Report HW1

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Introduction

We conducted the following exercises with 10^7 samples for each experiment.

We also used 5 different RNGs: (`MT19937`, `PCG64`, `PCG64DXSM`, `Philox`, `SFC64`) provided by the `numpy` library.

Exercise 1

Let X be the random variable representing the drawn number from one of the four Gaussians, and let Y be the random variable representing the Gaussian from which the number is drawn.

The expectation of X can be calculated as follows:

$$\begin{aligned}
\mathrm{E}\left[X\right] &= \mathrm{E}\left[X \mid Y = 1\right] P(Y = 1) + \mathrm{E}\left[X \mid Y = 2\right] P(Y = 2) \\
&\quad + \mathrm{E}\left[X \mid Y = 3\right] P(Y = 3) + \mathrm{E}\left[X \mid Y = 4\right] P(Y = 4) \\
&= 7.95.
\end{aligned}$$

The variance of X can be calculated as follows:

$$\begin{aligned}
\mathrm{Var}\left[X\right] &= \mathrm{E}\left[\mathrm{Var}\left[X \mid Y\right]\right] + \mathrm{Var}\left[\mathrm{E}\left[X \mid Y\right]\right] \\
&= \sigma_1^2 \cdot P(Y = 1) + \sigma_2^2 \cdot P(Y = 2) + \sigma_3^2 \cdot P(Y = 3) + \sigma_4^2 \cdot P(Y = 4) \\
&\quad + \mathrm{E}\left[\mathrm{E}\left[X \mid Y\right]^2\right] - \mathrm{E}\left[X\right]^2 \\
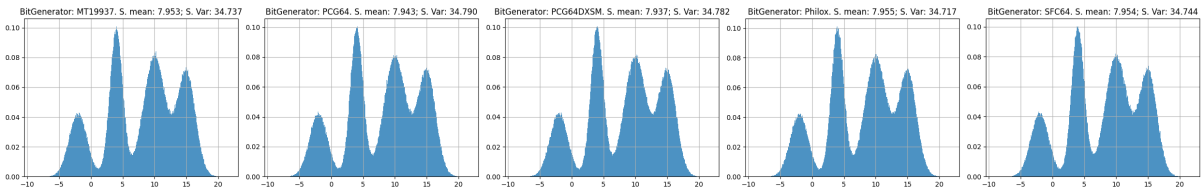&= 34.7475.
\end{aligned}$$

Figure 1: Density plot of the samples drawn from the four Gaussians.

Exercise 2

Let $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim U(a, b)$. We wish to calculate the probability:

$$P(X > Y).$$

Since X and Y are independent, we can write:

$$P(X > Y) = \int_a^b P(X > y \mid Y = y) f_Y(y) \, dy,$$

where $f_Y(y)$ is the density of the uniform distribution on $[a, b]$:

$$f_Y(y) = \frac{1}{b-a}, \quad \text{for } y \in [a, b].$$

For an exponential random variable X with parameter λ, the survival function is:

$$P(X > y) = \int_y^\infty \lambda e^{-\lambda x} \, dx = e^{-\lambda y}, \quad \text{for } y \geq 0.$$

Substituting the expression for $P(X > y)$ into the integral:

$$P(X > Y) = \frac{1}{b-a} \int_a^b e^{-\lambda y} \, dy.$$

We compute the integral:

$$\int_a^b e^{-\lambda y} \, dy = \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_a^b = \frac{1}{\lambda} \left(e^{-\lambda a} - e^{-\lambda b} \right).$$

We finally obtain:

$$P(X > Y) = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda (b-a)}.$$

Thus, the final expression is:

$$\boxed{P(X > Y) = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda (b-a)}.}$$

Setting $\lambda = 1$, $a = 0$, and $b = 5$, we have:

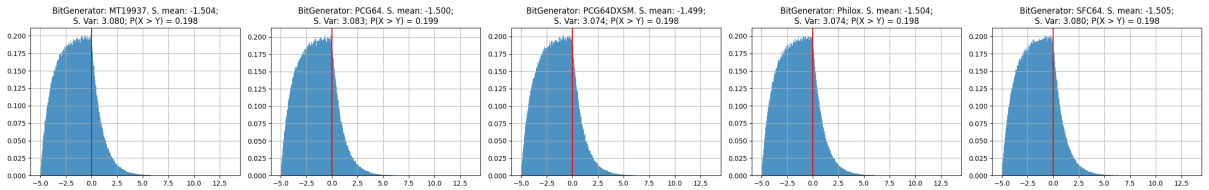$$P(X > Y) = \frac{1 - e^{-5}}{5} \approx 0.1987.$$

Figure 2: Density plot of the difference between the exponential and uniform distributions.