Report HW1

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1 Exercise 1

Gaussians:

N°	Mean	Variance	Probability
1	-2	2	0.15
2	4	1	0.25
3	10	3	0.35
4	15	2	0.25

Let X be the random variable representing the drawn number from one of the four Gaussians. Let Y be the random variable representing the Gaussian from which the number is drawn.

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[X|Y=1]\,P(Y=1) + \mathrm{E}[X|Y=2]\,P(Y=2) + \mathrm{E}[X|Y=3]\,P(Y=3) + \mathrm{E}[X|Y=4]\,P(Y= \\
&= -2 \cdot 0.15 + 4 \cdot 0.25 + 10 \cdot 0.35 + 15 \cdot 0.25 \\
&= -0.3 + 1 + 3.5 + 3.75 \\
&= 7.95
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X|Y]] + \mathrm{Var}[\mathrm{E}[X|Y]] \\
&= \sigma_1^2 \cdot P(Y=1) + \sigma_2^2 \cdot P(Y=2) + \sigma_3^2 \cdot P(Y=3) + \sigma_4^2 \cdot P(Y=4) \\
&+ \mathrm{E}\left[\mathrm{E}[X|Y]^2\right] - \mathrm{E}[X]^2 \\
&= 2 \cdot 0.15 + 1 \cdot 0.25 + 3 \cdot 0.35 + 2 \cdot 0.25 \\
&+ 0.15 \cdot (-2)^2 + 0.25 \cdot 4^2 + 0.35 \cdot 10^2 + 0.25 \cdot 15^2 \\
&- 7.95^2 \\
&= 0.3 + 0.25 + 1.05 + 0.5 \\
&+ 0.6 + 4 + 35 + 56.25 \\
&- 63.2025 \\
&= 2.1 + 95.85 - 63.2025 \\
&= 34.7475
\end{aligned}
$$

2 Exercise 2

Let $X \sim \text{Exp}(\lambda)$ and $Y \sim U(a, b)$. We wish to calculate the probability:

$$P(X > Y).$$

Since X and Y are independent, we can write:

$$P(X > Y) = \int_a^b P(X > y \mid Y = y) f_Y(y)\, dy,$$

where $f_Y(y)$ is the density of the uniform distribution on $[a, b]$:

$$f_Y(y) = \frac{1}{b-a}, \quad \text{for } y \in [a, b].$$

For an exponential random variable X with parameter λ, the survival function is:

$$P(X > y) = \int_y^\infty \lambda e^{-\lambda x}\, dx = e^{-\lambda y}, \quad \text{for } y \geq 0.$$

Substituting the expression for $P(X > y)$ into the integral:

$$P(X > Y) = \frac{1}{b-a} \int_a^b e^{-\lambda y}\, dy.$$

We compute the integral:

$$\int_a^b e^{-\lambda y}\, dy = \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_a^b = \frac{1}{\lambda} \left(e^{-\lambda a} - e^{-\lambda b} \right).$$

We finally obtain:

$$P(X > Y) = \frac{1}{b-a} \cdot \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda} = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda(b-a)}.$$

Thus, the final expression is:

$$\boxed{P(X > Y) = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda(b-a)}.}$$

Setting $\lambda = 1$, $a = 0$, and $b = 5$, we have:

$$P(X > Y) = \frac{e^{-1 \cdot 0} - e^{-1 \cdot 5}}{1 \cdot (5-0)} = \frac{1 - e^{-5}}{5} \approx 0.1987.$$