the speed of light (in vacuum), G the Newtonian gravitational constant, R the Ricci scalar, Λ the cosmological constant and $\mathcal{L}_{\mathrm{M}}$ the lagrange density of matter fields.
With the action principle, the variation $\delta S[g_{\mu\nu}]$ of the Einstein-Hilbert action with respect to the metric leads to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.24}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor and $T_{\mu\nu}$ the energy-momentum tensor.
Without getting too much into details about the single components of this equation, we keep in mind that the left hand side of (1.24) describes how spacetime behaves, while the right hand side of (1.24), how matter behaves.

1.2.2.1 The Cosmological Constant Λ

Originally, Einstein formulated his field equations without the cosmological constant Λ. Since Einstein believed in a static universe and found that his equations can not hold for a static universe (it would have collapsed due to the gravity of matter), he introduced the Λ term that acts repulsive towards the attraction of gravity, so that a static universe as he proposed would be possible.

[6] For a proper and detailed derivation, I recommend subsection 4.3 "*Lagrangian Formulation*" in [Car19, p. 159]

CHAPTER 2

THE STANDARD MODEL OF COSMOLOGY

The standard model of cosmology describes an expanding universe. Its basic assumptions are, that the cosmological principle is valid and that general relativity can describe nature on cosmological scales.

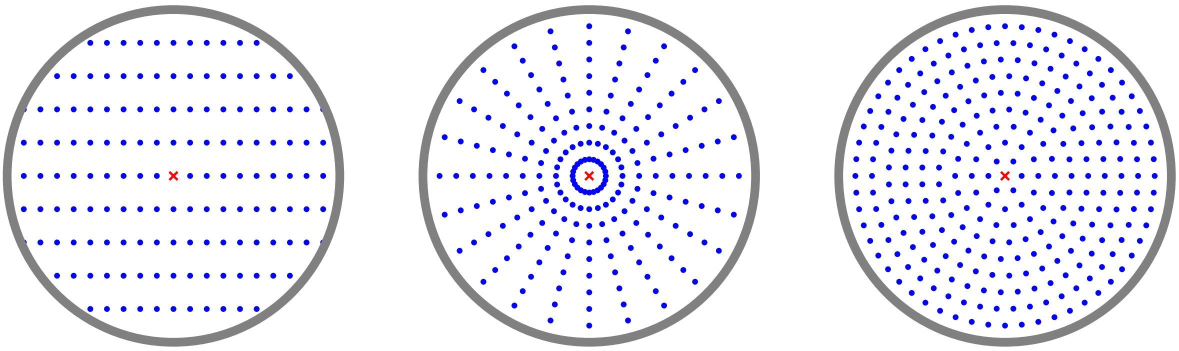
The dynamics of the expansion is guided by the cosmological constant Λ in Einstein's field equations, since this term was initally introduced to act repulsive towards gravity, so that the universe does not collapse due to the gravity of matter it contains. The drive behind the *accelerated* expansion was given the name "dark energy", which refers in the ΛCDM-model to the cosmological constant Λ.
Nevertheless, it should be mentioned that the term "dark energy" used in cosmology is not restricted to the cosmological constant. Its meaning depends on the particular cosmological model that is considered (see subsection 2.4.6 "*Dark energy*" in [DS20, p. 50] and [FTH08]).

The "CDM“ in "ΛCDM" stands for "cold dark matter". There are several indications that lead to the assumption of the existence of dark matter. The first and probably most dominant indication is by observing the velocity of stars while surrounding their galactic center. While, according to classical mechanics, the rotation velocity of an object in a gravitational field should decrease the further the object is to the center of mass, it is observed that the rotation curve (the distance-velocity-relation, see [Sch06, p. 64]) in spiral galaxies does not decrease, which leads to the assumption that there has to be some mass in galactic halos that holds the galactic disk due to its gravity together (otherwise, galaxies should fall apart due to the high velocity of the stars that surround the galactic nucleus).
Apart from the absence of other interactions than gravitational, dark matter behaves according to the ΛCDM-model like ordinary (brayonic) matter. While theories on "hot" dark matter assume small masses for the particles that dark matter consists of, so that they behave more like relativistic, hot gas, the ΛCDM-model assumes "cold" dark matter, which means in this context heavy particles that behave more classically.

For the purpose of this thesis, the distinction between ordinary, baryonic matter and dark matter can be neglected.

2.1 The Cosmological Principle

The cosmological principle states that, assuming the universe does not prefer any direction in space, i.e. the universe is isotropic for *every* observer, the larger scales we consider, the smaller the variance and thus the more homogeneous the matter distribution appears.
A mathematically more rigor definition of the cosmological principle can be found in [Bar19, p. 5] and [MTW17, p. 713/714].



(a) (b) (c)

Figure 2.1: The cosmological principle visualized.
The matter distribution (a) is homogeneous, but not isotropic.
The matter distribution (b) is for an observer at the center isotropic, but not homogeneous.
The matter distribution (c) is for an observer at the center isotropic *and* homogeneous.

Obviously, the universe is not perfectly homogeneous (especially not on small scales, for example our galactic neighbourhood). But keep in mind that the cosmological principle does *not* postulate a perfect homogeneous and isotropic universe.
The cosmological principle is statement about the *statistics* of matter distribution.
The *larger* the scales an observer considers at *any* point in the universe, the *more* homogeneous the matter distribution appears.

The strongest evidence at current state that supports the cosmological principle is the measurement of the cosmic microwave background, which radiation spectrum follows the planck distribution of a black body radiator with $T \approx 2.736\,\mathrm{K}$ as most accurate as ever measured in nature ([Pee93, p. 131/132] and [Whi99]).

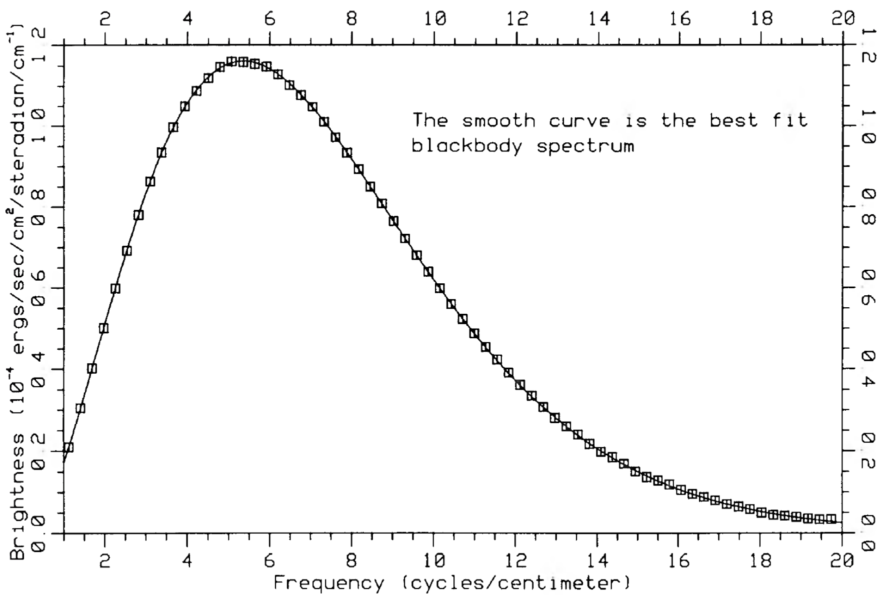


Figure 2.2: The spectrum of the microwave background radiation, measured by "COBE" (Cosmic Background Explorer) in 1990. For this exploration, John C. Mather and George F. Smoot were awarded in 2006 with the Nobel Prize in Physics.
Source: [Mat+90, Figure 2]

2.1.1 FLRW-Metric

Motivated by the cosmological principle, there are mathematically three possible geometries for the four-dimensional spacetime that satisfy spatial homogeneity and isotropy of the metric:

- a four-dimensional sphere $\mathcal{S}^3$ with positive curvature,
- a four-dimensional flat space $\mathcal{F}^3$ with zero curvature,
- a four-dimensional hyperboloid $\mathcal{H}^3$ with negative curvature.

We can parameterize these geometries by

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \overset{\mathcal{S}^3}{=} \begin{pmatrix} \sin(\psi)\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta)\sin(\phi) \\ \sin(\psi)\cos(\theta) \\ \cos(\psi) \end{pmatrix} \overset{\mathcal{F}^3}{=} \begin{pmatrix} \psi\sin(\theta)\cos(\phi) \\ \psi\sin(\theta)\sin(\phi) \\ \psi\cos(\theta) \\ 1 \end{pmatrix} \overset{\mathcal{H}^3}{=} \begin{pmatrix} \sinh(\psi)\sin(\theta)\cos(\phi) \\ \sinh(\psi)\sin(\theta)\sin(\phi) \\ \sinh(\psi)\cos(\theta) \\ \cosh(\psi) \end{pmatrix}$$
$$= \begin{pmatrix} r\sin(\theta)\cos(\phi) \\ r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \\ \frac{\partial r}{\partial\psi} \end{pmatrix} \quad \text{with} \quad r = \begin{cases} \sin(\psi) & \text{for} \quad \mathcal{S}^3 \\ \psi & \text{for} \quad \mathcal{F}^3 \\ \sinh(\psi) & \text{for} \quad \mathcal{H}^3 \end{cases}. \tag{2.1}$$

The metric for the spatial line element $\mathrm{d}\ell$ is

$$\mathrm{d}\ell^2 = \begin{cases} \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 & \text{for} \quad \mathcal{S}^3 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & \text{for} \quad \mathcal{F}^3 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}w^2 & \text{for} \quad \mathcal{H}^3 \end{cases}. \tag{2.2}$$

With the parameterization in equation (2.1), we can write for the infinitesimals

$$\begin{aligned} \mathrm{d}x &= \sin(\theta)\cos(\phi)\,\mathrm{d}r + r\cos(\theta)\cos(\phi)\,\mathrm{d}\theta - r\sin(\theta)\sin(\phi)\,\mathrm{d}\phi\,, \\ \mathrm{d}y &= \sin(\theta)\sin(\phi)\,\mathrm{d}r + r\cos(\theta)\sin(\phi)\,\mathrm{d}\theta + r\sin(\theta)\cos(\phi)\,\mathrm{d}\phi\,, \\ \mathrm{d}z &= \cos(\theta)\,\mathrm{d}r - r\sin(\theta)\,\mathrm{d}\theta\,, \\ \mathrm{d}w &= \frac{\partial w}{\partial r}\,\mathrm{d}r = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial\psi}\right)\mathrm{d}r\,. \end{aligned}$$

The components $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ lead to

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\underbrace{\left[\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\phi^2\right]}_{=:\mathrm{d}\Omega^2} = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,,$$

where $\mathrm{d}\Omega$ is the infinitesimal spatial angle.
Let us have a closer look on $\mathrm{d}w$. We obtain

$$\mathrm{d}w = \mathrm{d}r \begin{cases} \frac{\partial}{\partial r}\cos(\psi) = \frac{\partial}{\partial r}\cos(\arcsin(r)) = -\frac{r}{\sqrt{1-r^2}} & \text{for} \quad \mathcal{S}^3 \\ \frac{\partial}{\partial r}1 = 0 & \text{for} \quad \mathcal{F}^3 \\ \frac{\partial}{\partial r}\cosh(\psi) = \frac{\partial}{\partial r}\cosh(\operatorname{arsinh}(r)) = \frac{r}{\sqrt{1+r^2}} & \text{for} \quad \mathcal{H}^3 \end{cases}, \tag{2.3}$$

and hence for the spatial component of the metric (2.2)

$$\mathrm{d}\ell^2 = \begin{cases} \frac{1}{1-r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{S}^3 \\ \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{F}^3 \\ \frac{1}{1+r^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 & \text{for} \quad \mathcal{H}^3 \end{cases}. \tag{2.4}$$

If we define

$$k := \begin{cases} 1 & \text{for} \quad \mathcal{S}^3 \\ 0 & \text{for} \quad \mathcal{F}^3 \\ -1 & \text{for} \quad \mathcal{H}^3 \end{cases}, \tag{2.5}$$

we can write equation (2.4) in compact notation as

$$\mathrm{d}\ell^2 = \frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,. \tag{2.6}$$

Since the cosmological model describes an expanding universe, we introduce the scale factor $a(t)$, by which the spatial line element $\mathrm{d}\ell$ is scaled at time t.
Therefore, we finally obtain the Friedmann–Lemaître–Robertson–Walker metric of spacetime

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\ell^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{1}{1-kr^2}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\right]. \tag{2.7}$$

2.2 Friedmann Equations

Applying the FLRW-metric to Einstein's field equations leads to the fundamental equations – the Friedmann equations – that describe the dynamics of the expansion of the universe by the scale factor $a(t)$ with ([Bar19, p. 11])

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{2.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}. \tag{2.9}$$

Those equations contain the energy density $\rho(t) = \rho_{\mathrm{m}}(t) + \rho_{\mathrm{r}}(t)$ of matter and radiation, the curvature parameter k of the FLRW-metric, the pressure p that matter and radiation exerts as a perfect fluid and the cosmological constant Λ.

Taking the derivative of equation (2.8) with respect to time and inserting into equation (2.9) leads to the differential equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{2.10}$$

We define the Hubble parameter $H(t) := \dfrac{\dot{a}}{a}$ and $w := \dfrac{p}{\rho c^2}$, which leads with (2.10) to

$$\dot{\rho} + 3H\rho(1+w) = 0. \tag{2.11}$$

This equation (2.11) is sometimes called the continuity equation of cosmology. We obtain by integration the relation between density $\rho(t)$ and scale factor $a(t)$

$$\rho(a) = \rho_0 a^{-3(1+w)}(t) \tag{2.12}$$

with $\rho_0 := \rho(a(t_0))$.

2.2.1 Density Parameters

We are now interested in how the density components of the universe change as a function of the expansion of the universe scaled by $a(t)$.

In the case of matter at rest, the density of matter ρ_m shrinks by $a^3(t)$ when the volume that contains the matter is scaled by $a(t)$, so

$$\rho_\mathrm{m}(a) = \frac{E}{V(a)} = \frac{E}{V_0 a^3(t)} = \rho_{\mathrm{m},0} a^{-3}(t). \tag{2.13}$$

With comparison to equation (2.12) follows $w_\mathrm{m} = 0$.
In the case of radiation, we have to account that its wavelength λ is scaled by the factor $a(t)$, so $\lambda(a) = \lambda_0 a(t)$, which leads with $E(\lambda) = hc\frac{1}{\lambda}$ to the energy density

$$\rho_\mathrm{r}(a) = \frac{E(a)}{V(a)} = \frac{1}{V_0 a^3(t)} \frac{hc}{\lambda_0 a(t)} = \frac{hc}{V_0 \lambda_0} \frac{1}{a^4(t)} = \rho_{\mathrm{r},0} a^{-4}(t). \tag{2.14}$$

With comparison to equation (2.12) follows $w_\mathrm{r} = \frac{1}{3}$.
Now, let es rewrite the cosmological constant Λ by defining

$$\rho_\Lambda := \frac{\Lambda c^2}{8\pi G} \tag{2.15}$$

as the density that relates to the cosmological constant. As the name might implicate, this density remains constant as the universe expands and is therefore indipendent of the scale factor $a(t)$. This interpretation of Λ leads to the assumption, that ρ_Λ can be viewed as the energy density of vacuum. As vacuum increases exact the same way as space expands, its density remains constant, so

$$\rho_\Lambda(a) = \rho_{\Lambda,0}. \tag{2.16}$$

With comparison to equation (2.12) follows $w_\Lambda = -1$.
Our aim is now to write equation (2.8) as a sum of densities. We therefore define a “density” that would relate to the spatial curvature k,

$$\rho_k := -\frac{3}{8\pi G} k c^2 a^{-2}(t). \tag{2.17}$$

Further, we define the critical density ρ_cr as the density the universe would have if it is flat ($k = 0$) and has no component of Λ at the time t_0, so with $H_0 := H(t_0)$

$$\rho_\mathrm{cr} := \frac{3H_0^2}{8\pi G}. \tag{2.18}$$

Finally, we define

$$\Omega_i := \frac{\rho_i}{\rho_\mathrm{cr}} \tag{2.19}$$

where i ist the label for the density type. We can then write equation (2.8) as

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}\rho_{\mathrm{r+m}} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \\ &= \frac{8\pi G}{3}\big(\rho_\mathrm{r} + \rho_\mathrm{m} + \rho_k + \rho_\Lambda\big) \\ &= H_0^2 \frac{1}{\rho_\mathrm{cr}}\big(\rho_\mathrm{r} + \rho_\mathrm{m} + \rho_k + \rho_\Lambda\big) \\ &= H_0^2 \sum_{i \in \{\mathrm{r},\mathrm{m},k,\Lambda\}} \Omega_i. \end{aligned} \tag{2.20}$$

We clearly see, that the sum of all normalized density parameters $\Omega_{i,0}$ at time t_0 lead to

$$1 = \Omega_{\mathrm{r},0} + \Omega_{\mathrm{m},0} + \Omega_{k,0} + \Omega_{\Lambda,0}. \tag{2.21}$$

With the relations (2.13), (2.14), (2.16), (2.17) between the densities and the scale factor, we obtain

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}\Big(\rho_\mathrm{r} + \rho_\mathrm{m} + \rho_k + \rho_\Lambda\Big) \\ &= H_0^2 \frac{1}{\rho_\mathrm{cr}}\Big(\rho_{\mathrm{r},0} a^{-4}(t) + \rho_{\mathrm{m},0} a^{-3}(t) + \rho_{\mathrm{k},0} a^{-2}(t) + \rho_{\Lambda,0}\Big) \\ &= H_0^2 \Big(\Omega_{\mathrm{r},0} a^{-4}(t) + \Omega_{\mathrm{m},0} a^{-3}(t) + \Omega_{k,0} a^{-2}(t) + \Omega_{\Lambda,0}\Big) \\ &=: H_0^2 E^2(a), \end{aligned}$$

and we call

$$E(a) := \sqrt{\Omega_{\mathrm{r},0} a^{-4}(t) + \Omega_{\mathrm{m},0} a^{-3}(t) + \Omega_{k,0} a^{-2}(t) + \Omega_{\Lambda,0}} \tag{2.22}$$

the expansion function.

With the obtained values by the *Planck Collaboration* (see [Col+20, Table 7]), $\Omega_{\mathrm{m},0} \approx 0.3111 \pm 0.0056$, $\Omega_{\mathrm{r},0} = \frac{1}{1+z_\mathrm{eq}}\Omega_{\mathrm{m},0} \approx 9.812 \times 10^{-5}$ (at $z_\mathrm{eq} = 3387 \pm 21$) and $\Omega_{\Lambda,0} \approx 0.6889 \pm 0.0056$, we see that the radiation component was only relevant in the first $\sim 380\,000\,\mathrm{yr}$ of the universe until matter and radiation decoupled.

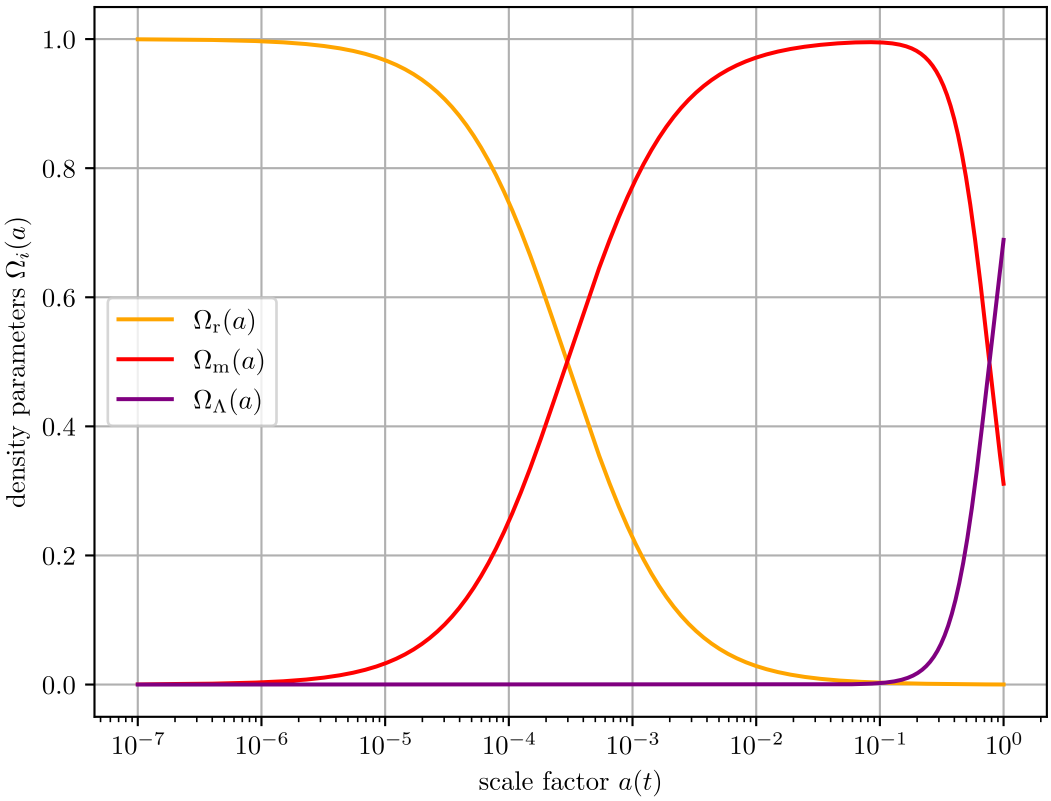


Figure 2.3: The normalized density parameters of radiation, matter and Λ in dependence of the scale factor $a(t)$ on logarithmic axis.

Since the radiation does not contribute significantly to the expansion of the universe, we assume for the rest of this thesis $\Omega_{\mathrm{r},0} = 0$.
Further, the universe seems to be almost flat ([Col+20, Table 7]) with a measured value of $\Omega_{k,0} \approx 0.0007 \pm 0.0019$.
Let us have a look, how $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ contribute to the expansion of a flat universe.

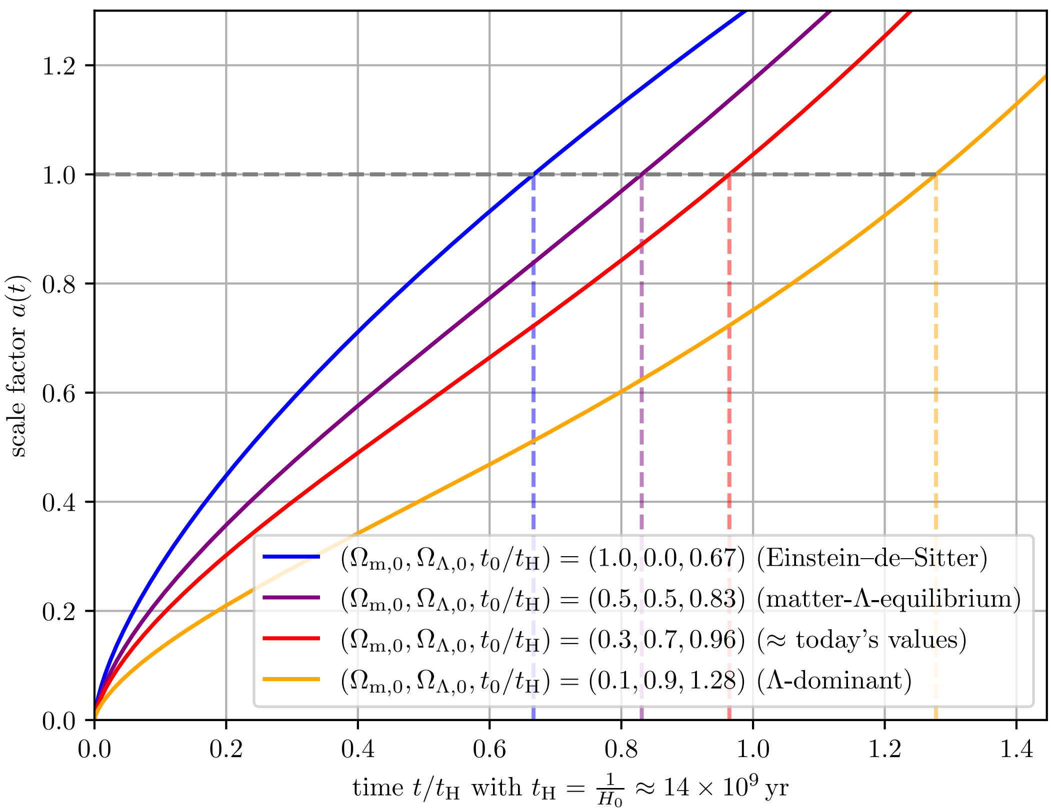


Figure 2.4: Contribution of $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ to the expansion in a flat universe ($k = 0$) without radiation $\Omega_{\mathrm{r},0} = 0$ for several cosmologies. $t_\mathrm{H} = \frac{1}{H_0} \approx 14 \times 10^9\,\mathrm{yr}$ is the Hubble time.

We notice from figure 2.4 that the higher the value of $\Omega_{\Lambda,0}$, the more it contributes to an accelerated expansion.

We define $a(t_0) := 1$ as the scale factor at the time $t_0 =: t_\mathrm{today}$.
With $\dfrac{\mathrm{d}a}{\mathrm{d}t} = H_0 a E(a)$, the age of the universe can be calculated by

$$t_0 = \frac{1}{H_0} \int_0^{a(t_0)} \mathrm{d}a\, \frac{1}{aE(a)} = \frac{1}{H_0} \int_0^1 \mathrm{d}a\, \frac{1}{\sqrt{\Omega_{\mathrm{r},0} a^{-2} + \Omega_{\mathrm{m},0} a^{-1} + \Omega_{k,0} + \Omega_{\Lambda,0} a^2}}. \tag{2.23}$$

For a flat universe without considering radiation ($\Omega_{\mathrm{r},0} = 0$, $\Omega_{k,0} = 0$), the expression (2.23) can be calculated analytically. With $\Omega_{\mathrm{m},0} = 0.3$ and therefore $\Omega_{\Lambda,0} = 1 - \Omega_{\mathrm{m},0} = 0.7$, we obtain

$$t_0 = \frac{2}{3H_0\sqrt{1 - \Omega_{\mathrm{m},0}}} \operatorname{arsinh}\left(\sqrt{\frac{1 - \Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0}}} a^{\frac{3}{2}}(t_0)\right) \approx 13.8 \times 10^9\,\mathrm{yr}. \tag{2.24}$$

2.3 Measures of Distance

Given the equations the standard model provides, we can derive measures of cosmological distances.
As already mentioned, the wavelength of light that travels to an observer is being redshifted due to the expansion of the universe. The relation between the scale factor and the redshift is given by (see [Bar19, p. 9])

$$a(z) = \frac{1}{1+z}. \tag{2.25}$$

For the path that light travels between two points (events) in spacetime (also called “lightlike geodesics”), we have $\mathrm{d}s = 0$, which leads with the FLRW-metric (see equation (2.7)) and isotropy ($\mathrm{d}\Omega = 0$) to

$$-\frac{c}{a(t)}\,\mathrm{d}t = \frac{1}{\sqrt{1-kr^2}}\,\mathrm{d}r\,. \tag{2.26}$$

For the left-hand-side of equation (2.26), we can write

$$-\frac{c}{a(t)}\,\mathrm{d}t = -\frac{c}{a}\frac{1}{\dot{a}}\,\mathrm{d}a = -\frac{c}{H_0}\frac{1}{a^2E(a)}\,\mathrm{d}a \tag{2.27}$$

and with $\mathrm{d}z = -\dfrac{1}{a^2}\,\mathrm{d}a$,

$$-\frac{c}{a(t)}\,\mathrm{d}t = -\frac{c}{H_0}\frac{1}{a^2E(a)}\,\mathrm{d}a = \frac{c}{H_0}\frac{1}{E(z)}\,\mathrm{d}z\,. \tag{2.28}$$

We define $d_\mathrm{H} := \dfrac{c}{H_0}$ as the Hubble distance.
For the right-hand-side of (2.26), we obtain for the integration

$$\int_0^{d_\mathrm{C}} \frac{1}{\sqrt{1-kr^2}}\,\mathrm{d}r =: S_k^{-1}(d_\mathrm{C}) = \begin{cases} \frac{1}{\sqrt{k}}\operatorname{arcsin}\left(\sqrt{k}d_\mathrm{C}\right) & \text{for} \quad k>0 \\ d_\mathrm{C} & \text{for} \quad k=0 \\ \frac{1}{\sqrt{|k|}}\operatorname{arsinh}\left(\sqrt{|k|}d_\mathrm{C}\right) & \text{for} \quad k<0 \end{cases}. \tag{2.29}$$

The substition $-k = \dfrac{1}{d_\mathrm{H}^2}\Omega_{k,0}$ (see (2.17) and (2.18)) leads to

$$S_k^{-1}\left(\tfrac{d_\mathrm{C}}{d_\mathrm{H}}\right) = d_\mathrm{H}\cdot\begin{cases} \frac{1}{\sqrt{|\Omega_{k,0}|}}\operatorname{arcsin}\left(\sqrt{|\Omega_{k,0}|}\frac{d_\mathrm{C}}{d_\mathrm{H}}\right) & \text{for} \quad \Omega_{k,0}<0 \\ \frac{d_\mathrm{C}}{d_\mathrm{H}} & \text{for} \quad \Omega_{k,0}=0 \\ \frac{1}{\sqrt{\Omega_{k,0}}}\operatorname{arsinh}\left(\sqrt{\Omega_{k,0}}\frac{d_\mathrm{C}}{d_\mathrm{H}}\right) & \text{for} \quad \Omega_{k,0}>0 \end{cases}. \tag{2.30}$$

By integration of (2.28) follows finally[1]

$$d_\mathrm{C} = S_k(I) = d_\mathrm{H}\cdot\begin{cases} \frac{1}{\sqrt{|\Omega_{k,0}|}}\sin\left(\sqrt{|\Omega_{k,0}|}I\right) & \text{for} \quad \Omega_{k,0}<0 \\ I & \text{for} \quad \Omega_{k,0}=0 \\ \frac{1}{\sqrt{\Omega_{k,0}}}\sinh\left(\sqrt{\Omega_{k,0}}I\right) & \text{for} \quad \Omega_{k,0}>0 \end{cases} \tag{2.31}$$

[1] I have explitly performed this derivation, since equations (2.31) and (2.32) are essential for the evaluation of supernovae data and implementation in the source codes in this thesis.

with

$$I := \int_0^z \mathrm{d}z' \, \frac{1}{E(z')}. \tag{2.32}$$

We call d_C the *comoving distance*. We define

$$d_\mathrm{A} := \frac{1}{1+z} d_\mathrm{C} \tag{2.33}$$

as the *angular diameter distance* and

$$d_\mathrm{L} := (1+z) d_\mathrm{C} \tag{2.34}$$

as the *luminosity distance*.

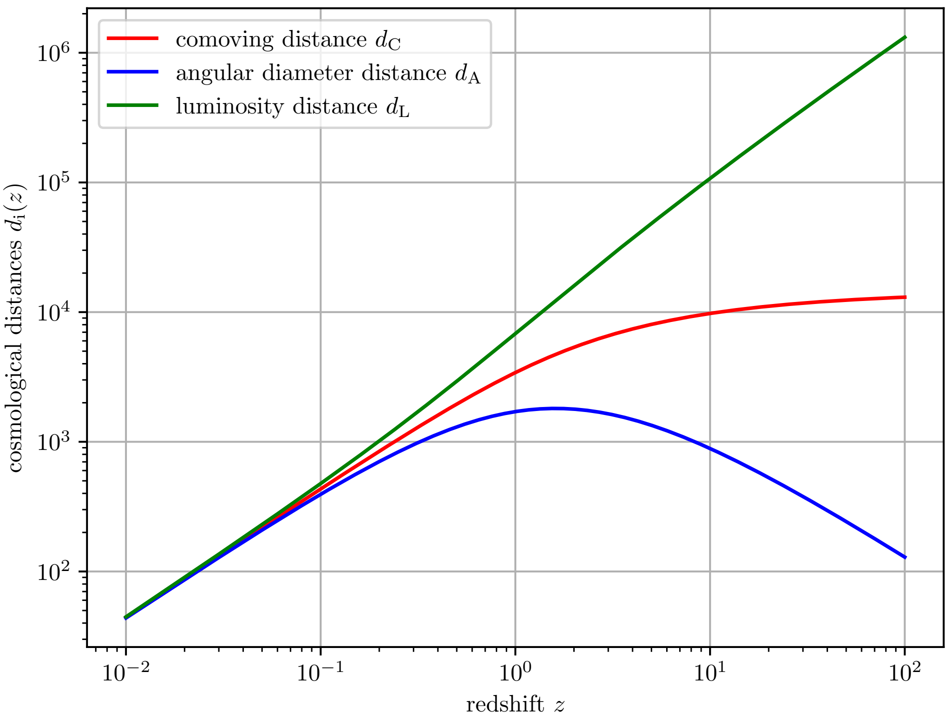


Figure 2.5: Comoving distance d_C, angular diameter distance d_A and luminosity distance d_L with respect to redshift z (logarithmic axis) for the ΛCDM-model with $\Omega_{\mathrm{m},0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$ and $H_0 = 67.66 \, \frac{\mathrm{km}}{\mathrm{s}} \mathrm{Mpc}^{-1}$.

We can clearly see in figure 2.5 the linear approximation for low redshifts, which explains the linear relation (1.12) that Edwin Hubble observed. Further, we see that those distances require observations at high redshift to distinguish them from each other.

The fact of the observed relation $\frac{d_\mathrm{L}}{d_\mathrm{A}} = (1+z)^2$ ([Wei08, p. 58]) and the mentioned slightly broadened light curve of supernovae type Ia due to the slowed supernovae explosion by the factor of $1+z$ ([Gol+01, p. 10/11]) shows, that the observed redshift is due to an expansion[2] of space, which supports the ΛCDM-model.

For this thesis, the most relevant distance is the luminosity distance d_L.

[2] Therefore, interpretations of observed redshift due to loss of energy by the received light in a static universe ("tired light"-theory) can be considered as ruled out.

CHAPTER 3

THE DVALI–GABADADZE–PORRATI-MODEL

The model proposed by Gia Dvali, Gregory Gabadadze and Massimo Porrati ([DGP00]) is an alternative cosmological model, in which our four-dimensional universe is embedded as a brane in a five-dimensional Minkowski spacetime.
The authors introduce a fourth spatial dimension y. While the electromagnetic, the weak and the strong nuclear force are limited to our four-dimensional world, gravity acts also onto the postulated extra spatial dimension.
The scale to which gravity acts familiar as the theory of general relativity predicts, but mimics dark energy as it acts onto the extra dimension, is introduced as the crossover scale r_c.
From the modified Einstein–Hilbert-action given by

$$S_\mathrm{DGP} = \frac{1}{r_\mathrm{c}} \frac{c^4}{8\pi G} \int \mathrm{d}^4x \, \mathrm{d}y \sqrt{\det\left(g^{(5)}_{AB}\right)} \mathcal{R} + \int \mathrm{d}^4x \sqrt{\det(g_{\mu\nu})} \left(\frac{c^4}{8\pi G} R + \mathcal{L}_\mathrm{SM} \right), \tag{3.1}$$

where $g^{(5)}_{AB}$ is the metric with $A, B \in \{0,1,2,3,4\}$ and $\mathcal{R}$ the ricci scalar on five-dimensional spacetime, follow the modified Einstein field equations

$$\frac{1}{r_\mathrm{c}} \mathcal{G}_{AB} + \delta(y) \delta^\mu_A \delta^\nu_B \left(G_{\mu\nu} - \frac{8\pi G}{c^4} T_{\mu\nu} \right), \tag{3.2}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor (left-hand-side of (1.24) without cosmological constant Λ) and $\mathcal{G}_{AB}$ its five-dimensional analogon. An ansatz as proposed in [DT03], the metric

$$\mathrm{d}s^2_{(5)} = f(y, H) \, \mathrm{d}s^2 - \mathrm{d}y^2 \,, \tag{3.3}$$

where $\mathrm{d}s^2$ is the four-dimensional maximally-symmetric FLRW-metric, H the four-dimensional Hubble parameter and $f(y, H)$ the so called warp-factor, leads to the modified Friedmann equation

$$H^2 \pm \frac{c}{r_\mathrm{c}} H = \frac{8\pi G}{3} \rho. \tag{3.4}$$

The positive sign accounts to a decelerated expansion, while a negative sign corresponds to an accelerated expansion. Thus, we consider in the following the case in which the sign in (3.4) is negative.

We are now introducing the parameter $\alpha \in \mathbb{R}$ which should interpolate between the DGP-model for $\alpha = 0$ and the ΛCDM-model in the case of neglecting radiation in a flat universe ($\Omega_{\mathrm{r},0} = 0, \Omega_{k,0} = 0$) for $\alpha = 1$. Therefore, we can rewrite (3.4) in the accelerated case as

$$H^2 - \left(\frac{c}{r_\mathrm{c}} \right)^{2-\alpha} H^\alpha = \frac{8\pi G}{3} \rho \tag{3.5}$$

and can conclude with equation (2.18), equation (2.13) and the crossover scale expressed by ([DT03, p. 3])

$$r_{\mathrm{c}} = \frac{c}{H_0}(1 - \Omega_{\mathrm{m},0})^{\frac{1}{\alpha-2}}, \tag{3.6}$$

the equation

$$E^2(z) - (1 - \Omega_{\mathrm{m},0})E^{\alpha}(z) - \Omega_{\mathrm{m},0}(1+z)^3 = 0, \tag{3.7}$$

where $E(z) = \frac{H(z)}{H_0}$ is the expansion function.

The DGP-model refrains from introducing a cosmological constant. The free parameters of the DGP-model are therefore $\Omega_{\mathrm{m},0}$ and α. We consider in this thesis the DGP-model of a flat universe. The definition of the luminosity distance d_{L} remains the same as in the ΛCDM-model (see equation (2.34)).
Since a deeper understanding of the mathematical formalism of the DGP-model and a detailed description of its phenomenological implications would go beyond the scope of this thesis, I recommend for further readings [Lue06].

CHAPTER 4

PARAMETER ESTIMATION

Let us now compare the ΛCDM- and DGP-model. To do this, we will consider the "Union2.1" SN Ia compilation and determine best-fit values for the parameter pairs $(\Omega_{\mathrm{m},0},\ \Omega_{\Lambda,0})$ in ΛCDM- and $(\Omega_{\mathrm{m},0},\ \alpha)$ in DGP-model by computing the χ^2-distribution, which is a measure for the likelihood.

4.1 Supernovae Type Ia "Union2.1" dataset

The SN Ia "Union2.1" dataset (see appendix A) used in this thesis contains the name, the redshift, the distance modulus and the distance modulus error of 580 supernovae, measured by the Hubble Space Telescope.

First, let us have a look at the dataset. We plot the distance modulus (see equation (1.7)) $m - M$ against the redshift z for the predicted luminosity distance d_L by the ΛCDM-model with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0.7)$ and DGP-model with $(\Omega_\mathrm{m}, \alpha) = (0.3, 0.0)$.

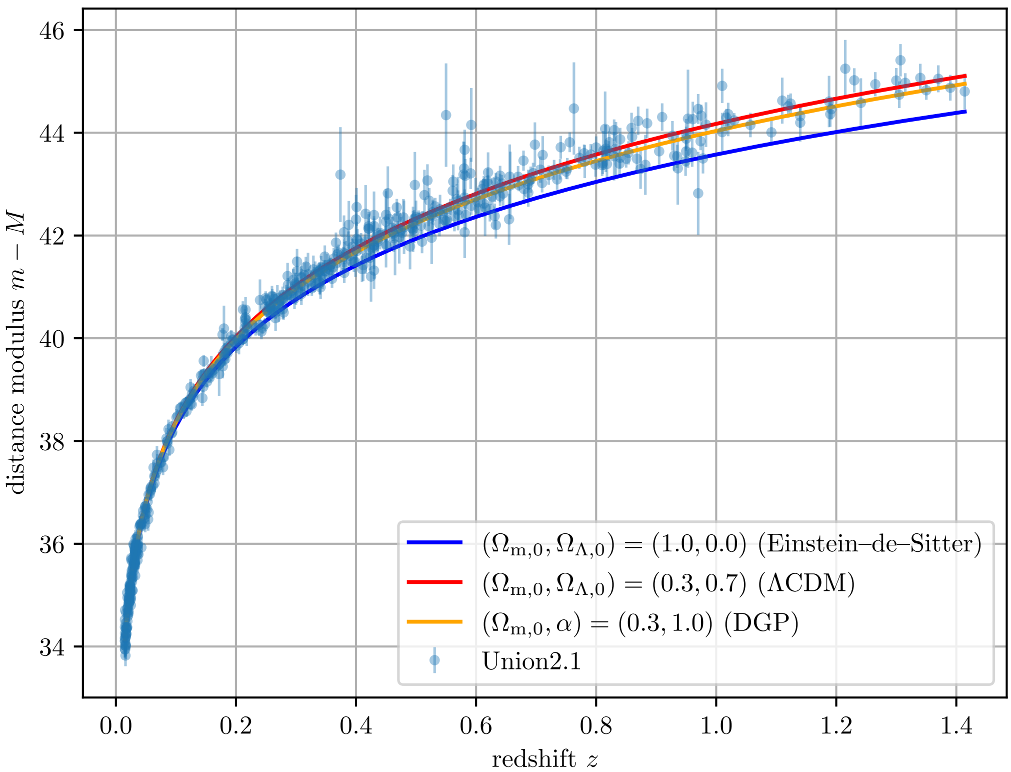


Figure 4.1: Distance modulus $m - M$ against redshift z for Einstein–de-Sitter-Model $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (1.0, 0.0)$, the ΛCDM-model with $(\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) = (0.3, 0.7)$ and DGP-model with $(\Omega_{\mathrm{m},0}, \alpha) = (0.3, 0.0)$.

At first sight, we see that a universe that contains only matter (Einstein–de–Sitter-model) cannot match the data, especially for high redshifts.
Further we can conclude, that the data points at high redshift ($z \gtrsim 0.8$) have a greater influence on the fit for the estimated parameters, since there is no significant fluctuation of data points at low redshifts.
The apparently large errors of some data points in the range of $0.35 \leq z \leq 1.0$ should not cause any concerns whether this data set is suitable for an adequate parameter estimation, since the error of a low-redshift data point does not impact the fit that much than the error of a high-redshift data point.

4.2 Statistical Analysis

To obtain values for the best-fit parameters, we consider the distance of every data point between its measured relative magnitude to the theoretical value the relative magnitude would have dependent on the parameters of the model.
From now on, we denote $\boldsymbol{\theta}$ as the parameter pairs of the model we want to estimate, so

$$\boldsymbol{\theta} = \begin{cases} (\Omega_{\mathrm{m},0}, \Omega_{\Lambda,0}) & \text{for } \Lambda\text{CDM-model} \\ (\Omega_{\mathrm{m},0}, \alpha) & \text{for DGP-model} \end{cases}. \tag{4.1}$$

The given dataset is a sample of $N = 580$ datapoints which contain for datapoint $i \in [1, N]$ the redshift z_i, the distance modulus and therefore implicitly[1] the relative magnitude m_i and its error σ_{m_i}.
Our goal is to express the quantities given *only* through the free parameters $\boldsymbol{\theta}$.
Now, let us consider the relative magnitude m, which is given by the distance modulus, see equation (1.7). First, we want to express the luminosity distance d_L in multiples of 1 Mpc, so we obtain

$$\begin{aligned} m &= M + 5\log_{10}\left(\frac{d_\mathrm{L}}{10\,\mathrm{pc}}\right) = M + 5\log_{10}\left(10^5 \frac{d_\mathrm{L}}{1\,\mathrm{Mpc}}\right) \\ &= M + 25 + 5\log_{10}\left(\frac{d_\mathrm{L}}{1\,\mathrm{Mpc}}\right). \end{aligned}$$

Since the luminosity distance d_L depends on the Hubble distance d_H (see equation (2.34) and (2.31)) and is therefore proportional to $d_\mathrm{L} \propto \frac{1}{H_0}$, we redefine the luminosity distance so that it is independent of the Hubble constant. Hence, we go on with

$$\begin{aligned} m &= M + 25 + 5\log_{10}\left(\frac{1}{H_0} H_0 d_\mathrm{L}(z, H_0, \boldsymbol{\theta})\mathrm{Mpc}^{-1}\right) \\ &= \underbrace{M + 25 - 5\log_{10}(H_0)}_{=:\mathcal{M}(H_0)} + 5\log_{10}\left(\underbrace{H_0 d_\mathrm{L}(z, H_0, \boldsymbol{\theta})}_{=:\mathcal{D}_\mathrm{L}(z,\boldsymbol{\theta})}\mathrm{Mpc}^{-1}\right) \\ &= \mathcal{M}(H_0) + 5\log_{10}(\mathcal{D}_\mathrm{L}(z, \boldsymbol{\theta})\mathrm{Mpc}^{-1}). \end{aligned} \tag{4.2}$$

With equation (4.2), the dependency on the Hubble constant is now in an additive constant $\mathcal{M}(H_0)$, which will be useful as we see later.

[1] Although the "Union2.1" SN Ia compilation contains the distance modulus $m - M$, we will handle it throughout the parameter estimation as if it only contains the relative magnitude m. Since we are going to marginalize the parameter M, it has no influence on the parameter estimation.

From now on, we are going to call the relative magnitude in equation (4.2) the *theoretical* relative magnitude m_{th}, since it contains the cosmological parameters $\boldsymbol{\theta}$.

For the best-fit parameter estimation, we consider the difference between the *measured* value of the relative magnitude m_i and the *theoretical* value of the relative magnitude m_{th}. Given the dataset $D := (z_i, m_i, \sigma_{m_i})$, where z_i is the measured redshift and σ_{m_i} the error of the relative magnitude m_i, we define

$$\chi^2(\mathcal{M}, \boldsymbol{\theta}|D) := \sum_{i=1}^{N} \left(\frac{m_i - m_{\mathrm{th}}(z_i, \mathcal{M}, \boldsymbol{\theta})}{\sigma_{m_i}} \right)^2 \tag{4.3}$$

as the χ^2-distribution of $\mathcal{M}$ and $\boldsymbol{\theta}$.
The likelihood, which is a probability density, is given by

$$L(\mathcal{M}, \boldsymbol{\theta}|D) := L_0 \exp\left(-\frac{1}{2} \chi^2(\mathcal{M}, \boldsymbol{\theta}|D) \right), \tag{4.4}$$

where L_0 is a normalization factor. With the likelihood L, it is possible to calculate the probability P to find $\mathcal{M} \in \mathcal{I}_{\mathcal{M}}$ and $\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}$ in a parameter intervall $\mathcal{I}_{\mathcal{M}}$ and $\mathcal{I}_{\boldsymbol{\theta}}$ with

$$P(\mathcal{M} \in \mathcal{I}_{\mathcal{M}}, \boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D) = \int\limits_{\mathcal{I}_{\mathcal{M}}} \mathrm{d}\mathcal{M} \int\limits_{\mathcal{I}_{\boldsymbol{\theta}}} \mathrm{d}\boldsymbol{\theta}\, L(\mathcal{M}, \boldsymbol{\theta}|D). \tag{4.5}$$

But this probability P still depends on $\mathcal{M}$, which is not measured. Since our goal is to find the best-fit values for $\boldsymbol{\theta}$ and therefore express the probability only through $\boldsymbol{\theta}$, we are going to *marginalize* over $\mathcal{M}$. We obtain the *marginalized* likelihood $\tilde{L}$ by integrating the likelihood L over all possible values that $\mathcal{M}$ could take. Assuming $\mathcal{M} \in (-\infty, \infty)$, it follows

$$\tilde{L}(\boldsymbol{\theta}|D) = \int\limits_{-\infty}^{\infty} \mathrm{d}\mathcal{M}\, L(\mathcal{M}, \boldsymbol{\theta}|D). \tag{4.6}$$

Thereby, we can calculate the probability $P(\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D)$ to find values for $\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}$, given the dataset D, but without any information on $\mathcal{M}$, which implicitly contains the Hubble constant H_0, so

$$P(\boldsymbol{\theta} \in \mathcal{I}_{\boldsymbol{\theta}}|D) = \int\limits_{\mathcal{I}_{\boldsymbol{\theta}}} \mathrm{d}\boldsymbol{\theta}\, \tilde{L}(\boldsymbol{\theta}|D). \tag{4.7}$$

Since $\mathcal{M}$ is only an additive constant (see equation (4.2)), this can be done analytically. We introduce the terms

$$c := \sum_{i=1}^{N} \frac{1}{\sigma_{m_i}^2}, \tag{4.8}$$

$$f_0 := \sum_{i=1}^{N} \frac{m_i - m_{\mathrm{th}}}{\sigma_{m_i}^2}, \tag{4.9}$$

$$f_1 := \sum_{i=1}^{N} \left(\frac{m_i - m_{\mathrm{th}}}{\sigma_{m_i}} \right)^2 \tag{4.10}$$

4.3 Computational Implementation

4.4 Results

CHAPTER 5

CONCLUSION

APPENDIX A

SN IA UNION2.1 DATA

The dataset used here is a reduced version of the full SN Ia “Union2.1” dataset.
Download: https://supernova.lbl.gov/Union/figures/SCPUnion2.1_mu_vs_z.txt
The full table of all supernovae contains additional data like B-band magnitude, stretch or color.
Description: https://supernova.lbl.gov/Union/descriptions.html#FullTable
Download full table as `.tex`-file: https://supernova.lbl.gov/Union/figures/SCPUnion2.1_AllSNe.tex

Here is the first appendix.

APPENDIX B

SOURCE CODES

All source codes that I wrote to produce the plots are open-source and available at the GitHub repository of this thesis: https://github.com/DaHaCoder/bachelor-thesis/tree/main/code.
Here is the list of the source code that correspond to following figures:

- Figure 2.3: `density-parameters_vs_scale-factor.py`
- Figure 2.4: `scale-factor_vs_time.py`
- Figure 2.5: `cosmological-distances_vs_redshift.py`
- Figure 4.1: `distance-modulus_vs_redshift.py`

BIBLIOGRAPHY

[DS20] Scott Dodelson and Fabian Schmidt. *Modern Cosmology*. Elsevier, 2020. ISBN: 978-0-12-815948-4. DOI: https://doi.org/10.1016/C2017-0-01943-2.

[Bar19] Matthias Bartelmann. *Das kosmologische Standardmodell*. Springer Verlag, 2019. ISBN: 978-3-662-59626-5. DOI: 10.1007/978-3-662-59627-2.

[CO07] Bradley W. Carroll and Dale A. Ostlie. *An Introduction to Modern Astrophysics*. Ed. by San Francisco: Pearson Addison-Wesley. 2nd (International). 2007. ISBN: 978-0-321-44284-0.

[Col+20] Planck Collaboration et al. “*Planck 2018 results: I. Overview and the cosmological legacy of Planck*”. en. In: *Astronomy & Astrophysics* 641 (Sept. 2020), A1. ISSN: 0004-6361, 1432-0746. DOI: 10.1051/0004-6361/201833880. URL: https://www.aanda.org/10.1051/0004-6361/201833880.

[LP12] Henrietta S. Leavitt and Edward C. Pickering. *Periods of 25 variable stars in the Small Magellanic Cloud*. Mar. 1912. URL: https://ui.adsabs.harvard.edu/link_gateway/1912HarCi.173....1L/ADS_PDF.

[Eng13] Philipp Engelmann. “*Cepheid Stars as standard candles for distance measurements*”. In: Sept. 2013. URL: https://www.haus-der-astronomie.de/3440685/04Engelmann.pdf.

[Mag17] Kate Maguire. “*Type Ia Supernovae*”. In: ed. by Athem W. Alsabti and Paul Murdin. Springer International Publishing, 2017, pp. 293–316. ISBN: 978-3-319-21845-8. DOI: 10.1007/978-3-319-21846-5_36. URL: https://link.springer.com/content/pdf/10.1007/978-3-319-21846-5_36.pdf.

[Spa17] Karl-Heinz Spatschek. *Astrophysik. Eine Einführung in Theorie und Grundlagen*. 2nd ed. Springer Spektrum Berlin, Heidelberg, Nov. 2017. ISBN: 978-3-662-55466-1. DOI: https://doi.org/10.1007/978-3-662-55467-8.

[Qui+18] Robert M. Quimby et al. “*Spectra of Hydrogen-poor Superluminous Supernovae from the Palomar Transient Factory*”. In: *The Astrophysical Journal* 855.1 (Feb. 2018), p. 2. DOI: 10.3847/1538-4357/aaac2f. URL: https://dx.doi.org/10.3847/1538-4357/aaac2f.

[PS03] Saul Perlmutter and Brian P. Schmidt. *Supernovae and Gamma-Ray Bursters*. Mar. 2003. Chap. Measuring Cosmology with Supernovae. DOI: 10.1007/3-540-45863-8_11. URL: https://arxiv.org/pdf/astro-ph/0303428.pdf.

[Phi93] M. M. Phillips. “*The Absolute Magnitudes of Type IA Supernovae*”. In: *The Astrophysical Journal* 413 (Aug. 1993). DOI: 10.1086/186970. URL: https://articles.adsabs.harvard.edu/pdf/1993ApJ...413L.105P.

[Val+21] Eleonora Di Valentino et al. “*In the realm of the Hubble tension—a review of solutions*”. In: *Classical and Quantum Gravity* 38.15 (July 2021), p. 153001. DOI: 10.1088/1361-6382/ac086d. URL: https://dx.doi.org/10.1088/1361-6382/ac086d.

[Hum36] Milton L. Humason. *"The Apparent Radial Velocities of 100 Extra-Galactic Nebulae"*. In: *Astrophysical Journal* 83 (Jan. 1936), p. 10. DOI: 10.1086/143696. URL: https://ui.adsabs.harvard.edu/abs/1936ApJ....83...10H.

[Hub29] Edwin Hubble. *"A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae"*. In: *Proceedings of the National Academy of Science* 15.3 (Mar. 1929), pp. 168–173. DOI: 10.1073/pnas.15.3.168. URL: https://ui.adsabs.harvard.edu/abs/1929PNAS...15..168H.

[HH31] Edwin Hubble and Milton L. Humason. *"The Velocity-Distance Relation among Extra-Galactic Nebulae"*. In: *Astrophysical Journal* 74 (July 1931), p. 43. DOI: 10.1086/143323. URL: https://ui.adsabs.harvard.edu/abs/1931ApJ....74...43H.

[Car19] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Cambridge University Press, Aug. 2019. ISBN: 978-1-108-48839-6. DOI: 10.1017/9781108770385.

[FTH08] Joshua A. Frieman, Michael S. Turner, and Dragan Huterer. *"Dark Energy and the Accelerating Universe"*. In: *Annual Review of Astronomy and Astrophysics* 46.1 (Sept. 2008), pp. 385–432. DOI: 10.1146/annurev.astro.46.060407.145243. URL: https://doi.org/10.48550/arXiv.0803.0982.

[Sch06] Peter Schneider. *Einführung in die Extragalaktische Astronomie und Kosmologie.* Springer Verlag, 2006. ISBN: 978-3-540-25832-2. DOI: 10.1007/3-540-30589-0.

[MTW17] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation.* Princeton University Press, 2017. ISBN: 978-0-691-17779-3. URL: https://press.princeton.edu/books/hardcover/9780691177793/gravitation.

[Pee93] P.J.E. Peebles. *Principles of Physical Cosmology.* 1993. DOI: 10.1515/9780691206721.

[Whi99] Martin White. *Anisotropies in the CMB.* 1999. DOI: 10.48550/ARXIV.ASTRO-PH/9903232. URL: https://arxiv.org/abs/astro-ph/9903232.

[Mat+90] J.C. Mather et al. *"A Preliminary Measurement of the Cosmic Microwave Background Spectrum by the Cosmic Background Explorer (COBE) Satellite"*. In: *Astrophysical Journal Letters* 354 (May 1990), p. L37. DOI: 10.1086/185717.

[Wei08] Steven Weinberg. *Cosmology.* Oxford University Press, 2008. ISBN: 978-0-19-852682-7.

[Gol+01] G. Goldhaber et al. *"Timescale Stretch Parameterization of Type Ia Supernova B-Band Light Curves"*. In: *The Astrophysical Journal* 558.1 (Sept. 2001), p. 359. DOI: 10.1086/322460. URL: https://dx.doi.org/10.1086/322460.

[DGP00] Gia Dvali, Gregory Gabadadze, and Massimo Porrati. *"4D gravity on a brane in 5D Minkowski space"*. In: *Physics Letters B* 485.1-3 (July 2000), pp. 208–214. DOI: 10.1016/s0370-2693(00)00669-9. URL: http://arxiv.org/abs/hep-th/0005016.

[DT03] Gia Dvali and Michael S. Turner. *Dark Energy as a Modification of the Friedmann Equation.* Jan. 2003. DOI: 10.48550/ARXIV.ASTRO-PH/0301510. URL: https://arxiv.org/abs/astro-ph/0301510.

[Lue06] Arthur Lue. *"The phenomenology of Dvali–Gabadadze–Porrati cosmologies"*. In: *Physics Reports* 423.1 (Jan. 2006), pp. 1–48. DOI: 10.1016/j.physrep.2005.10.007. URL: https://doi.org/10.1016%5C%2Fj.physrep.2005.10.007.

[Sch09] Bernard Schutz. *A First Course in General Relativity*. 2nd ed. Cambridge University Press, 2009. ISBN: 978-0-521-88705-2. DOI: 10.1017/CBO9780511984181.

DECLARATION OF AUTHORSHIP

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