SPATIAL DESCRIPTIONS AND TRANSFORMATION

COORDINATE SYSTEMS + TRANSFORMATION MATRICES

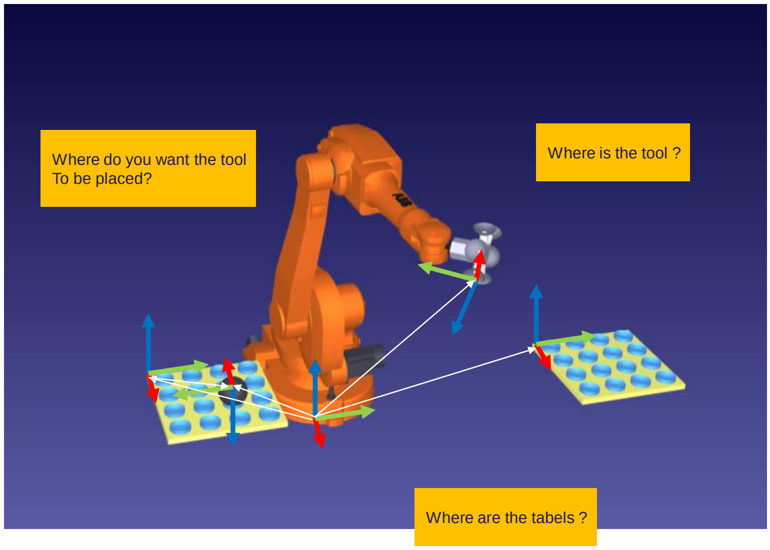
ROBOTICS 2023

OLE MADSEN
OM@M-TECH.AAU.DK

AALBORG UNIVERSITY
DENMARK

Lecture plan

No	Responsible	Contents
1	Ole Madsen	Introduction to: • the course; • robotics and robot terminology.
2	Ole Madsen	Spatial descriptions and transformation matrices
3 (FIB)	Ole Madsen + More	Practical exercise with the on-line programming (1.5 timer/gruppe).
4	Ole Madsen	Orientation
5	Ole Madsen	Forward Kinematics I
6	Ole Madsen	Forward Kinematics II (go though 6 DOF robot) – exercise, you go though your robot
7	Ole Madsen	Inverse kinematics I
8	Ole Madsen	Inverse kinematics II (go through 6DOF robot) – you start on your robot
9	Ole Madsen	Trajectory generation and control (joint)
10	Ole Madsen	Trajectory generation and control (cartesian)
11	Ole Madsen	Jacobian/Exam preparation



Agenda

1. Introduction
2. What is a coordinate system and how do we represent points and vectors (reminder from high school)
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7. Exercises

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A coordinate system (Frame)

To specify a coordinate system {A} we define:

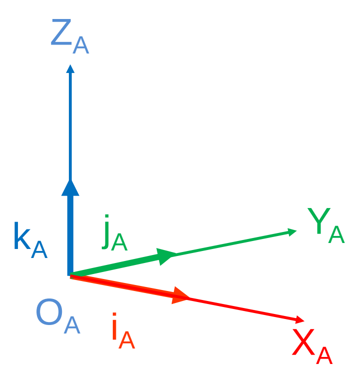
- Origin (O_A)
- Three orthogonal unit vectors:
  - i_A (defines the X axis)
  - j_A (defines the Y axis)
  - k_A (defines the Z axis)

And a few rules:

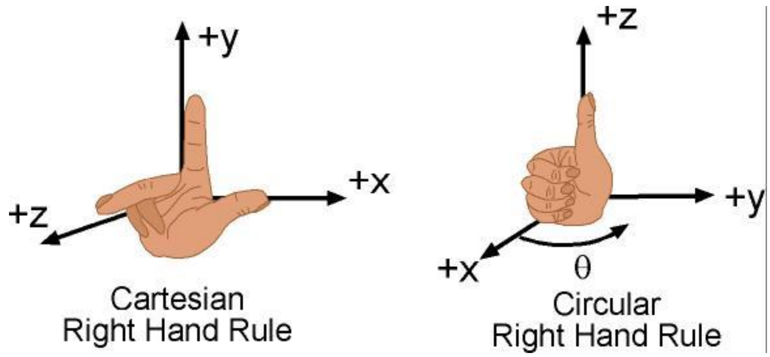
$$i_A \times j_A = k_A$$
$$j_A \times k_A = i_A$$
$$k_A \times i_A = j_A$$

$$i_A \cdot j_A = 0$$
$$j_A \cdot k_A = 0$$
$$k_A \cdot i_A = 0$$

$$i_A \cdot i_A = 1$$
$$j_A \cdot j_A = 1$$
$$k_A \cdot k_A = 1$$

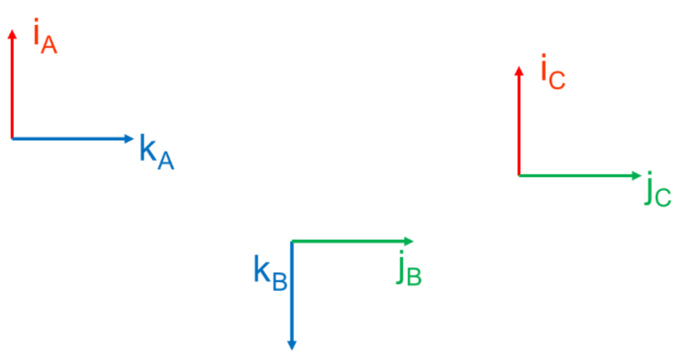


The secret hand signs !



A small exercise:

Find the missing axis:

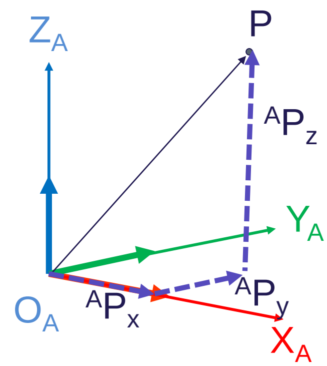


Specification of a position

P seen from {A}:

$$^{A}\overline{OP} = {}^{A}p_x \cdot \overline{i}_A + {}^{A}p_y \cdot \overline{j}_A + {}^{A}p_z \cdot \overline{k}_A$$

$$^{A}P = \begin{bmatrix} ^{A}p_x \\ ^{A}p_y \\ ^{A}p_z \end{bmatrix}$$

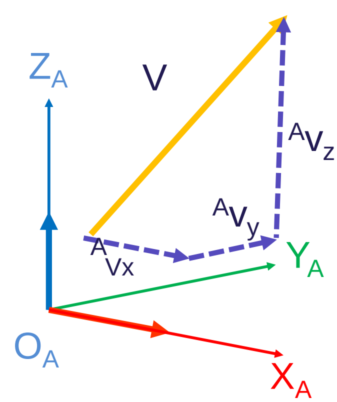


Specification of a vector

V seen from {A}:

$$\overline{V} = {}^{A}v_x \cdot \overline{i}_A + {}^{A}v_y \cdot \overline{j}_A + {}^{A}v_z \cdot \overline{k}_A$$

$$V = \begin{bmatrix} ^{A}v_x \\ ^{A}v_y \\ ^{A}v_z \end{bmatrix}$$



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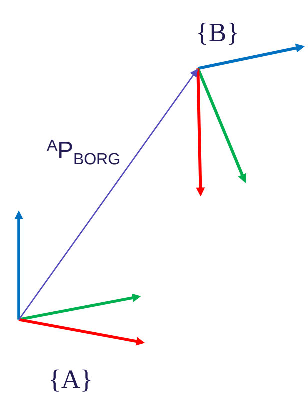
How can we specify the location of one coordinate system relatively to another ?

1. Position of origin of {B} seen from {A}
   - ${}^{A}P_{BORG}$
2. Coordinates of i_B, j_B and k_B seen from {A}:
   - ${}^{A}i_B$
   - ${}^{A}j_B$
   - ${}^{A}k_B$
3. Collected in a homogeneous transformation matrix

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}i_{B,x} & {}^{A}j_{B,x} & {}^{A}k_{B,x} & {}^{A}\mathrm{P}_{BORG,x} \\ {}^{A}i_{B,y} & {}^{A}j_{B,y} & {}^{A}k_{B,y} & {}^{A}\mathrm{P}_{BORG,y} \\ {}^{A}i_{B,z} & {}^{A}j_{B,z} & {}^{A}k_{B,z} & {}^{A}\mathrm{P}_{BORG,z} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Homogeneous transformation matrix

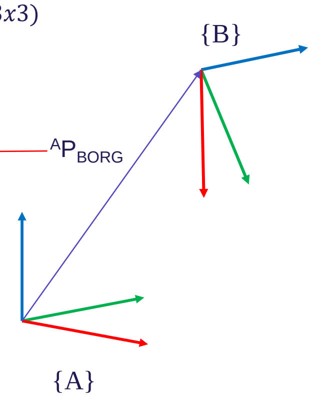
4x4 matrix

^A_BR $-$ *rotations matrix* (3x3)

$$^A_BT = \begin{bmatrix} \boxed{\begin{matrix} ^Ai_{B,X} & ^Aj_{B,X} & ^Ak_{B,X} \\ ^Ai_{B,Y} & ^Aj_{B,Y} & ^Ak_{B,Y} \\ ^Ai_{B,Z} & ^Aj_{B,Z} & ^Ak_{B,Z} \end{matrix}} & \boxed{\begin{matrix} ^AP_{BORG,X} \\ ^AP_{BORG,Y} \\ ^AP_{BORG,Z} \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 & 0 \end{matrix}} & \boxed{1} \end{bmatrix}$$

$^AP_{BORG}$

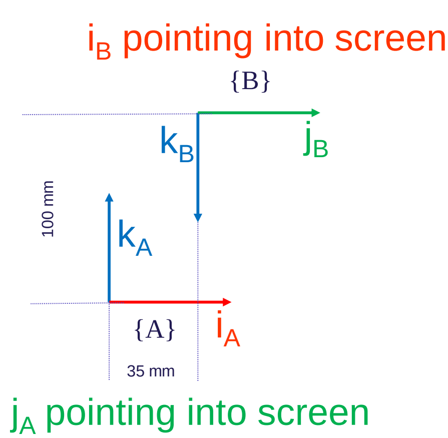
“0 0 0 1” added here



A small exercise:

Find the transformation matrix:

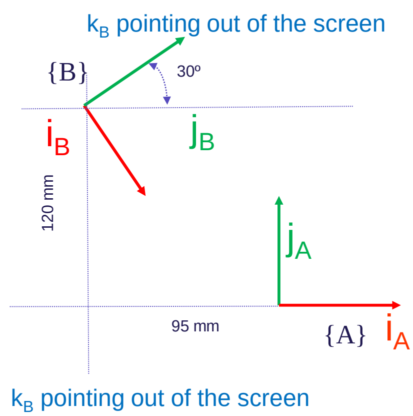
$$^A_BT = \begin{bmatrix} 0 & 1 & 0 & 35 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



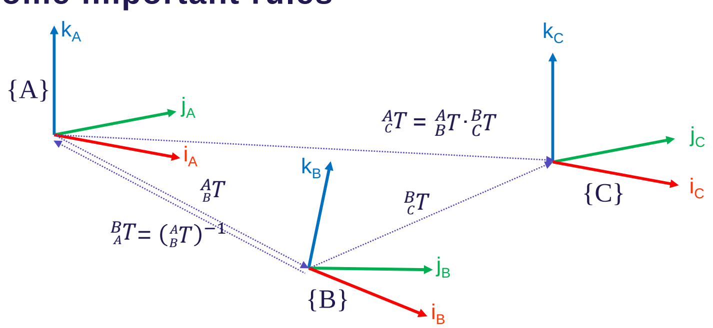
Another small exercise:

Find the transformation matrix:

$$
{}^{A}_{B}T = \begin{bmatrix} \sin(30°) & \cos(30°) & 0 & 95 \\ -\cos(30°) & sin(30°) & 0 & 120 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Some important rules



$$
{}^{A}_{C}T = {}^{A}_{B}T \cdot {}^{B}_{C}T
$$

$$
{}^{B}_{A}T = ({}^{A}_{B}T)^{-1}
$$

$$
{}^{A}_{C}T = {}^{A}_{B}T \cdot {}^{B}_{C}T \qquad {}^{A}_{C}R = {}^{A}_{B}R \cdot {}^{B}_{C}R
$$

$$
{}^{A}_{B}T = ({}^{B}_{A}T)^{-1} \qquad {}^{A}_{B}R = ({}^{B}_{A}R)^{-1} = ({}^{B}_{A}R)^{T}
$$

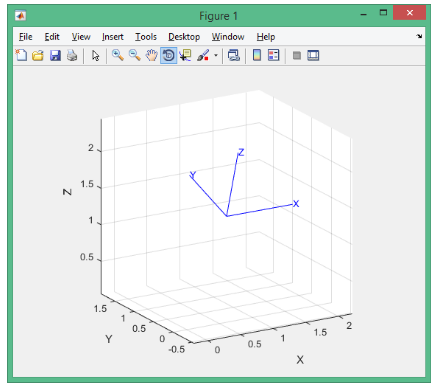
In robotic toolbox:

```
theta=pi;

T1=[cos(theta)  -sin(theta)   0   7;
      sin(theta)   cos(theta)   0   0;
         0              0        1   4;
         0              0        0   1] ;

%% Section 1
trplot(T1)

%% section 2
tranimate(T1)
```



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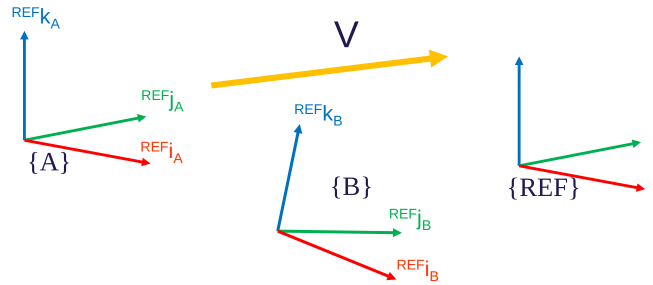


$$\bar{V} = {}^{A}V_x \cdot {}^{REF}\bar{i}_A + {}^{A}V_y \cdot {}^{REF}\overline{j_A} + {}^{A}V_z \cdot {}^{REF}\overline{k_A} = {}^{B}V_x \cdot {}^{REF}\bar{i}_B + {}^{B}V_y \cdot {}^{REF}\overline{j_B} + {}^{B}V_z \cdot {}^{REF}\overline{k_B}$$

$$\bar{V} \cdot {}^{REF}\bar{i}_A = {}^{A}V_x = {}^{B}V_x \cdot {}^{REF}\bar{i}_B \cdot {}^{REF}\bar{i}_A + {}^{B}V_y \cdot {}^{REF}\overline{j_B} \cdot {}^{REF}\bar{i}_A + {}^{B}V_z \cdot {}^{REF}\overline{k_B} \cdot {}^{REF}\bar{i}_A$$

$$\bar{V} \cdot {}^{REF}\overline{j_A} = {}^{A}V_y = {}^{B}V_x \cdot {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{j_A} + {}^{B}V_y \cdot {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{j_A} + {}^{B}V_z \cdot {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{j_A}$$

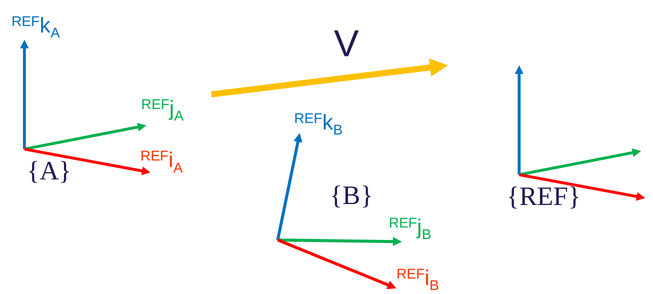
$$\bar{V} \cdot {}^{REF}\overline{k_A} = {}^{A}V_z = {}^{B}V_x \cdot {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{k_A} + {}^{B}V_y \cdot {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{k_A} + {}^{B}V_z \cdot {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{k_A}$$

$$\boxed{\begin{bmatrix} {}^{A}V_x \\ {}^{A}V_y \\ {}^{A}V_z \end{bmatrix}} = \boxed{\begin{bmatrix} {}^{REF}\bar{i}_B \cdot {}^{REF}\bar{i}_A & {}^{REF}\overline{j_B} \cdot {}^{REF}\bar{i}_A & {}^{REF}\overline{k_B} \cdot {}^{REF}\bar{i}_A \\ {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{j_A} & {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{j_A} & {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{j_A} \\ {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{k_A} & {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{k_A} & {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{k_A} \end{bmatrix}} \boxed{\begin{bmatrix} {}^{B}V_x \\ {}^{B}V_y \\ {}^{B}V_z \end{bmatrix}}$$

Vector Vs coordinates in A

Rotation matrix

Vector Vs coordinates in B



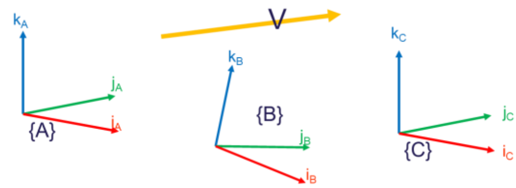
$$\begin{bmatrix} {}^{A}V_x \\ {}^{A}V_y \\ {}^{A}V_z \end{bmatrix} = \begin{bmatrix} {}^{REF}\bar{i}_B \cdot {}^{REF}\bar{i}_A & {}^{REF}\overline{j_B} \cdot {}^{REF}\bar{i}_A & {}^{REF}\overline{k_B} \cdot {}^{REF}\bar{i}_A \\ {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{j_A} & {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{j_A} & {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{j_A} \\ {}^{REF}\bar{i}_B \cdot {}^{REF}\overline{k_A} & {}^{REF}\overline{j_B} \cdot {}^{REF}\overline{k_A} & {}^{REF}\overline{k_B} \cdot {}^{REF}\overline{k_A} \end{bmatrix} \begin{bmatrix} {}^{B}V_x \\ {}^{B}V_y \\ {}^{B}V_z \end{bmatrix} = {}^{A}_{B}R \begin{bmatrix} {}^{B}V_x \\ {}^{B}V_y \\ {}^{B}V_z \end{bmatrix}$$

This is true for any REF. Hence if we set {REF}={A}, then:

$${}^{A}\bar{i}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, {}^{A}\overline{j_A} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, {}^{A}\overline{k_A} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, {}^{A}\bar{i}_B = \begin{bmatrix} {}^{A}i_{B,x} \\ {}^{A}i_{B,y} \\ {}^{A}i_{B,z} \end{bmatrix}, {}^{A}\overline{j_B} = \begin{bmatrix} {}^{A}j_{B,x} \\ {}^{A}j_{B,y} \\ {}^{A}j_{B,z} \end{bmatrix}, {}^{A}\overline{k_B} = \begin{bmatrix} {}^{A}k_{B,x} \\ {}^{A}k_{B,y} \\ {}^{A}k_{B,z} \end{bmatrix}$$

$${}^{A}_{B}R = \begin{bmatrix} {}^{A}i_{B,x} & {}^{A}j_{B,x} & {}^{A}k_{B,x} \\ {}^{A}i_{B,y} & {}^{A}j_{B,y} & {}^{A}k_{B,y} \\ {}^{A}i_{B,z} & {}^{A}j_{B,z} & {}^{A}k_{B,z} \end{bmatrix} = \begin{bmatrix} {}^{A}\bar{i}_B & {}^{A}\overline{j_B} & {}^{A}\overline{k_B} \end{bmatrix}$$

Rotation matrices – some nice properties



1. We know:
   ${}^A_B R$ and ${}^B_C R$

   What is ${}^A_C R$?

2. Answer: $\boxed{{}^A_C R = {}^A_B R \cdot {}^B_C R}$

3. Proof:
   We know:
   ${}^A V = {}^A_B R\, {}^B V$ and ${}^B V = {}^B_C R\, {}^C V$
   Combining these we get:
   ${}^A V = {}^A_B R\, {}^B_C R\, {}^C V = {}^A_C R\, {}^C V$

1. We know:
   ${}^A_B R$

   What is ${}^B_A R$?

2. Answer: $\boxed{{}^B_A R = ({}^A_B R)^{-1}}$

3. Proof:
   - We know:
   ${}^A V = {}^A_B R\, {}^B V$
   Taking $({}^A_B R)^{-1}$ on each side we get:
   $({}^A_B R)^{-1}\, {}^A V = ({}^A_B R)^{-1}\, {}^A_B R\, {}^B V = {}^B V = {}^B_A R\, {}^A V$

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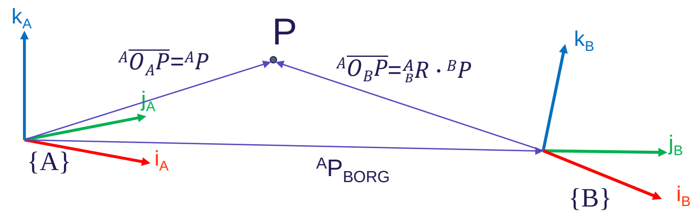


$$ {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG} $$

$$ \begin{bmatrix} {}^{A}P_x \\ {}^{A}P_y \\ {}^{A}P_z \end{bmatrix} = {}^{A}_{B}R \begin{bmatrix} {}^{B}P_x \\ {}^{B}P_y \\ {}^{B}P_z \end{bmatrix} + {}^{A}P_{BORG} $$

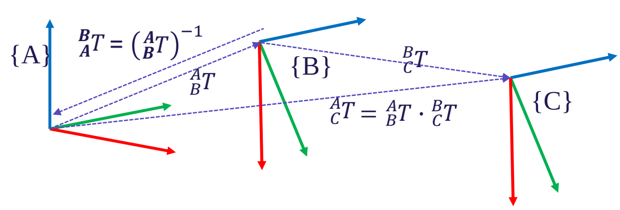
$$ \begin{bmatrix} {}^{A}P_x \\ {}^{A}P_y \\ {}^{A}P_z \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P_x \\ {}^{B}P_y \\ {}^{B}P_z \\ 1 \end{bmatrix} $$

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & & & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^{A}i_{B,X} & {}^{A}j_{B,X} & {}^{A}k_{B,X} & {}^{A}P_{BORG,X} \\ {}^{A}i_{B,Y} & {}^{A}j_{B,Y} & {}^{A}k_{B,Y} & {}^{A}P_{BORG,Y} \\ {}^{A}i_{B,Z} & {}^{A}j_{B,Z} & {}^{A}k_{B,Z} & {}^{A}P_{BORG,Z} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Homogeneous Transformation Matrix – some nice properties

1. Given ${}^{A}_{B}T$ and ${}^{B}_{C}T$ then ${}^{A}_{C}T$ can be found as:

$$ {}^{A}_{C}T = {}^{A}_{B}T \cdot {}^{B}_{C}T $$



2. Given ${}^{A}_{B}T$ then ${}^{B}_{A}T$ can be found as:

$$ {}^{B}_{A}T = \left({}^{A}_{B}T\right)^{-1} $$

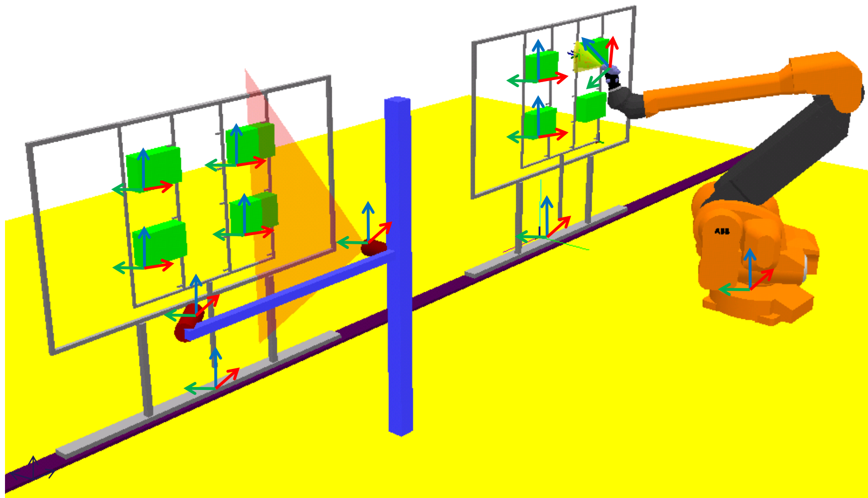
Proof:
We know: ${}^{A}P = {}^{A}_{B}T \cdot {}^{B}P$ and ${}^{B}P = {}^{B}_{C}T \cdot {}^{C}P$
Combining these we get: ${}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T \cdot {}^{C}P = {}^{A}_{C}T\,{}^{C}P$

Proof:
We know: ${}^{A}P = {}^{A}_{B}T \cdot {}^{B}P$
We now take $\left({}^{A}_{B}T\right)^{-1}$ on both sides:

$$ \left({}^{A}_{B}T\right)^{-1} {}^{A}P = \left({}^{A}_{B}T\right)^{-1} {}^{A}_{B}T \cdot {}^{B}P = {}^{B}P $$

Hence: ${}^{B}_{A}T = \left({}^{A}_{B}T\right)^{-1}$ since: ${}^{B}P = {}^{B}_{A}T \cdot {}^{A}P = \left({}^{A}_{B}T\right)^{-1} {}^{A}P$

A spray painting robot



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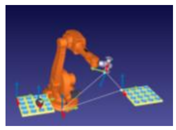
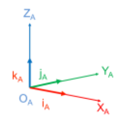
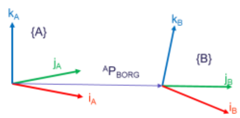
Summary

- Position and orientation of objects are represented by coordinate systems.





- The location (position and orientation) of a coordinate system {B} relatively to {A} can be represented by a transformation matrix



$$
{}^A_BT = \begin{bmatrix} {}^Ai_{B,X} & {}^Aj_{B,X} & {}^Ak_{B,X} & {}^AP_{BORG,X} \\ {}^Ai_{B,Y} & {}^Aj_{B,Y} & {}^Ak_{B,Y} & {}^AP_{BORG,Y} \\ {}^Ai_{B,Z} & {}^Aj_{B,Z} & {}^Ak_{B,Z} & {}^AP_{BORG,Z} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

${}^A_BR$ rotations matrix (3x3)

i_B seen from {A}

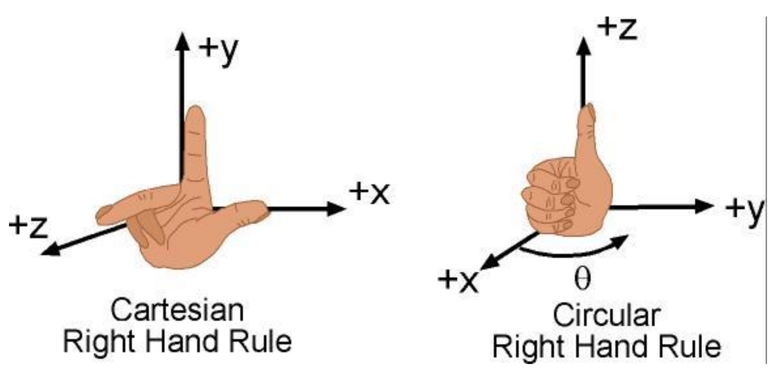
$^AP_{BORG}$

"0 0 0 1" always

- Some important rules.

$^AP = {}^A_BT \cdot {}^BP$	${}^B_AT = \left({}^A_BT\right)^{-1}$	${}^A_CT = {}^A_BT \cdot {}^B_CT$
$^AV = {}^A_BR \cdot {}^BV$	${}^B_AR = \left({}^A_BR\right)^{-1} = \left({}^A_BR\right)^T$	${}^A_CR = {}^A_BR \cdot {}^B_CR$

The secret hand signs !



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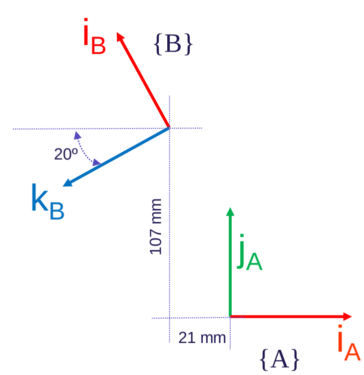
Test exercise:

Find the transformation matrix: ${}^{A}_{B}T = ???$



Hand-in solutions for this exercise on Moodle

:

Exercise 1

Given two coordinate systems ({A} and {B}), two points (P1 and P2) and two vectors (V1 and V2)

Also given:

$$ {}^{A}_{B}T = \begin{bmatrix} 0.7052 & -0.3749 & 0.6018 & 646.9 \\ 0.5546 & 0.8205 & -0.1387 & 103.8 \\ -0.4418 & 0.4316 & 0.7865 & -329.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Question 1.1:
Assume the coordinates of P1 and V1 relative to {A} are:

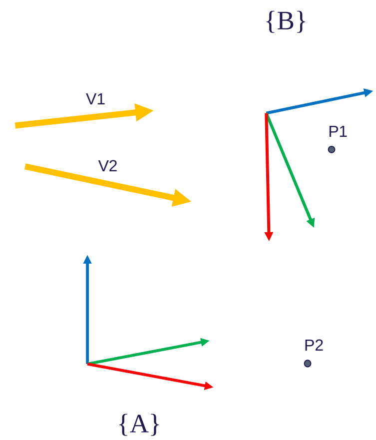
$$ {}^{A}P1 = \begin{bmatrix} 53 \\ 15 \\ -35 \\ 1 \end{bmatrix}, \ {}^{A}V1 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} $$

Compute the coordinates relative to {B}

Question 1.2:
Assume the coordinates of P2 and V2 relative to {B} are:

$$ {}^{B}P2 = \begin{bmatrix} 10 \\ -40 \\ 25 \\ 1 \end{bmatrix}, \ {}^{B}V2 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} $$

Compute the coordinates relative to {A}



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Exercise 2

The following frame definitions are given as known. Draw a diagram which quantitatively showing their location. Solve for ${}^{B}_{C}T$

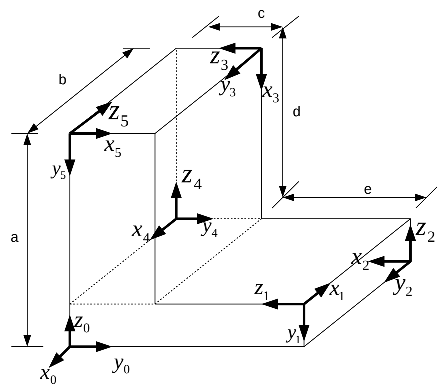
$$ {}^{U}_{A}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 11.0 \\ 0.500 & 0.866 & 0.000 & -1.0 \\ 0.000 & 0.000 & 1.000 & 8.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.0 \\ 0.000 & 0.866 & -0.500 & 10.0 \\ 0.000 & 0.500 & 0.866 & 20.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{C}_{U}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & -3.0 \\ 0.433 & 0.750 & -0.500 & -3.0 \\ 0.250 & 0.433 & 0.866 & 3.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Exercise 3

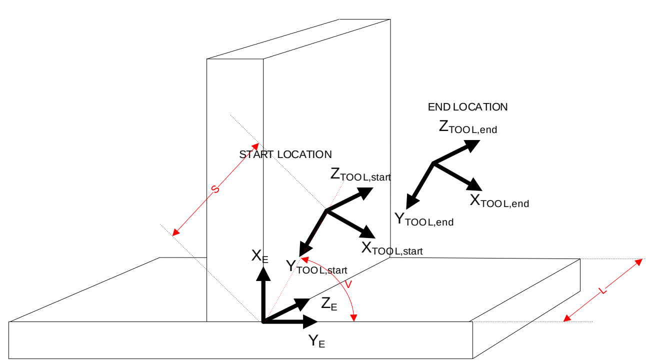
For the figure, find the 4x4 homogeneous transformation matrices ${}^{0}_{i}T$ and ${}^{i-1}_{i}T$ for i=1, 2, 3, 4, 5.



Exercise 4

For the welding task determine:

${}^{E}_{Tool,start}T$ and ${}^{E}_{Tool,end}T$



AALBORG UNIVERSITY
DENMARK