# QUADRATIC FORMS AND CLASS FIELDS II: LECTURE NOTES

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Contents



**Notations and conventions.** We denote by $\mathbb{N}$ the set of natural numbers. We write $\mathbb{N}^+$ for the proper subset of non-zero numbers. For a ring $R$, we denote by $R^\times$ the set of invertible elements of $R$; if $R$ is a field, then $R^\times = R \setminus \{0\}$.

**Acknowledgements.** The course follows to a large extent the exposition from Lam's book [Lam05]. For this introductory course, we focus on fields of characteristic different from 2, where the theory of quadratic forms is simpler than over fields of characteristic 2. The book of Elman, Karpenko and Merkurjev [EKM08]

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is a great reference for those who want to learn more about quadratic form theory over fields of arbitrary characteristic, and some parts of this course which hold in arbitrary characteristic, are inspired by their work. Finally, I gratefully acknowledge the inspiration taken from the course "Quadratic Forms" taught by Karim Johannes Becher at the University of Antwerp in Belgium, which has to a large extent shaped my vision on modern quadratic form theory.

## 1. Lecture 1

### 1.1. Bilinear and quadratic forms.

Let always $K$ be a field, $n \in \mathbb{N}$.

1.1.1. **Definition.** A *symmetric bilinear space over* $K$ is a pair $(V, B)$ where

- $V$ is a finite-dimensional vector space over $K$, and
- $B : V \times V \to K$ is a symmetric and bilinear map, i.e. for all $x, x', y \in V$ and $a \in K$ we have

$$\begin{aligned} B(x,y) &= B(y,x), \\ B(x+x',y) &= B(x,y) + B(x',y), \\ B(ax,y) &= aB(x,y). \end{aligned}$$

We call the map $B$ a *symmetric bilinear form on* $V$. We define the dimension of $(V, B)$ to be the dimension of $V$, and denote this by $\dim(V, B)$ or simply $\dim B$.

Let $n = \dim(V, B)$. Given a basis $\mathcal{B} = (e_1, \dots, e_n)$ of $V$, we define $\mathcal{M}_{\mathcal{B}}(B) = [B(e_j, e_i)]_{i,j=1}^n$, which we call *the matrix of* $(V, B)$ *with respect to* $\mathcal{B}$.

1.1.2. **Proposition.** *Let $V = K^n$ and let $\mathfrak{B} = (e_1, \dots, e_n)$ be the canonical basis. Let $B$ be a symmetric bilinear form on $V$. For column vectors $x = [x_1 \dots x_n]^T$ and $y = [y_1, \dots, y_n]^T$ we have*

$$B(x,y) = x^T \mathcal{M}_{\mathcal{B}}(B) y.$$

*Proof.* This is clear from the bilinearity of $B$. □

1.1.3. **Definition.** A *quadratic space over* $K$ is a pair $(V, q)$ where

- $V$ is a finite-dimensional vector space over $K$, and
- $q : V \to K$ is a map satisfying the following:
  (1) $\forall a \in K, \forall x \in V : q(ax) = a^2 q(x)$,
  (2) the map

$$\mathfrak{b}_q : V \times V \to K : (x,y) \mapsto q(x+y) - q(x) - q(y)$$

is a symmetric bilinear form on $V$.

We call the map $q$ a *quadratic form on* $V$, and $\mathfrak{b}_q$ its *polar form*. We define the dimension of $(V, q)$ to be the dimension of $V$, and denote this by $\dim(V, q)$ or simply $\dim q$.

1.1.4. **Definition.** Let $(V, B)$ and $(V', B')$ be symmetric bilinear spaces over $K$. An isomorphism of $K$-vector spaces $I : V \to V'$ is called an *isometry between* $(V, B)$ *and* $(V', B')$ if, for all $v, w \in V$, one has $B(v, w) = B'(I(v), I(w))$. Similarly, given quadratic spaces $(V, q)$ and $(V', q')$ over $K$, an isomorphism of $K$-vector spaces $I : V \to V'$ is called an *isometry between* $(V, q)$ *and* $(V', q')$ if, for all $v \in V$, one has $q(v) = q'(I(v))$.

We call two symmetric bilinear spaces $(V, B)$ and $(V, B')$ (respectively two quadratic spaces $(V, q)$ and $(V', q')$) *isometric*, which we denote by $(V, B) \cong (V', B')$ (respectively $(V, q) \cong (V', q')$) if there exists an isometry between them.

Traditionally, a quadratic form over $K$ is often defined to be a homogeneous polynomial of degree 2 over $K$. Definition 1.1.3 can be seen as a coordinate-free version of this, as the following proposition indicates.

1.1.5. **Proposition.** *Let $n \in \mathbb{N}$ and let $f \in K[X_1, \ldots, X_n]$ be a homogeneous polynomial of degree 2. The map*

$$q_f : K^n \to K : (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$$

*is a quadratic form on $K^n$.*

*Conversely, given a quadratic space $(V, q)$ of dimension $n$, there exists a homogeneous degree 2 polynomial $f \in K[X_1, \ldots, X_n]$ such that $(V, q)$ is isometric to $(K^n, q_f)$.*

*Proof.* For the first part of the statement, one verifies that the defined map satisfies the conditions stated in Definition 1.1.3.

The second part of the statement is left as an exercise. □

1.1.6. **Proposition.** *Let $f, g \in K[X_1, \ldots, X_n]$ be homogeneous polynomials of degree 2. The quadratic spaces $(K^n, q_f)$ and $(K^n, q_g)$ are isometric if and only if there exists $C \in \mathrm{GL}_n(K)$ such that*

$$f\left([x_1 \ldots x_n]^T\right) = g\left(([x_1 \ldots x_n]\, C)^T\right)$$

*for all $x_1, \ldots, x_n \in K$.*

*Proof.* Exercise. □

1.1.7. *Example.* Suppose $\mathrm{char}(K) \neq 2$. Let $f(X_1, X_2) = X_1 \cdot X_2$ and $g(X_1, X_2) = X_1^2 - X_2^2$. We observe that

$$g\left(\frac{X_1 + X_2}{2}, \frac{X_1 - X_2}{2}\right) = f(X_1, X_2)$$

and thus, in view of Proposition 1.1.6, that $(K^2, q_f) \cong (K^2, q_g)$, with

$$C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}.$$

We saw that, to a quadratic form $q$, one can associate a symmetric bilinear form $\mathfrak{b}_q$ on the same space. It is also possible to obtain a quadratic form from a symmetric bilinear form: if $(V, B)$ is a symmetric bilinear space, then

$$q_B : V \to K : v \mapsto B(v, v)$$

is easily seen to be a quadratic form. If $\mathrm{char}(K) \neq 2$, then these two operations are each others inverses (up to scaling by $\frac{1}{2}$), and hence the studies of quadratic and symmetric bilinear forms over $K$ are essentially the same:

1.1.8. **Proposition.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V, q)$ *be a quadratic space. Then* $q$ *is equal to the quadratic form associated to the form* $\frac{1}{2}\mathfrak{b}_q$. *Conversely, if* $(V, B)$ *is a symmetric bilinear space, then* $B$ *is equal to* $\frac{1}{2}\mathfrak{b}_q$ *where* $q = q_B$.

*Proof.* This is a straightforward computation. □

Over fields of characteristic 2, one can still associate to each quadratic form a symmetric bilinear form and to each symmetric bilinear form a quadratic form as before, but these operations are not invertible. In fact, one needs to make an entirely separate study of quadratic and symmetric bilinear forms! We refer the interested reader to [EKM08, Chapters I and II].

We now go on to study basic properties of quadratic forms.

1.1.9. **Definition.** Let $(V, q)$ be a quadratic space over $K$.

- We call $q$ *isotropic* if there exists $v \in V \setminus \{0\}$ such that $q(v) = 0$, or *anisotropic* otherwise.
- Given $a \in K^\times$, we say that $q$ *represents* $a$ if $\exists v \in V$ with $a = q(v)$. We write
$$D_K(q) = \{a \in K^\times \mid \exists v \in V : a = q(v)\}.$$
If $D_K(q) = K^\times$, we say that $q$ is *universal*.

1.1.10. *Examples.*

(1) Let $f(X_1, X_2) = X_1 \cdot X_2$. Then $q_f$ is isotropic, since $f(1, 0) = 0$. $q_f$ is also universal, since, $f(1, a) = a$ for any $a \in K^\times$.
(2) Let $f(X_1, X_2) = X_1^2 + X_2^2$. $q_f$ is isotropic if and only if $-1$ is a square in $K$. $D_K(q_f)$ is the set of elements of $K$ which are a sum of two squares.
(3) Let $f(X_1, X_2) = (X_1 + X_2)^2$. Then $q_f$ is isotropic since $f(1, -1) = 0$. $D_K(q_f)$ consists of those elements of $K$ which are squares.

The last example is somewhat peculiar: the quadratic form $q_f$ with $f(X_1, X_2) = (X_1 + X_2)^2$ is of dimension 2, but after a base change, one of the variables disappears. Indeed,

$$f(X_1 - X_2, X_2) = X_1^2.$$

We will often want to exclude from our study quadratic forms which have this property.

For a $K$-vector space $V$, we denote by $V^*$ the dual space, i.e. the space of linear maps $V \to K$. Recall that $\dim(V^*) = \dim(V)$.

1.1.11. **Proposition.** *Let $(V, B)$ be a symmetric bilinear space. Let $\mathcal{B}$ be a basis for $V$. The following are equivalent.*
*(a) $\forall v \in V \setminus \{0\}$, $\exists w \in V$ : $B(v, w) \neq 0$,*
*(b) The map $V \to V^* : v \mapsto (w \mapsto B(v, w))$ is a $K$-isomorphism.*
*(c) The matrix $M_{\mathcal{B}}(B)$ is invertible.*

*Proof.* Exercise. □

1.1.12. **Definition.** We call a symmetric bilinear space $(V, B)$ *nonsingular* if the above equivalent conditions hold. We call a quadratic space $(V, q)$ nonsingular if its polar form is nonsingular. Otherwise, we call the space *singular*. We use the same terminology for the symmetric bilinear and quadratic forms themselves.

We now show that, at least over fields of characteristic not 2, singular forms are precisely those for which, after a base change, one of the variables disappears.

1.1.13. **Proposition.** *Let $(V, q)$ be a quadratic space over $K$ and $v \in V$. Consider the statements*
*(a) $\mathfrak{b}_q(v, w) = 0$ for all $w \in V$,*
*(b) for all $w \in V$ we have $q(w + v) = q(w)$.*
*We have that* (b) $\Rightarrow$ (a) *in general. If* $\mathrm{char}(K) \neq 2$*, then* (a) *and* (b) *are equivalent.*

*In particular, it follows that, if* $\mathrm{char}(K) \neq 2$*, a quadratic space $(V, q)$ is singular if and only if there exists $v \in V \setminus \{0\}$ such that for all $w \in V$ we have $q(w+v) = q(w)$.*

*Proof.* If (b) holds, then $q(v) = q(v + 0) = q(0) = 0$, whence for any $w \in V$ we have $\mathfrak{b}_q(v, w) = q(v + w) - q(v) - q(w) = 0$.

Now assume that $\mathrm{char}(K) \neq 2$ and (a) holds. Then in particular $0 = \mathfrak{b}_q(v, v) = 2q(v)$ and thus $q(v) = 0$. It follows that, for any $w \in V$, we have $q(v + w) = q(v) + q(w) + \mathfrak{b}_q(v, w) = q(v)$, so (b) holds. □

If $\mathrm{char}(K) \neq 2$, a nonsingular quadratic form over $K$ is also called *regular* or *nondegenerate*. Note that, if $\mathrm{char}(K) = 2$, these terms have more specialised, distinct meanings.

1.1.14. *Remark.* So far, I have been somewhat careful in making the distinction between a symmetric bilinear/quadratic *space* and a symmetric bilinear/quadratic *form*. This makes notation and speaking somewhat heavy. I will in the future often simply refer to the forms themselves, taking the convention that a symmetric bilinear/quadratic space 'knows' its domain.

## 1.2. Orthogonality and diagonalisation.

1.2.1. **Definition.** Let $(V, B)$ be a symmetric bilinear space. Let $v, w \in V$. We say that $v$ and $w$ are *orthogonal (with respect to $B$)* if $B(v, w) = 0$. We write $v \perp w$.

Let $v \in V$ and $M \subseteq V$. We say that $v$ is *orthogonal to $M$ (with respect to $B$)* if $B(v, w) = 0$ for all $w \in M$. We write $v \perp M$. Similarly, given $M' \subseteq V$, we say that $M$ is *orthogonal to $M'$ (with respect to $B$)* if $B(v, w) = 0$ for all $v \in M$ and $w \in M'$, and write $M \perp M'$.

We write

$$M^\perp = \{v \in V \mid \forall w \in M : B(v, w) = 0\}$$

and call it the *orthogonal space of $M$* - note that it is always a subspace of $V$. We write $v^\perp$ instead of $\{v\}^\perp$.

If $U \subseteq V$ is a subspace and $V = U \oplus U^\perp$, we call $U^\perp$ an *orthogonal complement of $U$ in $V$*.

Observe that a symmetric bilinear space $(V, B)$ is by definition nonsingular if and only if $V^\perp = \{0\}$.

1.2.2. **Proposition.** *Let $(V, B)$ be nonsingular, $U \subseteq V$ a subspace. Then*

$$\dim U + \dim U^\perp = \dim V \quad \textit{and} \quad (U^\perp)^\perp = U.$$

*Proof.* Consider the $K$-linear maps

$$\varphi_1 : U^\perp \to V^* : v \mapsto (w \mapsto B(v, w))$$
$$\varphi_2 : V^* \to U^* : f \mapsto f|_U.$$

We observe that $\varphi_1$ is injective by the nonsingularity of $(V, B)$, that $\varphi_2$ is surjective, and that the image of $\varphi_1$ is precisely the kernel of $\varphi_2$ by definition of $U^\perp$. As such, we compute that

$$\begin{aligned}\dim V = \dim V^* &= \dim(\operatorname{Ker}\varphi_2) + \dim(\operatorname{Im}\varphi_2) \\ &= \dim(\operatorname{Im}\varphi_1) + \dim U^* = \dim(U^\perp) + \dim(U)\end{aligned}$$

as desired.

For the second statement, observe that we trivially have $U \subseteq (U^\perp)^\perp$, but that, by the first claim, $\dim(U) = \dim((U^\perp)^\perp)$, whence $U = (U^\perp)^\perp$ as desired. □

We now define an operation on the set of quadratic spaces over $K$.

1.2.3. **Proposition.** *Let $(V_1, q_1)$ and $(V_2, q_2)$ be quadratic spaces over $K$. Let $V = V_1 \times V_2$ and consider the map*

$$q : V \to K : (x, y) \mapsto q_1(x) + q_2(y).$$

*Furthermore, consider the natural embeddings $\iota_1 : V_1 \to V : x \mapsto (x, 0)$ and $\iota_2 : V_2 \to V : x \mapsto (0, x)$. We have that $(V, q)$ is a quadratic space, and $q$ is nonsingular if and only if both $q_1$ and $q_2$ are. Furthermore, $i_1(V_1) \perp i_2(V_2)$ with respect to $\mathfrak{b}_q$.*

*Proof.* Easy verification. □

1.2.4. **Definition.** Let $(V_1, q_1)$ and $(V_2, q_2)$ be quadratic spaces over $K$. We call the space $(V, q)$ defined in Proposition 1.2.3 the *orthogonal sum of* $(V_1, q_1)$ *and* $(V_2, q_2)$ and we denote the form $q$ by $q_1 \perp q_2$.

1.2.5. **Proposition.** *Let* $(V_i, q_i)$ *and* $(V_i', q_i')$ *be quadratic spaces for* $i = 1, 2, 3$. *We have the following computation rules:*

- $\dim(q_1 \perp q_2) = \dim(q_1) + \dim(q_2)$.
- $q_1 \perp q_2 \cong q_2 \perp q_1$, *and* $q_1 \perp (q_2 \perp q_3) \cong (q_1 \perp q_2) \perp q_3$.
- *If* $q_1 \cong q_1'$ *and* $q_2 \cong q_2'$, *then* $q_1 \perp q_1' \cong q_2 \perp q_2'$.

*Proof.* Easy verifications. □

1.2.6. **Proposition.** *Let* $(V, q)$, $(V_1, q_1)$ *and* $(V_2, q_2)$ *be quadratic spaces over* $K$. *Then* $q \cong q_1 \perp q_2$ *if and only if there are* $K$*-subspaces* $W_1$ *and* $W_2$ *of* $V$ *with* $W_1 \perp W_2$ *with respect to* $\mathfrak{b}_q$, $V = W_1 \oplus W_2$, *and such that* $(W_i, q|_{W_i}) \cong (V_i, q_i)$ *for* $i = 1, 2$.

*Proof.* Suppose that $\iota$ is an isomorphism $q_1 \perp q_2 \to q$ and let $W_1$ and $W_2$ be the images under this isomorphism of $V_1 \times \{0\}$ and $\{0\} \times V_2$ respectively. One verifies easily that these are as desired.

Conversely, assume that $W_1$ and $W_2$ are subspaces of $V$ with $W_1 \perp W_2$, $V = W_1 \oplus W_2$, and such that $(W_i, q|_{W_i}) \cong (V_i, q_i)$ for $i = 1, 2$. Without loss of generality, we may assume that $V_i = W_i$ and $q_i = q|_{W_i}$. Let $\iota$ be the unique $K$-linear map $V \to V_1 \times V_2$ which maps a vector $w \in W_1$ to $(w, 0)$ and a vector $w \in W_2$ to $(0, w)$. Clearly this is an isomorphism of $K$-vector spaces. Consider an arbitrary vector in $V$, which we may write as $w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. Since $W_1 \perp W_2$, we have that $\mathfrak{b}_q(w_1, w_2) = 0$. We compute that

$$\begin{aligned} q(w_1 + w_2) &= q(w_1) + q(w_2) + \mathfrak{b}_q(w_1, w_2) = q(w_1) + q(w_2) \\ &= q_1(w_1) + q_2(w_2) = (q_1 \perp q_2)(w_1, w_2) = (q_1 \perp q_2)(\iota(w_1 + w_2)). \end{aligned}$$

Hence $\iota$ is the desired isometry. □

We now discuss a special class of quadratic forms called diagonal forms. As it will turn out, in characteristic different from 2, every quadratic form is isometric to a diagonal form (see Corollary 1.2.10).

1.2.7. **Definition.** Let $a_1, \ldots, a_n \in K$. We denote by $\langle a_1, \ldots, a_n \rangle_K$ the quadratic form

$$K^n \to K : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} a_i x_i^2.$$

We call such a form a *diagonal form*. If the field $K$ is clear from the context we might simply write $\langle a_1, \ldots, a_n \rangle$ instead of $\langle a_1, \ldots, a_n \rangle_K$.

Note that $\langle a_1, \ldots, a_n \rangle_K \cong \langle a_1 \rangle_K \perp \ldots \perp \langle a_n \rangle_K$.

1.2.8. **Proposition.** *Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in K$, let $q = \langle a_1, \dots, a_n \rangle$. If $\mathrm{char}(K) \neq 2$, then $q$ is singular if and only if $a_i = 0$ for some $i \in \{1, \dots, n\rangle$. If $\mathrm{char}(K) = 2$, then $q$ is singular as soon as $n \geq 2$.*

*Proof.* Exercise. □

1.2.9. **Proposition.** *Assume $\mathrm{char}(K) \neq 2$. Let $(V, q)$ be a quadratic space over $K$, $d \in K^\times$. Then $d \in D_K(q)$ if and only if $q \cong \langle d \rangle \perp (V', q')$ for some quadratic space $(V', q')$.*

*Proof.* Clearly $d = d \cdot 1^2 + q'(0) \in D_K(\langle d \rangle \perp (V', q'))$ for any quadratic space $(V', q')$.

Conversely, assume that $d \in D_K(q)$. Let $W$ be any subspace of $V$ such that $V = V^\perp \oplus W$. Then $(W, q|_W)$ is nonsingular, and, in view of Proposition 1.1.13, we have $D_K(q|_W) = D_K(q)$. We may thus restrict our quadratic form to $W$, and assume without loss of generality that $q$ is nonsingular.

Now take $v \in V$ with $q(v) = d$. Set $U = v^\perp$. We have $v \notin v^\perp$ (since $\mathfrak{b}_q(v, v) = 2d \neq 0$) and $\dim(U) = \dim(V) - 1$ by Proposition 1.2.2, hence $V = Kv \oplus U$. Clearly $q|_{Kv} \cong \langle d \rangle$, so $q \cong \langle d \rangle \perp (U, q|_U)$ in view of Proposition 1.2.6. □

1.2.10. **Corollary.** *Assume $\mathrm{char}(K) \neq 2$, let $(V, q)$ be a quadratic space over $K$ of dimension $n$. Then there exist $a_1, \dots, a_n \in K$ such that $q \cong \langle a_1, \dots, a_n \rangle$.*

*Proof.* Apply Proposition 1.2.9 inductively. □

1.3. **Exercises.**

(1) Complete the proofs of Proposition 1.1.5, Proposition 1.1.6, Proposition 1.1.11 and Proposition 1.2.8.
(2) Illustrate by an example that the implication (a) ⇒ (b) in Proposition 1.1.13 does not hold in general if $\mathrm{char}(K) = 2$.
(3) Consider the quadratic form on $\mathbb{Q}^3$ given by the following polynomial:
$$f(X_1, X_2, X_3) = 3X_1^2 + 6X_1X_2 + 3X_2^2 - X_2X_3.$$
Explicitly construct a diagonal quadratic form $q$ on $\mathbb{Q}^3$ such that $(\mathbb{Q}^3, q_f) \cong (\mathbb{Q}^3, q)$.

## 2. Lecture 2

Let always $K$ be a field.

2.0.1. **Definition.** Let $(V, q)$ be a quadratic space. If $W$ is a subspace of $V$, the quadratic space $(W, q|_W)$ is called a *subform* of $(V, q)$. By abuse of terminology, we will also call a quadratic space $(U, q')$ which is isometric to $(W, q|_W)$ for some subspace $W$ of $V$ a subform of $(V, q)$.

In this lecture, we will get closer to a classification of quadratic spaces over a given field, by decomposing quadratic spaces as orthogonal sums of subforms with specific properties.

2.1. **Isotropic, totally isotropic, and hyperbolic forms.** Recall from Definition 1.1.9 the definition of an isotropic quadratic form.

2.1.1. **Definition.** Let $(V, q)$ be a quadratic space. We call $(V, q)$ *totally isotropic* if $q(v) = 0$ for all $v \in V$. If $W$ is a subspace of $V$, we call $W$ totally isotropic if $(W, q|_W)$ is totally isotropic.

Observe that a non-zero totally isotropic space is always singular.

2.1.2. **Proposition.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V, q)$ *be a quadratic space. Then the map*

$$\overline{q} : V/V^\perp \to K : \overline{v} \mapsto q(v)$$

*is a well-defined nonsingular quadratic form.*

*Proof.* The well-definedness follows from the fact that, for $v \in V$ and $w \in V^\perp$, one has $q(v + w) = q(v)$ by Proposition 1.1.13. It is then easy to verify that the map is a quadratic form.

For the nonsingularity, consider $v \in V$ such that $\overline{v} \neq 0$, i.e. $v \notin V^\perp$. Then there exists $w \in V$ with $0 \neq \mathfrak{b}_q(v, w) = \mathfrak{b}_{\overline{q}}(\overline{v}, \overline{w})$, whereby $\overline{v} \notin (V/V^\perp)^\perp$. Hence $(V/V^\perp)^\perp = \{0\}$, and thus $(V/V^\perp, \overline{q})$ is nonsingular. □

The following observation was already used implicitly in the proof of Proposition 1.2.9.

2.1.3. **Proposition.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V, q)$ *be a quadratic space. Let* $W$ *be an orthogonal complement of* $V^\perp$. *We have that*

$$(V, q) \cong (V^\perp, q|_{V^\perp}) \perp (W, q|_W),$$

*that* $(V^\perp, q|_{V^\perp})$ *is totally isotropic, and that* $(W, q|_W) \cong (V/V^\perp, \overline{q})$.

*Proof.* The first isometry is immediate form Proposition 1.2.6. The fact that $(V^\perp, q|_{V^\perp})$ is totally isotropic follows from Proposition 1.1.13.

Finally, consider the map

$$\iota : W \to V/V^\perp : w \mapsto \overline{w}.$$

Since $W \cap V^\perp = \{0\}$ we have that $\iota$ is injective, hence by comparing dimensions, $\iota$ is bijective. Furthermore, by definition we have for $w \in W$ that $q(w) = \overline{q}(\overline{w}) = \overline{q}(\iota(w))$. Hence we have obtained the required isometry $(W, q|_W) \cong (V/V^\perp, \overline{q})$. □

We can thus, in characteristic away from 2, decompose any quadratic space into the orthogonal sum of a totally isotropic space and a nonsingular space, and this decomposition is unique up to isometry.

We now want to study nonsingular isotropic forms. Nonsingular one-dimensional quadratic forms are always anisotropic.

2.1.4. **Definition.** We call the quadratic form $(K^2, q_f)$ with $f(X_1, X_2) = X_1 \cdot X_2$ the *hyperbolic plane over* $K$ and denote it by $\mathbb{H}_K$.

2.1.5. **Proposition.** *Let $(V,q)$ be a nonsingular quadratic space over $K$. Let $v \in V \setminus \{0\}$ such that $q(v) = 0$. Then there is a subspace $W \subseteq V$ with $v \in W$ such that $(W, q|_W)$ is isometric to $\mathbb{H}_K$.*

*Proof.* Since $(V,q)$ is nonsingular, there exists $w \in V$ such that $a = \mathfrak{b}_q(v,w) \neq 0$. We may replace $w$ by $a^{-1}w$ and assume without loss of generality that $a = 1$. Observe that $w \notin Kv$, so that $W = Kv \oplus Kw$ is a 2-dimensional subspace of $V$. Consider the map

$$\iota : K^2 \to W : (x,y) \mapsto xv + y(w - q(w)v).$$

Clearly this is a $K$-isomorphism of vector spaces. We compute that, for $x, y \in K$, we have

$$\begin{aligned} q(\iota(x,y)) &= q(xv + y(w - q(w)v)) \\ &= (x - yq(w))^2 q(v) + y^2 q(w) + \mathfrak{b}_q((x - yq(w))v, yw) \\ &= 0 + y^2 q(w) + (x - yq(w))y\mathfrak{b}_q(v,w) = xy. \end{aligned}$$

Hence $(W, q|_W) \cong \mathbb{H}_K$. □

In particular, it follows from Proposition 2.1.5 that the hyperbolic plane is, up to isometry, the only two-dimensional nonsingular isotropic quadratic form over $K$. We also obtain the following

2.1.6. **Corollary.** *Every nonsingular isotropic quadratic space is universal.*

*Proof.* We know from Examples 1.1.10 that $\mathbb{H}_K$ is universal. But by Proposition 2.1.5 every nonsingular isotropic quadratic space contains $\mathbb{H}_K$ as a subspace, hence is also universal. □

2.1.7. **Corollary.** *Let $(V,q)$ be a nonsingular quadratic space and $d \in K^\times$. We have that $d \in D_K(q)$ if and only if $q \perp \langle -d \rangle_K$ is isotropic.*

*Proof.* Exercise. □

2.1.8. **Proposition.** *Let $(V,q)$ be a nonsingular quadratic space, $W$ a nonsingular subspace of $V$. Then $V = W \oplus W^\perp$, $(V,q) \cong (W, q|_W) \perp (W^\perp, q|_{W^\perp})$, and also $(W^\perp, q|_{W^\perp})$ is nonsingular.*

*Proof.* Since $(V,q)$ is nonsingular, we have $\dim W + \dim W^\perp = \dim V$ by Proposition 1.2.2. Since $(W, q\ |_W)$ is nonsingular, we further have $W \cap W^\perp = \{0\}$. Hence, we obtain $V = W \oplus W^\perp$, and the natural induced $K$-isomorphism $V \to W \times W^\perp$ gives the required isometry $(V,q) \cong (W, q|_W) \perp (W^\perp, q|_{W^\perp})$; see Proposition 1.2.6.

Finally, since $(W^\perp)^\perp = W$ by Proposition 1.2.2, we obtain $(W^\perp)^\perp \cap W^\perp = W \cap W^\perp = \{0\}$, whereby $(W^\perp, q|_{W^\perp})$ is nonsingular. □

In the sequel, we will use the following notation: for a quadratic space $(V,q)$ and $n \in \mathbb{N}$, we write

$$n \times (V,q) = (V^n, \underbrace{q \perp \ldots \perp q}_{n \text{ times}}).$$

We will denote the described quadratic form on $V^n$ simply by $n \times q$. By convention, $0 \times (V,q)$ denotes the unique zero-dimensional quadratic space over $K$.

2.1.9. **Proposition.** *Let $(V,q)$ be a nonsingular quadratic space, $n \in \mathbb{N}$. The following are equivalent.*

*(1) $V$ contains a totally isotropic subspace of dimension $n$,*
*(2) $V$ contains a subform isometric to $n \times \mathbb{H}_K$.*

*Proof.* For $n = 0$ there is nothing to show, assume from now on that $n \geq 1$.

Assume (2). Then $V$ has subspaces $W_1, \ldots, W_n$ such that $W_i \perp W_j$ and $W_i \cap W_j = \{0\}$ for any $i \neq j$ and such that $(W_i, q|_{W_i}) \cong \mathbb{H}_K$. Let $w_i \in W_i \setminus \{0\}$ be such that $q(w_i) = 0$. Then $Kw_1 \oplus \ldots \oplus Kw_n$ is an $n$-dimensional totally isotropic subspace of $V$.

Conversely, assume (1). We argue via induction on $n$ - recall that the case $n = 0$ is covered, so we assume $n \geq 1$. Let $W$ be a totally isotropic subspace of $V$ of dimension $n$ and let $v \in W \setminus \{0\}$. By Proposition 2.1.5 there exists $w \in V$ such that, for $W' = Kv \oplus Kw$, we have $(W', q|_{W'}) \cong \mathbb{H}_K$. By Proposition 2.1.8 this implies that $(V,q) \cong \mathbb{H}_K \perp (U, q|_U)$ for $U = (W')^\perp$, and furthermore $(U, q|_U)$ is nonsingular. Further, since $W \subseteq v^\perp$, we have

$$U \cap W = (W')^\perp \cap W = v^\perp \cap w^\perp \cap W = w^\perp \cap W$$

whereby $\dim(W \cap U) \geq n-1$. Hence $(U, q|_U)$ contains a totally isotropic subspace $W \cap U$ of dimension $n-1$. The statement now follows by the induction hypothesis. □

2.1.10. **Corollary.** *Let $(V,q)$ be a nonsingular quadratic space of dimension $2n$, where $n \in \mathbb{N}$. The following are equivalent.*

*(1) $V$ contains a totally isotropic subspace of dimension $n$,*
*(2) $(V,q) \cong n \times \mathbb{H}_K$.*

2.1.11. **Definition.** We say that a nonsingular quadratic space of dimension $2n$ (for some $n \in \mathbb{N}$) is *hyperbolic* if it contains a totally isotropic subspace of dimension $n$.

Given a quadratic space $(V,q)$, we define the *Witt index* of $(V,q)$ to be the maximal possible dimension of a totally isotropic subspace of $(V/V^\perp, \overline{q})$. We denote it by $i_W(V,q)$, or simply $i_W(q)$.

2.1.12. **Proposition.** *Let $(V,q)$ be a nonsingular quadratic space. Then $(V,q) \perp (V,-q)$ is hyperbolic.*

*Proof.* Let $n = \dim V$. Then $\dim(V \times V) = 2n$. Let $W = \{(v,v) \in V \times V \mid v \in V\}$. Then $W$ is a subspace of $V \times V$ of dimension $n$, and it is a totally isotropic

subspace of $(V,q) \perp (V,-q)$, since for any $v \in V$ we have $(q \perp -q)(v,v) = q(v) - q(v) = 0$.

Since $(V,q) \perp (V,-q)$ is nonsingular (by Proposition 1.2.3) and has a totally isotropic subspace of dimension $n$, it is hyperbolic. □

2.2. **Witt's Theorems.** We are now in a position to prove the two most important structure theorems on quadratic forms, named after Ernst Witt. We will prove them, as Witt did in the 1930'ies, under the assumption that $\mathrm{char}(K) \neq 2$. Versions in characteristic 2 exist and can be proven with extra assumptions and a lot more work, see [EKM08, Section 8].

2.2.1. **Lemma.** *Assume that* $\mathrm{char}(K) \neq 2$. *Let* $(V,q)$ *be a quadratic space, and let* $v, w \in V$ *be such that* $q(v) = q(w) \neq 0$. *There exists an isometry* $\tau : (V,q) \to (V,q)$ *such that* $\tau(x) = y$.

*Proof.* One computes that $q(v+w) + q(v-w) = 4q(v) \neq 0$, so at least one of $q(v+w)$ and $q(v-w)$ is non-zero. Replacing $w$ by $-w$ if necessary, we may assume that $q(v-w) \neq 0$. Now consider the map

$$\tau : V \to V : u \mapsto u - \frac{\mathfrak{b}_q(u, v-w)}{q(v-w)}(v-w).$$

One verifies that $\tau$ gives an isometry $(V,q) \to (V,q)$, and that $\tau(v) = w$, as desired; see Exercise (2). □

2.2.2. **Theorem** (Witt Cancellation Theorem)**.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V,q)$, $(V_1,q_1)$ *and* $(V_2,q_2)$ *be quadratic spaces. If* $(V,q) \perp (V_1,q_1) \cong (V,q) \perp (V_2,q_2)$, *then* $(V_1,q_1) \cong (V_2,q_2)$.

*Proof.* We first reduce to the case where all involved quadratic spaces are nonsingular. To this end, use Proposition 2.1.3 to write $(V,q) \cong (V^\perp, q|_{V^\perp}) \perp (W, q|_W)$, $(V_1,q_1) \cong (V_1^\perp, q_1|_{V_1^\perp}) \perp (W_1, q|_{W_1})$ and $(V_2,q_2) \cong (V_2^\perp, q_2|_{V_2^\perp}) \perp (W_2, q|_{W_2})$ where $q|_W$, $q_1|_{W_1}$ and $q_2|_{W_2}$ are nonsingular. The hypothesis can be rewritten as

$$\begin{aligned}&((V \perp V_1)^\perp, (q \perp q_1)|_{(V \perp V_1)^\perp}) \perp (W \perp W_1, (q \perp q_1)|_{W \perp W_1}) \\ \cong\ &((V \perp V_2)^\perp, (q \perp q_2)|_{(V \perp V_2)^\perp}) \perp (W \perp W_2, (q \perp q_2)|_{W \perp W_2}),\end{aligned}$$

using that $V^\perp \perp V_1^\perp = (V \perp V_1)^\perp$ and similarly $V^\perp \perp V_2^\perp = (V \perp V_2)^\perp$. We further have by Proposition 1.2.3 that $(W \perp W_1, (q \perp q_1)|_{W \perp W_1})$ and $(W \perp W_2, (q \perp q_2)|_{W \perp W_2})$ are nonsingular. In view of Proposition 2.1.3 we have

$$\begin{aligned}&(W \perp W_1, (q \perp q_1)|_{W \perp W_1}) \cong ((V \perp V_1)/(V \perp V_1)^\perp, \overline{q \perp q_1}) \\ \cong\ &((V \perp V_2)/(V \perp V_2)^\perp, \overline{q \perp q_2}) \cong (W \perp W_2, (q \perp q_2)|_{W \perp W_2}),\end{aligned}$$

We conclude that we may assume for the remainder of the proof that $(V,q)$, $(V_1,q_1)$ and $(V_2,q_2)$ are nonsingular.

By Corollary 1.2.10 we may assume that $(V,q) \cong \langle a_1, \ldots, a_n \rangle$ for some $a_1, \ldots, a_n \in K^\times$. By inducting on $n$, we reduce to the situation $n = 1$. Let $\iota : \langle a \rangle_K \perp$

$(V_1, q_1) \to \langle a \rangle_K \perp (V_2, q_2)$ be an isometry. Let $v = \iota(1,0)$. We have $(\langle a \rangle_K \perp q_2)(v) = (\langle a \rangle_K \perp q_1)(1,0) = a \cdot 1^2 = a = (\langle a \rangle_K \perp q_2)(1,0)$.

By Lemma 2.2.1 there exists an isometry $\tau : \langle a \rangle_K \perp (V_2, q_2) \to \langle a \rangle_K \perp (V_2, q_2)$ with $\tau(v) = (1,0)$. Thus, $\tau \circ \iota$ gives an isometry $\langle a \rangle_K \perp (V_1, q_1) \to \langle a \rangle_K \perp (V_2, q_2)$ mapping $(1,0)$ to $(1,0)$. Furthermore, since $(K \times \{0\}) \perp (\{0\} \times V_1)$ (in $(K \times V_1, \langle a \rangle_K \perp q_1)$) and isometries preserve orthogonality, we obtain $(K \times \{0\}) \perp (\tau \circ \iota)(\{0\} \times V_1)$ (in $(K \times V_2, \langle a \rangle_K \perp q_2)$). So, we must have $(\tau \circ \iota)(\{0\} \times V_1) = \{0\} \times V_2$, whereby $\tau \circ \iota$ induces an isometry $(V_1, q_1) \to (V_2, q_2)$, as desired. □

2.2.3. **Theorem** (Witt Decomposition Theorem)**.** *Assume* $\operatorname{char}(K) \neq 2$. *Let* $(V, q)$ *be a quadratic space. There exist quadratic spaces* $(V_t, q_t)$, $(V_h, q_h)$ *and* $(V_a, q_a)$ *such that*

$$(V, q) \cong (V_t, q_t) \perp (V_h, q_h) \perp (V_a, q_a)$$

*where*

- $(V_t, q_t)$ *is totally isotropic,*
- $(V_h, q_h)$ *is hyperbolic (or zero),*
- $(V_a, q_a)$ *is anisotropic.*

*Furthermore, each of these spaces is determined up to isometry by* $(V, q)$*. In fact,* $(V_t, q_t)$ *is the unique totally isotropic space of dimension* $\dim V^\perp$*, and* $(V_h, q_h)$ *is the unique hyperbolic space of dimension* $2i_W(q)$*.*

*Proof.* We first prove the existence of the required spaces. By Proposition 2.1.3 we can write $(V, q) \cong (V_t, q_t) \perp (V', q')$ where $(V_t, q_t)$ is totally isotropic of dimension $\dim V^\perp$ and $(V', q')$ is nonsingular. Let $m = i_W(V, q)$. By Proposition 2.1.9 and Proposition 2.1.8 we can write $(V', q') \cong (V_h, q_h) \perp (V_a, q_a)$ where $(V_h, q_h)$ is hyperbolic of dimension $2m$. $(V_a, q_a)$ must be nonsingular, and in fact it is anisotropic, since otherwise one could find a totally isotropic subspace of $(V', q')$ of dimension $m + 1$, contradicting the choice of $m$. This concludes the existence part of the proof.

For the uniqueness, assume that

$$(V, q) \cong (V_t, q_t) \perp (V_h, q_h) \perp (V_a, q_a) \cong (V_t', q_t') \perp (V_h', q_h') \perp (V_a', q_a')$$

where $(V_t', q_t')$ is totally singular, $(V_h', q_h')$ is hyperbolic, and $(V_a', q_a')$ is anisotropic. Since $(V_t', q_t')$ is totally isotropic and $(V_h', q_h') \perp (V_a', q_a')$ is nonsingular, we must have

$$\dim V_t' = \dim V^\perp = \dim V_t.$$

Since $(V_t, q_t)$ and $(V_t', q_t')$ are totally isotropic of the same dimension, they must be isometric. By Theorem 2.2.2 we obtain that $(V_h, q_a) \perp (V_a, q_a) \cong (V_h', q_h') \perp (V_a', q_a')$. Similarly, since $(V_h', q_h')$ is hyperbolic and $(V_a', q_a')$ is anisotropic, we must have $\dim V_h' = 2m = \dim V_h$, whereby $(V_h, q_h)$ and $(V_h', q_h')$ are hyperbolic forms of the same dimension and hence isometric. Finally, applying Theorem 2.2.2 again, we obtain $(V_a, q_a) \cong (V_a', q_a')$. □

2.3. **Exercises.**

(1) Complete the proof of Corollary 2.1.7.
(2) Let $(V, q)$ be a quadratic space, and consider for $v \in V$ with $q(v) \neq 0$ the map
$$\tau_v : V \to V : w \mapsto w - \frac{\mathfrak{b}_q(w, v)}{q(v)} v.$$
Show the following for any $v \in V$ with $q(v) \neq 0$:
   (a) $\tau_v$ is an isometry $(V, q) \to (V, q)$,
   (b) $\tau_v(v) = -v$, and for $w \in v^\perp$ we have $\tau_v(w) = w$,
   (c) If $w \in V$ is such that $q(v) = q(w)$ and $q(v-w) \neq 0$, then $\tau_{v-w}(v) = w$.
(3) Show that the following are equivalent for a field $K$ with $\mathrm{char}(K) \neq 2$:
   (a) Any two nonsingular quadratic spaces over $K$ of the same dimension are isometric.
   (b) Every element of $K$ is a square.
(4) Let $(V, q)$ be a nonsingular isotropic space. Show that $V$ has a basis consisting of isotropic vectors.
(5) Let $(V, q)$ be a nonsingular quadratic space, set $n = \dim V$ and $m = i_W(q)$. Show that every subform of $(V, q)$ of dimension greater than $n - m$ is isotropic.
(6) Assume $\mathrm{char}(K) \neq 2$ and let $(V_1, q_1)$ and $(V_2, q_2)$ be nonsingular quadratic spaces over $K$. Show that $(V_2, q_2)$ is a subform of $(V_1, q_1)$ if and only if $i_W((V_1, q_1) \perp (V_2, -q_2)) \geq \dim V_2$.
(7) Let $K = \mathbb{F}_2$, the field with two elements. Consider the quadratic form
$$q : K^2 \to K : (x, y) \mapsto x^2 + xy + y^2.$$
Show that $q \perp \langle 1 \rangle_K \cong \mathbb{H}_K \perp \langle 1 \rangle_K$, but $q \ncong \mathbb{H}_K$. Conclude that Theorem 2.2.2 does not hold as stated without the assumption $\mathrm{char}(K) \neq 2$.

## 3. Lecture 3

3.1. **Tensor products of symmetric bilinear spaces.** In this section, we will define the tensor product (sometimes called Kronecker product) of two symmetric bilinear spaces. First, we define the tensor product of two $K$-vector spaces.

Let $V$ and $W$ be $K$-vector spaces. Denote by $K^{(V \times W)}$ the free $K$-vector space over the set $V \times W$. That is, for each $(v, w) \in V \times W$ we fix an element $e_{(v,w)} \in K^{(V \times W)}$, and then $\{e_{(v,w)} \mid (v, w) \in V \times W\}$ is a basis of $K^{(V \times W)}$. Let $A$ be the subspace of $K^{(V \times W)}$ generated by elements of the form
$$e_{(v+av',w)} - e_{(v,w)} - ae_{(v',w)} \quad \text{or} \quad e_{(v,w+aw')} - a_{(v,w)} - ae_{(v,w')}$$
for $v, v' \in V$, $w, w' \in W$ and $a \in K$.

3.1.1. **Definition.** With the notations from above, we call the quotient space $K^{(V \times W)}/A$ the *tensor product of $V$ and $W$*, which we denote by $V \otimes W$ - or $V \otimes_K W$ if we want to stress the underlying field. For $v \in V$ and $w \in W$ we

denote by $v \otimes w$ the class of $e_{(v,w)}$ in this quotient space. We call an element of $V \otimes_K W$ of the form $v \otimes w$ for $v \in V$ and $w \in W$ an *elementary tensor*.

3.1.2. *Remark.* Be careful! Not every element of $V \otimes W$ is of the form $v \otimes w$ for $v \in V$ and $w \in W$, i.e. not every element of $V \otimes W$ is an elementary tensor. However, every element of $V \otimes W$ is a sum of elementary tensors - although this decomposition is not unique.

The tensor product $V \otimes W$ is best understood through the following fundamental property.

3.1.3. **Proposition** (Universal property of tensor products)**.** *Let $V$ and $W$ be $K$-vector spaces. The map $V \times W \to V \otimes W : (v, w) \mapsto v \otimes w$ is a bilinear map, and its image generates $V \otimes W$.*

*For any $K$-vector space $U$ and any bilinear map $B : V \times W \to U$, there exists a unique linear map $\overline{B} : V \otimes W \to U$ such that $B(v, w) = \overline{B}(v \otimes w)$ for all $v \in V$, $w \in W$.*

*Proof.* The bilinearity of the map $V \times W \to V \otimes W : (v, w) \mapsto v \otimes w$ follows from the construction of $V \otimes W$: we have for any $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $a, b \in K$ that

$$(v_1 + av_2) \otimes (w_1 + bw_2) = (v_1 \otimes w_1) + a(v_2 \otimes w_1) + b(v_1 \otimes w_2) + ab(v_2 \otimes w_2).$$

The image of the map consists of elementary tensors, which by construction generate $V \otimes W$.

Now consider any bilinear map $B : V \times W \to U$. Since $\{e_{(v,w)} \mid (v, w) \in V \times W\}$ form a basis of $K^{(V \times W)}$, there is a unique $K$-linear map $\hat{B} : K^{(V \times W)} \to U$ mapping $e_{(v,w)}$ to $B(v, w)$ for $(v, w) \in V \times W$. By the bilinearity of $B$, we compute that for $v_1, v_2 \in V$ and $w_1, w_2 \in W$ we have

$$\begin{aligned} \hat{B}(e_{(v_1+av_2, w_1+bw_2)}) &= B(v_1 + av_2, w_1 + bw_2) \\ &= B(v_1, w_1) + aB(v_2, w_1) + bB(v_1, w_2) + abB(v_2, w_2) \\ &= \hat{B}(e_{(v_1,w_1)} + ae_{(v_2,w_1)} + be_{(v_1,w_2)} + abe_{(v_2,w_2)}). \end{aligned}$$

As such, $\mathrm{Ker}(\hat{B})$ contains all elements given as generators for the subspace $A$ of $K^{(V \times W)}$, whereby $A \subseteq \mathrm{Ker}(\hat{B})$. Recalling that $V \otimes W = K^{(V \times W)}/A$, we conclude that there exists a unique linear map $\overline{B} : V \otimes W \to U$ such that $\overline{B}(v \otimes w) = \hat{B}(e_{(v,w)}) = B(v, w)$ for all $(v, w) \in V \times W$. □

3.1.4. **Proposition.** *Let $U$, $V$ and $W$ be $K$-vector spaces. The tensor product satisfies the following properties.*

1. *There is a unique $K$-isomorphism $V \otimes W \to W \otimes V$ such that $v \otimes w \mapsto w \otimes v$ for $v \in V$ and $w \in W$.*
2. *There is a unique $K$-isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ such that $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ for $u \in U$, $v \in V$ and $w \in W$.*

(3) *There is a unique $K$-isomorphism $(U \times V) \otimes W \to (U \otimes W) \times (V \otimes W)$ such that $((u \otimes v), w) \mapsto ((u \otimes w), (v \otimes w))$ for $u \in U$, $v \in V$ and $w \in W$.*
(4) *Let $\mathfrak{B}_V$ and $\mathfrak{B}_W$ be bases for $V$ and $W$ respectively. Then*

$$\{v \otimes w \mid v \in \mathfrak{B}_B, w \in \mathfrak{B}_W\}$$

*is a basis for $V \otimes W$. In particular,* $\dim(V \otimes W) = \dim(V)\dim(W)$.

*Proof.* Each of these can be proven by making use of Proposition 3.1.3. □

We can now define the tensor product of symmetric bilinear spaces.

3.1.5. **Proposition.** *Let $(V_1, B_1)$ and $(V_2, B_2)$ be symmetric bilinear spaces. There exists a unique $K$-bilinear form $B$ on $V_1 \otimes V_2$ such that*

$$B(v_1 \otimes v_2, w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2)$$

*for all $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$.*

*Proof.* The uniqueness is clear, since $V \otimes W$ is generated by elementary tensors; furthermore, since such a bilinear map would by definition be symmetric on elementary tensors, it is automatically symmetric. It thus suffices to show the existence of such a bilinear map $B$.

Consider first for $(v_1, v_2) \in V_1 \times V_2$ the map

$$V_1 \times V_2 \to K : (w_1, w_2) \mapsto B_1(v_1, w_1) \cdot B_2(v_2, w_2).$$

This map is bilinear, hence by Proposition 3.1.3 induces a linear map $B_{(v_1,v_2)} : V_1 \otimes V_2 \to K$ such that $B_{(v_1,v_2)}(w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2)$ for $w_1 \in V_1$ and $w_2 \in V_2$. The map

$$B^* : V_1 \times V_2 \to (V_1 \otimes V_2)^* : (v_1, v_2) \mapsto B_{(v_1,v_2)}$$

is also bilinear, hence, again by Proposition 3.1.3, it induces a linear map $\overline{B^*} : V_1 \otimes V_2 \to (V_1 \otimes V_2)^*$ such that $\overline{B^*}(v_1 \otimes v_2) = B_{(v_1,v_2)}$ for $(v_1, v_2) \in V_1 \times V_2$.

Finally, consider the bilinear map

$$B : (V_1 \otimes V_2) \times (V_1 \otimes V_2) : (\alpha, \beta) \mapsto \overline{B^*}(\alpha)(\beta).$$

We compute that, for $v_1, w_1 \in V_1$ and $v_2, w_2 \in V_2$, we have

$$\begin{aligned} B(v_1 \otimes v_2, w_1 \otimes w_2) &= \overline{B^*}(v_1 \otimes v_2)(w_1 \otimes w_2) \\ &= B_{(v_1,v_2)}(w_1 \otimes w_2) = B_1(v_1, w_1) \cdot B_2(v_2, w_2). \end{aligned}$$

Hence, $B$ is as desired. □

3.1.6. **Definition.** Given symmetric bilinear spaces $(V_1, B_1)$ and $(V_2, B_2)$, we call the symmetric bilinear space constructed in Proposition 3.1.5 the *tensor product* of $(V_1, B_1)$ and $(V_2, B_2)$. We denote it by $(V_1 \otimes V_2, B_1 \otimes B_2)$.

Over fields of characteristic different from 2, we will also consider the tensor product of quadratic spaces; this is by definition the quadratic space corresponding to the tensor product of the underlying symmetric bilinear spaces, see Proposition 1.1.8. That is, for quadratic spaces $(V_1, q_1)$ and $(V_2, q_2)$, we define

$$q_1 \otimes q_2 : V_1 \otimes V_2 \to K : \alpha \mapsto \frac{(B_{q_1} \otimes B_{q_2})(\alpha)}{4}.$$

In the following proposition stating some computation rules, in the interest of brevity, we represent a quadratic space just by its quadratic form.

3.1.7. **Proposition.** *Assume* $\mathrm{char}(K) \neq 2$. *For quadratic forms* $q_1, q_2, q_3$ *over* $K$ *we have*

$$\begin{aligned} q_1 \otimes q_2 &\cong q_2 \otimes q_1 \\ (q_1 \otimes q_2) \otimes q_3 &\cong q_1 \otimes (q_2 \otimes q_3) \\ (q_1 \perp q_2) \otimes q_3 &\cong (q_1 \otimes q_3) \perp (q_2 \otimes q_3) \end{aligned}$$

*Proof.* Each of these follows by checking that the isomorphism of vector spaces established in Proposition 3.1.4 induces isometries of quadratic (/symmetric bilinear) spaces. □

3.1.8. **Corollary.** *Let* $m, n \in \mathbb{N}$ *and let* $a_1, \ldots, a_m, b_1, \ldots, b_n \in K$. *We have*

$$\langle a_1, \ldots, a_m \rangle_K \otimes \langle b_1, \ldots b_n \rangle_K \cong \langle a_1 b_1, \ldots, a_i b_j, \ldots, a_m b_n \rangle_K$$

*Proof.* This follows by Proposition 3.1.7 and the easy observation that $\langle a \rangle_K \otimes \langle b \rangle_K \cong \langle ab \rangle_K$ for $a, b \in K$. □

3.1.9. **Corollary.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V_1, q_1), (V_2, q_2)$ *be nonsingular quadratic spaces. Then* $(V_1 \otimes V_2, q_1 \otimes q_2)$ *is nonsingular.*

*Proof.* By Corollary 1.2.10 and Proposition 1.2.8 both $(V_1, q_1)$ and $(V_2, q_2)$ are isometric to diagonal forms where all entries are non-zero. By Corollary 3.1.8 the same holds for $(V_1 \otimes V_2, q_1 \otimes q_2)$, whence this form is also nonsingular. □

3.1.10. **Corollary.** *Assume* $\mathrm{char}(K) \neq 2$. *Let* $(V, q)$ *be a nonsingular quadratic space. Then* $(V, q) \otimes \mathbb{H}_K$ *is hyperbolic.*

*Proof.* We have $\mathbb{H}_K \cong \langle 1, -1 \rangle_K$ (see Example 1.1.7). Hence, by Proposition 3.1.7,

$$(V, q) \otimes \mathbb{H}_K \cong (V, q) \otimes \langle 1, -1 \rangle_K \cong (V, q) \perp (V, -q)$$

which is hyperbolic by Proposition 2.1.12. □

3.2. **Exercises.**

(1) Prove Proposition 3.1.4 and Proposition 3.1.7.

## 4. Lecture 4

4.1. **Witt equivalence and the Witt ring.** Throughout this subsection, all quadratic spaces are consider over a fixed field $K$, and we assume $\mathrm{char}(K) \neq 2$.

4.1.1. **Definition.** Let $(V^{(1)}, q^{(1)})$ and $(V^{(2)}, q^{(2)})$ be quadratic spaces. In view of Theorem 2.2.3 we may write

$$(V^{(1)}, q^{(1)}) \cong (V_t^{(1)}, q_t^{(1)}) \perp (V_h^{(1)}, q_h^{(1)}) \perp (V_a^{(1)}, q_a^{(1)})$$
$$(V^{(2)}, q^{(2)}) \cong (V_t^{(2)}, q_t^{(2)}) \perp (V_h^{(2)}, q_h^{(2)}) \perp (V_a^{(2)}, q_a^{(2)})$$

where

- $(V_t^{(1)}, q_t^{(1)})$ and $(V_t^{(2)}, q_t^{(2)})$ are totally isotropic,
- $(V_h^{(1)}, q_h^{(1)})$ and $(V_h^{(2)}, q_h^{(2)})$ are hyperbolic (or zero),
- $(V_a^{(1)}, q_a^{(1)})$ and $(V_a^{(2)}, q_a^{(2)})$ are anisotropic.

We say that $(V^{(1)}, q^{(1)})$ and $(V^{(2)}, q^{(2)})$ are *Witt equivalent* if $\dim V_t^{(1)} = \dim V_t^{(2)}$ and $(V_a^{(1)}, q_a^{(1)}) \cong (V_a^{(2)}, q_a^{(2)})$. We denote this by $(V^{(1)}, q^{(1)}) \equiv (V^{(2)}, q^{(2)})$.

Theorem 2.2.3 yields that this is indeed a well-defined equivalence relation on the class of quadratic spaces over $K$. One has the following easy observations.

4.1.2. **Proposition.** *Let $(V_1, q_1)$ and $(V_2, q_2)$ be quadratic spaces.*

*(1) $(V_1, q_1) \cong (V_2, q_2)$ if and only if $(V_1, q_1) \equiv (V_2, q_2)$ and $\dim V_1 = \dim V_2$.*
*(2) In every Witt equivalence class, there is up to isometry a unique anisotropic quadratic space. In particular, if $(V_1, q_1) \equiv (V_2, q_2)$ and both are anisotropic, then $(V_1, q_1) \cong (V_2, q_2)$.*

For a quadratic space $(V, q)$, let us denote by $[(V, q)]$ its Witt equivalence class. Let us denote by $W(K)$ the set of equivalence classes of nonsingular quadratic spaces up to Witt equivalence. We will see now that this set can naturally be given a ring structure.

4.1.3. **Theorem.** *The rules*

$$\perp\colon W(K) \times W(K) \to W(K) : ([(V_1, q_1)], [(V_2, q_2)]) \to [(V_1 \times V_2, q_1 \perp q_2)] \text{ and}$$
$$\otimes : W(K) \times W(K) \to W(K) : ([(V_1, q_1)], [(V_2, q_2)]) \to [(V_1 \otimes V_2, q_1 \otimes q_2)]$$

*are well-defined binary operations on $W(K)$, making $W(K)$ into a commutative ring with addition $\perp$ and multiplication $\otimes$. The class of the zero-dimensional form $[\langle\rangle_K]$ is a neutral element for $\perp$, and $[\langle 1\rangle_K]$ is a neutral element for $\otimes$. Given $[(V, q)] \in W(K)$, its additive inverse is given by $[(V, -q)]$.*

*Proof.* We first prove the well-definedness. That is, assume $(V_1, q_1), (V_1', q_1'), (V_2, q_2), (V_2, q_2')$ are such that $(V_1, q_1) \equiv (V_1', q_1')$ and $(V_2, q_2) \equiv (V_2', q_2')$, we need to show that $(V_1 \times V_2, q_1 \perp q_2) \equiv (V_1' \times V_2', q_1' \perp q_2')$ and $(V_1 \otimes V_2, q_1 \otimes q_2) \equiv (V_1' \otimes V_2', q_1' \otimes q_2')$. Since nonsingular quadratic spaces are Witt equivalent if and only if they are

isometric after adding a number of copies of the hyperbolic plane to one of them, it suffices to consider the case $(V_1, q_1) = (V_1', q_1')$ and $(V_2', q_2') = (V_2, q_2) \perp \mathbb{H}_K$.

We compute that

$$(V_1, q_1) \perp ((V_2, q_2) \perp \mathbb{H}_K) \cong ((V_1, q_1) \perp (V_2, q_2)) \perp \mathbb{H}_K \equiv (V_1, q_1) \perp (V_2, q_2)$$

as desired. Similarly

$$\begin{aligned}(V_1, q_1) \otimes ((V_2, q_2) \perp \mathbb{H}_K) &\cong (V_1, q_1) \otimes (V_2, q_2) \perp (V_1, q_1) \otimes \mathbb{H}_K \\ &\cong (V_1, q_1) \otimes (V_2, q_2) \perp \dim(V_1) \times \mathbb{H}_K \\ &\equiv (V_1, q_1) \otimes (V_2, q_2)\end{aligned}$$

where the second isometry follows from Corollary 3.1.10. This shows that the operations $\perp$ and $\otimes$ are well-defined on $W(K) \times W(K)$. The associativity, commutativity and distributivity are immediate from the corresponding properties for $\perp$ and $\otimes$ on quadratic spaces. That $[\langle\rangle_K]$ is a neutral element for $\perp$ and $[\langle 1\rangle_K]$ is a neutral element for $\otimes$, is readily verified. Finally, that $[(V, -q)] = -[(V, q)]$ is a reformulation of Proposition 2.1.12. □

4.1.4. **Definition.** The set $W(K)$ endowed with the ring structure described in Theorem 4.1.3 is called the *Witt ring of* $K$.

4.1.5. **Proposition.** *$W(K)$ has a unique ideal of index 2, which is given by*

$$I(K) = \{[(V, q)] \mid \dim V \text{ even}\}.$$

*Proof.* Observe that, if two nonsingular quadratic spaces are Witt equivalent, then their dimensions differ by an even number. In particular, if one of them has even dimension, then the other too. It is easy to see that $I(K)$ is an ideal. Furthermore, it has index 2, because for any quadratic space $(V, q)$, either $[(V, q)] \in I(K)$, or $[(V, q) \perp \langle 1\rangle_K] \in I(K)$.

Assume that $J$ is another ideal of $W(K)$ of index 2. For $a, b \in K^\times$, we have that $[\langle a\rangle_K], [\langle b\rangle_K] \in W(K)^\times \subseteq W(K) \setminus J$, hence $[\langle a, b\rangle_K] \in J$. In view of Corollary 1.2.10, we conclude that $J$ contains all classes of quadratic spaces of even dimension, hence $I(K) \subseteq J$. But then $I(K) = J$. □

4.1.6. **Definition.** The ideal $I(K)$ described in Proposition 4.1.5 is called the *fundamental ideal of* $W(K)$.

4.1.7. *Remark.* Over a field $K$ with $\mathrm{char}(K) = 2$, the situation is more subtle. There are natural operations $\perp$ and $\otimes$ on the class of *symmetric bilinear spaces* over $K$, and this allows one to define a Witt ring $W(K)$ of nonsingular symmetric bilinear forms. On the class of quadratic spaces over $K$ there is no natural notion of tensor product, but one can still define a group operation $\perp$, and one obtains a different object from $W(K)$: the quadratic Witt group $I_q(K)$. While $I_q(K)$ is not a ring, it does carry an action by $W(K)$: $I_q(K)$ is a $W(K)$-module. See [EKM08, Sections 2, 8] for more on this.

4.2. **Determinants and discriminants.** We briefly introduce the concept of the determinant of a symmetric bilinear form. This allows us to simplify certain computations with small-dimensional quadratic forms.

4.2.1. **Proposition.** *Let $(V_1, B_1)$ and $(V_2, B_2)$ be isometric symmetric bilinear spaces with bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Then* $\det(M_{\mathfrak{B}_1}(B_1)) \equiv \det(M_{\mathfrak{B}_2}(B_2)) \bmod K^{\times 2}$.

*Proof.* It suffices to consider the case $V_1 = V_2 = K^n$ for $n = \dim(V_1)$, and where $\mathfrak{B}_1$ is the canonical basis $\{e_1, \dots, e_n\}$. Let $C \in \mathbb{M}_n(K)^\times$ be the base change matrix between $\mathfrak{B}_1$ and $\mathfrak{B}_2$, i.e. such that $\mathfrak{B}_2 = \{Ce_1, \dots, Ce_n\}$. We see that for column vectors $v, w \in K^n$ we have

$$v^T C^T M_{\mathfrak{B}_1}(B) C w = B(Cv, Cw) = v^T M_{\mathfrak{B}_2}(B) w$$

whence $M_{\mathfrak{B}_2}(B) = C^T M_{\mathfrak{B}_1}(B) C$ and hence $\det(M_{\mathfrak{B}_2}(B)) = \det(M_{\mathfrak{B}_1}(B)) \det(C)^2 \equiv \det(M_{\mathfrak{B}_2}(B)) \bmod K^{\times 2}$ as desired. □

4.2.2. **Definition.** For a nonsingular symmetric bilinear space $(V, B)$, we define the *determinant* of $(V, B)$ (or simply of $B$) to be the equivalence class of $\det(M_{\mathfrak{B}}(B))$ in $K^\times / K^{\times 2}$, where $\mathfrak{B}$ is any basis of $V$. We denote it simply by $\det(V, B)$.

If $\operatorname{char}(K) \neq 2$ and $(V, q)$ is a quadratic space over $K$, we define its determinant as the determinant of $(V, \frac{\mathfrak{b}_q}{2})$.

For the rest of this subsection, assume that all quadratic spaces are considered over a field $K$ with $\operatorname{char}(K) \neq 2$.

4.2.3. **Proposition.** *We have the following properties.*

1. *For nonsingular quadratic spaces $(V_1, q_1)$ and $(V_2, q_2)$ we have* $\det((V_1, q_1) \perp (V_2, q_2)) = \det(V_1, q_1) \cdot \det(V_2, q_2)$.
2. *For $a_1, \dots, a_n \in K^\times$ we have* $\det(\langle a_1, \dots, a_n \rangle_K) \equiv a_1 \cdots a_n \bmod K^{\times 2}$.
3. $\det(\mathbb{H}_K) \equiv -1 \bmod K^{\times 2}$.

*Proof.* These can be verified easily via the definition. □

As announced, determinants are a useful invariant of quadratic spaces which can help to simplify certain calculations. We give an important example.

4.2.4. **Proposition.** *Let $a, b, c \in K^\times$ and assume that $c \in D_K(\langle a, b \rangle_K)$. Then $\langle a, b \rangle_K \cong \langle c, abc \rangle_K$.*

*Proof.* By Proposition 1.2.9 we have $\langle a, b \rangle_K \cong \langle c, d \rangle_K$ for some $d \in K^\times$. But since $cd \equiv \det(\langle c, d \rangle_K) \equiv \det(\langle a, b \rangle_K) \equiv ab \bmod K^\times$, we must have $d \equiv abc \bmod K^{\times 2}$, whereby $\langle c, d \rangle_K \cong \langle c, abc \rangle_K$. This concludes the proof. □

4.3. **Multiplicative forms.** When $(V, q)$ is a quadratic space, the set $D_K(q)$ of elements of $K^\times$ represented by $q$ is in general just a subset of $K^\times$. We now consider a class of quadratic forms where this is in fact a subgroup.

4.3.1. **Definition.** Let $(V,q)$ be a quadratic space. We call the set

$$G_K(q) = \{a \in K^\times \mid (V,q) \cong (V,aq)\}$$

the set of *similarity factors of* $(V,q)$.

By a *multiplicative form over* $K$ (some books use the term *round form*) we mean a nonsingular quadratic form $q$ for which $D_K(q) = G_K(q)$.

4.3.2. *Example.* Every hyperbolic form is multiplicative, see Corollary 2.1.10.

4.3.3. **Proposition.** *Let $(V,q)$ be a nonsingular quadratic space over $K$.*

*(1) $G_K(q)$ is a subgroup of $K^\times$ that contains $K^{\times 2}$.*
*(2) $G_K(q) \cdot D_K(q) = D_K(q)$.*

*Proof.* The first part is clear. For the second part, consider $a \in G_K(q)$ and $d \in D_K(q)$, then $ad \in D_K(aq) = D_K(q)$. □

For the rest of this subsection, assume $\mathrm{char}(K) \neq 2$.

4.3.4. **Theorem** (Witt)**.** *Let $q$ be a multiplicative form over $K$ and $a \in K^\times$. Then the form $\langle 1, a\rangle_K \otimes q$ is multiplicative. Moreover, if $q$ is anisotropic, then $\langle 1, a\rangle_K \otimes q$ is either anisotropic or hyperbolic.*

*Proof.* Let $q' = \langle 1, a\rangle_K \otimes q$. We have $1 \in G_K(q) = D_K(q) \subseteq D_K(q')$ and hence $G_K(q') \subseteq D_K(q')$ by Proposition 4.3.3. Further, observe that $D_K(q) \cup aD_K(q) = G_K(q) \cup aG_K(q) \subseteq G_K(q')$. Now consider $c \in D_K(q') \setminus (D_K(q) \cup aD_K(q))$ arbitrary. Then there exist $s,t \in D_K(q) = G_K(q)$ such that $c \in D_K(\langle s, at\rangle_K)$. By Proposition 4.2.4 it follows that $\langle s, at\rangle_K \cong \langle c, acst\rangle_K$. We now compute that

$$\begin{aligned} q' &\cong q \perp aq \cong sq \perp atq \cong \langle s, at\rangle_K \otimes q \cong \langle c, acst\rangle_K \otimes q \\ &\cong cq \perp acstq \cong cq \perp acq \cong cq' \end{aligned}$$

whereby $c \in G_K(q')$. Since $c \in D_K(q')$ was chosen arbitrarily, we conclude that $q'$ is multiplicative.

For the second part, assume that $q$ is anisotropic and $q'$ is isotropic. Then there exist $s,t \in D_K(q) = G_K(q)$ with $\langle s, at\rangle_K \cong \mathbb{H}_K$. We compute that

$$q' \cong q \perp aq \cong sq \perp atq \cong \langle s, at\rangle_K \otimes q \cong \mathbb{H}_K \otimes q$$

which is hyperbolic by Corollary 3.1.10. □

4.3.5. **Definition.** For $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in K^\times$, we use the notation

$$\langle\langle a_1, \ldots, a_n\rangle\rangle_K = \langle 1, -a_1\rangle \otimes \ldots \otimes \langle 1, -a_n\rangle_K.$$

In particular, $\langle\langle\rangle\rangle_K = \langle 1\rangle_K$, and $\langle\langle a_1\rangle\rangle_K = \langle 1, -a_1\rangle_K$. We call a form which is isometric to $\langle\langle a_1, \ldots, a_n\rangle\rangle_K$ for some $a_1, \ldots, a_n \in K^\times$ an *$n$-fold Pfister form*.

4.3.6. **Theorem** (Pfister)**.** *Let $q$ be a Pfister form over $K$. Then $q$ is multiplicative, and either anisotropic or hyperbolic.*

*Proof.* Assume that $q$ is an $n$-fold Pfister form; we proceed by induction on $n$. For $n = 0$ we have $q \cong \langle 1 \rangle_K$; this form is anisotropic and $D_K(q) = K^{\times 2} = G_K(q)$. Assume now $n > 0$. We have that $q \cong \langle 1, -a \rangle_K \otimes q'$ for some $(n-1)$-fold Pfister form $q'$ over $K$. If $q'$ is anisotropic, then by induction hypothesis, $q'$ is multiplicative, and by Theorem 4.3.4 also $q$ is multiplicative and either anisotropic or hyperbolic. If $q'$ is isotropic, then by induction hypothesis it is hyperbolic, and then also $q$ is hyperbolic by Corollary 3.1.10. □

We mention the following partial converse to Theorem 4.3.6, the proof of which is outside the scope of this course. We will not use this result in the sequel. For a quadratic form $q$ over $K$ and a field extension $L/K$, we denote by $q_L$ the quadratic form over $L$ obtained by extending scalars from $K$ to $L$ (we will see a formal definition later, see **??**).

4.3.7. **Theorem** (Pfister)**.** *Let $q$ be an anisotropic quadratic form over $K$. The following are equivalent.*

*(1) $q$ is a Pfister form,*
*(2) $D_L(q_L)$ is a subgroup of $L^\times$ for every field extension $L/K$,*
*(3) $1 \in D_K(q)$ and for every field extension $L/K$ we have that $q_L$ is either anisotropic or hyperbolic.*

*Proof.* See [EKM08, Theorem 23.2 and Corollary 23.4]. □

4.3.8. *Remark.* Over a field $K$ of characteristic 2, one can define a notion of Pfister form both for bilinear forms and for quadratic forms. As usual, we refer to [EKM08, Sections 7 and 9] for a characteristic-free exposition. An example of a 1-fold quadratic Pfister form is given by $X^2+XY+aY^2$ for $a \in K$. These quadratic Pfister forms still satisfy the properties of Theorem 4.3.6 in characteristic 2.

4.4. **Exercises.** In all exercises, assume $K$ is a field with $\mathrm{char}(K) \neq 2$.

(1) Compute the Witt ring of $\mathbb{C}$ and $\mathbb{R}$.
(2) Let $K$ be finite. Show the following:
   (a) $|K^\times / K^{\times 2}| = 2$,
   (b) Every nonsingular 2-dimensional quadratic form over $K$ is universal.
   (c) Assume $d \in K^\times \setminus K^{\times 2}$. Every anisotropic quadratic form over $K$ is isometric to precisely one of the following forms:
   $$\langle\rangle_K \qquad \langle 1 \rangle_K \qquad \langle d \rangle_K \qquad \langle 1, -d \rangle_K$$
   (d) If $|K| \equiv 1 \bmod 4$, then $-1 \in K^{\times 2}$ and $WK \cong (\mathbb{Z}/2\mathbb{Z})[T]/(T^2+1)$.
   (e) If $|K| \equiv 3 \bmod 4$, then $-1 \notin K^{\times 2}$ and $WK \cong \mathbb{Z}/4\mathbb{Z}$.
(3) Show that for $a, b \in K^\times$ and a Pfister form $q$ over $K$ we have $\langle\langle a \rangle\rangle_K \otimes q \cong \langle\langle b \rangle\rangle_K \otimes q$ if and only if $ab \in D_K(q)$.
(4) Let $q$ be a 4-dimensional nonsingular quadratic form over $K$ with $1 \in D_K(q)$ and $\det(q) \equiv 1 \bmod K^{\times 2}$. Show that $q$ is a Pfister form.
(5) Let $q$ be a universal 3-dimensional quadratic form over $K$. Show that $q$ is isotropic.

(6) Show that $D_{\mathbb{Q}}(\langle 1,1\rangle_{\mathbb{Q}})$ is a subgroup of $\mathbb{Q}^{\times}$. Is the same true for $D_{\mathbb{Q}}(\langle 1,1,1\rangle_{\mathbb{Q}})$?
(7) Give an example of an anisotropic quadratic form which is multiplicative but not a Pfister form.
(8) Let $n \in \mathbb{N}$ and suppose that $-1$ is a sum of $2^{n+1}-1$ squares in $K$. Show that $-1$ is a sum of $2^n$ squares in $K$.

# Index

## References

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