I. CODE

The script *meanField.py* computes the transformation matrix $\mathcal{J}^{-1}$ which defines the transformation $\tilde{\psi} = \mathcal{J}\psi$ where $\psi^\dagger = (a_1^\dagger, a_2^\dagger \ldots a_N^\dagger, a_1, a_2 \ldots a_N)$ are the Holstein-Primakoff bosons and $\tilde{\psi}^\dagger = (\gamma_1^\dagger, \gamma_2^\dagger \ldots \gamma_N^\dagger, \gamma_1, \gamma_2 \ldots \gamma_N)$ the magnons.

You can pass the size of the system when running it

```
python3 meanField.py Lx Ly stopRatio(optional=1)
```

with Lx, Ly the size in x, y directions and *stopRatio* a float between 0 and 1(default) which defines the Hamiltonian parameters: 0 is pure staggered field and 1 is pure XX model.

NB: the script computes $\mathcal{J}^{-1}$, look at equation (2) to get $\mathcal{J}$.

NB2: whenever we tilt the quantization axis ($\theta \neq 0$) the spectrum becomes gapless. This gives a singular behavior when computing the zero-energy mode (see derivation). This is a inherent problem of the mean-field calculation. I am removing this mode at the end of the calculation

```
if theta!=0:          # Remove the 0-energy eigenvector
    U_[:,0] *= 0
    V_[:,0] *= 0
```

II. DETAILS OF CALCULATION

When diagonalizing the Hamiltonian in real space we do not have access to the analytical solution. We then have to resort to a numerical diagonalization, paying attention to respect the commutation relations of the resulting bosons diagonalizing the Hamiltonian. The case we have at hand was described by Colpa [1] in Section 4, as the case of a general boson Hamiltonian. Since here we only have real coefficients in the Hamiltonian parameters we resort to the method described in Section 5, which we recap here in order to have a closed description.

The Hamiltonian can be separated into

$$\mathcal{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \tag{1}$$

and we call the transformation matrix $\tilde{\psi} = \mathcal{J}\psi$ where $\psi^\dagger = (a_1^\dagger, a_2^\dagger \ldots a_N^\dagger, a_1, a_2 \ldots a_N)$ are the Holstein-Primakoff bosons and $\tilde{\psi}^\dagger = (\gamma_1^\dagger, \gamma_2^\dagger \ldots \gamma_N^\dagger, \gamma_1, \gamma_2 \ldots \gamma_N)$ the magnons. This transformation matrix $\mathcal{J}$ performs a so-called para-unitary transformation on the system. We further divide into

$$\mathcal{J}^{-1} = \begin{pmatrix} U & V \\ V & U \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} U^\dagger & -V^\dagger \\ -V^\dagger & U^\dagger \end{pmatrix} \tag{2}$$

The form of $\mathcal{J}$ ensures that the bosons γ respect canonical commutation relations. The transformation can be written as

$$\left(\mathcal{J}^\dagger\right)^{-1} \mathcal{H} \mathcal{J}^{-1} = \mathrm{diag}(\omega_1, \ldots \omega_{N_s}, \omega_1, \ldots \omega_{N_s}) \tag{3}$$

We start by writing the eigenvalue equation, using as u_r, v_r the columns of U and V

$$Au_r + Bv_r = u_r\omega_r \tag{4}$$
$$Bu_r + Av_r = -v_r\omega_r \tag{5}$$

Adding and subtracting we write in terms of $\phi_r = u_r + v_r$ and $\psi_r = u_r - v_r$

$$(A+B)\phi_r = \omega_r\psi_r \tag{6}$$
$$(A-B)\psi_r = \omega_r\phi_r \tag{7}$$

By multiplying by $A - B$ and $A + B$ we get the eigenvalue equations

$$(A-B)(A+B)\phi_r = \omega_r^2\phi_r \tag{8}$$
$$(A+B)(A-B)\psi_r = \omega_r^2\psi_r \tag{9}$$

We can then start our procedure by first performing a Cholesky decomposition of $A - B = K^\dagger K$, so that

$$K^\dagger \left[K(A+B)K^\dagger - \omega_r^2 \mathcal{I} \right] (K^\dagger)^{-1} \phi_r = 0 \tag{10}$$

We then diagonalize $K(A+B)K^\dagger$ which will have eigenvalues ω_r^2 and eigenvectors $\tilde{\chi}_r$. We normalize these eigenvectors in order to get $\chi_r^\dagger \chi_r = \omega_r^{-1}$ and finally obtain

$$\phi_r = K^\dagger \chi_r \tag{11}$$

$$\psi_r = \frac{1}{\omega_r}(A+B)K^\dagger \chi_r \tag{12}$$

From these we derive the matrices U and V and finally the transformation matrix $\mathcal{J}^{-1}$.

[1] J. H. P. Colpa, Diagonalization of the quadratic boson hamiltonian, Physica A: Statistical Mechanics and its Applications **93**, 327 (1978).