Machine Learning

Answer Sheet for Homework 6

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Problem 1

With the definition of z_n, rewrite the equation

$$\min_{A,B} F\left(A,B\right) = \frac{1}{N}\sum_{n=1}^{N} \ln\left(1+\exp\left(-y_n\left(Az_n+B\right)\right)\right) \tag{1}$$

So

$$\frac{\partial F}{\partial A} = \frac{1}{N}\sum_{n=1}^{N} -y_n \left(\frac{\exp\left(-y_n\left(Az_n+B\right)\right)}{1+\exp\left(-y_n\left(Az_n+B\right)\right)}\right)^T z_n = \frac{1}{N}\sum_{n=1}^{N} -y_n p_n^T z_n \tag{2}$$

$$\frac{\partial F}{\partial B} = \frac{1}{N}\sum_{n=1}^{N} -y_n \left(\frac{\exp\left(-y_n\left(Az_n+B\right)\right)}{1+\exp\left(-y_n\left(Az_n+B\right)\right)}\right)^T = \frac{1}{N}\sum_{n=1}^{N} -y_n p_n^T \tag{3}$$

Hence

$$\nabla F\left(A,B\right) = \frac{1}{N}\sum_{n=1}^{N} \left[-y_n z_n p_n, -y_n p_n\right]^T \tag{4}$$

$\square$

Problem 2

Use the result of Problem 1 and define $\exp\left(-y_n\left(Az_n+B\right)\right)=\exp\left(\xi_n\right)$, we have

$$\begin{aligned}
\frac{\partial^2 F}{\partial A^2} &= \frac{\partial}{\partial A}\left(\frac{1}{N}\sum_{n=1}^{N}-y_n p_n^T z_n\right) &(5)\\
&= \frac{1}{N}\sum_{n=1}^{N}-y_n\left(\frac{-y_n\exp\left(\xi_n\right)\left(1+\exp\left(\xi_n\right)\right)z_n-y_n\left(\exp\left(\xi_n\right)\right)^2 z_n}{\left(1+\exp\left(\xi_n\right)\right)^2}\right)z_n &(6)\\
&= \frac{1}{N}\sum_{n=1}^{N}-y_n\left(\frac{-y_n\exp\left(\xi_n\right)z_n}{\left(1+\exp\left(\xi_n\right)\right)^2}\right)z_n &(7)\\
&= \frac{1}{N}\sum_{n=1}^{N}\left(y_n\right)^2\left(\frac{\exp\left(\xi_n\right)}{1+\exp\left(\xi_n\right)}\left(1-\frac{\exp\left(\xi_n\right)}{1+\exp\left(\xi_n\right)}\right)\right)z_n^2 &(8)\\
&= \frac{1}{N}\sum_{n=1}^{N}z_n^2 p_n\left(1-p_n\right) &(9)
\end{aligned}$$

where $y_n^2=1$ since $y_n\in\{-1,+1\}$.
The other term is

$$\begin{aligned}
\frac{\partial^2 F}{\partial A\partial B} &= \frac{\partial}{\partial A}\left(\frac{1}{N}\sum_{n=1}^{N}-y_n p_n^T\right)=\frac{1}{N}\sum_{n=1}^{N}-y_n\left(\frac{-y_n\exp\left(\xi_n\right)z_n}{\left(1+\exp\left(\xi_n\right)\right)^2}\right) &(10)\\
&= \frac{1}{N}\sum_{n=1}^{N}z_n p_n\left(1-p_n\right) &(11)\\
\frac{\partial^2 F}{\partial B\partial A} &= \frac{\partial}{\partial B}\left(\frac{1}{N}\sum_{n=1}^{N}-y_n p_n^T z_n\right)=\frac{1}{N}\sum_{n=1}^{N}-y_n\left(\frac{-y_n\exp\left(\xi_n\right)}{\left(1+\exp\left(\xi_n\right)\right)^2}\right)z_n &(12)\\
&= \frac{1}{N}\sum_{n=1}^{N}z_n p_n\left(1-p_n\right) &(13)\\
\frac{\partial^2 F}{\partial B^2} &= \frac{\partial}{\partial B}\left(\frac{1}{N}\sum_{n=1}^{N}-y_n p_n^T\right)=\frac{1}{N}\sum_{n=1}^{N}-y_n\left(\frac{-y_n\exp\left(\xi_n\right)}{\left(1+\exp\left(\xi_n\right)\right)^2}\right) &(14)\\
&= \frac{1}{N}\sum_{n=1}^{N}p_n\left(1-p_n\right) &(15)
\end{aligned}$$

Hence, we have

$$H\left(F\right)=\frac{1}{N}\sum_{n=1}^{N}\begin{bmatrix} z_n^2 p_n\left(1-p_n\right) & z_n p_n\left(1-p_n\right)\\ z_n p_n\left(1-p_n\right) & p_n\left(1-p_n\right)\end{bmatrix} \tag{16}$$

$\square$

Problem 3

As $\gamma \to \infty$, we have

$$\lim_{\gamma\to\infty} \exp\left(-\gamma \left\|\mathbf{x}_n - \mathbf{x}_m\right\|^2\right) = 0 \tag{17}$$

So K should be a zero matrix with size $N \times N$, which is $\mathbf{0}_{N\times N}$.
And $\boldsymbol{\beta}$ is

$$\boldsymbol{\beta} = (\lambda I + K)^{-1} \mathbf{y} = \lambda^{-1}\mathbf{y} \tag{18}$$

□

Problem 4

If $\left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| \geq \epsilon$, then

$$\begin{cases} \left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| - \epsilon = \xi_n^{\wedge} \text{ and } \xi_n^{\vee} = 0, & \text{if } y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b > 0 \\ \\ \left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| - \epsilon = \xi_n^{\vee} \text{ and } \xi_n^{\wedge} = 0, & \text{if } y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b < 0 \end{cases} \tag{19}$$

and if $\left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| < \epsilon$, then $\xi_n^{\wedge} = 0$ and $\xi_n^{\vee} = 0$. Hence, we have

$$\left(\xi_n^{\wedge}\right)^2 + \left(\xi_n^{\vee}\right)^2 = \left(\max\left(0, \left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| - \epsilon\right)\right)^2 \tag{20}$$

So P_2 is equivalent to

$$\min_{b,\mathbf{w}} \left(\frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_{n=1}^{N}\left(\max\left(0, \left|y_n - \mathbf{w}^T\phi\left(\mathbf{x}_n\right) - b\right| - \epsilon\right)\right)^2\right) \tag{21}$$

with no constraints.

□

Problem 5

The first term is of course

$$\frac{\partial}{\partial\beta_m}\left(\frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}\beta_n\beta_m K\left(\mathbf{x}_n, \mathbf{x}_m\right)\right) = \sum_{n=1}^{N}\beta_n K\left(\mathbf{x}_n, \mathbf{x}_m\right) \tag{22}$$

With $\mathbf{w}_*$, rewrite the result of Problem 4,

$$\text{something} = C\sum_{n=1}^{N}\left(\max\left(0, \left|y_n - \sum_{m=1}^{N}\beta_m K\left(\mathbf{x}_n, \mathbf{x}_m\right) - b\right| - \epsilon\right)\right)^2 \tag{23}$$

$$= C\sum_{n=1}^{N}\left(\max\left(0, \left|y_n - s_n\right| - \epsilon\right)\right)^2 \tag{24}$$

Consider the following cases.

1. $|y_n - s_n| \geq \epsilon$.

   Then we have

$$\begin{aligned}
\frac{\partial}{\partial \beta_m}(\text{something}) &= \frac{\partial}{\partial \beta_m}\left(C\left(|y_n - s_n| - \epsilon\right)^2\right) && (25)\\
&= \left(2C\left(|y_n - s_n| - \epsilon\right)\right)\frac{\partial}{\partial \beta_m}|y_n - s_n| && (26)\\
&= -2C\left(|y_n - s_n| - \epsilon\right)\operatorname{sign}\left(y_n - s_n\right)\frac{\partial s_n}{\partial \beta_m} && (27)\\
&= -2C\left(|y_n - s_n| - \epsilon\right)\operatorname{sign}\left(y_n - s_n\right)K\left(\mathbf{x}_n, \mathbf{x}_m\right) && (28)
\end{aligned}$$

2. $|y_n - s_n| < \epsilon$.

   Then we have

$$\frac{\partial}{\partial \beta_m}(\text{something}) = 0 \tag{29}$$

So we have

$$\frac{\partial F\left(b, \boldsymbol{\beta}\right)}{\partial \beta_m} = \sum_{n=1}^{N}\beta_n K\left(\mathbf{x}_n, \mathbf{x}_m\right) - 2C\sum_{n=1}^{N}[\![|y_n - s_n| \geq \epsilon]\!]\left(|y_n - s_n| - \epsilon\right)\operatorname{sign}\left(y_n - s_n\right)K\left(\mathbf{x}_n, \mathbf{x}_m\right) \tag{30}$$

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Problem 6

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Problem 7

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Problem 8

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Problem 9

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Problem 10

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Problem 11

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Problem 12

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Problem 13

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Problem 14

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Problem 15

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Problem 16

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Problem 17

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Problem 18

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Problem 19

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Problem 20

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Reference

[1] Lecture Notes by Hsuan-Tien LIN, Department of Computer Science and Information Engineering, National Taiwan University, Taipei 106, Taiwan.