Machine Learning

Answer Sheet for Homework 7

Da-Min HUANG

R04942045

Graduate Institute of Communication Engineering, National Taiwan University

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Problem 1

Set $\mu_- = 1 - \mu_+$, we have

$$1 - \mu_+^2 - \mu_-^2 = 1 - \mu_+^2 - (1-\mu_+)^2 = (1-\mu_+)(1+\mu_+) - (1-\mu_+)^2 \tag{1}$$

$$= 2\mu_+(1-\mu_+) = -2\mu_+^2 + 2\mu_+ = -2\left(\mu_+ - \frac{1}{2}\right)^2 + \frac{1}{2} \tag{2}$$

$$\leq \frac{1}{2} \tag{3}$$

Hence, if $\mu_+ = 1/2 \in [0,1]$, then the maximum value of Gini index is $1/2$.

$\square$

Problem 2

The normalized Gini index is

$$\frac{\left(1 - \mu_+^2 - \mu_-^2\right)}{\left(\frac{1}{2}\right)} = 2\left(1 - \mu_+^2 - \mu_-^2\right) \tag{4}$$

The squared error can be rewritten as

$$\mu_+\left(1 - (\mu_+ - \mu_-)\right)^2 + \mu_-\left(-1 - (\mu_+ - \mu_-)\right)^2 = 4\mu_+(1-\mu_+)^2 + 4\mu_+^2(1-\mu_+) \tag{5}$$

$$= 4\mu_+(1-\mu_+) \leq 4 \times \frac{1}{4} = 1 \tag{6}$$

Hence the normalized squared error is

$$4\mu_+ (1-\mu_+) = 2\left(2\mu_+ (1-\mu_+)\right) = 2\left((1-\mu_+)(1+\mu_+) - (1-\mu_+)^2\right) \tag{7}$$
$$= 2\left(1-\mu_+^2-\mu_-^2\right) \tag{8}$$

which is equal to normalized Gini index.

□

Problem 3

The probability of one example not sampled is

$$\left(1-\frac{1}{N}\right)^{pN} = \frac{1}{\left(\frac{N}{N-1}\right)^{pN}} = \frac{1}{\left(1+\frac{1}{N-1}\right)^{pN}} = \left(\frac{1}{\left(1+\frac{1}{N-1}\right)^{N}}\right)^{p} \tag{9}$$

As $N \to \infty$, we have

$$\lim_{N\to\infty}\left(\frac{1}{\left(1+\frac{1}{N-1}\right)^{N}}\right)^{p} = \left(\lim_{N\to\infty}\frac{1}{\left(1+\frac{1}{N-1}\right)^{N}}\right)^{p} = \left(\frac{1}{e}\right)^{p} = e^{-p} \tag{10}$$

So there approximately $e^{-p} \cdot N$ of the examples not sampled.

□

Problem 4

Since $G = \text{Uniform}\left(\{g_k\}_{k=1}^3\right)$, so if at least two terms of $\{g_k\}_{k=1}^3$ ouput wrong result, then G outputs wrong result. Let $\{E_k\}_{k=1}^3$ be the set of examples that $\{g_k\}_{k=1}^K$ got wrong results. Apparently $|E_3| > |E_2| > |E_1|$ and $|E_1| + |E_2| > |E_3|$. So

1. Maximum of $E_{\text{out}}(G)$ happens at $E_3 \subset (E_1 \cup E_2)$. Then G outputs wrong result in the region of E_3 with $E_{\text{out}}(G) = 0.35$.

2. Minimum of $E_{\text{out}}(G)$ happens at $E_i \cap E_j = \phi$, $i \neq j$ and $1 \leq i, j \leq 3$ with $i, j \in \mathbb{N}$. Then G always ouputs the correct result since $(E_1 \cup E_2 \cup E_3) \subset \{\text{all examples}\}$.

Hence, $0 \leq E_{\text{out}}(G) \leq 0.35$.

□

Problem 5

Since $G = \text{Uniform}\left(\{g_k\}_{k=1}^K\right)$, so if at least $(K+1)/2$ terms of $\{g_k\}_{k=1}^K$ ouput wrong result, then G outputs wrong result. Let $\{E_k\}_{k=1}^K$ be the set of examples that $\{g_k\}_{k=1}^K$ got wrong results.

If G outputs wrong result on some example $\mathbf{x}$, then we have

$$\mathbf{x} \in \bigcap_{i=1}^{\left(\frac{K+1}{2}\right)+m} E_{\alpha_i} \tag{11}$$

where $\alpha_i \in \{1, 2, \ldots, K\}$ satisfies $1 \leq \alpha_i \leq K$ and $m \in (\mathbb{N} \cup \{0\})$ with $0 \leq m < K + 1/2$. And

$$\left| \bigcap_{i=1}^{\left(\frac{K+1}{2}\right)+m} E_{\alpha_i} \right| \leq \frac{2}{K+1+2m} \sum_{k=1}^{K} e_k \leq \frac{2}{K+1} \sum_{k=1}^{K} e_k \tag{12}$$

(12) holds due to

$$\bigcap_{i=1}^{\left(\frac{K+1}{2}\right)+m} E_{\alpha_i} \subseteq E_\beta \text{ with } |E_\beta| \leq |E_{\alpha_i}|,\ \forall i \tag{13}$$

where β is some index such that $|E_\beta| = \min_{\alpha_i} |E_{\alpha_i}|$. So

$$\left(\frac{K+1}{2} + m\right) \left| \bigcap_{i=1}^{\left(\frac{K+1}{2}\right)+m} E_{\alpha_i} \right| \leq \left(\frac{K+1}{2} + m\right) |E_\beta| \leq \sum_{i=1}^{\left(\frac{K+1}{2}\right)+m} |E_{\alpha_i}| \leq \sum_{k=1}^{K} |E_k| \tag{14}$$

(14) holds since size of E_β is the samllest among $\left(\left(\frac{K+1}{2}\right) + m\right)$ terms and $\sum_{k=1}^{K} |E_k|$ must contains the $\left(\left(\frac{K+1}{2}\right) + m\right)$ terms. Hence, we have

$$E_{\text{out}}(G) \leq \frac{2}{K+1+2m} \sum_{k=1}^{K} e_k \leq \frac{2}{K+1} \sum_{k=1}^{K} e_k \tag{15}$$

where $|E_k| = e_k$ by definition.

$\square$

Problem 6

By the definition of U_t, we have

$$
\begin{aligned}
U_{t+1} &= \frac{1}{N}\sum_{n=1}^{N}\exp\left(-y_n\sum_{\tau=1}^{t}\alpha_\tau g_\tau\left(\mathbf{x}_n\right)\right) && (16)\\
&= \frac{1}{N}\sum_{n=1}^{N}\exp\left(-y_n\sum_{\tau=1}^{t-1}\alpha_\tau g_\tau\left(\mathbf{x}_n\right) - y_n\alpha_t g_t\left(\mathbf{x}_n\right)\right) && (17)\\
&= \frac{1}{N}\sum_{n=1}^{N}\exp\left(-y_n\sum_{\tau=1}^{t-1}\alpha_\tau g_\tau\left(\mathbf{x}_n\right)\right)\exp\left(-y_n\alpha_t g_t\left(\mathbf{x}_n\right)\right) && (18)\\
&= \sum_{n=1}^{N}u_n^{(t)}\exp\left(-y_n\alpha_t g_t\left(\mathbf{x}_n\right)\right) && (19)\\
&= \sum_{\substack{n\\ y_n\neq g_t(\mathbf{x}_n)}} u_n^{(t)}\exp\left(-y_n\alpha_t g_t\left(\mathbf{x}_n\right)\right) + \sum_{\substack{n\\ y_n= g_t(\mathbf{x}_n)}} u_n^{(t)}\exp\left(-y_n\alpha_t g_t\left(\mathbf{x}_n\right)\right) && (20)\\
&= \sum_{\substack{n\\ y_n\neq g_t(\mathbf{x}_n)}} u_n^{(t)}\exp\left(\alpha_t\right) + \sum_{\substack{n\\ y_n= g_t(\mathbf{x}_n)}} u_n^{(t)}\exp\left(-\alpha_t\right) && (21)\\
&= \exp\left(\alpha_t\right)\left(\epsilon_t\right)\sum_{n=1}^{N}u_n^{(t)} + \exp\left(-\alpha_t\right)\left(1-\epsilon_t\right)\sum_{n=1}^{N}u_n^{(t)} && (22)\\
&= U_t\left(\exp\left(\alpha_t\right)\left(\epsilon_t\right) + \exp\left(-\alpha_t\right)\left(1-\epsilon_t\right)\right) = U_t\cdot 2\sqrt{\epsilon_t\left(1-\epsilon_t\right)} && (23)
\end{aligned}
$$

Since

$$
U_1 = \sum_{n=1}^{N}u_n^{(1)} = \sum_{n=1}^{N}\frac{1}{N} = 1 \tag{24}
$$

we have

$$
\begin{aligned}
U_3 &= U_2\cdot 2\sqrt{\epsilon_2\left(1-\epsilon_2\right)} = \left(U_1\cdot 2\sqrt{\epsilon_1\left(1-\epsilon_1\right)}\right)\cdot 2\sqrt{\epsilon_2\left(1-\epsilon_2\right)} && (25)\\
&= 4\sqrt{\epsilon_1\epsilon_2\left(1-\epsilon_1\right)\left(1-\epsilon_2\right)} && (26)
\end{aligned}
$$

which can be generalized as

$$
U_{T+1} = \prod_{t=1}^{T}\left(2\sqrt{\epsilon_t\left(1-\epsilon_t\right)}\right) \tag{27}
$$

□

Problem 7

To compute s_n, we need to find the optimal η of

$$\min_{\eta} \frac{1}{N} \sum_{n=1}^{N} \left((y_n - s_n) - \eta g_t(\mathbf{x}_n) \right)^2 := A \tag{28}$$

From $\partial A / \partial \eta = 0$, we have

$$\eta = \frac{\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s_n)}{\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)} \tag{29}$$

Now $s_n = 0$ and $g_1(\mathbf{x}) = 2$, so

$$\eta = \frac{2 \sum_{n=1}^{N} y_n}{4 \sum_{n=1}^{N}} = \frac{1}{2N} \sum_{n=1}^{N} y_n \tag{30}$$

Since $\eta = \alpha_1$, so

$$\alpha_1 g_1(\mathbf{x}_n) = \frac{2}{2N} \sum_{n=1}^{N} y_n = \frac{1}{N} \sum_{n=1}^{N} y_n = s_n \tag{31}$$

$\square$

Problem 8

From the equatio of optimal η, we have

$$\eta = \frac{\sum_{n=1}^{N} g_t(\mathbf{x}_n)(y_n - s'_n)}{\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)} = \frac{\sum_{n=1}^{N} y_n g_t(\mathbf{x}_n) - \sum_{n=1}^{N} s'_n g_t(\mathbf{x}_n)}{\sum_{n=1}^{N} g_t^2(\mathbf{x}_n)} = \alpha_t \tag{32}$$

so

$$\sum_{n=1}^{N} y_n g_t(\mathbf{x}_n) - \sum_{n=1}^{N} s'_n g_t(\mathbf{x}_n) = \alpha_t \sum_{n=1}^{N} g_t^2(\mathbf{x}_n) = \sum_{n=1}^{N} \alpha_t g_t^2(\mathbf{x}_n) = \sum_{n=1}^{N} (s_n - s'_n) g_t(\mathbf{x}_n) \tag{33}$$

where s'_n is defined as the s_n in iteration $(t-1)$ and $s_n = s'_n + \alpha_t g_t(\mathbf{x}_n)$, so

$$\sum_{n=1}^{N} s_n g_t(\mathbf{x}_n) = \sum_{n=1}^{N} y_n g_t(\mathbf{x}_n) \tag{34}$$

$\square$

Problem 9

$\text{OR}\,(x_1, x_2, \ldots, x_d)$ means outputs TRUE if one input is TRUE; outputs FALSE if all inputs are FALSE.

<u>Claim</u>: $(w_0, w_1, \ldots, w_d) = (d-1, 1, \ldots, 1)$ implements OR.

<u>Proof of Claim</u>:

1. If all $x_i = -1$, then we have

$$\text{sign}\left(\sum_{i=0}^{d} w_i x_i\right) = \text{sign}\left(d - 1 + \sum_{i=1}^{d}(-1)\right) = \text{sign}\,(-1) = \text{FALSE} \tag{35}$$

2. If some $x_i = +1$ and others are -1, we have

$$\text{sign}\left(\sum_{i=0}^{d} w_i x_i\right) = \text{sign}\,(d - 1 + 1 + (-1)\,(d-1)) = \text{sign}\,(+1) = \text{TRUE} \tag{36}$$

Hence, $(w_0, w_1, \ldots, w_d) = (d-1, 1, \ldots, 1)$ implements OR.

$\square$

Problem 10

<u>Claim</u>: $D \geq 5$.

<u>Proof of Claim</u>:

From the conclusion of Problem 21, we have $D = 5$.

$\square$

Problem 11

<u>Claim</u>: Only the gradient components with respect to $w_{01}^{(L)}$ may be non-zero, all other gradient components must be zero.

<u>Proof of Claim</u>:

Consider

$$\frac{\partial e_n}{\partial w_{i1}^{(L)}} = -2\left(y_n - s_1^{(L)}\right)\left(x_i^{(L-1)}\right) = -2\left(y_n - \sum_{j=0}^{d^{(L-2)}} w_{j1}^{(L)} x_j^{(L-1)}\right)\left(x_i^{(L-1)}\right) \tag{37}$$

Since all $w_{ij}^{(\ell)} = 0$, we have

$$\frac{\partial e_n}{\partial w_{i1}^{(L)}} = -2 y_n x_i^{(L-1)} \tag{38}$$

If $i \neq 0$, then

$$\frac{\partial e_n}{\partial w_{i1}^{(L)}} = -2y_n x_i^{(L-1)} = -2y_n \tanh\left(s_i^{(L-1)}\right) = -2y_n \tanh\left(\sum_{j=0}^{d^{(L-2)}} w_{ji}^{(L-1)} x_j^{(L-2)}\right) = 0 \quad (39)$$

since all $w_{ij}^{(\ell)} = 0$.
If $i = 0$, then

$$\frac{\partial e_n}{\partial w_{01}^{(L)}} = -2y_n x_0^{(L-1)} = -2y_n \quad (40)$$

If $y_n \neq 0$, then

$$\frac{\partial e_n}{\partial w_{01}^{(L)}} \neq 0 \quad (41)$$

Similarly, we have

$$\frac{\partial e_n}{\partial w_{0j}^{(\ell)}} = \delta_j^{(\ell)} \left(x_0^{(\ell-1)}\right) = \delta_j^{(\ell)} = \sum_k \left(\delta_k^{(\ell+1)}\right)\left(w_{jk}^{(\ell+1)}\right)\left(\frac{\partial \tanh\left(s_j^{(\ell)}\right)}{\partial s_j^{(\ell)}}\right) = 0 \quad (42)$$

since all $w_{ij}^{(\ell)} = 0$.

$\square$

Problem 12

For $\ell = 1$ and all $w_{ij}^{(\ell)}$ initialized as 1, we have

$$\eta x_i^{(0)} \delta_j^{(1)} = \eta x_i^{(0)} \sum_k \left(\delta_k^{(2)}\right)\left(w_{jk}^{(2)}\right)\left(\frac{\partial \tanh\left(s_j^{(1)}\right)}{\partial s_j^{(1)}}\right) \quad (43)$$

$$= \eta x_i^{(0)} \sum_k \left(\delta_k^{(2)}\right)\left(\frac{\partial \tanh\left(s_j^{(1)}\right)}{\partial s_j^{(1)}}\right) \quad (44)$$

and

$$s_j^{(1)} = \sum_{i=0}^{d^{(0)}} w_{ij}^{(1)} x_i^{(0)} = \sum_{i=0}^{d^{(0)}} x_i^{(0)} \quad (45)$$

So we have

$$s_1^{(1)} = s_2^{(1)} = \cdots = s_{d^{(1)}}^{(1)} \Rightarrow \delta_1^{(1)} = \delta_2^{(1)} = \cdots = \delta_{d^{(1)}}^{(1)} \quad (46)$$

Hence, all update term of $w_{ij}^{(1)}$ is the same for $1 \leq j \leq d^{(1)}$, so

$$w_{ij}^{(1)} = w_{i(j+1)}^{(1)} \tag{47}$$

for $1 \leq j \leq d^{(1)} - 1$.

$\square$

Problem 13

The rules of following tree are

1. (feature column, s, θ). The meaning of combination of numbers.

2. If the feature of $\mathbf{x}$ is smaller than θ, then go to the left tree; if not, then go to the right tree.

3. If go to left and there is no more node, then return $s \times (+1)$; if go to the right and there is no more node, return $s \times (-1)$, where s is from the last node.

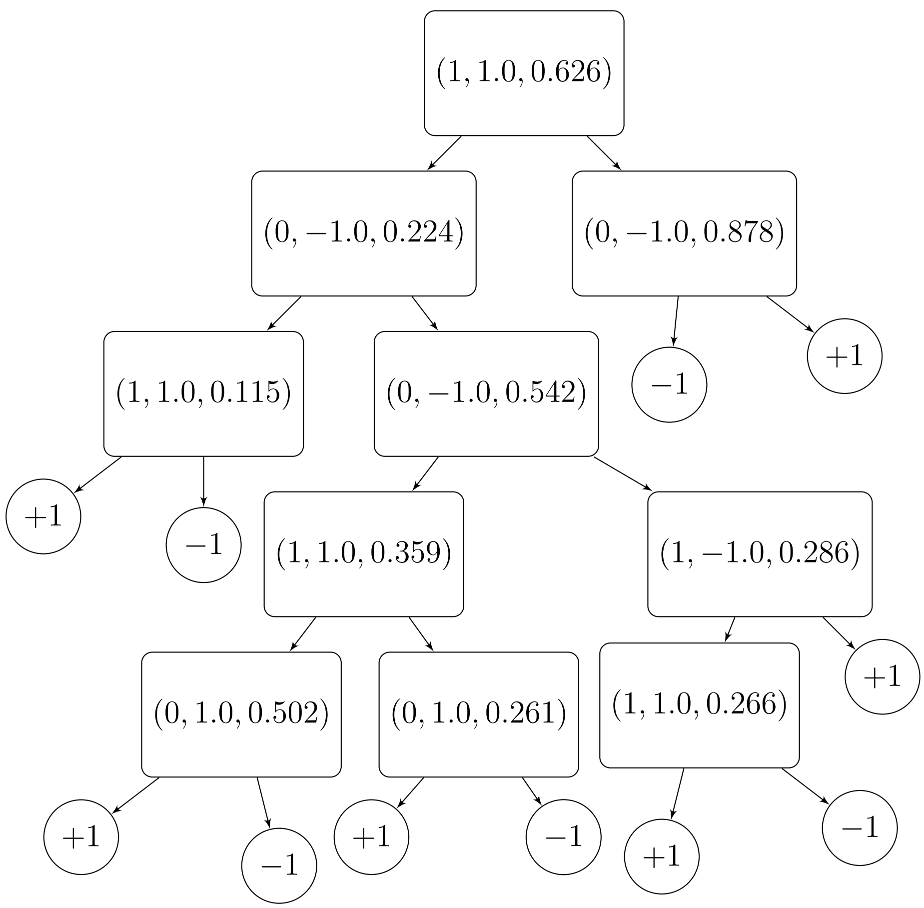


Figure 1: Tree Graph

□

Problem 14

$E_{\text{in}} = 0.0$.

□

Problem 15

$E_{\text{out}} = 0.126$.

□

Problem 16

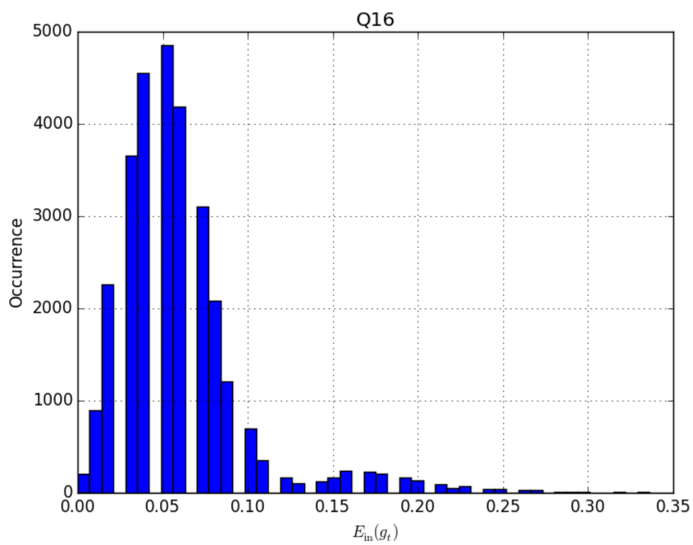


Figure 2: Q16

The average $E_{\text{in}}(g_t)$ of total 30,000 trees is 0.0593.

□

Problem 17

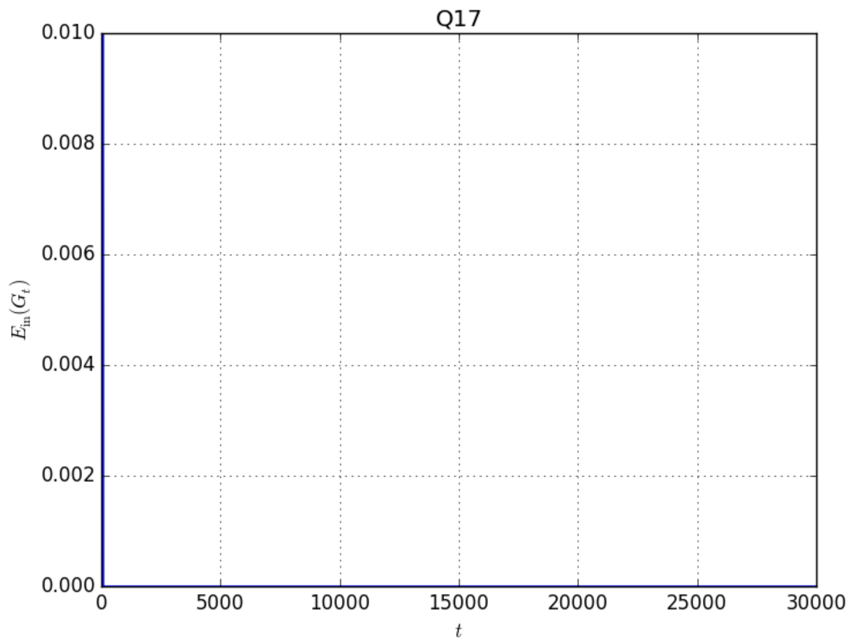


Figure 3: Q17

The average $E_{\mathrm{in}}(G_t)$ of total 30,000 trees is 0. Since most values are 0 so the figure seems nothing left.

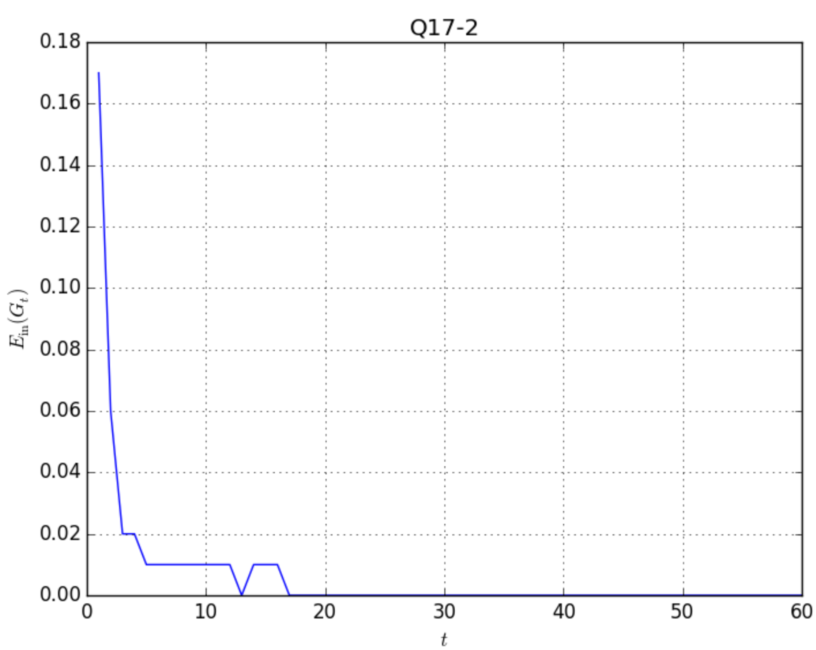


Figure 4: Q17 first 60 points

The figure of first 60 points. We can see that after the 20^{th} point, $E_{\text{in}}(G_t)$ is almost 0.

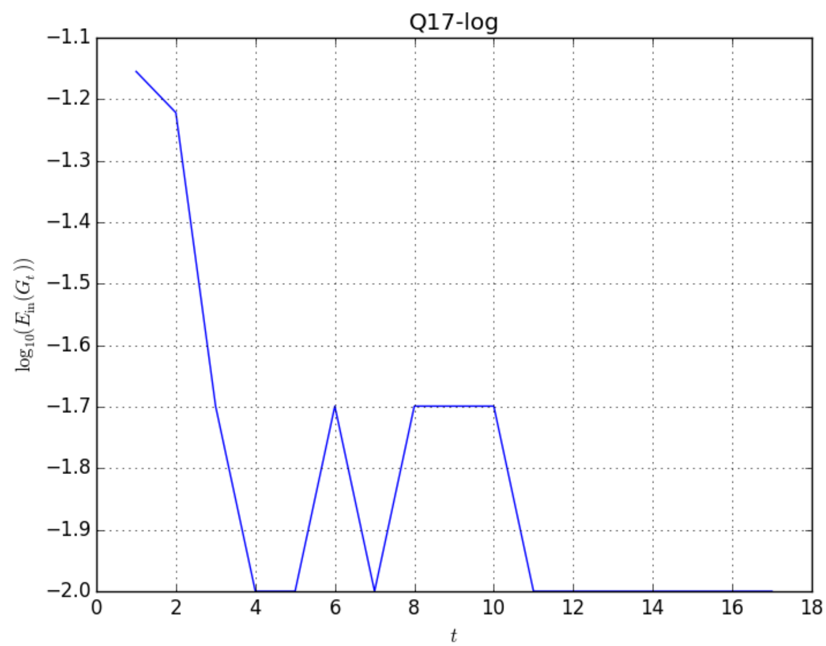


Figure 5: Q17 $\log_{10}(E_{\text{in}}(G_t))$

The figure of log value of $E_{\text{in}}(G_t)$ (removed $E_{\text{in}}(G_t) = 0$ points).
We can see that most of the $E_{\text{in}}(G_t) = 0$ (only 17 points left).

$\square$

Problem 18

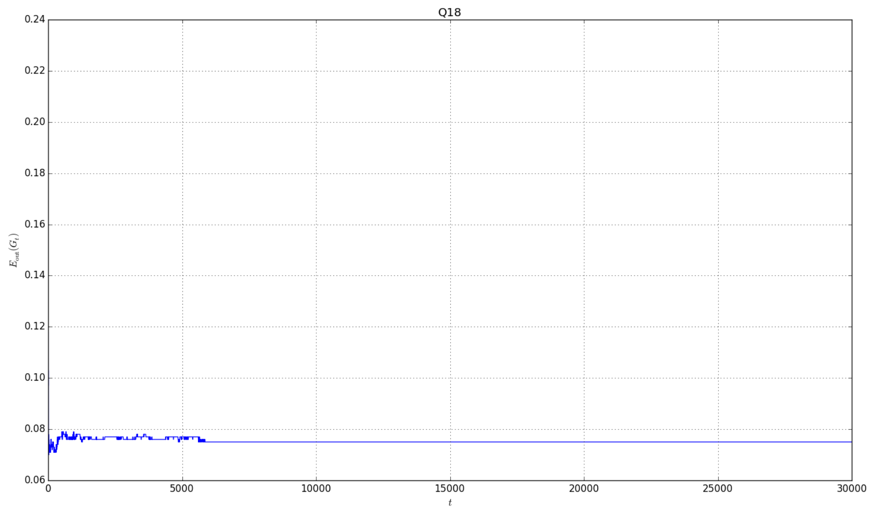


Figure 6: Q18

The average $E_{\text{out}}\left(G_t\right)$ of total 30,000 trees is 0.0753.
This curve approaches the average value as $t \to 30,000$. The curve of Problem 17 oscillates and most points are 0.

$\square$

Problem 19

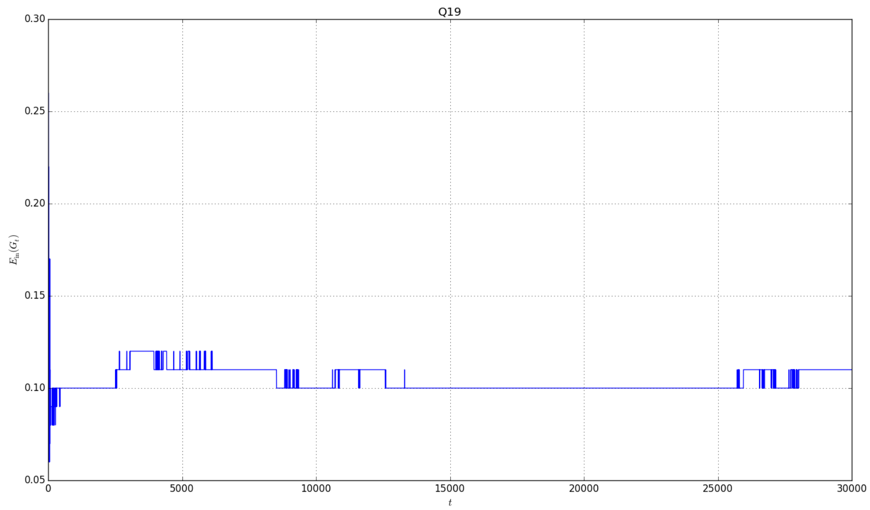


Figure 7: Q19

The average $E_{\text{in}}(G_t)$ of total 30,000 trees is 0.1042.

□

Problem 20

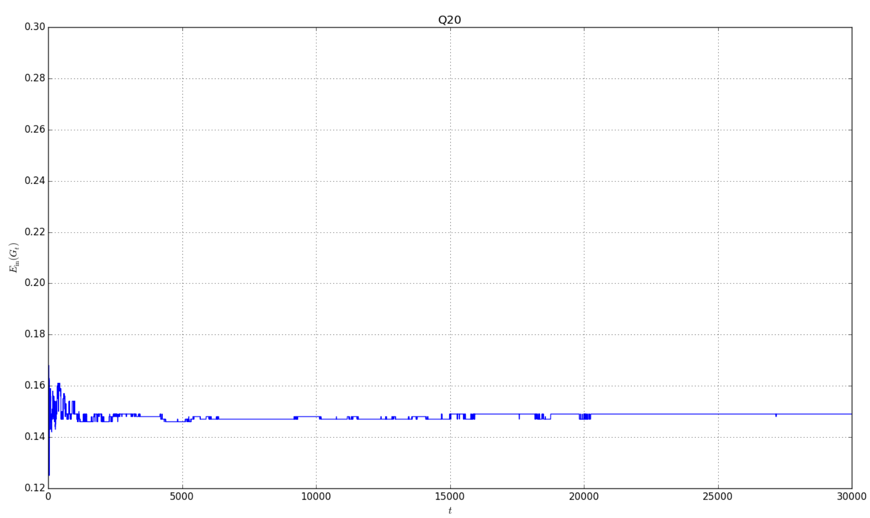


Figure 8: Q20

The average $E_{\text{out}}(G_t)$ of total 30,000 trees is 0.1483.

This curve approaches the average value as $t \to 30,000$. The curve of Problem 19 oscillates between the average value.

$\square$

Problem 21

Consider the projection of all convex points to diagonal line of N-dimensional hypercube, the value of each function is

1. 2-D: $(-1)\,(1,1)\,(-1)$
2. 3-D: $(-1)\,(1,1,1)\,(-1,-1,-1)\,(1)$
3. 4-D: $(-1)\,(1,1,1,1)\,(-1,-1,-1,-1,-1,-1)\,(1,1,1,1)\,(1)$

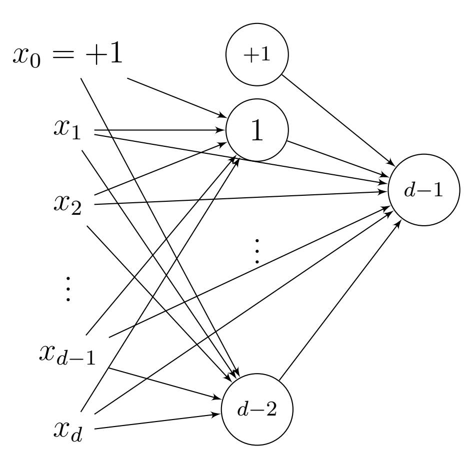
$\vdots$

the convex points are the input of neural network. The projections are linear transformation of these convex points, which can be implemented by d linear classfier (AND or OR). By uniform blending all these linear classfier and $\text{sign}\,(\{\text{all linear classifer}\})$, we implement the $\text{XOR}\,(x_1, x_2, \ldots, x_d)$.
The input layer to hidden layer $(d - D)$ is the linear transformation, the hidden layer to output layer $(D - 1)$ is the uniform blending of all these linear classifer.

$\square$

Problem 22

The answer is $D = d - 1$. Like



$\square$

Reference

[1] Lecture Notes by Hsuan-Tien LIN, Department of Computer Science and Information Engineering, National Taiwan University, Taipei 106, Taiwan.