Hilbert's 10th Problem and Decidability in Algebra and Number Theory

BMS Young Scholar Day

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Section 1

Hilbert's 10th Problem

Solving polynomial equations

Recall: for a univariate polynomial $f \in \mathbb{Z}[X]$, it is easy to find all of its integer and rational roots.

Theorem (Rational Root Theorem)

Consider a polynomial $f(X) = \sum_{i=0}^{n} a_i X^i \in \mathbb{Z}[X]$ for $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{Z}$ with $a_0, a_n \neq 0$.
All rational roots of $f(X)$ are of the form $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ such that $x \mid a_0$ and $y \mid a_n$.

In particular, there is an *algorithm* which can decide whether a univariate polynomial over $\mathbb{Z}$ has an integer root, and whether it has a rational root.

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For multivariate polynomials, it is much harder to decide whether there is an integer (respectively rational) zero.

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> *Can one find an algorithm which takes as input a polynomial equation with integer coefficients, and outputs YES if the equation is solvable over the integers, and NO otherwise ?*

This is now known as <u>Hilbert's 10th Problem</u>.

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Hilbert's 10th Problem is unsolvable!

That is, there can <u>never</u> be an algorithm which can decide whether a given polynomial equation with integer coefficients has an integer solution or not.
This was proven by Yuri Matiyasevich in 1970, building on work of Martin Davis, Hilary Putnam, and Julia Robinson.

Tarski's decision procedure

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Tarski's algorithm is of theoretical interest
$\rightarrow$ too unwieldy in practice, computational requirements grow superexponentially
$\rightarrow$ search for efficient algorithms in specific cases topic of ongoing research in real algebra

Hilbert's 10th problem over a ring

Definition

Let R be a commutative ring. We say that Hilbert's 10th Problem over R with coefficients is solvable if there exists an algorithm which takes as input a polynomial with integral coefficients and outputs YES if the polynomial has a zero in R, and NO otherwise. Otherwise, we say that Hilbert's 10th Problem over R with is unsolvable.

Let us abbreviate to "Hil10(R) is solvable/unsolvable". Examples:

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- Hil10($\mathbb{Z}$) is unsolvable (DPRM, 1970),
- Hil10($\mathbb{R}$) and Hil10($\mathbb{C}$) are solvable (Tarski, 1950).

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$\mathbb{R}, \mathbb{C}$, $\mathbb{F}_q$, $\mathbb{Q}_p$, $\mathbb{Z}_p$, the algebraic integers $\tilde{\mathbb{Z}}$, $\tilde{\mathbb{Z}} \cap \mathbb{R}$, $\tilde{\mathbb{Z}} \cap \mathbb{Q}_p$, ...	$\mathbb{Q}$, all number fields, $\mathbb{F}_q(X)$, $\mathbb{Q}^{\mathrm{ab}}$, $\mathbb{Z}^{\mathrm{ab}}$, Ω, ...	$\mathbb{Z}$, $\mathcal{O}_K$ for many number fields K (e.g. totally real or abelian), ...

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Variations of the question consider algorithms which take also other coefficients than integers.

Section 2

Positive-existentially definable subsets

Positive-existentially definable subsets

Let R be a commutative ring, $n \in \mathbb{N}$. A subset A of R^n is called positive-existentially definable ($\exists$-definable) if there exists a positive-existential formula $\varphi(X_1, \ldots, X_n)$ in the language of rings with parameters from R such that

$$A = \{(x_1, \ldots, x_n) \in R^n \mid R \models \varphi(x_1, \ldots, x_n)\}.$$

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Equivalently:

Proposition

Let R be a commutative ring, $n \in \mathbb{N}$. A subset $A \subseteq R^n$ is $\exists$-definable if and only if there exist $k, m \in \mathbb{N}$ and polynomials $f_1, \ldots, f_k \in R[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ such that

$$A = \{x \in R^n \mid \exists y \in R^m : f_1(x, y) = \ldots = f_k(x, y) = 0\}.$$

Usually one may assume wlog that $k = 1$ in the above definition (e.g. for $R = \mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$).

Positive-existentially definable subsets of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$

Which subsets of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are existentially definable?

- $\mathbb{C}$ (Tarski, I think): $\exists$-definable $=$ finite or cofinite
  E.g. $\{2,3,5\}$, $\mathbb{C} \setminus \{2,3,5\}$.

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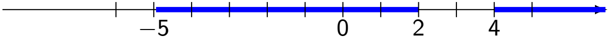

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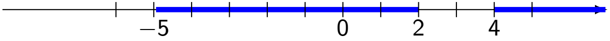

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  E.g. the set of prime numbers, the set of 2-powers, the set of factorials, ...
- $\mathbb{Q}$: Many $\exists$-definable subsets.
  E.g. the set of non-negative rational numbers

$$\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} \mid \exists y_1, \ldots, y_4 \in \mathbb{Q} : x - (y_1^2 + \ldots + y_4^2) = 0\}.$$

∃-definable sets and complexity

Vague, imprecise philosophy: For a commutative ring R, the following seem to correlate:

- more ∃-definable subsets,
- more and wilder obstructions to polynomials having zeros,
- less likely that Hil10(R) is solvable.

So, showing that Hil10(R) is unsolvable, is related to showing that many subsets of R (or R^n) are ∃-definable.

∃-definable sets and Hilbert 10

Theorem (M. Davis, H. Putnam, J. Robinson, Y. Matiyasevich)

Let $n \in \mathbb{N}$. Every recursively enumerable subset of $\mathbb{Z}^n$ is ∃-definable.

Corollary

Hil10($\mathbb{Z}$) *is unsolvable.*

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Proposition

Suppose $\mathbb{Z}$ is $\exists$-definable over $\mathbb{Q}$. Then every recursively enumerable subset of $\mathbb{Q}$ is $\exists$-definable and Hil10($\mathbb{Q}$) *is unsolvable.*

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Suppose $\mathbb{Z}$ is $\exists$-definable over $\mathbb{Q}$. Then every recursively enumerable subset of $\mathbb{Q}$ is $\exists$-definable and $\mathrm{Hil10}(\mathbb{Q})$ *is unsolvable.*

Question

Is $\mathbb{Z}$ an $\exists$-definable subset of $\mathbb{Q}$?

Section 3

∃-definability and subrings of fields

∃-definable subrings of $\mathbb{Q}$

It is possible for subrings (e.g. of $\mathbb{Q}$) to be ∃-definable subsets.
For a prime number p, consider the local ring

$$\mathbb{Z}_{(p)} = \left\{ \frac{x}{y} \;\middle|\; x \in \mathbb{Z}, y \in \mathbb{Z} \setminus p\mathbb{Z} \right\}.$$

This is always ∃-definable in $\mathbb{Q}$ ("essentially" due to Robinson, 1949).

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For example:

Proposition

Let p be a prime number, $p \equiv 3 \bmod 4$. Then

$$\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} \mid \exists y_1, y_2, y_3 \in \mathbb{Q} : 1 + (p-1)px^2 = y_1^2 + y_2^2 + py_3^2\}$$

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<u>Proof idea:</u>

- By Minkowski's Theorem, the quadratic form $Y_1^2 + Y_2^2 + pY_3^3$ represents an element of $\mathbb{Q}$ over $\mathbb{Q}$ if and only if it does so over $\mathbb{R}$ and $\mathbb{Q}_q$ for each prime q.
- The local cases can be understood entirely via Hensel's Lemma.

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- Builds on work of Poonen, 2009.
- Number theoretic/algebraic ingredients include: Minkowski's Theorem, quaternion algebras, Quadratic Reciprocity Law.
- Subsequently generalised to arbitrary global fields (Park 2013, Eisenträger-Morrison 2018, D. 2021).

Subrings of $\mathbb{Q}$

A concrete formula:

Theorem

Let $q \in \mathbb{Q}$. We have that $q \in \mathbb{Q} \setminus \mathbb{Z}$ if and only if

$$\exists a, b, t, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in \mathbb{Q} :$$

$$7\left(\left(\frac{a^2+1}{a}\right)^2 + \frac{(a^2+1)(b^2+1)}{ab} + \left(\frac{b^2+1}{b}\right)^2\right) + 2 = x_1^2 + x_2^2 + x_3^2,$$

$$(t^2 - (1+4a^2)y_1^2 - 2by_2^2 + (1+4a^2)2by_3^2 - 4)(7 + 4q^2(2 - y_1^2 - y_2^2 - y_3^2)) = 0, \ \textit{and}$$

$$\left(\frac{a^2}{q^2 - q - a^2}\left(\frac{16a^4}{1+4a^2} - \left(\frac{(b-1)^2}{b}\right)^2\right) - t\right)^2 - (1+4a^2)z_1^2 - 2bz_2^2 + (1+4a^2)2bz_3^2 = 4$$

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Question

How simple could the description of $\mathbb{Q} \setminus \mathbb{Z}$ in $\mathbb{Q}$ as an $\exists$-definable subset be? E.g. how few quantifiers?

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Let K be a field. Is there an algorithm which takes as input a polynomial with coefficients in $\mathbb{Z}[X]$ and outputs YES if the polynomial has a zero in $K(X)$, and NO otherwise?

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No examples known of fields K for which the answer is positive.
Most mysterious case: $K = \mathbb{C}$.

Valuation rings

Strategy: find existentially definable subrings of $K(X)$.
E.g. Consider in $K(X)$ the subring

$$K[X]_{(X)} = \left\{ \frac{f}{g} \;\middle|\; f, g \in K[X], X \nmid b \right\}.$$

Like $\mathbb{Z}_{(p)}$, this is a discrete valuation ring (= local principal ideal domain).

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- Technique pioneered by Denef in characteristic 0 (1978) and Pheidas in positive characteristic (1991), then subsequently generalised (char 0 Eisenträger and Moret-Bailly 2005-2007, pos. char Eisenträger-Shlapentokh, Pasten 2017)

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- Uses: elliptic curves, valuation theory, extensions of DPRM Theorem, ...

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- $\mathbb{Q} \subseteq K$, $K \cap \overline{\mathbb{Q}} \not\subseteq K^{(2)}$, and K is *large*[1] (Becher, D., and Dittmann 2023), e.g. $K = \mathbb{Q}_p^{\text{unr}}$.

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Conjecture

Let K be a field with $\mathbb{Q} \subseteq K$, $\overline{\mathbb{Q}} \not\subseteq K$. Then Geometric Hilbert's 10th Problem for $K(X)$ is unsolvable.

[1] large: Every smooth curve over K has either 0 or infinitely many rational points.

Thanks for your attention!

Nicolas Daans
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