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# Machine Learning

## Answer Sheet for Homework 3

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### Problem 1

Set  $\sigma = 0.1$  and  $d = 8$ , then we can rewrite the formula to be

$$\mathbb{E}_{\mathcal{D}}[E_{\text{in}}(\mathbf{w}_{\text{lin}})] = 0.01 \left(1 - \frac{9}{N}\right) > 0.008 \Rightarrow 0.2 > \frac{9}{N} \Rightarrow N > 45 \Rightarrow N \geq 46 \quad (1)$$

□

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### Problem 2

- (a)  $\mathbf{H}$  is positive semi-definite  $\Leftrightarrow$  All eigenvalues is non-negative. Refer to choice (c), we have shown the properties.
- (b) Refer to (e),  $\mathbf{H}$  is idempotent matrix. Suppose  $\mathbf{H}^{-1}$  exists, we have

$$\mathbf{H}^{-1}(\mathbf{H}^2) = \mathbf{H}^{-1}(\mathbf{H}) \Rightarrow \mathbf{H} = \mathbf{I} \quad (2)$$

Hence,  $\mathbf{H}$  is invertible if and only if  $\mathbf{H} = \mathbf{I}$ . Hence,  $\mathbf{H}$  is not always invertible.

- (c) Refer to choice (e), we have  $\mathbf{H}^2 = \mathbf{H}$ . Suppose  $\lambda$  is the eigenvalue of some non-zero vector  $\vec{v}$ ,

$$\mathbf{H}\vec{v} = \lambda\vec{v} = \mathbf{H}^2\vec{v} = \mathbf{H}(\lambda\vec{v}) = \lambda(\mathbf{H}\vec{v}) = \lambda^2\vec{v} \Rightarrow \lambda^2 = \lambda \quad (3)$$

Hence, the possible results of  $\lambda$  is 1 or 0.

- (d) For a symmetric and idempotent matrix  $\mathbf{H}$ ,  $\text{rank}(\mathbf{H}) = \text{trace}(\mathbf{H})$ , the number of non-zero eigenvalues of  $\mathbf{H}$ .

$$\text{trace}(\mathbf{H}) = \text{trace}\left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right) = \text{trace}\left((\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{X}\right) \quad (4)$$

$$= \text{trace}(\mathbf{I}_{d+1}) = d + 1 \quad (5)$$

where we have used  $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$ .

Since eigenvalue can only be 0 or 1, so there are  $d + 1$  eigenvalues of 1.

- (e) By the definition of  $\mathbf{H}$ , we have

$$\mathbf{H}^2 = \left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right)\left(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\right) \quad (6)$$

$$= \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\left((\mathbf{X}^T\mathbf{X})(\mathbf{X}^T\mathbf{X})^{-1}\right)\mathbf{X}^T \quad (7)$$

$$= \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T = \mathbf{H} \quad (8)$$

So

$$\mathbf{H}^2 = \mathbf{H} \Rightarrow \mathbf{H}^{1126} = \mathbf{H} \quad (9)$$

□

### Problem 3

If  $\text{sign}(\mathbf{w}^T\mathbf{x}) \neq y$ , then  $y\mathbf{w}^T\mathbf{x} < 0$  since the sign of  $y$  and  $\mathbf{w}^T\mathbf{x}$  are different. Similarly, if  $\text{sign}(\mathbf{w}^T\mathbf{x}) = y$ , then  $y\mathbf{w}^T\mathbf{x} \geq 0$ .

Claim:  $(\max(0, 1 - y\mathbf{w}^T\mathbf{x}))^2$  is an upper bound.

Proof of claim:

Consider the following cases.

$$1. [\text{sign}(\mathbf{w}^T\mathbf{x}) \neq y] = 0$$

Then  $y\mathbf{w}^T\mathbf{x} \geq 0$ . Hence  $(\max(0, 1 - y\mathbf{w}^T\mathbf{x}))^2 \geq 0$ , which bounds  $[\text{sign}(\mathbf{w}^T\mathbf{x}) \neq y]$ .

$$2. [\text{sign}(\mathbf{w}^T\mathbf{x}) \neq y] = 1$$

Then  $y\mathbf{w}^T\mathbf{x} < 0$ . Hence  $(\max(0, 1 - y\mathbf{w}^T\mathbf{x}))^2 = (1 - y\mathbf{w}^T\mathbf{x})^2 > 1$ , which bounds  $[\text{sign}(\mathbf{w}^T\mathbf{x}) \neq y]$ .

□

## Problem 4

Set  $y\mathbf{w}^T\mathbf{x} := z$ . Consider  $\max(0, -y\mathbf{w}^T\mathbf{x}) = \max(0, -z) := f(z)$ . We have  $f(z) = -z$  if  $z \leq 0$ , else  $f(z) = 0$ . So

$$\lim_{z \rightarrow 0^-} \frac{f(z) - f(0)}{z - 0} = \frac{-z - 0}{z - 0} = -1, \quad \lim_{z \rightarrow 0^+} \frac{f(z) - f(0)}{z - 0} = \frac{0 - 0}{z - 0} = 0 \quad (10)$$

Hence,  $f(z)$  is not differentiable at  $z = 0$ .

□

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## Problem 5

Calim:  $\max(0, -y\mathbf{w}^T\mathbf{x})$  results in PLA.

Proof of claim:

Consider following cases.

1.  $y = \text{sign}(\mathbf{w}^T\mathbf{x})$ . PLA update term is 0.

Then we have  $y\mathbf{w}^T\mathbf{x} > 0$ . So

$$\max(0, -y\mathbf{w}^T\mathbf{x}) = 0 \quad (11)$$

2.  $y \neq \text{sign}(\mathbf{w}^T\mathbf{x})$ . PLA update term is  $y\mathbf{x}$ .

Then we have  $y\mathbf{w}^T\mathbf{x} < 0$ . So

$$\max(0, -y\mathbf{w}^T\mathbf{x}) = -y\mathbf{w}^T\mathbf{x} \Rightarrow -\nabla_{\mathbf{w}} \max(0, -y\mathbf{w}^T\mathbf{x}) = y\mathbf{x} \quad (12)$$

□

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## Problem 6

$$\nabla E(0,0) = \left( \frac{\partial E}{\partial u}, \frac{\partial E}{\partial v} \right) \Big|_{(0,0)} \quad (13)$$

$$= (e^u + ve^{uv} + 2u - 2v - 3, 2e^{2v} + ue^{uv} - 2u + 4v - 2) \Big|_{(0,0)} \quad (14)$$

$$= (-2, 0) \quad (15)$$

□

## Problem 7

I write a program Q07.py to calculate the result by using

$$(u_{t+1}, v_{t+1}) = (u_t, v_t) - 0.01 \nabla E(u_t, v_t) \quad (16)$$

iteratively.

$$(u_1, v_1) = (0, 0) - 0.01 \nabla E(0, 0) = (0.02, 0) \quad (17)$$

$$(u_2, v_2) = (0.02, 0) - 0.01 \nabla E(0.02, 0) \approx (0.039398, 0.0002) \quad (18)$$

$$(u_3, v_3) \approx (0.039398, 0.0002) - 0.01 \nabla E(0.039398, 0.0002) \quad (19)$$

$$\approx (0.0582102, 0.000577975) \quad (20)$$

$$(u_4, v_4) \approx (0.0764524, 0.00111381) \quad (21)$$

$$(u_5, v_5) \approx (0.09414, 0.00178911) \quad (22)$$

$$E(u_5, v_5) \approx 2.825 \quad (23)$$

□

## Problem 8

$$\nabla E(0, 0) = \left( \frac{\partial E}{\partial u}, \frac{\partial E}{\partial v} \right) \quad (24)$$

$$= (e^u + v e^{uv} + 2u - 2v - 3, 2e^{2v} + u e^{uv} - 2u + 4v - 2) \quad (25)$$

From this we compute the Hessian matrix

$$\nabla^2 E(u, v) = \begin{pmatrix} e^u + v^2 e^{uv} + 2 & (uv + 1) e^{uv} - 2 \\ (uv + 1) e^{uv} - 2 & 4e^{2v} + u^2 e^{uv} + 4 \end{pmatrix} \quad (26)$$

So

$$\hat{E}(\Delta u, \Delta v) = E(0, 0) + \nabla E(0, 0) \cdot (\Delta u, \Delta v) + \frac{1}{2} (\Delta u, \Delta v) \nabla^2 E(0, 0) \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \quad (27)$$

$$= 3 - 2\Delta u + \frac{1}{2} (\Delta u, \Delta v) \begin{pmatrix} 3 & -1 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \quad (28)$$

$$= \frac{3}{2} (\Delta u)^2 + 4 (\Delta v)^2 - \Delta u \Delta v - 2\Delta u + 0\Delta v + 3 \quad (29)$$

□

## Problem 9

Claim:  $-(\nabla^2 E(u, v))^{-1} \nabla E(u, v)$  is the Newton direction.

Proof of claim:

$$\frac{\partial \hat{E}(\Delta u, \Delta v)}{\partial (\Delta u, \Delta v)} = \nabla E(u, v) + \nabla^2 E(u, v) (\Delta u, \Delta v) = 0 \quad (30)$$

$$\Rightarrow (\Delta u, \Delta v) = -(\nabla^2 E(u, v))^{-1} \nabla E(u, v) \quad (31)$$

□

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## Problem 10

I write a program Q10.py to calculate the result by using

$$(u_{t+1}, v_{t+1}) = (u_t, v_t) - (\nabla^2 E(u_t, v_t))^{-1} \nabla E(u_t, v_t) \quad (32)$$

iteratively.

$$(u_1, v_1) \approx (0.695652173913, 0.0869565217391) \quad (33)$$

$$(u_2, v_2) \approx (0.613762221112, 0.0711078990173) \quad (34)$$

$$(u_3, v_3) \approx (0.611812859879, 0.0705000613365) \quad (35)$$

$$(u_4, v_4) \approx (0.611811717261, 0.0704995471019) \quad (36)$$

$$(u_5, v_5) \approx (0.61181171726, 0.0704995471016) \quad (37)$$

$$E(u_5, v_5) \approx 2.36082334564 \quad (38)$$

This equals to the value of Problem 7 after 746 updates.

□

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## Problem 11

Write a program Q11.py to test,  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6\}$  is the biggest subset that can be shattered by the union of quadratic, linear, or constant hypotheses of  $\mathbf{x}$  by feature form of

$$\Phi(\mathbf{x}) = (1, x_1, x_2, x_1^2, x_1 x_2, x_2^2) \quad (39)$$

Then I ran the PLA and all cases of  $\{+1, -1\}^6$  are stasified. The  $\mathbf{w}$  for  $2^6 = 64$  cases are

$$\begin{array}{llll}
[0, 0, 0, 0, 0, 0], & [0, 1, 0, 1, 0, -4], & [1, 0, 1, -2, -1, -1], & [2, 0, 0, 0, 0, -2], \\
[-1, -2, 0, 0, -2, 0], & [0, 0, 1, 2, -1, -3], & [1, -1, 2, -1, -2, 0], & [2, -1, 1, 1, -1, -3], \\
[0, 0, -2, 0, 2, 0], & [0, 0, -3, 2, 3, -3], & [1, -1, -2, -1, 2, 0], & [2, 0, -2, 0, 2, -2], \\
[-1, -4, 0, 0, 0, 0], & [0, -2, -1, 4, 1, -3], & [1, -2, -1, 0, 1, 1], & [3, -2, 0, 2, 0, -4], \\
[-1, 1, 0, -1, -2, 0], & [0, 2, 0, 0, -2, 0], & [1, 1, -2, -3, -2, 0], & [1, 1, -1, -1, -3, -1], \\
[-1, 0, 1, 0, -3, 1], & [0, 1, 0, 1, -4, 0], & [1, -1, 1, -1, -3, 1], & [1, 0, 0, 0, -4, 0], \\
[0, 0, -3, 0, 1, 1], & [0, 1, -4, 1, 0, 0], & [1, -1, -3, -1, 1, 1], & [1, 0, -4, 0, 0, 0], \\
[0, -1, -2, 1, 0, 2], & [0, 0, -3, 2, -1, 1], & [1, -1, -1, -1, -1, 1], & [1, -1, -3, 1, -1, 1], \\
[0, 0, 2, 0, 2, 0], & [0, 2, 3, 0, 1, -1], & [1, 0, 3, -2, 1, -1], & [1, 1, 3, -1, 1, -1], \\
[-1, 0, 4, 0, 0, 0], & [0, 1, 4, 1, 0, 0], & [1, -1, 4, -1, 0, 0], & [1, 0, 4, 0, 0, 0], \\
[0, 0, 0, 0, 4, 0], & [0, 1, 0, 1, 4, 0], & [1, -1, 0, -1, 4, 0], & [1, 0, 0, 0, 4, 0], \\
[0, -1, 1, 1, 3, 1], & [0, 0, 1, 2, 3, 1], & [1, -2, 1, 0, 3, 1], & [1, -1, 1, 1, 3, 1], \\
[-2, 2, 0, -2, 0, 4], & [0, 4, 0, 0, 0, 0], & [1, 2, 1, -4, -1, 3], & [2, 4, 0, 0, 0, 0], \\
[-1, 0, 2, 0, -2, 2], & [0, 1, 3, 1, -1, 1], & [1, 0, 3, -2, -3, 3], & [1, 1, 3, 1, -1, 1], \\
[-1, 1, -1, -1, 1, 3], & [0, 3, -1, 1, 1, 1], & [1, 0, -1, -2, 1, 3], & [1, 2, 0, 0, 2, 0], \\
[-1, 0, 0, 0, 0, 2], & [0, 1, 1, 1, 1, 1], & [1, -1, 0, -1, 0, 4], & [1, 1, 1, 1, 1, 1]
\end{array}
\tag{40}$$

□

## Problem 12

By the transform, we have  $(\Phi(\mathbf{x}))_i = z_i = \left(0, \dots, \underbrace{1}_{i\text{-th term}}, \dots, 0\right)$ . To shatter the original  $N$  points, we can assign  $w_i$  to be positive or negative to get  $\mathbf{x}_i \circ$  or  $\times$ .

So, this transform shatter any  $N$  points. Hence  $d_{vc}(\mathcal{H}_\Phi) = \infty$ .

□

## Problem 13

The average  $E_{\text{in}}$  is 0.503979.

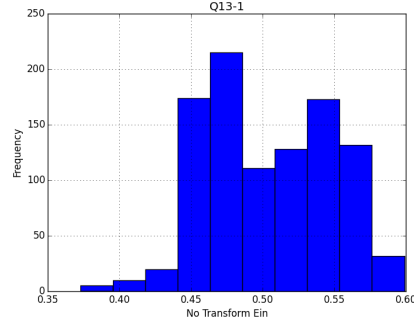


Figure 1: Q13 histogram

After feature transform, we have

$$\tilde{\mathbf{w}} = [-0.991720295, -3.00273423 \times 10^{-4}, -1.31851902 \times 10^{-3}, 1.09524038 \times 10^{-3}, 1.55703088, 1.55594342] \quad (41)$$

with new average  $E_{\text{in}} = 0.124126$ .

□

## Problem 14

The average  $\tilde{\mathbf{w}}_3 = 0.06306984$ .

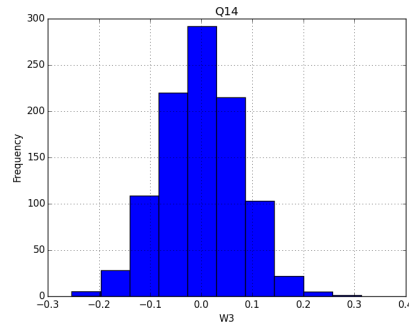


Figure 2: Q14 histogram

□

## Problem 15

The average  $E_{\text{out}} = 0.126195$ .

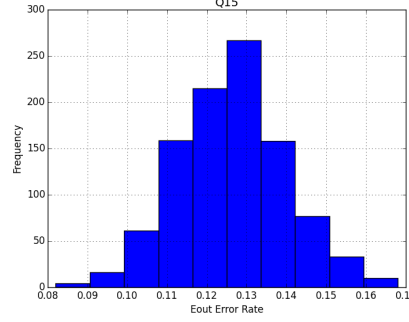


Figure 3: Q15 histogram

□

## Problem 16

Sum the minimized negative log likelihood  $h_y$ , which is  $\min_y (-\ln(h_y))$ , we have

$$E_{\text{in}} = \frac{1}{N} \sum_{n=1}^N \left( -\ln \left( \frac{\exp(\mathbf{w}_{y_n}^T \mathbf{x}_n)}{\sum_{i=1}^K \exp(\mathbf{w}_i^T \mathbf{x}_n)} \right) \right) \quad (42)$$

$$= \frac{1}{N} \sum_{n=1}^N \left( \ln \left( \sum_{i=1}^K \exp(\mathbf{w}_i^T \mathbf{x}_n) \right) - \mathbf{w}_{y_n}^T \mathbf{x}_n \right) \quad (43)$$

□

## Problem 17

$$\frac{\partial E_{\text{in}}}{\partial \mathbf{w}_i} = \frac{1}{N} \sum_{n=1}^N \frac{\partial}{\partial \mathbf{w}_i} \left( \ln \left( \sum_{i=1}^K \exp(\mathbf{w}_i^T \mathbf{x}_n) \right) - \mathbf{w}_{y_n}^T \mathbf{x}_n \right) \quad (44)$$

$$= \frac{1}{N} \sum_{n=1}^N \left( \frac{1}{\sum_{i=1}^K \exp(\mathbf{w}_i^T \mathbf{x}_n)} \frac{\partial}{\partial \mathbf{w}_i} \exp(\mathbf{w}_i^T \mathbf{x}_n) - \frac{\partial}{\partial \mathbf{w}_i} \mathbf{w}_{y_n}^T \mathbf{x}_n \right) \quad (45)$$

$$= \frac{1}{N} \sum_{n=1}^N \left( \frac{\exp(\mathbf{w}_i^T \mathbf{x}_n)}{\sum_{i=1}^K \exp(\mathbf{w}_i^T \mathbf{x}_n)} \mathbf{x}_n - \mathbb{I}[y_n = i] \mathbf{x}_n \right) \quad (46)$$

$$= \frac{1}{N} \sum_{n=1}^N (h_i(\mathbf{x}_n) - \mathbb{I}[y_n = i]) \mathbf{x}_n \quad (47)$$



□

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## Problem 18

The  $E_{\text{out}} = 0.475$  with

$$\begin{aligned}\tilde{\mathbf{w}} = & [0.01878417, -0.01260595, 0.04084862, -0.03266317, 0.01502334, \\ & -0.03667437, 0.01255934, 0.04815065, -0.02206419, 0.02479605, \\ & 0.06899284, 0.0193719, -0.01988549, -0.0087049, 0.04605863, \\ & 0.05793382, 0.061218, -0.04720391, 0.06070375, -0.01610907, -0.03484607]\end{aligned}\tag{48}$$

□

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## Problem 19

The  $E_{\text{out}} = 0.220$  with

$$\begin{aligned}\tilde{\mathbf{w}} = & [-0.00385379, -0.18914564, 0.26625908, -0.35356593, 0.04088776, \\ & -0.3794296, 0.01982783, 0.33391527, -0.26386754, 0.13489328, \\ & 0.4914191, 0.08726107, -0.25537728, -0.16291797, 0.30073678, \\ & 0.40014954, 0.43218808, -0.46227968, 0.43230193, -0.20786372, -0.36936337]\end{aligned}\tag{49}$$

□

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## Problem 20

The  $E_{\text{out}} = 0.473$  with

$$\begin{aligned}\tilde{\mathbf{w}} = & [0.01826899, -0.01308051, 0.04072894, -0.03295698, 0.01498363, \\ & -0.03691042, 0.01232819, 0.04791334, -0.02244958, 0.02470544, \\ & 0.06878235, 0.01897378, -0.02032107, -0.00901469, 0.04589259, \\ & 0.05776824, 0.06102487, -0.04756147, 0.06035018, -0.01660574, -0.03509342]\end{aligned}\tag{50}$$

□

## Problem 21

$$\mathbf{h}^T \mathbf{y} = \sum_{i=1}^N h(\mathbf{x}_i) y_i \quad (51)$$

We just need two times queries to obtain  $\mathbf{h}^T \mathbf{y}$ .

First, take some  $h'$  such that  $h'(\mathbf{x}_i) = 0, \forall i$ . Then we have

$$\text{RMSE}(h') = \sqrt{\frac{1}{N} \sum_{i=1}^N (y_i - h'(\mathbf{x}_i))^2} = \sqrt{\frac{1}{N} \sum_{i=1}^N y_i^2} \Rightarrow \sum_{i=1}^N y_i^2 = N (\text{RMSE}(h'))^2 \quad (52)$$

Second, query for some  $h$ ,

$$\text{RMSE}(h) = \sqrt{\frac{1}{N} \sum_{i=1}^N (y_i - h(\mathbf{x}_i))^2} \quad (53)$$

$$N (\text{RMSE}(h))^2 = \sum_{i=1}^N (y_i - h(\mathbf{x}_i))^2 = \sum_{i=1}^N y_i^2 - 2h(\mathbf{x}_i) y_i + h^2(\mathbf{x}_i) \quad (54)$$

$$\Rightarrow \mathbf{h}^T \mathbf{y} = -\frac{1}{2} \left( N (\text{RMSE}(h) - \text{RMSE}(h'))^2 - \sum_{i=1}^N h^2(\mathbf{x}_i) \right) \quad (55)$$

where we have known the value of  $\sum_{i=1}^N h^2(\mathbf{x}_i)$  since we have already known all  $\mathbf{x}_i$ .

□

## Problem 22

To find  $\min_{\mathbf{w}} \text{RMSE}(H)$ , we need to find  $\nabla \text{RMSE}(H) = 0$ .

$$\nabla \text{RMSE}(H) = 0 \Rightarrow \frac{\partial}{\partial w_k} \sum_{i=1}^N \left( y_i - \sum_{k=1}^K w_k h_k(\mathbf{x}_i) \right)^2 = 0, \forall k \quad (56)$$

$$\Rightarrow 2 \sum_{i=1}^N \left( h_k(\mathbf{x}_i) y_i - h_k(\mathbf{x}_i) \sum_{k=1}^K w_k h_k(\mathbf{x}_i) \right) = 0, \forall k \quad (57)$$

Hence, we need to know the value of  $\mathbf{h}_k^T \mathbf{y}$  for all  $k$ . Hence, follow the conclusion above, we need to query for  $K + 1$  times, where the  $+1$  is for querying the value of  $\sum_{i=1}^N y_i^2$ . The derivation steps are

1. First, use some hypothesis  $h'$  such that  $h'(\mathbf{x}_i) = 0, \forall i$  to get the value of  $\sum_{i=1}^N y_i^2$ .
2. Second, query for  $\text{RMSE}(h_k), \forall k$ , costs  $K$  times query. Then we have all  $\mathbf{h}_k^T \mathbf{y}$ .

Hence, we have

$$\mathbf{h}_k^T \mathbf{y} - \sum_{i=1}^N \left( h_k(\mathbf{x}_i) \sum_{k=1}^K w_k h_k(\mathbf{x}_i) \right) = 0, \quad \forall k \quad (58)$$

Then we can solve the value of  $\mathbf{w}$  to get the minimized value.

□

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## Reference

- [1] Lecture Notes by Hsuan-Tien LIN, Department of Computer Science and Information Engineering, National Taiwan University, Taipei 106, Taiwan.