Report HW1

Simone De Carli
`simone.decarli@studenti.unitn.it`

Damiano Salvaterra
`damiano.salvaterra@studenti.unitn.it`

Introduction

We conducted the following exercises with 10^6 samples for each experiment.

We also used 5 different RNGs: (`MT19937`, `PCG64`, `PCG64DXSM`, `Philox`, `SFC64`) provided by the `numpy` library.

Exercise 1

Let X be the random variable representing the drawn number from one of the four Gaussians, and let Y be the random variable representing the Gaussian from which the number is drawn.

The expectation of X can be calculated as follows:

$$
\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[X \mid Y=1]\, P(Y=1) + \mathrm{E}[X \mid Y=2]\, P(Y=2) \\
&\quad + \mathrm{E}[X \mid Y=3]\, P(Y=3) + \mathrm{E}[X \mid Y=4]\, P(Y=4) \\
&= 7.95.
\end{aligned}
$$

The variance of X can be calculated as follows:

$$
\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid Y]] + \mathrm{Var}[\mathrm{E}[X \mid Y]] \\
&= \sigma_1^2 \cdot P(Y=1) + \sigma_2^2 \cdot P(Y=2) + \sigma_3^2 \cdot P(Y=3) + \sigma_4^2 \cdot P(Y=4) \\
&\quad + \mathrm{E}\left[\mathrm{E}[X \mid Y]^2\right] - \mathrm{E}[X]^2 \\
&= 34.7475.
\end{aligned}
$$

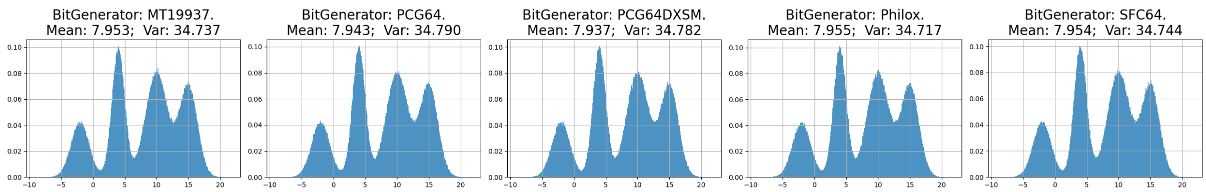

Figure 1: Density plot of the samples drawn from the four Gaussians.

Exercise 2

Let $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim U(a,b)$. We wish to calculate the probability: $P(X > Y)$.

Since X and Y are independent, we can write:

$$P(X > Y) = \int_a^b P(X > y \mid Y = y) f_Y(y)\, dy$$

where $f_Y(y)$ is the density of the uniform distribution on $[a, b]$:

$$f_Y(y) = \frac{1}{b-a}, \qquad \text{for } y \in [a, b]$$

For an exponential random variable X with parameter λ, the survival function is:

$$P(X > y) = \int_y^\infty \lambda e^{-\lambda x}\, dx = e^{-\lambda y}, \qquad \text{for } y \geq 0$$

Substituting the expression for $P(X > y)$ into the integral:

$$P(X > Y) = \frac{1}{b-a} \int_a^b e^{-\lambda y}\, dy. = \int_a^b e^{-\lambda y}\, dy = \left[-\frac{1}{\lambda} e^{-\lambda y} \right]_a^b = \frac{1}{\lambda} \left(e^{-\lambda a} - e^{-\lambda b} \right) = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda (b-a)}$$

Thus, the final expression is:

$$\boxed{P(X > Y) = \frac{e^{-\lambda a} - e^{-\lambda b}}{\lambda (b-a)}.}$$

$$P(X > Y) = \frac{1 - e^{-5}}{5} \approx 0.1987. \qquad \text{for } \lambda = 1, a = 0, b = 5$$

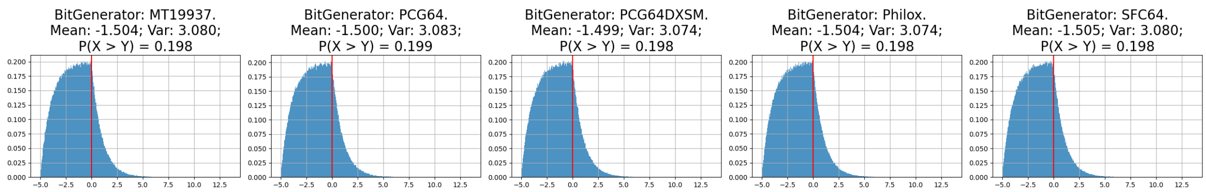

Figure 2: Density plot of the difference between the exponential and uniform distributions.